GRAVIMETRIC MEASUREMENTS ON MOVING AND NON-INERTIAL PLATFORMS

By

Haydey Alvarez Allende, Elias Huchim

Lakshmi Sankar, Seonjeong Lee

Yao Li, Raul Quiroga

and

Eli Vanney Roblero

IMA Preprint Series # 2335

(September 2010)

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455–0436

Phone: 612-624-6066 Fax: 612-626-7370

URL: http://www.ima.umn.edu

Gravimetric Measurements on Moving and Non-inertial Platforms

Haydey Alvarez Allende[1], **Elias Huchim**[2], **Lakshmi Sankar**[3], **Seonjeong Lee**[4], **Yao Li**[5], **Raul Quiroga**[6], **Eli Vanney Roblero**[7]

Abstract

Airborne gravimetry is a fairly recent technique to measure the strength of the gravitational field using a gravimeter mounted on an aircraft. In a moving non-inertial platform, like an aircraft, many factors could affect the measurement of gravimeter. In our project, we study corrections to measurement of gravimeter in the theoretical framework of classical mechanics, special relativity and general relativity.

1 Introduction

The gravitational field of Earth is not uniform and reveals the inner structure of Earth locally, which is helpful in finding oil and mineral deposits. The strength of Earth's gravity varies with latitude, altitude, local topography and geology. Gravimetry is the measurement of the strength of a gravitational field. It is a fairly recent technique to perform gravimetric measurements on moving platforms, for example with gravimeters mounted on aircrafts and satellites. A gravimeter is an accelerometer designed for measuring the constant downward acceleration of gravity.

The principle of kinematic gravimetry is based on Newton's equation of motion. The kinematic acceleration $\vec{a}$ is the sum of specific force (measured vector of acceleration) $\vec{f}$ and gravity $\vec{g}$. In a moving non-inertial platform many factors, like the orientation of the gravity sensor and non-gravitational accelerations could affect the measurement of a gravimeter. As the platform moves with velocity $\vec{v}$ there will be centrifugal force and coriolis force which need to be taken into account.

In section 2 we analyze corrections to the measurement of gravimeter in the theoretical framework of classical mechanics. Classical Mechanics is used to describe

[1] Applied Mathematics Department, Center for Research in Mathematics (CIMAT)
[2] Department of Mathematics, Center for Research in Mathematics (CIMAT)
[3] Department of Mathematics and Statistics, Mississippi State University
[4] Department of Mathematical Sciences, Seoul National University
[5] Department of Mathematics, Georgia Institute of Technology
[6] Department of Mathematics, Center for Research in Mathematics (CIMAT)
[7] Department of Mathematics, Center for Research in Mathematics (CIMAT)

the motion of macroscopic objects such as spacecrafts, planets and stars. Time is considered to be an absolute, which means people thought that a big clock measured time for the entire universe. We find correction vectors to local gravity (which comes just from the mass) and local gravitational acceleration. We choose appropriate coordinate systems to understand the physical phenomena better and also to simplify the calculations. The initial position vector of the moving platform will be given in geographic coordinate system and the coordinate frame on the platform is assumed to be the Frenet frame. Since working in geocentric equatorial inertial frame (GEI) makes the calculations easier we transform the coordinates into GEI using rotation matrices. All coordinate frames and transformations we use are explained in this section.

Next section is about the special relativistic correction to gravimetry. Classical mechanics is an approximation of the real world only when the relative speed is slow and the local gravitational field is weak. Contradicting the classical notion of absolute time, in special relativity, time is a fourth component of the coordinate system (spacetime coordinates) with semi-Riemannian metric tensor. This semi-Riemannian space is called Minkowski spacetime. In Minkowski spacetime, the time is no longer absolute, instead, the speed of light is absolute with respect to any choice of coordinates. We first discuss the corrections in the framework of Einstein's special theory of relativity and find the correction coefficient. In fact, when the moving platform is an airplane, the effect of special relativity could be ignored, while if the moving platform is a satellite, the effect of special relativity could not be ignored since it is in the accuracy of current gravimeter.

In the last section we discuss corrections in the framework of general relativity. Einstein's General relativity theory is a generalization of his own special relativity in which the curvature is no longer 0. In general relativity, the curvature of space replaces the notion of gravitational field. So the freely falling particle in gravitational field moves along the geodesic on the spacetime, and the structure of the spacetime is described by the Einstein's equation.

Generally, the Einstein's equation is difficult to solve. However, under some assumption, we can solve the Einstein's equation exactly, which is called the Schwarzschild solution. So we can compute the geodesic curve numerically, and which will give us the difference of the proper time between the observer on a moving platform and the observer on the ground. In fact, the effect of general relativity can not be ignored with current high accuracy of gravimeter. And the effect of general relativity is more significant than the effect of special relativity.

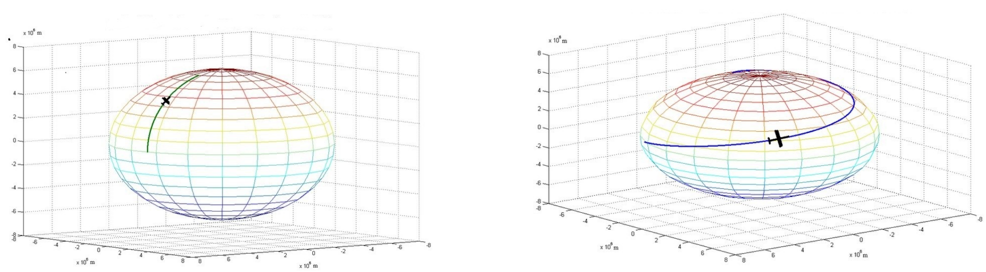
2 Correction to Gravimetry in Classical Mechanics

We start this section by describing the coordinate frames which we will be using. For example we use geographic coordinate frame (GEO) to describe the position vector of the platform according to an observer standing on the Earth. The coordinate frame on the platform is assumed to be Frenet frame. We also explain the time system we use , and then discuss the numerical results we have.

2.1 Coordinate Systems of Earth

Geographic coordinate frame (GEO) is a coordinate system that enables every location on the earth to be specified by a set of numbers. A common choice of coordinates is latitude (ϕ), longitude (λ) and ellipsoid height (h). Every point that is expressed in ellipsoidal coordinates can be also expressed as $x\ y\ z$ cartesian coordinates. The origin is usually the center of mass of the earth, the Z-axis is the Earth's rotation axis and the X-axis is towards the intersection of the Equator and the Greenwich Meridian.
Remark: GEO is a non-inertial reference frame.

Geocentric Equatorial coordinate frame (GEI) functions by projecting Earth's geographic poles and equator onto the celestial sphere. The Z axis is the Earth's rotation axis and its X axis towards the first point of Aries. This system is fixed with respect to distant stars.
Remark: GEI is an inertial reference frame. Hence centrifugal and coriolis forces are absent.



