PERFECTLY PROPER FRIENDLY EQUILIBRIA

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Perfectly Proper Friendly Equilibria

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Abstract

This paper introduces the perfectly proper frendly equilibrium, a strict refinement of the proper friendly equilibrium and of the perfectly proper equilibrium, and proves that every finite non-cooperative game with friends that satisfies certain conditions has at least one perfectly proper friendly equilibrium.

Keywords:n-person game, non-cooperative game, friendly equilibrium, perfectly proper equilibrium.

Classification AMS:90D06, 90D10, 90D99.

1 INTRODUCTION

For a strategy combination to be a plausible solution of a non-cooperative game it must be a Nash equilibrium (Nash 1951), but more refined concepts must be appealed to in order to determine a solution from among the multiplicity of Nash equilibria that generally exist. Marchi (1991) suggested a refinement in which the strategy played by each player i at an equilibrium mixed strategy profile $s = (s_i, s_{-i})$ is not only one that maximizes that player's expected payoff (conditional on the other players sticking to the strategies assigned them by this profile), but is also one that, among the set of such best responses to s_{-i} , maximizes the payoff of his or her best friend, or *first friendly successor*; and in case of there being more than one such strategy, it is one of those that also maximize the payoff of his or her second best friend, or *second friendly successor*; and so on. This solution concept, the friendly equilibrium, admits further refinement as the perfect friendly equilibrium (Marchi 1991), which also refines Selten's (1975) concept of perfect equilibrium; and as the proper friendly equilibrium (Marchi 1991), which also refines Myerson's (1978) concept of proper equilibrium.

In this paper we present the *perfectly proper friendly equilibrium* (PPFE), which refines both the proper friendly equilibrium and the concept of perfectly proper equilibrium, according to which the players who lose most by unilateral deviation from an equilibrium should be the least likely to deviate (García-Jurado 1989).

In section 2 of this paper we establish notation and state relevant concepts and results due to Marchi (1991). In section 3 we introduce the new solution concept and prove some of its properties.

2 Notation and background

Let Γ be a finite n-person non-cooperative game in normal form,

$$\Gamma = \{\Sigma_i,\ H_i,\ i \in \mathcal{N} = \{1, ..., n\}\}$$

where Σ_i is the finite nonempty set of *pure strategies* of player i and $H_i : \prod_{i=1}^{n} \Sigma_i \to R$ is his *payoff function.* Let S_i be the set of mixed strategies of player i, i.e.

$$S_i = \left\{ s_i \in R^{\Sigma_i} : \forall \sigma_i \in \Sigma_i \qquad s_i(\sigma_i) \geq 0, \quad \sum_{\sigma_i \in \Sigma_i} s_i(\sigma_i) = 1 \right\}$$

If $s_i \in S_i$, then $C(s_i)$ denotes the *Carrier* of s_i, i.e.

$$C(s_i) = \{\sigma_i \in \Sigma_i : s_i(\sigma_i) > 0\}.$$

s_i is said to be *completely mixed* if $C(s_i) = \Phi_i$.

Each $s = (s_1, ..., s_n) \in S = S_1 \times ... \times S_n$ is called a combination of strategies, and for $s_i^* \in S_i$ the combination of strategies $(s_1, ..., s_{i-1}, s_i^*, s_{i+1}, ..., s_n)$ is denoted by $s \backslash\ s_i^*$.

The extension of $H_i : S = \prod_{i=1}^{n} S_i \to R$ is defined by

$$H_i(s) = \sum_{\sigma \in \Sigma} H_i(\sigma).s(\sigma)$$

where $\sigma = (\sigma_1, ..., \sigma_n) \in \Sigma = \prod_{i=1}^{n} \Sigma_i$ and $s(\sigma) = \prod_{i=1}^{n} s_i(\sigma_i)$, with $s_i \in S_i$ for all i.

