Conformal transformations of leading-twist operators in QCD

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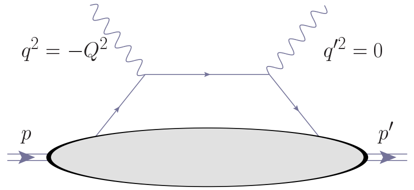
Hard exclusive scattering in Bjorken limit

- **E.g., Deeply Virtual Compton Scattering:** $\gamma^* N \to \gamma N$

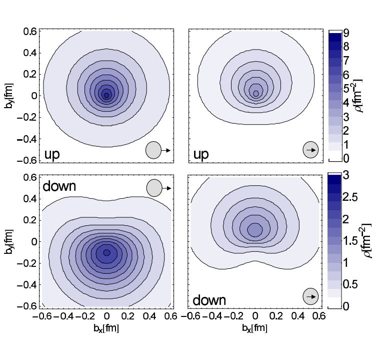


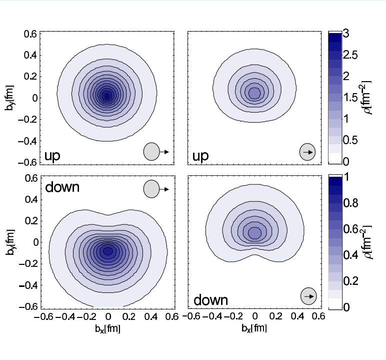
- Arguing for EIC as "exclusive" machine, need "gold plated" processes where QCD description can and will match the accuracy achieved in DIS and jet physics.
  DVCS can play this role.

- **Framework: collinear factorization, non-forward matrix elements**

Nucleon Tomography

access to three-dimensional picture of the nucleon (M. Burkardt)





$\hookrightarrow$ **first two moments of transverse spin parton density**

computer simulations: M. Göckeler *et al.*, Phys.Rev.Lett. 98 (2007) 222001

Multiplicatively renormalizable leading-twist operators

- **One loop:**
  anomalous dimensions $+$ conformal symmetry $\to$ full anomalous dimension matrix

$$\mathcal{O}_N \sim (\partial_{z_1}+\partial_{z_2})^N C_N^{3/2}\left(\frac{\partial_{z_1}-\partial_{z_2}}{\partial_{z_1}+\partial_{z_2}}\right)\bar{q}(z_1 n)\gamma_+ q(z_2 n)\Big|_{z_{1,2}\to 0}$$

Makeenko, 1980

- **D. Müller, '94-'00**

  — off-diagonal elements of the n-loop mixing matrix are determined by the $(n-1)$-loop conformal anomaly

- **Two loops:**

  Belitsky, Mueller, 2000

- **Three loops:**

  Braun, Manashov, Moch, Strohmaier, work in progress

Different approach:

Instead of studying consequences of conformal symmetry breaking in QCD we make use of *exact* conformal symmetry of a modified theory:

Large N_f QCD in $4-\epsilon$ dimension at critical coupling

QCD in $d = 4 - 2\epsilon$

Consider renormalized QCD in $d = 4 - 2\epsilon$ dimensions. In MS-like schemes

$$\beta^{QCD}(a_s) = 2a_s\left[-\epsilon - \beta_0 a_s + \ldots\right] \qquad Z = 1 + \sum_{j=1}^{\infty} \epsilon^{-j} \sum_{k=j}^{\infty} a_s^k \mathbb{Z}_{jk}$$

- scale and conformal invariance at the critical point — Banks, Zaks, '82

$$a_s^* = -4\pi\epsilon/\beta_0 + \ldots \qquad \beta^{QCD}(a_s^*) = 0$$

- Z_{jk} do not depend on ϵ by construction, thus

$$\boxed{\mathbb{H}(a_s^*) = a_s^*\,\mathbb{H}^{(1)} + (a_s^*)^2\,\mathbb{H}^{(2)} + \ldots} \qquad \Big(\mu\partial_\mu \; + \; \mathbb{H}(a_s^*)\Big)[\mathcal{O}](z_1, z_2) = 0$$