(a) GEO

(b) GEI

Figure 1: Path of platform in GEO and GEI coordinates

2.2 Frenet frame

A Frenet frame is a moving reference frame of n orthonormal vectors which are used to describe a curve locally at each point and it describes local properties in terms of a local reference system. Let $\gamma(t)$ be a trajectory of the platform moving through a 3-dimensional space. Then Frenet frame is defined by the following 3 orthonormal vectors $T(t),\ N(t),\ B(t)$.

$$
\begin{aligned}
T(t) &= \frac{\gamma'(t)}{\|\gamma'(t)\|} \\
N(t) &= \frac{T'(t)}{\|T'(t)\|} \\
B(t) &= T(t) \times N(t).
\end{aligned}
$$

2.3 Ellipsoidal Coordinate system

We use ellipsoidal coordinates systems as it closely approximate Earth's surface. A rotational ellipsoid is selected which is flattened at the poles and is created by rotating the meridian ellipse about its minor axis b. The transformation equation

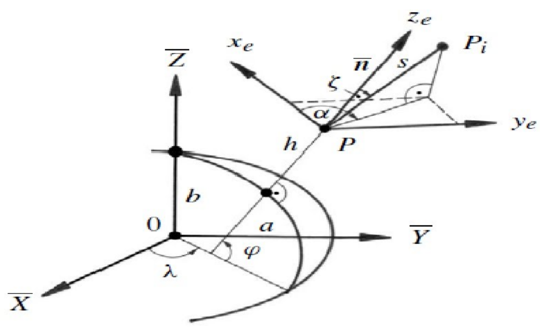


Figure 2: Ellipsoidal coordinate system

between the geographic ellipsoidal coordinates ϕ, λ, h and Cartesian coordinates is:

$$
\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} (\bar{N}+h)\cos\phi\cos\lambda \\ (\bar{N}+h)\cos\phi\sin\lambda \\ ((1-e^2)\bar{N}+h)\sin\phi \end{pmatrix} \tag{2.1}
$$

$\bar{N}$ is the radius of curvature in the prime vertical given by:

$$
\bar{N} = \frac{a}{\sqrt{1-e^2\sin^2\phi}}
$$

where a is the semi-major axis and $e = \sqrt{\frac{a^2-b^2}{a^2}}$ is the first numerical eccentricity.

2.4 The time system

In pre-Einstein relativity time is considered to be an absolute. The time interval between any pair of events is same for all observers. Classical mechanics assumes Euclidean geometry for the structure of space and absolute time. The Greenwich sidereal time (θ) is the angle between the Greenwich meridian and the first point of Aries measured eastward from the first point of Aries in the Earth's equator and can be calculated as follows:

$$\theta = 100.461 + 36000.770\, T_0 + 15.04107\, UT \tag{2.2}$$

where

$$T_0 = \frac{MJD - 51544.5}{36525}$$

with MJD the Modified Julian Date (ie. Julian Date -2400000.5) and UT (universal time) is the local time of the Greenwich meridian; and the Julian date (JD) is the interval of time in days and fractions of a day since January 1, 4713 BC Greenwich noon. Note that the angle θ is a function of the time of day and time of year.

2.5 Procedure

Let (ϕ, λ, h) be the initial position vector given in geographic coordinate system. We can get the corresponding cartesian coordinate (X, Y, Z) in GEO as explained in (2.1). Since centrifugal and coriolis forces are absent in GEI we will use a transformation described below to go from GEO to GEI which will make the calculations easier. The rotation matrix R_{oi} for this transformation is calculated as:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

where θ is the Greenwich sidereal time. Hence the corresponding position vector in GEI is :

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{GEI} = R_{oi} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{GEO}.$$

We first consider local gravity which comes just from the mass. A correction vector (say, CR1) which is the acceleration of platform is calculated in GEI coordinates. Suppose the measured acceleration (specific force) is $\vec{f}$, then the local gravity on the platform, $g_h = R_{fi}\vec{f} - CR1$ where R_{fi} is the rotation matrix for the transformation from Frenet frame to GEI coordinate system. Note that g_h will be in GEI coordinate system. Then the local gravity on the surface of earth (say g_0) is given by $g_0 = g_h(\frac{r+h}{r})^2$ where r is the mean radius of the Earth and h is the height of the platform from the sea level.

Next we consider the local gravitational acceleration, where local gravity and the centrifugal force due to Earth's rotation has to be considered. Here we calculate a correction vector (say CR2) which is the sum of CR1 and the centrifugal force. Then the local gravitational acceleration on the platform, $g'_h = R_{fi}\vec{f} - CR2$ and the local gravitational acceleration on Earth's surface (g'_0) is given by

$$g'_0 = \left(\frac{r+h}{r}\right)^2 g'_h - (\omega \times v) + (\omega \times (v+h))(\frac{r+h}{r})^2$$

where v is the velocity vector and ω is the angular velocity of Earth's rotation. Figure 3 demonstrates the local Frenet frame $\{T, B, N\}$ and the gravity vector G at time $t = 1000s$. Norms of correction vectors CR1, CR2 are described in figures below.

When the acceleration of the moving platform is not significant, the majority of

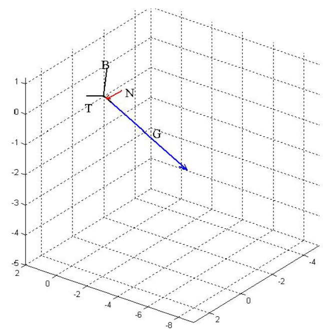
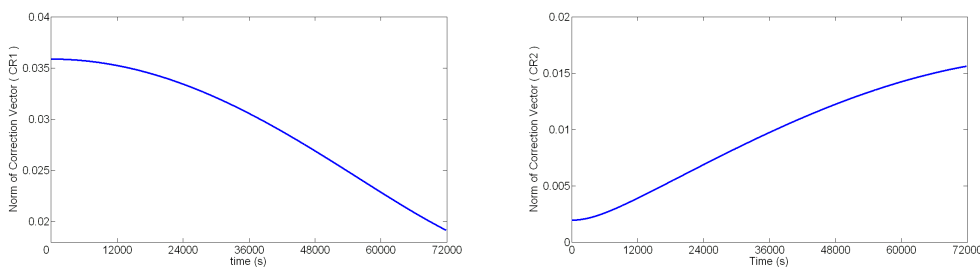


Figure 3: The local Frenet frame and the gravity vector.



(a) CR1

(b) CR2

Figure 4: Norms of corrections CR1 and CR2.

correction $CR1$ is the centrifugal force, so when the airplane is flying to high latitude area, the norm of correction $CR1$ decreased. Also, the majority of correction $CR2$ is the Coriolis force, so when the airplane is flying to high latitude area, the norm of correction $CR2$ increased. We can also find this on the figure 4.

3 Correction to Gravimetry in Special Relativity

In previous section, we discussed the gravimetric measurement problem on moving platform, and gave correction of gravimetry under the theoretical framework to classical mechanics. As explained in the introduction if we consider the effect of the relative speed as well as the gravitational field, then the physical laws in Newton's mechanics must be modified.