For each $i \in \mathcal{N}$ we assume given a finite sequence $f_{(i)}^1 = i,\ f_{(i)}^2, ...,\ f_{(i)}^{k_i}$ $(1 \leq k_i \leq n)$ of *friendly successors* of player i. For each $\overline{s} \in S$ we define the sets

$$\begin{aligned}
\Psi_i^1(\overline{s}) &= \{s_i \in S_i : H_i(\overline{s}/s_i) \geq H_i(\overline{s}/s_i^{,}) \qquad s_i^{,} \in S_i\} \\
\Psi_i^2(\overline{s}) &= \left\{s_i \in \Psi_i^1(\overline{s}) : H_{f_{(i)}^2}(\overline{s}/s_i) \geq H_{f_{(i)}^2}(\overline{s}/s_i^{,}) \qquad s_i^{,} \in \Psi_i^1(\overline{s})\right\} \\
&\vdots \\
\Psi_i^{k_i}(\overline{s}) &= \left\{s_i \in \Psi_i^{k_i-1}(\overline{s}) : H_{f_{(i)}^{k_i}}(\overline{s}/s_i) \geq H_{f_{(i)}^{k_i}}(\overline{s}/s_i^{,}) \qquad s_i^{,} \in \Psi_i^{k_i-1}(\overline{s})\right\}.
\end{aligned} \tag{1}$$

Definition 1 *Given Γ and sequences $f_{(i)}^1,\ f_{(i)}^2, ..., f_{(i)}^{k_i}$ as above, a strategy combination $s \in S$ is a friendly equilibrium if $s_i \in \Psi_i^{k_i}(s)$ for all $i \in \mathcal{N}$.*

Theorem 2 *(Marchi 1991). Given Γ and sequences $f_{(i)}^1,\ f_{(i)}^2, ..., f_{(i)}^{k_i}$ such that, for all i, the point-to-set correspondence $\Psi_i^{k_i}$ defined by equations* (1) *is upper semicontinuous, then there exists a friendly equilibrium.*

The concepts of $\varepsilon-$perfect, $\varepsilon-$proper, perfect and proper equilibria are made friendly in ways analogous to that in which the Nash equilibrium is made friendly by Definition 1, as follows.

Definition 3 *Given Γ, sequences $f_{(i)}^1,\ f_{(i)}^2, ..., f_{(i)}^{k_i}$ and $\varepsilon > 0$. A strategy combination $s \in S$ is a $\varepsilon-$perfect friendly equilibrium if each s_i is completely mixed and, for all $i \in \mathcal{N}$, satisfies the conditions:*

i) $\forall \sigma_i, \sigma_i^{,} \in \Sigma_i \qquad H_i(s/\sigma_i) < H_i(s/\sigma_i^{,}) \Longrightarrow s_i(\sigma_i) \leq \varepsilon;$

ii) $\forall \sigma_i, \sigma_i^{,} \in \Psi_i^1(s) \qquad H_{f_{(i)}^2}(s/\sigma_i) < H_{f_{(i)}^2}(s/\sigma_i^{,}) \Longrightarrow s_i(\sigma_i) \leq \varepsilon;$

$\vdots$

ki) $\forall \sigma_i, \sigma_i^{,} \in \Psi_i^{k_i - 1}(s) \qquad H_{f_{(i)}^{k_i}}(s/\sigma_i) < H_{f_{(i)}^{k_i}}(s/\sigma_i^{,}) \Longrightarrow s_i(\sigma_i) \leq \varepsilon.$

$s \in S$ is a $\varepsilon-$proper friendly equilibrium if it is completely mixed and, for all $i \in \mathcal{N}$, satisfies the conditions:

i) $\forall \sigma_i, \sigma_i^{,} \in \Sigma_i \qquad H_i(s/\sigma_i) < H_i(s/\sigma_i^{,}) \Longrightarrow s_i(\sigma_i) \leq \varepsilon . s_i(\sigma_i^{,});$

ii) $\forall \sigma_i, \sigma_i^{,} \in \Psi_i^1(s) \qquad H_{f_{(i)}^2}(s/\sigma_i) < H_{f_{(i)}^2}(s/\sigma_i^{,}) \Longrightarrow s_i(\sigma_i) \leq \varepsilon . s_i(\sigma_i^{,});$

