A MATRIX FORMULATION OF THE NEWTON DYNAMICS FOR THE FREE FLIGHT OF AN INSECT

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A matrix formulation of the Newton dynamics for the free flight of an insect

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Abstract

An insect flies and maneuvers by coupling its Newton dynamics and aerodynamics. In this note, we establish a matrix formulation of the Newton dynamics for the free flight of an insect. The flight can be driven by two means. One is to prescribe the kinematics of each wing relative to the body, and the other is to prescribe the torque exerted by the body on each wing. Our formulation unifies the two and is very easy to use. It is useful for studying the stability and maneuverability in insect flight.

Nomenclature

$\vec{\alpha}$	Tait-Bryan angles between two reference frames.
$\vec{\tau}$	Fluid or internal torque
$\vec{\Omega}$	Angular velocity of an insect body or wing relative to the lab frame.
$\vec{\Pi}$	Angular velocity of an insect wing relative to the body.
$\vec{c}$	Muscle torque actively exerted on an insect wing by the body.
$\vec{f}$	Fluid, gravitational, or intenal force

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m	Mass of an insect, an insect body, or wing
$\vec{x}$	Displacement of an insect body, wing, or body-wing hinge.
E	3×3 indentity matrix.
H	Skew-symmetric matrix formed from the coordinates of a body-wing hinge.
I	Moment of inertia matrix of an insect body or wing.
K	Matrix relating angular velocity with rate of change of Tait-Bryan angles.
R	Matrix of transformation between two reference frames.

1 Introduction

A flying insect is a multibody system consisting of a body and multiple wings. The physical connections between its body and wings are physical constrains in the system. To study the stability and maneuverability of free flight, we need to couple the Newton dynamics and aerodynamics of this multibody system. Specifically, the Newton dynamics needs the aerodynamic force and torque from the aerodynamics, and the aerodynamics needs the velocity and acceleration from the Newton dynamics.

In this note, we establish a formulation of the Newton dynamics for the free flight of an insect. General formulations to write general codes are available for the Newton dynamics of multibody systems [1]. Equations specific for the Newton dynamics for insect flight were also presented in [2, 3]. However, how to implement these formulations and equations in a simple and straightforward manner is not always clear to the readers. The objective of this note is to provide a neat and concise formulation which is very easy to use. In particular, we establish a matrix formulation by choosing suitable reference frames and unknown variables and manipulating dynamical equations and physical constraints. The formulation unifies two cases. In one case, the kinematics of each wing relative to the body is prescribed. In the other, the torque exerted by the body on each wing is prescribed. This formulation is neat, concise, and simple. It would be useful in studying insect flight or aquatic animal swimming. In particular, it is useful for coupling the Newton dynamics and aerodynamics of insect flight in CFD or for analyzing insect flight with reduced fluid force and torque models.

In Section 2, we prepare the notations for the presentation of the formulation. In Section 3, we present the complete formulation. In Section 4, we give detailed derivations of the formulation. In Section 5, we show some tests on the formulation. In Appendix, we list the formulas for calculating some transformation matrices. The information given in Sections 2 and 3 and Appendix is sufficient for just implementing the formulation.

2 Preparations

2.1 Reference frames

A flying insect has a body and multiple wings. Each wing is connected to the body at a hinge. Multiple reference frames (coordinate systems) shown in Fig. 1 are used to describe the positions and orientations of the body and wings. They are

- $\vec{x}^l = (x^l, y^l, z^l)$, the static lab frame used for the motion of the whole insect (the system);
- $\vec{x}^b = (x^b, y^b, z^b)$, the body frame attached to the insect body such that the origin B is the center of mass (CM) of the body, and x^b, y^b, and z^b are the principle axes of the

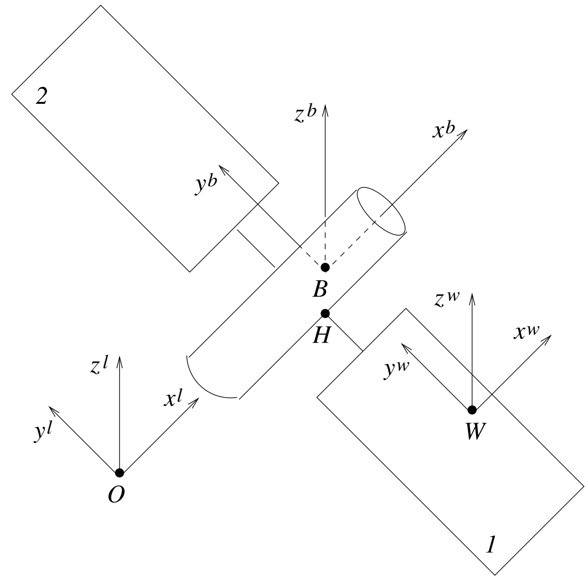

Figure 1: Multiple reference frames (coordinate systems) for a free-flying insect.

moments of inertia I_B of the body;

- $\vec{x}^w = (x^w, y^w, z^w)$, a wing frame attached to each insect wing such that the origin W is the CM of the wing, and x^w, y^w, and z^w are the principle axes of the moments of inertia I_W of the wing.

Hereafter, we use lowercase superscripts l, b, and w to specify the lab frame, the body frame, and a wing frame, respectively, capital subscripts B, W, H to specify the body, a wing, and a hinge, respectively, and one dot and two dots atop a symbol to denote first and second order time derivatives, respectively. For example, $\ddot{\vec{x}}^l_W$ denotes the translational acceleration of the CM of a wing in the lab frame.