Nowadays, the accuracy of gravimeter is fairly high (about $10^{-9}Gal$), which makes additional corrections necessary. In this section, as a basic theoretical background, the semi-Riemannian geometry will be reviewed. Then we will introduce the

Minkowski space, which is an important semi-Riemannian space, and the foundation of special relativity. Next, we will deduce the remarkable Lorentz transformation. Finally, from Lorentz transformation, we could give the formula about correction in gravimetry when consider the effect of special relativity.

3.1 Review: Semi-Riemannian Geometry

In differential geometry, a semi-Riemannian manifold, which is also called pseudo-Riemannian manifold, is a generalization of a Riemannian manifold. The key difference between a Riemannian manifold and a semi-Riemannian manifold is that on a semi-Riemannian manifold the Riemannian metric tensor need not be positive-definite. Some fundamental theorems of Riemannian geometry can be generalized to the pseudo-Riemannian case, for example the Levi-Civita connection is also well defined on a semi-Riemannian manifold, which means we could associate the curvature tensor on a semi-Riemannian manifold. Meanwhile, some significant differences between semi-Riemannian manifold and Riemannian manifold also exist, for example, a submanifold of a semi-Riemannian manifold need not be a pseudo-Riemannian manifold.

Definition 3.1. Let M be a differential manifold with dimension n, a metric tensor g_p on M is a family of symmetric nondegenerate (0,2) tensor field of constant index

$$g_p : T_pM \times T_pM \to \mathbb{R}, \qquad p \in M$$

such that for all differentiable vector fields X, Y on M,

$$p \longmapsto g_p(X(p), Y(p))$$

defines a smooth function $M \to \mathbb{R}$.

Definition 3.2. A semi-Riemannian manifold is a differential manifold with a metric tensor g.

Remark 3.3. A Riemannian manifold is a special case of the semi-Riemannian manifold.

If the local coordinate system on M is given as $\{x_1, \cdots, x_n\}$, as in Riemannian geometry, we can also write the metric tensor g as g_{ij} relative to the local coordinate system.

Example 3.4. *Let $\{\partial_i, \cdots, \partial_n$ be the natural coordinates of $T_p\mathbb{R}^n$, and denote $v_p = sum_{i=1}^n v_i \partial_i$ for $v \in T_p\mathbb{R}^n$. Then if the metric tensor is given as*

$$g(v_p, w_p) = -\sum_{i=1}^{v} v_i w_i + \sum_{i=v+1}^{n} v_i w_i$$

of some index $1 \leq v \leq n$, this semi-Riemannian space is called semi-Euclidean space $\mathbb{R}^n_v$.

Since the metric tensor g is not positive definite as in Riemannian geometry, we have the following definitions:

Definition 3.5. For $v \in T_pM$, we have

$$\begin{array}{lll} v \text{ is spacelike} & \text{if} & g_p(v,v) > 0 \text{ or } v = 0, \\ v \text{ is lightlike} & \text{if} & g_p(v,v) = 0 \text{ and } v \neq 0, \\ v \text{ is timelike} & \text{if} & g_p(v,v) < 0. \end{array}$$

In the following of this section, we will review an important property of semi-Riemannian manifold – the Levi-Civita connection. From now on, we denote the set of smooth vector field on M by $\mathfrak{X}(M)$, and denote the set of smooth real-valued functions on M by $\mathfrak{F}(M)$. Before introducing the Levi-Civital connection, we have the following definitions.

Definition 3.6. If $u_1, \cdots, u_n$ are natural coordinates on $\mathbb{R}^n_v$, and $V = \sum V_i \partial_i$, $W = \sum W_i \partial_i$ are two vector fields. Then the vector field

$$D_v W = \sum V(W_i) \partial_i$$

is called the natural covariant coordinate of W with respect to V.

Then we can extend the definition of covariant coordinate from natural coordinates to the general semi-Riemannian manifold.

Definition 3.7. A connection D is a map $D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

- $D_V W$ is $\mathfrak{F}(M)$ linear in V
- $D_V W$ is $\mathbb{R}$ linear in W
- $D_V(fW) = (Vf)W + fD_v W$ for $f \in \mathfrak{F}(M)$.

Then we have the fundamental result about the existence and uniqueness of the Levi-Civital connection. This result gives a semi-Riemannian manifold a series of remarkable similarity as a Riemannian manifold.

Theorem 3.8. *On a semi-Riemannian manifold (M, g), there exist a unique connection D such that*

- $[V, W] = D_V W - D_W V$
- $Xg(V, W) = g(D_X V, W) + g(V, D_X W)$.

Proof. See [1]. □

From now on, without specially mentioned, the term $D_V W$ means the Levi-Civital connection of W with respect to V. With the Levi-Civital connection, we can define the Christoffel symbol, the parallel transport, the geodesic and the curvature tensor on a semi-Riemannian manifold.

Definition 3.9. Let $u_1, \cdots, u_n$ be the local coordinate on a neighborhood U in a semi-Riemannian manifold M, then the following function Γ_{ij}^k is called the Christoffel symbol:

$$\Gamma_{ij}^k = (D_{\partial_i}\partial_j)_k$$

Definition 3.10. Let $\gamma : I \to M$ be a smooth curve on M and V be a vector field along γ, then V is called parallel if

$$D_{\gamma'(t)}V = 0 \qquad \forall t \in I.$$

So the definition of geodesic could be given as,

Definition 3.11. A geodesic is a curve γ on M whose vector field $\gamma'(t)$ is parallel.

Intuitively, the geodesic is a constant speed curve on a manifold.
Different from the Riemannian geometry, the speed of geodesic could be 0.

Definition 3.12. The null-geodesic is a geodesic $\gamma(t)$ with $g(\gamma', \gamma') = 0$ for each t.

And we have the following theorem about the geodesic equation, which is same as the geodesic equation in Riemannian geometry.

Theorem 3.13. *Let $u_1, \cdots, u_n$ be the local coordinates on $U \subset M$, a curve $\gamma(t)$ is a geodesic if and only if $\gamma(t)$ satisfies the following equation*

$$\frac{d^2\gamma_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt} \tag{3.1}$$

where γ_k is the k-th coordinate on $\gamma(t)$.

Moreover, we can give the definition of the Riemannian curvature tensor.

Definition 3.14. Let (M, g) be a semi-Riemannian manifold with Levi-Civita connection D, then the following $(3, 1)$ tensor R is called the Riemannian curvature tensor:

$$R : \mathfrak{X}^3(M) \to \mathfrak{X}(M)$$

$$R_{XY}Z = D_{[X,Y]}Z - [D_X, D_Y]Z. \tag{3.2}$$

Remark 3.15. With the Riemannian curvature tensor R, we can have the $(4, 0)$ Riemannian curvature, the sectional curvature, the scalar curvature as well as the Ricci curvature as in the Riemannian geometry.