$\Longleftrightarrow$

$$\boxed{\mathbb{H}(a_s) = a_s\,\mathbb{H}^{(1)} + a_s^2\,\mathbb{H}^{(2)} + \ldots} \qquad \Big(\mu\partial_\mu + \beta(g)\partial_g + \mathbb{H}(a_s)\Big)[\mathcal{O}(z_1, z_2)] = 0$$

“Hidden” conformal invariance of QCD RG equations in MS-like schemes

$$\Big(\mu\partial_\mu + \beta(g)\partial_g + \mathbb{H}(a_s)\Big)[\mathcal{O}(z_1,z_2)] = 0$$

- Conformal symmetry implies existence of three generators that satisfy usual SL(2) relations and commute with the renormalization kernel

$$[S_k, \mathbb{H}] = 0$$

$$[S_+, S_-] = 2S_0 \qquad [S_0, S_+] = S_+ \qquad [S_0, S_-] = -S_-$$

- True to all orders in perturbation theory (in MS-like schemes)
- Complete RG kernel in $d = 4$, a digression to $d = 4 - \epsilon$ is an intermediate step

In a free theory, in coordinate representation

generators $j=1$ for quarks

$$\boxed{\begin{aligned} S_+ &= z^2\partial_z + 2jz \\ S_0 &= z\partial_z + j \\ S_- &= -\partial_z \end{aligned}}$$

$SL(2)$ algebra

$$\boxed{\begin{aligned} [S_+, S_-] &= 2S_0 \\ [S_0, S_+] &= S_+ \\ [S_0, S_-] &= -S_- \end{aligned}}$$

- In the interacting theory the generators are modified by quantum corrections

$$\boxed{\begin{aligned} S_+ &= S_+^{(0)} + a_s^* S_+^{(1)} + (a_s^*)^2 S_+^{(2)} + \ldots \\ S_0 &= S_0^{(0)} + a_s^* S_0^{(1)} + (a_s^*)^2 S_0^{(2)} + \ldots \\ S_- &= S_-^{(0)} \end{aligned}}$$

- General structure

$$
\begin{aligned}
S_- &= S_-^{(0)}\,,\\
S_0 &= S_0^{(0)} - \epsilon + \frac{1}{2}\mathbb{H}(a_s^*)\,,\\
S_+ &= S_+^{(0)} + (z_1+z_2)\Big(-\epsilon + \frac{1}{2}\mathbb{H}\Big) + (z_1-z_2)\Delta_+\,,
\end{aligned}
\qquad \boxed{a_s = \frac{\alpha_s^*}{4\pi}}
$$

where

$$
\begin{aligned}
\mathbb{H}(a_s^*) &= a_s^*\,\mathbb{H}^{(1)} + (a_s^*)^2\,\mathbb{H}^{(2)} + (a_s^*)^3\,\mathbb{H}^{(3)} + \ldots\\
\Delta_+(a_s^*) &= a_s^*\,\Delta_+^{(1)} + (a_s^*)^2\,\Delta_2^{(2)} + (a_s^*)^3\,\Delta_3^{(3)} + \ldots
\end{aligned}
$$

- Modification of S_0 can be written in terms of the evolution kernel

...but $\Delta_+(a_s^*)$ requires explicit calculation

- Light-ray operator representation *Balitsky, Braun '89*

$$[\mathcal{O}](z_1,z_2) \quad\equiv\quad [\bar q(z_1 n)\not{n} q(z_2 n)] \;\equiv\; \sum_{m,k}\frac{z_1^m z_2^k}{m!k!}[(D_n^m \bar q)(0)\not{n}(D_n^k q)(0)]$$

$$\mathbb{H}[\mathcal{O}](z_1,z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, h(\alpha,\beta)\,[\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

$$\boxed{\begin{array}{l} z_{12}^\alpha \equiv z_1\bar\alpha + z_2\alpha \\ \bar\alpha = 1-\alpha \end{array}}$$

one-loop result Braun, Manashov, PLB **734**, 137 (2014)

$$\Delta_+^{(1)}[\mathcal{O}](z_1,z_2) = -2C_F\int_0^1 d\alpha\left(\frac{\bar\alpha}{\alpha}+\ln\alpha\right)\Big[[\mathcal{O}](z_{12}^\alpha,z_2) - [\mathcal{O}](z_1,z_{21}^\alpha)\Big]$$

