THE GREEN WAVE IN A GRID OF AVENUES

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Abstract

In this paper we follow the material presented by Marchi [12] in order to obtain a sufficient and necessary condition to have a green way in a city whose design is a grid. The known fact and where each street has two directions. Barceló [1] states that transportation sources must be strongly relay on a suitable mathematical model.

Key words: Traffic Lights, Green Wave, Synchronization, Avenue, Grid & Cities, Manhattan Problem.

1 Introduction

The called, "The Manhattan Problem" that was covered by Chapad, Duponts Lethi [4] has not attracted a lot of attention among the operation researchers and engineers in the last ten years. One is holding to the fact that if perhaps the engineering to apply the analytic results which generally are provided by more analytical analysis.

Overwhelming for a great sense of the real words which definitively is the important one in the applications and understanding the industrial, and material world.

In a recent paper Marchi [12] studied the problem of the synchronization of lights for the two way avenues.

Here in these notes we extend it to a rectangular arrangement of a city having a grid of avenues.

The literature of traffic flow and the obtaining of the result is vast. However, the real problem of it is extremely complex, and the advance of a real solution is in its beginnings. We have various cases of considering the problem of traffic flow in a city, that is to say when the traffic is dense, semi-dense, and fluid. A pioneer work beginning with the understanding of the problem was given by Charnes & Cooper in [3]. In the instance of the last fluid case, the systematic study of it for an avenue with two directions, has been studied by Marchi [12] very recently. The final overall output of the paper is rather deep, obtaining necessary and sufficient conditions for the existence and real computation of the green way. This is done by means of elementary mathematical methods.

As we wish to keep the material in a consistent way, and since it would need to have all the study material presented in [12], in a particular way. Thus we will repeat it for better consistency for the reader, and we need it for the purpose of this paper.

Many important studies have to be presented in traffic flow. We mention only some of them, with the purpose to mention from our opinion, the more important. When the cars

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or vehicles move in with constant speed in different, we have that this is valid for piece-wise linear case. Moreover, it is valid for movements described by a strictly increasing function. This is a trivial thing but it was pointed out by J. M. Alonso of Stockholm University.

With this precise aim, we consider that our contribution Marchi [12], is right in this way.

On the other hand the results obtained by Katsualos A. and M. Papagorgion [], Barceló et al. [1] and Quadrat et [7] and [8] all, belong to the class of important contributors in this area.

Deganzo in [5] has made a important study for general theory of transport and traffic operation. He presents several cases where the vehicles or platoons go in a straight line. Following his ideas we, in Marchi [12], obtain after some previous attempts a suitable approach for the better understanding, and real possible application in the world of traffic flow. These results are related to some previous works of consideration made by the team lead by Quadrat [7].

By the way Quadrat is one of the pioneers of the introduction of max-plus algebra which is very important in several subjects, for example in the study of delay of the train systems, as presented on Heidergott et al. [6]. Another important part of our study is to relate the theory of Heidergott [6] with ours. We expect shortly to allow to incorporate combination of both theories. Our intuition is that better results might be obtained. By the way the max-plus algebra is important in the tropical mathematics as well as for some studies of DNA and cladisties.

2 Previous Studies: About which decisions are to be made

Intelligent Transport Systems (ITS) which apply combined, advanced detection, communication and computer sciences, technology to traffic and transportation systems are important examples of such quantitative decisions support systems.

In concrete, in the report we consider the problem of a grid composed with a grid of two direction where each avenue have two directions and the cars or platoons represented by a point go either in a linear way, piece-wise straight line. It was pointed out by Alonso of Stockholm University, the all material presented here is general to be valid. If we replace the linearity by strictly increasing functions which we know they have inverse.

Another way that it is interesting to say is that by any suitable contraction or deformation, taking intact the topology of the system, as in algebra topology the theory is kept intact.

Further Barceló J. et al. [1] in a very recent article published in the SIAM News Nov. 2007, says that quantitative decision-making relies on appropriate mathematical models of the system about which decisions are to be made. Intelligent transport systems (ITS), which apply combined advanced detection, communication and computer sciences, technologies to traffic and transport systems, are important examples of such quantitative decisions support systems.

Other studies related with the mathematics of this paper are engaged to the developing of simple indices in order to characterize the two flow of vehicles which are moving on an avenue with two directions having traffic lights systems.

Finally we in [9] introduce a study called "The Manhattan Problem", called in this way by Chapad, Duponts & Luthi in [4].

To my knowledge there are very few studies that deal with the topics we have studies here (see Quadrat et al. [7]). However, in our paper we present one of the most general cases of co-ordination in a two way avenue in which the vehicles move in opposite directions.

We would like to emphasise from an elementary point of view that in the papers of Marchi [8] and Marchi & Tarazaga [9] we have obtained an implicit solution of this problem in one and two dimensions.

However, in this paper we write and illustrate the positive aspects of the LAUMAR systems.

3 The One-dimensional Model

Consider a rectangular arrangement having bi-directional traffic flow. Each avenue can be viewed as a segment of the straight line from 0 to $n+1$ avenues as is shown in the figure:

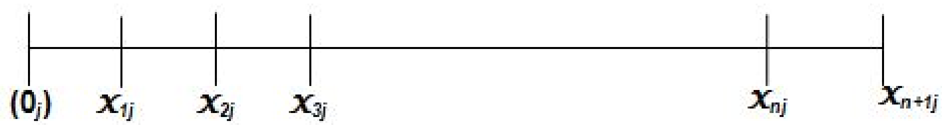


Figure 1

The first block in the avenue is described as having the starting point at 0_j and the end at x_{1j}.

The second block is described from x_{1j} to x_{2j} and so on, until the last once which stops at x_{n+1}. At x_{1j} it is not necessary to assume that the perpendicular street has a non zero width. Here we consider that this assumption is made to reduce the complexity of the theory, and the rather complex terminology which will appear in the last part. We assume that the perpendicular streets have zero width. At the end of this paper we discuss how one overcomes this trivial obstacle. Moreover the straight line might be considered curve as a manifold of one dimension.

Here we indicate in a similar way that for the other co-ordinates the introduction of the straight is analogous and is clear, and for this we do not include it explicitly.

Now the light at each intersection $x_{1j}, i = 1, 2, \quad m.j = 1, 2, ..., n$ where this is the amount of space in the manifold j which goes from the origin to the street j, can be introduced in different ways. One of them is a mechanism which at the intersections (ij) from the times a_{ijk} to a_{ijk+1} is with k odd a green light in the positive direction, which is to say from left to right. On the other hand for the perpendicular direction through the axis j, during such times the light is red in both directions, namely from north-south and south-north. On the reverse direction right to left it is also green. During the times from a_{ijk} to a_{ijk+1} with j even the light is red.

From a physical engineering and practical point of view, a light can be described by a vector.

$$v_{ij}(t) = \begin{cases} 1 & \text{if } a_{ijk} < t < a_{ijk+1} \qquad k \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

where t is the time variable. Here the 1 means that the vehicle in front sees the light, it green and 0 red, respectively.

In the Figure 1, remembering that each west-east street which is "parallel" to it, we consider that at each intersection (i, j) the crossing street has zero widht. This fact is not a restriction in our study since in the last part of this work we say how to handle the general situation and, we present the study in this way only for simplicity.

Furthermore a light can be viewed in the corresponding axis t as a partition of segments shown as in the next figure.

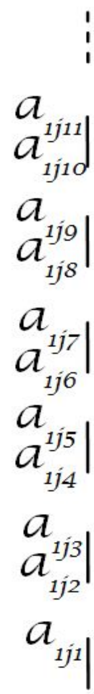


Figure 2

Where the i-fixed since it represents the avenue given by i.

Where we have solid lines indicating that the light is red and gaps where the light is green.

We have shown the first light at (ij).

In this way we can write in the coordinate system (x, t) the following arrangement with the different light. We can take the system (xy) in the same way as that considered in Deganzo [5].