3.2 The Minkowski Spacetime

We reviewed the semi-Riemannian geometry in the previous subsection. Now let's introduce a special case of semi-Riemannian geometry – the Minkowski spacetime. Minkowski spacetime is a "flat" semi-Riemannian geometry with zero curvature, we can investigate the effect of special relativity on this semi-Riemannian space.

Definition 3.16. If in some coordinates, the metric tensor have the form

$$g(v_p, w_p) = -\sum_{i=1}^{v} v_i w_i + \sum_{i=v+1}^{n} v_i w_i$$

for each $p \in M$, then the common value v is called the index of M.

Definition 3.17. If M is a semi-Riemannian manifold with index $v = 1$ and $\dim M \geq 2$, then M is called a Lorentz manifold.

And we have the following definitions about the Minkowski spacetime.

Definition 3.18. A spacetime is a connected time-oriented 4-d Lorentz manifold

Definition 3.19. A Minkowski spacetime M is a space that is isometric to $\mathbb{R}^4_1$.

It is known that in a Newtonian spacetime, the time is independent with space, and time is "absolute". However, a Minkowski spacetime M is just a space that is isometric to $\mathbb{R}^4_1$, so the choice of time is not unique. Moreover, the time on a Minkowski spacetime is not independent with space. Recall that on $\mathbb{R}^4_1$, the metric tensor is given as

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2,$$

which gives the relation between time and space in a Minkowski spacetime.

Then let's define some concepts that describe the motion in a Minkowski spacetime.

Definition 3.20. The positive direction of M is called future, the negative direction of M is called past.

Definition 3.21. For $p \in M$, the future time cone is the set

$$\{q \in M; \vec{pq} \text{ is time like}\},$$

the future lightcone is the set

$$\{q \in M; \vec{pq} \text{ is light like}\}.$$

Definition 3.22. A material particle in M is a time like future pointing curve $\alpha : I \mapsto M$, s.t. $|\alpha'(\tau)| = 1, \forall \tau \in I$.

Definition 3.23. A lightlike particle is a future-pointing null-geodesic $\gamma : I \mapsto M$.

Definition 3.24. A particle in M that is a geodesic is said to be freely falling.

Different from the Newtonian spacetime $\mathbb{R} \times \mathbb{R}^3$, we don't have canonical time function on M. Instead, we have following definition about the time on a Minkowski spacetime:

Definition 3.25. The proper time of a material particle in M is the parameter τ.

The freely falling particles are not unique, and the proper time is also not unique. In fact, for every freely falling particle, we can find a coordinate system corresponding to its proper time, which is called Lorentz coordinate system.

Definition 3.26. A Lorentz coordinate system in M is a time orientation preserving isometry.

$$\xi : M \to \mathbb{R}^4_1$$

Remark 3.27. In Lorentz coordinate system, the inner product is

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Remark 3.28. For simplicity, from now on we will rescale the coordinate such that the light speed c is the unit speed. And the inner product becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

At the last of this subsection, we want to claim that in Minkowski space, time is relative, but the light speed is absolute. No matter which reference frame we would choose, the light speed is invariant. Actually, we have the following result in [1].

Theorem 3.29. *The light has the same constant speed* 1 *relative to every Lorentz coordinates.*

3.3 Lorentz Transformation and Some Effects in Special Relativity

In the last subsection, we reviewed the notation and some basic properties of the Minkowski spacetime. At this point, we will demonstrate the coordinate transformation in the Minkowski spacetime, which is called Lorentz transformation. The Lorentz transformation describes how, according to the theory of special relativity, two observers' varying measurements of space and time can be converted into each other's frames of reference. It reflects the surprising fact that observers moving at different velocities may measure different distances, elapsed times, and even different orderings of events.

First, we have the following result from the definition of Lorentz coordinates:

Theorem 3.30. $\xi : M \mapsto \mathbb{R}^4_1$ *is Lorentz coordinate system if and only if* $g_{ij} = \delta_{ij}\epsilon_j$, *where* $\epsilon = (-1, 1, 1, 1)$.

For simplicity, first we will consider the Lorentz coordinates on $\mathbb{R}^2_1$. Suppose (x, t) is the coordinate axis of a Lorentz coordinate system ξ, and ξ' is another Lorentz coordinate system corresponding to a freely falling particle, the relative speed is v, then clearly the time axis of ξ' has direction vector $(1, v)$. From the theorem 10, the space axis of ξ' should have direction vector $(1/v, 1)$. (See figure 10)

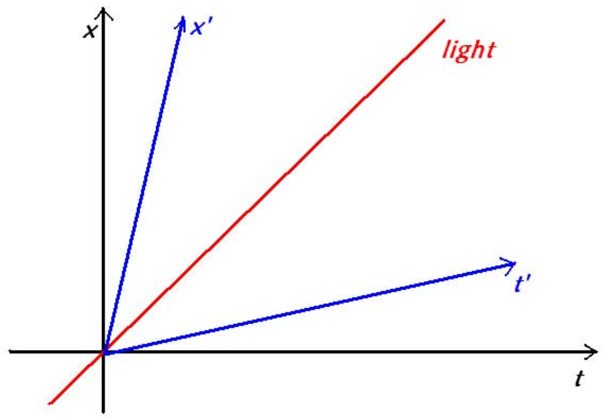


Figure 5: Lorentz Transformation

Then suppose an event σ has Lorentz coordinates (x, t) in the coordinate system ξ, suppose (x', t') are the coordinates of this event in the coordinate system ξ'. Then we have

$$(x, t) = x'\vec{e_x} + t'\vec{e_t} \tag{3.3}$$

where $\vec{e_x}$ and $\vec{e_t}$ is the unit vector with direction $(1/v, 1)$ and $(1, v)$. From the rule of the inner product, we have

$$\vec{e_x} = (\frac{1}{\sqrt{1-v^2}}, \frac{v}{\sqrt{1-v^2}}), \quad \vec{e_t} = (\frac{v}{\sqrt{1-v^2}}, \frac{1}{\sqrt{1-v^2}}).$$

Solving the equation (3.3), we can find the coordinate (x', t').

$$\begin{array}{rcl} x' & = & \frac{1}{\sqrt{1-v^2}}(x - vt) \\ t' & = & \frac{1}{\sqrt{1-v^2}}(t - vx). \end{array}$$

Remark 3.31. For the coordinates (x, t) and (x', t') without rescaling, we set $\gamma = 1/\sqrt{1-v^2}$ as the Lorentz factor, then we have the following expression:

$$\left\{ \begin{array}{rcl} x' & = & \gamma(x - vt) \\ t' & = & \gamma(t - \frac{vx}{c^2}). \end{array} \right.$$

For the Minkowski space, we can find the Lorentz transformation based on the previous discussions. Now suppose ξ is a Lorentz coordinate system, σ is a freely falling particle with speed v in ξ, then we can change the coordinate of the space variables to let the projection of v on $\mathbb{R}^3$ is the x-axis, and denote the coordinates in ξ as (t, x, y, z). Then we can consider the Lorentz coordinates corresponding to the particle σ. By similar computation, we will get the following Lorentz transformation:

$$\begin{cases} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - \frac{vx}{c^2}). \end{cases}$$