Evolution equations from operator algebra

- **Expanding the commutation relations in powers of a_s^***

$$[S_+^{(0)}, \mathbb{H}^{(1)}] = 0\,,$$
$$[S_+^{(0)}, \mathbb{H}^{(2)}] = [\mathbb{H}^{(1)}, S_+^{(1)}]\,,$$
$$[S_+^{(0)}, \mathbb{H}^{(3)}] = [\mathbb{H}^{(1)}, S_+^{(2)}] + [\mathbb{H}^{(2)}, S_+^{(1)}]\,, \qquad \text{etc.}$$

- A nested set of inhomogenious first order differential equations for $\mathbb{H}^{(k)}$
  Their solution determines $\mathbb{H}^{(k)}$ up to an $SL(2)$-invariant term
- The r.h.s. involves $\mathbb{H}^{(k)}$ and $S_+^{(m)}$ at one oder less compared to the l.h.s. D.Müller

One loop

$$\mathbb{H}[\mathcal{O}](z_1,z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, h(\alpha,\beta)\, [\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

$$z_{12}^\alpha \equiv z_1\bar\alpha + z_2\alpha$$
$$\bar\alpha = 1-\alpha$$

$$\boxed{[S_+^{(0)}, \mathbb{H}^{(1)}] = 0} \qquad \Longrightarrow \qquad h^{(1)}(\alpha,\beta) = h^{(1)}\left(\frac{\alpha\beta}{\bar\alpha\bar\beta}\right) = h^{(1)}(\tau)$$

i.e. function of two variables reduces to a function of one variable
$\longrightarrow$ can be restored from anomalous dimensions

$$\gamma_N = \int_0^1 d\alpha \int_0^1 d\beta\, (1-\alpha-\beta)^{N-1} h(\alpha,\beta)$$

Balitsky, Braun, 1989

$$\boxed{h^{(1)}(\alpha,\beta) = -4C_F\left[\delta_+(\tau) + \theta(1-\tau) - \frac{1}{2}\delta(\alpha)\delta(\beta)\right]}$$

- **Combined DGLAP, ERBL and GPD evolution equations in a compact form**

Two loops (I)

Braun, Manashov, PLB734 (2014) 137

have to solve

$$[S_+^{(0)},\mathbb{H}^{(2)}] = [\mathbb{H}^{(1)},S_+^{(1)}] = [\mathbb{H}^{(1)},z_1+z_2]\left(\beta_0+\frac{1}{2}\mathbb{H}^{(1)}\right)+[\mathbb{H}^{(1)},\Delta_+^{(1)}]$$

$$\boxed{\mathbb{H}^{(2)} = \mathbb{T}^{(1)}\left(\beta_0+\frac{1}{2}\mathbb{H}^{(1)}\right)+[\mathbb{H}^{(1)},\mathbb{X}^{(1)}]+\mathbb{H}^{(2)}_{inv}}$$

$$\mathbb{X}^{(1)}f(z_1,z_2) = 2C_F\int_0^1\frac{d\alpha}{\alpha}\,\ln\alpha\Big[2f(z_1,z_2)-f(z_1,z_{21}^\alpha)-f(z_{12}^\alpha,z_2)\Big]$$

$$\mathbb{T}^{(1)}f(z_1,z_2) = 4C_F\bigg\{\int_0^1\frac{d\alpha}{\alpha}\,\bar\alpha\ln\bar\alpha\Big[2f(z_1,z_2)-f(z_1,z_{21}^\alpha)-f(z_{12}^\alpha,z_2)\Big]$$
$$-\int_0^1 d\alpha\int_0^{\bar\alpha}d\beta\,\ln(1-\alpha-\beta)f(z_{12}^\alpha,z_{21}^\beta)\bigg\}$$

