ANALYSIS OF GENERALIZED FORCHHEIMER FLOWS OF COMPRESSIBLE FLUIDS IN POROUS MEDIA

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ANALYSIS OF GENERALIZED FORCHHEIMER FLOWS OF COMPRESSIBLE FLUIDS IN POROUS MEDIA

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Abstract. This work is focused on the analysis of non-linear flows of slightly compressible fluids in porous media not adequately described by Darcy's law. We study a class of generalized nonlinear momentum equations which covers all three well-known Forchheimer equations, the so-called two-term, power, and three-term laws. The non-linear Forchheimer equation is inverted to a non-linear Darcy equation with implicit permeability tensor depending on the pressure gradient. This results in a degenerate parabolic equation for the pressure. Two classes of boundary conditions are considered, given pressure and given total flux. In both cases they are allowed to be unbounded in time. The uniqueness, Lyapunov and asymptotic stabilities, and other long-time dynamical features of the corresponding initial boundary value problems are analyzed. The results obtained in this paper have clear hydrodynamic interpretations and can be used for quantitative evaluation of engineering parameters. Some numerical simulations are also included.

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1. Introduction

The mathematical modeling and analysis of non-linear flows in porous media is quickly becoming a key to solving many challenging problems in engineering and applied sciences. Most of the studies in porous media are based on Darcy's law, which describes a linear relationship between the pressure gradient and the fluid velocity. "*Darcy's equation has become the model of choice for the study of the flow of fluids through porous solids due to pressure gradients, so much that it has now been elevated to the status of a law in physics*"[35]. However, almost immediately after Darcy's discovery, his student Dupuit observed on the field data that this linear relation is no longer valid for flows with high values of velocity. Later, Forchheimer in his book [19] reported a number of experimental data underlining these discrepancies, and constituted three different empirical formulae to interpret these results. Nowadays researchers and engineers recognize that non-Darcy effects are very important in many applications [41, 8, 24, 31, 32].

By analogy to pipe flows, it was originally assumed that "convective" forces are responsible for non-linear deviations from laminar flow associated with Darcy's equation. Later in the 1950s and 1960s (c.f. [10, 22] and references therein) it was observed that the Darcy law is valid as long as the Reynolds number (Re) does not exceed some characteristic value between 1 and 10. Unlike pipe flows, where the deviation from linearity is associated with turbulence at high Re numbers, in porous media it occurs at low Re numbers. Yet, the actual nature of this phenomenon is not fully understood. In recent experiments [38, 1] it was observed that in samples of the porous media containing fractures Darcy's law does not hold even for $Re \approx 1$. The latest research suggests that even for low velocity flows, Darcy's law needs to be revised (see e.g. [9, 22]).

Almost all off-the-shelf industrial simulators of the process of filtration in porous media utilize the linear Darcy's approximation of the momentum equation [40]. In order to capture physical phenomena lost in the linear approximation, researchers have been recently directed to the mathematical and numerical modeling of Forchheimer flows, (c.f. [8, 17, 15, 26] and references therein). In those papers, the continuity and the Forchheimer-Darcy's momentum equations are treated separately as a coupled system of first order PDE. The Forchheimer equation can also be considered as the limiting case of Brinkman-Forchheimer equations. There are a large number of research on Brinkman-Forchheimer equations and Forchheimer equation in this connection with the former one for *incompressible fluids* [27, 28, 29, 30, 18, 39, 12].

A different approach to study analytically the long-time dynamics of the flow was initiated in our previous works [3, 4, 7, 5, 6] for *compressible fluids*. Namely, to constitute a non-linear momentum equation with permeability tensor dependent on the pressure gradient. This leads to the reduction of the original system to one PDE for the pressure function only (see also [15]). Therefore ones can explore the equivalent problem within the framework of degenerate elliptic and parabolic PDE [20, 14].

In those papers, we mainly focus on the two-term Forchheimer law and the equilibrium states called *pseudo-steady states* (PSS). The PSS are defined by solutions of auxiliary boundary value problems and are proved to be stable in the class of solutions of IBVP with constant total flux on the boundary. Also, the pressure gradient is assumed to be uniformly bounded for all time. The study there then is used to analyze the productivity index/diffusive capacity in different industrial problems. Those assumptions on the pressure are quite severe from theoretical and practical points of view and leave much to be desired:

(a) Latest theoretical research (see [9]) indicates that even for low Reynolds numbers the pressure gradient can be a cubic function of the velocity. On the other hand experimental and field data suggest different functional relations between gradient of the pressure and velocity. Therefore there is a need to introduce a generic Forchheimer law, which covers all polynomial dependence of the gradient of the pressure on the velocity.

(b) The above assumption on the boundedness of the pressure gradient excludes sharp non-homogeneity in porous media, which often leads to deviation from Darcy's law (see [38, 1]).

(c) In practice, the production rate may vary in time and/or the pressure distribution on the well can become relatively large as time evolves (see [11]).

In the current paper, we investigate a class of general g-Forchheimer equations which cover all three classical Forchheimer laws, without any *a priori* assumption on the hydrodynamic parameters (such as boundness of the pressure gradient). We prove that the g-Forchheimer equation is equivalent to non-linear Darcy equation with permeability tensor $K(\cdot)$ depending on the pressure gradient. It is then shown that the corresponding non-linear field $K(y)y$ acquires important *monotonicity* properties. Moreover we introduce a class of g-Forchheimer equations consisting of generalized polynomials with positive coefficients (GPPC). For such equations, we show that for large $|y|$ the non-linear permeability $K(\cdot)$ satisfies the following asymptotic relation: $K(y)y \cdot y \approx |y|^{(2-a)}$, where $a \in [0, 1)$ depends on the degree of g-Forchheimer polynomial. Using these features, we develop a machinery to analyze the behavior of non-linear hydrodynamic systems of Forchheimer type, dealing with the change in physical parameters.

To model the regime of the filtration we consider two types of the boundary conditions: given pressure or given total flux on the accessible boundary. To derive *a priori* estimates for the solutions to these IBVP, we introduce the function $H = H(x, t)$, defined in terms of the pressure gradient ∇p, whose integral plays the role of a Lyapunov function. The L^1 norm of H is equivalent to a "weighted norm" of $|\nabla p|$, and for the class of (GPPC) it is comparable with Sobolev norms of $p(x, t)$ in $W^{1,q}$ where q explicitly depends on the g-Forchheimer polynomial. We investigate qualitative properties of the solutions and their long-time dynamics. In particular, the established monotonicity of the vector field $K(y)y$ results in the Lyapunov stability of the solutions. Moreover, the asymptotic stability is proved by utilizing the *a priori* estimates to balance the degeneracy of the parabolic equation. Concerning the structural stability of the problems, we prove the continuous dependence of the solutions on the boundary data. This requires suitable trace estimates. We also obtain effective comparisons between solutions with two types of boundary conditions: given pressure or given total flux.

Though problems discussed in this paper originate from hydraulic and reservoir engineering [10, 34], their mathematical studies may have wider applications. For instance, they can be adopted in biomathematics to investigate conjugate blood flows in the lumen and arterial wall (see [2, 33] and references therein).

The paper is organized as follows:

In Section 2 we introduce the generalized formulation of the Forchheimer's laws for slightly compressible fluids. Also, different boundary conditions are described, namely, the Dirichlet and the total flux boundary conditions. In Section 3, the resulting *implicit* non-linear Darcy equations are derived from the generalized Forchheimer equations. Using those equations, the dynamics of the system can be described by a non-linear degenerate parabolic equation for the pressure only. Such reduction is valid under the G-Conditions (see (3.3)). Primary properties of the equations are studied, particularly, the monotonicity under the Lambda-Condition (see (3.25)). We introduce the class of "generalized polynomials with positive coefficients" as the main model for our study and applications. In Section 4, two initial boundary value problems (IBVP) corresponding to two types of boundary conditions are introduced. The uniqueness and Lyapunov stability of their solutions are studied. In Section 5, we focus on special time-dependent solutions, called pseudo-steady state solutions, which generate time-invariant velocity fields. Their *a priori* estimates and Lipschitz continuity on the total flux are established. In Section 6, we derive several *a priori* estimates for solutions of IBVP with boundary data split in time and spatial variables (see Definition 6.1). In Section 7, we obtain the asymptotic stability of the above IBVP. In Section 8, we study both IBVP with perturbed boundary data. We evaluate deviation between solutions with respect to deviation of the boundary data. In particular, we estimate their asymptotic deviations. In Section 9, numerical computations are presented for different cases of generalized Forchheimer equations and boundary data to illustrate the preceding theoretical study.

2. Formulation of the Problem

2.1. **General Forchheimer equations.** Darcy's law is commonly related to viscous fluid laminar flows in porous media and is characterized by the permeability coefficient, which is obtained empirically in order to match the linear relation between velocity vector and pressure gradient. Darcy's equation has also been obtained rigorously within the context of homogenization and other averaging/up-scaling techniques [35, 37]. From hydrodynamic point of view, the Darcy's equation is interpreted as the momentum equation. The Darcy's equation, the continuity equation and the equation of state serve as the framework to model processes in reservoirs [25, 13]. For a slightly compressible fluid, the original PDE system reduces to a scalar linear second order parabolic equation for the pressure only. The pressure function is a major feature of the oil or gas filtration in porous media, which is bounded by the well surface and the exterior reservoir boundary. Different boundary conditions on the well correspond to different regimes of production, while the condition on the exterior boundary models flux or absence of flux into the drainage area. All together, the linear parabolic equation, boundary conditions and some assumptions or guesses about the initial pressure distribution form the IBVP.

There are different approaches for modeling non-Darcy's phenomena [17, 19, 41, 27, 36]. It can be derived from the more general Brinkman-Forchheimer's equation [27, 12], or from mixture theory assuming certain relations between velocity field and "drag-like" forces due to fluid to solid friction in the porous media [35]. It can be also derived using homogenization arguments [37], or assuming some functional relation and then match the experimental data. In the current paper we will just postulate a general constitutive equation bounding the velocity vector field and the pressure gradient. We will introduce constraints on the momentum equation and on the fluid density. This will allow the reduction of the original system to a scalar quasi-linear parabolic equation for the pressure only.

Hereafter, the following notation and basic definitions are used:

- $u(x,t)$ is the velocity field; $x \in \mathbb{R}^d, d = 2,3$ spatial variable; t time; $p(x,t)$ pressure distribution; $y \in \mathbb{R}^d$ variable vectors related to ∇p; s, ξ scalar variables;
- Π dimensionless (normalized) permeability tensor - positive definite, symmetric matrix; it may depend on spacial variable, and is subjected to conditions

$$k_1 \geq \ (\Pi\, y, y)/|y|^2 \geq \ k_0, \tag{2.1}$$

here $(\cdot\,,\cdot)$ is the scalar product in the Euclidean space, and $|y|$ is the corresponding norm $|y| = (\sum_{i=1}^d y_i^2)^{1/2}$.
- The notations $C, C_0, C_1, C_2, \ldots$ denote generic positive constants not depending on the solutions.
- When not specified, $\|\cdot\|_{L^q}$ and $\|\cdot\|_{W^{r,q}}$ denote the norms over the domain U, i.e., $\|\cdot\|_{L^q(U)}$ and $\|\cdot\|_{W^{r,q}(U)}$, respectively. Here the domain $U \subset \mathbb{R}^d$ of interest is fixed in the subsection 2.2 below.

In studies of flows in porous media, the three Forchheimer's laws (two-term, power, and three-term) are widely used. Darcy and Forchheimer laws can be written in the vector forms as follows:

- The Darcy law

$$\alpha u = -\Pi \nabla p, \tag{2.2}$$

where $\alpha = \dfrac{\mu}{k}$ with k, in general, being the permeability non-homogeneous function depending on x subjected to the condition: $k_2^{-1} \geq \ k \geq \ k_2$, $1 \geq k_2 > 0$. Here, the constant μ is the viscosity of the fluid.
- The Forchheimer two-term law

$$\alpha u + \beta\sqrt{(Bu,u)}\,u = -\Pi \nabla p, \tag{2.3}$$

where $\beta = \dfrac{\rho F \Phi}{k^{1/2}}$, F is the Forchheimer's coefficient, Φ is the porosity, and ρ is the density of the fluid.
- The Forchheimer power law

$$au + c^n\sqrt{(Bu,u)^{n-1}}u = -\Pi \nabla p, \tag{2.4}$$

where n is a real number belonging to the interval $[1,2]$. The strictly positive and bounded functions c and a are found empirically, or can be taken as $c = (n-1)\sqrt{\beta}$ and $a = \alpha$. By this way, $n = 1$ and $n = 2$ reduce the power law (2.4) to Darcy's law and to Forchheimer two-term law, respectively.
- The Forchheimer three-term law

$$\mathcal{A}u + \mathcal{B}\,\sqrt{(Bu,u)}\,u + \mathcal{C}(Bu,u)u = -\Pi \nabla p. \tag{2.5}$$

Here $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical constants.

We now introduce a general form of Forchheimer equations.

Definition 2.1 (g-Forchheimer Equations)**.**

$$g(x,|u|_B)\,u = -\Pi\nabla p, \tag{2.6}$$

here $g(x,s) > 0$ for all $s \geq 0$ and $|u|_B = \sqrt{(Bu,u)}$, where $B = B(x)$ is a positive definite tensor with bounded entries depending, in general, on the spatial variable.

We will refer to (2.6) as g-Forchheimer (momentum) equations.

Under isothermal condition the state equation relates the density $\rho = \rho(p)$ with the pressure only. For slightly compressible fluid it takes the form:

$$\frac{d\rho}{dp} = \frac{1}{\kappa}\rho, \tag{2.7}$$

where $1/\kappa$ is the compressibility of the fluid. Substituting the last equation in the continuity equation

$$\frac{d\rho}{dt} = -\nabla\cdot(\rho u), \tag{2.8}$$

yields

$$\frac{d\rho}{dp}\frac{dp}{dt} = -\rho\nabla\cdot u - \frac{d\rho}{dp}u\cdot\nabla p,$$

$$\frac{dp}{dt} = -\kappa\nabla\cdot u - u\cdot\nabla p. \tag{2.9}$$

Since κ is large for most slightly compressible fluids in porous media, following engineering tradition we drop the last term in (2.9) and study the reduced equation:

$$\frac{dp}{dt} = -\kappa\nabla\cdot u\,. \tag{2.10}$$

2.2. **Boundary conditions.** Let $U \subset \mathbb{R}^d$ be a domain modeling the drainage area in the porous media (reservoir), bounded by two boundaries: the exterior boundary Γ_e, and the accessible boundary Γ_i.

The exterior boundary Γ_e models the geometrical limit of the well impact on the flow filtration and is often considered impermeable. This yields the boundary condition:

$$u\cdot N|_{\Gamma_e} = 0, \tag{2.11}$$

where N is the outward normal vector on the boundary $\Gamma = \Gamma_i \cup \Gamma_e$. Other types of boundary conditions on the exterior boundary are discussed in [3].

The accessible boundary Γ_i models the well and defines the regime of filtration inside the domain. On Γ_i the data are given rate of production $Q(t)$, or given pressure value $p = \varphi(x,t)$, or a combination of both. It is very important from a practical point of view to build some “base line” solutions capturing significant features of the well capacity and analyze the impact of the boundary conditions on these solutions. This analysis will be used to forecast the well performances and tune the model to the actual data.

On the boundary Γ_i it is of particular interest the “split” condition of the following type

$$p = \psi(x,t) = \gamma(t) + \varphi(x), \tag{2.12}$$

where the time and space dependence of p are separated. This type of condition models wells which have conductivity much higher than the conductivity inside the reservoir. The limiting homogeneous case $\psi(x) = const$ corresponds to the case of infinite conductivity on the well.

In case the flow is controlled by given production rate $Q(t)$, the solution is not unique. We will restrict the class of solutions by imposing the split boundary constraint (2.12) on the well, where only $\varphi(x)$ is known and $\gamma(t)$ is determined by $Q(t)$ (see Section 6).

Two important cases are:

(a) pressure distribution of the form $-At + \varphi(x)$, and

(b) constant total flux $\int_{\Gamma_i} u \cdot N d\sigma = Q = const$.

The particular solutions of IBVP with boundary conditions (a) and (b) are "time-invariant" (see Section 5) and are used actively by engineers in their practical work.

3. Non-Linear Darcy Equation and Monotonicity

In order to make further constructions let the porous media be homogeneous and isotropic and the function g in (2.6) be independent of the spatial variables. Thus one has

$$\Pi(x) = I,\ B(x) = I,\ g(x, |u|) = g(|u|), \tag{3.1}$$

where I is the identity matrix. From (2.6) one has

$$G(|u|) = g(|u|)|u| = |\nabla p|, \quad \text{where} \quad G(s) = sg(s), \quad \text{for} \quad s \geq 0. \tag{3.2}$$

Henceforth in this section the following notation for function G and its inverse G^{-1} are used: $G(s) = sg(s) = \xi$, and $s = G^{-1}(\xi)$. To make sure one can solve (3.2) for $|u|$, we impose the following conditions.

