WEAK SOLUTION TO COMPRESSIBLE HYDRODYNAMIC FLOW OF LIQUID CRYSTALS IN DIMENSION ONE

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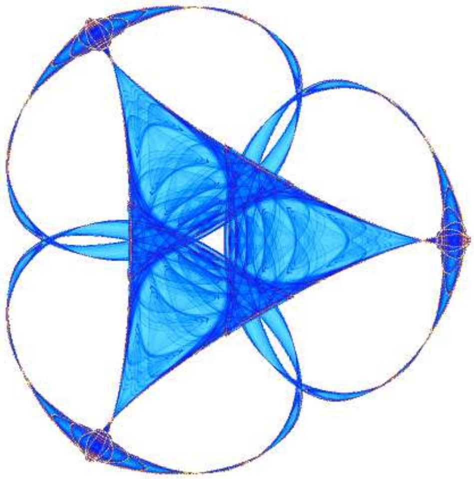

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Weak solution to compressible hydrodynamic flow of liquid crystals in dimension one

Shijin Ding* Changyou Wang† Huanyao Wen*

Abstract

We consider the equation modeling the compressible hydrodynamic flow of liquid crystals in one dimension. In this paper, we establish the existence of a weak solution (ρ, u, n) of such a system when the initial density function $0 \le \rho_0 \in L^\gamma$ for $\gamma > 1$, $u_0 \in L^2$, and $n_0 \in H^1$. This extends a previous result by [12], where the existence of a weak solution was obtained under the stronger assumption that the initial density function $0 < c \le \rho_0 \in H^1$, $u_0 \in L^2$, and $n_0 \in H^1$.

Key Words: Liquid crystal, compressible hydrodynamic flow, global weak solution.

1 Introduction

In this paper, we consider the one dimensional initial-boundary value problem:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + a(\rho^\gamma)_x = \mu u_{xx} - \lambda(|n_x|^2)_x, \\ n_t + u n_x = \theta(n_{xx} + |n_x|^2 n), \end{cases} \tag{1.1}$$

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for $(x,t) \in (0,1) \times (0,+\infty)$, with the initial condition:

$$(\rho,\ \rho u,\ n)\big|_{t=0} = (\rho_0,\ m_0,\ n_0) \text{ in } [0,1], \tag{1.2}$$

where $n_0 : [0,1] \to S^2$ and the boundary condition:

$$(u,\ n_x)\big|_{\partial[0,1]} = (0,\ 0),\ t > 0, \tag{1.3}$$

where $\rho : [0,1] \times [0,+\infty) \to \mathbb{R}_+$ is the density function, $u : [0,1] \times [0,+\infty) \to \mathbb{R}$ is the scalar-valued velocity field in dimension one, $n : [0,1] \times [0,+\infty) \to S^2$ is the optical axis vector of the liquid crystal, with $S^2 = \{y \in \mathbb{R}^3 :\ |y| = 1\}$ the unit sphere in $\mathbb{R}^3$, the constants $\mu > 0, \lambda > 0, \theta > 0$ are the fluid viscosity, competition between kinetic and potential energy, and microscopic elastic relaxation time respectively, and $\gamma > 1$ and $a > 0$ are given constants.

The hydrodynamic flow of compressible (or incompressible) liquid crystals was first derived by Ericksen [1] and Leslie [2] in 1960's. However, its rigorous mathematical analysis was not taken place until 1990's, when Lin [3] and Lin-Liu [4, 5, 6] made some important progress towards the existence of global weak solutions and partial regularity of the incompressible hydrodynamic flow equation of liquid crystals.

When the Ossen-Frank energy configuration functional reduces to the Dirichlet energy functional, the hydrodynamic flow equation of liquid crystals in $\Omega \subset \mathbb{R}^d$ can be written as follows (see Lin [3]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + a\nabla(\rho^\gamma) = \mu\Delta u - \lambda \operatorname{div}\left(\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2}\mathrm{I}_d\right), \\ n_t + u \cdot \nabla n = \theta(\Delta n + |\nabla n|^2 n), \end{cases} \tag{$\star$}$$

where $\rho : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is the density function, $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ is the velocity field, $n : \Omega \times \mathbb{R}_+ \to S^2$ is the director field, $u \otimes u$ is the matrix of order d, whose (i,j)-th entry is $u^i u^j$ for $1 \le i,j \le d$, and $\nabla n \odot \nabla n$ is the matrix of order d whose (i,j)-th entry is $n_{x_i} \cdot n_{x_j}$ for $1 \le i,j \le d$.

Observe that for $d = 1$, since $\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2}\mathrm{I}_d = \frac{1}{2}|n_x|^2$, the system ($\star$) reduces to (1.1) with λ replaced by 2λ. If the density function ρ is a positive constant, then ($\star$) becomes the hydrodynamic flow equation of incompressible liquid crystals (i.e., div $u = 0$). In a series of papers, Lin [3] and Lin-Liu [4, 5, 6] addressed the existence and partial regularity theory of suitable weak solution to the incompressible hydrodynamic flow of liquid crystals of variable length. More precisely, they considered the approximate equation of incompressible hydrodynamic flow of liquid crystals in which $\rho = 1$ and $|\nabla n|^2 n$ in $(\star)_3$ is replaced by $\dfrac{(1-|n|^2)n}{\epsilon^2}$, and proved in [4], among other results, both the existence of local classical solutions and the global existence of weak solutions in dimension two and three. For any fixed $\epsilon > 0$, they also showed the existence and uniqueness of global classical solution either in dimension two or dimension three when the fluid viscosity μ is sufficiently large; in [6], Lin and Liu extended the classical theorem by Caffarelli-Kohn-Nirenberg [7] on the Navier-Stokes equation that asserts the one dimensional parabolic Hausdorff measure of the singular set of any *suitable* weak solution is zero. See also [8, 9] for relevant results. For the incompressible case $\rho = 1$ and div $u = 0$, it remains to be an open problem that for $\epsilon \downarrow 0$ whether a sequence of solutions (u_ϵ, n_ϵ) to the approximate equation converges to a solution of the original equation $(\star)$. It is also an open problem to ask for $d = 3$, whether there exists a global weak solution to the incompressible hydrodynamic flow equation $(\star)$ similar to the Leray-Hopf type solutions in the context of Naiver-Stokes equation. We answered this later question for $d = 2$ in [10]. For $\rho \geq 0$, div$u = 0$, and $d = 2$ or 3, Ding and Wen showed in [11] (i) the existence of a unique local strong solution to $(\star)$, and (ii) for $d = 2$ if initial density $\rho_0 \geq c > 0$, then there exists a unique global strong solution for small initial data.