$\vdots$

ki) $\forall \sigma_i, \sigma_i^{,} \in \Psi_i^{k_i - 1}(s) \qquad H_{f_{(i)}^{k_i}}(s/\sigma_i) < H_{f_{(i)}^{k_i}}(s/\sigma_i^{,}) \Longrightarrow s_i(\sigma_i) \leq \varepsilon . s_i(\sigma_i^{,}).$

Definition 4 *Given Γ and sequences $f_{(i)}^1, f_{(i)}^2, ..., f_{(i)}^{k_i}$, a strategy combination $\overline{s} \in S$ is a perfect(proper) friendly equilibrium if there exists sequences $\{\varepsilon_k\}_{k \in \mathrm{N}}$ and $\{s^k\}_{k \in \mathrm{N}}$ such that:*

$\forall k \in N \qquad \varepsilon_k > 0, \quad \lim_{k \to \infty} \varepsilon_k = 0.$

$\forall k \in N \qquad s^k$ is an ε_k-perfect (proper) friendly equilibrium.

$\lim_{k \to \infty} s^k = \overline{s}.$

Remark 1 *If for each i $k_i = 1$, the friendly concepts defined in Definitions 1, 3 and 4 reduce to the corresponding "friendless" concepts of Nash equilibrium, $\varepsilon-$perfect equilibrium, $\varepsilon-$proper equilibrium, perfect equilibrium and proper equilibrium.*

Remark 2 *It is clear from Definition 4 that all proper friendly equilibria are also perfect friendly equilibria, but the converse does not necessarily hold.*

3 Perfectly proper friendly equilibria

Definition 5 *Given a finite n-person non-cooperative game in normal form Γ, sequences $f_{(i)}^1, f_{(i)}^2, ..., f_{(i)}^{k_i}$ and $\varepsilon > 0$, then $s \in S$ is a $\varepsilon-$perfectly proper friendly equilibrium ($\varepsilon-$PPFE) if it is completely mixed and, for all $i, j \in \mathcal{N}$, satisfies the following conditions:*

i) $\forall \sigma_i \in \Psi_i^1(s), \forall \sigma_j \in \Psi_j^1(s), \forall \sigma_i^{`} \in \Sigma_i, \forall \overline{\sigma}_j \in \Sigma_j$

if $H_i(s/\sigma_i) - H_i(s/\sigma_i^{`}) < H_j(s/\sigma_j) - H_j(s/\overline{\sigma}_j)$ then
$s_j(\overline{\sigma}_j) \leq \varepsilon.s_i(\sigma_i^{`})$;

ii) $\forall \sigma_i \in \Psi_i^2(s), \forall \sigma_j \in \Psi_j^2(s), \forall \sigma_i^{`} \in \Psi_i^1(s), \forall \overline{\sigma}_j \in \Psi_j^1(s)$

if $H_{f_{(i)}^2}(s/\sigma_i) - H_{f_{(i)}^2}(s/\sigma_i^{`}) < H_{f_{(j)}^2}(s/\sigma_j) - H_{f_{(j)}^2}(s/\overline{\sigma}_j)$ then
$s_j(\overline{\sigma}_j) \leq \varepsilon.s_i(\sigma_i^{`})$;

$\vdots$

ki) $\forall \sigma_i \in \Psi_i^{k_i}(s), \forall \sigma_j \in \Psi_j^{k_j}(s), \forall \sigma_i^{`} \in \Psi_i^{k_i-1}(s), \forall \overline{\sigma}_j \in \Psi_j^{k_i-1}(s)$

if $H_{f_{(i)}^{k_i}}(s/\sigma_i) - H_{f_{(i)}^{k_i}}(s/\sigma_i^{`}) < H_{f_{(j)}^{k_j}}(s/\sigma_j) - H_{f_{(j)}^{k_j}}(s/\overline{\sigma}_j)$ then
$s_j(\overline{\sigma}_j) \leq \varepsilon.s_i(\sigma_i^{`})$.

If the friendly successors of i are $f_{(i)}^1, f_{(i)}^2, ..., f_{(i)}^m$, and those of j $f_{(j)}^1, f_{(j)}^2, ...,$ $f_{(j)}^n$, with $m < n$, then we define $f_{(i)}^r = f_{(i)}^m$ for all r, $r \in \{m+1, ..., n\}$.

