ON DISPERSIVE EFFECT OF THE CORIOLIS FORCE FOR THE STATIONARY NAVIER-STOKES EQUATIONS

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Abstract. The dispersive effect of the Coriolis force for the stationary Navier-Stokes equations is investigated. The effect is of a different nature than the one shown for the non-stationary case by J. Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier. Existence of a unique solution is shown for arbitrary large external force provided the Coriolis force is large enough. The analysis is carried out in a new framework of the Fourier-Besov spaces. In addition to the stationary case counterparts of several classical results for the non-stationary Navier-Stokes problem have been proven.

MSC: 35Q30, 75D05

Key words: Coriolis force, stationary Navier-Stokes equations, dispersive effect, large data

1 Introduction

We consider the stationary 3D-Navier-Stokes equations with the Coriolis force:

$$(v\cdot\nabla)v+\Omega e_3\times v-\Delta v+\nabla p=F,\quad \nabla\cdot v=0\quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $v=v(x)=(v^1(x),v^2(x),v^3(x))$ is the unknown velocity vector field and $p=p(x)$ is the unknown scalar pressure at the point $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ in space, and F is a given external force. Here $\Omega\in\mathbb{R}$ is the Coriolis parameter, which is twice the angular velocity of the rotation around the vertical unit vector $e_3=(0,0,1)$, the kinematic viscosity coefficient is normalized by one. By $\times$ we denote the exterior product, and hence, the Coriolis term is represented by $e_3\times u=Ju$ with the corresponding skew-symmetric 3×3 matrix J.

Problems concerning large-scale atmospheric and oceanic flows are known to be dominated by rotational effects. Almost all of the models of oceanography and meteorology dealing with large-scale phenomena include the Coriolis force. For example, oceanic circulation featuring a hurricane is caused by the large rotation. There is no doubt that other physical effects are of similar significance like salinity, natural boundary conditions and so on. However the first step in the study of more complex model is to understand the behavior of rotating fluids. To this end, we treat in a standard manner the Navier-Stokes equations with the Coriolis force.

Let us look back on the history of the Coriolis force. In 1868 Kelvin observed that a sphere moving along the axis of uniformly rotating water takes with it a column of liquid as if this were a rigid mass (see [9] for references). After that, Hough [16], Taylor [20] and Proudmann [19] made important contributions. Mathematically it was investigated by Poincaré [19], more recently, Babin, Mahalov and Nicolaenko [1, 2] considered non-stationary Navier-Stokes equations with Coriolis force in periodic case. The periodicity is extended to the almost periodic case by several authors. For the results of local existence of non-stationary rotating Navier- Stokes equations with spatially almost periodic data and its properties, see [10, 13, 14]. Moreover, for the results of global existence and long time existence in the almost periodic setting, see [11, 12, 21] for example.

On the other hand, Chemin, Desjardins, Gallagher and Grenier (CDGG)[7] considered decaying data case. CDGG derived dispersion estimates on a linearized version of the 3D-Navier-Stokes equations with the Coriolis force to show existence of global solution to the non-stationary rotating Navier-Stokes system. To construct such estimate, they handled eigenvalues and eigenfunctions of the Coriolis operator.

The main result of this paper is to show existence of the solution to the stationary Navier-Stokes equations with the Coriolis force for arbitrary large external force provided that the Coriolis force is sufficiently large (compare it with results for the Navier-Stokes equations (1.1) with $\Omega = 0$, for example [18] for the case of exterior domain). To do so, we handle new type of function spaces, namely, Fourier Besov spaces (FB) which are designed to present in a clear way how the Coriolis force has influence on the solution to the considered system. A similar approach to introduce function spaces which make analysis of specific features of a system much easier has been shown in a paper by the first author and P. B. Mucha in [17], where they investigate asymptotic structure of solution to the stationary Navier-Stokes equations in $\mathbb{R}^2$.

In FB spaces, we cannot expect to use energy type estimates and the structure of Hilbert spaces as CDGG used. The main motivation to introduce those spaces is that in this framework we are able to present directly dispersive effect of the Coriolis force (see Proposition 2.4), which is in principle different from the dispersive effect from CDGG.

To show usefulness of introduced spaces we prove existence to the non-stationary Navier-Stokes-Coriolis system in function spaces which are counterparts for well known classical results in the Navier-Stokes theory (see [3, 5, 6]). Moreover we can considerably simplify other results for the Navier-Stokes-Coriolis system, like recent results by Giga, Inui, Mahalov and Saal [12].

1.1 Preliminaries

In this section we would like to recall basic facts of Littlewood-Paley theory. We denote by $\varphi \in \mathcal{S}(\mathbb{R}^3)$ a radially symmetric supported in $\{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for all } \xi \neq 0.$$

We also introduce the following functions:

$$\varphi_j(\xi) = \varphi(2^{-j}\xi), \quad \psi_j(\xi) = \sum_{k\leq j-1} \varphi_k(\xi).$$

Now we define standard localization operators:

$$\Delta_j f = \varphi_j f, \qquad S_j f = \sum_{k\leq j-1} \Delta_k f = \psi_j f, \qquad \text{for } j \in \mathbb{Z}. \tag{1.2}$$

It is then easy to verify the following identities:

$$\Delta_j \Delta_k f = 0 \qquad \text{if } |j-k| \geq 2, \tag{1.3}$$

$$\Delta_j (S_{k-1} f \Delta_k f) = 0 \qquad \text{if } |j-k| \geq 5. \tag{1.4}$$

Moreover one can follow Bony (see [4]) and introduce the following decomposition:

$$fg = T_f g + T_g f + R(f,g), \tag{1.5}$$

where

$$T_f g = \sum_{j\in\mathbb{Z}} S_{j-1} f \Delta_j g, \quad R(f,g) = \sum_{j\in\mathbb{Z}} \Delta_j f \tilde{\Delta}_j g, \quad \tilde{\Delta}_j g = \sum_{|j'-j|\leq 1} \Delta_{j'} g. \tag{1.6}$$