G-Conditions: The function g belongs to $C([0,\infty))$ and $C^1((0,\infty))$, and satisfies

$$g(0) > 0, \quad \text{and} \quad g'(s) \geq 0 \text{ for all } s \geq 0. \tag{3.3}$$

Under the G-Conditions, one has $G'(s) = sg'(s) + g(s) \geq g(0) > 0$. Note also $G(0) = 0$. Hence G is a one-to-one mapping from $[0,\infty)$ onto $[0,\infty)$, therefore one can find $|u|$ as a function $|\nabla p|$

$$|u| = G^{-1}(|\nabla p|) \tag{3.4}$$

Substituting equation (3.4) into (2.6) one obtains the following alternative form of the g-Forchheimer momentum equation (2.6):

Definition 3.1. *(Non-linear Darcy Equation)*

$$u = \frac{-\nabla p}{g(G^{-1}(|\nabla p|))} = -K(|\nabla p|)\nabla p, \tag{3.5}$$

where the function $K : [0,\infty) \to [0,\infty)$ is defined by

$$K(\xi) = K_g(\xi) = \frac{1}{g(G^{-1}(\xi))}, \quad \xi \geq 0. \tag{3.6}$$

Note that

$$G^{-1}(0) = 0, \quad K(0) = \frac{1}{g(0)} > 0. \tag{3.7}$$

Substituting (3.5) for u into (2.10) one derives the degenerate parabolic equation for the pressure:

$$\frac{dp}{dt} = \kappa \nabla \cdot (K(|\nabla p|)\nabla p)\,. \tag{3.8}$$

Next, we will rewrite Eqs. (3.5) and (3.8) in their dimensionless form. Let $1/\kappa$, Q and $|U|$ be some reference values for the compressibility, the total production rate and the reservoir volume. Hence $L = |U|^{1/d}$ is the reference length and $T = |U|/Q$ is the reference time. The dimensionless pressure and velocity p^* and u^* are defined as

$$p^* = \frac{p}{\kappa}, \tag{3.9}$$

$$u^* = \frac{L^{d-1}}{Q}\, u, \tag{3.10}$$

respectively. We also define the dimensionless nonlinear function

$$K^*(\xi^*) = \frac{\kappa L^{d-2} K(\xi)}{Q} = \frac{\kappa L^{d-2} K(\frac{\kappa}{L}\xi^*)}{Q}.$$

Eq. (3.5) can be rewritten as

$$\frac{Q}{L^{d-1}}\, u^* = -K(|\nabla^*\, (\kappa/L\; p^*)\,|)\nabla^*(\kappa/L\; p^*), \tag{3.11}$$

or the same

$$u^* = -\frac{\kappa L^{d-2} K(|\nabla^*\, (\kappa/L\; p^*)\,|)}{Q}\nabla^* p^* = -K^*(|\nabla^* p^*|)\nabla^* p^*. \tag{3.12}$$

Similarly Eq. (3.8) can be rewritten as

$$\frac{dp^*}{dt^*} = \frac{\kappa L^{d-2} K(|\nabla^*\, (\kappa/L\; p^*)\,|)}{Q}\nabla^* p^* = \nabla^* \cdot (K^*(|\nabla^* p^*|)\nabla^* p^*)\,. \tag{3.13}$$

For sake of notation, we drop the $*$ apex, keeping in mind that all the quantities are dimensionless:

$$u = -K(|\nabla p|)\nabla p, \tag{3.14}$$

$$\frac{dp}{dt} = \nabla \cdot (K(|\nabla p|)\nabla p)\,. \tag{3.15}$$

Some properties of the function K are stated in the following lemma.

Lemma 3.2. *Let $g(s)$ satisfy the G-Conditions.*

(i) The function $K = K_g$ in (3.6) is well-defined, belongs to $C^1([0,\infty))$ and is decreasing. Moreover, for any $\xi \ge 0$, let $s = G^{-1}(\xi)$, then one has

$$K'(\xi) = -K(\xi)\frac{g'(s)}{\xi g'(s) + g^2(s)} \le 0. \tag{3.16}$$

(ii) For any $n \ge 1$, the function $K(\xi)\xi^n$ is increasing and satisfies

$$(K(\xi)\xi^n)' = K(\xi)\xi^{n-1}\left(n - \frac{\xi g'(s)}{\xi g'(s) + g^2(s)}\right) \ge 0, \quad s = G^{-1}(\xi). \tag{3.17}$$

(ii) The function $y \in \mathbb{R}^d \to K(|y|)y$ belongs to $C^1(\mathbb{R}^d \setminus \{0\})$ and for $y \ne 0$

$$\nabla(K(|y|)y) = K(|y|)\left(I - \frac{g'(|y|)}{|y|(|y|g'(s) + g^2(s))} y \otimes y\right), \tag{3.18}$$

where $s = G^{-1}(|y|)$.

Proof. Eq. (3.16) follows from the chain rule

$$K'(\xi) = -\frac{1}{g^2(s)} g'(s) \frac{1}{G'(s)} = -K(\xi) \frac{g'(s)}{g(s)} \frac{1}{sg'(s) + g(s)},$$

while Eq. (3.17) is obtained from (3.16) by direct substitution.

For $y \in \mathbb{R}^d, y \neq 0$, elementary calculations with the use of (3.16) give

$$\frac{\partial}{\partial y_j}(K(|y|)y_i) = K(|y|)\left(\delta_{ij} - \frac{g'(|y|)}{|y|} \frac{y_i y_j}{|y|g'(s) + g^2(s)}\right), \tag{3.19}$$

for $i, j = 1, \ldots, d$, where $s = G^{-1}(|y|)$. This proves (3.19). □

It turns our that the function $y \to K(|y|)y$ associated with non-linear potential field on the RHS of equation (3.5) possesses a monotone property. This monotonicity and other monotone properties are crucial in the study of the uniqueness and qualitative behavior of the the solutions of initial value problems (see e.g. [16]).

Definition 3.3. *Let F be a mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$.*

- *F is monotone if*

$$(F(y') - F(y)) \cdot (y' - y) \geq 0, \text{ for all } y', y \in \mathbb{R}^d. \tag{3.20}$$

- *F is strictly monotone if there is $c > 0$ such that*

$$(F(y') - F(y)) \cdot (y' - y) \geq c|y' - y|^2, \text{ for all } y', y \in \mathbb{R}^d. \tag{3.21}$$

- *F is strictly monotone on bounded sets if for any $R > 0$, there is a positive number $c_R > 0$ such that*

$$(F(y') - F(y)) \cdot (y' - y) \geq c_R|y' - y|^2, \text{ for all } |y'| \leq R,\ |y| \leq R. \tag{3.22}$$

To connect the above monotonicity and Eq. (3.5), we define the function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$\Phi(y, y') = (K(|y'|)y' - K(|y|)y) \cdot (y' - y), \quad \text{for} \quad y, y' \in \mathbb{R}^d. \tag{3.23}$$

Proposition 3.4. *Let $g(s)$ satisfy the G-Conditions. Then $F(y) = K(|y|)y$ is monotone, hence*

$$\Phi(y, y') \geq 0 \quad \text{for all} \quad y, y' \in \mathbb{R}^d. \tag{3.24}$$

The proof of Proposition 3.4 will be given below with that of Proposition 3.6.

For stronger monotone properties, we impose an extra condition on $g(s)$.

Lambda-Condition: There is $\lambda > 0$ such that

$$g(s) \geq \lambda s g'(s), \quad \text{for all} \quad s > 0. \tag{3.25}$$

Note that this condition is satisfied for any polynomial $g(s)$ with positive coefficients and positive exponents.

Lemma 3.5. *Let $g(s)$ satisfy the G-Conditions and the Lambda-Condition then*

$$0 \geq K'(\xi) \geq -\frac{1}{\lambda + 1} \frac{K(\xi)}{\xi}, \tag{3.26}$$

and

$$(K(\xi)\xi^n)' \geq K(\xi)\xi^{n-1}\left(n - \frac{1}{\lambda + 1}\right) \geq 0 \text{ for } n \geq 1. \tag{3.27}$$

Proof. Let $s = G^{-1}(\xi)$. If $g'(s) = 0$, then $K'(\xi) = 0$ and one can easily verify that inequalities (3.26) and (3.27) hold. We now assume $g'(s) \neq 0$. Inequality (3.26) follows by using Lambda-Condition (3.25) in (3.16)

$$K'(\xi) = -K(\xi)\frac{g'(s)}{\xi g'(s) + g^2(s)} \geq$$
$$-K(\xi)\frac{g'(s)}{\xi g'(s) + g(s)\lambda s g'(s)} = -\frac{1}{\lambda+1}\frac{K(\xi)}{\xi}.$$

Inequality (3.27) follows at once from (3.26)

$$(K(\xi)\xi^n)' = K'(\xi)\xi^n + nK(\xi)\xi^{n-1} \geq$$
$$-\frac{1}{\lambda+1}\frac{K(\xi)}{\xi}\xi^n + nK(\xi)\xi^{n-1} = K(\xi)\xi^{n-1}\left(n - \frac{1}{\lambda+1}\right) \geq 0$$

for $n \geq 1$. □

Proposition 3.6. *Let $g(s)$ satisfy the G-Conditions and the Lambda-Condition. Then $F(y) = K(|y|)y$ is strictly monotone on bounded sets. More precisely,*

$$\Phi(y, y') \geq \frac{\lambda}{\lambda+1}K(\max\{|y|, |y'|\})|y'-y|^2, \quad \textit{for all} \quad y, y' \in \mathbb{R}^d. \tag{3.28}$$

Proofs of Propositions 3.4 and 3.6. We consider the following two cases:

Case 1: The origin does not belong to $[y, y']$. Here $[y, y']$ is the line segment connecting y and y'. Let $z = y' - y$, and let $\gamma(t) = (ty' + (1-t)y)$, $t \in [0,1]$, be the parameterization of $[y, y']$. Define $h(t) = (K(|\gamma(t)|)\gamma(t)) \cdot z$, for $t \in [0,1]$.

By the Mean Value Theorem, there is $t_0 \in [0,1]$ with $y_0 = \gamma(t_0) \neq 0$ such that

$$\Phi(y, y') = h(1) - h(0) = h'(t_0) = (\nabla(K(|y_0|)y_0)(y'-y)) \cdot (y'-y).$$

Recollecting identity (3.19) one gets:

$$\Phi(y, y') = K(|y_0|)|y'-y|^2 - K(|y_0|)\frac{g'(s)}{|y_0|}\frac{\sum_{i,j} y_{0i}y_{0j}(y'_j - y_j)(y'_i - y_i)}{|y_0|g'(s) + g^2(s)}$$
$$= K(|y_0|)|z|^2 - K(|y_0|)\frac{g'(s)}{|y_0|}\frac{|y_0 \cdot z|^2}{(|y_0|g'(s) + g^2(s))},$$

where $s = G^{-1}(|y_0|)$. Applying the Cauchy-Schwarz inequality to $|y_0 \cdot z|$ yields

$$\Phi(y, y') \geq K(|y_0|)|z|^2\Big(1 - \frac{|y_0|g'(s)}{|y_0|g'(s) + g^2(s)}\Big) \geq 0. \tag{3.29}$$

This proves (3.24).

In case $g(s)$ satisfies the Lambda-Condition, noting that $|y_0| = sg(s)$, one has

$$|y_0|g'(s) + g^2(s) \geq |y_0|g'(s) + g(s)\left(\lambda s g'(s)\right) = |y_0|g'(s) + \lambda|y_0|g'(s).$$

Hence

$$\Phi(y, y') \geq K(|y_0|)|z|^2\left(1 - \frac{|y_0|g'(s)}{(1+\lambda)|y_0|g'(s)}\right) = K(|y_0|)|z|^2\frac{\lambda}{1+\lambda}. \tag{3.30}$$

Since $|y_0|$ is between $|y|$ and $|y'|$, and $K(\cdot)$ is decreasing, the last inequality implies (3.28).

Case 2: The origin belongs to $[y, y']$. We replace y' by some $y_\varepsilon \neq 0$ so that $0 \notin [y, y_\varepsilon]$, and $y_\varepsilon \to 0$ as $\varepsilon \to 0$. Apply the above inequality for y and y_ε, then let $\varepsilon \to 0$.

Finally, (3.28) and the fact K is decreasing clearly imply that $K(|y|)y$ is strictly monotone on bounded sets. $\square$

To illustrate the monotonicity properties, we consider the particular case of two-term Forchheimer's equation. In this case function K can be calculated explicitly.

Example 3.7. For the Forchheimer two-term law (2.3), let $g(s) = \alpha + \beta s$, then one has $G(s) = \beta s^2 + \alpha s$ and $s = G^{-1}(\xi) = \frac{-\alpha+\sqrt{\alpha^2+4\beta\xi}}{2\beta}$. Thus

$$K(\xi) = \frac{1}{\alpha + \beta G^{-1}(\xi)} = \frac{2}{\alpha + \sqrt{\alpha^2 + 4\beta\xi}}.$$

One can easily verify that (3.25) holds with $\lambda = 1$. Proposition 3.4 then yields

$$\Phi(y, y') \geq \frac{1}{2} K(\max\{|y|, |y'|\})|y' - y|^2.$$

The Lambda-Condition (3.25) imposes an exponential upper bound for $g(s)$:

$$g(s) \leq A + Bs^{1/\lambda}, \quad \forall s \geq 0, \quad \text{some} \quad A, B > 0. \tag{3.31}$$

It is not difficult to see that all three Forchheimer equations (2.3), (2.4), and (2.5) satisfies the G-conditions and Lambda-Condition. Based on those three models (2.3), (2.4), (2.5) and the constraint (3.31), we introduce the following "generalized polynomials with positive coefficients" (GPPC).

Definition 3.8. *We say that a function $g(s)$ is a GPPC if*

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_k s^{\alpha_k} = \sum_{j=0}^{k} a_j s^{\alpha_j}, \tag{3.32}$$

where $k \geq 0$, the exponents satisfy $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k$, and the coefficients $a_0, a_1, \ldots, a_k$ are positive.

The largest exponent α_k is the degree of g and is denoted by $\deg(g)$.

Class (GPPC) is defined as the collection of all GPPC.

If the function g in Definition 2.1 belongs to class (GPPC) then we call it the g-Forchheimer polynomial.

Lemma 3.9. *Let $g(s)$ be a function of class (GPPC). Then g satisfies G-Conditions and Lambda-Condition. Consequently, $F(y) = K_g(|y|)y$ is strictly monotone on bounded sets and Ineq. (3.28) holds for $K = K_g$.*

Proof. Obviously, $g(s)$ satisfies the G-Conditions. We now check the Lambda-Condition. The case $\alpha_k = 0$, (3.25) holds trivially with any $\lambda > 0$. We consider the case $\alpha_k > 0$. One has

$$sg'(s) = \sum_{j=1}^{k} \alpha_j a_j s^{\alpha_j} \leq \alpha_k \sum_{j=1}^{k} a_j s^{\alpha_j} \leq \alpha_k \sum_{j=0}^{k} a_j s^{\alpha_j} = \alpha_k g(s).$$

Thus the Lambda-Condition holds with $\lambda = 1/\alpha_k$. $\square$

Lemma 3.10. *Let $g(s)$ be a function of class (GPPC) as in (3.32). Then $K(\xi) = K_g(\xi)$ is well-defined, is decreasing and satisfies*

$$\frac{C_0}{(1+\xi)^a} \leq K(\xi) \leq \frac{C_1}{(1+\xi)^a}, \ \forall \xi \geq 0, \tag{3.33}$$

where $a = \alpha_k/(\alpha_k + 1) \in [0, 1)$*, and* C_0 *and* C_1 *are positive numbers depending on* a_j*'s and* α_j*'s. Subsequently*

$$C_2\xi^{2-a} - 1 \le K(\xi)\xi^2 \le C_1\xi^{2-a},\ \forall \xi \ge 0, \tag{3.34}$$

where $C_2 = \min\left(\frac{C_0}{2}, 1\right)$.

Proof. To prove inequalities in (3.33), one first notes that

$$\xi + 1 = sg(s) + 1 = 1 + a_0 s + \ldots + a_k s^{\alpha_k+1} \ge C_3(1+s)^{\alpha_k+1},$$

$$\xi + 1 = sg(s) + 1 = 1 + a_0 s + \ldots + a_k s^{\alpha_k+1} \le C_4(1+s)^{\alpha_k+1},$$

and

$$g(s) = a_0 + \ldots + a_k s^{\alpha_k} \le C_5(1+s)^{\alpha_k},$$

$$g(s) = a_0 + \ldots + a_k s^{\alpha_k} \ge C_6(1+s)^{\alpha_k},$$

where positive numbers C_3 ,C_4, C_5 and C_6 depend on coefficients a_j's and α_j's. Hence

$$K(\xi) = \frac{1}{g(s)} \ge \frac{1}{C_5(1+s)^{\alpha_k}} \ge \frac{1}{C_5[\frac{1}{C}_3(1+\xi)]^{\alpha_k/(\alpha_k+1)}} = \frac{C_0}{(1+\xi)^a},$$

and

$$K(\xi) = \frac{1}{g(s)} \le \frac{1}{C_6(1+s)^{\alpha_k}} \le \frac{1}{C_6[\frac{1}{C}_4(1+\xi)]^{\alpha_k/(\alpha_k+1)}} = \frac{C_1}{(1+\xi)^a}.$$

To prove the left inequality in (3.34), one considers the two cases:

$$K(\xi)\xi^2 \ge 0 \ge \xi^{2-a} - 1, \text{ for } \xi \le 1,$$

$$K(\xi)\xi^2 \ge \frac{C_0\xi^2}{(1+\xi)^a} \ge \frac{C_0\xi^2}{(2\xi)^a} = \frac{C_0}{2}\xi^{2-a} \ge \frac{C_0}{2}\xi^{2-a} - 1, \text{ for } \xi > 1,$$

which can be reduced to the left inequality in (3.34) for all $\xi \ge 0$. To prove the right inequality in (3.34) one considers

$$K(\xi)\xi^2 \le \frac{C_1\xi^2}{(1+\xi)^a} \le \frac{C_1\xi^2}{\xi^a} = C_1\xi^{2-a},\ \forall \xi \ge 0\,.$$

The proof is complete. □

As a consequence of the monotonicity, we have the following estimates which will be used repeatedly in the next sections.

Lemma 3.11. *Let the function* g *be of the class (GPPC). For any functions* f*,* p_1 *and* p_2*, and for* $1 \le q < 2$*, one has*

$$\left(\int_U |f|^q dx\right)^{2/q} \le C\left(\int_{U_1} K(|\nabla p_1|)\,|f|^2 dx + \int_{U_2} K(|\nabla p_2|)\,|f|^2 dx\right) \times \left\{1 + \max\left(\|\nabla p_1\|_{L^{\frac{aq}{2-q}}(U)}, \|\nabla p_2\|_{L^{\frac{aq}{2-q}}(U)}\right)\right\}^a, \tag{3.35}$$

where

$$U_1 = \{x : |\nabla p_1(x)| \ge |\nabla p_2(x)|\}, \quad U_2 = \{x : |\nabla p_1(x)| < |\nabla p_2(x)|\}.$$

Consequently

$$\int_U \Phi(\nabla p_1, \nabla p_2)dx \geq C\left(\int_U |\nabla(p_1-p_2)|^q dx\right)^{2/q} \times \left\{1+\max\left(\|\nabla p_1\|_{L^{\frac{aq}{2-q}}(U)}, \|\nabla p_2\|_{L^{\frac{aq}{2-q}}(U)}\right)\right\}^{-a}. \tag{3.36}$$

Proof. First, $\int_U |f|^q dx = \int_{U_1} |f|^q dx + \int_{U_2} |f|^q dx = J_1 + J_2$, hence

$$\left(\int_U |f|^q dx\right)^{2/q} \leq C(J_1^{2/q} + J_2^{2/q}). \tag{3.37}$$

By Holder inequality we have

$$J_1 = \int_{U_1} |f|^q dx \leq \left(\int_{U_1} |f|^{qr} \cdot (K(|\nabla p_1|))^{\beta r}\, dx\right)^{\frac{1}{r}} \cdot \left(\int_{U_1} (K(|\nabla p_1|))^{-\beta s}\, dx\right)^{\frac{1}{s}}, \tag{3.38}$$

where $\frac{1}{r} + \frac{1}{s} = 1$ and $\beta > 0$ is a free parameter.