Remark 3 *For $i = j$, Definition 5 is the definition of a $\varepsilon-$proper friendly equilibrium.*

Remark 4 *If $k_i = 1$ for all i, Definition 5 is the definition of a $\varepsilon-$perfectly proper equilibrium.*

Definition 6 *Given a finite n-person non-cooperative game in normal form Γ and sequences $f_{(i)}^1, f_{(i)}^2, ..., f_{(i)}^{k_i}$, a strategy combination $\overline{s} \in S$ is a perfectly proper friendly equilibrium (PPFE) if there exists sequences $\{\varepsilon_k\}_{k \in \mathrm{N}}$ and $\{s^k\}_{k \in \mathrm{N}}$ such that:*

$\forall k \in N \qquad \varepsilon_k > 0, \quad \lim_{k \to \infty} \varepsilon_k = 0.$

$\forall k \in N \qquad s^k$ is an ε_k-perfectly proper friendly equilibrium.

$\lim_{k \to \infty} s^k = \overline{s}.$

Theorem 7 .

a) *All perfectly proper friendly equilibria are perfectly proper equilibria.*

b) *All perfectly proper friendly equilibria are proper friendly equilibria.*

Proof. By the definitions and Remarks 3 and 4. ■

The following examples show that the converses of the statements of Theorem 7 do not necessarily hold.

Example 8 *Let player 2 be the friendly successor of player 1 in the following game:*

2, 2	1, 0	1, 0
2, 2	0, 1	2, 1
1, 2	1, 301	1, 2
1, 2	1, 1	1, 3
2, 2	1, 0	1, 0

In this game (α_2, β_1) and, for all $\mu \in [0,1]$, $(\mu\alpha_1 + (1-\mu)\,\alpha_5, \beta_1)$ are all perfectly proper equilibria, but only (α_2, β_1) is a PPFE, as may be seen by considering, for (α_2, β_1), the sequences $\{\varepsilon_k\}_{k \in N}$ and $\{s^k\}_{k \in N}$ defined by

$\forall k \quad \varepsilon_k = \frac{1}{k+2}.$

$s_1^k(\alpha_1) = \frac{1}{2(k+2)}, \quad s_1^k(\alpha_3) = \frac{1}{300(k+2)^3}, \quad s_1^k(\alpha_4) = \frac{1}{2(k+2)^3}, \quad s_1^k(\alpha_5) = \frac{1}{2(k+2)^2}$

$s_1^k(\alpha_2) = 1 - \frac{150(k+2)^2 + 150(k+2) + 151}{300(k+2)^3}.$

$s_2^k(\beta_1) = 1 - \frac{1+(k+2)^2}{150(k+2)^7}, \quad s_2^k(\beta_2) = \frac{1}{150(k+2)^7}, \quad s_2^k(\beta_3) = \frac{1}{150(k+2)^5}.$

and for $(\mu\alpha_1 + (1-\mu)\,\alpha_5, \beta_1)$ the sequences defined by

$\forall k \quad \varepsilon_k = \frac{1}{k+2}.$

$s_1^k(\alpha_2) = \frac{1}{k+2} =, \quad s_1^k(\alpha_3) = s_1^k(\alpha_4) = \frac{1}{(k+2)[300(k+2)-1]},$

$s_1^k(\alpha_1) = s_1^k(\alpha_5) = \frac{1}{2}\left[1 - \frac{1}{k+2} - \frac{2}{(k+2)[300(k+2)-1]}\right].$

$s_2^k(\beta_1) = 1 - \frac{k+3}{(k+2)^3[300(k+2)-1]}, \quad s_2^k(\beta_2) = \frac{1}{(k+2)^2[300(k+2)-1]},$

$s_2^k(\beta_3) = \frac{1}{(k+2)^3[300(k+2)-1]}.$

Example 9 *Let player 2 be the friendly successor of player 1 in the following 3-player game:*

1, 1, 1	0, 0, 1
0, 0, 1	0, 0, 1

α_1^3

0, 0, 0	0, 0, 0
0, 0, 0	1, 1, 0

α_2^3

In this game $(\alpha_2^1, \alpha_2^2, \alpha_1^3)$ and $(\alpha_1^1, \alpha_1^2, \alpha_1^3)$ are both proper friendly equilibria, but $(\alpha_2^1, \alpha_2^2, \alpha_1^3)$ is not a PPFE.