For the compressible hydrodynamic flow equation $(\star)$ in dimension $d = 1$, in a previous work [12] Ding-Lin-Wang-Wen obtained both the existence and uniqueness of a global strong solution (ρ, u, n) when the initial data $\rho_0 \in H^1([0,1])$ has a positive lower bound $\rho_0 \geq c_0 > 0$, and $u_0 \in H^1([0,1])$, $n_0 \in H^2([0,1], S^2)$. Moreover, by suitable approximation the method in [12] can yield the existence of a global weak solution under the assumption that $0 < c_0 \leq \rho_0 \in H^1([0,1]), u_0 \in L^2([0,1]), n_0 \in H^1([0,1], S^2)$. Based on the energy inequality of $(\star)$, we conjectured in [12] (see [12] Remark 1.1) the existence of a global weak solution (ρ, u, n) of $(\star)$ whenever $(\rho_0, u_0, n_0) \in L^\gamma([0,1] \times L^2([0,1]) \times H^1([0,1], S^2)$. The main purpose of this paper is to answer this question by adopting the ideas of weak convergence and compensated compactness by Feireisl-Navotný-Petzeltová [15] on the compressible isentropic Navier-Stokes equation.

In fact, when the optical axis n is a constant unit vector, $(\star)$ reduces to the compressible isentropic Navier-Stokes equation and there have been many works on the existence of weak solutions to the compressible isentropic Navier-Stokes equation. For example, P.L. Lions obtained in [13] the existence of a global weak solution for $\gamma \geq \frac{9}{5}$ and $d = 3$. S. Jiang and P. Zhang obtained in [14] the existence of a global weak solution to the Cauchy problem for spherically symmetric initial data $\rho_0 \in L^\gamma$ for any $\gamma > 1$ in dimensions $d = 2$ or 3. For general initial data ρ_0 and $d = 3$, E. Feireisl et al in [15] extended the earlier work by P.L. Lions in [13] to the cases $\gamma > \frac{3}{2}$.

While our ideas were originated mainly from [15], the proof is simpler, since we exploit such of the one-dimensional features and use integrals instead of commutators. Moreover, our argument works for all $\gamma > 1$.

Since the exact values of constants a and μ, λ, θ in (1.1) don't play any role in the analysis, we assume henceforth that

$$\mu = \lambda = \theta = a = 1.$$

Notations:

(1) $I = (0, 1)$, $\partial I = \{0, 1\}$, $Q_T = I \times (0, T)$ for $T > 0$.

(2) $\widehat{f}$: $\widehat{f}(x) = f(x)$ for $x \in I$, and $\widehat{f}(x) = 0$ for $x \in \mathbb{R} \setminus I$.

(3) $\eta_\sigma(\cdot) = \frac{1}{\sigma^d}\eta(\frac{\cdot}{\sigma})$, where η is a standard mollifier.

(4) $C([0, T]; X - \omega)$: $f \in C([0, T]; X - \omega) \Leftrightarrow \forall g \in X'$, $\langle f(t), g\rangle_{X \times X'} \in C([0, T])$.

(5) $H^1(I, S^2) = \{v \in H^1(I, \mathbb{R}^3) : \ |v(x)| = 1 \text{ a.e. } x \in I\}$.

(6) $\mathcal{D}'(Q_T) = (C_0^\infty(Q_T))'$ is the dual space of $C_0^\infty(Q_T)$.

Definition 1.1 *We call* $(\rho, u, n) : Q_\infty \to \mathbb{R}_+ \times \mathbb{R} \times S^2$ *a global weak solution of*

(1.1)-(1.3) if for any $0 < T < +\infty$,

$$\text{(1)}\qquad \rho \in L^\infty(0,T;L^\gamma(I)),\ \rho u^2 \in L^\infty(0,T;L^1(I)),\ \rho \geq 0\ a.e.\ in\ Q_T,$$

$$u \in L^2(0,T;H_0^1(I)),\ n \in L^\infty(0,T;(H^1(I))^3) \cap L^2(0,T;(H^2(I))^3),$$

$$n_t \in L^2(0,T;(L^2(I))^3),\ |n| = 1\ in\ Q_T,$$

$$(\rho,\rho u)(x,0) = (\rho_0(x), m_0(x)),\ weakly\ in\ L^\gamma(I) \times L^{\frac{2\gamma}{\gamma+1}}(I),$$

$$n(x,0) = n_0(x)\ in\ \overline{I},\ (n_x(0,t), n_x(1,t)) = 0\ a.e.\ in\ (0,T).$$

(2) $(1.1)_1$, $(1.1)_2$ *are satisfied in* $\mathscr{D}'(Q_T)$, *and* $(1.1)_3$ *holds a.e. in* Q_T.

$$\text{(3)}\qquad \int_I \left(\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma-1} + |n_x|^2\right)(t) + \int_{I\times[0,t]} \left(u_x^2 + 2\left|n_{xx} + |n_x|^2 n\right|^2\right)$$
$$\leq \int_I \left(\frac{m_0^2}{2\rho_0} + \frac{\rho_0^\gamma}{\gamma-1} + |(n_0)_x|^2\right),\ for\ a.e.\ t \in (0,T). \tag{1.4}$$

Our main result is as follows

Theorem 1.1 *If* $\rho_0 \geq 0$, $\rho_0 \in L^\gamma(I)$, $\frac{m_0}{\sqrt{\rho_0}} \in L^2(I)$, *and* $n_0 \in H^1(I,S^2)$, *then there exists a global weak solution* $(\rho,u,n) : I \times [0,+\infty) \to \mathbb{R}_+ \times \mathbb{R} \times S^2$ *to (1.1)-(1.3) such that for any* $T > 0$,

$$\int_{Q_T} \rho^{2\gamma} \leq c(E_0,T),$$

where

$$E_0 := \int_I \left(\frac{m_0^2}{2\rho_0} + \frac{\rho_0^\gamma}{\gamma-1} + |(n_0)_x|^2\right)$$

is the total energy of the initial data.

The rest of the paper is organized as follows. In section 2, we present some useful Lemmas which will be needed. In section 3, we derive some a priori estimates for the approximate solutions of (1.1)-(1.3), and prove the existence of weak solution.

2 Preliminaries

Lemma 2.1 *([16]). Assume* $X \subset E \subset Y$ *are Banach spaces and* $X \subset E$ *is compact. Then*

(i) $\left\{\varphi : \varphi \in L^q(0,T;X), \dfrac{\partial\varphi}{\partial t} \in L^1(0,T;Y)\right\} \subset L^q(0,T;E)$ is compact for $q \geq 1$,

(ii) $\left\{\varphi : \varphi \in L^\infty(0,T;X), \dfrac{\partial\varphi}{\partial t} \in L^r(0,T;Y)\right\} \subset C([0,T];E)$ is compact for $r > 1$.