The coordinates of a body-wing hinge in the body frame are denoted as $\vec{x}^b_H$. For any vector $\vec{p}$, we can write $\vec{x}^b_H \times \vec{p}$ as

$$\vec{x}^b_H \times \vec{p} = H_b \vec{p}, \tag{1}$$

where the skew-symmetric matrix H_b is

$$H_b = \begin{pmatrix} 0 & -z_H^b & y_H^b \\ z_H^b & 0 & -x_H^b \\ -y_H^b & x_H^b & 0 \end{pmatrix}. \tag{2}$$

The coordinates of a body-wing hinge in the wing frame are denoted as $\vec{x}_H^w$. Similarly, we can write $\vec{x}_H^b \times \vec{p}$ as

$$\vec{x}_H^w \times \vec{p} = H_w \vec{p}. \tag{3}$$

Let $\vec{x}$ be the displacement and $\vec{q}$ be the velocity, angular velocity, force, or torque. Then $\vec{x}$ and $\vec{u}$ transform between different reference frames as follows.

$$\vec{x}^b = R_{lb}(\vec{x}^l - \vec{x}_B^l), \quad \vec{q}^b = R_{lb}\vec{q}^l, \tag{4}$$

$$\vec{x}^w = R_{bw}(\vec{x}^b - \vec{x}_W^b), \quad \vec{q}^w = R_{bw}\vec{q}^b, \tag{5}$$

$$\vec{x}^w = R_{lw}(\vec{x}^l - \vec{x}_W^l), \quad \vec{q}^w = R_{lw}\vec{q}^l, \tag{6}$$

where R_{lb}, R_{bw}, and R_{lw} are the lab-to-body, body-to-wing, and lab-to-wing transformation matrices, respectively. They are orthogonal matrices and are given in Appendix.

2.2 Angular velocity

We denote the angular velocity of the body relative to the lab frame as $\vec{\Omega}_B$ and the angular velocity of a wing relative to the body as $\vec{\Pi}_W$. Then the angular velocity of the wing relative to the lab frame is

$$\vec{\Omega}_W = \vec{\Omega}_B + \vec{\Pi}_W. \tag{7}$$

In the wing frame, it reads

$$\vec{\Omega}^w_W = \vec{\Omega}^w_B + \vec{\Pi}^w_W = R_{bw}\vec{\Omega}^b_B + \vec{\Pi}^w_W. \tag{8}$$

2.3 Force and torque

We denote the aerodynamic force and torque on an insect wing as $\vec{f}_{fW}$ and $\vec{\tau}_{fW}$, respectively. In the lab frame, they can be written as [5]

$$\vec{f}^l_{fW} = \vec{f}^l_{0W} + A_{fW}\dot{\vec{\Omega}}^l_W + m_{fW}\ddot{\vec{x}}^l_W, \tag{9}$$

$$\vec{\tau}^l_{fW} = \vec{\tau}^l_{0W} + \vec{\tau}^l_{1W} + A_{\tau W}\dot{\vec{\Omega}}^l_W, \tag{10}$$

where $\vec{f}^l_{0W}$ and $\vec{\tau}^l_{0W}$ are proportional to the fluid viscosity, $\vec{\tau}^l_{1W}$ depends only on the shape, position, and the angular velocity of the wing, the matrices A_{fW} and $A_{\tau W}$ depends only on the shape and position of the wing, and m_{fW} is the mass of the fluid displaced by the wing. The translational and angular accelerations of the wing are $\ddot{\vec{x}}^l_W$ and $\dot{\vec{\Omega}}^l_W$, respectively, and their contributions to the aerodynamic force and torque are separated in the above formulas. Please refer to [4, 5] for the derivation of the formulas. Similarly, the aerodynamic force and torque on the insect body can be written as

$$\vec{f}^l_{fB} = \vec{f}^l_{0B} + A_{fB}\dot{\vec{\Omega}}^l_B + m_{fB}\ddot{\vec{x}}^l_B, \tag{11}$$

$$\vec{\tau}^l_{fB} = \vec{\tau}^l_{0B} + \vec{\tau}^l_{1B} + A_{\tau B}\dot{\vec{\Omega}}^l_B. \tag{12}$$

We use $\vec{f}_{gW}$, $\vec{f}_{gB}$, and $\vec{f}_{gS}$ to denote the gravitational force (due to weight and buoyancy) on a wing, the body, and the insect respectively. In the lab frame, they are given by

$$\vec{f}^{l}_{gW} = (m_W - m_{fW})\vec{g}^{l}, \tag{13}$$

$$\vec{f}^{l}_{gB} = (m_B - m_{fB})\vec{g}^{l}, \tag{14}$$

$$\vec{f}^{l}_{gB} = \vec{f}^{l}_{gB} + \sum_{W} \vec{f}^{l}_{gW}, \tag{15}$$

where m_W and m_B are the mass of the wing and the body, respectively, and $\vec{g}^{l}$ is the gravational acceleration.

3 Matrix formulation

In this section, We present the complete matrix formulation of the Newton dynamics for the free flight of an insect. We unify two cases in our formulation. In the first case, the kinematics of each wing relative to the body is prescribed, and $\vec{\Pi}^{w}_{W}$ (the angular velocity of the wing relative to the body) is thus a known function of the time. In the second case, the torque exerted by the body on each wing (denoted as $\vec{c}^{b}_{bW}$) is prescribed in the body frame. To unify the two cases into the formulation, we introduce

$$\vec{a}_W = \delta_{i2}\dot{\vec{\Pi}}^{w}_{W} + \delta_{i1}\vec{c}^{b}_{bW}, \quad \vec{b}_W = \delta_{i1}\dot{\vec{\Pi}}^{w}_{W} + \delta_{i2}\vec{c}^{b}_{bW}, \tag{16}$$

where δ_{ij} is the Kronecker delta. The first and second cases above correspond to $i = 1$ and $i = 2$, respectively. Hence, $\vec{a}_W$ is unknown, $\vec{b}_W$ is known, and we have

$$\dot{\vec{\Pi}}^{w}_{W} = \delta_{i2}\vec{a}_W + \delta_{i1}\vec{b}_W, \quad \vec{c}^{b}_{bW} = \delta_{i1}\vec{a}_W + \delta_{i2}\vec{b}_W. \tag{17}$$

