THE FACE PROJECTION IN LINEAR PROGRAMMING

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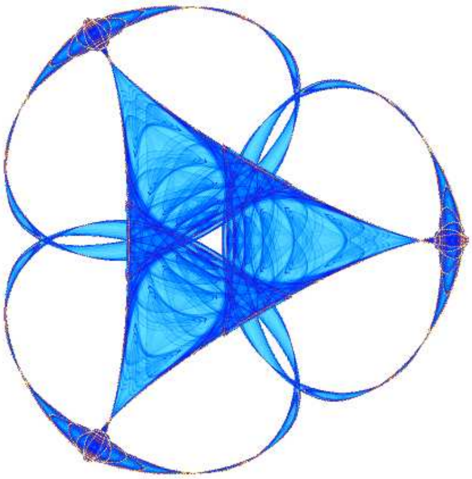

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THE FACE PROJECTION IN LP
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Abstract: in this short paper we introduce a new method for linear programming which is an interior one which works on face projections.

Keyworks: face projection, linear programming, interior point.

Introduction

The landmark in the subject of allocation of resources and its optimization has been introduced by Kantorovich [3]. He came up with the technique of linear programming. After that, during the 40´s decade, Dantzig[1] has introduced the famous simplex method. This still plays an extreme important role in the actual days. Even though Klee and Minty [6] provided an example where the simplex method is not efficient. Independently, Khachiyan [4] after several years introduced another method consisting of a novel algorithm for LP. Finally, Karmakar [4] was the first mathematician introducing a new polynomial time algorithm for LP. This is related with interior points methods.

In this short paper we present a novel method, called the face projection method. This, in the first step works as an interior method. But it considers the gradient or the objective function until it reaches the first hyperplane. At this point we consider the cone of the normal vectors of all hyperplanes containing the point. If the objective vector belongs to such a cone, the problem is solved and it reaches a global maximum. If not, we project the objective vector onto the linear manifold which is obtained by the intersection of the hyperplanes mentioned above. It turns out that if this happens and the objective vector is not zero, the objective function at this last projection is strictly greater than before. Therefore, we have derived a new method since the problem is finite. By the last remark the method cannot enter into a cycle.

The Face Projection Method

Consider a linear programming problem given by:

$$\begin{aligned} &\max\ cx \\ &Ax \le b \end{aligned} \qquad (1)$$

where A is a $m \times n$ matrix, $b \in R^m$, $c \in \mathrm{R}^n$ and $x \in \mathrm{R}^n$. Let $X = \{x \in \mathrm{R}^n / Ax \le b\}$ which is considered to be non-empty, convex and compact. Let ∂X be the boundary of X. Consider an interior point $x_0 \in X - \partial X$. At this interior point x_0, the objective function is cx_0. Therefore consider the ray $x_0 + \lambda' c$ with $\lambda' \in \mathrm{R}$. By hypothesis there are negative and positive λ's such that the point $x_0 + \lambda' c$ reaches ∂X. Consider:

$$\bar{\lambda}' = \min_{\lambda_i > 0} \{ a_i (x_0 + \lambda_i c) = b_i \}$$

The point $x_0 + \bar{\lambda}' c$ belongs to an hyperplane H_1 which has some region in the boundary. At that point we have

$$c\left(x_0+\bar{\lambda}'c\right)=cx_0+\bar{\lambda}'cc>cx_0$$

since $cc>0$, $\bar{\lambda}'>0$ and $c\neq 0$. Thus, the objective function has increased the value. At the point $x_0+\bar{\lambda}'c\in\partial X$ consider all the hyperplanes H_i, $i=r_1,\dots,r_s$, containing such a point. Take the cone of normal vectors a_i, $i=r_1,\dots,r_s$. Since they are the components of the incidence matrix A. If c belongs to the cone of H_i, that is $c=\sum_{j=1}^{s}\mu_j a_j$ with $\mu_j\geq 0$, then since $\bar{x}=x_0+\bar{\lambda}'c$ is the point that belongs to the π_i hyperplanes

$$c\left(x_0+\bar{\lambda}'c\right)=c\bar{x}=\sum_{j=1}^{s}\mu_{r_j}a_{r_j}\bar{x}=\sum_{j=1}^{s}\mu_{r_j}b_{r_j}\geq\sum_{j=1}^{s}\mu_{r_j}a_{r_j}x=cx$$

for all $x\in X$. Therefore the point $\bar{x}$ is a global solution of the linear programming.