Lemma 2.2 *([15]). Let* $\rho \in L^2(\Omega \times (0,T))$ *and* $u \in L^2(0,T; H_0^1(\Omega;\mathbb{R}^d))$ *solve*

$$\rho_t + \operatorname{div}(\rho u) = 0 \ in \ \mathscr{D}'(\Omega \times (0,T)). \tag{2.1}$$

Then

$$\partial_t(b(\rho)) + \operatorname{div}(b(\rho)u) + [b'(\rho)\rho - b(\rho)]divu = 0, \ in \ \mathscr{D}'(\Omega \times (0,T)), \tag{2.2}$$

for any $b \in C^1(\mathbb{R})$ *such that* $b'(z) \equiv 0$ *for all* $z \in \mathbb{R}$ *large enough.*

Lemma 2.3 *([20]). There exists* $C > 0$ *such that for any* $\rho \in L^2(\mathbb{R}^d)$ *and* $u \in H^1(\mathbb{R}^d)$*,*

$$\|\eta_\sigma * \operatorname{div}(\rho u) - \operatorname{div}(u(\rho * \eta_\sigma))\|_{L^1(\mathbb{R}^d)} \leq C\|u\|_{H^1(R^d)}\|\rho\|_{L^2(\mathbb{R}^d)}.$$

In addition,

$$\eta_\sigma * \operatorname{div}(\rho u) - \operatorname{div}(u(\rho * \eta_\sigma)) \to 0 \ in \ L^1(\mathbb{R}^d), \ as \ \sigma \to 0.$$

Lemma 2.4 *([19]). For a bounded Lipschitz domain* $\Omega \subset \mathbb{R}^d$*, let* $\rho \in L^2(\Omega \times (0,T))$ *and* $u \in L^2(0,T; H_0^1(\Omega;\mathbb{R}^d))$ *solve (2.1). Then* (ρ, u) *solve (2.1) in* $\mathscr{D}'(\mathbb{R}^d \times (0,T))$ *provided* (ρ, u) *were extended to be zero outside* Ω*.*

Lemma 2.5 *([19]). Let* $\overline{O} \subset \mathbb{R}^M$ *be a compact set and* X *be a separable Banach space. Assume that* $v_m : \overline{O} \to X^*$*,* $m \in \mathbb{Z}_+$*, is a sequence of measurable functions such that*

$$\operatorname*{esssup}_{t \in \overline{O}} \|v_m(t)\|_{X^*} \leq N, \ uniformly \ in \ m.$$

Moreover, let the family of functions

$$\langle v_m, \Phi \rangle : t \to \langle v_m(t), \Phi \rangle, \ t \in \overline{O}$$

be equi-continuous for any Φ *belonging to a dense subset in* X*. Then* $v_m \in C(\overline{O}; X - \omega)$ *for* $m \in \mathbb{Z}_+$*, and there exists* $v \in C(\overline{O}; X - \omega)$ *such that after taking possible subsequences,*

$$v_m \to v \ in \ C(\overline{O}; X - \omega), \ as \ m \to \infty.$$

Lemma 2.6 *([19]). Let $O \subset \mathbb{R}^N$ be a measurable set and $v_m \in L^1(O; \mathbb{R}^M)$, $m \in \mathbb{Z}_+$, be such that*

$$v_m \to v \text{ weakly in } L^1(O; \mathbb{R}^M).$$

Let $\Phi : \mathbb{R}^M \to (-\infty, \infty]$ be a lower semi-continuous convex function such that $\Phi(v_m) \in L^1(O)$ for any m, and

$$\Phi(v_m) \to \overline{\Phi(v)}, \text{ weakly in } L^1(O).$$

Then

$$\Phi(v) \leq \overline{\Phi(v)}, \text{ a.e. in } O.$$

3 Existence of weak solution

In this section, we approximate the initial data (ρ_0, u_0, n_0) by a sequence of smooth initial data $(\rho_{0\delta}, u_{0\delta}, n_{0\delta})$ such that $\rho_{0\delta}$ has positive lower bounds, solve (1.1) with these smooth initial data to get a sequence of classical solutions $(\rho_\delta, u_\delta, n_\delta)$, and then derive some a priori estimates of such solutions. The main difficulty is to show the convergence of the pressure functions ρ_δ^γ, which is achieved by Lemmas 3.2-3.4.

By the Sobolev's extension theorem (see [18]), there exists $\widetilde{n}_0 \in H^1(\mathbb{R}) \cap C_0(\mathbb{R})$ such that $\widetilde{n}_0 = n_0$ in I. We mollify the initial data as follows.

$$\begin{cases} \rho_{0\delta} = \eta_\delta \star \widehat{\rho}_0 + \delta, \\ u_{0\delta} = \dfrac{1}{\sqrt{\rho_{0\delta}}} \eta_\delta \star (\widehat{\dfrac{m_0}{\sqrt{\rho_0}}}), \\ n_{0\delta} = \dfrac{\eta_\delta \star \widetilde{n}_0}{|\eta_\delta \star \widetilde{n}_0|}. \end{cases}$$

Then $\rho_{0\delta} \geq \delta > 0$, $(\rho_{0\delta}, u_{0\delta}, n_{0\delta}) \in C^{2+\alpha}(\overline{I})$ for $0 < \alpha < 1$, and

$$\begin{cases} \rho_{0\delta} \to \rho_0, \text{ in } L^\gamma(I), \\ \sqrt{\rho_{0\delta}} u_{0\delta} \to \frac{m_0}{\sqrt{\rho_0}} \text{ in } L^2(I), \\ \rho_{0\delta} u_{0\delta} \to m_0 \text{ in } L^{\frac{2\gamma}{\gamma+1}}(I), \\ n_{0\delta} \to n_0 \text{ in } H^1(I), \end{cases} \tag{3.1$'$}$$

as $\delta \to 0$. From [12], there exists a sequence of global classical solutions $(\rho_\delta, u_\delta, n_\delta)$ to

$$\begin{cases} (\rho_\delta)_t + (\rho_\delta u_\delta)_x = 0, \ \rho_\delta > 0, \\ (\rho_\delta u_\delta)_t + (\rho_\delta u_\delta^2)_x + (\rho_\delta^\gamma)_x = (u_\delta)_{xx} - (|(n_\delta)_x|^2)_x, \\ (n_\delta)_t + u_\delta (n_\delta)_x = (n_\delta)_{xx} + |(n_\delta)_x|^2 n_\delta, \ |n_\delta| = 1, \end{cases} \tag{3.1}$$

for $(x,t) \in [0,1] \times (0,+\infty)$, with the initial and boundary conditions:

$$(\rho_\delta, \ u_\delta, \ n_\delta)\big|_{t=0} = (\rho_{0\delta}, \ u_{0\delta}, \ n_{0\delta}) \text{ in } [0,1],$$

$$(u_\delta, \ \partial_x n_\delta)\big|_{\partial I} = (0, \ 0).$$

For such solutions, the following Lemma has been proven by [12].

