MULTISCALE ANALYSIS OF HETEROGENEOUS MEDIA IN THE PERIDYNAMIC FORMULATION

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Multiscale Analysis of Heterogeneous Media in the Peridynamic Formulation*

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Abstract

A rigorous multiscale method is presented for modeling the dynamics of fiber-reinforced composite structures using the peridynamic formulation. The multiscale analysis delivers a new multiscale numerical method that captures the dynamics at structural length scales while at the same time is capable of resolving the dynamics at the length scales of the fiber reinforcement. The new numerical method is able extract this information at a cost that is anticipated to be far less than the direct numerical simulation of structural components made from multiple plys containing thousands of fibers.

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1 Introduction

The peridynamic formulation introduced in [21] is a nonlocal continuum theory for deformable bodies that does not use the spatial derivatives of the displacement field. Interactions between material particles are characterized by a pairwise force field that acts across a finite *horizon*, see Section 1.1. The same equations of motion are applicable over the entire body and no special treatment is required near or at defects. These properties make it a powerful tool to model problems that involve cracks, interfaces, and other defects, see [2, 3, 14, 22, 23, 24]. This work focuses on the multiscale analysis of heterogeneous media using the peridynamic formulation. The objective is to provide numerical methods that capture the dynamics inside composites at both the structural scale and the microscopic scale with a cost far below that of direct numerical simulation.

We consider particle or fiber reinforced composites. Here the characteristic length scale of the particle or fiber reinforced geometry is assumed to be very small relative to the length scale of the applied loads. The length scale of the microstructure is denoted by ε. We study three peridynamic models of fiber-reinforced materials. In the first model, which we call "the short-range bond model", the peridynamic horizon is of the same length scale as that of the microstructure and the horizon approaches zero as ε goes to zero. In the second

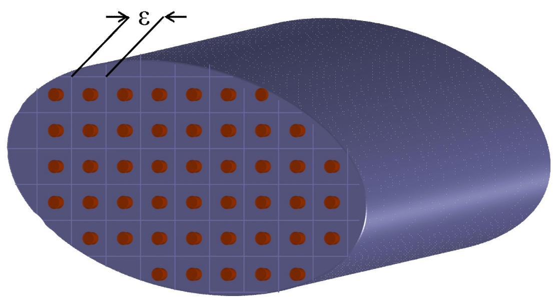

Figure 1: Fiber-reinforced composite.

model, a long-range ε-independent pairwise force is added to the short-range pairwise force of the first model. Here the long-range pairwise force depends only on the relative position of the two particles and the associated peridynamic horizon is fixed and independent of the microstructure length scale ε. We will refer to the second model as "the short-range and long-range bond model". In the third model, we consider a long-range pairwise force that fluctuates with the microstructure. The peridynamic horizon in this model is fixed and independent of ε. This model will be called "the fluctuating long-range bond model". In all of these models, the peridynamic initial value problem is a partial integro-differential equation with rapidly-oscillating coefficients supplemented with initial conditions.

For the first two models the concept of two-scale convergence, introduced by Nguetseng [18] and Allaire [1], is used as a tool to identify both the macroscopic and microscopic dynamics inside the composite. A downscaling method obtained through the use of Semigroup theory provides a strong approximation for capturing the mirco-level fluctuations about the macroscopic displacement field. The multiscale approximation obtained for the first two models are shown to be good approximations to the actual solution in the L^p norm when the microstructure is sufficiently fine. Explicit error estimates are provided for sufficiently regular initial and loading data for the first model. This multiscale analysis provides the theoretical framework for a new multiscale numerical method for computing the deformation of fiber-reinforced composites in the presence of residual forces. The multiscale numerical method delivered here captures the dynamics at structural length scales while at the same time is capable of resolving the dynamics at the length scales of the fiber reinforcement. The new numerical method is able extract this information at a cost that is anticipated to be far less than the direct numerical simulation of structural components made from multiple plys containing thousands of fibers.

For the third model, the Semigroup theory of linear operators [12, 13] is utilized to identify both the macroscopic and microscopic dynamics of the composite. These are used to develop an approximation to the actual solution that is shown to be a good approximation to the actual solution in the L^p norm when the microstructure is sufficiently fine. Explicit error estimates for the approximation are provided for this model. Last, the corresponding multiscale numerical scheme is presented.