By Lemma 3.10

$$\int_{U_1} (K(|\nabla p_1|))^{-\beta s}\, dx \leq C_0 \int_{U_1} (1+|\nabla p_1|)^{\beta s a} dx \leq C_1(1 + \int_{U_1} |\nabla p_1|^{\beta s a} dx)\,. \tag{3.39}$$

Set $\beta\, r = 1$, $q\, r = 2$, then

$$r = \frac{2}{q},\ \beta = \frac{q}{2},\ s = \frac{2}{2-q},\ \beta s = \frac{q}{2-q} = \frac{s}{r}.$$

Hence it follows from (3.38) that

$$\begin{aligned} J_1^{2/q} &= \Big(\int_{U_1} |f|^p dx\Big)^{2/q} \\ &\leq \left(\int_{U_1} |f|^2 \cdot K(|\nabla p_1|)dx\right) C_2 \left\{1 + \left(\int_{U_1} |\nabla p_1|^{\frac{ap}{2-q}} dx\right)^{\frac{2-q}{q}}\right\} \\ &\leq C_3 M \int_{U_1} |f|^2 \cdot K(|\nabla p_1|)dx, \end{aligned}$$

where $M = \left\{1+\max\left(\|\nabla p_1\|_{L^{\frac{aq}{2-q}}(U)}, \|\nabla p_2\|_{L^{\frac{aq}{2-q}}(U)}\right)\right\}^a$.

Similarly, one obtains the estimate for J_2:

$$J_2^{2/q} \leq C_4 M \int_{U_2} |f|^2 \cdot K(|\nabla p_2|)dx.$$

Combining the above estimates of $J_1^{2/q}$ and $J_2^{2/q}$ with (3.37), one derives

$$\left(\int_U |f|^q dx\right)^{2/q} \leq C_5 M\Big(\int_{U_1} |f|^2 \cdot K(|\nabla p_1|)dx \int_{U_2} |f|^2 \cdot K(|\nabla p_2|)dx\Big)$$

which yields (3.35).

To prove (3.36), one applies inequality (3.28) in Proposition 3.4 to have

$$\int_U \Phi(\nabla p_1, \nabla p_2)dx \geq \frac{\lambda}{\lambda+1}\left[\int_{U_1} K(|\nabla p_1|)|\nabla z|^2 dx + \int_{U_2} K(|\nabla p_2|)|\nabla z|^2 dx\right], \tag{3.40}$$

where $z = p_1 - p_2$. Then apply Ineq. (3.35) with $f = \nabla z$. □

In our subsequent sections, we always assume that the function $g(s)$ satisfies the G-Conditions. Therefore the function $K(\xi) = K_g(\xi)$ and the equation (3.15) are well-defined.

4. Initial Boundary Value Problem and Uniqueness

In this section we consider two IBVP for solutions of the equation (3.15). The flow is subjected to the non-flow condition on exterior boundary Γ_e. On the accessible boundary Γ_i, there are to two types of boundary conditions: (1) given pressure distribution, and (2) given total flux. For general non-linear function $g(s)$ satisfying the G-Conditions, we will prove the uniqueness of the IBVP for case (1) without any restriction, and for case (2) under additional constraint on the behavior of the solutions on Γ_i. Furthermore, under the Lambda-Condition (3.25) on the function $g(s)$, we will show that solutions of both IBVP are asymptotically and exponentially stable (with respect to initial data), if the pressure gradients are bounded for all time.

We will study below two IBVP, namely, IBVP-I and IBVP-II, corresponding to the Dirichlet and total flux conditions on Γ_i, respectively.

Definition 4.1. *(IBVP-I) The function $p(x,t)$ is a solution of the IBVP-I if $p(x,t)$ satisfies:*

$$\begin{cases} \frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p)\,, \quad \text{in} \quad D = U \times (0,\infty), \\ p(x,0) = p_0(x), \quad \text{in} \quad U, \\ \frac{\partial p}{\partial N} = 0 \quad \text{on} \quad \Gamma_e \times (0,\infty), \\ p(x,t) = \psi(x,t) \quad \text{on} \quad \Gamma_i \times (0,\infty), \end{cases} \tag{4.1}$$

where $p_0(x)$ is the given the initial pressure, and $\varphi(x,t)$ is the prescribed pressure distribution on Γ_i.

Definition 4.2. *(IBVP-II) The function $p(x,t)$ is a solution of the IBVP-II if $p(x,t)$ satisfies:*

$$\begin{cases} \frac{dp}{dt} = \nabla \cdot K(\nabla p)\nabla p, \quad \text{in} \quad D = U \times (0,\infty), \\ p(x,0) = p_0(x), \quad \text{in} \quad U, \\ \frac{\partial p}{\partial N} = 0 \quad \text{on} \quad \Gamma_e \times (0,\infty), \\ -\int_{\Gamma_i} K(\nabla p(x,t))\nabla p(x,t) \cdot N = Q(t) \quad \text{on} \quad \Gamma_i \times (0,\infty), \end{cases} \tag{4.2}$$

where $p_0(x)$ is the given initial pressure, and $Q(t)$ is the prescribed total flux.

The solutions can be either the classical solutions or, more generally as studied in this paper, the weak ones. For the latter class of solutions, one needs the following assumptions:

- $p(x,t) \in L^2_{loc}(0,\infty; W^{2,2}(U))$ and $\frac{\partial p}{\partial t}(x,t) \in L^2_{loc}(0,\infty; W^{1,2}(U))$,
- $p(x,t)$ satisfies the first equation in (4.1) in the distributional sense,
- $p(x,t)$ satisfies the boundary conditions and initial data in the sense of conventional traces.

We start with some primary properties of solutions of (3.15), which is the leading differential equation in (4.1) and (4.2).

Lemma 4.3. *Let $p_i(x,t), i = 1,2$ be two solution of the (3.15), satisfying impermeable condition (2.10) on Γ_e. Let $z(x,t) = p_1(x,t) - p_2(x,t)$ and function $\Phi(\nabla p_1, \nabla p_2)$ be defined by (3.23). Then*

(4.3)
$$\frac{1}{2}\frac{d}{dt}\int_U z^2 dx = -\int_U \Phi(\nabla p_1, \nabla p_2)dx + \int_{\Gamma_i} z(K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2)\cdot N d\sigma,$$

(4.4)
$$\int_U |z(x,t)|^2 dx \le \int_U |z(x,0)|^2 dx + 2\int_0^t \int_{\Gamma_i} z(K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2)\cdot N d\sigma d\tau,$$

$$\frac{\partial}{\partial t}\int_U z dx = \int_{\Gamma_i} (K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2)\cdot N d\sigma, \tag{4.5}$$

(4.6)
$$\int_U z(x,t)dx = \int_U z(x,0)dx + \int_0^t \int_{\Gamma_i} (K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2)\cdot N d\sigma d\tau.$$

Proof. First observe that difference $z(x,t) = p_1(x,t) - p_2(x,t)$ satisfies

$$\begin{cases} \frac{dz}{dt} = \nabla\cdot(K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2), \\ \frac{\partial z}{\partial N} = 0 \quad \text{on} \quad \Gamma_e \end{cases} \tag{4.7}$$

By multiplying LHS and RHS of the equation (4.7) by $z(x,t)$, integrating over domain U, and applying Green's formula to the RHS of the resulting equation, one obtains identity (4.3).

Integrating (4.3) from 0 to t and using the monotonicity property (3.24), which gives $\Phi(\nabla p_1, \nabla p_2) \ge 0$, one obtains inequality (4.4).

Next integrating the first equation in (4.7) over the domain, and applying the Green formula to RHS yields (4.5), and consequently identity (4.6). □

Proposition 4.4. *Let p_1 and p_2 are two solutions of IBVP-I (4.1). Then one has for all $t \ge 0$ that*

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \le \int_U |p_1(x,0) - p_2(x,0)|^2 dx. \tag{4.8}$$

Subsequently, if $p_1(x,0) = p_2(x,0) \in L^2(U)$ then $p_1(x,t) = p_2(x,t)$ for all t.

Assume, in addition, that $g(s)$ satisfies the Lambda-Condition (3.25), and

$$\nabla p_1, \nabla p_2 \in L^\infty(0,\infty, L^\infty(U)). \tag{4.9}$$

Then

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \le e^{-c_1 K(M)t}\int_U |p_1(x,0) - p_2(x,0)|^2 dx, \tag{4.10}$$

for all $t \ge 0$, where

$$M = \max\{\|\nabla p_1\|_{L^\infty(0,\infty,L^\infty(U))}, \|\nabla p_2\|_{L^\infty(0,\infty;L^\infty(U))}\}. \tag{4.11}$$

Consequently,

$$\lim_{t\to\infty}\int_U |p_1(x,t) - p_2(x,t)|^2 dx = 0. \tag{4.12}$$

Proof. Let $z(x,t) = p_1(x,t) - p_2(x,t)$. Since function $z(x,t)$ vanishes on Γ_i, the integral over the boundary Γ_i in (4.4) in the Lemma 4.3 is equal zero, and therefore

$$\int_U z^2(x,t)dx \le \int_U z^2(x,0)dx. \tag{4.13}$$

Similarly we will obtain

$$\frac{1}{2}\frac{d}{dt}\int_U z^2 dx = -\int_U \Phi(\nabla p_1, \nabla p_2)dx. \tag{4.14}$$

By the monotonicity (3.28), the fact that $|\nabla p_1|, |\nabla p_2| \le M$, and the function K is decreasing (Lemma 3.2), it follows that

$$\Phi(\nabla p_1, \nabla p_2) \ge \frac{\lambda}{1+\lambda} K(M)|\nabla p_1 - \nabla p_2|^2. \tag{4.15}$$

Then from (4.14) follows

$$\frac{1}{2}\frac{d}{dt}\int_U z^2 dx \le -\int_U \frac{\lambda}{1+\lambda} K(M)|\nabla z|^2 dx. \tag{4.16}$$

Applying Poincare's inequality to RHS of the equation above one can get

$$\frac{1}{2}\frac{d}{dt}\int_U z^2 dx \le -C\frac{\lambda}{1+\lambda} K(M)\int_U z^2 dx. \tag{4.17}$$

Finally using Gronwall's inequality, we get (4.10). □

Proposition 4.5. *Let p_1 and p_2 be two solutions of IBVP-II (4.2). Assume the difference $(p_1 - p_2)$ on Γ_i is independent of spatial variable x. Then*

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \le \int_U |p_1(x,0) - p_2(x,0)|^2 dx, \quad t \ge 0. \tag{4.18}$$

If $g(s)$ satisfies the Lambda-Condition (3.25) and p_1, p_2 satisfy (4.9), then

$$\int_U |p_1(x,t) - p_2(x,t) - A_0|^2 dx \le e^{-CK(M)t}\int_U |p_1(x,0) - p_2(x,0) - A_0|^2 dx, \tag{4.19}$$

for $t \ge 0$, where $A_0 = \int_U (p_1(x,0) - p_2(x,0))dx$, $C > 0$, and M is defined by (4.11).

Proof. Similar to Proposition 4.4, let $z = p_1 - p_2$. The function $z(x,t)$ on Γ_i is spatially homogeneous, and total fluxes on the accessible boundary Γ_i for both IBVP (4.2) are the same. Therefore, the integral over the boundary Γ_i in (4.3) becomes $z(Q(t) - Q(t)) = 0$. Hence one finds

$$\frac{1}{2}\frac{d}{dt}\int_U z^2 dx = -\int_U \Phi(\nabla p_1, \nabla p_2)dx \le 0, \tag{4.20}$$

the last inequality is due to the monotonicity (3.24). Clearly, (4.18) and the uniqueness of the solution of IBVP-I (4.2) follow.

Next assume $g(s)$ satisfies the Lambda-Condition. Let $\overline{z} = p_1 - p_2 - A_0$. The function $\overline{z}$ solves (4.7), and hence equation (4.3) holds for $\overline{z}(x,t)$:

$$\frac{1}{2}\frac{d}{dt}\int_U \overline{z}^2 dx = -\int_U \Phi(\nabla p_1, \nabla p_2)dx + \int_{\Gamma_i} \overline{z}(K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2)\cdot N d\sigma. \tag{4.21}$$

In addition, $\overline{z}(x,t)$ is spatially independent on the boundary Γ_i, and similar to the above argument, the boundary term in (4.21) is equal to zero. Therefore (4.20) holds for function $\overline{z}$:

$$\frac{1}{2}\frac{d}{dt}\int_U \overline{z}^2 dx = -\int_U \Phi(\nabla p_1, \nabla p_2)dx. \tag{4.22}$$

By virtue of Ineq. (3.28) in Proposition 3.6, we have

$$\frac{1}{2}\frac{d}{dt}\int_U \overline{z}^2 dx \leq -\frac{\lambda}{1+\lambda}K(M)\int_U |\nabla\overline{z}|^2 dx. \tag{4.23}$$

Applying Poincare inequality for RHS of the inequality (4.23) one gets

$$\frac{1}{2}\frac{d}{dt}\int_U \overline{z}^2 dx \leq -\frac{1}{2}c_1 K(M)\int_U \overline{z}^2 dx + c_1\left(\int_U \overline{z}dx\right)^2. \tag{4.24}$$

Since the total fluxes on Γ_i of both solutions are the the same, the integral over Γ_i in identity (4.6) is equal to zero. Therefore

$$\int_U \overline{z}(x,t)dx = \int_U \overline{z}(x,0)dx = 0.$$

Then estimate (4.19) follows from inequality (4.24) and Gronwall's inequality. □

Remark 4.6. As one can see from the above, the uniqueness for both IBVP follows from simple monotonicity (3.20) of the vector field $K(|\nabla p|)\nabla p$, i.e. the non-negativity of $\Phi(\nabla p_1, \nabla p_2)$. However asymptotic stability requires a stronger condition on $K(|\nabla p|)\nabla p$, provided by strict monotonicity on the bounded sets. To guarantee this condition we imposed a constraint on the gradient of the solutions to be bounded uniformly in time. This assumption is very restrictive. We will drop this assumption in Sec. 7 for g belonging to class (GPPC), by utilizing the *a priori* estimates of the pressure gradients in Sec. 6.

5. Pseudo Steady State Solutions

Often in engineering and physics it is essential to identify special time-dependent pressure distributions that generate flows which are time-invariant. In this section we will introduce a class of the so-called pseudo-steady state (PSS) solutions which is used frequently by reservoir and hydraulic engineers to evaluate "capacity" of the well (see. [3, 7, 21] and references therein).

Definition 5.1. *A solution $\overline{p}(x,t)$ of the equation (3.15) in domain U, satisfying the Neumann condition on Γ_e is called the pseudo steady state (PSS) with respect to constant A if*

$$\frac{\partial\overline{p}(x,t)}{\partial t} = const. = -A \quad \textit{for all} \quad t. \tag{5.1}$$

Note: In practice, the constant A in the above definition is conventionally assumed to be positive. However, we will not impose that condition on A in our study.

Equation (3.15) then reduces to

$$\frac{\partial\overline{p}(x,t)}{\partial t} = -A = \nabla\cdot(K(|\nabla\overline{p}|)\nabla\overline{p}). \tag{5.2}$$

Using Green's formula and the Neumann boundary condition on Γ_e one derives

$$A|U| = -\int_{\Gamma_i} (K(|\nabla p|)\nabla p)\cdot N d\sigma = \int_{\Gamma_i} u\cdot N d\sigma = Q(t).$$

Therefore, the total flux of a PSS solution is time-independent

$$Q(t) = A|U| = Q = const., \quad \text{for all} \quad t. \tag{5.3}$$

The PSS solutions inherit two important features of IBVP-I and IBVP-II, which we will explore further. On one hand, the total flux is defined by stationary equation (5.2) and is given. On the other hand, the trace of the solution on the boundary is split *a priori*. Namely re-writing the PSS solution as

$$\overline{p}(x,t) = -At + h(x), \tag{5.4}$$

one has $\nabla p = \nabla h$, hence h and p satisfy the same boundary condition on Γ_e. On Γ_i, in general, we consider

$$\overline{p}(x,t) = -At + \varphi(x), \quad \text{on} \quad \Gamma_i, \tag{5.5}$$

where $\varphi(x)$ is given. Therefore $h(x)$ satisfies

$$-A = \nabla\cdot K(|\nabla h|)\nabla h, \tag{5.6}$$

$$\frac{\partial h}{\partial N} = 0 \quad \text{on} \quad \Gamma_e, \tag{5.7}$$

$$h = \varphi \quad \text{on} \quad \Gamma_i. \tag{5.8}$$

Of particular interest, we consider the case $\varphi(x) = const.$ From physical point of view, it relates to the constraint that conductivity inside well is non-comparably higher than in the porous media. By shifting the values of $\varphi(x)$ and $h(x)$ by a constant, one has

$$\overline{p}(x,t) = -At + B + W(x), \tag{5.9}$$

where A and B are two numbers, and $W(x)$ satisfies

$$-A = \nabla\cdot K(|\nabla W|)\nabla W, \tag{5.10}$$

$$\frac{\partial W}{\partial N} = 0 \quad \text{on} \quad \Gamma_e, \tag{5.11}$$

$$W = 0 \quad \text{on} \quad \Gamma_i, \tag{5.12}$$

A solution $h(x)$ of BVP (5.6), (5.7) and (5.8), considered in this study, is a function in $W^{2,2}(U)$ that satisfies (5.6) in the distributional sense and satisfies the boundary conditions (5.7) and (5.8) with its traces on Γ.