Theorem 10 *Every finite non-cooperative game* Γ *with friends for which the* $\Psi_i^r(s)$ *are upper semicontinuous has at least one PPFE.*

Proof.

Let $\gamma = \frac{\varepsilon^{(\alpha_1+\alpha_2+\ldots+\alpha_r)\sum\limits_{i=1}^{n} m_i}}{\sum\limits_{i=1}^{n} m_i}$ where $\varepsilon \in (0,1)$, $m_i = |\Sigma_i|$ and

$r = |\{1, ..., k_i\}|$, $\alpha_k = \left(1 + \sum\limits_{i=1}^{n} m_i\right)^{r-k}$ $(1 \leq k \leq r)$.

$\forall i$, let $S_i(\gamma) = \{s_i \in S_i / \forall \sigma_i \in \Sigma_i \quad s_i(\sigma_i) \geq \gamma\}$; and let

$S(\gamma) = S_1(\gamma) \times ... \times S_n(\gamma)$. $S(\gamma)$ is compact, convex and non-empty.

Consider the correspondence $F : S(\gamma) \to \mathcal{P}(S(\gamma))$such that, $\forall s' \in S(\gamma)$

$F\left(s^{,}\right) = \{s \in S\left(\gamma\right) / \ \forall i, j \in N,$

a) $\forall \sigma_i \in \Psi_i^1\left(s^{,}\right), \sigma_j \in \Psi_j^1\left(s^{,}\right), \ \forall \sigma_i^{,} \in \Sigma_i, \overline{\sigma}_j \in \Sigma_j$

if $H_i\left(s^{,}/\sigma_i\right) - H_i\left(s^{,}/\sigma_i^{,}\right) < H_j\left(s^{,}/\sigma_j\right) - H_j\left(s^{,}/\overline{\sigma}_j\right)$ then $s_j\left(\overline{\sigma}_j\right) \leq \varepsilon.s_i\left(\sigma_i^{,}\right);$

b) $\forall \sigma_i \in \Psi_i^2\left(s\right), \sigma_j \in \Psi_j^2\left(s\right), \ \forall \sigma_i^{,} \in \Psi_i^1\left(s\right), \overline{\sigma}_j \in \Psi_j^1\left(s\right)$

if $H_{f_{(i)}^2}\left(s/\sigma_i\right) - H_{f_{(i)}^2}\left(s/\sigma_i^{,}\right) < H_{f_{(j)}^2}\left(s/\sigma_j\right) - H_{f_{(j)}^2}\left(s/\overline{\sigma}_j\right)$ then

$s_j\left(\overline{\sigma}_j\right) \leq \varepsilon.s_i\left(\sigma_i^{,}\right);$

$\vdots$

r) $\forall \ \sigma_i \in \Psi_i^r\left(s\right), \sigma_j \in \Psi_j^r\left(s\right), \ \forall \sigma_i^{,} \in \Psi_i^{r-1}\left(s\right), \overline{\sigma}_j \in \Psi_j^{r-1}\left(s\right)$

if $H_{f_{(i)}^r}\left(s/\sigma_i\right) - H_{f_{(i)}^r}\left(s/\sigma_i^{,}\right) < H_{f_{(j)}^r}\left(s/\sigma_j\right) - H_{f_{(j)}^r}\left(s/\overline{\sigma}_j\right)$ then

$s_j\left(\overline{\sigma}_j\right) \leq \varepsilon.s_i\left(\sigma_i^{,}\right) \}.$

We shall show that F complies with the requirements of the Kakutani fixed point theorem. We first note that $\forall \ s \in S\left(\gamma\right), \ F\left(s\right)$ is convex and compact. We now show that $\forall \ s \in S\left(\gamma\right)$, $F\left(s\right) \neq \emptyset$.