This report is organized as follows. Section 1.1 provides an overview of the peridynamic formulation of continuum mechanics. In Section 1.2, we introduce three peridynamic models of fiber-reinforced composites. The results for the first two models are discussed and derived in Sections 2-5. In Section 2, we present a multiscale analysis method for these two models. Section 3 provides uniqueness and existence results for the linear peridynamic initial-value problem (1.10)-(1.12). In Section 4, we review two-scale convergence and then use it to identify the two-scale asymptotic limit of (1.10)-(1.12). In Section 5, we build on the analysis provided in Section 4 to justify the results of Section 2. Section 6 is devoted to the third peridynamic model of fiber-reinforced composites. A multiscale analysis method is presented and justified for this model.

1.1 The Peridynamic Formulation of Continuum Mechanics

In the peridynamic theory, the time evolution of the displacement vector field u, in a heterogeneous medium, is given by the partial integro-differential equation

$$\rho(x)\,\partial_t^2 u(x,t) = \int_{H_x} f(u(\hat{x},t)-u(x,t),\hat{x}-x,x)\,d\hat{x} + b(x,t), \quad (x,t)\in\Omega\times(0,T) \tag{1.1}$$

where H_x is a neighborhood of x, ρ is the mass density, b is a prescribed loading force density field, and Ω is a bounded set in $\mathbb{R}^3$. Here f denotes the pairwise force field whose value is the force vector (per unit volume squared) that the particle at $\hat{x}$ exerts on the particle at x. For a homogeneous medium f is of the form $f(u(\hat{x},t)-u(x,t),\hat{x}-x)$, i.e., it depends only on the relative position of the two particles. We will often refer to f as a *bond force.* Equation (1.1) is supplemented with initial conditions for $u(x,0)$ and $\partial_t u(x,0)$. For the sake of simplicity, we assume constant mass density given by $\rho(x)=1$. However, the removal of this hypothesis presents no barrier to the subsequent analysis. For the purposes of discussion it will be convenient to set

$$\xi = \hat{x} - x,$$

which represents the relative position of these two particles in the reference configuration, and

$$\eta = u(\hat{x},t) - u(x,t),$$

which represents their relative displacement (see Figure 2). In the peridynamic formulation, it is assumed that for a given material there is a positive number δ, called the horizon, such that

$$f(\eta,\xi,x) = 0, \text{ for } |\xi| > \delta.$$

The pairwise force field f is required to satisfy the following properties:

$$f(-\eta,-\xi,x+\xi) = -f(\eta,\xi,x) \tag{1.2}$$

which assures conservation of linear momentum, and

$$(\xi+\eta)\times f(\eta,\xi,x) = 0$$

which assures conservation of angular momentum.

A material is said to be *microelastic* if the pairwise force is derivable from a scalar *micropotential* ω

$$f(\eta,\xi,x) = \frac{\partial\omega}{\partial\eta}(\eta,\xi,x).$$

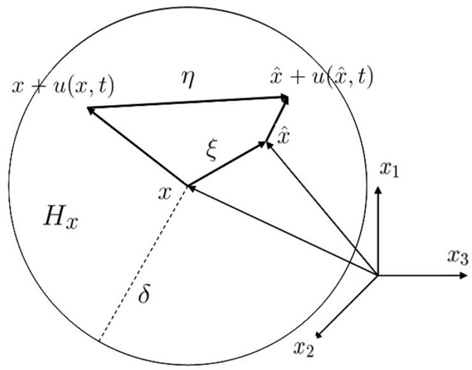

Figure 2: Deformation of a bond within the peridynamic horizon.

It can be shown that for a microelastic material the pairwise force is of the form (see [21])

$$f(\eta, \xi, x) = H(|\xi + \eta|, \xi, x)(\xi + \eta),$$

where H is a real-valued function. Finally, a material is linear if the associated bond force $f(\eta, \xi, x)$ is linear in η.

In this treatment, all materials will be taken to be microelastic and linear.