We choose $\ddot{\vec{x}}^{l}_{B}$ and $\dot{\vec{\Omega}}^{b}_{B}$ as unknown variables for the body B and $\ddot{\vec{x}}^{l}_{W}$, $\dot{\vec{\Omega}}^{w}_{W}$, $\vec{f}^{l}_{fW}$, and $\vec{a}_W$ as unknown variables for a wing W. If there is only one wing, the matrix equation for the unknown variables reads

$$\begin{pmatrix} C^{11} & C^{12} & C_W^{13} & 0 & C^{15} & 0 \\ 0 & C^{22} & C_W^{23} & 0 & C^{25} & C^{26} \\ 0 & 0 & C_W^{33} & C_W^{34} & C_W^{35} & C_W^{36} \\ C^{41} & C^{42} & C^{43} & C_W^{44} & 0 & 0 \\ 0 & C_W^{52} & 0 & C^{54} & 0 & C^{56} \\ 0 & 0 & C_W^{63} & C_W^{64} & C^{65} & 0 \end{pmatrix} \begin{pmatrix} \ddot{\vec{x}}_B^l \\ \dot{\vec{\Omega}}_B^b \\ \ddot{\vec{x}}_W^l \\ \dot{\vec{\Omega}}_W^w \\ \vec{f}_{fW}^l \\ \vec{a}_W \end{pmatrix} = \begin{pmatrix} \vec{d}^1 \\ \vec{d}^2 \\ \vec{d}_W^3 \\ \vec{d}_W^4 \\ \vec{d}_W^5 \\ \vec{d}_W^6 \end{pmatrix}, \tag{18}$$

where, with E denoting the 3×3 identity matrix, the entries in the coefficient matrix and the right hand side are

$$C^{11} = (m_B - m_{fB})E, \quad C^{12} = -A_{fB}R_{lb}^T, \quad C_W^{13} = m_W E, \quad C^{15} = -E,$$

$$C^{22} = I_B - R_{lb}A_{\tau B}R_{lb}^T, \quad C_W^{23} = m_W H_b R_{lb}, \quad C^{25} = -H_b R_{lb}, \quad C^{26} = \delta_{i1}E,$$

$$C_W^{33} = -m_W H_w R_{lw}, \quad C_W^{34} = I_W - R_{lw}A_{\tau W}R_{lw}^T, \quad C_W^{35} = H_w R_{lw}, \quad C_W^{36} = -\delta_{i1}R_{bw},$$

$$C^{41} = -E, \quad C^{42} = R_{lb}^T H_b, \quad C^{43} = E, \quad C_W^{44} = -R_{lw}^T H_w,$$

$$C_W^{52} = -R_{bw}, \quad C^{54} = E, \quad C^{56} = -\delta_{i2}E,$$

$$C_W^{63} = -m_{fW}E, \quad C_W^{64} = -A_{fW}R_{lw}^T, \quad C^{65} = E,$$

and

$$
\begin{aligned}
\vec{d}^1 &= \vec{f}^l_{0B} + \vec{f}^l_{gS},\\
\vec{d}^2 &= -\vec{\Omega}^b_B \times (I_B\vec{\Omega}^b_B) + R_{lb}(\vec{\tau}^l_{0B} + \vec{\tau}^l_{1B}) + \sum_W(-\delta_{i2}\vec{b}_W + H_b R_{lb}\vec{f}^l_{gW}),\\
\vec{d}^3_W &= \delta_{i2}R_{bw}\vec{b}_W - \vec{\Omega}^w_W \times (I_W\vec{\Omega}^w_W) + R_{lw}(\vec{\tau}^l_{0W} + \vec{\tau}^l_{1W}) - H_w R_{lw}\vec{f}^l_{gW},\\
\vec{d}^4_W &= -R^T_{lb}(\|\vec{\Omega}^b_B\|^2_2\vec{x}^b_H - (\vec{\Omega}^b_B\cdot\vec{x}^b_H)\vec{\Omega}^b_B) + R^T_{lw}(\|\vec{\Omega}^w_W\|^2_2\vec{x}^w_H - (\vec{\Omega}^w_W\cdot\vec{x}^w_H)\vec{\Omega}^w_W),\\
\vec{d}^5_W &= \delta_{i1}\vec{b}_W - R_{bw}((R^T_{bw}\vec{\Pi}^w_W)\times\vec{\Omega}^b_B),\\
\vec{d}^6_W &= \vec{f}^l_{0W}.
\end{aligned}
$$

The coefficient matrix only depends on the shape, position, and mass distribution of the wing and the body. Besides the shape, position, and mass distribution, the right hand side also depends on the kinematic.

The position and kinematics of the body and the wing is fully described by

$$\mathbf{w} = [\vec{x}^l_B, \dot{\vec{x}}^l_B, \vec{\alpha}_{lb}, \vec{\Omega}^b_B, \vec{x}^l_W, \dot{\vec{x}}^l_W, \vec{\alpha}_{bw}, \vec{\Pi}^w_W]^T, \tag{19}$$

where $\vec{\alpha}_{lb}$ is the vector formed by Tait-Bryan angles between the lab frame and the body frame, and and $\vec{\alpha}_{bw}$ between the lab frame and the wing frame. Definitions of these angles are given in Appendix. We can numerically integrate in time the following first order differential equations

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} = \left[\dot{\vec{x}}^l_B, \ddot{\vec{x}}^l_B, K_B^{-1}\vec{\Omega}^b_B, \dot{\vec{\Omega}}^b_B, \dot{\vec{x}}^l_W, \ddot{\vec{x}}^l_W, K_W^{-1}\vec{\Pi}^w_W, \dot{\vec{\Pi}}^w_W\right]^T, \tag{20}$$

where the matrices K_B and K_W are given in Appendix. The dynamic entries at the right hand side of Eq. (20) are obtained by solving Eq. (18).