Lemma 3.1 *([12]) For any $T > 0$ and $0 \le t \le T$, it holds*

$$\begin{aligned} &\int_I \left(\frac{\rho_\delta u_\delta^2}{2} + \frac{\rho_\delta^\gamma}{\gamma - 1} + |(n_\delta)_x|^2 \right)(t) + \int_0^t \int_I \left(|(u_\delta)_x|^2 + 2\left|(n_\delta)_{xx} + |(n_\delta)_x|^2 n_\delta\right|^2 \right) \\ = &\int_I \left(\frac{\rho_{0\delta} u_{0\delta}^2}{2} + \frac{\rho_{0\delta}^\gamma}{\gamma - 1} + |(n_{0\delta})_x|^2 \right), \end{aligned} \tag{3.2}$$

and

$$\int_{Q_T} \left(|n_\delta)_t|^2 + |(n_\delta)_{xx}|^2 \right) \le c(E_0, T). \tag{3.3}$$

From (3.2), we have $\rho_\delta^\gamma \in L^\infty(0,T; L^1(I))$. To take limits of ρ_δ^γ as $\delta \to 0$, we need more regularity of ρ_δ with respect to the space variable. More precisely, we have

Lemma 3.2

$$\int_{Q_T} \rho_\delta^{2\gamma} \le c(E_0, T).$$

Proof. Multiplying $(3.1)_2$ by $\left(\int_0^x \rho_\delta^\gamma - x \int_I \rho_\delta^\gamma\right)$, integrating the resulting equation over Q_T, and using integration by parts, we get

$$\begin{aligned} \int_{Q_T} \rho_\delta^{2\gamma} &= \int_I \rho_\delta u_\delta \left(\int_0^x \rho_\delta^\gamma - x \int_I \rho_\delta^\gamma \right) \Big|_0^T - \int_0^T \int_I \rho_\delta u_\delta \left[\int_0^x (\rho_\delta^\gamma)_t - x \int_I (\rho_\delta^\gamma)_t \right] \\ &- \int_0^T \int_I (\rho_\delta u_\delta^2) \left(\rho_\delta^\gamma - \int_I \rho_\delta^\gamma \right) + \int_0^T \left(\int_I \rho_\delta^\gamma \right)^2 + \int_0^T \int_I (u_\delta)_x \left(\rho_\delta^\gamma - \int_I \rho_\delta^\gamma \right) \\ &- \int_0^T \int_I |(n_\delta)_x|^2 \left(\rho_\delta^\gamma - \int_I \rho_\delta^\gamma \right) \\ &= I + II + III + IV + V + VI. \end{aligned}$$

$$
\begin{aligned}
I &= \int_I \rho_\delta u_\delta \left(\int_0^x \rho_\delta^\gamma - x \int_I \rho_\delta^\gamma \right) \Big|_0^T \\
&\le c \sup_{0\le t\le T} \left(\int_I \rho_\delta |u_\delta| \int_I \rho_\delta^\gamma \right) \\
&\le c \sup_{0\le t\le T} \left(\int_I \rho_\delta u_\delta^2 \int_I \rho_\delta^\gamma \right) + c \sup_{0\le t\le T} \left(\int_I \rho_\delta \int_I \rho_\delta^\gamma \right) \\
&\le c(E_0),
\end{aligned}
$$

where we have used (3.2). To estimate II, we multiply $(3.1)_1$ by $\gamma\rho_\delta^{\gamma-1}$ and get

$$
(\rho_\delta^\gamma)_t + (\rho_\delta^\gamma u_\delta)_x + (\gamma-1)\rho_\delta^\gamma (u_\delta)_x = 0. \tag{3.4}
$$

Therefore, we have from (3.4) that

$$
\begin{aligned}
II &= \int_0^T \int_I \rho_\delta u_\delta \int_0^x \left[(\rho_\delta^\gamma u_\delta)_x + (\gamma-1)\rho_\delta^\gamma (u_\delta)_x \right] \\
&- \int_0^T \int_I x \rho_\delta u_\delta \int_I \left[(\rho_\delta^\gamma u_\delta)_x + (\gamma-1)\rho_\delta^\gamma (u_\delta)_x \right] \\
&= \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + (\gamma-1) \int_0^T \int_I \rho_\delta u_\delta \int_0^x \rho_\delta^\gamma (u_\delta)_x \\
&- (\gamma-1) \int_0^T \int_I x \rho_\delta u_\delta \int_I \rho_\delta^\gamma (u_\delta)_x \\
&\le \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + c \int_0^T \int_I \rho_\delta |u_\delta| \int_I \rho_\delta^\gamma |(u_\delta)_x| \\
&\le \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + c \int_0^T \int_I \rho_\delta^\gamma |(u_\delta)_x| \int_I (\rho_\delta + \rho_\delta u_\delta^2) \\
&\le \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + c(E_0) \int_0^T \int_I \rho_\delta^\gamma |(u_\delta)_x|.
\end{aligned}
$$

By Cauchy's inequality, Hölder's inequality, and (3.2), we have

$$
\begin{aligned}
II &\le \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + c(E_0) \int_0^T \|\rho_\delta\|_{L^{2\gamma}(I)}^\gamma \|(u_\delta)_x\|_{L^2(I)} \\
&\le \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + \frac{1}{4} \int_{Q_T} \rho_\delta^{2\gamma} + c(E_0).
\end{aligned}
$$

$$
\begin{aligned}
III + IV &= -\int_0^T \int_I (\rho_\delta u_\delta^2) \left(\rho_\delta^\gamma - \int_I \rho_\delta^\gamma \right) + \int_0^T \left(\int_I \rho_\delta^\gamma \right)^2 \\
&= -\int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + \int_0^T \int_I \rho_\delta u_\delta^2 \int_I \rho_\delta^\gamma + \int_0^T \left(\int_I \rho_\delta^\gamma \right)^2 \\
&\le -\int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + c(E_0, T).
\end{aligned}
$$

$$
\begin{aligned}
V &= \int_0^T \int_I (u_\delta)_x \rho_\delta^\gamma - \int_0^T \int_I (u_\delta)_x \int_I \rho_\delta^\gamma \\
&\le \int_0^T \|(u_\delta)_x\|_{L^2(I)} \|\rho_\delta\|_{L^{2\gamma}(I)}^\gamma + \frac{1}{2} \sup_{0\le t\le T} \int_I \rho_\delta^\gamma \int_{Q_T} (|(u_\delta)_x|^2 + 1) \\
&\le \frac{1}{4} \int_{Q_T} \rho_\delta^{2\gamma} + c(E_0, T).
\end{aligned}
$$

$$
\begin{aligned}
VI &= -\int_0^T \int_I |(n_\delta)_x|^2 \left(\rho_\delta^\gamma - \int_I \rho_\delta^\gamma \right) \\
&\le \sup_{0\le t\le T} \int_I \rho_\delta^\gamma \int_0^T \int_I |(n_\delta)_x|^2 \\
&\le c(E_0, T).
\end{aligned}
$$

Putting all these inequalities together, we have

$$
\begin{aligned}
\int_{Q_T} \rho_\delta^{2\gamma} &= I + II + III + IV + V + VI \\
&\le \frac{1}{2} \int_{Q_T} \rho_\delta^{2\gamma} + c(E_0, T).
\end{aligned}
$$

This completes the proof. □

It follows from Lemma 3.1 and 3.2 that there exists a subsequence of $(\rho_\delta, u_\delta, n_\delta)$, still denoted by $(\rho_\delta, u_\delta, n_\delta)$, such that for any $T > 0$, as $\delta \to 0$ it holds

$$
\rho_\delta \to \rho \;\; \textit{weak} \star \;\; \textit{in} \;\; L^\infty(0,T; L^\gamma(I)), \;\; \textit{and weakly in} \;\; L^{2\gamma}(Q_T), \tag{3.5}
$$

$$
\rho_\delta^\gamma \to \overline{\rho^\gamma} \;\; \textit{weakly in} \;\; L^2(Q_T), \tag{3.6}
$$

$$
u_\delta \to u \;\; \textit{weakly in} \;\; L^2(0,T; H_0^1(I)), \tag{3.7}
$$

$$
n_\delta \to n \;\; \textit{weak} \star \;\; \textit{in} \;\; L^\infty(Q_T), \tag{3.8}
$$

$$
(n_\delta)_x \to n_x \;\; \textit{weak} \star \;\; \textit{in} \;\; L^\infty(0,T; L^2), \tag{3.9}
$$

$$
((n_\delta)_t, \; (n_\delta)_{xx}) \to (n_t, \; n_{xx}) \;\; \textit{weakly in} \;\; L^2(Q_T). \tag{3.10}
$$