1.2 Three Peridynamic Models of Fiber-Reinforced Materials

To fix ideas, we consider a periodic medium of unidirectional fiber-reinforced material. Here the pairwise force is given by the linearized version of the *bond-stretch model* proposed in [24]

$$f(\eta, \xi, x) = \alpha(x, x + \xi)\, \frac{\xi \otimes \xi}{|\xi|^3} \eta, \quad \text{for } \xi \in H_x.$$

Here α is a real-valued function satisfying $\alpha(x, \hat{x}) = \alpha(\hat{x}, x)$. We will study three different peridynamic models for this composite. These models are distinct in the way the coefficient α and the neighborhood set H_x are defined. We start by providing the mathematical description of the periodic microgeometry.

Let $Y \subset \mathbb{R}^3$ be a unit cube and the local coordinates inside Y are denoted by y with the origin at the center of the unit cube. The unit cube is composed of a fiber which is surrounded by a second material called the matrix material, see Figure 3. Let χ_{f} denote the indicator function of the set occupied by the fiber material and χ_{m} denote the the indicator function of the set occupied by the matrix material. Here χ_{f} is given by

$$\chi_{\text{f}}(y) = \begin{cases} 1, & \text{y is in the fiber phase,} \\ 0, & \text{otherwise,} \end{cases}$$

and χ_{m} is given by

$$\chi_{\text{m}}(y) = 1 - \chi_{\text{f}}(y).$$

We extend the functions χ_{f} and χ_{m} to $\mathbb{R}^3$ by periodicity. For future reference, we denote by θ_{f} and θ_{m} the volume fractions of the fiber material and the matrix material, respectively.

Here $\theta_{\rm f} = \int_Y \chi_{\rm f}(y)dy$ and $\theta_{\rm m} = 1 - \theta_{\rm f}$. Also, we let n denote a unit vector parallel to the fiber direction.

In the first model, the short-range pairwise force is given by

$$f_{\rm short}(\eta_y, \xi_y, y) = \begin{cases} \alpha(y, y+\xi_y)\dfrac{\xi_y \otimes \xi_y}{|\xi_y|^3}\eta_y, & |\xi_y| \le \delta \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

where $y \in Y$, $\xi_y = \hat{y} - y$, $\eta_y = u(\hat{y},t) - u(y,t)$, and α is given by

$$\alpha(y,\hat{y}) = C_{\rm f}\,\chi_{\rm f}(y)\chi_{\rm f}(\hat{y}) + C_{\rm m}\,\chi_{\rm m}(y)\chi_{\rm m}(\hat{y}) + C_{\rm i}\left(\chi_{\rm f}(y)\chi_{\rm m}(\hat{y}) + \chi_{\rm m}(y)\chi_{\rm f}(\hat{y})\right). \tag{1.4}$$

We note that (1.3)-(1.4) give the pairwise force on $\mathbb{R}^3$ associated with a unit periodic geometry. In summary, the function α in (1.4) is given by

$$\alpha(y,\hat{y}) = \begin{cases} C_{\rm f}, & \text{if } y \text{ and } \hat{y} \text{ are in the fiber phase} \\ C_{\rm m}, & \text{if } y \text{ and } \hat{y} \text{ are in the matrix phase} \\ C_{\rm i}, & \text{otherwise.} \end{cases}$$

In equation (1.3), the peridynamic horizon δ is chosen to be smaller than the fiber thickness in the unit cell. The material parameters $C_{\rm f}$ and $C_{\rm m}$ are intrinsic to each phase and can be determined through experiments. Bonds connecting particles in the different materials are characterized by $C_{\rm i}$, which can be chosen such that $C_{\rm f} > C_{\rm i} > C_{\rm m} > 0$, see [24].

The microgeometry associated with the length scale ε is obtained by rescaling the bond force $f_{\rm short}$ as follows. For $x \in \Omega$,

$$f^{\varepsilon}_{\rm short}(\eta, \xi, x) = \begin{cases} \dfrac{1}{\varepsilon^2}\,\alpha\left(\dfrac{x}{\varepsilon}, \dfrac{x+\xi}{\varepsilon}\right)\,\dfrac{\xi \otimes \xi}{|\xi|^3}\eta, & |\xi| \le \varepsilon\delta \\ 0\,, & \text{otherwise.} \end{cases}$$