Multiple wings can be included easily. For example, if there are two wings W_1 and W_2, then we have

$$
\begin{pmatrix} C_1 & C_2^{12} \\ C_2^{21} & C_2^{22} \end{pmatrix}
\begin{pmatrix} \mathbf{w}_1 \\ \ddot{\vec{x}}^l_{W_2} \\ \dot{\vec{\Omega}}^w_{W_2} \\ \vec{f}^l_{fW_2} \\ \vec{a}_{W_2} \end{pmatrix}
=
\begin{pmatrix} \mathbf{d}_1 \\ \vec{d}^3_{W_2} \\ \vec{d}^4_{W_2} \\ \vec{d}^5_{W_2} \\ \vec{d}^6_{W_2} \end{pmatrix}, \tag{21}
$$

where C_1, $\mathbf{w}_1$, and $\mathbf{d}_1$ are the coefficient matrix, unknown vector, and right hand side of Eq. (18) for the first wing W_1, and the sub-matrices C_2^{21}, C_2^{22}, and C_2^{12} are associated with the second wing W_2 and are given by

$$
C_2^{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ C^{41} & C^{42} & 0 & 0 & 0 & 0 \\ 0 & C^{52}_{W_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad
C_2^{22} = \begin{pmatrix} C^{33}_{W_2} & C^{34}_{W_2} & C^{35}_{W_2} & C^{36}_{W_2} \\ C^{43} & C^{44}_{W_2} & 0 & 0 \\ 0 & C^{54} & 0 & C^{56} \\ C^{63}_{W_2} & C^{64}_{W_2} & C^{65} & 0 \end{pmatrix}.
$$

$$
C_2^{12} = \begin{pmatrix} C^{13}_{W_2} & 0 & C^{15} & 0 \\ C^{23}_{W_2} & 0 & C^{25} & C^{26} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{22}
$$

4 Derivation

First, we introduce a few relations which will be used later in our derivation.

Let the vector $\vec{q}$ be velocity, angular velocity, force, or torque. We have

$$\dot{R}_{lb}^T R_{lb} \vec{q}^l = \vec{\Omega}_B^l \times \vec{q}^l, \tag{23}$$

$$\dot{R}_{lw}^T R_{lw} \vec{q}^l = \vec{\Omega}_W^l \times \vec{q}^l, \tag{24}$$

$$\dot{R}_{bw}^T R_{bw} \vec{q}^b = \vec{\Pi}_W^b \times \vec{q}^b. \tag{25}$$

Using Eqs. (23) and (24), we have

$$\dot{R}_{lb}^T \vec{\Omega}_B^b = (\dot{R}_{lb}^T R_{lb})(R_{lb}^T \vec{\Omega}_B^b) = (\dot{R}_{lb}^T R_{lb})\vec{\Omega}_B^l = \vec{\Omega}_B^l \times \vec{\Omega}_B^l = 0, \tag{26}$$

$$\dot{R}_{lw}^T \vec{\Omega}_W^w = (\dot{R}_{lw}^T R_{lw})(R_{lw}^T \vec{\Omega}_W^w) = (\dot{R}_{lw}^T R_{lw})\vec{\Omega}_W^l = \vec{\Omega}_W^l \times \vec{\Omega}_W^l = 0. \tag{27}$$

Let $\vec{\beta}$ and $\vec{\gamma}$ be any vectors and R be a 3×3 orthogonal matrix, we have

$$(R\vec{\beta}) \times (R\vec{\gamma}) = R(\vec{\beta} \times \vec{\gamma}). \tag{28}$$

We can convert a cross product to a matrix-vector product as

$$\vec{\beta} \times \vec{\gamma} = A\vec{\gamma}, \tag{29}$$

where the skew-symmetric matrix A is defined using the three components of $\vec{\beta}$ as in Eq. (2).

4.1 Geometric and kinematic relations

The body and a wing of an insect are connected at a hinge, denoted as H in Fig. 1. By Eqs. (4) and (6), we have

$$\vec{x}_H^l - \vec{x}_B^l = R_{lb}^T \vec{x}_H^b, \tag{30}$$

$$\vec{x}_H^l - \vec{x}_W^l = R_{lw}^T \vec{x}_H^w. \tag{31}$$

The geometric relation for the body-wing constraint is obtained by subtracting the above two equations. It reads

$$\vec{x}^l_W - \vec{x}^l_B = R^T_{lb}\vec{x}^b_H - R^T_{lw}\vec{x}^w_H. \tag{32}$$

The coordinates of the hinge, denoted as $\vec{x}^b_H$ in the body frame and $\vec{x}^w_H$ in the wing frame, are time-independent if both the body and the wing are rigid.

Using Eqs. (23), (24), (28), and (29), we can obtain the first order time derivative of Eq. (32) as the following

$$\begin{aligned}
&\dot{\vec{x}}^l_W - \dot{\vec{x}}^l_B = \dot{R}^T_{lb}\vec{x}^b_H - \dot{R}^T_{lw}\vec{x}^w_H = \dot{R}^T_{lb}R_{lb}R^T_{lb}\vec{x}^b_H - \dot{R}^T_{lw}R_{lw}R^T_{lw}\vec{x}^w_H \\
&= \vec{\Omega}^l_B \times (R^T_{lb}\vec{x}^b_H) - \vec{\Omega}^l_W \times (R^T_{lw}\vec{x}^w_H) \\
&= (R^T_{lb}\vec{\Omega}^b_B) \times (R^T_{lb}\vec{x}^b_H) - (R^T_{lw}\vec{\Omega}^w_W) \times (R^T_{lw}\vec{x}^w_H) \\
&= R^T_{lb}(\vec{\Omega}^b_B \times \vec{x}^b_H) - R^T_{lw}(\vec{\Omega}^w_W \times \vec{x}^w_H).
\end{aligned} \tag{33}$$

Then we can obtain the second order time derivative as

$$\begin{aligned}
&\ddot{\vec{x}}^l_W - \ddot{\vec{x}}^l_B = \dot{R}^T_{lb}(\vec{\Omega}^b_B \times \vec{x}^b_H) - \dot{R}^T_{lw}(\vec{\Omega}^w_W \times \vec{x}^w_H) \\
&+R^T_{lb}(\dot{\vec{\Omega}}^b_B \times \vec{x}^b_H) - R^T_{lw}(\dot{\vec{\Omega}}^w_W \times \vec{x}^w_H) \\
&= \vec{d}^4_W - R^T_{lb}H_b\dot{\vec{\Omega}}^b_B + R^T_{lw}H_w\dot{\vec{\Omega}}^w_W,
\end{aligned} \tag{34}$$