Finally, when the projection of the velocity of a freely falling particle on $\mathbb{R}^3$ is (v_x, v_y, v_z), we have:

$$\beta_x = \frac{v_x}{c}; \quad \beta_y = \frac{v_y}{c}; \quad \beta_z = \frac{v_z}{c}$$

and

$$\beta = \frac{|v|}{c} = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

and the Lorentz transformation is given by

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta_x\gamma & -\beta_y\gamma & -\beta_z\gamma \\ -\beta_x\gamma & 1 + (\gamma - 1)\frac{\beta_x^2}{\beta^2} & (\gamma - 1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma - 1)\frac{\beta_x\beta_z}{\beta^2} \\ -\beta_y\gamma & (\gamma - 1)\frac{\beta_y\beta_x}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_y^2}{\beta^2} & (\gamma - 1)\frac{\beta_y\beta_z}{\beta^2} \\ -\beta_z\gamma & (\gamma - 1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma - 1)\frac{\beta_z\beta_y}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_z^2}{\beta^2} \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}.$$

From the Lorentz transformation, we can summarize the effects of special relativity

- Time dilation: Suppose we have two observers A and B, A is on the ground, B is moving with speed v, then the time interval observed by B is dilated by coefficient γ compared with that observed by A.

- Length contraction: Suppose we have two observers A and B, A is on the ground, B is moving with speed v. Let an object moving with B has length L, then from the observation of A, the length is L/γ.

- Moving Mass: For the object with rest mass m_0, the moving mass is

$$m = \gamma m_0$$

- Relativistic mechanics: In special relativity, Newton's second law does not hold in its form $\vec{F} = m\vec{v}$, but it does if it is expressed as

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{dt}.$$

Notice that $m = \gamma m_0$, we have

$$\vec{F} = \frac{\gamma^3 m_0 (\vec{v} \cdot \vec{a})}{c^2} \vec{v} + \gamma m_0 \vec{a}.$$

So in special relativity, the acceleration does not parallel with the Force.

3.4 Our Results and Correction Formulas

Now we can consider the gravimetry problem. When the moving speed of the platform is high, from the Lorentz transformation, the time dilation may cause additional error on the gravimeter. Here we have an additional assumption:

Assumption: The direction of gravity and the direction of velocity of platform is almost orthogonal.

Remark 3.32. The reason why we have this assumption is that the speed could cause not only time dilation but also length contraction. If the length on the direction of gravity is also contracted, then from an observer on the ground, the acceleration on the ground and the acceleration on the platform is different, which does not make sense.

According to the Lorentz transformation, from the observer on the ground, the time on moving platform is dilated by coefficient γ:

$$\mathrm{d}t' = \gamma \mathrm{d}t$$

$$a' = \frac{\mathrm{d}^2 s}{\mathrm{d}t'} = \gamma^{-2} \frac{\mathrm{d}^2 s}{\mathrm{d}t} = \gamma^{-2} a$$

so the special relativistic correction is γ^2.

Now suppose the measured acceleration is $\vec{f}$, and the correction formula from classical mechanics is

$$\vec{g} = R_{fi} \vec{f} - CR$$

where CR is the correction, and R_{fi} is the rotation matrix for the transformation from Frenet frame to GEI coordinates, then considering the special relativity, the new correction formula becomes

$$\vec{g} = R_{fi} \gamma^2 \vec{f} - CR.$$

Discussion

Suppose the moving platform is an airplane with speed $400 km/h$, then the correction coefficient is

$$\gamma^2 = 1.000000000000137174 = 1 + 1.37174 \times 10^{-13}$$

which is beyond the current accuracy of the gravimeter, so the effect of special relativity can be ignored.

Suppose the moving platform is a satellite with speed $7.5km/s$, then the correction coefficient is

$$\gamma^2 = 1.000000000693444 = 1 + 6.93444 \times 10^{-10}.$$

Since the gravity is about $980gal$ and the current accuracy of the gravimeter could be up to $10^{-9}gal$, the effect of special relativity can not be ignored in this case. So for the gravimetry on satellite, we must take the correction from special relativity into consideration.

4 Correction to Gravimetry in General Relativity

General relativity or the general theory of relativity is the geometric theory of gravitation published by Albert Einstein in 1915. It is the current description of gravitation in modern physics. It generalises special relativity and Newton's law of universal gravitation, providing a unified description of gravity as a geometric property of space and time, or spacetime.

General relativity's predictions have been confirmed in all observations and experiments to date. We are interested in finding and describing the ways in which particles move around stars and planets.

Physics tells us that particles in the universe move describing "the shortest path". In the standard reference frame of classical mechanics, objects in free motion move along straight lines at constant speed. But, in spacetime the particles (freely falling) have a "different" movement. This freely falling particles are modeled by timelike geodesic in spacetime. The gravity bends the geodesic on which particles are moving on the spacetime but does not bend the worldline of the particles.

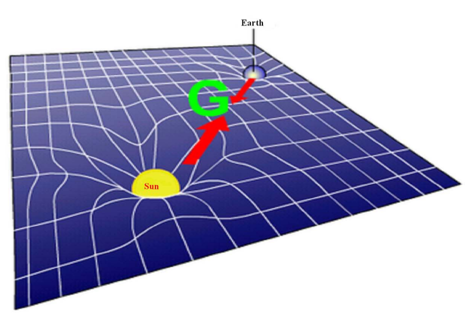


Figure 6: Gravity bends geodesic

The relative position of particles in motion (these particles are freely falling) around a freely falling particle moving on a trajectory γ_0 are given by Jacobi vector fields Y on γ_0.

Definition 4.1. If γ is a geodesic, a vector field Y on γ that satisfies the *Jacobi differential equation* $Y'' = R_{Y\gamma'}(\gamma')$ (where R is the *Ricci curvature tensor*) is called a *Jacobi vector field.*

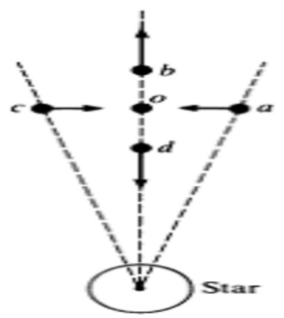


Figure 7: Relative accelerations observed by O

If we suppose these particles have unit mass we can read the Jacobi equation $V'' = R_{V\gamma'}(\gamma')$ as Newton's second law with the curvature vector $R_{V\gamma'}(\gamma')$ in the role of force, so-called *tidal force*.

When a body (body 1) is affected by the gravity of another body (body 2), the field can vary significantly on body 1 between the side of the body facing body 2 and the side facing away from body 2. Figure 8 shows the differential force of gravity on a spherical body (body 1) exerted by another body (body 2). These so called *tidal forces* cause strains on both bodies and may distort them or even, in extreme cases, break one or the other apart.