We see from (1.4) that $\alpha(\frac{x}{\varepsilon}, \frac{\hat{x}}{\varepsilon})$ is given by

$$\alpha\left(\frac{x}{\varepsilon}, \frac{\hat{x}}{\varepsilon}\right) = C_{\rm f}\,\chi^{\varepsilon}_{\rm f}(x)\chi^{\varepsilon}_{\rm f}(\hat{x}) + C_{\rm m}\,\chi^{\varepsilon}_{\rm m}(x)\chi^{\varepsilon}_{\rm m}(\hat{x}) + C_{\rm i}\left(\chi^{\varepsilon}_{\rm f}(x)\chi^{\varepsilon}_{\rm m}(\hat{x}) + \chi^{\varepsilon}_{\rm m}(x)\chi^{\varepsilon}_{\rm f}(\hat{x})\right), \tag{1.5}$$

where $\chi^{\varepsilon}_{\rm f}(x) := \chi_{\rm f}(\frac{x}{\varepsilon})$ and $\chi^{\varepsilon}_{\rm m}(x) := \chi_{\rm m}(\frac{x}{\varepsilon})$.

The peridynamic equation of motion for this model is given by

$$\partial^2_t u^{\varepsilon}(x,t) = \int_{H_{\varepsilon\delta}(x)} \frac{1}{\varepsilon^2}\alpha\left(\frac{x}{\varepsilon}, \frac{\hat{x}}{\varepsilon}\right)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(u^{\varepsilon}(\hat{x},t) - u^{\varepsilon}(x,t)\right)d\hat{x} + b\left(x, \frac{x}{\varepsilon}, t\right) \tag{1.6}$$

supplemented with initial conditions

$$u^{\varepsilon}(x,0) = u^0\left(x, \frac{x}{\varepsilon}\right), \tag{1.7}$$

$$\partial_t u^{\varepsilon}(x,0) = v^0\left(x, \frac{x}{\varepsilon}\right). \tag{1.8}$$

In what follows, we will denote by s a real number such that $\frac{3}{2} < s < \infty$. In (1.6)-(1.8), $b(x,y,t)$ is in $C([0,T]; L^s(\Omega \times Y)^3)$ and Y-periodic in y and $u^0(x,y)$ and $v^0(x,y)$ are in $L^s(\Omega \times Y)^3$ and Y-periodic in y.

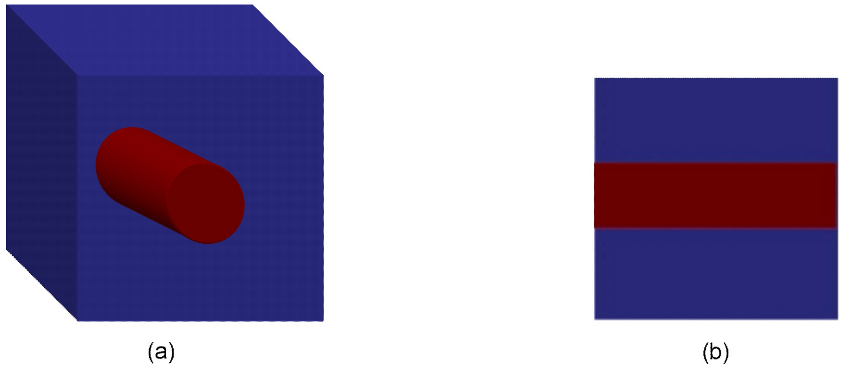

Figure 3: (a) Composite cube Y. (b) Cross-section of Y along the fiber direction.

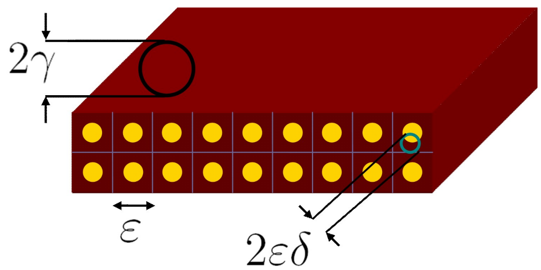

Figure 4: Long-range bonds (horizon γ) and short-range bonds (horizon $\varepsilon\delta$).