Since $\rho_\delta \in L^{2\gamma}(Q_T), u_\delta \in L^2(0,T; H_0^1(I)) \subset L^2(0,T; L^\infty(I))$, we have

$$
\rho_\delta u_\delta \in L^{\frac{2\gamma}{\gamma+1}}(0,T; L^{2\gamma}(I)).
$$

Therefore, $\partial_t \rho_\delta = -(\rho_\delta u_\delta)_x \in L^{\frac{2\gamma}{\gamma+1}}(0,T; H^{-1}(I))$. Since $\frac{2\gamma}{\gamma+1} > 1$, $\rho_\delta \in L^\infty(0,T; L^\gamma(I))$, and $L^\gamma \subset H^{-1}(I)$ is compact, Lemma 2.1 and Lemma 2.5 imply

$$
\rho_\delta \to \rho \;\; \textit{in} \;\; C([0,T]; L^\gamma - \omega), \tag{3.11}
$$

$$\rho_\delta \to \rho \ in\ C([0,T];H^{-1}). \tag{3.12}$$

(3.7) and (3.12) imply

$$\rho_\delta u_\delta \to \rho u \ in\ \mathscr{D}'(Q_T). \tag{3.13}$$

Hence,

$$\rho_t + (\rho u)_x = 0 \ in\ \mathscr{D}'(Q_T). \tag{3.14}$$

Moreover, $\sqrt{\rho_\delta}u_\delta \in L^\infty(0,T;L^2)$ and $\sqrt{\rho_\delta} \in L^\infty(0,T;L^{2\gamma})$ imply

$$\rho_\delta u_\delta \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}).$$

From $(3.1)_2$, we get

$$(\rho_\delta u_\delta)_t = -(\rho_\delta u_\delta^2)_x - (\rho_\delta^\gamma)_x + (u_\delta)_{xx} - (|(n_\delta)_x|^2)_x \in L^2(0,T;W^{-1,\frac{2\gamma}{\gamma+1}}).$$

By (3.13), Lemma 2.1 and Lemma 2.5, we obtain

$$\rho_\delta u_\delta \to \rho u \ in\ C([0,T];L^{\frac{2\gamma}{\gamma+1}} - \omega), \tag{3.15}$$

$$\rho_\delta u_\delta \to \rho u,\ in\ C([0,T];H^{-1})\ (also\ weak \star\ in\ L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}})). \tag{3.16}$$

From (3.7) and (3.16), we have

$$\rho_\delta u_\delta^2 \to \rho u^2 \ in\ \mathscr{D}'(Q_T)\ (also\ weakly\ in\ L^2(0,T;L^{\frac{2\gamma}{\gamma+1}})). \tag{3.17}$$

Similar to the above argument, (3.8)-(3.10) and Lemma 2.1 imply

$$n_\delta \to n \ in\ C(\overline{Q}_T), \tag{3.18}$$

$$n_\delta \to n \ in\ L^2(0,T;C^1([0,1])). \tag{3.19}$$

This, together with (3.6), (3.7), (3.9), (3.10), (3.13), and (3.17), implies

$$(\rho u)_t + (\rho u^2)_x + (\overline{\rho^\gamma})_x = u_{xx} - (|n_x|^2)_x \ in\ \mathscr{D}'(Q_T), \tag{3.20}$$

$$n_t + u n_x = n_{xx} + |n_x|^2 n \ in\ L^2(0,T;L^2). \tag{3.21}$$

It follows from $(3.1)'_1$, $(3.1)'_3$, (3.11), and (3.15) that

$$(\rho, \rho u)(x,0) = (\rho_0(x), m_0(x))\ weakly\ in\ L^\gamma(I) \times L^{\frac{2\gamma}{\gamma+1}}(I).$$

By $(3.1)'_4$ and (3.18), and $|n_\delta| = 1$, we have

$$n(x,0) = n_0(x) \ in \ [0,1] \ and \ |n| = 1 \ in \ \overline{Q}_T.$$

(1.3) follows from (3.7) and (3.19). Since $\rho_\delta > 0$ in Q_T, (3.5) implies

$$\int_{Q_T} \rho f = \lim_{\delta\to 0} \int_{Q_T} \rho_\delta f \geq 0,$$

for any nonnegative $f \in C_0^\infty(Q_T)$. Since $f \geq 0$ is arbitrary, we have

$$\rho \geq 0 \ a.e. \ in \ Q_T.$$

From (3.17), we have

$$\begin{aligned} \frac{1}{\epsilon}\int_t^{t+\epsilon}\int_I \rho u^2 &= \frac{1}{\epsilon}\lim_{\delta\to 0}\int_t^{t+\epsilon}\int_I \rho_\delta u_\delta^2 \\ &\leq \frac{1}{\epsilon}\int_t^{t+\epsilon}\overline{\lim_{\delta\to 0}}\int_I \rho_\delta u_\delta^2(s), \end{aligned}$$

for $t \in (0,T)$ and $\epsilon > 0$. Let $\epsilon \to 0^+$ and apply Lebesgue's density theorem, we get

$$\int_I \rho u^2(t) \leq \overline{\lim_{\delta\to 0}}\int_I \rho_\delta u_\delta^2(t) \ for \ a.e. \ t \in \ Q_T.$$

This, together with $(3.1)'$, (3.2), and the lower semi-continuity, implies the energy inequality (1.4). □

We need to prove $\overline{\rho^\gamma} = \rho^\gamma$. This follows from the following lemmas.

Lemma 3.3 *As $\delta \to 0$, we have*

$$\left[(u_\delta)_x - \rho_\delta^\gamma\right]\rho_\delta \to (u_x - \overline{\rho^\gamma})\,\rho \ in \ \mathscr{D}'(Q_T).$$

Proof. For any $\varphi \in C_0^\infty((0,T))$, $\phi \in C_0^\infty((0,1))$, multiplying $(3.1)_2$ by $\varphi\phi\int_0^x \rho_\delta$, integrating the resulting equation over Q_T, and using integration by parts, we have

$$\begin{aligned} &\int_{Q_T} \varphi(t)\phi(x)\left[(u_\delta)_x - \rho_\delta^\gamma\right]\rho_\delta \\ =& \int_{Q_T} \varphi'(t)\phi(x)\rho_\delta u_\delta \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi(x)\rho_\delta u_\delta \left(\int_0^x \rho_\delta\right)_t + \int_{Q_T} \varphi(t)\phi(x)\rho_\delta^2 u_\delta^2 \\ +& \int_{Q_T} \varphi(t)\phi'(x)\rho_\delta u_\delta^2 \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi'(x)\rho_\delta^\gamma \int_0^x \rho_\delta - \int_{Q_T} \varphi(t)\phi'(x)(u_\delta)_x \int_0^x \rho_\delta \\ +& \int_{Q_T} \varphi(t)\phi'(x)|(n_\delta)_x|^2 \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi(x)|(n_\delta)_x|^2 \rho_\delta \\ =& \int_{Q_T} \varphi'(t)\phi(x)\rho_\delta u_\delta \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi'(x)\rho_\delta u_\delta^2 \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi'(x)\rho_\delta^\gamma \int_0^x \rho_\delta \\ -& \int_{Q_T} \varphi(t)\phi'(x)(u_\delta)_x \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi'(x)|(n_\delta)_x|^2 \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi(x)|(n_\delta)_x|^2 \rho_\delta, \end{aligned}$$

where we have used $(3.1)_1$.