The framework for our results is determined by the Fourier-Besov spaces defined as follows:

Definition 1.1 *We introduce the following homogeneous function spaces (called Fourier-Besov spaces):*

- $F\dot{B}^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}' : \hat{f} \in L^1_{\text{loc}}, \|f\|_{F\dot{B}^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{Z}} 2^{ksq} \|\varphi_k \hat{f}\|^q_{L_p(\mathbb{R}^n)}\right)^{1/q} < \infty\}$, $1 \leq p \leq \infty$, $1 \leq q < \infty$,
- $F\dot{B}^s_{p,\infty}(\mathbb{R}^n) = \{f \in \mathcal{S}' : \|f\|_{F\dot{B}^s_{p,\infty}(\mathbb{R}^n)} = \sup_{k\in\mathbb{Z}} 2^{ks} \|\varphi_k \hat{f}\|_{L_p(\mathbb{R}^n)} < \infty\}$.

In our considerations we are using results for the Stokes problem with the Coriolis force:

$$\begin{cases} u_t - \nu\Delta u + \Omega e_3 \times u + \nabla p = F, \\ \text{div } u = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1.7}$$

For this system one has the following formula for the solution (see [10]):

$$\hat{u}(t,\xi) = \cos\left(\Omega \frac{\xi_3}{|\xi|} t\right) e^{-\nu|\xi|^2 t} I \hat{u}_0(\xi) + \sin\left(\Omega \frac{\xi_3}{|\xi|} t\right) e^{-\nu|\xi|^2 t} R(\xi) \hat{u}_0(\xi), \quad t \geq 0, \xi \in \mathbb{R}^3, \tag{1.8}$$

where I is the identity matrix and

$$R(\xi) = \begin{pmatrix} 0 & \frac{\xi_3}{|\xi|} & -\frac{\xi_2}{|\xi|} \\ -\frac{\xi_3}{|\xi|} & 0 & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & -\frac{\xi_1}{|\xi|} & 0 \end{pmatrix}. \tag{1.9}$$

An important observation is that

$$|\hat{u}(t,\xi)| \leq 2e^{-\nu|\xi|^2 t} |\hat{u}_0(\xi)|, \quad t \geq 0, \xi \in \mathbb{R}^3. \tag{1.10}$$

2 Main results

In this section we formulate our main results for the non-stationary and stationary Navier-Stokes equations with the Coriolis force. We would like to mention that it is not difficult to obtain also other results (like stability of solutions to the non-stationary case) in this framework. We refer readers to the paper by Cannone and Karch [5] as a reference for what can be expected. We do not prove those results to keep the paper more readable.

2.1 Non-stationary case

In the following theorem we consider mild solutions to the following non-stationary Navier-Stokes system with the Coriolis force:

$$u_t - \nu\Delta + \Omega e_3 \times u + u \cdot \nabla u + \nabla p = 0, \tag{2.1}$$

$$\text{div } u = 0, \tag{2.2}$$

$$u(0,x) = u_0(x). \tag{2.3}$$

Theorem 2.1 *Let $\Omega \in \mathbb{R}$ be an arbitrary constant. Let $u_0 \in X_0$ and $\|u_0\|_{X_0}$ be small enough (independently of Ω). Then there exists a unique global in time solution $u \in Y$ to problem (2.1)-(2.3), where X_0 and Y one can take as follows:*

- $X_0 = F\dot{B}^{2-3/p}_{p,\infty}$, $Y = C^w([0,\infty); F\dot{B}^{2-3/p}_{p,\infty}) \cap L^\infty(0,\infty; F\dot{B}^{2-3/p}_{p,\infty})$, *where* $3 < p \le \infty$,
- $X_0 = F\dot{B}^{2-3/p}_{p,p}$, $Y = C([0,\infty); F\dot{B}^{2-3/p}_{p,p}) \cap L^\infty(0,\infty; F\dot{B}^{2-3/p}_{p,p})$, *where* $3 < p < \infty$,
- $X_0 = F\dot{B}^{-1}_{1,1} \cap F\dot{B}^{0}_{1,1}$, $Y = C([0,\infty); F\dot{B}^{0}_{1,1}) \cap L^2(0,\infty; F\dot{B}^{-1}_{1,1} \cap F\dot{B}^{0}_{1,1})$.

Moreover the following case is valid:

- $X_0 = F\dot{B}^{2-3/p}_{p,\infty}$,
  $Y = L^\infty(0,\infty; F\dot{B}^{2-3/p}_{p,\infty}) \cap L^1(0,\infty; F\dot{B}^{4-3/p}_{p,\infty}) \cap C^w([0,\infty); F\dot{B}^{2-3/p}_{p,\infty})$, *for* $1 < p \le \infty$,

Note: The mentioned cases have their counterparts in the current literature for the non-stationary Navier-Stokes equations. For example the case $F\dot{B}^{2}_{\infty,\infty}$ was considered by Cannone and Karch in [5], the case $\dot{B}^{3/p-1}_{p,\infty}$ (which is a counterpart for $F\dot{B}^{2-3/p}_{p,\infty}$) in the paper [6] by Cannone. The case $F\dot{B}^{2-3/p}_{p,p}$ was treated by Biswas and Swanson for periodic case in [3]. Their result covers the whole range $1 < p \le \infty$ due to the periodicity – more precisely in their case the authors do not have problems with integrability (summability) close to 0 in the Fourier space. In our case analysis close to 0 in the Fourier space requires the assumption $p > 3$. An analogue of the case $F\dot{B}^{-1}_{1,1} \cap F\dot{B}^{0}_{1,1}$, that is $FM_0^{-1} \cap FM_0$ spaces, has been published recently by Giga, Inui, Mahalov and Saal in [12].