Let $s \in S\left(\gamma\right). \ \forall i \in \mathcal{N}, \ \sigma_i^{,} \in \Sigma_i$ we define

$$A_i\left(s/\sigma_i^{,}\right) = \sum_{j=1}^{n} \left|\{\overline{\sigma}_j \in \Sigma_j \ /\forall \sigma_i \in \Psi_i^1\left(s\right), \forall \sigma_j \in \Psi_j^1\left(s\right)\right.$$

$$\left. H_j\left(s/\sigma_j\right) - H_j\left(s/\overline{\sigma}_j\right) < H_i\left(s/\sigma_i\right) - H_i\left(s/\sigma_i^{,}\right)\}\right|;$$ and $\forall \ k = \{2, ..., r\}$,

$\forall \sigma_i^{,} \in \Psi_i^{k-1}\left(s\right)$ we define

$$A_{f_{(i)}^k}\left(s/\sigma_i^{,}\right) = \sum_{j=1}^{n} \left|\left\{\overline{\sigma}_j \in \Psi_j^{k-1}\left(s\right) /\forall \sigma_i \in \Psi_i^k\left(s\right), \forall \sigma_j \in \Psi_j^k\left(s\right)\right.\right.$$

$$\left.\left. H_{f_{(j)}^k}\left(s/\sigma_j\right) - H_{f_{(j)}^k}\left(s/\overline{\sigma}_j\right) < H_{f_{(i)}^k}\left(s/\sigma_i\right) - H_{f_{(i)}^k}\left(s/\sigma_i^{,}\right)\right\}\right|.$$

Consider $\overline{s} = (\overline{s}_1, ..., \overline{s}_n)$, defined by

$$\overline{s}_i(\sigma_i) = \frac{\varepsilon^{\sum_{k=1}^{r} \alpha_k A_{f_{(i)}^k}(s/\sigma_i)}}{\sum_{u=1}^{n} \sum_{\sigma'_u \in \Sigma_u} \varepsilon^{\sum_{k=1}^{r} \alpha_k A_{f_{(u)}^k}(s/\sigma'_u)}} \qquad \forall \sigma_i \notin \Psi_i^r(s)$$

$$\overline{s}_i(\sigma_i) = \frac{1 - \sum_{\sigma'_i \notin \Psi_i^r(S)} s_i(\sigma'_i)}{|\Psi_i^r(s)|} \qquad \forall \sigma_i \in \Psi_i^r(s).$$

$\overline{s} \in S$, because $\forall i \in \mathcal{N}$, $\sigma_i \in \Sigma_i$

$\overline{s}_i(\sigma_i) \geq 0$, and $\sum_{\sigma_i \in \Sigma_i} \overline{s}_i(\sigma_i) = 1$.

Moreover, $\overline{s} \in S(\gamma)$

i.e. $\forall \sigma_i \in \Sigma_i \quad \overline{s}_i(\sigma_i) \geq \gamma$: for

a) if $\sigma_i \notin \Psi_i^r(s)$, then $\overline{s}_i(\sigma_i) \geq \gamma$ follows from

$$\sum_{k=1}^{r} \alpha_k A_{f_{(i)}^k}(s/\sigma_i) \leq \sum_{k=1}^{r} \alpha_k \sum_{i=1}^{n} m_i$$

and

$$\sum_{u=1}^{n} \sum_{\sigma'_u \in \Sigma_u} \varepsilon^{\sum_{k=1}^{r} \alpha_k A_{f_{(u)}^k}(s/\sigma'_u)} \leq \sum_{u=1}^{n} m_u$$

and **b)** if $\sigma_i \in \Psi_i^r(s)$, then $\overline{s}_i(\sigma_i) \geq \gamma$ follows from

$$1 - \sum_{\sigma'_i \notin \Psi_i^r(S)} s_i(\sigma'_i) = \frac{\sum_{\substack{u=1 \\ u \neq i}}^{n} \sum_{\sigma'_u \in \Sigma_u} \varepsilon^{\sum_{k=1}^{r} \alpha_k A_{f_{(u)}^k}(s/\sigma'_u)} + \sum_{\sigma'_i \in \Psi_i^r(S)} \varepsilon^{\sum_{k=1}^{r} \alpha_k A_{f_{(u)}^k}(s/\sigma'_u)}}{\sum_{u=1}^{n} \sum_{\sigma'_u \in \Sigma_u} \varepsilon^{\sum_{k=1}^{r} \alpha_k A_{f_{(u)}^k}(s/\sigma'_u)}}$$