We call $h(x)$ the profile of PSS corresponding to A and the boundary profile $\varphi(x)$. The solution $W(x)$ of BVP (5.10), (5.11) and (5.12) is called the *basic PSS profile* corresponding to A.

Remark 5.2. Note that for a PSS as in (5.4),

$$\frac{1}{|U|}\int_U \overline{p}(x,t)dx - \frac{1}{|\Gamma_i|}\int_{\Gamma_i}\overline{p}(x,t)d\sigma = \frac{1}{|U|}\int_U h(x)dx - \frac{1}{|\Gamma_i|}\int_{\Gamma_i}\varphi(x)d\sigma, \tag{5.13}$$

that is, the difference between averages in the domain and on the boundary Γ_i is independent of time. Engineers widely utilize this property in their routine to calculate productivity index of the well, and sometimes use it as the definition of PSS regime itself (see [3, 7, 21]). However we will not investigate the concept of productivity index for general g-Forchheimer flows in this article.

Proposition 5.3 (Uniqueness of PSS profile)**.** *Let the function $g(s)$ satisfy the Lambda-Condition (3.25).*

(i) Then for each number A and boundary profile φ, there is at most one PSS profile $h(x)$ corresponding to A and $\varphi(x)$.

(ii) Consequently, if $\bar{p}_1$ and $\bar{p}_2$ are two PSS solutions satisfying $\bar{p}_1|_{\Gamma_i} = \bar{p}_2|_{\Gamma_i}$ then $\bar{p}_1(x,t) = \bar{p}_2(x,t)$ for all $x \in U$ and t.

Proof. (i) Let h_1, h_2 be two solutions of the equation (5.6). Then by virtue of the boundary conditions, one has

$$\begin{aligned} 0 &= \int_U (K(|\nabla h_1|)\nabla h_1 - F(|\nabla h_2|)\nabla h_2)\cdot(\nabla h_1 - \nabla h_2)dx \\ &\geq C\int_U K(|\nabla h_1| + |\nabla h_2|)|\nabla h_1 - \nabla h_2|^2 dx. \end{aligned}$$

The inequality above comes from Lemma 3.6. Since $K(|\nabla h_1(x)| + |\nabla h_2(x)|) > 0$ a.e. one has $\nabla(h_2(x) - h_1(x)) = 0$ a.e. in U. Therefore the fact that $(h_2 - h_1)|_{\Gamma_i} = 0$ implies $h_2 - h_1 = 0$ in U.

(ii) Now, suppose $\bar{p}_k(x,t) = -A_k t + h_k(x)$ and $\bar{p}_k(x,t)|_{\Gamma_i} = -A_k t + \varphi_k(x)$ and $p_1 = p_2$ on Γ_i. Obviously, $\varphi_1(x) = \varphi_2(x)$ and $A_1 = A_2$. Part (i) then implies $h_1(x) = h_2(x)$ and hence $\bar{p}_1(x,t) = \bar{p}_2(x,t)$. □

We now focus on the study of the basic profile $W(x)$. Applying Green's formula to (5.10), one easily obtains the following identities:

$$A = -\int_{\Gamma_i} K(|\nabla W|)\nabla W \cdot N d\sigma, \tag{5.14}$$

$$A\int_U W(x)dx = \int_U K(|\nabla W|)|\nabla W|^2 dx. \tag{5.15}$$

First, we derive an *a priori* estimate for $W(x)$ with respect to constant A.

Theorem 5.4. *Let the function $g(s)$ be of class (GPPC) and $a = \deg(g)/(1 + \deg(g))$. Then for any number A, the corresponding basic profile $W(x)$ satisfies*

$$\|\nabla W\|_{L^{2-a}(U)} \leq C\,(|A| + 1)^{1/(1-a)}. \tag{5.16}$$

Proof. From (3.34) and (5.15) one can have

$$\int_U |\nabla W|^{2-a} dx \leq C_1 \int_U K(|\nabla W|)|\nabla W|^2 dx + C_2 \leq C_1 \cdot A \int_U |W| dx + C_2.$$

Applying Poincare inequality to RHS and the Young's inequality we get

$$\begin{aligned} \|\nabla W\|^{2-a}_{L^{(2-a)}} &= \int_U |\nabla W|^{2-a} dx \leq CA\|\nabla W\|_{L^1} + C \leq CA\|\nabla W\|_{L^{(2-a)}} + C \\ &\leq \varepsilon\|\nabla W\|^{2-a}_{L^{(2-a)}} + C|A|^q + C, \end{aligned}$$

where $q = (2-a)/(1-a)$. Taking $\varepsilon = 1/2$ one obtains

$$\|\nabla W\|^{2-a}_{L^{(2-a)}} \leq C|A|^q + C \leq C(1 + |A|)^q.$$

Then (5.16) follows. □

Furthermore, basic profiles are continuous in A, hence in the total flux Q as shown below.

Theorem 5.5. *Let $g(s)$ be of class (GPPC). Let $W_1(x)$ and $W_2(x)$ be two basic profiles corresponding A_1 and A_2, respectively. Then there exists a constant C such that*

$$\|\nabla(W_1 - W_2)\|_{L^{2-a}(U)} \le C\, M\, |A_1 - A_2|, \tag{5.17}$$

where $a = \deg(g)/(1+\deg(g))$, $M = (\max(|A_1|, |A_2|) + 1)^{a/(1-a)}$.

Consequently, for $1 \le q \le (2-a)^* = d(2-a)/(d-(2-a))$, one has

$$\|W_1 - W_2\|_{L^q(U)} \le C\, M\, |A_1 - A_2|. \tag{5.18}$$

Proof. Denote $W = W_1 - W_2$. Using (3.36), one has

$$\begin{aligned}(A_2 - A_1)\int_U W(x)dx &= \int_U \Phi(\nabla W_2, \nabla W_1)dx\\ &\ge C\left(\int_U |\nabla W|^p dx\right)^{2/p}\Big[1 + \max\Big(\|\nabla W_1\|_{L^{\frac{ap}{2-p}}}, \|\nabla W_2\|_{L^{\frac{ap}{2-p}}}\Big)\Big]^{-a}.\end{aligned}$$

Hence

$$\left(\int_U |\nabla W|^p dx\right)^{2/p} \le C|A_2 - A_1|\Big[1 + \max\Big(\|\nabla W_1\|_{L^{\frac{ap}{2-p}}}, \|\nabla W_2\|_{L^{\frac{ap}{2-p}}}\Big)\Big]^{a}\int_U |W(x)|dx.$$

Let $\widetilde{M_1} = (|A_1| + 1)^{a/(1-a)}$, $\widetilde{M_2} = (|A_2| + 1)^{a/(1-a)}$ and $\widetilde{M} = \max(\widetilde{M_1}, \widetilde{M_2})$.

We take p so that $ap/(2-p) = 2-a$ which implies $p = 2-a > 1$. Therefore

$$\left(\int_U |\nabla W|^{2-a} dx\right)^{2/(2-a)} \le C\widetilde{M}|A_2 - A_1|\int_U |W(x)|dx.$$

which yields

$$\left(\int_U |\nabla(W_1 - W_2)|^{2-a}\, dx\right)^{\frac{2}{2-a}} \le C\, M\, |A_1 - A_2|\int_U |W_1 - W_2|dx, \tag{5.19}$$

From (5.19) and Poincare inequality we have

$$\begin{aligned}\left(\int_U |\nabla W|^{2-a} dx\right)^{\frac{2}{2-a}} &\le C\, M\, |A_1 - A_2|\int_U |W|dx\\ &\le C\, M\, |A_1 - A_2|\left(\int_U |\nabla W|^{2-a} dx\right)^{\frac{1}{2-a}},\end{aligned}$$

hence yielding (5.17).

Then (5.18) follows from (5.17) and Poincare-Sobolev's inequality. $\square$

Remark 5.6. The result obtained in Theorem 5.5 has a clear engineering interpretation and can be applied to evaluating the productivity index (PI) of a well. To illustrate this point, suppose that the flow of slightly compressible fluid is subject to g-Forchheimer momentum equation (2.6), and all assumptions used to derive equation (3.15) hold. In previous work ([7]), productivity index of the well for pseudo-steady state regime with constant total rate Q is calculated as

$$PI = \frac{Q|U|}{\int_U W(x)dx} \tag{5.20}$$

It is clear in case of linear Darcy flow that the PI does not depend on rate Q. On contrary, the PI of non-linear Forchheimer flows depends on Q and this fact must be taken into account. The result in Theorem 5.5 allows ones to explicitly estimate the PI of the well with respect to perturbation in Q. Let PI_1 and PI_2 are productivity indices corresponding to $Q_1 = Q$ and $Q_2 = Q + \Delta_Q$, with "relatively" small Δ_Q. Then we have

$$|PI_1 - PI_2| \le C(W, Q, |U|)|\Delta_Q|.$$

We will not study here applications of the developed framework to PI analysis, leaving this topic for a separate article.

6. Bounds for the Solutions

In the previous section we studied the PSS solutions which is reduced to (time-independent) elliptic BVP. Here we are investigating solutions of the (evolution) parabolic equations with two types of time-dependent boundary conditions. Namely we will consider the IBVP with: (1) given pressure values (Dirichlet data) on Γ_i, and (2) given total flux on Γ_i. The second problem does not, in general, has a unique solution. Therefore we will restrain the boundary data to a certain class. We will derive *a priori* bounds for ∇p in appropriate L^q norms, where the exponent q explicitly depends on the degree of the function g. This study is important by itself and it will also be used in subsequent sections in the analysis of long-time dynamics of the non-linear process in porous media flows.

Consider a solution $p(x,t)$ to either IBVP-I (4.1) or IBVP-II (4.2) as in Sec. 4. For our convenience, we recall the equations that $p(x,t)$ satisfies

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p), \quad \text{in} \quad U, \quad t > 0, \tag{6.1}$$

$$p(x,0) = p_0(x), \quad \text{in} \quad U, \tag{6.2}$$

$$\frac{\partial p}{\partial N} = 0 \quad \text{on} \quad \Gamma_e. \tag{6.3}$$

For IBVP-I we study the following particular Dirichlet data on Γ_i:

$$p(x,t) = \gamma(t) + \varphi(x) \quad \text{on} \quad \Gamma_i, \quad t > 0, \tag{6.4}$$

where the function $\varphi(x)$ is defined for $x \in \Gamma_i$ and satisfies

$$\int_{\Gamma_i} \varphi(x) d\sigma = 0. \tag{6.5}$$

We call $(\gamma(t), \varphi(x))$ the boundary profile with temporal component $\gamma(t)$ and spatial component $\varphi(x)$.

We say that $\gamma(t)$ is a PSS temporal profile if $\gamma(t) = -At + B$ for some numbers A, B.

For IBVP-II, the total flux condition is

$$-\int_{\Gamma_i} K(|\nabla p|)\nabla p \cdot N ds = Q(t). \tag{6.6}$$

Note that the condition (6.5) is imposed to guarantee the uniqueness of the splitting (6.4).

By virtue of the boundary constraints (6.4) and (6.5) one has for $t > 0$ that

$$\gamma(t) = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} \gamma(t) d\sigma = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} p(x,t) - \varphi(x) d\sigma = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} p(x,t) d\sigma. \tag{6.7}$$

The function $\gamma(t)$ considered in this and next sections is assumed to satisfy:

$$\gamma(t) \in C([0,\infty)) \quad \text{and} \quad \gamma'(t) \in L^2_{loc}([0,\infty)). \tag{6.8}$$

Definition 6.1. *Depending on what data are available we classify the solutions as follows.*

- *We say that $p(x,t)$ is a solution of IBVP-I type (S), or **IBVP-I(S)**, if it satisfies (6.1), (6.2), (6.3), (6.4) and (6.5) with given $p_0(x)$, $\gamma(t)$ and $\varphi(x)$.*
- *We say that $p(x,t)$ is a solution of IBVP-II type (S), or **IBVP-II(S)**, if it satisfies (6.1), (6.2), (6.3), and (6.6), with given $p_0(x)$ and $Q(t)$; also the values of $p(x,t)$ on Γ_i have the form (6.4) and (6.5), where $\gamma(t)$ and $\varphi(x)$ are not necessarily given.*
- *We say that $p(x,t)$ is a solution of IBVP type (S), or **IBVP-(S)**, if it is a solution of either IBVP-I(S) or IBVP-II(S).*

6.1. **Solutions of IBVP-I type (S).** We will derive *a priori* estimate for solutions of IBVP-I(S). The following function $H(x,t)$ is used in the derivation.

Definition 6.2. *For any function $p(x,t)$ we define $H[p](x,t)$ by:*

$$H[p](x,t) = \int_0^{|\nabla p(x,t)|^2} K(\sqrt{s}) ds, \tag{6.9}$$

for $(x,t) \in U \times [0,\infty)$.

The function $H[p]$ can be compared with $|\nabla p|$ as follows.

Claim: For any (x,t) one has

$$K(|\nabla p(x,t)|)|\nabla p(x,t)|^2 \le H[p](x,t) \le 2K(|\nabla p(x,t)|)|\nabla p(x,t)|^2. \tag{6.10}$$

Indeed, on one hand, the function $K(\sqrt{s})$ is decreasing, by (3.16), hence one has

$$H[p](x,t) \ge \int_0^{|\nabla p(x,t)|^2} K(|\nabla p(x,t)|) ds = K(|\nabla p(x,t)|)|\nabla p(x,t)|^2.$$

On the other hand, by setting the variable $\xi = \sqrt{s}$ in (6.9) and using the increasing property of $K(\xi)\xi$ (see (3.17)) one has

$$\begin{aligned} H[p](x,t) &= \int_0^{|\nabla p(x,t)|} 2\xi K(\xi) d\xi \le \int_0^{|\nabla p(x,t)|} 2|\nabla p(x,t)| K(|\nabla p(x,t)|) ds \\ &\le 2K(|\nabla p(x,t)|)|\nabla p(x,t)|^2. \end{aligned}$$

Note: Also, the decrease of $K(s)$ directly implies $H[p](x,t) \le K(0)|\nabla p(x,t)|^2$.

Moreover, if $g(s)$ satisfies the Lambda-Condition then by Ineq. (6.10) above and Ineq. (3.34) in Lemma 3.10, there are positive constants C_0 and C_1 such that

$$C_0|\nabla p(x,t)|^{2-a} - 1 \le H[p](x,t) \le C_1|\nabla p(x,t)|^{2-a}. \tag{6.11}$$

Theorem 6.3. *Let $p(x,t)$ be a solution of IBVP-I(S) with the boundary profile $(\gamma(t),\varphi(x))$. Then one has for all $t \geq 0$ that*

$$\int_U K\left(|\nabla p(x,t)|\right)|\nabla p(x,t)|^2\,dx \leq 2\int_U K\left(|\nabla p(x,0)|\right)|\nabla p(x,0)|^2\,dx + |U|\int_0^t \left(\gamma\prime(\tau)\right)^2 d\tau - \frac{1}{|U|}\int_0^t Q^2(\tau)d\tau, \tag{6.12}$$

where $Q(t)$ is defined by (6.6).

If, in addition, $g(s)$ belongs to class (GPPC), then one has

$$\int_U |\nabla p(x,t)|^{2-a}dx \leq C_1 + C_2\int_U |\nabla p(x,0)|^{2-a}dx + C_3\int_0^t \left(\gamma\prime(\tau)\right)^2 d\tau - C_4\int_0^t Q^2(\tau)d\tau. \tag{6.13}$$

Proof. Multiply Eq. (6.1) by $\frac{\partial p}{\partial t}$ and integrate over the domain U:

$$\int_U \left(\frac{\partial p}{\partial t}\right)^2 dx = -\int_U K(|\nabla p|)\nabla p\cdot\frac{\partial}{\partial t}(\nabla p)dx + \int_{\Gamma_i} K(|\nabla p|)(\nabla p\cdot N)\frac{\partial p}{\partial t}ds$$
$$= -\int_U K(|\nabla p|)\nabla p\cdot\frac{\partial}{\partial t}(\nabla p)dx - Q(t)\cdot\gamma\prime(t). \tag{6.14}$$

Note that

$$K(|\nabla p|)\nabla p\cdot\frac{\partial}{\partial t}(\nabla p) = \frac{1}{2}\frac{\partial}{\partial t}\int_0^{|\nabla p|^2} K(\sqrt{s})ds = \frac{1}{2}\frac{\partial}{\partial t}H(x,t), \tag{6.15}$$

where $H = H[p]$ defined by (6.9).

Integrating Eq. (6.1) over U, one finds the relation

$$\frac{d}{dt}\int_U p(x,t)dx = \int_U \frac{\partial P(x,t)}{\partial t}dx = -Q(t). \tag{6.16}$$

By Holder's inequality

$$Q^2(t) = \left(\int_U \frac{\partial p}{\partial t}dx\right)^2 \leq |U|\int_U \left(\frac{\partial p}{\partial t}\right)^2 dx. \tag{6.17}$$

It follows from (6.14), (6.15) and (6.17) that

$$\frac{1}{2}\int_U \frac{\partial}{\partial t}H(t,x)dx = -\int_U \left(\frac{\partial p}{\partial t}\right)^2 dx - \gamma\prime(t)\cdot Q(t) \leq -\frac{Q^2(t)}{|U|} - \gamma\prime(t)\cdot Q(t). \tag{6.18}$$

Applying Cauchy's inequality to $\gamma\prime(t)\cdot Q(t)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_U H(t,x)dx \leq \frac{|U|}{2}\cdot|\gamma\prime(t)|^2 - \frac{Q^2(t)}{2|U|}, \tag{6.19}$$

and thus

$$\int_U H(t,x)dx \leq \int_U H(0,x)dx + |U|\int_0^t |\gamma\prime(\tau)|^2 d\tau - \frac{1}{|U|}\int_0^t Q^2(\tau)d\tau. \tag{6.20}$$

Using (6.10) to estimate $H(x,t)$ and $H(x,0)$ in (6.20), one obtains (6.12).
On the other hand, using (6.11) instead of (6.10) in (6.20) yields (6.13). □

6.2. **Solutions of IBVP-II type (S).** As a consequence of Proposition 4.5 in Sec. 4, each solution of IBVP-II(S) is unique. Here we estimate $\int_U |\nabla p(x,t)|^{2-a}dx$ in terms of $Q(t)$ but not $\gamma(t)$.