In the second model, the following long-range pairwise force is added to the short-range pairwise force of the first model (see Figure 4)

$$f_{\text{long}}(\eta,\xi) = \begin{cases} \lambda(\xi)\dfrac{\xi\otimes\xi}{|\xi|^3}\eta, & |\xi|\leq\gamma \\ 0, & \text{otherwise,} \end{cases}$$

where γ is a prescribed peridynamic horizon. Here λ is a real-valued function defined by

$$\lambda(\xi) = \begin{cases} C_{\text{f}}^{\text{M}}, & \nu_\xi \leq \dfrac{\pi}{2}\,\theta_{\text{f}}, \\ C_{\text{m}}^{\text{M}}, & \text{otherwise,} \end{cases} \tag{1.9}$$

where ν_ξ denotes the angle between ξ and a line parallel to the fiber direction, with $0 \leq \nu_\xi \leq \frac{\pi}{2}$. The constants C_{f}^{M} and C_{m}^{M} are macroscopic parameters determined through experiments, see [24, 10].

Now the peridynamic equation of motion associated with the total pairwise force is given by

$$\begin{aligned} \partial_t^2 u^\varepsilon(x,t) &= \int_{H_\gamma(x)} \lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(u^\varepsilon(\hat{x},t)-u^\varepsilon(x,t)\right)d\hat{x} \\ &+ \int_{H_{\varepsilon\delta}(x)} \frac{1}{\varepsilon^2}\,\alpha\left(\frac{x}{\varepsilon},\frac{\hat{x}}{\varepsilon}\right)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(u^\varepsilon(\hat{x},t)-u^\varepsilon(x,t)\right)d\hat{x} \\ &+ b\left(x,\frac{x}{\varepsilon},t\right), \end{aligned} \tag{1.10}$$

supplemented with initial conditions

$$u^\varepsilon(x,0) = u^0\left(x,\frac{x}{\varepsilon}\right), \tag{1.11}$$

$$\partial_t u^\varepsilon(x,0) = v^0\left(x,\frac{x}{\varepsilon}\right). \tag{1.12}$$

Remark 1. The first model follows from the second model on setting $\lambda = 0$. Thus in Sections 2-5, we will often present our results and analysis for the second model only.

In the third model, the pairwise force is given by

$$f(\eta_y,\xi_y,y) = \begin{cases} \alpha_L(y,y+\xi_y)\dfrac{\xi_y\otimes\xi_y}{|\xi_y|^3}\eta_y, & |\xi_y|\le\delta \\ 0, & \text{otherwise,} \end{cases}$$

where $y\in Y$ and δ is a prescribed peridynamic horizon, and α_L is given by

$$\alpha_L(y,y+\xi_y) = \begin{cases} C_{\rm f}\,|\xi_y|\,\delta_n(\xi_y), & \text{if } y \text{ and } y+\xi_y \text{ are in the fiber phase,} \\ & \text{and } \xi_y \text{ is parallel to } n, \\ C_{\rm m}\,|\xi_y|\,, & \text{otherwise.} \end{cases}$$

Here δ_n is the Dirac delta distribution concentrated at a line parallel to n. The function α_L can be written in terms of $\chi_{\rm f}$ as follows

$$\alpha_L(y,y+\xi_y) = C_{\rm f}\,|\xi_y|\,\delta_n(\xi_y)\,\chi_{\rm f}(y)\chi_{\rm f}(y+\xi_y) + C_{\rm m}\,|\xi_y|\,\left(1-\delta_n(\xi_y)\,\chi_{\rm f}(y)\chi_{\rm f}(y+\xi_y)\right). \tag{1.13}$$

We note that in equation (1.13), $\chi_{\rm f}(y) = \chi_{\rm f}(y+\xi_y)$ because y and $y+\xi_y$ both lie on a line parallel to the fiber direction n.

The the pairwise force defined on Ω is given by

$$f^\varepsilon(\eta,\xi,x) = \begin{cases} \alpha_L^\varepsilon(x,x+\xi)\,\dfrac{\xi\otimes\xi}{|\xi|^3}\eta, & |\xi|\le\delta \\ 0\,, & \text{otherwise,} \end{cases}$$

where α_L^ε is defined by

$$\alpha_L^\varepsilon(x,x+\xi) = C_{\rm f}\,|\xi|\,\delta_n(\xi)\,\chi_{\rm f}^\varepsilon(x) + \varepsilon\, C_{\rm m}\,|\xi|\,\left(1-\delta_n(\xi)\,\chi_{\rm f}^\varepsilon(x)\right). \tag{1.14}$$

The peridynamic equation of motion for this model is given by

$$\partial_t^2 u^\varepsilon(x,t) = \int_{H_\delta(x)} \alpha^\varepsilon(x,\hat{x})\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(u^\varepsilon(\hat{x},t)-u^\varepsilon(x,t)\right)d\hat{x} \tag{1.15}$$

supplemented with initial data

$$u^\varepsilon(x,0) = u^0(x)\,, \tag{1.16}$$

$$\partial_t u^\varepsilon(x,0) = v^0(x)\,. \tag{1.17}$$

Here the initial data u^0 and v^0 are in $L^p(\Omega)^3$ with $1\le p<\infty$ and the loading force in equation (1.15) is zero.