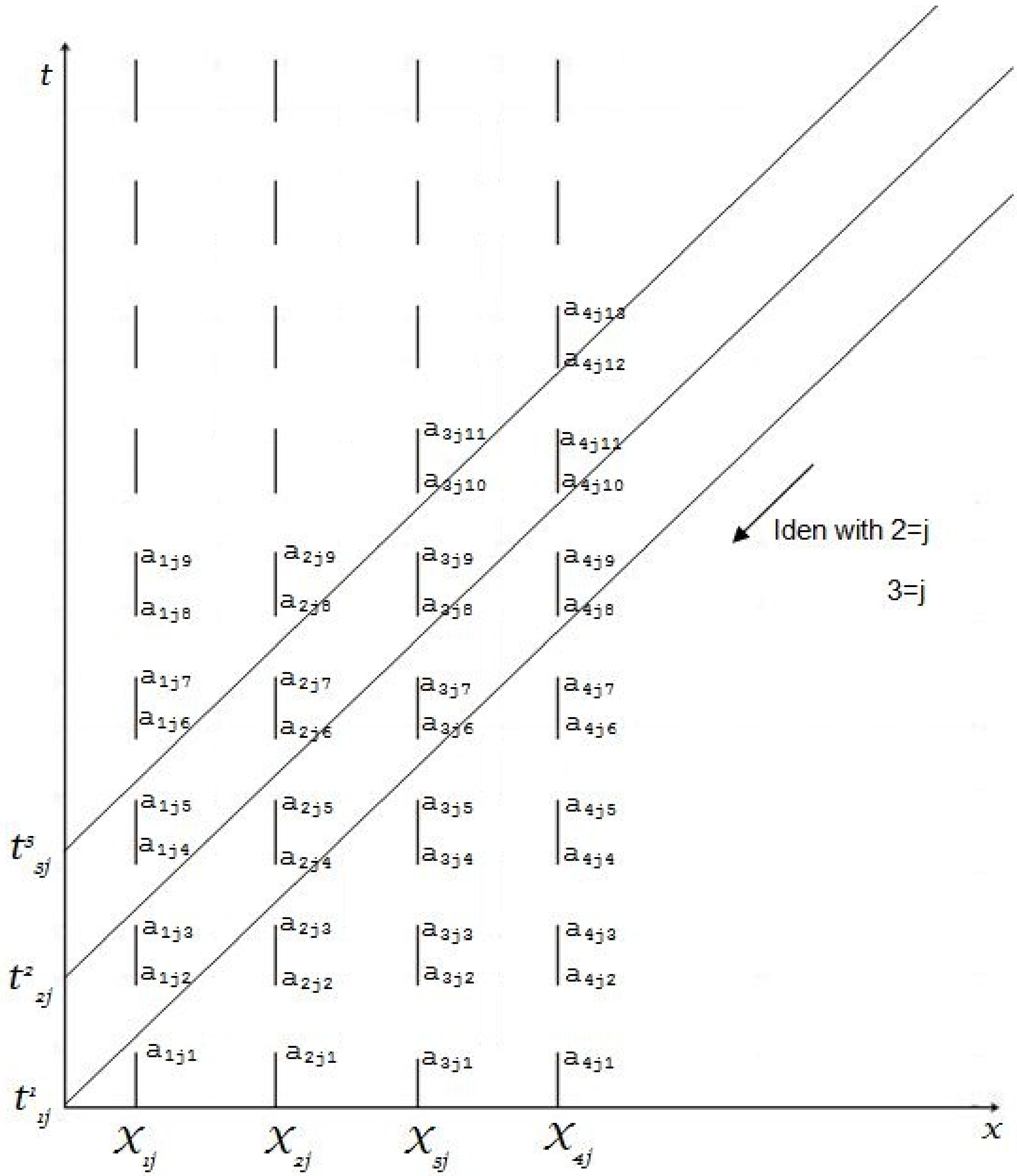


Figure 3

Where now the co-ordinate x_{ij} it meets the intersection the place (ij). Our synchronization system, will work by the fundamental diagram.

Now we will consider a first vehicle, which for reason of simplicity will be considered a point traveling from left to right, having the easiest movement law, which is to say

$$x = at + b \qquad a > 0$$

We arrange it in a suitable way for the (t, x) axis system

$$t = \frac{1}{a}x - \frac{b}{a} = \alpha x + \beta \qquad \alpha = \frac{1}{a} \quad \beta = \frac{b}{a} \qquad a > 0$$

This fact from a real point of view is important since we consider it as stationary regime. The vehicle or cars, if not stopped by a light, move in a straight line.

Consider a first vehicle moving with respect to the following law that is to say the straight lines unity for simplicity of the presentation will keep constant. This is the speed, true also if the functions are piece-like.

$$t_{ij} = \alpha_{ij}\, x_{ij} + \beta_{ij}$$

this is piece-way linear where α_{ij} is the inverse of the velocity of a car i, given a vehicle in the corresponding block.

Then it passed at x_{1j} at time

$$t_{1j}^1 = a_{1j}^1\, x_{1j} + \beta_{1j}^1$$

this is for the plan necessary and sufficient condition that such a vehicle passes the first light on green is given by

$$a_{1j1} < t_{1j}^1 = a_{1j}^1\, x_{1j} + \beta_{ij}^1 < a_{1j2}$$

In the second and subsequent light we have

$$\begin{aligned}
a_{2j3} &< t_{2j}^1 = a_{2j}^1\, x_{2j} + \beta_{2j}^1 < a_{2j4}\\
a_{3j5} &< t_{3j}^1 = a_{3j}^1\, x_{3j} + \beta_{3j}^1 < a_{3j6}\\
a_{4j7} &< t_{4j}^1 = a_{4j}^1\, x_{4j} + \beta_{4j}^1 < a_{4j8}
\end{aligned}$$

And in general

$$a_{ij2i-1} < t_{ij}^1 = a_{ij}^1\, x_{ij} + \beta_{ij}^1 < a_{ij2i} \qquad i = 1, 2, ...$$

It is important to emphasize the model that from light ij to light $i+1j$ is expressed by the law of the passing cars, which is called jumping the two time periods, since we consider that the first car and the subsequent cars are going through from a_{1j1} to a_{2j2} for each j. This assumption is restrictive in this model, and it will derive rather a theory for slow cars. We study it in order to have a simpler understanding for the existence and real computation of the green wave. We study it also where the next car indexed with $2j$ as shown in the figure, we will have for the constrained of the passing through the lights without stopping

$$\begin{aligned}
a_{1j3} &< t_{1j}^2 = a_{1j}^2\, x_{1j} + \beta_{1j}^2 < a_{1j4}\\
a_{2j5} &< t_{2j}^2 = a_{2j}^2\, x_{2j} + \beta_{2j}^2 < a_{2j6}\\
a_{3j7} &< t_{3j}^2 = a_{3j}^2\, x_{3j} + \beta_{3j}^2 < a_{3j8}\\
a_{4j9} &< t_{4j}^2 = a_{4j}^2\, x_{4j} + \beta_{4j}^2 < a_{4j1\,0}
\end{aligned}$$