In this paper consider those results in our setting, which seems to be more suitable for the Navier-Stokes equations with the Coriolis force. Unfortunately using these methods we were not able to include the case $p = 2, q = 2$ which has been recently proven by Hieber and Shibata in [15].

2.2 Stationary case

In the following theorem we consider mild solutions to the Navier-Stokes system with the Coriolis force (1.1).

Definition 2.2 *For the sake of the stationary case with Coriolis force we introduce the following function space for $1 \le p \le \infty$,*

$$X^p_{\mathcal{C},\Omega} = \{f \in \mathcal{S}' : \|f\|_{X^p_{\mathcal{C},\Omega}} = \|w_1(\cdot)\hat{f}(\cdot)\|_{L^p} + \|w_2(\cdot)\hat{f}(\cdot)\|_{L^p} < \infty\}, \tag{2.4}$$

where

$$w_1(\xi) = \frac{|\xi|^{6-3/p}}{|\xi|^6 + \Omega^2|\xi_3|^2}, \quad w_2(\xi) = \frac{\Omega|\xi_3||\xi|^{3-3/p}}{|\xi|^6 + \Omega^2|\xi_3|^2} R(\xi),$$

and $R(\xi)$ *is the matrix (1.9).*

The following theorem is the main result of our paper.

Theorem 2.3 ***Stationary case.*** *Let* $3 < p \leq \infty$. *Then for all* $F \in X^p_{\mathcal{C},\Omega}$ *such that* $\|F\|_{X^p_{\mathcal{C},\Omega}}$ *is small enough there exists a unique solution* u *to the problem (1.1) such that* $u \in \dot{FB}^{2-3/p}_{p,p}$ *and the following estimate is valid:*

$$\|u\|_{\dot{FB}^{2-3/p}_{p,p}} \leq C\|F\|_{X^p_{\mathcal{C},\Omega}}. \tag{2.5}$$

Note: *Analogous result holds also for the spaces* $\dot{FB}^{2-3/p}_{p,\infty}$.

Remark: An important fact about the space $X^p_{\mathcal{C},\Omega}$ is that $\dot{FB}^{-3/p}_{p,p} \subsetneq X^p_{\mathcal{C},\Omega}$ for $\Omega \neq 0$ (see the proof of Proposition 2.4). This means that the Coriolis force not only helps to weaken smallness assumptions on the force F (see [5] and Lemma 2.4 below) but extends considerably the class of admissible external forces (for which we have existence result). For example the following function (its Fourier transform):

$$\frac{|\xi|^6 + \Omega^2\xi_2^2}{|\xi|^6} \cdot \left(\frac{\xi_1\xi_3}{|\xi|^2}, \frac{\xi_2\xi_3}{|\xi|^2}, \frac{\xi_3^2}{|\xi|^2}\right) \tag{2.6}$$

is an element of $X^\infty_{\mathcal{C},\Omega}$ for which (up to a constant) we have existence. This function, however, is not an element of the space of pseudo-measures $\mathcal{PM} = \dot{FB}^0_{\infty,\infty}$ from the paper [5].

In the case $F \in \dot{FB}^{-3/p}_{p,p}$ we can remove the smallness assumption provided that the Coriolis parameter Ω is large enough. This is being precised in the following Proposition (compare it with the case $F \in \dot{FB}^{-3/p}_{p,\infty}$ in Remark 2 after the proof of the Proposition).

Proposition 2.4 *Let* $3 < p < \infty$. *Then for any given function* $F \in \dot{FB}^{-3/p}_{p,p}$ *there exists* Ω_0 *such that for all* $\Omega \in \mathbb{R}$ *satisfying* $|\Omega| \geq \Omega_0$ *there exists the unique solution* u *to problem (1.1) such that* $u \in \dot{FB}^{2-3/p}_{p,p}(\mathbb{R}^3)$.

Proof of the Proposition. First we will show that for each $F \in \dot{FB}^{-3/p}_{p,p}$ and for all ϵ there exists Ω_0 such that for all $|\Omega| \geq \Omega_0$,

$$\|F\|_{X^p_{\mathcal{C},\Omega}} \leq \epsilon\|F\|_{\dot{FB}^{-3/p}_{p,p}}. \tag{2.7}$$

This fact together with Theorem 2.3 proves the Proposition.

First we have $\dot{FB}^{-3/p}_{p,p} \subset X^p_{\mathcal{C},\Omega}$. This is a simple observation since:

$$\frac{|\xi|^4}{|\xi|^6 + \Omega^2|\xi_3|^2} = \int_0^\infty e^{-t|\xi|^2} \cos(\Omega\frac{\xi_3}{|\xi|}t)dt \leq \int_0^\infty e^{-t|\xi|^2} dt = |\xi|^{-2} \tag{2.8}$$

and

$$\frac{\Omega\xi_3|\xi|}{|\xi|^6+\Omega^2|\xi_3|^2} = \int_0^\infty e^{-t|\xi|^2}\sin(\Omega\frac{\xi_3}{|\xi|}t)dt \le \int_0^\infty e^{-t|\xi|^2}dt = |\xi|^{-2}. \tag{2.9}$$

The proof of (2.7) is fairly simple. First we decompose $\mathbb{R}^3$ into three regions: $\mathbb{R}^3 = A_\delta + B_\delta + C_\delta$, where $A_\delta = \{\xi : |\xi_3| > \delta \text{ and } \delta < |\xi| < \frac{1}{\delta}\}$, $B_\delta = \{\xi : |\xi_3| > \delta \text{ and } |\xi| > \frac{1}{\delta}\}$ and $C_\delta = \{\xi : |\xi_3| < \delta\}$.