On the other hand from a geometrical point of view it happens that the hyperplane H_c with normal c separates all the convex polyhedron X. In the case that c does not belongs to the cone, then at least one μ_{r_i} is negative and therefore H_c does not separate the set X.

Let π_i, i : r_i ,.., r_s the subspaces with normal vectors a_i at $x_0+\bar{\lambda}'c$. Then consider

$$\bigcap_{i=1}^{s}\pi_i=\text{gen}\left\{z_1,\dots,z_s\right\}$$

where the z_i determine a orthonormal base. Then the projection of c onto the face $\bigcap_{i=1}^{s}\pi_i$ may be written as

$$\text{proj}_{\bigcap_{i=1}^{s}\pi_i}(c)=\sum_{i=1}^{s}\frac{z_i c}{z_i z_i}z_i$$

and the objective function on it is

$$c\left(x_1+\text{proj}_{\bigcap_{i=1}^{s}\pi_i}(c)\right).$$

Consider the point $x_1+\bar{\lambda}^2\text{proj}_{\bigcap_{i=1}^{s}\pi_i}(c)$, where

$$\bar{\lambda}^2=\min_{\lambda_j>0}\left\{\lambda_j : a_j\left(x_j+\lambda_j\text{proj}_{\bigcap_{i=1}^{s}\pi_i}(c)\right)=b_j, j\notin I_{x_1}\right\}$$

where I_{x_1} is the set of i such that $i\neq r_1,\dots,r_s$. By finiteness of the number n of hyperplanes and by construction, it is clear that, $\bar{\lambda}^2>0$. This is because X is bounded. Therefore

$$c\left(x_1+\bar{\lambda}^2\text{proj}_{\bigcap_{i=1}^{s}\pi_i}(c)\right)=cx_1+\bar{\lambda}^2c\sum_{i=1}^{s}\frac{z_i c}{z_i z_i}z_i=$$

$$cx_1+\bar{\lambda}^2c\sum_{i=1}^{s}\frac{|z_i|^2|c|^2}{|z_i|}\cos^2\theta_i=cx_1+\bar{\lambda}^2|c|^2\sum_{i=1}^{s}\cos^2\theta_i$$

where the cosine is determined by z_i and c. The term with the cosines is strictly greater that zero since at least one $\cos^2 \theta_i$ is strictly greater than zero, and $\overline{\lambda}^2 > 0$ and $|c| > 0$ if $c \neq 0$. Thus we obtain another point $x_2 = x_1 + \overline{\lambda}^2 \operatorname{proj}_{\bigcap_{i=1}^{s} \pi_i}(c)$, where the objective function is strictly greater than in x_1. In this way we repeat the same procedure at x_2 and then by a finiteness argument we obtain the global maximum in a finite number of steps. Therefore, it converges. Thus, the face projection method proposed by us take a form, since, at each projection the value of the objective function strictly increases.

If $x_0 \in \partial X$, we are in the case that the point belongs to an hyperplane and therefore we are in the inductive step described above.

Comment

We have implemented several examples in low dimensional problems and our method is much better than the simplex method.

We would like to say that the purpose of the context of this note is to present a new method in linear programming without considering the algorithm complexity. We would like to say that in this way there are many facts to be considered. As for example, the comparison with other methods, as well as the relation with the Dantzig-Wolfe Theorem. Surely the geometrical part of it appears to be interesting.

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