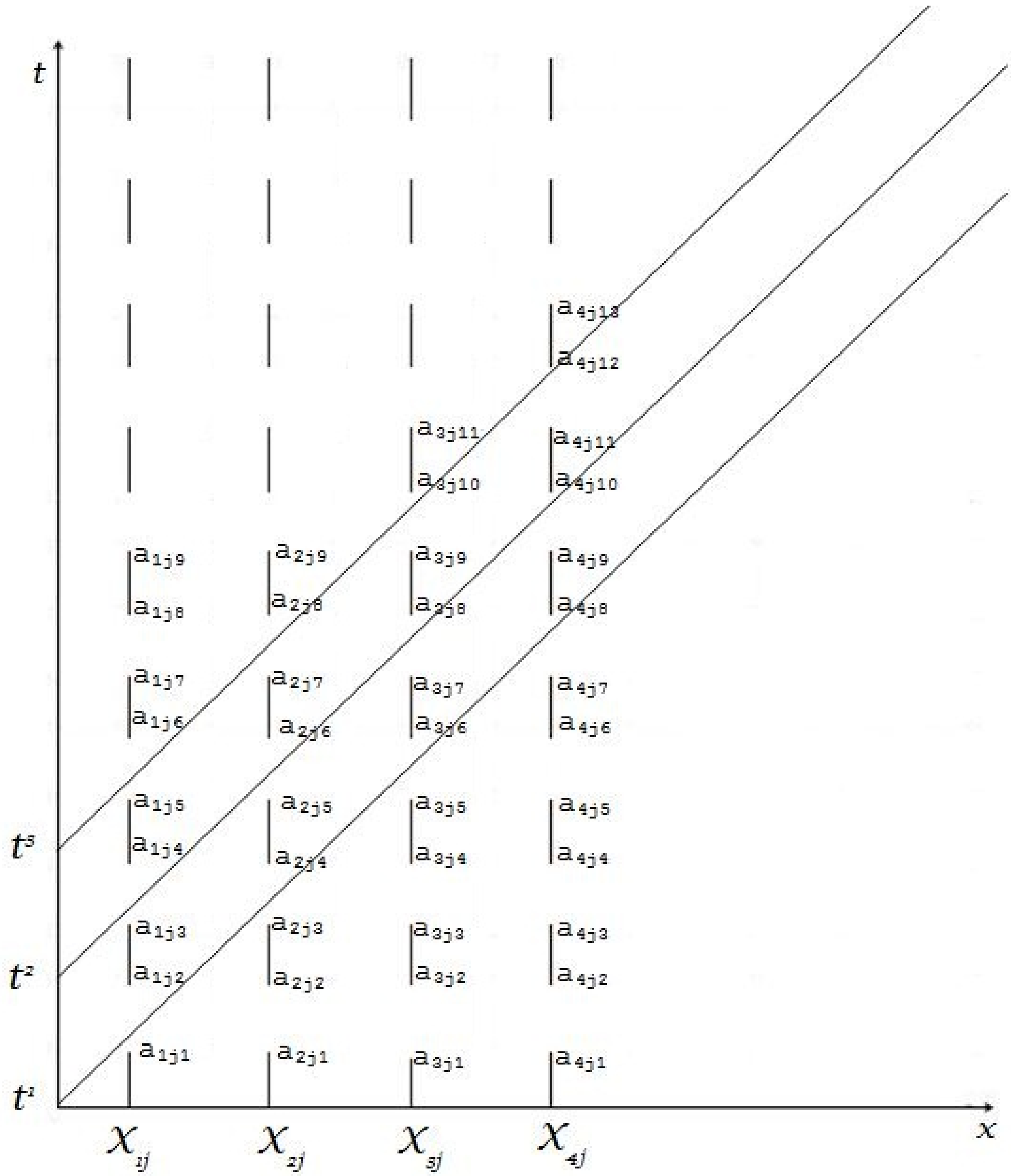


Figure 4: We emphasize the fact that the number a_{ijk} is the time running at the position.

Where now coordinate x_{ij} means the amount of space from the first street to the intersection of the place (ij).

Our synchronization system, will work by the fundamental diagram.

Now we will consider a first vehicle, which for reason of simplicity will be considered a point going from left to right, assuming that it has the easiest movement law, which is given by the law

$$x = at + b \qquad a > 0$$

We arrange it in a suitable way for the (t, x) axis system

$$t = \frac{1}{a}x - \frac{b}{a} = ax + \beta \qquad a = \frac{1}{a} \qquad \beta = \frac{b}{a} \qquad a > 0$$

This fact for the real point of view is important since we consider a stationary regime. The vehicles or cars if not stopped by a light, move in a straight line, or piece-like linear.

Consider a first vehicle moving by the equation

$$t^1 = a^1 x + \beta^1$$

where a^1 is the velocity of a given vehicle in the corresponding block. In order to study the existence of a green wave in the whole rectangular area, we must study in a systematic way the problem in a phase. For this reason we begin to realize that all the north-south straight lines can be considered in a parallel way to be equal, in the sense that everything that happens in one, happens in all of them. For example and particulary the movement of the cars or platoons follow the same scheme. In other words, they are almost parallel. This part which is a rather trivial observation, it is the essential point of our theory.

Thus using the previous remarks we seek to see what happens in the avenue marked by $(1, x)$ and $(1, y)$.

Then it passes at x_{1j} at time is t^1_{1j} where the upper index indicate the first car or platoon and the sub-index are for the first light in the intersection.

$$t^1_{1j} = a^1_{1j}\, x_{1j} + \beta^1_{1j}$$

where it is considered as piece-like linear, and continuous.

We remember that x_{1j} is the amount of space in the manifold j which goes from the origin to street i.

Therefore it is necessary and sufficient condition that such a vehicle should pass the first light on green as given by

$$a_{1j1} < t^1_{1j} = \alpha^1_{1j}\, x_{1j} + \beta^1_{1j} < a_{1j2}$$

For the second and subsequent light

$$\begin{aligned} a_{2j3} &< t^1_{2j} = \alpha^1_{2j}\, x_{2j} + \beta^1_{2j} < a_{2j4} \\ a_{3j5} &< t^1_{3j} = \alpha^1_{3j}\, x_{3j} + \beta^1_{3j} < a_{3j6} \\ a_{4j7} &< t^1_{4j} = \alpha^1_{4j}\, x_{4j} + \beta^1_{4j} < a_{4j8} \end{aligned}$$

and in general

$$a_{ij2i-1} < t^1_{ij} = \alpha^1_{ij}\, x_{ij} + \beta^1_{ij} < a_{ij2i} \qquad i = 1, 2...$$

Where here we consider a linear piece-wise such as which can be obtained much better way as it was presented in the previous material.

As we have said, it is possible to have the piece-linear condition by obtaining the next conditions. We assume then implicitly.

Take

$$t_{1j}^1 = \alpha_{1j}^1 \, x_{1j} + \beta_{1j}^1$$

and

$$t_{2j}^1 = \alpha_{2j}^1 \, x_{2j} + \beta_{2j}^1$$

and in the middle point $\frac{x_{2j}+x_{1j}}{2}$ they have to be equals, namely

$$\alpha_{1j}^1 \left(x_{1j} + \frac{x_{2j} - x_{1j}}{2} \right) + \beta_{1j}^1 = \alpha_{2j}^1 \left(x_{2j} - \frac{x_{2j} - x_{1j}}{2} + \beta_{2j}^1 \right)$$

or making some manipulations

$$\left(\alpha_{1j}^1 - \alpha_{2j}^1\right) \left(x_{1j} + x_{2j}\right) = 2 \left(\beta_{2j}^1 - \beta_{1j}^1\right).$$

We consider this conditions implicitly all over the paper without tell it in any step.

At this point we emphasize that we have considered the model that shows from the light (i, j) to the light $(i + 1, j)$. Thus the law of passing cars is called jumping the two time periods, since we consider that the first car and the subsequent cars are going through from a_{1j1} to a_{1j2}. This assumption is restrictive in this model and it will derive a theory for slow cars. The reason to study it is because we wish to have a simpler understanding for the existence and real computation of the green wave. In a separate study we will consider it elsewhere. Moreover, with this result, it is possible to attack the problem where the traffic is dense and semi-dense. Indeed we are beginning to study with this kind of tools the case of dense, and semi-dense fluid of traffic flow.

For the next car indexed with 2 as shown in the figure we will have for the constrained of the passing through the light without stopping

$$\begin{aligned}
a_{1j3} &< t_{1j}^2 = \alpha_{1j}^2 \, x_{1j} + \beta_{1j}^2 < a_{1j4} \\
a_{2j5} &< t_{2j}^2 = \alpha_{2j}^2 \, x_{2j} + \beta_{2j}^2 < a_{2j6} \\
a_{3j7} &< t_{3j}^2 = \alpha_{3j}^2 \, x_{3j} + \beta_{3j}^2 < a_{3j8} \\
a_{4j9} &< t_{4j}^2 = \alpha_{4j}^2 \, x_{4j} + \beta_{4j}^2 < a_{4j10}
\end{aligned}$$

and in general we will have

$$a_{ij2i+1} < t_{ij}^2 = \alpha_{ij}^2 \, x_{ij} + \beta_{ij}^2 < a_{ij2i+2}$$

In order to keep the material and the presentation in the paper elemental, we will consider the third car and the fourth and then will derive the general relation among the constraints for the validation of the green wave:

$$
\begin{aligned}
a_{1j5} &< t^3_{1j} = \alpha^3_{1j}\, x_{1j} + \beta^3_{1j} < a_{1j6} \\
a_{2j7} &< t^3_{2j} = \alpha^3_{2j}\, x_{2j} + \beta^3_{2j} < a_{2j8} \\
a_{3j9} &< t^3_{3j} = \alpha^3_{3j}\, x_{3j} + \beta^3_{3j} < a_{3j10} \\
a_{4j11} &< t^3_{4j} = \alpha^3_{4j}\, x_{4j} + \beta^3_{4j} < a_{4j12}
\end{aligned}
$$

and in general we will have

$$
a_{ij2i+3} < t^3_{ij} = \alpha^3_{ij}\, x_{ij} + \beta^3_{ij} < a_{ij2i+4}
$$

From here, we obtain for the fourth vehicle and in general the following inequalities

$$
\begin{aligned}
a_{1j7} &< t^4_{1j} = \alpha^4_{1j}\, x_{1j} + \beta^4_{1j} < a_{1j8} \\
a_{2j9} &< t^4_{2j} = \alpha^4_{2j}\, x_{2j} + \beta^4_{2j} < a_{2j10} \\
a_{3j11} &< t^4_{3j} = \alpha^4_{3j}\, x_{3j} + \beta^4_{3j} < a_{3j12} \\
a_{4j13} &< t^4_{4j} = \alpha^4_{4j}\, x_{4j} + \beta^4_{4j} < a_{4j14} \\
a_{5j15} &< t^4_{5j} = \alpha^4_{5j}\, x_{5j} + \beta^4_{5j} < a_{5j16}
\end{aligned}
$$

and in general for this platoon or car we have

$$
a_{ij2i+k+1} < t^k_{ij} = \alpha^k_{ij}\, x_{ij} + \beta^k_{ij} < a_{ij2i+k+2}
$$

4 Negative Direction in The Same East-West Way

Next we consider the negative direction. That is to say, the movement of the cars coming from the right to the left and the cars coming from the front or south to the north, the corresponding inequalities relating the movements of the cars going from right to left. In order to keep this study simple, we begin with the first car from right to left beginning with the number one. The second the following one and so on.