where $\vec{d}_W^4$ is defined and computed as

$$\begin{aligned}
\vec{d}_W^4 &= \dot{R}_{lb}^T(\vec{\Omega}_B^b \times \vec{x}_H^b) - \dot{R}_{lw}^T(\vec{\Omega}_W^w \times \vec{x}_H^w) \\
&= -\dot{R}_{lb}^T R_{lb} R_{lb}^T(\vec{x}_H^b \times \vec{\Omega}_B^b) + \dot{R}_{lw}^T R_{lw} R_{lw}^T(\vec{x}_H^w \times \vec{\Omega}_W^w) \\
&= -\vec{\Omega}_B^l \times (R_{lb}^T(\vec{x}_H^b \times \vec{\Omega}_B^b)) + \vec{\Omega}_W^l \times (R_{lw}^T(\vec{x}_H^w \times \vec{\Omega}_W^w)) \\
&= -(R_{lb}^T\vec{\Omega}_B^b) \times (R_{lb}^T(\vec{x}_H^b \times \vec{\Omega}_B^b)) + (R_{lw}^T\vec{\Omega}_W^w) \times (R_{lw}^T(\vec{x}_H^w \times \vec{\Omega}_W^w)) \\
&= -R_{lb}^T(\vec{\Omega}_B^b \times (\vec{x}_H^b \times \vec{\Omega}_B^b)) + R_{lw}^T(\vec{\Omega}_W^w \times (\vec{x}_H^w \times \vec{\Omega}_W^w)) \\
&= -R_{lb}^T(\|\vec{\Omega}_B^b\|_2^2\vec{x}_H^b - (\vec{\Omega}_B^b \cdot \vec{x}_H^b)\vec{\Omega}_B^b) + R_{lw}^T(\|\vec{\Omega}_W^w\|_2^2\vec{x}_H^w - (\vec{\Omega}_W^w \cdot \vec{x}_H^w)\vec{\Omega}_W^w).
\end{aligned} \tag{35}$$

Eq. (34) cooresponds to the fourth equation in Eq. (18) after it is written as

$$C_W^{41}\ddot{\vec{x}}_B^l + C_W^{42}\dot{\vec{\Omega}}_B^b + C_W^{43}\ddot{\vec{x}}_W^l + C_W^{44}\dot{\vec{\Omega}}_W^w + 0\vec{f}_{fW}^l + 0\vec{a}_W = \vec{d}_W^4. \tag{36}$$

As $R_{bw}R_{bw}^T = E$, we have $\dot{R}_{bw}R_{bw}^T = -R_{bw}\dot{R}_{bw}^T$. Using Eq. (25), we then obtain

$$\begin{aligned}
&\dot{R}_{bw}\vec{\Omega}_B^b = \dot{R}_{bw}R_{bw}^T R_{bw}\vec{\Omega}_B^b = -R_{bw}\dot{R}_{bw}^T R_{bw}\vec{\Omega}_B^b = \\
&-R_{bw}\vec{\Pi}_W^b \times \vec{\Omega}_B^b = -R_{bw}((R_{bw}^T\vec{\Pi}_W^w) \times \vec{\Omega}_B^b).
\end{aligned} \tag{37}$$

With the subsitution of Eq. (17), the first order time derivative of Eq. (8) is therefore given by

$$\begin{aligned}
\dot{\vec{\Omega}}_W^w &= \dot{R}_{bw}\vec{\Omega}_B^b + R_{bw}\dot{\vec{\Omega}}_B^b + \dot{\vec{\Pi}}_W^w \\
&= -R_{bw}((R_{bw}^T\vec{\Pi}_W^w) \times \vec{\Omega}_B^b) + R_{bw}\dot{\vec{\Omega}}_B^b + \dot{\vec{\Pi}}_W^w \\
&= -R_{bw}((R_{bw}^T\vec{\Pi}_W^w) \times \vec{\Omega}_B^b) + R_{bw}\dot{\vec{\Omega}}_B^b + \delta_{i2}\vec{a}_W + \delta_{i1}\vec{b}_W,
\end{aligned} \tag{38}$$

which cooresponds to the fifth equation in Eq. (18) after it is written as

$$0\ddot{\vec{x}}_B^l + C_W^{52}\dot{\vec{\Omega}}_B^b + 0\ddot{\vec{x}}_W^l + C_W^{54}\dot{\vec{\Omega}}_W^w + 0\vec{f}_{fW}^l + C_W^{56}\vec{a}_W = \vec{d}_W^5. \tag{39}$$

4.2 Translational dynamics

In the lab frame, the Newton equation for the CM of the whole free-flying insect reads

$$m_S \ddot{\vec{x}}^l_S = \vec{f}^l_{gS} + \vec{f}^l_{fB} + \sum_W \vec{f}^l_{fW}, \tag{40}$$

where m_S is the mass of the insect, $\vec{x}^l_S$ is the coordinates of its CM in the lab frame, $\vec{f}_{gS}$ is the gravitational force on the insect, $\vec{f}_{fB}$ is the fluid force on its body, and $\vec{f}_{fW}$ is the fluid force on its one wing. We have

$$m_S \ddot{\vec{x}}^l_S = m_B \ddot{\vec{x}}^l_B + \sum_W m_W \ddot{\vec{x}}^l_W. \tag{41}$$

Using Eqs. (9) and (26), we can write $\vec{f}^l_{fB}$ as

$$\begin{aligned}\vec{f}^l_{fB} = \vec{f}^l_{0B} + A_{fB} \dot{\vec{\Omega}}^l_B + m_{fB} \ddot{\vec{x}}^l_B = \\ \vec{f}^l_{0B} + A_{fB} (\dot{R}^T_{lb} \vec{\Omega}^b_B + R^T_{lb} \dot{\vec{\Omega}}^b_B) + m_{fB} \ddot{\vec{x}}^l_B = \\ \vec{f}^l_{0B} + A_{fB} R^T_{lb} \dot{\vec{\Omega}}^b_B + m_{fB} \ddot{\vec{x}}^l_B.\end{aligned} \tag{42}$$

