EXTREME POINT OF CONVEX SET IN A HILBERT SPACE

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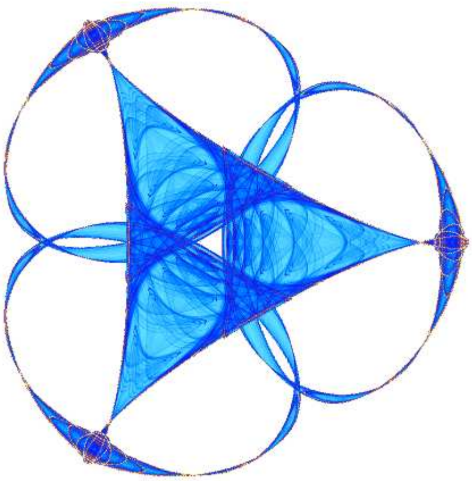

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Extreme Point of convex set in a Hilbert space

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Abstract: This short note deals with the characterization of an extreme point of convex set in a Hilbert space which appear as solutions of linear inequalities.

Key words: Convex set, extreme point, Hilbert space.

Characterization of extreme points.

Let H be a Hilbert space, T a set of indexes, $\{a_t/t \in T\} \subset H$ and $\{b_t/t \in T\} \subset R$.

For a system $\sigma = \{a_t x \geq b_t/t \in T\}$, denotes the set of solutions of feasible solutions $F(\sigma)$, that is to say:

$$F(\sigma) = \{x \in H/a_t x \geq b_t; t \in T\}$$

Definition: A point $\overline{x} \in F(\sigma)$ is a *extreme point* of $F(\sigma)$ if it is impossible to represent $\overline{x}$ as a middle point a segment with its extreme point belonging to $F(\sigma)$ and both different to $\overline{x}$.

Theorem: $\overline{x} \in F(\sigma)$ is a extreme point of $F(\sigma)$ if and only if:

a) There exists a subset $S \subset T$ such that $\{a_t/t \in S\}$ is an orthonormal complete system and $a_t\overline{x} = b_t$ for all $t \in S$.

or

b)For each $n \in N$, there exist a set T_n, $S_n \subset T_n$ and an element $x_n \in F(\sigma)$; $\sigma_n = \{a_t x \geq b_t/t \in T\}$, such that:

i) $F(\sigma) \subset F(\sigma_n)$
ii) $\{a_t/t \in S\}$ is an orthonormal complete system.
iii) $a_t x_n = b_t$, for all $t \in S_n$

and $(x_n)_{n\geq 1}$ converges to $\overline{x}$.

Note: The conditions i) can be replaced by i') $F(\sigma_n) \subset F(\sigma)$.

<u>**Proof**</u>

(Necessity) Let $\overline{x}$ be an extremal point of $F(\sigma)$, and assume that every sequence $(x_n)_{n\geq 1}$ such that for each n it verifies i), ii) and iii), does not converge to $\overline{x}$, this would imply that such sequences do not accumulate to $\overline{x}$, too.

Let $I = \{t \in T / a_t \overline{x} = b_t\}$. Then $a_t \overline{x} > b_t$ for $t \in T - I$.

If $\{a_t / t \in I\}$ does not contain an orthonormal complete system, there exists an $a \in H$, $a \neq 0$ such that $a.a_t = 0$ for all $t \in T$.

Since b) is not fulfilled there exists a ball $B(\overline{x}, \delta)$ such that if $x \in B(\overline{x}, \delta)$ then $a_t x > b_t$ for each $t \in T - I$.

Let the variety

$$V_I = \{x \in H / a_t x = b_t, t \in I\}$$

V_I is not H since that $\{a_t / t \in I\}$ is not complete. Besides $v = \overline{x} + a \in V_I$ because: $a_t(\overline{x} + a) = a_t \overline{x} + a_t a = b_t$ for all $t \in I$. Calling $\lambda v = \overline{x} + \lambda a$ it results $\lambda v \in V_I$ and λv is a one-dimensional variety contained in V_I which determines a segment in $B(\overline{x}, \delta)$ which has to $\overline{x}$ as a middle point, therefore $\overline{x}$ would not be extremal.

(Sufficiency)

1^{st} case) Suppose that a) is fulfilled, that is to say there exists a $S \subset T$ such that $\{a_t / t \in S\}$ is an orthonormal complete system and $a_t \overline{x} = b_t$ for each $t \in S$. If $\overline{x} = \frac{1}{2}x_1 + \frac{1}{2}x_2$ for some $x_1, x_2 \in F(\sigma)$ then $b_t = a_t \overline{x} = \frac{1}{2}a_t x_1 + \frac{1}{2}a_t x_2 \geq \frac{1}{2}b_t + \frac{1}{2}b_t = b_t$ and $a_t x_1 = a_t x_2 = b_t$ for $t \in S$ and therefore $a_t(x_1 - x_2) = 0$ for all $t \in S$ (complete).

Then $x_1 = x_2 = \overline{x}$ and $\overline{x}$ is extremal of $F(\sigma)$.

2^{st}case) Assume that there exists a sequence $(x_n)\, n \geq 1$ where for each n, there exist a T_n, $S_n \subset T_n$ such that $\{a_t / t \in S_n\}$ is a complete orthonormal system and $a_t x_n = b_t$ and $(x_n)\, n \geq 1$ converges to $\overline{x}$.

For each n is valid what it was proved the 1^{st} case), that is to say, if $x_n = \frac{1}{2}x_{1n} + \frac{1}{2}x_{2n}$ then $x_n = x_{1n} = x_{2n}$. Because $\lim\limits_{n \to \infty} x_n = \overline{x}$, $\overline{x}$ cannot be of the form $\overline{x} = \frac{1}{2}x_1 + \frac{1}{2}x_2$ with $x_1, x_2 \in F(\sigma)$ and $x_1 \neq x_2 \neq \overline{x}$.

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