Now we present in the next figure the timing of the corresponding $a's$ in the right part of the avenue.

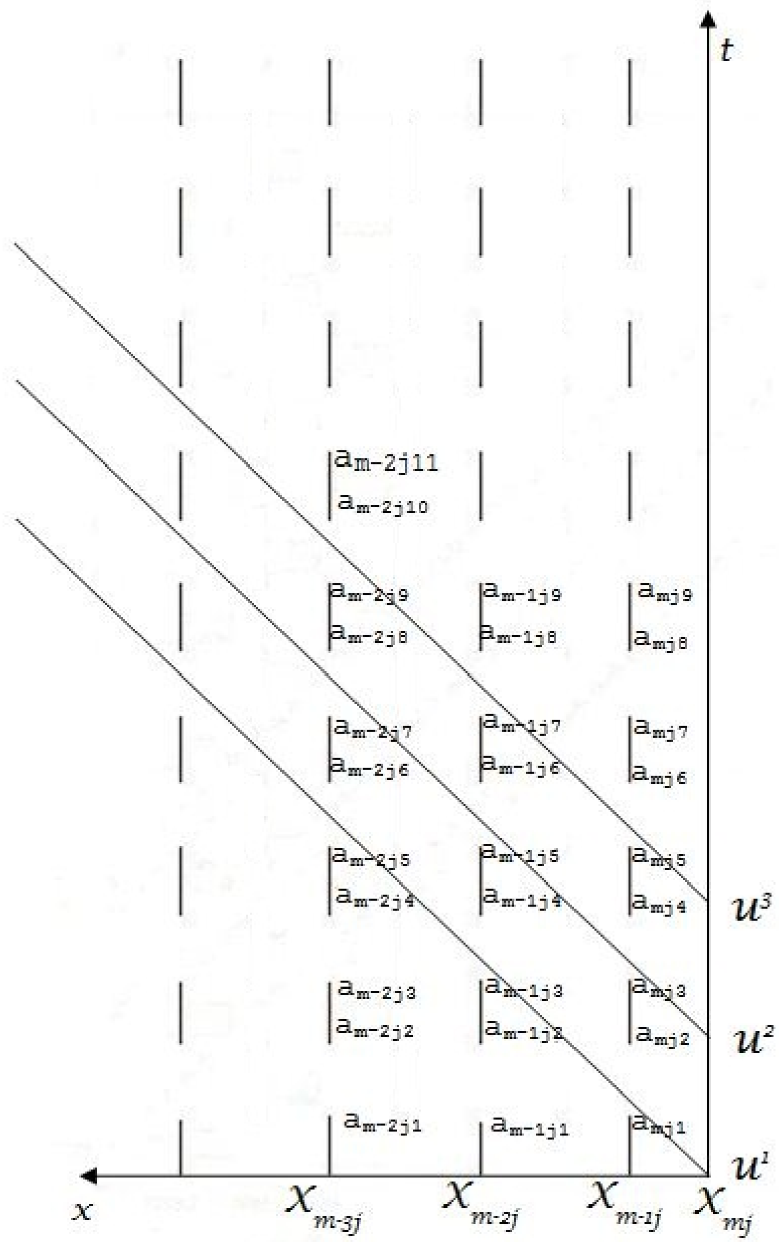


Figure 5

Here, we have that for cars coming from the right to the left or in the contrary or negative direction, it is written by the equation

$$u = \gamma x + \delta$$

where u here is time and $\gamma < 0$ since the displacement of the cars is from right to left, which it is the inverse of the velocity:

$$x = \frac{u-\delta}{\gamma} = \frac{u}{\gamma} - \frac{\delta}{\gamma} = cu + d \qquad \frac{1}{\gamma} = c < 0, \qquad -\frac{\delta}{\gamma} = d$$

Then we have for the first car the inequalities

$$u^k_{m-i+1,j} = \gamma^k_{m-i+1,j}\ x_{ij} + \delta^k_{m-i+1,j}$$

$$\begin{aligned}
a_{mj1} &< u^1_{1j} = \gamma^1_{1j}\ x_{mj} + \delta^1_{1j} < a_{mj2} \\
a_{m-1j3} &< u^1_{2j} = \gamma^1_{2j}\ x_{m-1j} + \delta^1_{2j} < a_{m-1j4} \\
a_{m-2j5} &< u^1_{3j} = \gamma^1_{3j}\ x_{m-2j} + \delta^1_{3j} < a_{m-2j6} \\
a_{m-3j7} &< u^1_{4j} = \gamma^1_{4j}\ x_{m-3j} + \delta^1_{4j} < a_{m-3j8} \\
a_{m-4j9} &< u^1_{5j} = \gamma^1_{5j}\ x_{m-4j} + \delta^1_{5j} < a_{m-4j10}
\end{aligned}$$

and in general we have

$$a_{n-i+1j2i-1} < u^1_{m-i+1,j} = \gamma^1_{m-i+1,j}\ x_{m-i+1,j} + \delta^1_{m-i+1,j} < a_{m-i+1,j,2i}$$

For the second car we have

$$\begin{aligned}
a_{mj3} &< u^2_{1j} = \gamma^2_{1j}\, x_{mj} + \delta^2_{1j} < a_{mj4} \\
a_{m-1j5} &< u^2_{2j} = \gamma^2_{2j}\, x_{m-1j} + \delta^2_{2j} < a_{m-1j6} \\
a_{m-2j7} &< u^2_{3j} = \gamma^2_{3j}\, x_{m-2j} + \delta^2_{3j} < a_{m-2j8} \\
a_{m-3j9} &< u^2_{4j} = \gamma^2_{4j}\, x_{m-3j} + \delta^2_{4j} < a_{m-3j10} \\
a_{m-4j11} &< u^2_{5j} = \gamma^2_{5j}\, x_{m-4j} + \delta^2_{5j} < a_{m-4j12}
\end{aligned}$$

and from here in general we derive the next general inequalities.

$$a_{m-i+1,j,2i+1} < u^2_{ij} = \gamma^2_{ij}\, x_{m-i+1,j} + \delta^2_{ij} < a_{m-i+1j2i+2}$$

In this way we have obtained the entire inequality for the second platoon or car. For the third one we will have a_{ijk}.

We emphasize the fact that the number a_{ijk} is the time running at the position (ij), in the period k.

$$\begin{aligned}
a_{mj5} &< u^3_{ij} = \gamma^3_{ij}\, x_{mj} + \delta^3_{ij} < a_{mj6} \\
a_{m-1j7} &< u^3_{2j} = \gamma^3_{2j}\, x_{m-1j} + \delta^3_{2j} < a_{m-1j8} \\
a_{m-2j9} &< u^3_{3j} = \gamma^3_{3j}\, x_{m-2j} + \delta^3_{3j} < a_{m-2j10} \\
a_{m-3j11} &< u^3_{4j} = \gamma^3_{4j}\, x_{m-3j} + \delta^3_{4j} < a_{m-3j12}
\end{aligned}$$

and in general for this platoon or car

$$a_{m-i+1,j,2i+3} < u^3_{ij} = \gamma^3_{ij}\, x_{m-i+1,j} + \delta^3_{ij} < a_{m-i+1,j,2i+4}$$

For the next car or platoon we have the following inequalities

$$\begin{aligned}
a_{mj7} &< u^4_{ij} = \gamma^4_{ij}\, x_{mj} + \delta^4_{ij} < a_{mj8} \\
a_{m-1j9} &< u^4_{2j} = \gamma^4_{2j}\, x_{m-1j} + \delta^4_{2j} < a_{m-1j10} \\
a_{m-2j11} &< u^4_{3j} = \gamma^4_{3j}\, x_{m-2j} + \delta^4_{3j} < a_{m-2j12} \\
a_{m-3j13} &< u^4_{4j} = \gamma^4_{4j}\, x_{m-3j} + \delta^4_{4j} < a_{m-3j14} \\
a_{m-4j15} &< u^4_{5j} = \gamma^4_{5j}\, x_{m-4j} + \delta^4_{5j} < a_{m-4j16}
\end{aligned}$$

and for arbitrary i which is in general for this platoon or car we have

$$a_{m-i+1,j,2i+5} < u^4_{ij} = \gamma^4_{ij}\, x_{m-i+1,j} + \delta^4_{ij} < a_{m-i+1,j,2i+6.}$$

In general for an arbitrary k, we have that the inequality to be considered is

$$a_{m-i+1,j,2i+2k-3} < u^k_{ij} = \gamma^k_{ij}\, x_{m-i+1,j} + \delta^k_{ij} < a_{m-i+1,j,2i+2k-2}$$

5 The Perpendicular Direction

This perpendicular direction can be given by considering the plane fixing i. The first thing to do is considering that also here we have linear functions or piece-wise linear in the time and the space.