Since $\int_0^x \rho_\delta \in L^\infty(0,T;W^{1,\gamma})$, $\partial_t(\int_0^x \rho_\delta) = -\rho_\delta u_\delta \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}})$, Lemma 2.1 and (3.5) imply

$$\int_0^x \rho_\delta \to \int_0^x \rho \ in\ C(\overline{Q}_T),\ as\ \delta \to 0. \tag{3.22}$$

This, combined with (3.5)-(3.7), (3.16), (3.17), and (3.19), gives

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{Q_T} \varphi(t)\phi(x)[(u_\delta)_x - \rho_\delta^\gamma]\rho_\delta \qquad (3.23)\\
= &\int_{Q_T} \varphi'(t)\phi(x)\rho u \int_0^x \rho + \int_{Q_T} \varphi(t)\phi'(x)\rho u^2 \int_0^x \rho + \int_{Q_T} \varphi(t)\phi'(x)\overline{\rho^\gamma} \int_0^x \rho\\
- &\int_{Q_T} \varphi(t)\phi'(x)u_x \int_0^x \rho + \int_{Q_T} \varphi(t)\phi'(x)|n_x|^2 \int_0^x \rho + \int_{Q_T} \varphi(t)\phi(x)|n_x|^2\rho.
\end{aligned}$$

To complete the proof, it suffices to show that the right side of (3.23) is equal to $\int_{Q_T} \varphi(t)\phi(x)(u_x - \overline{\rho^\gamma})\rho$. The main difficulty is $\rho u \notin L^2(Q_T)$. To overcome it, take $\varphi\phi\int_0^x \langle\widehat{\rho}\rangle_\sigma$ as a test function of (3.20), where $\langle\widehat{\rho}\rangle_\sigma = \eta_\sigma * \widehat{\rho}$, we have

$$\begin{aligned}
&\int_{Q_T} \varphi(t)\phi(x)(u_x - \overline{\rho^\gamma})\langle\widehat{\rho}\rangle_\sigma\\
&= \int_{Q_T} \varphi'(t)\phi(x)\rho u \int_0^x \langle\widehat{\rho}\rangle_\sigma + \int_{Q_T} \varphi(t)\phi(x)\rho u(\int_0^x \langle\widehat{\rho}\rangle_\sigma)_t +\\
&\int_{Q_T} \varphi(t)\phi(x)\rho u^2\langle\widehat{\rho}\rangle_\sigma + \int_{Q_T} \varphi(t)\phi'(x)\rho u^2 \int_0^x \langle\widehat{\rho}\rangle_\sigma +\\
&\int_{Q_T} \varphi(t)\phi'(x)\overline{\rho^\gamma} \int_0^x \langle\widehat{\rho}\rangle_\sigma - \int_{Q_T} \varphi(t)\phi'(x)u_x \int_0^x \langle\widehat{\rho}\rangle_\sigma +\\
&\int_{Q_T} \varphi(t)\phi'(x)|n_x|^2 \int_0^x \langle\widehat{\rho}\rangle_\sigma + \int_{Q_T} \varphi(t)\phi(x)|n_x|^2\langle\widehat{\rho}\rangle_\sigma.
\end{aligned} \tag{3.24}$$

Since $\rho \in L^2(Q_T), u \in L^2(0,T;H_0^1(I))$, Lemma 2.4 implies

$$(\widehat{\rho})_t + (\widehat{\rho}\widehat{u})_x = 0 \ in\ \mathscr{D}'(\mathbb{R}\times(0,T)). \tag{3.25}$$

Denote $r_\sigma = (\langle\widehat{\rho}\rangle_\sigma\widehat{u})_x - \langle(\widehat{\rho}\widehat{u})_x\rangle_\sigma$. It follows from Lemma 2.3 that $r_\sigma \in L^1(\mathbb{R}\times(0,T))$, and

$$r_\sigma \to 0 \ in\ L^1(\mathbb{R}\times(0,T)),\ as\ \sigma \to 0. \tag{3.26}$$

Take $\eta_\sigma(x-\cdot)$ as a test function of (3.25), we have

$$(\langle\widehat{\rho}\rangle_\sigma)_t + (\langle\widehat{\rho}\rangle_\sigma\widehat{u})_x = r_\sigma \ a.e.\ in\ \mathbb{R}\times(0,T). \tag{3.27}$$

Integrating (3.27) over $(0,x)$, for $0 < x \le 1$, we have

$$(\int_0^x \langle\widehat{\rho}\rangle_\sigma)_t = -\langle\widehat{\rho}\rangle_\sigma \widehat{u} + \int_0^x r_\sigma.$$

Therefore we obtain

$$\begin{aligned}
& \int_{Q_T} \varphi(t)\phi(x)\rho u(\int_0^x \langle\widehat{\rho}\rangle_\sigma)_t + \int_{Q_T} \varphi(t)\phi(x)\rho u^2 \langle\widehat{\rho}\rangle_\sigma \\
= & -\int_{Q_T} \varphi(t)\phi(x)\rho u \langle\widehat{\rho}\rangle_\sigma \widehat{u} + \int_{Q_T} \varphi(t)\phi(x)\rho u \int_0^x r_\sigma + \int_{Q_T} \varphi(t)\phi(x)\rho u^2 \langle\widehat{\rho}\rangle_\sigma \\
= & \int_{Q_T} \varphi(t)\phi(x)\rho u \int_0^x r_\sigma,
\end{aligned}$$

where we have used $\widehat{u} = u$ in Q_T. This, together with (3.24), implies

$$\begin{aligned}
& \int_{Q_T} \varphi(t)\phi(x)(u_x - \overline{\rho^\gamma})\langle\widehat{\rho}\rangle_\sigma \\
= & \int_{Q_T} \varphi'(t)\phi(x)\rho u \int_0^x \langle\widehat{\rho}\rangle_\sigma + \int_{Q_T} \varphi(t)\phi(x)\rho u \int_0^x r_\sigma \\
+ & \int_{Q_T} \varphi(t)\phi'(x)\rho u^2 \int_0^x \langle\widehat{\rho}\rangle_\sigma + \int_{Q_T} \varphi(t)\phi'(x)\overline{\rho^\gamma} \int_0^x \langle\widehat{\rho}\rangle_\sigma \\
- & \int_{Q_T} \varphi(t)\phi'(x)u_x \int_0^x \langle\widehat{\rho}\rangle_\sigma + \int_{Q_T} \varphi(t)\phi'(x)|n_x|^2 \int_0^x \langle\widehat{\rho}\rangle_\sigma \\
+ & \int_{Q_T} \varphi(t)\phi(x)|n_x|^2 \langle\widehat{\rho}\rangle_\sigma. \qquad (3.28)
\end{aligned}$$