2 Multiscale Analysis and the Numerical Scheme for the Short-Range and Long-Range Bond Model

In this section, we present the multiscale analysis method for computing the deformation of fiber-reinforced composites modeled by the peridynamic formulation. This is done for the Short-Range and Long-Range Bond model described in Section 1.2. The method delivers a computationally inexpensive multiscale numerical scheme for the analysis of these peridynamic models of fiber-reinforced materials. It consists of the following three steps.

1. **Macroscopic Equation**
   Compute the macroscopic or average displacement field $u^H(x,t)$ by solving a peridynamic macroscopic equation.

2. **Cell–Problem**
   Compute the micro-level displacement field $r(y,t)$ by solving a peridynamic problem on a single period cell.

3. **Downscaling**
   The displacement field of the oscillatory peridynamic equation is given approximately by superimposing the rescaled micro-level mechanical responses over the average displacement field, i.e., $u^\varepsilon_{Approx} = u^H(x,t) + r(x/\varepsilon, t)$. The error in this approximation is shown to converge in norm to zero, i.e., $\|u^\varepsilon(x,t) - u^\varepsilon_{Approx}(x,t)\| \to 0$ as $\varepsilon \to 0$.

In the following subsections, we consider four cases of initial and loading conditions. For each case, we present the macroscopic equation, the cell-problem, and the associated approximation. The results provided in this section are justified in Section 5.

For convenience, we introduce the following notation for the average of a periodic function. Let a function of the form $p(y)$, $p(x,y)$, or $p(x,y,t)$ be Y-periodic in the variable y. Its average over Y is denoted by

$$\begin{aligned} \bar{p} &= \int_Y p(y)\,dy, \\ \bar{p}(x) &= \int_Y p(x,y)\,dy, \text{ or} \\ \bar{p}(x,t) &= \int_Y p(x,y,t)\,dy, \end{aligned}$$

respectively. For future reference, we let

$$K = \int_{H_\gamma(x)} \lambda(\hat{x} - x) \frac{(\hat{x} - x) \otimes (\hat{x} - x)}{|\hat{x} - x|^3}\, d\hat{x}\ . \tag{2.1}$$

By the change of variables $\xi = \hat{x} - x$, it is easy to see that K is a constant matrix, which depends on the macroscopic parameters γ, C^M_{matrix}, and C^M_{fiber}.

For future reference, we will adopt the notation $L^p_{per}(Y)$ for the space of Lebesgue p-integrable functions which are Y-periodic. Similarly, $C_{per}(Y)$ denotes the space of continuous Y-periodic functions. Also we denote by $C^{0,\beta}(\bar{\Omega})$ the space of Hölder continuous functions with exponent β, where $0 < \beta \leq 1$.

2.1 First Case

In this section, the loading force and initial data are given by

$$b\left(x, \frac{x}{\varepsilon}, t\right) = l(x,t) + R\left(\frac{x}{\varepsilon}\right), \tag{2.2a}$$

$$u^0\left(x, \frac{x}{\varepsilon}\right) = u_0(x) + u_1\left(\frac{x}{\varepsilon}\right), \tag{2.2b}$$

$$v^0\left(x, \frac{x}{\varepsilon}\right) = v_0(x) + v_1\left(\frac{x}{\varepsilon}\right), \tag{2.2c}$$

where $l \in C([0,T];\, L^s(\Omega)^3)$, R is in $L^s_{per}(Y)^3$ with $\bar{R} = 0$, u_0 and v_0 are in $L^s(\Omega)^3$, and u_1 and v_1 are in $L^s_{per}(Y)^3$ with $\bar{u}_1 = \bar{v}_1 = 0$. Here, $R(\frac{x}{\varepsilon})$ can be interpreted as a residual force. For example, such forces can arise from the differences in thermal expansion between the two materials.