For fixed F there exists a compact set $K \subset \mathbb{R}^3$ such that $\||\xi|^{-3/p}F\|_{L^p(\mathbb{R}^3\setminus K)} \le \epsilon/3$ and by (2.8) and (2.9) we have the following estimates, uniform with respect to Ω:

$$\left(\frac{|\xi|^{6-3/p}}{|\xi|^6+\Omega^2|\xi_3|^2}|\hat{F}|\right)^p \le \left(|\xi|^{-3/p}|\hat{F}|\right)^p \tag{2.10}$$

and

$$\left(\frac{\Omega\xi_3|\xi|^{3-3/p}}{|\xi|^6+\Omega^2|\xi_3|^2}|R(\xi)||\hat{F}|\right)^p \le \left(|\xi|^{-3/p}|\hat{F}|\right)^p. \tag{2.11}$$

From the definition of B_δ and C_δ we get that $|K\cap(B_\delta\cup C_\delta)| \to 0$ as $\delta \to 0$, hence for δ small enough we have $\|F\|_{X^p_{\mathcal{C},\Omega}(K\cap(B_\delta\cup C_\delta))} \le \epsilon/2$. Once δ is fixed we get back to the integral over $K\cap A_\delta$:

$$\left(\int_{X^p_{\mathcal{C},\Omega}(K\cap A_\delta)}\left(\frac{|\xi|^{6-3/p}}{|\xi|^6+\Omega^2|\xi_3|^2}|\hat{F}|\right)^p d\xi\right)^{1/p} \le \frac{(1/\delta)^6}{\delta^6+\Omega^2\delta^2}\|F\|_{\dot{FB}^{-3/p}_{p,p}} \le \epsilon/4, \tag{2.12}$$

for Ω large enough (depending on ϵ, δ and $\|F\|$). Similarly

$$\left(\int_{X^p_{\mathcal{C},\Omega}(K\cap A_\delta)}\left(\frac{\Omega\xi_3|\xi|^{3-3/p}}{|\xi|^6+\Omega^2|\xi_3|^2}|\hat{F}|\right)^p d\xi\right)^{1/p} \le \frac{\Omega(1/\delta)^4}{\delta^6+\Omega^2\delta^2}\|F\|_{\dot{FB}^{-3/p}_{p,p}} \le \epsilon/4, \tag{2.13}$$

for Ω large enough.

This completes the proof. □

Remark 2: The counterpart of Proposition 2.4 for the case when $F \in \dot{FB}^{-3/p}_{p,\infty}$ requires additional assumptions on F. Method which we presented in the previous proof requires smallness assumptions of the following form: there exists a number K such that

$$\sup_{|k|\ge K} 2^{-3k/p}\left(\|\varphi_k w_1\hat{F}\|_{L^p} + \|\varphi_k w_2\hat{F}\|_{L^p}\right) \quad \text{is small enough,} \tag{2.14}$$

where $w_1(\xi)$ and $w_2(\xi)$ are weights from the definition (2.4) of the space $X^p_{\mathcal{C},\Omega}$. In particular this condition allows one to have $\|F\|_{\dot{FB}^{-3/p}_{p,\infty}}$ arbitrary large not only in frequences within the region $[-K,K]$ but also for $|k| \ge K$ provided weights w_1 and w_2 make them small enough. The proof of this fact is analogous to the proof of Proposition 2.4.

3 Proofs of main results

3.1 Proof of Theorem 2.1

We use a rather standard approach to show existence, namely via the following Banach fixed point theorem ([5]):

Lemma 3.1 *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space and $B : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ a bounded bilinear form satisfying $\|B(x_1, x_2)\|_{\mathcal{X}} \le \eta \|x_1\|_{\mathcal{X}} \|x_2\|_{\mathcal{X}}$ for all $x_1, x_2 \in \mathcal{X}$ and a constant $\eta > 0$. Then if $0 < \epsilon < 1/(4\eta)$ and if $y \in \mathcal{X}$ such that $\|y\|_{\mathcal{X}} < \epsilon$, the equation $x = y + B(x,x)$ has a solution in $\mathcal{X}$ such that $\|x\|_{\mathcal{X}} \le 2\epsilon$. This solution is the only one in the ball $\overline{B}(0, 2\epsilon)$. Moreover, the solution depends continuously on y in the following sense: if $\|\tilde{y}\|_{\mathcal{X}} \le \epsilon$, $\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x})$, and $\|\tilde{x}\|_{\mathcal{X}} \le 2\epsilon$ then*

$$\|x - \tilde{x}\|_{\mathcal{X}} \le \frac{1}{1 - 4\eta\epsilon} \|y - \tilde{y}\|_{\mathcal{X}}.$$

In our case the bilinear form B is defined as follows:

$$B(u,v)(t) = -\int_0^t \mathcal{G}(t-\tau)\mathbf{P}\mathrm{div}(u \otimes v) d\tau, \tag{3.1}$$

where $\mathcal{G}$ was defined in (3.5).

It is then straightforward that in order to prove existence we have to prove corresponding estimates in all cases of space X.

- In case $X_0 = \dot{FB}^{2-3/p}_{p,\infty}$, where $3 < p \le \infty$ we use Lemma 3.5 with $r = \infty$ to get:

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{L^\infty_T(\dot{FB}^s_{p,\infty})} \le \frac{1}{\nu} \|f\|_{L^\infty_T(\dot{FB}^{s-2}_{p,\infty})}. \tag{3.2}$$

  and then for $f = \mathrm{div}(u \otimes v)$ we use inequality (3.13).
  Estimate for convolution with initial data u_0 comes from Lemma 3.4.

- In the case $X_0 = \dot{FB}^{2-3/p}_{p,p}$, where $3 < p < \infty$ we use Lemma 3.10 to estimate the bilinear form. Initial data u_0 estimates trivially.