Two loops (II)

Braun, Manashov, PLB **734** (2014) 137

invariant part

$$\mathbb{H}^{(2)}_{inv}(N) = \mathbb{H}^{(2)}(N) - T_1(N)\Big(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}(N)\Big)$$

$$\begin{aligned}\mathbb{H}^{(2)}_{inv} = 4C_F\Big\{&\beta_0\left(\frac{13}{12} + \frac{5}{3}\mathcal{H}_{\langle+\rangle} - \frac{11}{3}\mathcal{H}_{\langle 1\rangle}\right) + 2C_A\left(\frac{19}{48} - \frac{1}{3}\mathcal{H}_{\langle+\rangle} - \frac{2}{3}\mathcal{H}_{\langle 1\rangle} - \frac{1}{4}\mathcal{H}^2_{\langle 1\rangle}\right)\\ &+ \frac{2}{N_c}\Big[\left(3\zeta(3) - \frac{\pi^2}{3} + \frac{11}{16}\right) - \frac{\pi^2-6}{6}\left(\mathcal{H}_{\langle+\rangle} - \mathcal{H}_{\langle 1\rangle}\right) + \mathcal{H}_{\langle\frac{1}{\tau}\ln\bar\tau\rangle} - \frac{3}{4}\mathcal{H}^2_{\langle 1\rangle} - \mathcal{H}^3_{\langle 1\rangle}\\ &- \frac{1}{2}\mathbb{P}_{12}\Big(\mathcal{H}_{\langle\ln^2\bar\tau\rangle} - 2\mathcal{H}_{\langle\tau\ln\bar\tau\rangle}\Big)\Big]\Big\}.\end{aligned}$$

compare LO

$$\mathbb{H}^{(1)} = 4C_F\Big\{\mathcal{H}_{\langle+\rangle} - \mathcal{H}_{\langle 1\rangle} + \frac{1}{2}\Big\}$$

Conformal Ward Identity (I)

- Conformal symmetry at the critical coupling implies

$$\Big(S_+^{(z)} - \frac{1}{2}x^2(\bar n\partial_x)\Big)\big\langle[\mathcal{O}_n](z_1,z_2),[\mathcal{O}_{\bar n}](x,w_1,w_2)\big\rangle = 0 \qquad \boxed{(nx) = (\bar n x) = 0}$$

- To find explicit expression for S_+, consider Ward identity

$$\big\langle\delta_+[\mathcal{O}^{(n)}](z)\,[\mathcal{O}^{(\bar n)}](x,w)\big\rangle + \big\langle[\mathcal{O}^{(n)}](z)\,\delta_+[\mathcal{O}^{(\bar n)}](x,w)\big\rangle - \big\langle\delta_+ S_R\,[\mathcal{O}^{(n)}](z)\,[\mathcal{O}^{(\bar n)}](x,w)\big\rangle = 0$$

Then

$$\begin{aligned}\delta_+[\mathcal{O}^{(\bar n)}](x;w_1,w_2) &= -x^2(\bar n\partial_x)[\mathcal{O}^{(\bar n)}](x;w_1,w_2)\\ \delta_+[\mathcal{O}^{(n)}](0;z_1,z_2) &= 2(n\bar n)\Big(S_+^{(0)} - \epsilon(z_1+z_2) - \frac{a_s}{2}[\mathbb{H}^{(1)}, z_1+z_2)]\Big)[\mathcal{O}^{(n)}](0;z_1,z_2)\end{aligned}$$

and for the last term

$$\delta_+ S^{QCD} = 4\epsilon\int d^dx(x\bar n)\mathcal{L}^{QCD} + 2(d-2)\bar n^\mu\int d^dx\,\delta_{BRST}(\bar c^a A^a_\mu).$$

Conformal Ward Identity (II)

- $\epsilon\mathcal{L}^{QCD}$ can be reexpanded in terms of renormalized operators