By the regularities of (ρ, u, n), (3.26), Lebesgue's Dominated convergence theorem, (3.28) implies, after taking $\sigma \to 0$,

$$\begin{aligned}
& \int_{Q_T} \varphi(t)\phi(x)(u_x - \overline{\rho^\gamma})\rho \qquad (3.29) \\
= & \int_{Q_T} \varphi'(t)\phi(x)\rho u \int_0^x \rho + \int_{Q_T} \varphi(t)\phi'(x)\rho u^2 \int_0^x \rho + \int_{Q_T} \varphi(t)\phi'(x)\overline{\rho^\gamma} \int_0^x \rho \\
- & \int_{Q_T} \varphi(t)\phi'(x)u_x \int_0^x \rho + \int_{Q_T} \varphi(t)\phi'(x)|n_x|^2 \int_0^x \rho + \int_{Q_T} \varphi(t)\phi(x)|n_x|^2 \rho.
\end{aligned}$$

The conclusion now follows from (3.23) and (3.29). This completes the proof. □

Lemma 3.4 *The following holds*

$$\overline{\lim_{\delta\to 0}} \int_{Q_T} \rho_\delta (u_\delta)_x \le \int_{Q_T} \rho u_x.$$

Proof. Since $\rho \in L^{2\gamma}(Q_T), u \in L^2(0,T;H_0^1)$, we replace b in (2.2) by b_j^l $(j,l \in Z_+)$, where $b_j^l \in C^1(\mathbb{R})$ is given by

$$\begin{aligned} b_j^l(z) &= (z+\frac{1}{l})\log(z+\frac{1}{l}), \text{ for } 0 \le z \le j, \\ &= (j+1+\frac{1}{l})\log(j+1+\frac{1}{l}), \text{ for } z \ge j+1. \end{aligned}$$

Since $\rho \in L^\infty(0,T;L^\gamma)$, we have $\rho < +\infty$ a.e. in Q_T. This implies

$$b_j^l(\rho) \to (\rho+\frac{1}{l})\log(\rho+\frac{1}{l}) \text{ a.e. in } Q_T, \text{ as } j \to \infty.$$

Let $j \to \infty$ in (2.2), the Lebesgue's Dominated convergence theorem implies

$$\partial_t\left[(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})\right]+\left[(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})u\right]_x+\rho u_x-\frac{1}{l}u_x\log(\rho+\frac{1}{l}) = 0 \; in \; \mathscr{D}'(Q_T). \tag{3.30}$$

Since $\rho \in L^{2\gamma}(Q_T)$, we have $(\rho+\frac{1}{l})\log(\rho+\frac{1}{l}) \in L^2(Q_T)$. Similar to (3.25)-(3.27), we extend ρ, u in (3.30) to be zero outside I, mollify (3.30), integrate the resulting equation over Q_T, and take limits, we obtain

$$\begin{aligned} \int_{Q_T}\rho u_x &= \int_I(\rho_0+\frac{1}{l})\log(\rho_0+\frac{1}{l})-\int_I(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})(T) \\ &+ \frac{1}{l}\int_{Q_T}u_x\log(\rho+\frac{1}{l}). \end{aligned} \tag{3.31}$$

Since $(3.1)_1$ is valid in the classical sense, a direct calculation gives

$$\partial_t\left[(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})\right]+\left[(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})u_\delta\right]_x+\rho_\delta(u_\delta)_x-\frac{1}{l}(u_\delta)_x\log(\rho_\delta+\frac{1}{l}) = 0. \tag{3.32}$$

Integrating (3.32) over Q_T, we have

$$\begin{aligned} \int_{Q_T}\rho_\delta(u_\delta)_x &= \int_I(\rho_{0\delta}+\frac{1}{l})\log(\rho_{0\delta}+\frac{1}{l})-\int_I(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})(T) \\ &+ \frac{1}{l}\int_{Q_T}(u_\delta)_x\log(\rho_\delta+\frac{1}{l}) \\ &\le \int_I(\rho_{0\delta}+\frac{1}{l})\log(\rho_{0\delta}+\frac{1}{l})-\int_I(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})(T) \\ &+ \frac{1}{l}\|(u_\delta)_x\|_{L^2(Q_T)}\|\rho_\delta+1\|_{L^2(Q_T)} \\ &\le \int_I(\rho_{0\delta}+\frac{1}{l})\log(\rho_{0\delta}+\frac{1}{l})-\int_I(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})(T) \\ &\quad +\frac{1}{l}c(E_0,T), \end{aligned} \tag{3.33}$$

where we have used Hölder inequality, Lemma 3.1, and Lemma 3.2.

Since $\rho_\delta \in L^\infty(0,T;L^\gamma)$, we have

$$(\rho_\delta + \frac{1}{l})\log(\rho_\delta + \frac{1}{l}) \in L^\infty(0,T;L^\tau), \tag{3.34}$$

for some $\tau > 1$. From (3.32), we get

$$\partial_t[(\rho_\delta + \frac{1}{l})\log(\rho_\delta + \frac{1}{l})] \in L^{\frac{2\gamma}{\gamma+1}}(0,T;W^{-1,\frac{2\gamma}{\gamma+1}}). \tag{3.35}$$

(3.34), (3.35), and Lemma 2.5 give

$$(\rho_\delta + \frac{1}{l})\log(\rho_\delta + \frac{1}{l}) \to \overline{(\rho + \frac{1}{l})\log(\rho + \frac{1}{l})} \ in\ C([0,T];L^\tau - \omega), \text{ as } \delta \to 0.$$

This implies

$$\lim_{\delta\to 0}\int_I (\rho_\delta + \frac{1}{l})\log(\rho_\delta + \frac{1}{l})(T) = \int_I \overline{(\rho + \frac{1}{l})\log(\rho + \frac{1}{l})}(T).$$

Since the function $(z+\frac{1}{l})\log(z+\frac{1}{l})$ is convex for $z \ge 0$, Lemma 2.6 implies

$$(\rho + \frac{1}{l})\log(\rho + \frac{1}{l}) \le \overline{(\rho + \frac{1}{l})\log(\rho + \frac{1}{l})} \ a.e.\ in\ Q_T.$$

Therefore,

$$\lim_{\delta\to 0}\int_I (\rho_\delta + \frac{1}{l})\log(\rho_\delta + \frac{1}{l})(T) \ge \int_I (\rho + \frac{1}{l})\log(\rho + \frac{1}{l})(T). \tag{3.36}$$