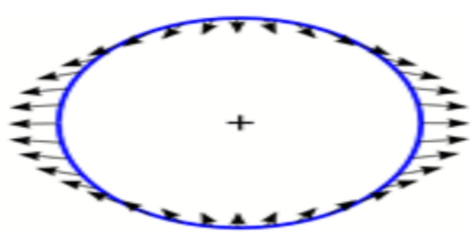


Figure 8

.

4.1 Einstein Equation

"Matter" is an undefined term that we use intuitively to mean all the stuff of the universe. In Newtonian physics the unique source of gravitation is the *mass* of matter. Relativistically, gravitation springs from the *energy-momentum* (a relation between the energy, momentum and the mass of a body) of matter, to which mass is but one contributor. For a particular form of matter modeled in a spacetime M the energy-momentum content is described infinitesimally by a *stress-energy tensor field* T on M. And the conservation of energy-momentum is expressed infinitesimally by $\operatorname{div} T = 0$.

The flow of a fluid could be described literally by a vast swarm of particles in a spacetime M. Instead of this discrete model it is easier to deal with a smooth model, where the 4-velocity of the flow is given by a timelike unit vector field U on M. Intuitively, the integral curves of U are the average wordlines of the "molecules" of the fluid.

The classical stress tensor measures internal forces in a body in space by giving at each point m the forces across all the surface elements through m. To motivate the definition of *stress-energy tensor field* we apply this scheme heuristically to the spacetime flow of a so-called *perfect fluid*.

Fix $m \in M$. If $v \in T_m M$ is a unit vector, then in a hypersurface through m perpendicular to v let, $B\,(v)$ be a small coordinate cube centered at m. Let P_B^+ be the total energy-momentum of molecules in $B\,(v)$ that are crossing from the $-v$ to the $+v$ side of $B\,(v)$; correspondingly let P_B^- derive from those crossing from the $+v$ to the $-v$. Then for another unit vector $w \in T_m\,(M)$, *let* $T\,(v, w)$ *be the limit as* $\mathrm{vol}B\,(v) \to 0$ *of the* w *component of* $P_B = P_B^+ - P_B^-$.

Now, let $u = U_m$ and consider the following choices of v, w.

(1) $T(u, u) = \rho(m)$, *energy density* at m.

The infinitesimal observer u, riding with the flow can consider $B(u)$ as a local restspace since its tangent space at m is $u^\perp$. Then $P_B^- = 0$, and $P_B^+ = P_B = E_B u + \vec{P}_B$ gives the energy E_B and momentum $\vec{P}_B$ of the box B as measured by u. Then by the definition above,

$$T(u, u) = \lim_{volB \to 0} \frac{E_B}{volB},$$

which is the energy density of the fluid as measured by u.

(2) If $x, y \in u^\perp$, then $T(x, y) = \wp(m)\langle x, y\rangle$, where $\wp(m)$ is the *pressure* at m.

Since $x \perp u$, the box $B(x)$ is a three dimensional spacetime. Let $B(x) = \Sigma \times I$, where Σ is a spacelike patch of surface through m and I is a time interval of length Δt. Then if A is the area of Σ,

$$T(x, y) = \lim_{A \to 0} \frac{1}{A} \left[\lim_{\Delta t \to 0} \frac{\langle P_B, y\rangle}{\Delta t} \right].$$

The second limit shows that $T(x, y)$ is the y component of *stress* (force per unit area) across Σ in the x direction. Thus T restricted to $u^\perp$ is the classical stress tensor measured by u in his restspace $u^{\perp} \approx B(u)$. The pressure $\wp_x$ is the same in all space directions, hence $T(x, y) = \wp(m)\langle x, y\rangle$.

(3) If $x \in u^{\perp}$, then $T(x, u) = T(u, x) = 0$.

Definition 4.2. A **perfect fluid** on a spacetime M is a triple $(U, \rho, \wp)$ where:
(1) U is a timelike future-pointing unit vector field on M called the *flow vector field.*
(2) $\rho \in \mathfrak{F}(M)$ is the *energy density function*; $\wp \in \mathfrak{F}(M)$ is the *pressure function.*
(3) The *stress-energy tensor* is

$$T = (\rho + \wp)\, U^{*} \otimes U^{*} + \wp \mathbf{g}$$

where U^{*} is the one-form metrically equivalent to U.

Evidently this formula for T is equivalent to the three equations found above for $X, Y \perp U$, namely:

$$T\,(U, U) = \rho, \qquad T\,(X, U) = T\,(U, X) = 0, \qquad T\,(X, Y) = \wp\langle X, Y\rangle.$$

Matter is gravitationally significant only as a carrier of energy-momentum, so for its effect as a source of gravitation (known as curvature) we must look to the strees-energy tensor T. But how is T related to the curvature tensor? Einstein proposed the formula $G = kT$, where G is some variant of Ricci curvature and k is constant. He tried several possibilities for G.

Definition 4.3. *The Einstein gravitational tensor G of a spacetime M is*

$$G = Ric - \frac{1}{2} S\mathbf{g}.$$

If M is a spacetime containing matter with stress-energy tensor T, then G can be expressed by

$$G = 8\pi T,$$

and it tells how matter determines Ricci curvature. $\operatorname{div} T = 0$ tells how Ricci curvature moves this matter. If $T = 0$, that is, if M is Ricci flat, then M is said to be a *vacuum.*

4.2 Schwarzschild Solution

The Einstein field equations are nonlinear and very difficult to solve. Einstein used approximation methods in working out initial predictions of the theory. But as early as 1916, the astrophysicist Karl Schwarzschild found the first non-trivial exact solution to the Einstein field equations, the so-called Schwarzschild metric. This solution laid the groundwork for the description of the final stages of gravitational collapse, and the objects known today as black holes.

Schwarzschild spacetime will emerge naturally from the physical conditions given below.

(1) *Static* : The spacetime is to be static relative to observers comparable to Newtonian observers at rest in Euclidean 3-space. Let the spacetime be the manifold $\mathbb{R}^1 \times \mathbb{R}^3$ with line element of the form

$$A(x)dt^2 + \mathbf{q} \quad (x \in \mathbb{R})$$

where $\mathbf{q}$ is lifted from $\mathbb{R}^3$.

(2) *Spherical Symmetry* : Since the star and hence the resulting space-time are to be spherically symmetric, for each $\phi \in O(3)$ the map

$$(t, x) \to (t, \phi x)$$

must be an isometry. Thus it is natural to give a spherical description of $\mathbb{R}^3$ (minus the origin) as $\mathbb{R}^+ \times S^2$, where $\mathbb{R}^+ = \{\rho \in \mathbb{R} : \rho > 0\}$ and S^2 is the unit 2-sphere. Spherical symmetry implies that the line element $\mathbf{q}$ on $\mathbb{R}^+ \times S^2 \approx \mathbb{R}^3 - 0$ can be written as $B(\rho)d\rho^2 + C(\rho)d\sigma^2$, where $d\sigma^2$ is the line element standard on the unit sphere. For every $\phi \in O(3)$ the differential map $id \times \phi$ carries ∂_t to ∂_t, hence the coefficient function $A(x)$ of dt^2 actually depends only on ρ. Thus the line element on $\mathbb{R}^1 \times \mathbb{R}^+ \times S^2 \approx \mathbb{R}^1 \times (\mathbb{R}^3 - 0)$ becomes

$$A(\rho)dt^2 + B(\rho)d\rho^2 + C(\rho)d\sigma^2.$$

(3) *Normalization* : A change of variable in $\mathbb{R}$ replaces $C(\rho)$ by r^2, so the line element is now

$$E(r)dt^2 + G(r)dr^2 + r^2 d\sigma^2.$$

(4) *Vacuum and Minkowski at infinity* : The only source of gravitation in the Schwarzschild universe is the star itself -which we do not model. Thus the spacetime must be a *vacuum*, that is, Ricci flat.