In the most general easiest way, say in a linear way

$$w = ev + \omega$$

Where w is the time and $e > 0$ because the movements of the cars in this direction in south-north. Taking the inverse of the value by

$$v = \frac{w - \omega}{e} = \frac{w}{e} - \frac{\omega}{e} = \delta w + f \qquad \frac{1}{e} = \delta > 0 \qquad -\frac{\omega}{e} = f$$

Then as we have done until now with the same methodology we have that the platoon or car indexed by $k = 1$ in order to pass all the green light which on the directions are green or it is the same to say that in the perpendicular direction the same light is red.

Then for the indexed car we have the passing free law is depending on the red or green of a light.

Here the situation of passing the light is different from the first part. In the first we have had that on a light if in the direction of the plane j, which is west-east or east-west the green light was designed as follows in the next comparative figures

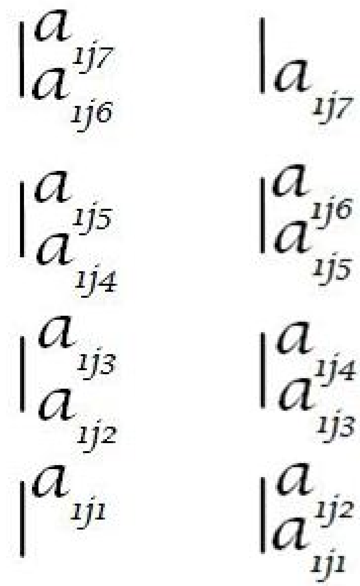


Figure 6

The first one shows how a signal switching between red and green the corresponding green in that direction appears in the segments a_{ij2k+1} to a_{ij2k+2}. On the other hand during these hours the signals in the perpendicular direction are red and the green appear between the times a_{ij2k} to a_{ij2k+1}. Therefore the theory for the perpendicular direction is similar for the previous one with the same nomenclature switching by one. But for sake of presentation we are going to present it accordingly.

Now we will consider a first vehicle which for reason of simplicity will be considered a point going south-north having the easiest analytical movement law

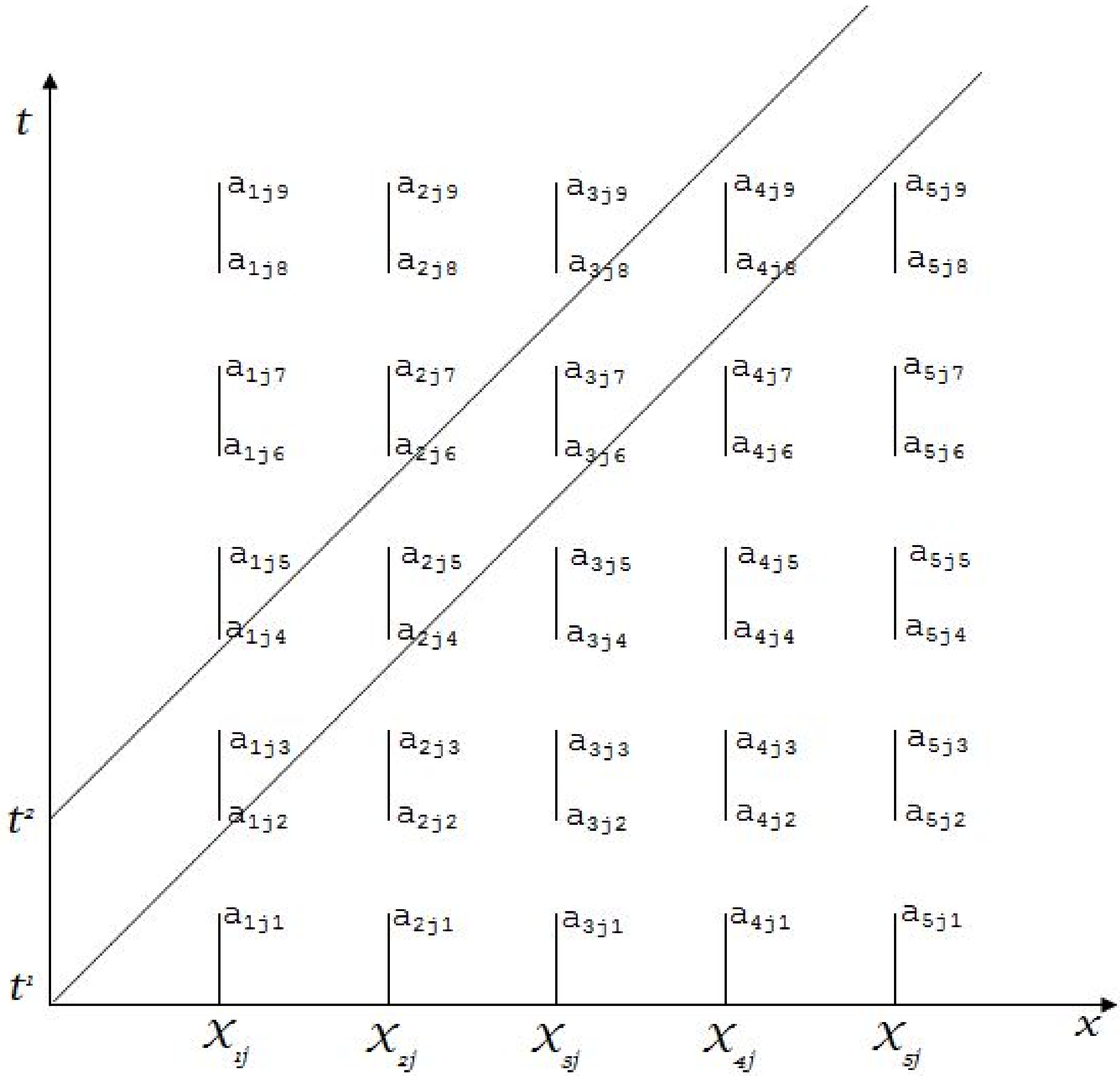
$$\overline{x} = \alpha t + \beta \qquad \alpha > 0$$

which says that the variable moves in a linear dependence with respect the variable $\overline{x}$. We rearrange in a more useful way for the diagram $(\overline{x}, t)$ axis system

$$\begin{aligned} t &= \frac{1}{\alpha}\overline{x} + \beta = \gamma\overline{x} + \delta \\ \gamma &= \frac{1}{\alpha} > 0 \qquad \frac{\beta}{\alpha} = \delta \end{aligned}$$

We are going to indicate the two main diagrams which take into consideration the whole plan of coordination of the green wave in the whole rectangular system.