$$\geq \frac{\sum\limits_{\sigma_i^{,}\in\Psi_i^r(s)} \varepsilon^{\sum\limits_{k=1}^{r}\alpha_k A_{f_{(u)}^k}(s/\sigma_u^{,})}}{\sum\limits_{u=1}^{n}\sum\limits_{\sigma_u^{,}\in\Sigma_u} \varepsilon^{\sum\limits_{k=1}^{r}\alpha_k A_{f_{(u)}^k}(s/\sigma_u^{,})}} \geq \left|\Psi_i^r(s)\right|.\gamma.$$

We now show that $\overline{s} \in F(s)$.

$\forall t,\ t \in \{1, ..., r\}$, consider any $\widetilde{\sigma}_i \in \Psi_i^{t-1}(s), \overline{\sigma}_j \in \Psi_j^{t-1}(s)$ such that $\forall \sigma_i \in \Psi_i^t(S), \sigma_j \in \Psi_j^t(S)$

$$H_{f_{(i)}^t}(s/\sigma_i) - H_{f_{(i)}^t}(s/\widetilde{\sigma}_i) < H_{f_{(j)}^t}(s/\sigma_j) - H_{f_{(j)}^t}(s/\overline{\sigma}_j)$$

Clearly, $\overline{\sigma}_j \notin \Psi_j^t(s)$ and

$A_{f_{(j)}^t}(s/\overline{\sigma}_j) \geq 1 + A_{f_{(i)}^t}(s/\widetilde{\sigma}_i)$.

Now either **a)** $\widetilde{\sigma}_i \notin \Psi_i^t(s)$ or **b)** $\widetilde{\sigma}_i \in \Psi_i^t(s)$.

a) If $\widetilde{\sigma}_i \notin \Psi_i^t(s)$ then $\widetilde{\sigma}_i \notin \Psi_i^r(s)$, whence

$$\overline{s}_j(\overline{\sigma}_j) = \frac{\varepsilon^{\sum\limits_{k=1}^{r}\alpha_k A_{f_{(j)}^k}(s/\overline{\sigma}_j)}}{\sum\limits_{u=1}^{n}\sum\limits_{\sigma_u^{,}\in\Sigma_u} \varepsilon^{\sum\limits_{k=1}^{r}\alpha_k A_{f_{(u)}^k}(s/\sigma_u^{,})}} \leq \frac{\varepsilon^{\sum\limits_{\substack{k=1\\k\neq i}}\alpha_k A_{f_{(j)}^k}(s/\overline{\sigma}_j)+\alpha_t\left(1+A_{f_{(i)}^t}(s/\widetilde{\sigma}_i)\right)}}{\sum\limits_{u=1}^{n}\sum\limits_{\sigma_u^{,}\in\Sigma_u} \varepsilon^{\sum\limits_{k=1}^{r}\alpha_k A_{f_{(u)}^k}(s/\sigma_u^{,})}}$$

$$\leq \frac{\varepsilon^{1+(\alpha_t-1)+\alpha_t A_{f_{(i)}^t}(s/\widetilde{\sigma}_i)}}{\sum\limits_{u=1}^{n}\sum\limits_{\sigma_u^{,}\in\Sigma_u} \varepsilon^{\sum\limits_{k=1}^{r}\alpha_k A_{f_{(u)}^k}(s/\sigma_u^{,})}} = \varepsilon\frac{\varepsilon^{(\alpha_t-1)+\alpha_t A_{f_{(i)}^t}(s/\widetilde{\sigma}_i)}}{\sum\limits_{u=1}^{n}\sum\limits_{\sigma_u^{,}\in\Sigma_u} \varepsilon^{\sum\limits_{k=1}^{r}\alpha_k A_{f_{(u)}^k}(s/\sigma_u^{,})}}$$