- In the case $X_0 = \dot{FB}^{-1}_{1,1} \cap \dot{FB}^{0}_{1,1}$ we make two steps. First we use Lemma 3.9 with $s = 0$ combined with Lemma 3.8 (inequality (3.20)) to estimate bilinear form $B(u,v)$ in the space $L^2([0,\infty); \dot{FB}^0_{1,1})$ and Lemma 3.7 with $s = 0$ to estimate initial data u_0. This gives us the unique solution in the space $L^2([0,\infty); \dot{FB}^0_{1,1})$. In the second step we notice that using inequality (3.21) and again Lemma 3.9 with $s = 1$ we obtain that the solution is in fact in the space $L^2([0,\infty); \dot{FB}^1_{1,1} \cap \dot{FB}^0_{1,1})$. This improved regularity is essential to show (in an elementary way) strong continuity of the solution, i.e. $u \in C([0,\infty); \dot{FB}^0_{1,1})$.

To prove the second part of Theorem 2.1, that is for $1 < p \le \infty$ one uses the same results as in the case $3 < p \le \infty$ but with estimate (3.12). Since this cases are of less interest to us (our paper focuses on the stationary case) we do not include more details in order to keep the paper more consistent.

3.2 Proof of Theorem 2.3

To prove existence results in the stationary case one may use the results from Theorem 2.1 in case $3 < p \leq \infty$ and $X = F\dot{B}^{2-3/p}_{p,\infty}$ or $X = F\dot{B}^{2-3/p}_{p,p}$ and repeat reasoning from the paper by Cannone and Karch [5]. The authors there use the following Lemma which is essential to obtain this result:

Proposition 3.2 *The following two facts are equivalent*

- $u = u(x)$ *is a stationary mild solution to the problem (2.1)-(2.2), that is*

$$u = \mathcal{G}(t)u - \int_0^t \mathcal{G}(t-\tau)\mathbf{P}\mathrm{div}\ (u \otimes u)d\tau + \int_0^t \mathcal{G}(\tau)\mathbf{P}Fd\tau \tag{3.3}$$

  for every $t > 0$.

- $u = u(x)$ *satisfies the integral equation*

$$u = -\int_0^\infty \mathcal{G}(\tau)\mathbf{P}\mathrm{div}\ (u \otimes u)d\tau + \int_0^\infty \mathcal{G}(\tau)\mathbf{P}Fd\tau, \tag{3.4}$$

where $\mathbf{P}$ *is the Helmholtz projection.*

Using this proposition and results for non-stationary case we see that in order to obtain existence of solution using a fixed point argument we just need to obtain estimates for the term with the force F. We use the formula for the Stokes-Coriolis semigroup, that is:

$$\hat{\mathcal{G}}(t) = \cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-\nu|\xi|^2 t}I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-\nu|\xi|^2 t}R(\xi). \tag{3.5}$$

Integrating this formula with respect to t from 0 to ∞ we get:

$$\int_0^\infty \hat{\mathcal{G}}(t)dt = \frac{|\xi|^4}{|\xi|^6 + \xi_3^2\Omega^2}I + \frac{\xi_3|\xi|}{|\xi|^6 + \xi_3^2\Omega^2}R(\xi) \tag{3.6}$$

It is then straightforward (from the definition of $\mathcal{X}^p_{\mathcal{C},\Omega}$) that

$$\left\|\int_0^\infty \mathcal{G}Fdt\right\|_{F\dot{B}^{2-3/p}_{p,p}} \leq \|F\|_{\mathcal{X}^p_{\mathcal{C},\Omega}}. \tag{3.7}$$

3.3 Main estimates

Lemma 3.3 *For* $1 \leq q \leq p \leq \infty$ *and any multiindex* γ *the following inequalities are valid:*

- $\mathrm{supp}\ \hat{f} \subset \{|\xi| \leq A2^j\} \Rightarrow \|(i\xi)^\gamma \hat{f}\|_{L^q(\mathbb{R}^n)} \leq C2^{j|\gamma| + nj(\frac{1}{q}-\frac{1}{p})}\|\hat{f}\|_{L^p(\mathcal{R}^n)}$.
- $\mathrm{supp}\ \hat{f} \subset \{B_1 2^j \leq |\xi| \leq B_2 2^j\} \Rightarrow \|\hat{f}\|_{L^q(\mathbb{R}^n)} \leq C2^{-j|\gamma|}\sup_{|\beta|=|\gamma|}\|(i\xi)^\beta \hat{f}\|_{L^p(\mathcal{R}^n)}$.

Lemma 3.4 *For $p \in [1,\infty]$ and $u_0 \in F\dot{B}^{2-3/p}_{p,\infty}$ one has:*

$$\|\mathcal{G}(t)u_0\|_{L_\infty(0,T;F\dot{B}^{2-3/p}_{p,\infty}\cap L_1(0,T;F\dot{B}^{4-3/p}_{p,\infty}))} \leq \max{(1,\frac{1}{\nu})}\|u_0\|_{F\dot{B}^{2-3/p}_{p,\infty}}. \tag{3.8}$$

Moreover one also has:

$$\|\mathcal{G}(t)u_0\|_{L^\infty_T(F\dot{B}^s_{p,p})} \leq \|u_0\|_{F\dot{B}^s_{p,p}}. \tag{3.9}$$

Proof . While the second estimate is straightforward let us focus on the first inequality. We consider the case $p < \infty$. The case $p = \infty$ can be obtained analogously. Let us first estimate the norm $\|\mathcal{G}(t)u\|_{L^\infty_T(F\dot{B}^{2-3/p}_{p,\infty})}$:

$$\|\mathcal{G}(t)u\|_{L^\infty_T(F\dot{B}^{2-3/p}_{p,\infty})} \leq \sup_{0\leq t<T}\sup_k 2^{k(2-3/p)}\|\varphi_k\hat{u_0}\|_{L^p} \leq \|u_0\|_{F\dot{B}^{2-3/p}_{p,\infty}}$$