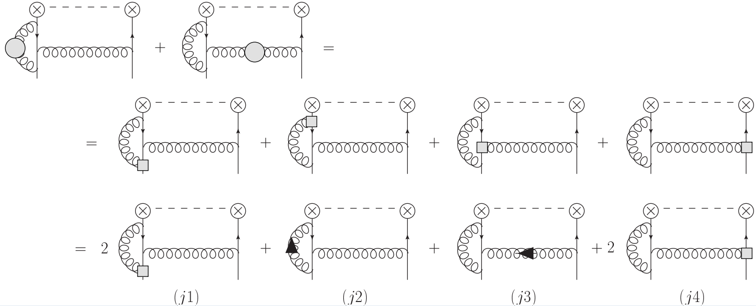
$$\epsilon\mathcal{L}^{QCD} = -\frac{\beta(a_s)}{a_s}[\mathcal{L}^{\text{YM+gf}}] + \text{ EOM} + \text{ BRST}\,, \qquad \beta(a_s^*) = 0$$

leading to

$$\boxed{\Delta_+^{(z)}\Big\langle[\mathcal{O}_n](0,\underline{z}),[\mathcal{O}_{\bar{n}}](x,\underline{w})\Big\rangle = \text{KR'}\left\{\left\langle\int d^dy\,(\bar{n}y)\mathcal{L}^{\text{YM+gf}}(y)[\mathcal{O}_n](0,\underline{z}),[\mathcal{O}_{\bar{n}}](x,\underline{w})\right\rangle\right\}}$$

where one has to pick simple residues in ϵ

Diagrammatics of special vertex insertions (I)

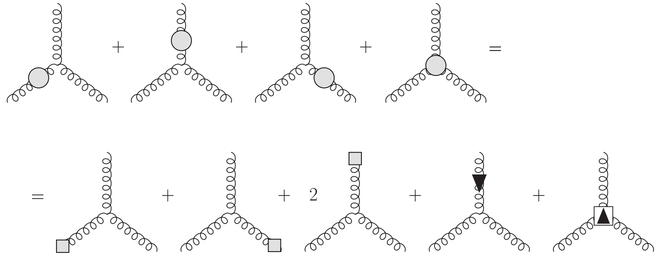


The first and the last terms, (j1) and (j4), combine to

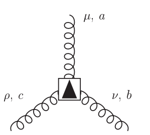
$$\left(\delta S_{+}^{(2)}\right)_{j1+j4} = \frac{1}{2}\mathbb{H}^{(2)}_{(j)}(z_1+z_2)$$

Diagrammatics of special vertex insertions (II)

Rearrangement of three-gluon vertex insertions combined with effective propagators



Special vertex



$$gf^{abc}\left(g^{\mu\rho}\bar{n}^{\nu} - g^{\mu\nu}\bar{n}^{\rho}\right)$$

Two-loop conformal anomaly

- two-loop result Braun, Manashov, Moch, Strohmaier: JHEP **1603** (2016) 142

$$
\begin{aligned}
[\Delta_+^{(2)}\mathcal{O}](z_1,z_2) &= \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \Big[\omega(\alpha,\beta) + \omega^{\mathbb{P}}(\alpha,\beta)\mathbb{P}_{12}\Big]\Big[\mathcal{O}(z_{12}^\alpha, z_{21}^\beta) - \mathcal{O}(z_{12}^\beta, z_{21}^\alpha)\Big] \\
&\quad + \int_0^1 du \int_0^1 dt\, \varkappa(t)\, \Big[\mathcal{O}(z_{12}^{ut}, z_2) - \mathcal{O}(z_1, z_{21}^{ut})\Big].
\end{aligned}
$$

$$
\boxed{\omega^{\mathbb{P}} = 2\Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\Big[\Big(\bar\alpha - \frac{1}{\bar\alpha}\Big)\Big[\mathrm{Li}_2\Big(\frac{\alpha}{\bar\beta}\Big) - \mathrm{Li}_2(\alpha) - \ln\bar\alpha\ln\bar\beta\Big] + \alpha\bar\tau\ln\bar\tau + \frac{\beta^2}{\bar\beta}\ln\bar\alpha\Big]}
$$