$$\leq \varepsilon\frac{\varepsilon^{\alpha_t A_{f_{(i)}^t}(s/\widetilde{\sigma}_i)+\alpha_{t+1}A_{f_{(i)}^{t+1}}(s/\widetilde{\sigma}_i)+...+\alpha_r A_{f_{(i)}^r}(s/\widetilde{\sigma}_i)}}{\sum\limits_{u=1}^{n}\sum\limits_{\sigma_u^{,}\in\Sigma_u} \varepsilon^{\sum\limits_{k=1}^{r}\alpha_k A_{f_{(u)}^k}(s/\sigma_u^{,})}}$$

and since $A_{f_{(i)}^t}(s/\widetilde{\sigma}_i) = 0$ when $\widetilde{\sigma}_i \in \Psi_i^{t-1}(s)$

$$\overline{s}_j\left(\overline{\sigma}_j\right) \leq \varepsilon.\overline{s}_i\left(\widetilde{\sigma}_i\right).$$

b) If $\widetilde{\sigma}_i \in \Psi_i^t(s)$, then either

i) $\exists\ p \in \{t+1, ..., r\}$ such that $\widetilde{\sigma}_i \notin \Psi_i^p(s)$ in which case an argument analogous that developed in **(a)** leads to the conclusion

$$\overline{s}_j\left(\overline{\sigma}_j\right) \leq \varepsilon.\overline{s}_i\left(\widetilde{\sigma}_i\right);$$

or **ii)** $\widetilde{\sigma}_i \in \Psi_i^r(s)$, in which case

$A_{f_{(i)}^r}\left(s/\widetilde{\sigma}_i\right) = ... = A_i\left(s/\widetilde{\sigma}_i\right) = 0$, whence

$$\varepsilon.s_i\left(\widetilde{\sigma}_i\right) = \varepsilon.\frac{1 - \sum\limits_{\sigma_i^{,} \notin \Psi_i^r(s)} s_i\left(\sigma_i^{,}\right)}{\left|\Psi_i^r(s)\right|} \geq \varepsilon \frac{\sum\limits_{\sigma_i^{,} \in \Psi_i^r(s)} \varepsilon^{\sum_{k=1}^{r} \alpha_k A_{f_{(u)}^k}\left(s/\sigma_u^{,}\right)}}{\left|\Psi_i^r(s)\right| \sum\limits_{u=1}^{n} \sum\limits_{\sigma_u^{,} \in \Sigma_u} \varepsilon^{\sum_{k=1}^{r} \alpha_k A_{f_{(u)}^k}\left(s/\sigma_u^{,}\right)}}$$

$$\geq \varepsilon \frac{\sum\limits_{\sigma_i^{,} \in \Psi_i^r(s)} \varepsilon^{\alpha_t A_{f_{(j)}^t}\left(s/\overline{\sigma}_j\right) - (\alpha_t - 1) - 1}}{\left|\Psi_i^r(s)\right| \sum\limits_{u=1}^{n} \sum\limits_{\sigma_u^{,} \in \Sigma_u} \varepsilon^{\sum_{k=1}^{r} \alpha_k A_{f_{(u)}^k}\left(s/\sigma_u^{,}\right)}} \geq \overline{s}_j\left(\overline{\sigma}_j\right).$$

Hence $\overline{s}_j\left(\overline{\sigma}_j\right) \leq \varepsilon.\overline{s}_i\left(\widetilde{\sigma}_i\right)$, whence $\overline{s} \in F(s)$ and $F(s)$ is non-empty.

Since F is also upper semicontinuous (because, by hypothesis, the $\Psi_i^k(s)$ are), the conditions of the Kakutani fixed point theorem are satisfied and F has a fixed point, which is therefore an $\varepsilon-$PPFE. Hence, given $\{\varepsilon_k\}_{k \in N}$ such that $\lim\limits_{k \to \infty} \varepsilon_k = 0$, there is a sequence of $\varepsilon-$PPFE $\left\{s^k\right\}_{k \in N}$; and since S is compact, this sequence has a limit point, which is therefore a PPFE. ■

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