2.1.1 The Macroscopic Equation

The macroscopic or homogenized peridynamic equation is given by

$$\partial_t^2 u^H(x,t) = \int_{H_\gamma(x)} \lambda(\hat{x} - x) \frac{(\hat{x} - x) \otimes (\hat{x} - x)}{|\hat{x} - x|^3} \left(u^H(\hat{x},t) - u^H(x,t)\right) d\hat{x} + l(x,t), \tag{2.3}$$

supplemented with initial data

$$u^H(x,0) = u_0(x), \quad \partial_t u^H(x,0) = v_0(x). \tag{2.4}$$

Here the macroscopic displacement u^H is the weak limit of the sequence of displacements u^ε. This is described by the following theorem.

Theorem 2.1. *Let u^ε be the solution of (1.10)-(1.12), where b, u^0, and v^0 are given by (2.2). Then as $\varepsilon \to 0$*

$$u^\varepsilon(x,t) \to u^H(x,t) \quad \textit{weakly in } L^s(\Omega \times (0,T))^3,$$

where $u^H \in C^2([0,T];\, L^s(\Omega)^3)$ is the unique solution of (2.3)-(2.4).

Moreover, assume that $l \in C([0,T];\, C(\bar{\Omega})^3)$, and u_0 and v_0 are in $C(\bar{\Omega})^3$. Then u^H is in $C^2([0,T];\, C(\bar{\Omega})^3)$.

2.1.2 The Cell–Problem

The cell-problem or the micro-level peridynamic equation is given by

$$\begin{aligned} \partial_t^2 r(y,t) &= \int_{H_\delta(y)} \alpha(y,\hat{y}) \frac{(\hat{y} - y) \otimes (\hat{y} - y)}{|(\hat{y} - y)|^3} \left(r(\hat{y},t) - r(y,t)\right) d\hat{y} \\ &\quad -K\, r(y,t) + R(y), \end{aligned} \tag{2.5}$$

supplemented with initial conditions

$$r(y,0) = u_1(y), \quad \partial_t r(y,0) = v_1(y). \tag{2.6}$$

The matrix K is given by (2.1).

2.1.3 Downscaling

The macroscopic displacement u^H together with the rescaled solution of the cell problem provide the approximation to the actual solution u^ε given by $u^\varepsilon_{Approx} = u^H(x,t) + r(x/\varepsilon, t)$. This is expressed in the following theorem.

Theorem 2.2. *Let u^ε be the solution of (1.10)-(1.12), where b, u^0, and v^0 are given by (2.2). Assume that $l \in C([0,T];\, C(\bar{\Omega})^3)$, and u_0 and v_0 are in $C(\bar{\Omega})^3$. Then for almost every $t \in (0,T)$,*

$$\lim_{\varepsilon\to 0} \left\| u^\varepsilon(x,t) - \left(u^H(x,t) + r\left(\frac{x}{\varepsilon}, t\right)\right)\right\|_{L^s(\Omega)^3} = 0, \tag{2.7}$$

where $r \in C^2([0,T];\, L^s_{per}(Y)^3)$ is the unique solution of (2.5)-(2.6).

Moreover, assume that $\lambda = 0$ in equation (1.10). Then, for $t \in (0,T)$ and u_0, v_0, and $l(\cdot,t)$ in $C^{0,\beta}(\bar{\Omega})^3$, the error in (2.7) is estimated by

$$\left\| u^\varepsilon(x,t) - \left(u^H(x,t) + r\left(\frac{x}{\varepsilon}, t\right)\right)\right\|_{L^s(\Omega)^3} \leq M_1(t)\varepsilon^\beta, \tag{2.8}$$

where $M_1(t)$ is independent of ε. The function $M_1(t)$ is given explicitly in Section 5.2.1.

2.2 Second Case

In this section, the loading force and initial data are given by

$$\begin{aligned} b\left(x, \frac{x}{\varepsilon}, t\right) &= F\left(\frac{x}{\varepsilon}, t\right) h(x), && (2.9a)\\ u^0\left(x, \frac{x}{\varepsilon}\right) &= 0, && (2.9b)\\ v^0\left(x, \frac{x}{\varepsilon}\right) &= 0, && (2.9c) \end{aligned}$$

where $F \in C([0,T];\, L^s_{per}(Y)^{3\times 3})$ and $h \in L^s(\Omega)^3$.