Consider the following figures which we call the fundamental of the swithching system of the city or grid



t
a_{1j9} a_{2j9} a_{3j9} a_{4j9} a_{5j9}
a_{1j8} a_{2j8} a_{3j8} a_{4j8} a_{5j8}
a_{1j7} a_{2j7} a_{3j7} a_{4j7} a_{5j7}
a_{1j6} a_{2j6} a_{3j6} a_{4j6} a_{5j6}
a_{1j5} a_{2j5} a_{3j5} a_{4j5} a_{5j5}
a_{1j4} a_{2j4} a_{3j4} a_{4j4} a_{5j4}
a_{1j3} a_{2j3} a_{3j3} a_{4j3} a_{5j3}
t^2
a_{1j2} a_{2j2} a_{3j2} a_{4j2} a_{5j2}
a_{1j1} a_{2j1} a_{3j1} a_{4j1} a_{5j1}
t^1
X_{1j} X_{2j} X_{3j} X_{4j} X_{5j} x

Figure 7

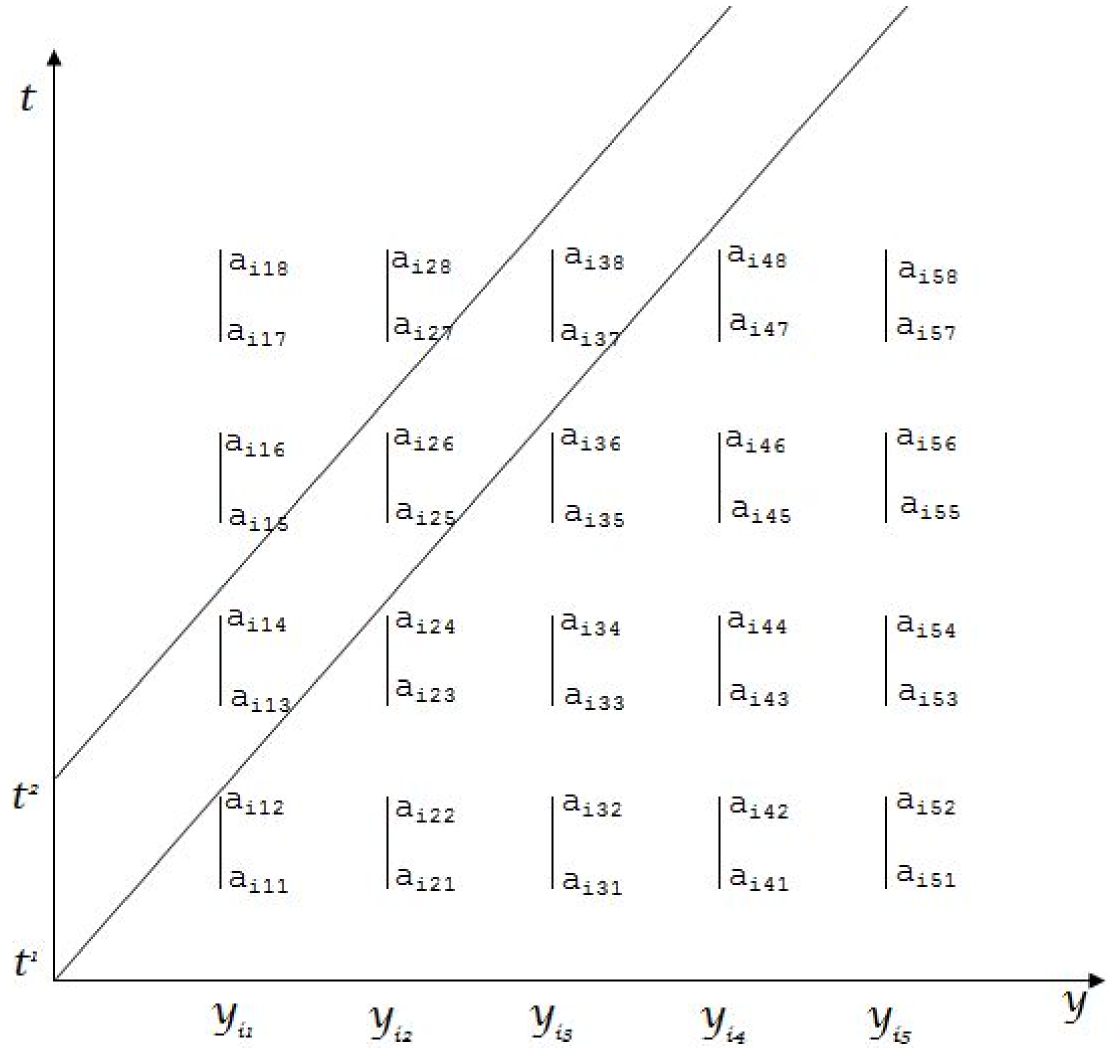


Figure 8

where the element yij or $\overline{x}ij$ mean the amount of space in the street i which goes from the origin of the street to the street j.

At this time we like to take some break analysing these two fundamental diagrams. We should like to emphasize that without the understanding of them, it would be impossible to have the theory built in the rectangular city or grid. First of all we wish to say that the full and empty intervals a taken in a simple and particular way and they are not forced to be as the arrangement shown. From the beginning we have said that we do not necessarily have a engineering period for light. The reason is that the main purpose of the paper is in which easiest ways the green wave in the grid exists. We are interested in the mathematical aspect because as was mention by Barcelo [1] the engineering and real aspects came after it.

In order to understand the fundamental diagrams that give the right structure of the light in the rectangular city, consider an example at one intersection of it say (3,2,5). This means that at the intersection $i = 3$, $j = 2$ and at time $t = 5$ we have from the first diagram

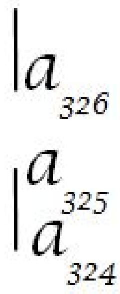


Figure 9

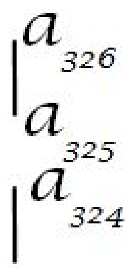


Figure 10

The Figure 9 tells you that from West-East and East-West the permitted right of way is between the times

$$a_{325} \text{ to } a_{326}$$

and between the times

$$a_{324} \text{ to } a_{325}$$

they are close to the transient in this direction.

And in the perpendicular direction South-North and North-South, during these times the right of way is not allowed to pass because the light is red.

On the other hand between

$$a_{324} \text{ to } a_{325}$$

the light is green, in the direction South-North and North-South.

With this simple example we understand that both figure are complimentary and they define completely the light system for the system.

Indeed the first figure is useful for the wave in the west-east and east-west directions one will allow us to have the relation for the wave north-south and south-north. Since we have obtained the relation for east-west now we will obtain the constrain for the last case.

In the first case we take as the model indicate the straight line or part-wise linear: is the easiest behaviour for studying the problem

$$\overline{x} = c^1 t^1 + d^1 \qquad c^1 > 0$$

Where t^1indicates the time of the first car leaving the origin of the corresponding street in the direction south-north that we consider positive. It has to be connected with the first car or platoon in the perpendicular way east-west.

In a convenient way it can be written as

$$t^1 = \gamma^1 \overline{x} + \delta^1 \qquad \gamma > 0$$

The passing permit allows us to write the following inequalities

$$\begin{aligned}
a_{i12} &< t^1_{i1} = \gamma^1_{i1}\,\overline{x}_{i1} + \delta^1_{i1} < a_{i13} \\
a_{i24} &< t^1_{i2} = \gamma^1_{i2}\,\overline{x}_{i2} + \delta^1_{i2} < a_{i25} \\
a_{i36} &< t^1_{i3} = \gamma^1_{i3}\,\overline{x}_{i3} + \delta^1_{i3} < a_{i37} \\
a_{i48} &< t^1_{i4} = \gamma^1_{i4}\,\overline{x}_{i4} + \delta^1_{i4} < a_{i49}
\end{aligned}$$

and from here in general for the first car or platoon

$$a_{ij2j} < t^1_{ij} = \gamma^1_{ij}\,\overline{x}_{ij} + \delta^1_{ij} < a_{ij2j+1}$$

For the second car or platoon we have

$$\begin{aligned}
a_{i14} &< t^2_{i1} = \gamma^2_{i1}\,\overline{x}_{i1} + \delta^2_{i1} < a_{i15} \\
a_{i26} &< t^2_{i2} = \gamma^2_{i2}\,\overline{x}_{i2} + \delta^2_{i2} < a_{i27} \\
a_{i38} &< t^2_{i3} = \gamma^2_{i3}\,\overline{x}_{i3} + \delta^1_{i3} < a_{i39} \\
a_{i4\,10} &< t^2_{i4} = \gamma^2_{i4}\,\overline{x}_{i4} + \delta^2_{i4} < a_{i4\;11} \\
a_{i5\,12} &< t^2_{i5} = \gamma^2_{i5}\,\overline{x}_{i5} + \delta^2_{i5} < a_{i5\;13}
\end{aligned}$$

then in general we obtain by induction or other procedures that

$$a_{ij2j+2} < t^2_{ij} = \gamma^2_{ij}\,\overline{x}_{ij} + \delta^2_{ij} < a_{ij2j+3}$$

thus we have this general expression that determine the overall green wave in the south-north direction and in general we have

$$a_{ij2j+k} < t^k_{ij} = \gamma^k_{ij}\,x_{ij} + \delta^k_{ij} < a_{ij2j+k+1}$$

Therefore we have to compare just the equations determined the first avenue in the perpendicular directions; namely with $v = 1$ and $j = 1$.

They there exist at this stage the green ways in all the considered directions.

6 Negative Direction

Finally we have to study the contrary direction in south-north context. For this let us compare the corresponding graphical or figures providing the fundamental figures in the negative directions. These are presented in the following draws, as the Figure 4 and it's complementary to be constructed. The Figure 9 and Figure 10 possess all the information that we need to obtain as well we have in the positive direction. What we need in order to prove the existence of the green wave in the whole rectangular city.