The influence of gravitation becomes arbitrarily small if the particle is sufficiently far away from the source of gravitation. Hence we require that, as r approaches infinity, the Schwarzschild metric tensor approaches the Minkowski metric of empty spacetime, which in spherical term is

$$-dt^2 + dr^2 + r^2 d\sigma^2.$$

Thus $E(r) \to -1$, and $G(r) \to +1$ as $r \to +\infty$.

The conditions above determine the functions E and G and thereby the Schwarzschild metric. Hence we have the warped product line element

$$-\mathbf{h} dt^2 + \mathbf{h}^{-1} dr^2 + r^2 d\sigma^2$$

with the *Schwarzschild function* $\mathbf{h}(r) = 1 - \frac{2M}{r}$.

4.3 Procedure

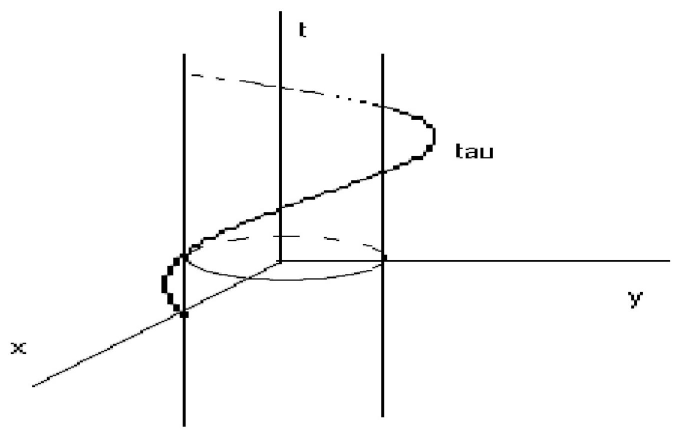


Figure 9

In general relativity there is no absolute time. Here we consider two different times. One is the time fixed on the stars (time in GEI frame), say t, and the other is the proper time on the platform, say τ. A path of the platform in the spacetime is given by

$$\gamma(\tau) = (x(\tau), y(\tau), z(\tau), t(\tau))$$

with velocity

$$\frac{d\gamma}{d\tau} = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, \frac{dt}{d\tau} \right).$$

In General Relativity $t = f(\tau)$,

$$\begin{aligned}
\frac{d^2x}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{dx}{d\tau}\right) \\
&= \frac{d}{d\tau}\left(\frac{dt}{d\tau}\frac{dx}{dt}\right) \\
&= \frac{dt}{d\tau}\frac{d}{d\tau}\left(\frac{dx}{dt}\right) + \frac{dx}{dt}\frac{d}{d\tau}\left(\frac{dt}{d\tau}\right) \\
&= \left(\frac{dt}{d\tau}\right)^2\left(\frac{d^2x}{dt^2}\right) + \frac{d^2t}{d\tau^2}\frac{dx}{dt}. \qquad (4.1)
\end{aligned}$$

Here the first term $\left(\frac{dt}{d\tau}\right)^2\left(\frac{d^2x}{dt^2}\right)$ is as we saw in the special relativity case and the second term $\frac{d^2t}{d\tau^2}\frac{dx}{dt}$ is due to the effect of general relativity. An example below shows the effect of the general relativity term on the measure of the gravity.

We consider the movement of a particle on the Earth (from the equator to the north pole in the meridian). Suppose that the Earth does not have any movement and the movement of the particle in the XZ plane is given by

$$\gamma_1(\tau) = (R_0\cos(a\tau), 0, R_0\sin(a\tau), t = f(\tau))$$

with linear angular velocity a and constant R_0.

To make the calculations of general and special relativity terms in the Schwarzschild metric easier, we need to convert γ_1 (cartesian coordinates) to γ_2 (spherical coordinates). Then γ_2 can be calculated as

$$\gamma_2(\tau) = (R_0, \theta_0, \varphi(\tau), t = f(\tau))$$

with

$$\gamma_2' = \frac{d}{d\tau}\gamma_2(\tau) = \left(\frac{dR_0}{d\tau}, \frac{d\theta_0}{d\tau}, \frac{d\varphi(\tau)}{d\tau}, \frac{df(\tau)}{d\tau}\right) = \left(0, 0, \frac{d\varphi}{d\tau}, \frac{df}{d\tau}\right).$$

Also we need

$$\langle\gamma_2'(\tau), \gamma_2'(\tau)\rangle_{Sch} = -1 \qquad (4.2)$$

in the Schwarzschild inner product where the Schwarzschild line element is given by

$$ds^2 = r^2 d\sigma^2 + \frac{1}{1-\frac{2M}{r}}dr^2 - \left(1 - \frac{2M}{r}\right)dt^2.$$

Then (4.2) can be calculated as follows :

$$
\begin{aligned}
-1 &= \left(r^2 d\sigma^2 + \frac{1}{1-\frac{2M}{r}} dr^2\right)\left(\left(0,0,\frac{d\varphi}{d\tau}\right),\left(0,0,\frac{d\varphi}{d\tau}\right)\right) - \left(1-\frac{2M}{R_0}\right)\left(\frac{df}{d\tau}\right)^2 \\
&= \left[\frac{1}{1-\frac{2M}{r}} dr^2 + r^2\left(d\theta^2 + \sin^2(\theta) d\varphi^2\right)\right]\left(\left(0,0,\frac{d\varphi}{d\tau}\right),\left(0,0,\frac{d\varphi}{d\tau}\right)\right) - \left(1-\frac{2M}{R_0}\right)\left(\frac{df}{d\tau}\right)^2 \\
&= \frac{1}{1-\frac{2M}{r}}\cdot 0^2 + R_0^2 \cdot 0^2 + R_0^2 \sin^2(\theta_0)\left(\frac{d\varphi}{d\tau}\right)^2 - \left(1-\frac{2M}{R_0}\right)\left(\frac{df}{d\tau}\right)^2 \\
&= R_0^2 \sin^2(\theta_0)\left(\frac{d\varphi}{d\tau}\right)^2 - \left(1-\frac{2M}{R_0}\right)\left(\frac{df}{d\tau}\right)^2 .
\end{aligned}
$$

Finally we can get

$$\left(\frac{df}{d\tau}\right)^2 = \left(1-\frac{2M}{R_0}\right)^{-1}\left[1 + R_0^2 \sin^2(\theta_0)\left(\frac{d\varphi}{d\tau}\right)^2\right]. \tag{4.3}$$

Remark 4.4. In the above computation, we assume the coordinates are rescaled, which means the light speed is the unit speed. So in the following computation, R_0 will become R_0/c, where c is the light speed.