The second part estimates as follows:

$$\begin{aligned}\|\mathcal{G}(t)u\|_{L^1_T(F\dot{B}^{4-3/p}_{p,\infty})} &\leq \int_0^T \sup_k 2^{k(4-3/p)}e^{-\nu t 2^{2k}}\|\varphi_k\hat{u_0}\|_{L^p}dt \\ &\leq \sup_k \frac{1}{\nu}2^{-2k}2^{k(4-3/p)}\|\varphi_k\hat{u_0}\|_{L^p} \leq \frac{1}{\nu}\|u_0\|_{F\dot{B}^{2-3/p}_{p,\infty}}. \end{aligned}\tag{3.10}$$

This finishes the proof of this Lemma. □

Lemma 3.5 *Let $s \in \mathbb{R}$, $p, q \in [1,\infty]$ and $f \in L^r_T(F\dot{B}^s_{p,\infty})$. Then the following estimate is valid:*

$$\left\|\int_0^t \mathcal{G}(t-\tau)f(\tau)d\tau\right\|_{L^q_T(F\dot{B}^s_{p,\infty})} \leq \frac{1}{\nu}\|f\|_{L^r_T(F\dot{B}^{s-2-2/q+2/r}_{p,\infty})}. \tag{3.11}$$

Proof. Since

$$\left\|\int_0^t \mathcal{G}(t-\tau)f(\tau)d\tau\right\|_{L^q_T(F\dot{B}^s_{p,\infty})} = \sup_k 2^{sk}\left\|\int_0^t \|\hat{\mathcal{G}}(t-\tau)\hat{f}(\tau)\varphi_k\|_{L^p}d\tau\right\|_{L^q_T}$$

we may fix k and estimate the corresponding term:

$$2^{ks}\left\|\int_0^t \|\hat{\mathcal{G}}(t-\tau)\hat{f}(\tau)\varphi_k\|_{L^p}d\tau\right\|_{L^q_T} \leq 2\cdot 2^{ks}\left\|\int_0^t e^{(t-\tau)2^{2k}}\|\hat{f}(\tau)\varphi_k\|_{L^p}d\tau\right\|_{L^q_T}$$

Using Young's inequality with $\tilde{q}$ such that $1+\frac{1}{q} = \frac{1}{\tilde{q}}+\frac{1}{r}$, that is: $\frac{1}{\tilde{q}} = 1+\frac{1}{q}-\frac{1}{r}$ we get:

$$\begin{aligned}2^{ks}\left\|\int_0^t e^{(t-\tau)2^{2k}}\|\hat{f}(\tau)\varphi_k\|_{L^p}d\tau\right\|_{L^q_T} &\leq 2^{ks}\|e^{t2^{2k}}\|_{L^{\tilde{q}}_T}\|\hat{f}(t)\varphi_k\|_{L^r_T(L^p)} \\ &\leq 2^{k(s-2-\frac{2}{q}+\frac{2}{r})}\|\hat{f}(t)\varphi_k\|_{L^r_T(L^p)}.\end{aligned}$$

Taking supremum over all $k \in \mathbb{Z}$ one obtains the desired estimate. □

Lemma 3.6 *The following estimates are valid:*

- *For* $1 < p \le \infty$ *and* $V = L^\infty_T(0,T;\dot{FB}^{2-3/p}_{p,\infty}(\mathbb{R}^3)) \cap L^1_T(0,T;\dot{FB}^{4-3/p}_{p,\infty}))$ *(introduced here for readability):*

$$\|uv\|_{L^1_T(\dot{FB}^{3-\frac{3}{p}}_{p,\infty})} \le C\|u\|_V\|v\|_V. \tag{3.12}$$

- *For* $p > 3$*:*

$$\|uv\|_{L^\infty_T(\dot{FB}^{1-\frac{3}{p}}_{p,\infty})} \le C\|u\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,\infty})}\|v\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,\infty})}. \tag{3.13}$$

Proof. In the following proof we follow in principle the reasoning from [8]. Let us focus on the first inequality. From the definition we have:

$$\|uv\|_{L^1_T(\dot{FB}^{3-\frac{3}{p}}_{p,\infty})} = \int_0^T \sup_j 2^{j(3-\frac{3}{p})}\|\widehat{\Delta_j(uv)}\|_{L_p}dt. \tag{3.14}$$

For $\Delta_j(uv)$ we use decomposition (1.5), that is:

$$\Delta_j(uv) = \sum_{|k-j|\le 4}\Delta_j(S_{k-1}u\Delta_k v) + \sum_{|k-j|\le 4}\Delta_j(S_{k-1}v\Delta_k u) + \sum_{k\ge j-2}\Delta_j(\Delta_k u\tilde{\Delta}_k v), \tag{3.15}$$

and denote each corresponding integral from (3.14) as I_j, II_j and III_j.

$$I_j = \int_0^T 2^{j(3-3/p)}\|\sum_{|k-j|\le 4}\varphi_j(\psi_{k-1}\hat{u} * \varphi_k\hat{v})\|_{L^p}dt \le \int_0^T 2^{j(3-3/p)}\sum_{|k-j|\le 4}\|\psi_{k-1}\hat{u}\|_{L^1}\|\varphi_k\hat{v}\|_{L^p}dt. \tag{3.16}$$