After substituting Eqs. (41) and (42) into Eq. (40), we obtain

$$\begin{aligned}m_B \ddot{\vec{x}}^l_B + \sum_W m_W \ddot{\vec{x}}^l_W = \\ \vec{f}^l_{gS} + \vec{f}^l_{0B} + A_{fB} R^T_{lb} \dot{\vec{\Omega}}^b_B + m_{fB} \ddot{\vec{x}}^l_B + \sum_W \vec{f}^l_{fW},\end{aligned} \tag{43}$$

which corresponds to the first equation in Eq. (18) after it is written as

$$C^{11} \ddot{\vec{x}}^l_B + C^{12} \dot{\vec{\Omega}}^b_B + \sum_W \left(C^{13}_W \ddot{\vec{x}}^l_W + 0 \dot{\vec{\Omega}}^w_W + C^{15}_W \vec{f}^l_{fW} + 0 \vec{a}_W \right) = \vec{d}^1. \tag{44}$$

Using Eqs. (9) and (27), we can write $\vec{f}^l_{fW}$ as

$$\begin{aligned}\vec{f}^l_{fW} = \vec{f}^l_{0W} + A_{fW} \dot{\vec{\Omega}}^l_W + m_{fW} \ddot{\vec{x}}^l_W = \\ \vec{f}^l_{0W} + A_{fW} (\dot{R}^T_{lw} \vec{\Omega}^w_W + R^T_{lw} \dot{\vec{\Omega}}^w_W) + m_{fW} \ddot{\vec{x}}^l_W = \\ \vec{f}^l_{0W} + A_{fW} R^T_{lw} \dot{\vec{\Omega}}^w_W + m_{fW} \ddot{\vec{x}}^l_W,\end{aligned} \tag{45}$$

which corresponds to the sixth equation in Eq. (18) after it is written as

$$0\ddot{\vec{x}}^l_B + 0\dot{\vec{\Omega}}^b_B + C^{63}_W\ddot{\vec{x}}^l_W + C^{64}_W\dot{\vec{\Omega}}^w_W + C^{65}_W\vec{f}^l_{fW} + 0\vec{a}_W = \vec{d}^6_W. \tag{46}$$

In the lab frame, the Newton equation for the CM of a wing reads

$$m_W\ddot{\vec{x}}^l_W = \vec{f}^l_{gW} + \vec{f}^l_{fW} + \vec{f}^l_{bW}, \tag{47}$$

where $\vec{f}_{bW}$ is the force exerted by the body on the wing at the hinge. By Newton's third law, the force exerted by this wing on the body at the hinge is thereforce given by

$$\vec{f}_{wB} = -\vec{f}_{bW} = \vec{f}^l_{gW} + \vec{f}^l_{fW} - m_W\ddot{\vec{x}}^l_W. \tag{48}$$

4.3 Rotational dynamics

In the body frame, the Euler equation for the body reads

$$I_B\dot{\vec{\Omega}}^b_B + \vec{\Omega}^b_B \times (I_B\vec{\Omega}^b_B) = \vec{\tau}^b_{fB} + \sum_W \vec{\tau}^b_{wB}, \tag{49}$$

where the diagonal matrix I_B is the moments of inertia of the body, $\vec{\tau}_{fB}$ is the fluid torque on the body, and $\vec{\tau}_{wB}$ is the torque on the body received from a hinge. Using Eqs. (10) and (26), we have

$$\begin{aligned}\vec{\tau}^b_{fB} &= R_{lb}\vec{\tau}^l_{fB} = R_{lb}(\vec{\tau}^l_{0B} + \vec{\tau}^l_{1B} + A_{\tau B}\dot{\vec{\Omega}}^l_B)\\ &= R_{lb}(\vec{\tau}^l_{0B} + \vec{\tau}^l_{1B} + A_{\tau B}R^T_{lb}\dot{\vec{\Omega}}^b_B).\end{aligned} \tag{50}$$

Using Eq. (48), we have

$$\begin{aligned}\vec{\tau}^b_{wB} &= \vec{c}^b_{wB} + \vec{x}^b_H \times \vec{f}^b_{wB} = \vec{c}^b_{wB} + H_b\vec{f}^b_{wB} = \vec{c}^b_{wB} + H_bR_{lb}\vec{f}^l_{wB},\\ &= -\vec{c}^b_{bW} + H_bR_{lb}(\vec{f}^l_{gW} + \vec{f}^l_{fW} - m_W\ddot{\vec{x}}^l_W),\end{aligned} \tag{51}$$

where $\vec{c}_{wB} = -\vec{c}_{bW}$ and $\vec{c}_{bW}$ is the muscle torque actively exerted by the body on this wing. After substituting Eqs. (50), (51), (17) into Eq. (49), we have

$$I_B\dot{\vec{\Omega}}^b_B + \vec{\Omega}^b_B \times (I_B\vec{\Omega}^b_B) = R_{lb}(\vec{\tau}^l_{0B} + \vec{\tau}^l_{1B} + A_{\tau B}R^T_{lb}\dot{\vec{\Omega}}^b_B) + \sum_W \left(-(\delta_{i1}\vec{a}_W + \delta_{i2}\vec{b}_W) + H_b R_{lb}(\vec{f}^l_{gW} + \vec{f}^l_{fW} - m_W\ddot{\vec{x}}^l_W)\right), \tag{52}$$

which cooresponds the second equation in Eq. (18) after it is written as

$$0\ddot{\vec{x}}^l_B + C^{22}\dot{\vec{\Omega}}^b_B + \sum_W \left(C^{23}_W\ddot{\vec{x}}^l_W + 0\dot{\vec{\Omega}}^w_W + C^{25}_W\vec{f}^l_{fW} + C^{26}_W\vec{a}_W\right) = \vec{d}^2. \tag{53}$$

In the wing frame, the Euler equation for a wing reads

$$I_W\dot{\vec{\Omega}}^w_W + \vec{\Omega}^w_W \times (I_W\vec{\Omega}^w_W) = \vec{\tau}^w_{fW} + \vec{\tau}^w_{bW}, \tag{54}$$

where the diagonal matrix I_W is the moments of inertia of the wing, $\vec{\tau}_{fW}$ is the fluid torque on the wing, and $\vec{\tau}_{bW}$ is the torque on the wing received from the hinge. Using Eqs. (10) and (27), we have

$$\begin{aligned}\vec{\tau}^w_{fW} &= R_{lw}\vec{\tau}^l_{fW} = R_{lw}(\vec{\tau}^l_{0W} + \vec{\tau}^l_{1W} + A_{\tau W}\dot{\vec{\Omega}}^l_W) \\ &= R_{lw}(\vec{\tau}^l_{0W} + \vec{\tau}^l_{1W} + A_{\tau W}R^T_{lw}\dot{\vec{\Omega}}^w_W).\end{aligned} \tag{55}$$