Theorem 6.4. *Let $p(x,t)$ be a solution of IBVP-II(S) with total flux $Q(t)$. Assume that $Q(t) \in C^1([0,\infty))$. Then for any $\delta > 0$, one has*

(6.21)
$$\int_U |\nabla p(x,t)|^{2-a}dx \le e^{\delta t}\int_U |\nabla p(x,0)|^{2-a}dx + \int_0^t e^{\delta(t-\tau)}\Big(\Lambda^*(\tau) - C_1 h_2(\tau)\Big)d\tau,$$

for any $t \ge 0$, where

(6.22)
$$\Lambda^*(t) = C_2 L_2 + C_3|Q(t)|^{\frac{2-a}{1-a}} + L_0 h_0(t) + L_1 h_1(t) + 2h_0(t)h_1(t) + C_\delta h_3(t),$$

the functions $h_i(t)$, $i = 0,1,2,3$ are defined by

(6.23)
$$h_0(t) = \int_0^t |Q(\tau)|d\tau, \quad h_1(t) = \int_0^t |Q'(\tau)|d\tau,$$
$$h_2(t) = \int_0^t Q^2(\tau)d\tau, \quad h_3(t) = \int_0^t |Q'(\tau)|^{\frac{2-a}{1-a}}d\tau,$$

the positive numbers L_0, L_1, L_2 depend on the initial data and are given by

(6.24)
$$L_0 = |Q(0)|,\ L_1 = \Big|\int_U p(x,0)dx\Big|,\ L_2 = 1 + L_0^{\frac{2-a}{1-a}} + L_1^{2-a} + \int_U |\nabla p(x,0)|^{2-a}dx,$$

and C_1, C_2, C_3, C_δ are positive constants .

Proof. Let

(6.25)
$$I(t) = \frac{1}{2}\int_U H(x,t)dx, \quad J(t) = \int_U |\nabla p(x,t)|^{2-a}dx.$$

By (6.11), one has

(6.26)
$$C_0(J(t) - 1) \le I(t) \le C_1 J(t).$$

From (6.18) above:

(6.27)
$$\frac{d}{dt}I(t) \le -\frac{Q^2(t)}{|U|} - \gamma\prime(t)Q(t).$$

Integrating this inequality from 0 to t and then integrating by parts the last term give

(6.28)
$$I(t) - I(0) \le -\frac{\int_0^t Q^2(\tau)d\tau}{|U|} - \gamma(t)Q(t) + \gamma(0)Q(0) + \int_0^t \gamma(\tau)Q'(\tau)d\tau.$$

We need to estimate $\gamma(t)$ in terms of $Q(t)$. Using the formula of $\gamma(t)$ in (6.7) and applying Poincare's inequality, one obtains

$$|\gamma(t)| \le C\int_{\Gamma_i} |p(x,t)|ds \le C\int_U |\nabla p(x,t)|dx + C\int_U |p(x,t)|dx$$
$$\le C\int_U |\nabla p(x,t)|dx + C\left(\int_U |\nabla p(x,t)|dx + \Big|\int_U p(x,t)dx\Big|\right).$$

Clearly from (6.16), $\int_U p(x,t)dx = \int_U p(x,0)dx - \int_0^t Q(\tau)d\tau$. Then one continues the above estimate as

$$
\begin{aligned}
|\gamma(t)| &\le C\int_U |\nabla p(x,t)|dx + C\left|\int_U p(x,0)dx\right| + C\left|\int_0^t Q(\tau)d\tau\right| \\
&\le CJ(t)^{\frac{1}{2-a}} + C\ell_1 + Ch_0(t),
\end{aligned}
\tag{6.29}
$$

where $\ell_1 = \left|\int_U p(x,0)dx\right| = L_1$.

Combining this estimate of $\gamma(t)$ with (6.28) and (6.26), one obtains

$$
\begin{aligned}
J(t) \le -Ch_2(t) + C\ell_2 + C\Big(J(t)^{\frac{1}{2-a}} + \ell_1 + h_0(t)\Big)|Q(t)| \\
+ C\int_0^t \Big(J(\tau)^{\frac{1}{2-a}} + \ell_1 + h_0(\tau)\Big)|Q'(\tau)|d\tau,
\end{aligned}
$$

where $\ell_2 = I(0) + |\gamma(0)Q(0)| + 1$.

Note that one can estimate $|\gamma(0)|$ by using (6.29):

$$
|\gamma(0)| = \lim_{t\searrow 0}|\gamma(t)| \le CJ(0)^{\frac{1}{2-a}} + C\ell_1,
\tag{6.30}
$$

and hence

$$
\ell_2 \le C(I(0) + |Q(0)|(J(0)^{\frac{1}{2-a}} + \ell_1) + 1) \le C(|Q(0)|^{\frac{2-a}{1-a}} + I(0) + J(0) + \ell_1^{2-a} + 1).
$$

This yields $\ell_2 \le C\ell_3$ where $\ell_3 = |Q(0)|^{\frac{2-a}{1-a}} + J(0) + \ell_1^{2-a} + 1$.

Let $\delta > 0$ be fixed. By Young's inequality, one derives

$$
\begin{aligned}
J(t) \le -Ch_2(t) + C\ell_3 + \big(\frac{1}{2}J(t) + C|Q(t)|^{\frac{2-a}{1-a}}\big) + (\ell_1 + h_0(t))|Q(t)| \\
+ \int_0^t \delta J(\tau) + C_\delta |Q'(\tau)|^{\frac{2-a}{1-a}}d\tau + \int_0^t (\ell_1 + h_0(\tau))|Q'(\tau)|d\tau.
\end{aligned}
$$

Therefore one obtains

$$
J(t) \le -Ch_2(t) + \delta\int_0^t J(\tau)d\tau + \Lambda_*(t).
\tag{6.31}
$$

where

$$
\begin{aligned}
\Lambda_*(t) = C\ell_3 + C|Q(t)|^{\frac{2-a}{1-a}} + (\ell_1 + h_0(t))|Q(t)| \\
+ \int_0^t (\ell_1 + h_0(\tau))|Q'(\tau)|d\tau + C_\delta\int_0^t |Q'(\tau)|^{\frac{2-a}{1-a}}d\tau.
\end{aligned}
\tag{6.32}
$$

Note that

$$
\ell_1|Q(t)| \le \ell_1^{2-a} + |Q(t)|^{\frac{2-a}{1-a}},
\tag{6.33}
$$

$$
h_0(t)|Q(t)| = h_0(t)\Big|Q(0) + \int_0^t Q'(\tau)d\tau\Big| \le h_0(t)|Q(0)| + h_0(t)h_1(t),
\tag{6.34}
$$

$$
\int_0^t (\ell_1 + h_0(\tau))|Q'(\tau)|d\tau \le \ell_1 h_1(t) + h_0(t)h_1(t).
\tag{6.35}
$$

Hence $\Lambda_*(t) \le \Lambda^*(t)$, where $\Lambda^*(t)$ is defined by (6.22).

Applying Gronwall's inequality to (6.31) with $\Lambda^*(t)$ replacing $\Lambda_*(t)$ gives

$$
J(t) \le J(0)e^{\delta t} + \int_0^t e^{\delta(t-\tau)}\Big(\Lambda^*(\tau) - Ch_2(\tau)\Big)d\tau,
$$

which yields Ineq. (6.21). □

6.3. **Comparing solutions of IBVP type (S).** The estimate in the Sec. 6.2 is adequate to establish the dependence of the solutions to IBVP-II(S) on the total flux in *finite* time intervals (see Theorem 8.5 in Section 8.2 below). However, due to its exponential growth, it does not imply the asymptotic stability of the solutions. The estimate can be improved in some instances when additional information is provided, for example, when a "related" solution $\bar{p}(x,t)$ of IBVP-I(S) is known, and the total flux has some monotone properties.

Here we will estimate a solution $p(x,t)$ of IBVP-II(S) using a known solution $\bar{p}(x,t)$ of IBVP-I(S) having the same total flux $Q(t)$. The solution $\bar{p}(x,t)$ is called base line solution to IBVP-II(S) with respect to $Q(t)$.

Theorem 6.5. *Let $g(s)$ be of class (GPPC). Let $p_\gamma(x,t)$ be a solution of IBVP-I(S) with known total flux $Q(t)$ and known boundary profile $(\gamma(t),\varphi(x))$. Let $p(x,t)$ be a solution of IBVP-II(S) with total flux $Q(t)$ and boundary profile $(B(t),\varphi(x))$, where $B(t)$ is not given but bounded from above. Suppose that $Q\in C^1([0,\infty))$, $Q'(t)\ge 0$ and $B(t)\le B_0<\infty$. Then*

$$\int_U |\nabla p(x,t)|^{2-a}dx \le -C_1h_2(t) + C_2\Big(\int_U |\nabla p_\gamma(x,t)|^{2-a}dx + |Q(t)|^{\frac{2-a}{1-a}} + |Q(t)||\gamma(t)|\Big) + C_3L_0, \tag{6.36}$$

where $h_2(t)$ is defined in (6.23) and

$$L_0 = 1+|B(0)-B_0||Q(0)|+|B_0|^{2-a}+\int_U |\nabla p(x,0)|^{2-a}dx+\Big|\int_U p(x,0)-p_\gamma(x,0)dx\Big|^{2-a}. \tag{6.37}$$

Consequently,

$$\int_U |\nabla p(x,t)|^{2-a}dx \le -C_4h_2(t) + C_5\Big(\int_0^t |\gamma'(\tau)|^2d\tau + |Q(t)|^{\frac{2-a}{1-a}} + |Q(t)||\gamma(t)|\Big) + C_6L_1, \tag{6.38}$$

where

$$L_1 = L_0 + \int_U |\nabla p_\gamma(x,0)|^{2-a}dx. \tag{6.39}$$

Proof. Let $I(t)$ and $J(t)$ be defined as in (6.25). Applying (6.28) to the solution $p(x,t)$ with $B(t)$ replacing $\gamma(t)$, one has

$$\begin{aligned} I(t) &\le I(0) - Ch_2(t) - B(t)Q(t) + B(0)Q(0) + \int_0^t B(\tau)\cdot Q'(\tau)d\tau \\ &\le I(0) - Ch_2(t) - B(t)Q(t) + B(0)Q(0) + \int_0^t B_0\cdot Q'(\tau)d\tau \\ &= I(0) - Ch_2(t) + (B_0-B(t))Q(t) + (B(0)-B_0)Q(0). \end{aligned}$$

Letting $L_2 = I(0) + |B(0)-B_0||Q(0)|$, one obtains

$$I(t) \le CL_2 - Ch_2(t) + (|B_0|+|B(t)|)|Q(t)|. \tag{6.40}$$

We now evaluate $|B(t)|$ through $\gamma(t)$, $\int_U |\nabla p|^{2-a}dx$. Applying the trace theorem and then Poincare's inequality, one gets

$$(6.41)\qquad |B(t)| = \frac{1}{|\Gamma_i|}\left|\int_{\Gamma_i} p(x,t)d\sigma\right| \le C_1\int_U |\nabla p|dx + C_2\left|\int_U pdx\right|.$$

Next, from Lemma 4.3 it follows that

$$(6.42)\qquad \left|\int_U p(x,t)dx\right| \le \left|\int_U p_\gamma(x,t)dx\right| + \left|\int_U (p(x,0)-p_\gamma(x,0))dx\right| \le B_1(t) + Z_1,$$

where $Z_1 = \left|\int_U (p(x,0)-p_\gamma(x,0))dx\right|$ and $B_1(t) = \int_U |p_\gamma(x,t)|dx$. Then

$$(6.43)\qquad |B(t)| \le C\int_U |\nabla p|dx + CB_1(t) + CZ_1 \le CJ(t)^{\frac{1}{2-a}} + CB_1(t) + CZ_1.$$

Combining this with (6.40) yields

$$(6.44)\qquad I(t) \le -Ch_2(t) + C(J(t)^{\frac{1}{2-a}} + B_1(t) + L_3)|Q(t)| + CL_2,$$

where $L_3 = Z_1 + |B_0|$. Thus by Young's inequality

$$(6.45)\qquad I(t) \le -Ch_2(t) + \varepsilon J(t) + C|Q(t)|^{\frac{2-a}{1-a}} + CB_1(t)|Q(t)| + CL_4,$$

where $L_4 = L_2 + L_3^{2-a}$. Then using Ineq. (6.26) and taking ε sufficiently small, one obtains

$$(6.46)\qquad J(t) \le -Ch_2(t) + C|Q(t)|^{\frac{2-a}{1-a}} + CB_1(t)|Q(t)| + CL_5,$$

where $L_5 = 1 + L_4$.

To estimate $B_1(t)$, one uses Poincare-Sobolev inequality (e.g. [23], the space $W^{1,2-a}(U)$ is compactly embedded into $L^1(U)$) and relation (6.7):

$$\begin{aligned} B_1(t) &\le C\int_U |p_\gamma|dx \le C\left(\int_U |\nabla p_\gamma|^{2-a}dx\right)^{\frac{1}{2-a}} + C\left|\int_{\Gamma_i} p_\gamma(x,t)d\sigma\right| \\ &\le C\left(\int_U |\nabla p_\gamma|^{2-a}dx\right)^{\frac{1}{2-a}} + C|\gamma(t)|. \end{aligned}$$

Hence

$$(6.47)\quad J(t) \le -Ch_2(t) + C|Q(t)|^{\frac{2-a}{1-a}} + C\|\nabla p_\gamma\|_{L^{2-a}}|Q(t)| + C|Q(t)||\gamma(t)| + CL_5.$$

Thus applying Young's inequality to the third term on the RHS yields

$$\begin{aligned}(6.48)\qquad \int_U |\nabla p(x,t)|^{2-a}dx \le -Ch_2(t) + C\int_U |\nabla p_\gamma(x,t)|^{2-a}dx + C|Q(t)|^{\frac{2-a}{1-a}} \\ + C|Q(t)\gamma(t)| + CL_5.\end{aligned}$$

Estimating L_5 gives $L_5 \le CL_0$, hence (6.36) follows (6.48). Then utilizing estimate (6.13) for $|\nabla p_\gamma|$ in (6.36), one obtains (6.38). □

In case $p_\gamma(x,t)$ is a PSS solution, a sharper estimate is obtained below.

Theorem 6.6. *Let $g(s)$ belong to class (GPPC). Let $p(x,t)$ be a solution of IBVP-II(S) with total flux $Q(t) \equiv \bar{Q} = const.$, with the boundary profile $(B(t),\varphi(x))$*

satisfying $\varphi(x) = 0$. *Assume the basic PSS profile* $W(x)$ *corresponding to* $\bar{Q}/|U|$ *exists. Then there is a positive constant* C *such that for all* $t \geq 0$ *one has*

$$\int_U K(|\nabla p(x,t)|)|\nabla p(x,t)|^2 dx \leq C\Big(1 + \int_U K(|\nabla p(x,0)|)|\nabla p(x,0)|^2 dx + \int_U K(|\nabla W(x)|)|\nabla W(x)|^2 dx\Big), \tag{6.49}$$

or equivalently,

$$\int_U |\nabla p(x,t)|^{2-a} dx \leq C\left(1 + \int_U |\nabla p(x,0)|^{2-a} dx + \int_U |\nabla W(x)|^{2-a} dx\right). \tag{6.50}$$

Consequently,

$$\int_U |\nabla p(x,t)|^{2-a} dx \leq C\left(1 + \int_U |\nabla p(x,0)|^{2-a} dx + |\bar{Q}|^{\frac{2-a}{1-a}}\right). \tag{6.51}$$

Proof. Let $\gamma(t) = -t\bar{Q}/|U|$ and let $p_\gamma(t) = \gamma(t) + W(x)$ be the corresponding PSS solution with the total flux $Q_\gamma(t) = Q(t) = \bar{Q}$.

As in the proof of Lemma 6.5, one has

$$0 \leq \int_U (\partial_t p - \partial_t p_\gamma)^2 dx = -\int_U (K(|\nabla p|)\nabla p - K(|\nabla p_\gamma|)\nabla p_\gamma)\cdot(\partial_t \nabla p - \partial_t \nabla p_\gamma)dx + \int_{\Gamma_i} (K(|\nabla p|)\nabla p - K(|\nabla p_\gamma|)\nabla p_\gamma)\cdot N(\partial_t p - \partial_t p_\gamma)d\sigma.$$

One easily gets

$$\begin{aligned} 0 \leq &-\int_U (K(|\nabla p|)\nabla p \cdot \partial_t \nabla p dx - \int_U K(|\nabla p_\gamma|)\nabla p_\gamma \cdot \partial_t \nabla p_\gamma dx \\ &+ \int_U (K(|\nabla p|)\nabla p \cdot \partial_t \nabla p_\gamma dx + \int_U K(|\nabla p_\gamma|)\nabla p_\gamma \cdot \partial_t \nabla p dx \\ &+ (B'(t) - \gamma'(t))(Q_\gamma(t) - Q(t)). \end{aligned} \tag{6.52}$$

Note that $\nabla p_\gamma = \nabla W$ and $\partial_t \nabla p_\gamma = 0$. Let $H(x,t) = H[p](x,t)$ and $H_\gamma(x,t) = H[p_\gamma](x,t)$ be defined as in (6.9). Then

$$\begin{aligned} 0 &\leq -\frac{1}{2}\partial_t \int_U H dx - 0 + 0 + \partial_t \int_U K(|\nabla W|)\nabla W \cdot \nabla p dx + 0 \\ &= -\frac{1}{2}\partial_t \int_U H(x,t)dx + \partial_t \int_U K(|\nabla W|)\nabla W \cdot \nabla p dx. \end{aligned}$$

Integrating this inequality from 0 to t, we obtain

$$\frac{1}{2}\int_U H(x,t)dx \leq \frac{1}{2}\int_U H(x,0)dx + \int_U K(|\nabla W|)\nabla W \cdot \nabla p(x,t)dx - \int_U K(|\nabla W|)\nabla W \cdot \nabla p(x,0)dx.$$

Applying Lemma 3.10 to function $H(x,t)$ in LHS of the inequality above and Young's inequality with to the term $K(|\nabla W|)\nabla W\cdot\nabla p$, one gets

$$\frac{1}{2}\int_U K(|\nabla p|)|\nabla p|^2dx \le \int_U K(|\nabla p(x,0)|)|\nabla p(x,0)|^2dx$$
$$+\varepsilon\int_U |\nabla p|^{2-a}dx + C\int_U |\nabla p(x,0)|^{2-a}dx + C\int_U |K(|\nabla W|)\nabla W|)^{\frac{2-a}{1-a}}\,dx.$$

By (3.34), one finally obtains

$$\begin{aligned}\frac{1}{2}\int_U K(|\nabla p|)|\nabla p|^2dx &\le C\int_U K(|\nabla p(x,0)|)|\nabla p(x,0)|^2dx + \varepsilon\int_U K(|\nabla p|)|\nabla p|^2dx\\ &\quad + C\int_U (|\nabla W|^{1-a})^{\frac{2-a}{1-a}}dx + C\\ &\le C\varepsilon\int_U K(|\nabla p|)|\nabla p|^2dx + C\int_U K(|\nabla p(x,0)|)|\nabla p(x,0)|^2dx\\ &\quad + C\int_U K(|\nabla W(x)|)|\nabla W(x)|^2dx + C.\end{aligned}$$

Letting ε be sufficiently small, one obtains (6.49). With (6.50), one uses (3.34) again to obtain (6.50).