Now using Lemma 3.3 we have the following inequality:

$$\|\psi_{k-1}\hat{u}\|_{L^1} \le \sum_{k'<k}\|\varphi_{k'}\hat{u}\|_{L^1} \le \sum_{k'<k}2^{k'3(1-1/p)}\|\varphi_{k'}\hat{u}\|_{L^p}, \tag{3.17}$$

which allows us to estimate I_j as follows:

$$\begin{aligned} I_j &\le \int_0^T 2^{j(3-3/p)}\sum_{|k-j|\le 4}\sum_{k'<k}2^{k'}2^{k'(2-3/p)}\|\varphi_{k'}\hat{u}\|_{L_p}\|\varphi_k\hat{v}\|_{L_p}dt \\ &\le \int_0^T 2^{j(3-3/p)}\sum_{|k-j|\le 4}2^k\sup_{k'}2^{k'(2-3/p)}\|\varphi_{k'}\hat{u}\|_{L_p}\|\varphi_k\hat{v}\|_{L_p}dt \\ &\le \int_0^T 2^{j(4-3/p)}\|\varphi_k\hat{v}\|_{L_p}dt\sup_{k'}2^{k'(2-3/p)}\|\varphi_{k'}\hat{u}\|_{L_p} \\ &\le \|v\|_{L^1_T(\dot{FB}^{4-3/p}_{p,\infty})}\|u\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,\infty})}, \end{aligned}$$

where we used the fact that since $|j-k| < 4$ then $2^j \sim 2^k$.

Integral II_j is easily estimated in the same way as I_j. We will now focus on integral III_j.

$$
\begin{aligned}
III_j &= \int_0^T 2^{j(3-3/p)} \sum_{k\geq j-2} \|\varphi_j \varphi_k u \tilde{\varphi}_k v\|_{L^p} dt \leq \int_0^T 2^{j(3-3/p)} \sum_{k\geq j-2} \|\varphi_k \hat{u}\|_{L^1} \|\varphi_k \hat{v}\|_{L^p} dt \\
&= \int_0^T \sum_{k\geq j-2} 2^{(j-k)(3-3/p)} \|\varphi_k \hat{u}\|_{L^p} 2^{k(2-3/p)} \|\tilde{\varphi}_k \hat{v}\|_{L^p} 2^{k(4-3/p)} dt \\
&\leq \sup_k \|\varphi_k \hat{u}\|_{L^p} 2^{k(2-3/p)} \int_0^T \sup_k \|\tilde{\varphi}_k \hat{v}\|_{L^p} 2^{k(4-3/p)} dt \leq \|u\|_{L^\infty_T(F\dot{B}^{2-3/p}_{p,\infty})} \|v\|_{L^1_T(F\dot{B}^{4-3/p}_{p,\infty})},
\end{aligned}
$$

where we again used Lemma 3.3.

In order to obtain estimate (3.13) one proceeds in a similar way as for the case of (3.12), applying proper changes like $3-3/p$ is replaced by $1-3/p$. The requirement that $p>3$ comes from estimate of III_j, that is in the case of (3.12) one has the term $\sum_{k\geq j-2} 2^{(j-k)(3-3/p)}$, which is finite for $p>1$, while in case of estimate (3.13) one encounters the term $\sum_{k\geq j-2} 2^{(j-k)(1-3/p)}$, which is finite for $p>3$.
□

In what follows we focus on estimates for the space $L^2_T(F\dot{B}^0_{1,1})$.

Lemma 3.7 *The following estimate is valid:*

$$
\|e^{t\Delta} u_0\|_{L^2_T(F\dot{B}^s_{1,1})} \leq \|u_0\|_{F\dot{B}^{s-1}_{1,1}}. \tag{3.18}
$$

Proof . This inequality is easily obtained:

$$
\|e^{t\Delta} u_0\|_{L^2_T(F\dot{B}^s_{1,1})} = \left(\int_0^T \left(\sum_k \int_{\mathbb{R}^3} \varphi_k e^{-t|\xi|^2} |\xi|^s u_0(\xi) d\xi \right)^2 d\tau \right)^{1/2}
$$

$$
\leq \sum_k \left(\int_0^T e^{-t2^{2k+1}} 2^{sk} \|\varphi_k u_0(\xi)\|^2_{L^1} d\tau \right)^{1/2} \leq \sum_k 2^{(s-1)k} \|\varphi_k u_0(\xi)\|_{L^1} = \|u_0\|_{F\dot{B}^{s-1}_{1,1}} \tag{3.19}
$$

□

Lemma 3.8 *The following estimate is valid:*

$$
\|uv\|_{L^1_T(F\dot{B}^0_{1,1})} \leq \|u\|_{L^2_T(F\dot{B}^0_{1,1})} \|v\|_{L^2_T(F\dot{B}^0_{1,1})}. \tag{3.20}
$$

Moreover if $u, v \in L^2_T(F\dot{B}^0_{1,1} \cap F\dot{B}^1_{1,1})$ then the following estimate is valid:

$$
\|uv\|_{L^1_T(F\dot{B}^1_{1,1})} \leq \|u\|_{L^2_T(F\dot{B}^0_{1,1} \cap F\dot{B}^1_{1,1})} \|v\|_{L^2_T(F\dot{B}^0_{1,1} \cap F\dot{B}^1_{1,1})}. \tag{3.21}
$$

Proof . First we note that $f \in F\dot{B}^0_{1,1} \Leftrightarrow \hat{f} \in L^1$. Then our inequality (3.20) is proven in the following way:

$$\|uv\|_{L^1_T(F\dot{B}^0_{1,1})} = \int_0^T \|\hat{u} * \hat{v}\|_{L^1} \leq \int_0^T \|\hat{u}\|_{L^1} \|\hat{v}\|_{L^1}$$
$$\leq \|\hat{u}\|_{L^2_T(L^1)} \|\hat{v}\|_{L^2_T(L^1)} = \|u\|_{L^2_T(F\dot{B}^0_{1,1})} \|v\|_{L^2_T(F\dot{B}^0_{1,1})}$$