Again as we presented the Figure 6 and 7 the segments mean that the car or platoon does not pass. Meanwhile the open gaps, in other words, when there are not obstacles the car or the platoon pass freely.

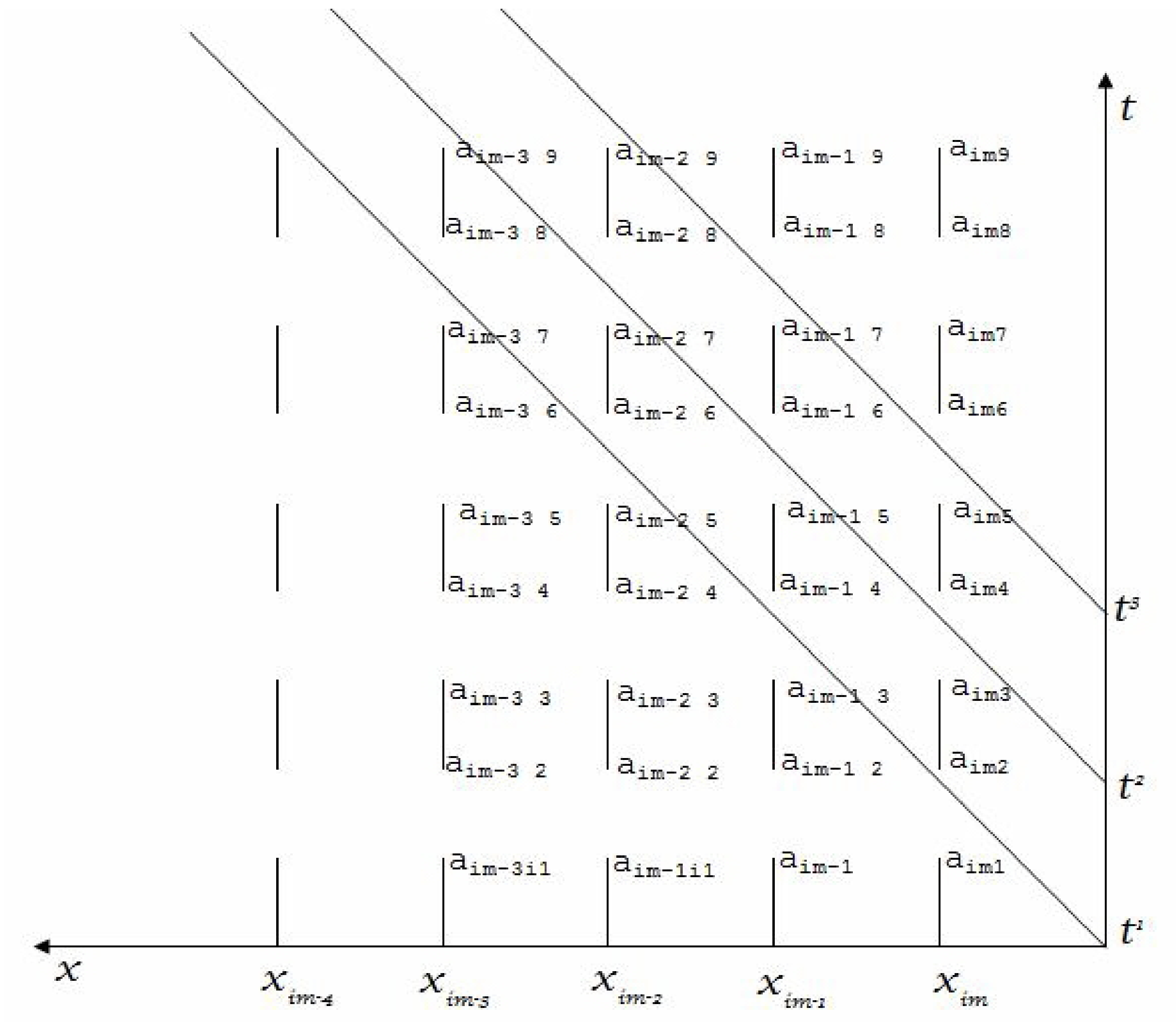


Figure 11

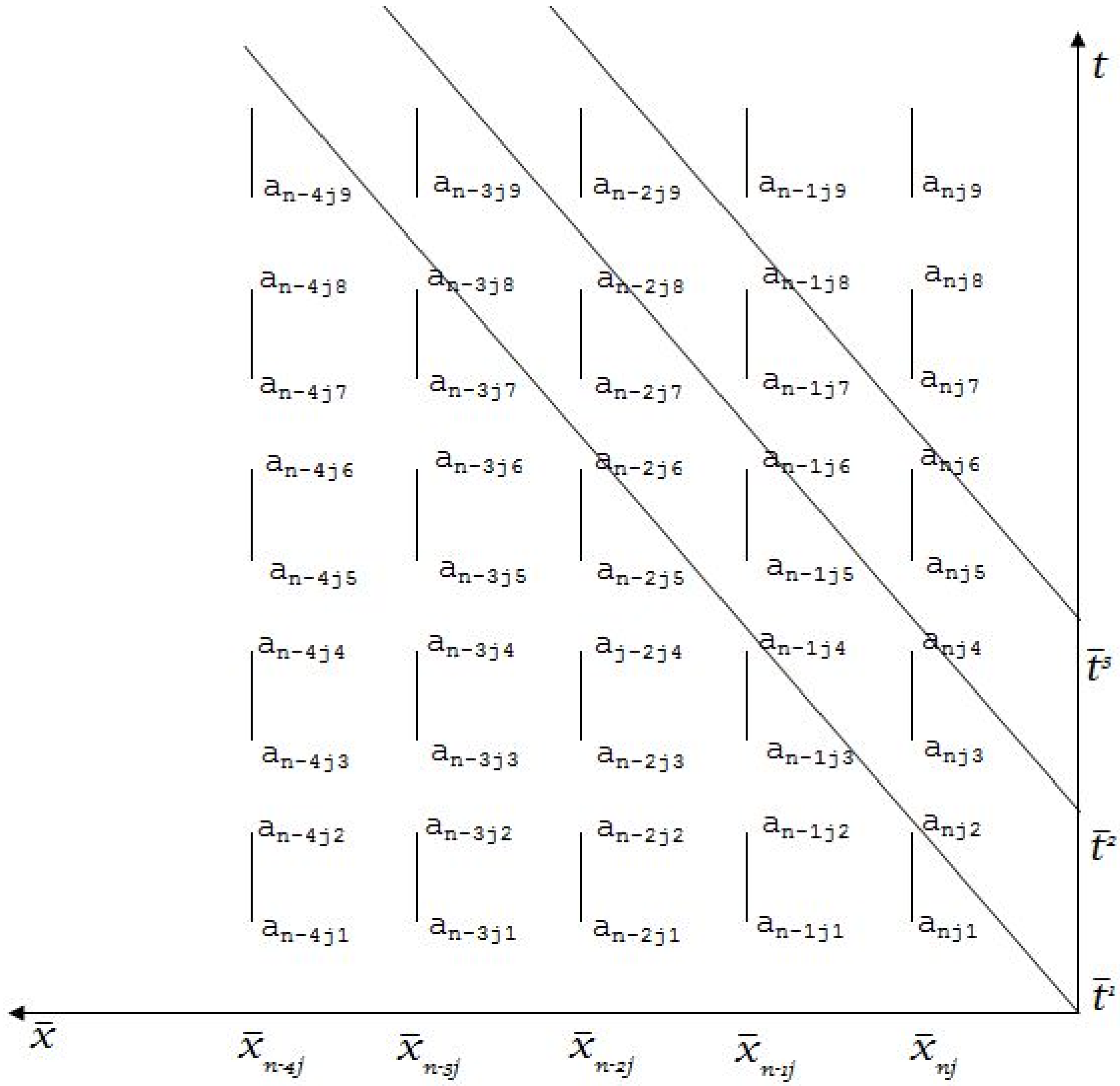


Figure 12

We see that they are consistent with each other as was the case of the previous two figures presenting the movements of the east-west and west-east car movement configuration. Now we will present the corresponding equations for the remaining part. The first equation that explains the first trajectory or linear functions are for the passing permit.