Two-loop conformal anomaly (II)

- two-loop result — Braun, Manashov, Moch, Strohmaier: JHEP **1603** (2016) 142

$$\omega(\alpha,\beta) = C_F^2\,\omega_{FF}(\alpha,\beta) + C_F C_A\,\omega_{FA}(\alpha,\beta)$$

$$\begin{aligned}
\omega_{FF} = 4\Big\{&\Big(\alpha-\frac{1}{\alpha}\Big)\Big[\mathrm{Li}_2\Big(\frac{\beta}{\bar\alpha}\Big)-\mathrm{Li}_2(\beta)-\mathrm{Li}_2(\alpha)-\frac{1}{4}\ln^2\bar\alpha\Big]-\alpha\Big[\mathrm{Li}_2(\alpha)-\mathrm{Li}_2(1)\Big]\\
&-\frac{\alpha+\beta}{2}\ln\alpha\ln\bar\alpha+\frac{1}{4}\Big[\beta\ln^2\bar\alpha-\alpha\ln^2\alpha\Big]-\frac{\alpha}{\tau}\Big[\tau\ln\tau+\bar\tau\ln\bar\tau\Big]\\
&+\frac{1}{4}\Big[\beta-2\bar\alpha+\frac{2\beta}{\alpha}\Big]\ln\bar\alpha+\frac{1}{2}\Big[\bar\alpha-\frac{\alpha}{\bar\alpha}-3\beta\Big]\ln\alpha-\frac{15}{4}\alpha\Big\},
\end{aligned}$$

$$\begin{aligned}
\omega_{FA} = 2\Big\{&\Big(\frac{1}{\alpha}-\alpha\Big)\Big[\mathrm{Li}_2\Big(\frac{\beta}{\bar\alpha}\Big)-\mathrm{Li}_2(\beta)-2\,\mathrm{Li}_2(\alpha)-\ln\alpha\ln\bar\alpha\Big]\\
&+\frac{\alpha}{\tau}\Big[\tau\ln\tau+\bar\tau\ln\bar\tau\Big]-\bar\beta\ln\alpha-\frac{\bar\alpha}{\alpha}\ln\bar\alpha\Big\},
\end{aligned}$$

$$\boxed{\tau=\frac{\alpha\beta}{\bar\alpha\bar\beta}}$$

- result for $\varkappa(t)$ of similar complexity

Three loop evolution equations

Braun, Manashov, Moch, Strohmaier: work in progress

have to solve

$$\boxed{[S_+^{(0)}, \mathbb{H}^{(3)}] = [\mathbb{H}^{(1)}, S_+^{(2)}] + [\mathbb{H}^{(2)}, S_+^{(1)}]}$$

$\Rightarrow$ three-loop evolution equation for light-ray operators

- **Alternative: find a transformation**

$$\mathcal{O}'(z_1,z_2) \to U\,\mathcal{O}(z_1,z_2)\,U^{-1} \qquad\qquad [S_+\mathcal{O}](z_1,z_2) \to S_+^{(0)}\mathcal{O}'(z_1,z_2)$$

$\Rightarrow$ transformation to the conformal scheme

Summary and outlook

- **QCD evolution equations in $\overline{MS}$ schemes have "hidden" conformal symmetry**

$$[S_k, \mathbb{H}] = 0 \qquad [S_+, S_-] = 2S_0 \qquad [S_0, S_+] = S_+ \qquad [S_0, S_-] = -S_-$$

- **The generators are deformed by quantum corrections; known to two-loop accuracy**
- **QCD at $d = 4 - \epsilon$ at critical coupling can be used to define**

$$\mathcal{Q} = \mathcal{Q}^{\text{conformal}} + \frac{\beta(g)}{g}\Delta\mathcal{Q}$$

- **Possible applications:**
  - — **Three-loop evolution equations in non-forward kinematics**
  - — **Generalized Crewther relation**
  - — **Large-N_f QCD in conformal window**
  - — $N = 4$ **SYM**