The Ineq. (6.51) simply follows (6.50) and the estimate (5.16) of the solution W in Section 5. □

As a consequence, we obtain an improvement of Theorem 6.3 for the special case of PSS boundary profile.

Corollary 6.7. *Let $p(x,t)$ be a solution to IBVP-I(S) with the PSS boundary profile, i.e., the boundary profile $(B(t),\varphi(x))$ satisfies $B'(t)=-A$ and $\varphi(x)=0$. Assume that the basic PSS solution corresponding to A exists. Then one has for any $t\ge 0$ that*

$$\int_U |\nabla p(x,t)|^{2-a}dx \le C\left(1+\int_U |\nabla p(x,0)|^{2-a}dx + |A|^{\frac{2-a}{1-a}}\right). \tag{6.53}$$

Proof. Proceed as in the proof of Theorem 6.6. Note that the last term in (6.52) vanishes because $B'(t)=\gamma'(t)=-A$. We omit the details. □

7. Asymptotic Stability

In this section we study the stability of IBVP-I(S) and IBVP-II(S). Their Lyapunov stability is already a consequence of Propositions 4.4 and 4.5 in Section 4.

Theorem 7.1. *The IBVP-I(S) and IBVP-II(S) are Lyapunov stable with respect to the L^2 norm. More specifically, if p_1 and p_2 are two solutions of the same IBVP-(S), then*

$$\|p_1(\cdot,t)-p_2(\cdot,t)\|_{L^2(U)} \le \|p_1(\cdot,0)-p_2(\cdot,0)\|_{L^2(U)}, \tag{7.1}$$

for all $t\ge 0$.

We now focus on the asymptotic stability. For this, unlike the Lyapunov stability in Theorem 7.1, the nonlinear function $g(s)$ will be restricted to the class (GPPC).

Let us start with notations and assumptions used henceforward:

The function $g(s)$ belongs to class (GPPC), $a = \deg(g)/(1 + \deg(g))$ and $b = a/(2 - a)$.

Two solutions $p_k(x,t)$, $(k = 1, 2)$, of IBVP-(S) have boundary profiles $(\gamma_k(t), \varphi_k(x))$, and the total flux $Q_k(t)$, with $\varphi_1(x) = \varphi_2(x) = \varphi(x)$.

For simplicity, we assume that

$$\gamma_k(t),\ Q_k(t) \in C^1([0,\infty)), \quad k = 1, 2. \tag{7.2}$$

The difference of two solutions is $z(x,t) = p_1(x,t) - p_2(x,t)$.

The differences of boundary data are denoted by:

$$\begin{aligned} \Delta_\gamma(t) &= \gamma_1(t) - \gamma_2(t), & \Delta'_\gamma(t) &= \gamma'_1(t) - \gamma'_2(t), \\ \Delta_Q(t) &= Q_1(t) - Q_2(t), & \Delta'_Q(t) &= Q'_1(t) - Q'_2(t). \end{aligned} \tag{7.3}$$

We will establish various estimates for $\int_U z^2(x,t)dx$ for $t \geq 0$ under different boundary conditions.

First, we derive a general differential inequality which will be applied to different scenarios both in this section and the next one.

Lemma 7.2. *One has for all $t \geq 0$,*

$$\frac{1}{2}\frac{d}{dt}\int_U z^2(x,t)dx \leq -C\left[\int_U |\nabla z(x,t)|^q dx\right]^{\frac{2}{q}} N(t)^{-b} - \Delta_\gamma(t)\Delta_Q(t), \tag{7.4}$$

where $1 \leq q \leq 2 - a$ and

$$N(t) = 1 + \int_U |\nabla p_1(x,t)|^{2-a}dx + \int_U |\nabla p_2(x,t)|^{2-a}dx. \tag{7.5}$$

Proof. First, one easily derives

$$\frac{1}{2}\frac{d}{dt}\int_U z^2 dx = -\int_U \Phi(\nabla p_1, \nabla p_2)dx - \Delta_\gamma(t)\Delta_Q(t). \tag{7.6}$$

Then applying Ineq. (3.36) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_U z^2 dx \\ &\leq -C\left[\int_U |\nabla z|^q dx\right]^{\frac{2}{q}} [1 + \max(\|\nabla p_1\|_{L^{ap/(2-p)}}, \|\nabla p_2\|_{L^{ap/(2-p)}})]^{-a} - \Delta_\gamma(t)\Delta_Q(t) \\ &\leq -C\left[\int_U |\nabla z|^q dx\right]^{\frac{2}{q}} \left[1 + \|\nabla p_1\|^{2-a}_{L^{2-a}} + \|\nabla p_2\|^{2-a}_{L^{2-a}}\right]^{\frac{-a}{2-a}} - \Delta_\gamma(t)\Delta_Q(t). \end{aligned}$$

which proves Ineq. (7.4). Above, we imposed the condition

$$aq/(2-q) \leq 2 - a, \quad \text{which is equivalent to,} \quad q \leq 2 - a. \tag{7.7}$$ □

As usual, we start with IBVP-I.

Theorem 7.3. *Assume that* $\deg(g) \leq \frac{4}{d-2}$. *Suppose $p_1(x,t)$, $p_2(x,t)$ are two solutions of IBVP-I(S) with the same boundary profile $(\gamma(t), \varphi(x))$. Then*

$$\|p_1(x,t) - p_2(x,t)\|_{L_2(U)} \leq \|p_1(x,0) - p_2(x,0)\|_{L_2(U)} \cdot \exp\left[-C\int_0^t \Lambda^{-b}(\tau)d\tau\right], \tag{7.8}$$

for all $t \geq 0$*, where*

$$\Lambda(t) = 1 + \int_U |\nabla p_1(x,0)|^{2-a}dx + \int_U |\nabla p_2(x,0)|^{2-a}dx + \int_0^t |\gamma'(\tau)|^2 d\tau.$$

Proof. First, $N(t)$ in (7.5) can be bounded by using the estimate (6.13) for each solution p_1, p_2:

$$N(t) \leq C\Lambda(t). \tag{7.9}$$

Then apply Lemma 7.2 with $\gamma_1(t) = \gamma_2(t) = \gamma(t)$ and use (7.9), one gets

$$\frac{1}{2}\frac{d}{dt}\int_U z^2 dx \leq -C\left[\int_U |\nabla z|^q dx\right]^{\frac{2}{q}} \Lambda(t)^{-\frac{a}{2-a}}, \tag{7.10}$$

where C is independent of the solutions p_1 and p_2.

Further we apply Sobolev's inequality (e.g. [23]) to function z with $z|_{\Gamma_i} = 0$ to have:

$$\int_U z^2 dx \leq C\left[\int_U |\nabla z|^q\right]^{\frac{2}{q}}, \tag{7.11}$$

with p satisfying:

$$2 = \frac{d\cdot p}{d-p}, \quad \text{equivalently,} \quad q = \frac{2d}{d-2}. \tag{7.12}$$

From (7.10)

$$\frac{1}{2}\frac{d}{dt}I(t) \leq -C\, I(t)\, \Lambda(t)^{-b}, \tag{7.13}$$

where $I(t) = \int_U z^2(x,t)dx$ and consequently

$$I(t) \leq I(0)\cdot \exp(-C\int_0^t \Lambda^{-b}(t)dt).$$

Now from the relations (7.12) and (7.7), on finds that

$$a \leq \frac{4}{d+2}, \quad \text{or equivalently,} \quad \deg(g) \leq \frac{4}{d-2}. \tag{7.14}$$

□

Corollary 7.4. *If* $\int_0^t |\gamma'(\tau)|^2 d\tau = O(t^r)$ *as* $t \to \infty$*, for* $k = 1, 2$*, and for some* $0 < r < 1/b$*, then*

$$\|z(x,t)\|_{L^2(U)} \leq C_1 e^{-C_2 t^{\varepsilon}}\|z(0)\|_{L^2(U)}, \tag{7.15}$$

where $\varepsilon = (1 - rb) > 0$*.*

Proof. By elementary calculations, one has $\Lambda(t) \leq C_1 t^{2\beta-1} + C_2$. Thus from Theorem 7.3 one obtains the desired result. □

In the following, we consider the case when $\gamma(t)$ is a generalized polynomial.

Example 7.5. Suppose $\gamma(t) = a_0 + a_1 t^{\beta}$, where $a_1 \neq 0$, for all $t > T$, where $T > 0$. Then

(1) if $\beta < 1/a$ then (7.15) holds for $\varepsilon = (1 - 2\beta)b + 1$;
(2) if $\beta = 1/a$ then $\|z(x,t)\|_{L^2} \leq C_2(1+t)^{-c}\|z(0)\|_{L^2}$ for some constant c.

In some cases, when either p_1 or p_2 is a known baseline solution, one can improve the above estimate.

Theorem 7.6. *Assume that* $\deg(g) \le \frac{4}{d-2}$. *Let* $p_\gamma(x,t)$ *be a known solution of IBVP-I(S) with the boundary profile* $(\gamma(t), \varphi(x))$ *and the total flux* $Q(t)$. *Let* $p(x,t)$ *be a solution of IBVP-II(S) with the boundary profile* $(B(t), \varphi(x))$ *and the total flux* $Q(t)$. *Assume* $B(t) \le B_0$ *and* $Q'(t) \ge 0$ *for all* $t \ge 0$.

(*i*) *One has for all* $t \ge 0$ *that*

$$\|p(t)-p_\gamma(t)-A_0\|_{L^2(U)} \le \|p(0)-p_\gamma(0)-A_0\|_{L^2(U)} \exp\left(-L\int_0^t \Psi^{-b}(\tau)d\tau\right), \tag{7.16}$$

where $A_0 = \int_U p(x,0)dx - \int_U p_\gamma(x,0)dx$, $L>0$ *depends on the initial data of the solutions* $p_\gamma(x,t)$ *and* $p(x,t)$, *and*

$$\Psi(t) = 1 + |Q(t)|^{\frac{2-a}{1-a}} + |Q(t)\gamma(t)| + \int_0^t |\gamma'(\tau)|^2 d\tau. \tag{7.17}$$

(*ii*) *If* p_γ *is a PSS solution, then one has*

$$\|p(t)-p_\gamma(t)-A_0\|_{L^2(U)} \le e^{-Lt}\|p_1(0)-p_\gamma(0)-A_0\|_{L^2(U)}. \tag{7.18}$$

Proof. Let $p_2(x,t) = p_\gamma(x,t)$ and $p_1(x,t) = p(x,t)$ then $z(x,t) = p(x,t) - p_\gamma(x,t)$.

(*i*) First we assume that $A_0 = 0$. Then from Lemma 4.3, one has $\int_U z(x,t)dx = 0$ for all $t \ge 0$.

In the below L_0, L_1, and L_2 are positive numbers depending on the initial data of the solutions $p_\gamma(x,t)$ and $p(x,t)$.

Using the estimates (6.13) for p_γ and (6.38) for p, one can bound $N(t)$ in (7.5) by: $N(t) \le L_0\Psi(t)$.

Then, applying Lemma 7.2 with $Q_1 = Q_2 = Q$, one has

$$\frac{1}{2}\frac{d}{dt}\int_U z^2 dx \le -L_1\left[\int_U |\nabla z|^q\right]^{\frac{2}{q}} \Psi(t)^{-b}, \tag{7.19}$$

where $q = 2d/(d-2)$. Clearly, q satisfies (7.7).

Applying Sobolev's inequality to $\bar z(x,t) = z(x,t) - (B(t) - \gamma(t))$ and noting that $\nabla \bar z = \nabla z$ and $\bar z|_{\Gamma_i} = 0$, one obtains

$$\int_U \Big|z(x,t) - (B(t)-\gamma(t))\Big|^2 dx = \int_U \bar z^2 dx \le C\Big(\int_U |\nabla \bar z|^q dx\Big)^{\frac{2}{q}} = C\Big(\int_U |\nabla z|^q dx\Big)^{\frac{2}{q}}.$$

Hence

$$\begin{aligned}\int_U z^2 dx &\le C\left(\int_U |\nabla z|^q dx\right)^{\frac{2}{q}} + 2(B(t)-\gamma(t))\int_U z(x,t)dx - (B(t)-\gamma(t))^2|U|\\ &= C\left(\int_U |\nabla z|^q dx\right)^{\frac{2}{q}} - (B(t)-\gamma(t))^2|U| \le C\left(\int_U |\nabla z|^q dx\right)^{\frac{2}{q}}.\end{aligned}$$

Therefore

$$\frac{d}{dt}\int_U z^2 dx \le -L_2\left(\int_U z^2 dx\right)\Psi^{-b}(t).$$

Hence Ineq. (7.16) follows by Gronwall's inequality.

For the general case, i.e. $A_0 \ne 0$, we replace p_γ by $p_\gamma + A_0$. Note that $Q(t)$ is the same, $\gamma(t)$ becomes $\gamma(t) + A_0$. All above estimates apply, with the constants now depending on A_0 as well. We omit the details.

(*ii*) Let p_γ be a PSS solution. Using Corollary 6.7, one estimates $N(t)$ in (7.5) and take $\Psi(t) = L$ instead of (7.17). Then (7.18) follows Ineq. (7.16). $\square$

8. Perturbed Boundary Value Problems

We consider the perturbed boundary problems of both IBVP-I(S) and IBVP-II(S). We will establish the continuous dependence of solutions on initial data and boundary data both on finite and infinite time intervals.

We use the same notation $g(s)$, a, b, $p_k(x,t)$, $\gamma_k(t)$, $Q_k(t)$, $(k = 1,2)$, and $\Delta_\gamma(t)$, $\Delta_Q(t)$, $z(x,t)$ as in the previous section. We will obtain the L^2 estimates which control the difference $(p_1 - p_2)$ in terms of the difference of boundary data, either $\Delta_\gamma(t)$ or $\Delta_Q(t)$. Under certain conditions on the boundary data, these deviations between two solutions with specific corrections due to boundary constraints are asymptotically small, and can vanish at infinity.

8.1. **IBVP-I type (S).**

Let p_1 and p_2 be two solutions of IBVP-I(S). We assume that for $k = 1, 2$:

$$\int_0^t |\gamma_k'(\tau)|^2 d\tau \le \lambda_0(t), \tag{8.1}$$

where $\lambda_0 \in C([0,\infty))$.

Under this condition, we first estimate the function $N(t)$ defined by (7.5) in terms of $\lambda_0(t)$ and initial data. Then from (7.5):

$$N(t) \le \Big(1 + \int_U |\nabla p_1(x,0)|^{2-a} dx + \int_U |\nabla p_2(x,0)|^{2-a} dx\Big)(1 + \lambda_0(t)), \tag{8.2}$$

hence

$$\frac{1}{N^b(t)} \ge \frac{A_1}{(1+\lambda_0(t))^b} = A_1 \Lambda_0(t)^{-1}, \tag{8.3}$$

where

$$A_1 = \Big(1 + \int_U |\nabla p_1(x,0)|^{2-a} dx + \int_U |\nabla p_2(x,0)|^{2-a} dx\Big)^{-b} \tag{8.4}$$

and

$$\Lambda_0(t) = (1 + \lambda_0(t))^b. \tag{8.5}$$

Let

$$Z(t) = \int_U z^2(x,t) dx, \quad F_1(t) = e^{-C_0 A_1 \int_0^t \Lambda_0^{-1}(\tau) d\tau}, \tag{8.6}$$

where $C_0 > 0$ is a constant independent of the solutions.

First, we estimate $Z(t)$ in terms of $\Delta_\gamma(t)$ and $\Delta_\gamma'(t)$.

Theorem 8.1. *Assume* $\deg(g) \le \frac{4}{d-2}$. *Let* $\overline{p}_k(x,t) = p_k(x,t) - \gamma_k(t)$ *for* $k = 1,2$. *Let*

$$\overline{z}(x,t) = \overline{p}_1(x,t) - \overline{p}_2(x,t), \quad \text{and} \quad \overline{Z}(t) = \int_U \overline{z}^2(x,t) dx. \tag{8.7}$$

Then one has for all $t \ge 0$ *that*

$$\overline{Z}(t) \le F_1(t)\overline{Z}(0) + C_1 A_1^{-1} F_1(t) \int_0^t \Lambda_0(\tau)(\Delta_\gamma'(\tau))^2 F_1^{-1}(\tau) d\tau. \tag{8.8}$$

Consequently,

$$Z(t) \le 2F_1(t)\overline{Z}(0) + 2C_2A_1^{-1}F_1(t)\int_0^t \Lambda_0(\tau)(\Delta'_\gamma(\tau))^2F_1^{-1}(\tau)d\tau + 2|\Delta_\gamma(t)|^2. \tag{8.9}$$

Proof. First, note that $\nabla\overline{p}_k = \nabla p_k$ and $\overline{z}|_{\Gamma_i} = 0$. Then similar to (7.6) one derives

$$\frac{1}{2}\frac{d}{dt}\int_U \overline{z}^2dx = -\int_U \Phi(\nabla\overline{p}_1, \nabla\overline{p}_2)dx - \Delta'_\gamma(t)\int_U \overline{z}dx. \tag{8.10}$$

Using Theorem 6.3 one can estimate $\int_U |\nabla\overline{p}_k(x,t)|^{2-a}\,dx = \int_U |\nabla p_k(x,t)|^{2-a}\,dx$.