Here we have that for the cars coming from the right to the left or in the contrary or negative direction, is written by the equation

$$u = \gamma y + \delta$$

where u here is time and $\gamma < 0$ since the displacement of the cars is from right to left, which it is the inverse of the velocity.

$$y = \frac{u-\delta}{\gamma} = \frac{u}{\gamma} - \frac{\delta}{\gamma} = cu + d, \qquad \frac{1}{\gamma} = c < 0, \quad -\frac{\delta}{\gamma} = d$$

Then we have for the first car the inequalities

$$\begin{aligned}
a_{im1} &< u^1_{i1} = \gamma^1_{i1}\, x_{im} + \delta^1_{i1} < a_{im2}\\
a_{im-13} &< u^1_{i2} = \gamma^1_{i2}\, x_{im-1} + \delta^1_{i2} < a_{im-14}\\
a_{im-25} &< u^1_{i3} = \gamma^1_{i3}\, x_{im-2} + \delta^1_{i3} < a_{im-26}\\
a_{im-37} &< u^1_{i4} = \gamma^1_{i4}\, x_{im-3} + \delta^1_{i4} < a_{im-38}\\
a_{im-49} &< u^1_{i5} = \gamma^1_{i5}\, x_{im-4} + \delta^1_{i4} < a_{im-4\;10}
\end{aligned}$$

and in general we have

$$a_{im-j+1,2j-1} < u^1_{ij} = \gamma^1_{ij}\, x_{im-j+1} + \delta^1_{ij} < a_{im-j+1,2j}$$

For the second car we have

$$\begin{aligned}
a_{im3} &< u^2_{i1} = \gamma^2_{i1}\, x_{im} + \delta^2_{i1} < a_{im4}\\
a_{im-15} &< u^2_{i2} = \gamma^2_{i2}\, x_{im-1} + \delta^2_{i2} < a_{im-16}\\
a_{im-27} &< u^2_{i3} = \gamma^2_{i3}\, x_{im-2} + \delta^2_{i3} < a_{im-28}\\
a_{im-39} &< u^2_{i4} = \gamma^2_{i4}\, x_{im-3} + \delta^2_{i4} < a_{im-3\;10}\\
a_{im-4\;11} &< u^2_{i5} = \gamma^2_{i5}\, x_{im-4} + \delta^2_{i5} < a_{im-4\;12}
\end{aligned}$$

and therefore in general

$$a_{im-i+1,2j+1} < u^2_{ij} = \gamma^2_{ij}\, x_{im-i+1} + \delta^2_{ij} < a_{im-i+1,2j+2}$$

In this way we have obtained the entire inequality for the second platoon or car. For the third one we will have

$$\begin{aligned}
a_{im5} &< u^3_{i1} = \gamma^3_{i1}\, x_{im} + \delta^3_{i1} < a_{im6}\\
a_{im-17} &< u^3_{i2} = \gamma^3_{i2}\, x_{im-1} + \delta^3_{i2} < a_{im-18}\\
a_{im-29} &< u^3_{i3} = \gamma^3_{i3}\, x_{im-2} + \delta^3_{i3} < a_{im-2\;10}\\
a_{im-3\;11} &< u^3_{i4} = \gamma^3_{i4}\, x_{im-3} + \delta^3_{i4} < a_{im-3\;12}\\
a_{im-4\;13} &< u^3_{i5} = \gamma^3_{i5}\, x_{im-4} + \delta^3_{i5} < a_{im-4\;14}
\end{aligned}$$

and in general we have

$$a_{im-i+1,2j+3} < u^3_{ij} = \gamma^3_{ij}\, x_{im-i+1} + \delta^3_{ij} < a_{im-i+1,2j+4}$$

For the fourth car we obtain the following system of inequalities

$$
\begin{aligned}
a_{im7} &< u^4_{i1} = \gamma^4_{i1}\, x_{im} + \delta^4_{i1} < a_{im8} \\
a_{im-19} &< u^4_{i2} = \gamma^4_{i2}\, x_{im-1} + \delta^4_{i2} < a_{im-1\,10} \\
a_{im-2\,11} &< u^4_{i3} = \gamma^4_{i3}\, x_{im-2} + \delta^4_{i3} < a_{im-2\,12} \\
a_{im-3\,13} &< u^4_{i4} = \gamma^4_{i4}\, x_{im-3} + \delta^4_{i4} < a_{im-3\,14}
\end{aligned}
$$

and in general it follows

$$
a_{i,m-j+1,2j+k} < u^k_{ij} = \gamma^k_{ij}\, x_{im-j+1} + \delta^k_{ij} < a_{i,m-j+1,2j+k+1}
$$

Next and the last consideration of the paper we consider the equations for the north-south and south-north directions. The times in both diagrams have to be related, otherwise you cannot compare the behaviour of the systems.

Then we go, for the first time $k = 1$, we have

$$
\begin{aligned}
a_{nj2} &< \bar{t}^1_{1j} = \bar{\gamma}^1_{1j}\, \bar{x}_{nj} + \bar{\delta}^1_{1j} < a_{nj3} \\
a_{n-1j4} &< \bar{t}^1_{2j} = \bar{\gamma}^1_{2j}\, \bar{x}_{n-1j} + \bar{\delta}^1_{2j} < a_{n-1j5} \\
a_{n-2j6} &< \bar{t}^1_{3j} = \bar{\gamma}^1_{3j}\, \bar{x}_{n-2j} + \bar{\delta}^1_{3j} < a_{n-2j7} \\
a_{n-3j8} &< \bar{t}^1_{4j} = \bar{\gamma}^1_{4j}\, \bar{x}_{n-3j} + \bar{\delta}^1_{4j} < a_{n-3j9} \\
a_{n-4j10} &< \bar{t}^1_{5j} = \bar{\gamma}^1_{5j}\, \bar{x}_{n-4j} + \bar{\delta}^1_{5j} < a_{n-4j11}
\end{aligned}
$$

Then in general

$$
a_{n-i+1,j,2i} < \bar{t}^1_{ij} = \bar{\gamma}^1_{ij}\, \bar{x}_{n-ij} + \bar{\delta}^1_{ij} < a_{n-i+1,j,2i+1}
$$

For second wave of cars or platoons we have the inequalities are:

$$
\begin{aligned}
a_{nj4} &< \bar{t}^2_{1j} = \bar{\gamma}^2_{1j}\, \bar{x}_{nj} + \bar{\delta}^2_{1j} < a_{nj5} \\
a_{n-1j6} &< \bar{t}^2_{2j} = \bar{\gamma}^2_{2j}\, \bar{x}_{n-1j} + \bar{\delta}^2_{2j} < a_{n-1j7} \\
a_{n-2j8} &< \bar{t}^2_{3j} = \bar{\gamma}^2_{3j}\, \bar{x}_{n-2j} + \bar{\delta}^2_{3j} < a_{n-2j9} \\
a_{n-3j10} &< \bar{t}^2_{4j} = \bar{\gamma}^2_{4j}\, \bar{x}_{n-3j} + \bar{\delta}^2_{4j} < a_{n-3j11} \\
a_{n-4j12} &< \bar{t}^2_{5j} = \bar{\gamma}^2_{5j}\, \bar{x}_{n-4j} + \bar{\delta}^2_{5j} < a_{n-4j13}
\end{aligned}
$$

and in general

$$
a_{n-i+1,j,2i+k} < \bar{t}^k_{ij} = \bar{\gamma}^k_{ij}\, \bar{x}_{n-ij} + \bar{\delta}^k_{ij} < a_{n-i+1,j,2i+k+1}
$$

Then, form these inequalities we might extrapolate to

Thus we have got a general necessary and sufficient condition in order to have the green wave for the cars coming from the right to the left, or the negative direction.

Final Remarks

First of all we would like to point out that in an easy way it is possible to introduce the directions of the perpendicular streets. Until now they were considered with zero wide. The only consideration that we have to do is to consider $x_i + L_i$ where L_i is the wideness of the i-th perpendicular street. The relation $x_i + L_i < x_{i+1}$ is obvious. Now here the different inequalities must be considered accordingly.

On the other hand we have the scheme that we have considered here is flexible in the sense that it is possible to consider the mechanism of the lights in the sense that the velocity in the different regions are different. This is done in such a way that the trajectories are piece-linear.

Another question is the aspect that instead of one single car, we have an entire platoon. In such a case the conditions of the platoons must give for the initial and the final cars of the platoon. This change the scheme but not the essential of the model.

Also it is possible to arrange the timing in order that two platoons separate by same distance become one platoon by a simple fusion.

Finally, it is possible to relate the material presented here with that study hoy the congestion begins at the intersection points. It is interesting to take into consideration in the future to avoid lights in the city. This might be obtained by the study of cities without lights as the first paper in the subject presented by Marchi [10]. By the way our system in operation satisfies Walrop principle.

One car connect this material with the important that just was established by Olster and all [6].

Lastly we would like to say that many aspects as for example it is possible to study similarity as in [6] the delay of a platoon as well if there is a tandem of cars. By the way or approach with some modification can be useful as the studies provided by Quadrat and Olsder.

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