4.4 Correction

Suppose that an airplane is moving with a speed of $400km/h$ and has an angular velocity of approximately $1.74 \times 10^{-5}(1/s)$. Let us consider the simplest case with

$$\varphi(\tau) = c\tau + d.$$

Then

$$\frac{d\varphi}{d\tau} = c \quad \text{with} \quad c = 5.8 \times 10^{-11}(1/km).$$

Also we assume that the Earth is a perfect sphere with radius $R_0 = 6371km$ and the plane flies at a height of $14km$. Then $R_p = 6385km$ and the Earth's mass $M = 5.9736 \times 10^{24}kg$, (so $GM/c^2 = 4.4463 \times 10^{-3}$). From equation (4.3) assuming $\theta_0 = \frac{\pi}{2}$, the proper time on the ground is τ_0, and the proper time on the platform is τ_p, then we have

$$\left(\frac{df}{d\tau_0}\right)^2 = \left(1-\frac{2GM}{c^2 R_0}\right)^{-1}\left[1 + R_0^2/c^2 \sin^2(\theta_0)\left(\frac{d\varphi}{d\tau_0}\right)^2\right]$$

and

$$\left(\frac{df}{d\tau_p}\right)^2 = \left(1-\frac{2GM}{c^2R_p}\right)^{-1}\left[1+R_p^2/c^2\sin^2(\theta_0)\left(\frac{d\varphi}{d\tau_p}\right)^2\right].$$

So we get

$$\frac{d\tau_p}{d\rho_0} = \frac{1-\frac{2GM}{c^2R_0}}{1-\frac{2GM}{c^2R_p}} \times \frac{1+R_p^2/c^2\sin^2(\theta_0)\left(\frac{d\varphi}{d\tau_p}\right)^2}{1+R_p^2/c^2\sin^2(\theta_0)\left(\frac{d\varphi}{d\tau_0}\right)^2}.$$

Under the previous setting, the term

$$\frac{1+R_p^2/c^2\sin^2(\theta_0)\left(\frac{d\varphi}{d\tau_p}\right)^2}{1+R_p^2/c^2\sin^2(\theta_0)\left(\frac{d\varphi}{d\tau_0}\right)^2} = 1+2\times10^{-15},$$

which is too small to be considered. So we only consider the term

$$\frac{1-\frac{2GM}{c^2R_0}}{1-\frac{2GM}{c^2R_p}},$$

which is the square of the ratio of the proper time on the platform and that on the ground. So we denote it as β^2, which means

$$\beta^2 = \frac{1-\frac{2GM}{c^2R_0}}{1-\frac{2GM}{c^2R_p}}.$$

Now suppose from the observer on the ground, the acceleration on the platform is f, while on the platform, the measured acceleration is f', then we have

$$f' = \frac{\mathrm{d}^2 s}{\mathrm{d}t'} = \beta^{-2}\frac{\mathrm{d}^2 s}{\mathrm{d}t} = \beta^{-2} f.$$

So the correction coefficient is β^2. In the previous setting, we have

$$\beta^2 = 0.99999999999847 = 1 - 1.53\times10^{-12}.$$

Now we find the correction taking into account the rotation of the Earth. In this case γ_2 is defined as

$$\gamma_2(\tau) = (R_0, \theta_0 + \omega_E \cdot f(\tau), \varphi(\tau), t = f(\tau))$$

also we assume that $\theta_0 = 0$. Then γ_2 will be

$$\gamma_2(\tau) = (R_0, \omega_E \cdot f(\tau), \varphi(\tau), t = f(\tau))$$

where $\omega_E = 7.29 \times 10^{-5} rad/s$ is the angular velocity of the Earth.

Following a similar procedure as before we obtain

$$\left(\frac{df}{d\tau}\right)^2 = \left(1 - \frac{2M}{R_0}\right)^{-1} \left[1 + R_0^2 \sin^2(\omega_E \cdot f(\tau)) \left(\frac{d\varphi}{d\tau}\right)^2\right].$$

Using the same values as before we get

$$\left(\frac{df}{d\tau}\right)^2 = \beta^2 \left[1 + (R_p/c)^2 \sin^2\left(\frac{7.29 \times 10^{-5} rad/s}{3 \times 10^5 km/s} \cdot f(\tau)\right) \left(5.8 \times 10^{-11}(1/km)\right)^2\right] \tag{4.4}$$

By solving this equation numerically for $\tau \in (0, 3600 \text{ sec})$ we obtain $\frac{dt}{d\tau} = \frac{df}{d\tau}$ and the results are shown in following figure :

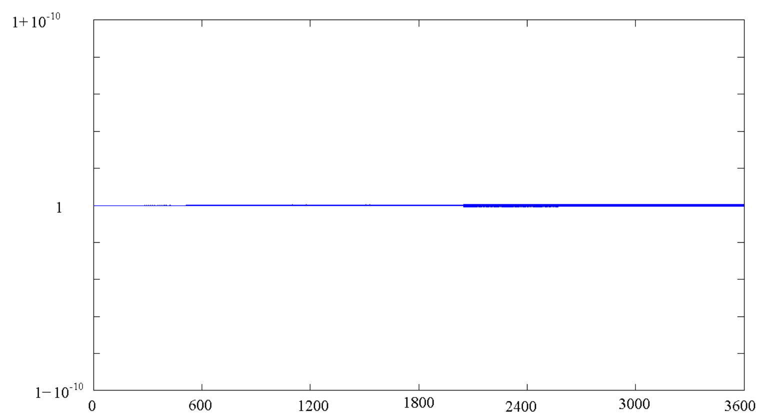


Figure 10: Time Contraction

From figure 10, we can find that the ratio is very near to one, which is beyond the accuracy of numerical solution of (4.4). Actually, in practice, the correction coefficient β^2 is accurate enough.

In practice, since the gravitation is around $980 Gal$, and the accuracy of current gravimeter could be up to $10^{-9} Gal$, the correction $\beta = 1 - 1.53 \times 10^{-12}$ can be used to modify the data of gravimeter. Also note that the effect of the general relativity is more significant than the effect of the special relativity.

References

[1] B. O'neill, Semi-Riemannian geometry: with applications to relativity, Academic Press, 1983

[2] Misner, Charles W.; Thorne, Kip S.; Wheeler, John Archibald, Gravitation, San Francisco: W. H. Freeman, 1973

[3] Gunter Seeber, Satellite geodesy, Walter de Gruyter Gmbh Co. Kg. 10875, Berlin, 2003

[4] Wolfgang Torge, Geodesy, 2001.

[5] Wikipedia.