Claim:

$$\frac{1}{2}\frac{d}{dt}\overline{Z} \le -\frac{CA_1}{\Lambda_0(t)}\overline{Z} + \int_U |\overline{z}\Delta'_\gamma(t)|dx. \tag{8.11}$$

The proof of (8.11) is similar to that of (7.13). Namely, first we apply Lemma 3.11 to the integral $\int_U \Phi(\nabla\overline{p}_1, \nabla\overline{p}_2)dx$ to obtain

$$\frac{1}{2}\frac{d}{dt}\overline{Z} \le -C\Big(\int_U |\nabla\overline{z}|^q dx\Big)^{2/q}N(t)^{-b} + \int_U |\overline{z}\Delta'_\gamma(t)|dx, \tag{8.12}$$

then estimate $N(t)^{-b}$ by using (8.3) and apply Poincare's inequality (7.11) to function $\overline{z}$.

Now, applying Cauchy's inequality to the last integral of (8.11) yields

$$\frac{1}{2}\frac{d}{dt}\overline{Z} \le -\frac{CA_1}{2\Lambda_0(t)}\overline{Z} + CA_1^{-1}(\Delta'_\gamma(t))^2\Lambda_0(t)|U|. \tag{8.13}$$

Thus, Ineq. (8.8) follows from Gronwall's inequality. Finally, Ineq. (8.9) follows from (8.8) and Cauchy-Schwarz inequality

$$Z(t) = \int_U (\overline{z} + \Delta\gamma(t))^2dx \le \int_U 2(\overline{z})^2 + 2(\Delta\gamma(t))^2dx. \tag{8.14}$$ □

We will use the estimate in Theorem 8.1 to obtain the global stability of the dynamical system with respect to perturbation of Dirichlet boundary data on Γ_i explicitly.

Corollary 8.2. *Assume that*

$$\int_0^\infty \Lambda_0^{-1}(\tau)d\tau = \infty, \tag{8.15}$$

$$\int_0^\infty \Lambda_0(\tau)(\Delta'_\gamma(\tau))^2F_1^{-1}(\tau)d\tau = \infty, \tag{8.16}$$

$$\lim_{t\to\infty} \Lambda_0(t)(\Delta'_\gamma(t)) = \lambda_1 \in \mathbb{R}. \tag{8.17}$$

Then

$$\overline{Z}(t) \le F_1(t)\overline{Z}(0) + C_1C_0^{-1}A_1^{-2}\lambda_1^2 + \epsilon(t), \tag{8.18}$$

where $\epsilon(t) \to 0$ as $t \to \infty$.

Consequently, if $\lambda_1 = 0$ then

$$\lim_{t\to\infty} \overline{Z}(t) = 0. \tag{8.19}$$

Proof. The conditions (8.15)–(8.17) allow one to apply the L'Hopital Rule to the integral term in (8.8), noting that $F_1^{-1}(t) \to \infty$ as $t \to \infty$ and $dF_1^{-1}/dt = C_0A_1\Lambda_0^{-1}F_1^{-1}$. We omit the details. □

Let us illustrate the above results with two examples where the temporal boundary profiles are polynomials.

Example 8.3. Suppose $\gamma_k(t) = a_{0,k} + a_{1,k}t^{\beta_k}$, where $a_{i,k} \neq 0$, for $i = 0,1$, and $k = 1,2$. Let $\beta = \max\{\beta_1, \beta_2\}$ and $\alpha = \deg(g)$. If $\beta < 2(\alpha+1)/(3\alpha+2)$ then $\lim_{t\to\infty} \overline{Z}(t) = 0$.

In Example 8.3, the coefficients and orders of $\gamma_k(t)$ can be different, and therefore, one cannot expect the L^2 norm of the difference between two solutions to decay to zero as $t \to \infty$. By virtue of Corollary 8.2, in such case, the difference between solutions shifted by $\Delta_\gamma(t)$ is vanishing at infinity, i.e. (8.19) holds, if the growth rate β of the boundary profile is “small”. For instance in the case of Darcy equation $\beta < 1$ (since $\alpha = 0$). In case the boundary profiles are the same, as seen in Example 7.5, (8.19) holds for larger growth rate β. For Darcy's law such β can be arbitrarily large.

In the following example, the two boundary profiles are different but have the same growth rate.

Example 8.4. Suppose $\gamma_1(t) = a_{0,1} + a_{1,1}t^\beta$ and $\gamma_2(t) = \gamma_1(t) + \Delta_\gamma(t)$, where $\Delta_\gamma(t) = O(t^r)$, with $\beta > r$. Then $\lim_{t\to\infty} \overline{Z}(t) = 0$ if $\beta \le 1/a$ and $r \le 1 - (2\beta - 1)a/(2-a)$.

8.2. **IBVP-II type (S).** Let p_1 and p_2 be two solutions of IBVP-II(S). Let $\delta > 0$ be fixed, and let

(8.20)
$$\Lambda_k^*(t) = \Big(1 + \int_0^t |Q_k(\tau)|d\tau\Big)\Big(1 + \int_0^t |Q_k'(\tau)|d\tau\Big) + |Q(t)|^{\frac{2-a}{1-a}} + \int_0^t |Q'(\tau)|^{\frac{2-a}{1-a}} d\tau.$$

We assume that

$$\int_0^t e^{-\delta\tau}\Lambda_k^*(\tau)d\tau \le \widetilde{\lambda}_0(t), \quad t \ge 0, \quad k = 1,2, \tag{8.21}$$

where the function $\widetilde{\lambda}_0(t)$ is known and belongs to $C([0,\infty))$.

Similar to Lemma 7.2, with the use of estimate (6.21) and assumption (8.21), one derives

$$\frac{1}{2}\frac{d}{dt}\int_U z^2(x,t)dx \le -L_0\left[\int_U |\nabla z(x,t)|^{2-a}dx\right]^{\frac{2}{2-a}} \widetilde{\Lambda}_0(t) - \Delta_\gamma(t)\Delta_Q(t), \tag{8.22}$$

where L_0 depends on initial data, and

$$\widetilde{\Lambda}_0(t) = e^{-b\,\delta\,t}(1 + \widetilde{\lambda}_0(t))^{-b}. \tag{8.23}$$

Similar to estimate (6.29) one has

$$|\Delta_\gamma(t)| \le C\Big(\int_U |\nabla z(x,t)|^{2-a}dx\Big)^{\frac{1}{2-a}} + C\int_U |z(x,0)|dx + C\int_0^t |\Delta_Q(\tau)|d\tau.$$

Therefore

$$|\Delta_\gamma(t)\Delta_Q(t)| \le \frac{L_0}{2}\left[\int_U |\nabla z(x,t)|^{2-a}dx\right]^{\frac{2}{2-a}} \widetilde{\Lambda}_0(t) + L_0^{-1}\widetilde{\Lambda}_0^{-1}(t)|\Delta_Q(t)|^2$$
$$+ C|\Delta_Q(t)|\Big(\int_U |z(x,0)|dx + C\int_0^t |\Delta_Q(\tau)|d\tau\Big).$$

Combining this with (8.22) yields

$$\frac{1}{2}\frac{d}{dt}\int_U z^2(x,t)dx \le L_0^{-1}\widetilde{\Lambda}_0^{-1}(t)|\Delta_Q(t)|^2 + C|\Delta_Q(t)|\Big(\int_U |z(x,0)|dx + C\int_0^t |\Delta_Q(\tau)|d\tau\Big). \tag{8.24}$$

Integrating the inequality from 0 to t, one obtains

$$\begin{aligned}\frac{1}{2}\int_U z^2(x,t)dx \le \frac{1}{2}\int_U z^2(x,0)dx + \int_0^t L_0^{-1}\widetilde{\Lambda}_0^{-1}(\tau)|\Delta_Q(\tau)|^2 d\tau\\ + C\int_0^t |\Delta_Q(\tau)|\Big(\int_U |z(x,0)|dx + C\int_0^\tau |\Delta_Q(\theta)|d\theta\Big)d\tau.\end{aligned} \tag{8.25}$$

Combining this inequality with the estimate in Theorem 6.4, one can establish the continuous dependence of the solutions of IBVP-II(S) on the total flux. Namely,

Theorem 8.5. *Given δ, T and a solution $p_1(x,t)$ with $Q_1, Q_1' \in L^\infty_{loc}([0,\infty))$. For any $\varepsilon > 0$, there is $\sigma > 0$ depending on δ, T, $\|Q_1\|_{L^\infty(0,T)}$, $\|Q_1'\|_{L^\infty(0,T)}$ and the initial data of p_1, such that if*

$$\int_U |z(x,0)|^2 dx, \int_U |\nabla z(x,0)|^{2-a}dx, \|\Delta_Q\|_{L^\infty(0,T)}, \|\Delta_Q'\|_{L^\infty(0,T)} < \sigma, \tag{8.26}$$

then

$$\int_U |z(x,t)|^2 dx < \varepsilon, \quad \textit{for all} \quad t\in[0,T]. \tag{8.27}$$

More specifically, there is $L > 0$ depending on δ, T, $\|Q_1\|_{L^\infty(0,T)}$, $\|Q_1'\|_{L^\infty(0,T)}$ and the initial data of p_1, such that

$$\sup_{t\in[0,T]}\int_U |p_1(x,t)-p_2(x,t)|^2 dx \le L\Big(\int_U |p_1(x,0)-p_2(x,0)|^2 dx + (\sup_{t\in[0,T]}|\Delta_Q(t)|)^2\Big). \tag{8.28}$$

The estimate (8.28) is in terms of total flux only, but number L grows exponentially in T. This exponential growth does not yield the asymptotic stability of IBVP-II(S). With additional information about the growth rates of $\gamma_1(t)$ and $\gamma_2(t)$, but not of their difference one can obtain better estimates than (8.28) and for all $t \ge 0$. These new estimates are used to track the asymptotic behaviors of the solutions to the IBVP-II(S).

Theorem 8.6. *Let p_1 and p_2 be two solutions of IBVP-II(S) satisfying condition (8.1). Assume $\deg(g) \le \frac{4}{d-2}$. Let*

$$\overline{p}_k(x,t) = p_k(x,t) + |U|^{-1}\int_0^t Q_k(\tau)d\tau \quad \textit{for} \quad k=1,2, \tag{8.29}$$

$$\overline{z}(x,t) = \overline{p}_1(x,t) - \overline{p}_2(x,t) - |U|^{-1}\int_U (p_1(x,0)-p_2(x,0))dx, \quad \textit{and} \tag{8.30}$$

$$\overline{Z}(t) = \int_U \overline{z}^2(x,t)dx. \tag{8.31}$$

Then one has for all $t \ge 0$ that

$$\overline{Z}(t) \le F_1(t)\overline{Z}(0) + C_2 A_1^{-1} F_1(t)\int_0^t \Lambda_0(\tau)(\Delta_Q(\tau))^2 F_1^{-1}(\tau)d\tau. \tag{8.32}$$

Proof. We have

$$\frac{\partial}{\partial t}\int_U \overline{p}_k(x,t)\,dx = \frac{\partial}{\partial t}\int_U p_k(x,t)\,dx + Q_k(t) = -Q_k(t) + Q_k(t) = 0. \tag{8.33}$$

Hence $\frac{\partial}{\partial t}\int_U \overline{z}_k(x,t)\,dx = 0$. Since $\int_U \overline{z}(x,0)dx = 0$ one has

$$\int_U \overline{z}(x,t)dx = 0 \quad \text{for all} \quad t \geq 0. \tag{8.34}$$

Note also that $\overline{z}$ on Γ_i is the function of t only.

Similar to (8.10) we obtain

$$\frac{1}{2}\frac{d}{dt}\overline{Z} = -\int_U \Phi(\nabla\overline{p}_1, \nabla\overline{p}_2)dx + \frac{1}{|U|}\Delta_Q(t)\int_U \overline{z}dx + B(t)\Delta_Q(t), \tag{8.35}$$

where $B(t) = \overline{z}|_{\Gamma_i}$. By (8.34), the second term on the RHS of equation above vanishes.

From (8.35) and (3.36) it follows

$$\frac{1}{2}\frac{d}{dt}\overline{Z} \leq -C\left(\int_U (\nabla(\overline{p}_1 - \overline{p}_2))^{2-a}dx\right)^{\frac{2}{2-a}} N(t)^{-b} + B(t)\Delta_Q(t), \tag{8.36}$$

or

$$\frac{1}{2}\frac{d}{dt}\overline{Z} \leq -C\left(\int_U (\nabla(\overline{z} - B(t)))^{2-a}dx\right)^{\frac{2}{2-a}} N(t)^{-b} + B(t)\Delta_Q(t). \tag{8.37}$$

Applying Poincare's inequality from (8.37) one can get

$$\frac{1}{2}\frac{d}{dt}\overline{Z} \leq -C\int_U (\overline{z} - B(t))^2 dx N(t)^{-b} + B(t)\Delta_Q(t), \tag{8.38}$$

or

$$\frac{1}{2}\frac{d}{dt}\overline{Z} \leq -C\int_U (\overline{z}^2 - 2\overline{z}B(t) + B^2(t))dx\, N(t)^{-b} + B(t)\Delta_Q(t). \tag{8.39}$$

Again, the second term on RHS of inequality (8.39) is zero. Applying Cauchy's inequality to the last term in (8.39), and using (8.3) we obtain

$$\frac{1}{2}\frac{d}{dt}\overline{Z} \leq -\frac{CA_1}{2\Lambda_0(t)}\overline{Z} + CA_1{}^{-1}1\Lambda_0(t)(\Delta_Q(t))^2. \tag{8.40}$$

Then inequality (8.32) follows by applying Gronwall's inequality. □

We call the $\bar{p}_k(x,t)$ in (8.29) above the *shifted solutions*, i.e., the solutions shifted by the accumulation in time of total flux.

Using (8.32) one can establish the asymptotic stability of these shifted solutions of IBVP-II(S) in this context for fast decaying $\Delta_Q(t)$. For example, the result obtained in Corollary 8.2 is valid for function $\overline{Z}(t)$ defined by (8.30) and $\Delta'_\gamma(t)$ replaced by $\Delta_Q(t)$.

The estimate in Theorems 8.6 for the shifted solutions also induces an estimate for the solutions themselves. Alternatively, we will derive a L^2 estimate for the difference between two solutions directly, which is slightly more accurate.

Define

$$I_Q(t) = \left(\int_0^t \Delta_Q(\tau)d\tau\right)^2 \quad \text{and} \quad I_z(t) = \left(\int_U z(x,t)dx\right)^2. \tag{8.41}$$

Theorem 8.7. *Assume* $\deg(g) < \frac{4}{d-2}$. *One has for all* $t \geq 0$ *that*

$$Z(t) \leq F_1(t)\Big[Z(0) + C_1 A_1 \int_0^t \frac{I_Q(\tau)}{F_1(\tau)\Lambda_0(\tau)} d\tau + C_2 A_1 I_z(0) \int_0^t \frac{1}{F_1(\tau)\Lambda_0(\tau)} d\tau + C_3 A_1^{-1} \int_0^t \frac{\Lambda_0(\tau)\Delta_Q^2(\tau)}{F_1(\tau)} d\tau\Big]. \tag{8.42}$$

Proof. Applying Lemma 7.2, one has

$$\frac{1}{2}\frac{d}{dt}\int_U z^2 dx \leq -C\left[\int_U |\nabla z|^{2-a} dx\right]^{\frac{2}{2-a}} \Lambda_0^{-1}(t) - \Delta_\gamma(t)\Delta_Q(t). \tag{8.43}$$

Let $\bar{z} = z - \Delta_\gamma$. Then $\int_{\Gamma_i} \bar{z} d\sigma = 0$. Applying the generalized Sobolev's inequality to $\bar{z}$, one has

$$\int_U \bar{z}^2 dx \leq C\left(\int_U |\nabla \bar{z}|^p dx\right)^{\frac{2}{p}} + C\left|\int_{\Gamma_i} \bar{z} d\sigma\right| = C\left(\int_U |\nabla \bar{z}|^p dx\right)^{\frac{2}{p}},$$

where $2 < dp/(d-p)$ and $p \leq (2-a)$. Equivalently, $2 - a \geq p > 2d/(d+2)$. Therefore $a < 4/(d+2)$, i.e., $\deg(g) < 4/(d-2)$.

Subsequently, we obtain

$$\begin{aligned}\int_U z^2 dx &\leq C\left(\int_U |\nabla z|^p dx\right)^{\frac{2}{p}} + 2(\Delta_\gamma(t))\int_U z(x,t)dx - (\Delta\gamma(t))^2|U| \\ &\leq C\left(\int_U |\nabla z|^p dx\right)^{\frac{2}{p}} + C\left(\int_U z(x,t)dx\right)^2 - (1/2)(\Delta\gamma(t))^2|U|.\end{aligned} \tag{8.44}$$

Thus

$$\int_U z^2 dx \leq C\left(\int_U |\nabla z|^{2-a} dx\right)^{\frac{2}{2-a}} + C\left(\int_U z(x,t)dx\right)^2 - (1/2)(\Delta\gamma(t))^2|U|. \tag{8.45}$$

One observes from Lemma 4.3 that

$$\int_U z(x,t)dx = \int_U z(x,0)dx + \int_0^t \Delta_Q(\tau)d\tau. \tag{8.46}$$

Hence it follows that

$$\int_U z^2 dx \leq C\left(\int_U |\nabla z|^{2-a} dx\right)^{\frac{2}{2-a}} + CI_Q(t) + CI_z(0) - (1/2)(\Delta_\gamma(t))^2|U|. \tag{8.47}$$

Substituting inequality (8.47) into the RHS of (8.43) one obtains

$$\frac{1}{2}\frac{d}{dt}\int_U z^2 dx \leq -\frac{CA_1}{\Lambda_0(t)}\left[\int_U z^2 dx - CI_Q(t) - CI_z(0) + (\Delta_\gamma(t))^2|U|/2\right] - \Delta_\gamma(t)\Delta_Q(t). \tag{8.48}$$

Once more applying Cauchy inequality to the term $\Delta_\gamma(t)\Delta_Q(t)$ one can get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_U z^2dx &\le -\frac{CA_1}{\Lambda_0(t)}\int_U z^2dx + \frac{CA_1 I_Q(t)+CA_1I_z(0)}{\Lambda_0(t)} - \frac{CA_1\Delta_\gamma^2(t)|U|}{4\Lambda_0(t)}\\ &\quad + CA_1^{-1}\Lambda_0(t)\Delta_Q^2(t)\\ &\le -\frac{CA_1}{\Lambda_0(t)}\int_U z^2dx + \frac{CA_1 I_Q(t)+CA_1I_z(0)}{\Lambda_0(t)} + CA_1^{-1}\Lambda_0(t)\Delta_Q^2(t).\end{aligned}$$

Then applying Gronwall's inequality gives (8.42). □

Similar to Corollary 8.2, the estimate in Theorem 8.7 can be simplified for large t by using L'Hopital's Rule.