Using Eq. (48), we have

$$\begin{aligned}\vec{\tau}^w_{bW} &= \vec{c}^w_{bW} + \vec{x}^w_H \times \vec{f}^w_{bW} = \vec{c}^w_{bW} + H_w\vec{f}^w_{bW} = R_{bw}\vec{c}^b_{bW} + H_w R_{lw}\vec{f}^l_{bW} \\ &= R_{bw}\vec{c}^b_{bW} + H_w R_{lw}(m_W\ddot{\vec{x}}^l_W - \vec{f}^l_{gW} - \vec{f}^l_{fW}).\end{aligned} \tag{56}$$

After substituting Eqs. (55), (56), and (17) into Eq. (54), we have

$$I_W\dot{\vec{\Omega}}^w_W + \vec{\Omega}^w_W \times (I_W\vec{\Omega}^w_W) = R_{lw}(\vec{\tau}^l_{0W} + \vec{\tau}^l_{1W} + A_{\tau W}R^T_{lw}\dot{\vec{\Omega}}^w_W) + R_{bw}(\delta_{i1}\vec{a}_W + \delta_{i2}\vec{b}_W) + H_w R_{lw}(m_W\ddot{\vec{x}}^l_W - \vec{f}^l_{gW} - \vec{f}^l_{fW}), \tag{57}$$

which cooresponds the third equation in Eq. (18) after it is written as

$$0\ddot{\vec{x}}^l_B + 0\dot{\vec{\Omega}}^b_B + C^{33}_W\ddot{\vec{x}}^l_W + C^{34}_W\dot{\vec{\Omega}}^w_W + C^{35}_W\vec{f}^l_{fW} + C^{36}_W\vec{a}_W = \vec{d}^3_W. \tag{58}$$

5 Tests

We perform two simple tests on our formulation for an insect model with two wings. The test code in MATLAB is posted on the author's webpage. One thing can always be checked in the tests is how well Eq. (32) (the constraint between the body and wings) is satisfied when Eq. (20) is numerically integrated to update the position and orientation of both the body and wings. In real applications, we should update only the orientation of the wings from numerical integration, and obtain the position of the wings directly from Eq. (32). Otherwise, numerical errors may break the constraint. Note that the initial conditions for integrating Eq. (20) must satisfy Eqs. (32) and (33).

In the first test, we examine the conservation of angular momentum of a torque free insect model. In this model, the two identical wings are fixed relative to the body with reflective symmetry through the $x^b - z^b$ plane, and the CM of the model falls on the x^b axis. If the initial angular velocity of the model has only the rolling component, then the model keeps rolling about the rolling x^b axis with the initial rate. The simulation verifies this conservation, as shown in Fig. 2. The constraint relation, Eq. (32), is also well satisfied in the simulation, as shown in Fig. 2.

In the second test, we examine Newton's second law for an insect model subject to an external force which has three non-zero components in the lab frame. The two wings flap with respect to the body under the action of prescribed body-to-wing torque. The CM of

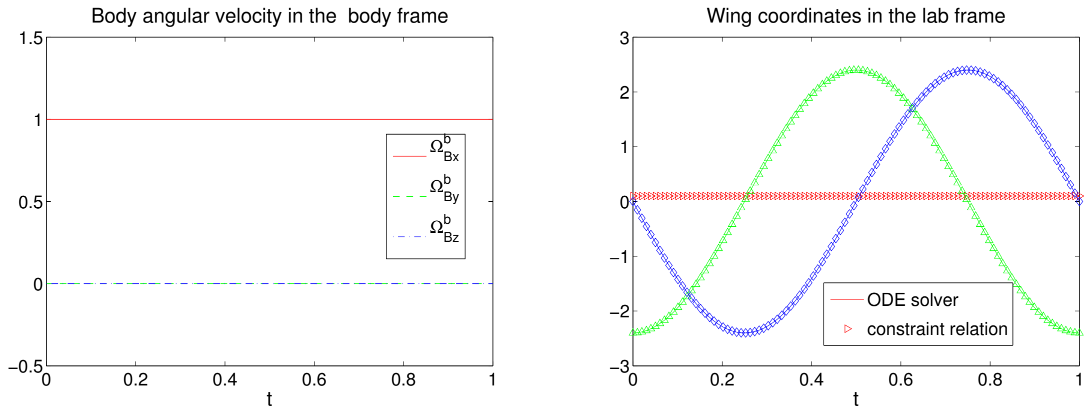

Figure 2:

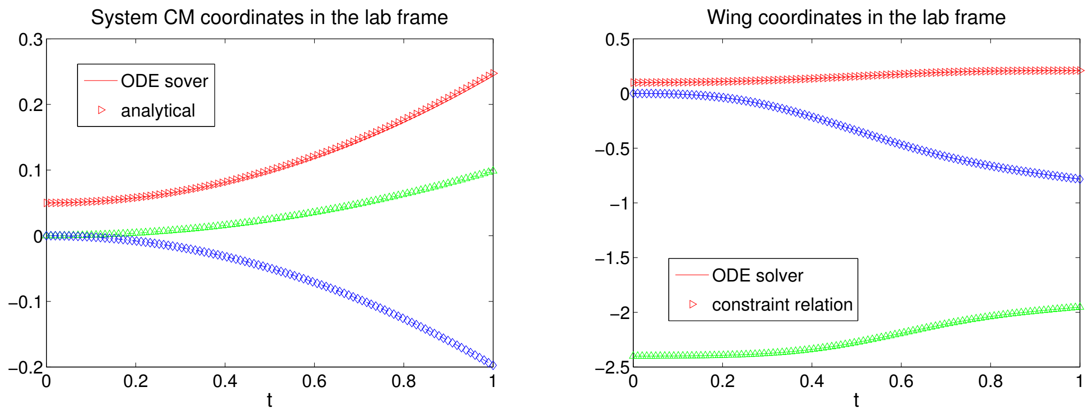

Figure 3:

the model follows a known trajectory according to Newton's second law, which is verified in Fig. 3. The constraint relation, Eq. (32), is well satisfied again in this simulation, as shown in Fig. 3.