Take $\overline{\lim\limits_{\delta\to 0}}$ in (3.33), and use (3.36), we get

$$\begin{aligned}
&\overline{\lim_{\delta\to 0}}\int_{Q_T} \rho_\delta (u_\delta)_x \\
\le\ & \int_I (\rho_0 + \frac{1}{l})\log(\rho_0 + \frac{1}{l}) - \underline{\lim_{\delta\to 0}}\int_I (\rho_\delta + \frac{1}{l})\log(\rho_\delta + \frac{1}{l})(T) + \frac{1}{l}c(E_0,T) \\
\le\ & \int_I (\rho_0 + \frac{1}{l})\log(\rho_0 + \frac{1}{l}) - \int_I (\rho + \frac{1}{l})\log(\rho + \frac{1}{l})(T) + \frac{1}{l}c(E_0,T) \\
=\ & \int_{Q_T} \rho u_x - \frac{1}{l}\int_{Q_T} u_x \log(\rho + \frac{1}{l}) + \frac{1}{l}c(E_0,T) \\
\le\ & \int_{Q_T} \rho u_x + \frac{1}{l}\|u_x\|_{L^2(Q_T)}\|\rho + 1\|_{L^2(Q_T)} + \frac{1}{l}c(E_0,T).
\end{aligned}$$

Since $u_x \in L^2(Q_T)$, and $\rho \in L^2(Q_T)$, sending $l \to \infty$ yields

$$\overline{\lim_{\delta\to 0}}\int_{Q_T} \rho_\delta (u_\delta)_x \le \int_{Q_T} \rho u_x.$$

The proof of the Lemma is complete. $\square$

Now we return to the proof of $\rho^\gamma = \overline{\rho^\gamma}$. Assume $\varphi_m \in C_0^\infty(0,T)$, $\phi_m \in C_0^\infty(0,1)$, $0 \le \varphi_m, \phi_m \le 1$, and $\varphi_m, \phi_m \to 1$ as $m \to \infty$. For any $\psi \in C_0^\infty(Q_T)$, denote $v = \rho - \epsilon\psi$ for $\epsilon > 0$, then

$$
\begin{aligned}
& \int_{Q_T} (\overline{\rho^\gamma} - v^\gamma)(\rho - v) \\
= & \int_{Q_T} \varphi_m\phi_m(\overline{\rho^\gamma} - v^\gamma)(\rho - v) + \int_{Q_T} (1-\varphi_m\phi_m)(\overline{\rho^\gamma} - v^\gamma)(\rho - v) \\
= & \int_{Q_T} \varphi_m\phi_m(\overline{\rho^\gamma}\rho - \overline{\rho^\gamma}v - v^\gamma\rho + v^{\gamma+1}) + \int_{Q_T} (1-\varphi_m\phi_m)(\overline{\rho^\gamma} - v^\gamma)(\rho - v) \\
= & \int_{Q_T} \varphi_m\phi_m(\overline{\rho^\gamma} - u_x)\rho + \int_{Q_T} (\varphi_m\phi_m - 1)\rho u_x + \int_{Q_T} \rho u_x \\
+ & \int_{Q_T} \varphi_m\phi_m(-\overline{\rho^\gamma}v - v^\gamma\rho + v^{\gamma+1}) + \int_{Q_T} (1-\varphi_m\phi_m)(\overline{\rho^\gamma} - v^\gamma)(\rho - v).
\end{aligned}
$$

Denote $A_m = \int_{Q_T}(\varphi_m\phi_m - 1)\rho u_x + \int_{Q_T}(1-\varphi_m\phi_m)(\overline{\rho^\gamma} - v^\gamma)(\rho - v)$. Together with Lemma 3.3 and 3.4, (3.5), and (3.6), we have

$$
\begin{aligned}
& \int_{Q_T} (\overline{\rho^\gamma} - v^\gamma)(\rho - v) \\
\ge & \overline{\lim_{\delta\to 0}} \int_{Q_T} \varphi_m\phi_m[\rho_\delta^\gamma - (u_\delta)_x]\rho_\delta + \overline{\lim_{\delta\to 0}} \int_{Q_T} \rho_\delta (u_\delta)_x \\
+ & \overline{\lim_{\delta\to 0}} \int_{Q_T} \varphi_m\phi_m(-\rho_\delta^\gamma v - v^\gamma\rho_\delta + v^{\gamma+1}) + A_m \\
\ge & \overline{\lim_{\delta\to 0}} [\int_{Q_T} \varphi_m\phi_m[\rho_\delta^\gamma - (u_\delta)_x]\rho_\delta + \int_{Q_T} \varphi_m\phi_m\rho_\delta(u_\delta)_x \\
+ & \int_{Q_T} \varphi_m\phi_m(-\rho_\delta^\gamma v - v^\gamma\rho_\delta + v^{\gamma+1})] - \overline{\lim_{\delta\to 0}} \int_{Q_T} \rho_\delta|1-\varphi_m\phi_m||(u_\delta)_x| + A_m \\
= & \overline{\lim_{\delta\to 0}} \int_{Q_T} \varphi_m\phi_m(\rho_\delta^\gamma - v^\gamma)(\rho_\delta - v) - \overline{\lim_{\delta\to 0}} \int_{Q_T} \rho_\delta|1-\varphi_m\phi_m||(u_\delta)_x| + A_m.
\end{aligned}
$$

By the monotonicity of z^γ, we have

$$\int_{Q_T} \varphi_m\phi_m(\rho_\delta^\gamma - v^\gamma)(\rho_\delta - v) \ge 0.$$

Therefore,

$$
\begin{aligned}
\int_{Q_T} (\overline{\rho^\gamma} - v^\gamma)(\rho - v) \ \ge\ & -\overline{\lim_{\delta\to 0}} \int_{Q_T} \rho_\delta|1-\varphi_m\phi_m||u_{\delta x}| + A_m \\
\ge\ & -\overline{\lim_{\delta\to 0}} \|1-\varphi_m\phi_m\|_{L^{\frac{2\gamma}{\gamma-1}}(Q_T)} \|\rho_\delta\|_{L^{2\gamma}(Q_T)} \|u_{\delta x}\|_{L^2(Q_T)} + A_m \\
\ge\ & -c(E_0,T)\|1-\varphi_m\phi_m\|_{L^{\frac{2\gamma}{\gamma-1}}(Q_T)} + A_m, \qquad (3.37)
\end{aligned}
$$

where we have used Hölder inequality, Lemma 3.1, and Lemma 3.2. By the Lebesgue's Dominated Convergence Theorem, we have

$$\|1-\varphi_m\phi_m\|_{L^{\frac{2\gamma}{\gamma-1}}(Q_T)} \to 0,\ A_m \to 0 \text{ as } m \to \infty.$$

Let $m \to \infty$ in (3.37), we get

$$\int_{Q_T} (\overline{\rho^\gamma} - v^\gamma)(\rho - v) \geq 0.$$

Since $v = \rho - \epsilon\psi$, and $\epsilon > 0$, we have

$$\int_{Q_T} [\overline{\rho^\gamma} - (\rho - \epsilon\psi)^\gamma]\psi \geq 0. \tag{3.38}$$

Sending $\epsilon \downarrow 0$ yields

$$\int_{Q_T} (\overline{\rho^\gamma} - \rho^\gamma)\psi \geq 0.$$

This clearly implies

$$\overline{\rho^\gamma} = \rho^\gamma \ a.e.\ in\ Q_T.$$

The proof of Theorem 1.1 is complete. □

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