Corollary 8.8. *Assume that*

$$\int_0^\infty \Lambda_0^{-1}(\tau)d\tau = \infty, \tag{8.49}$$

$$\int_0^\infty \Lambda_0(\tau)(\Delta_Q(\tau))^2F_1^{-1}(\tau)d\tau = \infty, \tag{8.50}$$

$$\lim_{t\to\infty}\Lambda_0(t)\Delta_Q(t) = \lambda_3\in\mathbb{R}, \tag{8.51}$$

$$\int_0^\infty I_Q(\tau)\Lambda_0^{-1}(\tau)F_1^{-1}(\tau)d\tau = \infty, \tag{8.52}$$

$$\lim_{t\to\infty} I_Q(t) = \lambda_2\in\mathbb{R}, \tag{8.53}$$

$$\int_0^\infty \Lambda_0^{-1}(\tau)F_1^{-1}(\tau)d\tau = \infty. \tag{8.54}$$

Then one has

$$Z(t)\le F_1(t)Z(0) + C_1C_0^{-1}\lambda_2 + C_2C_0^{-1}I_z(0) + C_3C_0^{-1}A_1^{-2}\lambda_3^2 + \epsilon(t), \tag{8.55}$$

where $\epsilon(t)\to 0$, *as* $t\to\infty$.

8.3. **IBVP-I type (S) with flux constraints.** The techniques used in the previous subsection to study IBVP-II(S) actually can be applied to IBVP-I(S). Of course, additional conditions on the relations between $\Delta_Q(t)$ and $\Delta_\gamma(t)$ are needed. With such, can improve the estimate in Theorem 8.1, which depends on both Δ_γ and Δ'_γ, and reduces the dependence to Δ_γ only.

Theorem 8.9. *Let* p_1 *and* p_2 *be two solutions to IBVP-I(S). Assume that*

$$\Delta_Q^2(t)\le q_0I_Q(t) + q_1\Delta_Q^2(0) + q_2,\quad \textit{some}\quad q_0,q_1,q_2\ge 0. \tag{8.56}$$

Then one has

$$\begin{aligned}Z(t)\le F_1(t)\Big[Z(0) + C_1(q_1\Delta_Q(0)+q_2+I_z(0))\int_0^t \frac{1}{F_1(\tau)\Lambda_0(\tau)}d\tau\\ + C_2\int_0^t\frac{\Lambda_0(\tau)\Delta_\gamma^2(\tau)}{F_1(\tau)}d\tau\Big].\end{aligned} \tag{8.57}$$

Proof. Applying Cauchy's inequality to the term $\Delta_\gamma(t)\int_U z(x,t)dx$ on the RHS of (8.44) gives

$$\int_U z^2dx \le C\left(\int_U |\nabla z|^{2-a}dx\right)^{\frac{2}{2-a}} + C(\Delta_\gamma(t))^2. \tag{8.58}$$

From inequalities (8.58), (8.43) and (8.56) it follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}Z(t) &\le -\frac{A_1}{\Lambda_0(t)}\left[CZ(t)-C(\Delta_\gamma(t))^2\right]+|\Delta_Q(t)|\,|\Delta_\gamma(t)| \\ &\le -\frac{CA_1Z(t)}{\Lambda_0(t)}+\varepsilon\frac{(\Delta_Q(t))^2}{\Lambda_0(t)}+C\Lambda_0(t)|\Delta_\gamma(t)|^2 \\ &\le -\frac{CZ(t)}{\Lambda_0(t)}+\varepsilon\frac{q_0I_Q(t)+q_1\Delta_Q(0)+q_2}{\Lambda_0(t)}+C\Lambda_0(t)|\Delta_\gamma(t)|^2. \end{aligned} \tag{8.59}$$

Then similar identities to those in Lemma 4.3 lead to

$$I_Q(t)\le CZ(t)+CI_z(0). \tag{8.60}$$

Hence

$$\frac{1}{2}\frac{d}{dt}Z(t)\le -\frac{CZ(t)}{2\Lambda_0(t)}+\frac{C(q_1\Delta_Q(0)+q_2+I_z(0))}{\Lambda_0(t)}+C\Lambda_0(t)\Delta_\gamma^2(t). \tag{8.61}$$

By Gronwall's inequality, one obtains (8.57). □

Remark 8.10. Similar to Corollaries 8.2 and 8.8, under appropriate conditions one can obtain the following explicit estimate of $Z(t)$ for large t:

$$Z(t)\le F_1(t)C_1Z(0)+C_3(q_1\Delta_Q(0)+q_2+I_z(0))+C_4\Lambda_0^2(t)\Delta_\gamma^2(t)+\epsilon(t), \tag{8.62}$$

where $\epsilon(t)\to 0$ as $t\to\infty$.

Remark 8.11. From physical point of view, condition (8.56) restricts the amplitude of possible spikes of the total flux from too large deviation, and this, in fact, is not stringent. Indeed , from (8.56) and (8.41) one has

$$|\Delta_Q(t)|\le C_1\int_0^t|\Delta_Q(\tau)|d\tau+C_2,$$

hence by Gronwall's inequality: $|\Delta_Q(t)|\le C_3e^{C_1t}+C_4$. It means that $|\Delta_Q(t)|$ cannot grow faster than exponential functions.

Remark 8.12. The results in this section can be interpreted as follows: Given a non-linear flow in porous media with pressure distribution $p(x,t)$ being the solution of the IBVP-I for some $\gamma(t)$ and initial data $p_0(x)$. Assume the hydrodynamic system is perturbed for all time by varying the parameters on the boundary. Let us consider two scenarios of the excitation of the system.

Case A: The prescribed/observed pressure on the accessible boundary Γ_i is perturbed by deviation $\Delta_\gamma(t)$.

Case B: The prescribed/observed total flux on the accessible boundary Γ_i is perturbed by deviation $\Delta_Q(t)$.

We proved above that the hydrodynamic system is "robust", that is, by monitoring both $\gamma(t)$ (excited and non-excited ones) the L^2 norm of the solution can be estimated for all time in terms of controllable parameters, $\Delta_\gamma(t)$ in Case A, and $\Delta_Q(t)$ in Case B.

9. Numerical Results

In this section we numerically investigate two major results obtained for the IBVP-I in Sections 6 and 7. First we will validate the *a priori* estimate in Theorem 6.3. We will show that inequality (6.12) is rather sharp independently from the type of non-linearity, $\deg(g)$, and boundary condition, $\gamma(t)$. Then we will validate the asymptotic stability result in Theorem 7.3. We will show that if $\gamma(t)$ is chosen to be the power function

$$\gamma(t) = Ct^{m+1}, \tag{9.1}$$

then, according to Corollary 7.4, there exists a threshold value M such that, if $m \le M$ then $\|p_1 - p_2\|_{L^2}$ in (7.8) decays exponentially. The value of M depends on the type of non-linearity, in particular $M = 1/\deg(g)$. We will show that for $\deg(g) = 1$ the threshold value M occurs exactly in the transition region between exponential and $1/t^p$ decay. In case of $\deg(g) = 2$, Ineq. (7.8) still holds, but there is still some space for improvement.

We consider a fully penetrated vertical well in a 3-D rectangular box. Because of the boundary conditions on the well and on the exterior boundary, the problem reduces to the 2-D geometry sketched in Fig. 1.

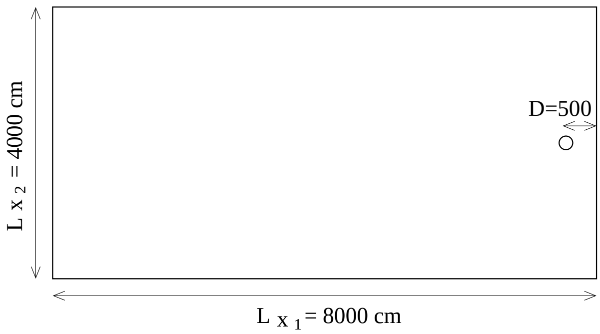

Figure 1. 2-D Scheme of the fully penetrated vertical well in rectangular reservoir.

The geometrical parameters are: $L_{x_1} = 8000$, $L_{x_2} = 4000$, $r_w = 30$, $D = 500$, where r_w is the radius of the well. The hydrodynamical parameters are: compressibility $1/\kappa = 1/15000$, and according to the definition of (GPPC) in Eq. (3.32)

$a_0 = 10; a_1 = 20; a_2 = 30;$

$\alpha_0 = 0; \alpha_1 = 1; \alpha_2 = 2;$

Two different polynomials are considered

$$g_1(u) = \sum_{j=0}^{1} a_j u^{\alpha_j} \text{ and } g_2(u) = \sum_{j=0}^{2} a_j u^{\alpha_j} .$$

Clearly $\deg(g_1) = 1$ and $\deg(g_2) = 2$.

The results for the g_1 polynomial are reported in Figs. 2 and 3. In Fig. 2 the time evolution of the ratio between the LHS of (6.12) and the leading positive term in RHS of inequality (6.12)

$$R(t) = \frac{\int_U K\left(|\nabla p(x,t)|\right)|\nabla p(x,t)|^2\,dx}{|U|\int_0^t \left(\gamma\prime(\tau)\right)^2 d\tau} \tag{9.2}$$

is given for different values of m in Eq. (9.1). From the top to the bottom on the y-axis the values of m are equal to $0., 0.5, 0.6, 0.8, 1, 1.5$, respectively. Clearly, for each case the denominator in (9.2) diverges. The $x - axis$ is in logarithmic scale. In the long time dynamics, only for $m = 0$ the ratio (9.2) converges to zero. This is justified to the fact that in this case the PSS solution is reached and the numerator converges to some constant value, while the denominator diverges. On the other hand for all the other values of m the ratio in (9.2) stabilizes to some value grater than zero but less than one. This shows that numerator and denominator in (9.2) diverge with the same speed, or the same LHS and RHS of (6.12) diverge with the same speed.

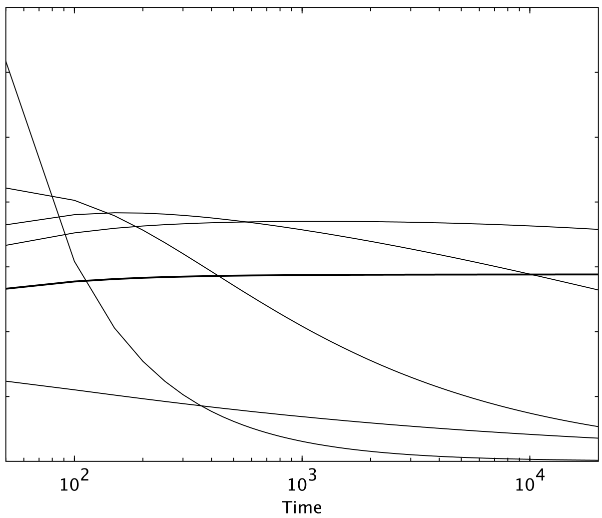

FIGURE 2. Time evolution of R(t) in Eq. (9.2), for g_2 and different values of m. From the top to the bottom on the y-axis m takes values $0., 0.5, 0.6, 0.8, 1, 1.5$.

In Fig. 3 the time evolution of the norm $\|p_1 - p_2\|_{L^2}$ is reported for the same values of m as before, while the order on the y-axis is reversed. Here $p_1(t,x)$ and $p_2(t,x)$ are two distinct solutions of the same IBVP-I with different initial pressure distributions $p_1(0,x)$ and $p_2(0,x)$, respectively. Both the x-axis and the y-axis are in logarithmic scale. According to Corollary 7.4 exponential convergence is expected for $m < 1$. From the picture, it is clear that for $m = 0., 0.5, 0.6, 0.8$ all the curves are concave down and $\ln \|p_1 - p_2\|_{L^2} \to -\infty$ as $\ln t \to -\infty$. This corresponds to $\|p_1 - p_2\|_{L^2} \le C_0 e^{-(t^p)}$, for some positive p. On the other hand for $m = 1$ the curve (the bold one) is a straight line and although it still diverges to $-\infty$, it diverges much more slowly: $\|p_1 - p_2\|_{L^2} \le C_0 t^{-p}$, for some positive p. For $m = 1.5$ the curve becomes concave up and it fails to diverge.

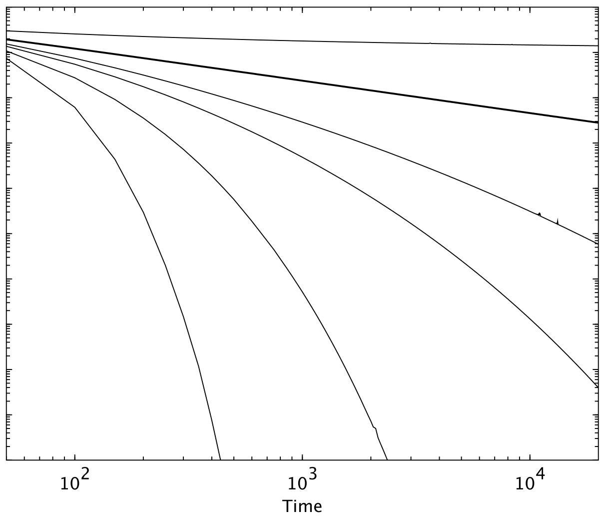

FIGURE 3. Time evolution of $\|p_1-p_2\|_{L^2}$ for g_2 and different values of m. From the bottom to the top on the y-axis, m takes values $0., 0.3, 0.5, 0.7, 0.8$.

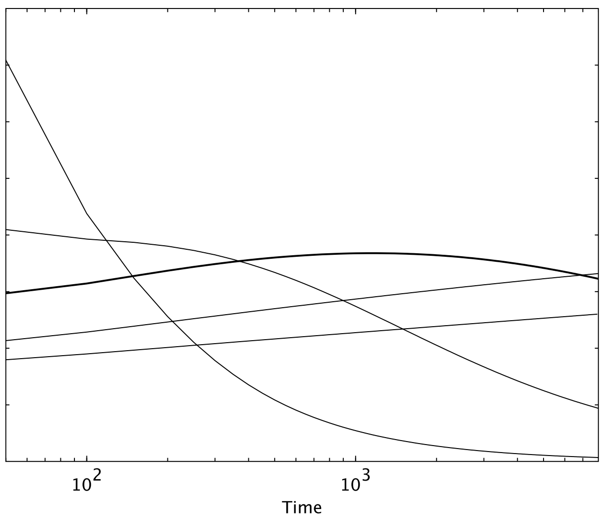

FIGURE 4. Time evolution of R(t) in Eq. (9.2), for g_2 and different values of m. From the top to the bottom on the y-axis m takes values $0., 0.3, 0.5, 0.7, 0.8$.

The results for the g_2 polynomial are reported in Figs. 4 and 5, and almost resemble the results for g_1. In Fig. 4 the time evolution of $R(t)$ in Eq. (9.2) is given for different values of m. From the top to the bottom on the y-axis the values of m are $0., 0.3, 0.5, 0.7, 0.8$, respectively. Again in the long time dynamics, only for

$m = 0$, $R(t)$ converges to zero. For all the other values of m the ratio in (9.2) stabilizes to some value grater than zero but less than one. This shows that even for the g_2 case the LHS and RHS of (6.12) diverge with the same speed.

In Fig. 5 the time evolution of $\|p_1 - p_2\|_{L^2}$ is reported for the same values of m. Here $p_1(t, x)$ and $p_2(t, x)$ are as before. According to Corollary 7.4 exponential convergence is expected for $m < 0.5$. From the picture, it is clear that for $m = 0., 0.3$ all the curves are concave down. This corresponds to $\|p_1 - p_2\|_{L^2} \leq C_0 e^{-(t^p)}$, for some positive p. In this case even for $m = 0.5, 0.7$ the graph are still concave down, and only for $m = 0.8$ a straight line is obtained. This shows that for g_2 the transition region occurs a little bit later that 0.5. This does not contradict Corollary 7.4, but only indicates that estimate (7.11) for g_2 is less sharper than same estimate for g_1.

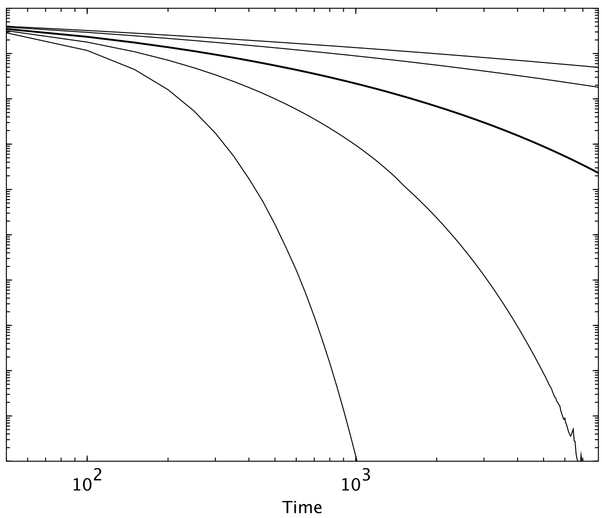

FIGURE 5. Time evolution of $\|p_1 - p_2\|_{L^2}$ for g_2 and different values of m. From the bottom to the top on the y-axis, m takes values $0., 0.5, 0.6, 0.8, 1, 1.5$.

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