Appendix

To write the explicit forms of the transformation matrices, we use the following Tait-Bryan angles (angles of roll, pitch, and yaw) to describe the orientation of one reference frame relative to another.

- ϕ_{lb}, θ_{lb}, and ψ_{lb}: the roll (around the x^b axis), pitch (around the y^b axis), and yaw (around the z^b axis) angles of the body relative to the lab frame, respectively;

- ϕ_{bw}, θ_{bw}, and ψ_{bw}: the roll (around the x^w axis), pitch (around the y^w axis), and yaw (around the z^w axis) angles of a wing relative to the body, respectively.

Applying the rotations in the order of yaw, pitch, and roll, we have

$$R_{lb} = R_{lb}^{roll} R_{lb}^{pitch} R_{lb}^{yaw}, \tag{59}$$

$$R_{bw} = R_{bw}^{roll} R_{bw}^{pitch} R_{bw}^{yaw}, \tag{60}$$

$$R_{lw} = R_{bw} R_{lb}, \tag{61}$$

where

$$R_{lb}^{roll} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{lb} & \sin\phi_{lb} \\ 0 & -\sin\phi_{lb} & \cos\phi_{lb} \end{pmatrix}, \quad R_{bw}^{roll} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{bw} & \sin\phi_{bw} \\ 0 & -\sin\phi_{bw} & \cos\phi_{bw} \end{pmatrix}, \tag{62}$$

$$R_{lb}^{pitch} = \begin{pmatrix} \cos\theta_{lb} & 0 & -\sin\theta_{lb} \\ 0 & 1 & 0 \\ \sin\theta_{lb} & 0 & \cos\theta_{lb} \end{pmatrix}, \quad R_{bw}^{pitch} = \begin{pmatrix} \cos\theta_{bw} & 0 & -\sin\theta_{bw} \\ 0 & 1 & 0 \\ \sin\theta_{bw} & 0 & \cos\theta_{bw} \end{pmatrix}, \tag{63}$$

$$R_{lb}^{yaw} = \begin{pmatrix} \cos\psi_{lb} & \sin\psi_{lb} & 0 \\ -\sin\psi_{lb} & \cos\psi_{lb} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{bw}^{yaw} = \begin{pmatrix} \cos\psi_{bw} & \sin\psi_{bw} & 0 \\ -\sin\psi_{bw} & \cos\psi_{bw} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{64}$$

The angular velocity $\vec{\Omega}_B$ in the body frame and the angular velocity $\vec{\Pi}_W$ in the wing frame are

$$\vec{\Omega}_B^b = \begin{pmatrix} \dot{\phi}_{lb} \\ 0 \\ 0 \end{pmatrix} + R_{lb}^{roll} \begin{pmatrix} 0 \\ \dot{\theta}_{lb} \\ 0 \end{pmatrix} + R_{lb}^{roll} R_{lb}^{pitch} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}_{lb} \end{pmatrix} = K_B \dot{\vec{\alpha}}_{lb}, \tag{65}$$

$$\vec{\Pi}_W^w = \begin{pmatrix} \dot{\phi}_{bw} \\ 0 \\ 0 \end{pmatrix} + R_{bw}^{roll} \begin{pmatrix} 0 \\ \dot{\theta}_{bw} \\ 0 \end{pmatrix} + R_{bw}^{roll} R_{bw}^{pitch} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}_{bw} \end{pmatrix} = K_W \dot{\vec{\alpha}}_{bw}, \tag{66}$$

where

$$K_B = \begin{pmatrix} 1 & 0 & -\sin\theta_{lb} \\ 0 & \cos\phi_{lb} & \sin\phi_{lb}\cos\theta_{lb} \\ 0 & -\sin\phi_{lb} & \cos\phi_{lb}\cos\theta_{lb} \end{pmatrix}, \quad \vec{\alpha}_{lb} = \begin{pmatrix} \phi_{lb} \\ \theta_{lb} \\ \psi_{lb} \end{pmatrix}, \tag{67}$$

$$K_W = \begin{pmatrix} 1 & 0 & -\sin\theta_{bw} \\ 0 & \cos\phi_{bw} & \sin\phi_{bw}\cos\theta_{bw} \\ 0 & -\sin\phi_{bw} & \cos\phi_{bw}\cos\theta_{bw} \end{pmatrix}, \quad \vec{\alpha}_{bw} = \begin{pmatrix} \phi_{bw} \\ \theta_{bw} \\ \psi_{bw} \end{pmatrix}. \tag{68}$$

Note that the initial conditions for integrating Eq. (20) must satisfy Eq. (66) when the wing kinematics (the Tait-Bryan angles of the wing relative to the body) is prescribed.

If the Tait-Bryan angles of the wing relative to the body are prescribed, then by Eq. (66), $\dot{\vec{\Pi}}_W^w$ can be calculated from

$$\dot{\vec{\Pi}}_W^w = \dot{K}_W \dot{\vec{\alpha}}_{bw} + K_W \ddot{\vec{\alpha}}_{bw}, \tag{69}$$

where $\dot{K}_W$ is

$$\dot{K}_W = \dot{\phi}_{bw} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi_{bw} & \cos\phi_{bw}\cos\theta_{bw} \\ 0 & -\cos\phi_{bw} & -\sin\phi_{bw}\cos\theta_{bw} \end{pmatrix} + \dot{\theta}_{bw} \begin{pmatrix} 0 & 0 & -\cos\theta_{bw} \\ 0 & 0 & -\sin\phi_{bw}\sin\theta_{bw} \\ 0 & 0 & -\cos\phi_{bw}\sin\theta_{bw} \end{pmatrix}. \tag{70}$$

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List of Figure Captions