To prove inequality (3.21) we proceed in a similar way:

$$\|uv\|_{L^1_T(F\dot{B}^1_{1,1})} = \int_0^T \int_{\mathbb{R}^n} |\xi| \int_{\mathbb{R}^n} \hat{u}(\xi-\eta,\tau)\hat{v}(\eta,\tau) d\eta d\xi d\tau \leq$$
$$\int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (|\xi-\eta| + |\eta|) \hat{u}(\xi-\eta,\tau)\hat{v}(\eta,\tau) d\xi d\eta d\tau$$
$$\leq \|\xi \hat{u}(\xi)\|_{L^2_T(L^1)} \|\hat{v}\|_{L^2_T(L^1)} + \|\hat{u}(\xi)\|_{L^2_T(L^1)} \|\eta \hat{v}(\eta)\|_{L^2_T(L^1)},$$

which finishes the proof of the Lemma 3.8. □

Lemma 3.9 *The following inequality is valid:*

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{L^2_T(F\dot{B}^s_{1,1})} \leq \frac{1}{\nu} \|f\|_{L^1_T(F\dot{B}^{s-1}_{1,1})}. \tag{3.22}$$

Proof . As previously we use triangle and Young's inequality to obtain:

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{L^2_T(F\dot{B}^s_{1,1})} \leq \left\| \sum_k \int_0^t e^{(t-\tau)2^{2k}} 2^{sk} \|\varphi_k f(\tau)\|_{L^1} d\tau \right\|_{L^2_T}$$
$$\leq \sum_k \left\| e^{t2^{2k}} \right\|_{L^2_T} 2^{sk} \|\varphi_k f(\tau)\|_{L^1_T(L^1)} = \sum_k 2^{(s-1)k} \|\varphi_k f(\tau)\|_{L^1_T(L^1)} = \|f\|_{L^1_T(F\dot{B}^{s-1}_{1,1})}.$$

□

In what follows we focus on estimates for the space $L^\infty_T(F\dot{B}^{2-3/p}_{p,p})$, where $p > 3$.

Lemma 3.10 *The following estimate is valid:*

$$\left\| \int_0^t \mathcal{G}(t-\tau) \nabla(u \otimes v) d\tau \right\|_{L^\infty_T(F\dot{B}^{2-3/p}_{p,p})} \leq \|u\|_{L^\infty_T(F\dot{B}^{2-3/p}_{p,p})} \|v\|_{L^\infty_T(F\dot{B}^{2-3/p}_{p,p})} \tag{3.23}$$

Proof . First let us estimate the convolution $\hat{u} * \hat{v}$. We do this as follows:

$$|\hat{u} * \hat{v}(\xi)| \leq \int_{\mathbb{R}^3} \frac{1}{|\eta|^{2-3/p}} \frac{1}{|\xi-\eta|^{2-3/p}} |\xi-\eta|^{2-3/p} |\hat{v}(\xi-\eta)| |\eta|^{2-3/p} |\hat{u}(\eta)| d\eta$$
$$\leq \left(\int_{\mathbb{R}^3} \left(\frac{1}{|\eta|^{2-3/p}} \frac{1}{|\xi-\eta|^{2-3/p}} \right)^{p'} d\eta \right)^{1/p'}$$
$$\cdot \left(\int_{\mathbb{R}^3} \left(|\xi-\eta|^{2-3/p} |\hat{v}(\xi-\eta)| |\eta|^{2-3/p} |\hat{u}(\eta)| \right)^p d\eta \right)^{1/p}.$$

Now in order to estimate the convolution $\frac{1}{|\xi|^{(2-3/p)\bar{p}}} * \frac{1}{|\xi|^{(2-3/p)p'}}$ we use the well known fact

$$\mathcal{F}(|\xi|^{-\alpha})(x) = C_{\alpha,n}|x|^{\alpha-n}, \tag{3.24}$$

for $0 < \alpha < n$. Taking the Fourier transform of this convolution we get:

$$\mathcal{F}\left(\frac{1}{|\xi|^{(2-3/p)p'}} * \frac{1}{|\xi|^{(2-3/p)p'}}\right)(x) = C_\alpha |x|^{2[(2-3/p)p'-3]}.$$

Now using inverse Fourier transform we get:

$$\frac{1}{|\xi|^{(2-3/p)p'}} * \frac{1}{|\xi|^{(2-3/p)p'}} \sim |\xi|^{-2[(2-3/p)p'-3]-3} = |\xi|^{-2p'(2-3/p)+3}.$$

This formula holds for $p > 3$ (in dimension 3) in order to satisfy (two times) condition for validity of (3.24). We thus obtained the following formula:

$$\left(\int_{\mathbb{R}^3}\left(\frac{1}{|\eta|^{2-3/p}}\frac{1}{|\xi-\eta|^{2-3/p}}\right)^{p'} d\eta\right)^{1/p'} \sim |\xi|^{-2(2-3/p)+3/p'}. \tag{3.25}$$

Going back to our main estimate:

$$\left\|\int_0^t \mathcal{G}(t-\tau)\nabla(uv)d\tau\right\|_{L^\infty_T(F\dot{B}^{2-3/p}_{p,p})} \le$$

$$\sup_t \left(\int_{\mathbb{R}^3} |\xi|^{(2-3/p)p}(\int_0^t e^{t\xi^2}dt)^p|\xi|^p|\hat{u}*\hat{v}(\xi)|^p d\xi\right)^{1/p}$$

$$\le \sup_t \left(\int_{\mathbb{R}^3} |\xi|^A|\xi|^B \left(\int_{\mathbb{R}^3}\left(|\eta|^{2-3/p}\hat{v}(\eta)|\xi-\eta|^{2-3/p}\hat{u}(\xi-\eta)\right)^p d\eta\right)^{p/p} d\xi\right)^{1/p},$$

where $A = 2p - 3 - 2p + p$ and $B = [-2(2-3/p) + 3/p'] \cdot p$.

It is not hard to notice that $A + B = 0$ and thus the proof of the lemma follows easily from integration of the last term first with respect to ξ and then η.

□

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