2.2.1 The Macroscopic Equation

The macroscopic peridynamic equation is given by

$$\partial_t^2 u^H(x,t) = \int_{H_\gamma(x)} \lambda(\hat{x}-x) \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3} \left(u^H(\hat{x},t) - u^H(x,t)\right) d\hat{x} + \bar{F}(t)h(x), \tag{2.10}$$

supplemented with initial data

$$u^H(x,0) = 0, \quad \partial_t u^H(x,0) = 0. \tag{2.11}$$

Here the macroscopic displacement u^H is the weak limit of the sequence of displacements u^ε. This is described by the following theorem.

Theorem 2.3. *Let u^ε be the solution of (1.10)-(1.12), where b, u^0, and v^0 are given by (2.9). Then as $\varepsilon \to 0$*

$$u^\varepsilon(x,t) \to u^H(x,t) \quad \text{weakly in } L^s(\Omega\times(0,T))^3,$$

where $u^H \in C^2([0,T];\, L^s(\Omega)^3)$ is the unique solution of (2.10)-(2.11).

Moreover, assume that $h \in C(\bar{\Omega})^3$. Then u^H is in $C^2([0,T];\, C(\bar{\Omega})^3)$.

2.2.2 The Cell–Problem

The micro-level peridynamics is given by the following equations. For $j = 1, 2, 3$,

$$\begin{aligned}\partial_t^2 r^j(y,t) &= \int_{H_\delta(y)} \alpha(y,\hat{y}) \frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|(\hat{y}-y)|^3} \left(r^j(\hat{y},t) - r^j(y,t)\right) d\hat{y} \\ &\quad -K\, r^j(y,t) + (F^j(y,t) - \bar{F}^j(t)),\end{aligned} \tag{2.12}$$

supplemented with initial conditions

$$r^j(y,0) = 0, \quad \partial_t r^j(y,0) = 0. \tag{2.13}$$

In (2.12), $F^j(y,t)$ and $\bar{F}^j(t)$ denote the j^{th} columns of the matrices $F(y,t)$ and $\bar{F}(t)$, respectively. The matrix K is given by (2.1).

2.2.3 Downscaling

The macroscopic displacement u^H together with the rescaled solution of the cell problem provide an approximation to the actual solution u^ε. This is expressed in the following theorem.

Theorem 2.4. *Let u^ε be the solution of (1.10)-(1.12), where b, u^0, and v^0 are given by (2.9). Assume that $h \in C(\bar{\Omega})^3$. Then for almost every $t \in (0,T)$,*

$$\lim_{\varepsilon\to 0} \left\| u^\varepsilon(x,t) - \left(u^H(x,t) + \sum_{j=1}^{3} r^j\left(\frac{x}{\varepsilon}, t\right) h_j(x) \right) \right\|_{L^s(\Omega)^3} = 0, \tag{2.14}$$

where $r^j \in C^2([0,T]; L^s_{per}(Y)^3)$ is the unique solution of (2.12)-(2.13).

Moreover, assume that $\lambda = 0$ in equation (1.10). Then, for $t \in (0,T)$ and $h \in C^{0,\beta}(\bar{\Omega})^3$, the error in (2.14) is estimated by

$$\left\| u^\varepsilon(x,t) - \left(u^H(x,t) + \sum_{j=1}^{3} r^j\left(\frac{x}{\varepsilon}, t\right) h_j(x) \right) \right\|_{L^s(\Omega)^3} \le M_2(t)\varepsilon^\beta, \tag{2.15}$$

where $M_2(t)$ is independent of ε. The function $M_2(t)$ is given explicitly in Section 5.2.2.

2.3 Third Case

In this section, the loading force and initial data are given by

$$b\left(x, \frac{x}{\varepsilon}, t\right) = 0, \tag{2.16a}$$

$$u^0\left(x, \frac{x}{\varepsilon}\right) = F\left(\frac{x}{\varepsilon}\right) h(x), \tag{2.16b}$$

$$v^0\left(x, \frac{x}{\varepsilon}\right) = 0, \tag{2.16c}$$

where $F \in L^s_{per}(Y)^{3\times 3}$ and $h \in L^s(\Omega)^3$.

2.3.1 The Macroscopic Equation

The macroscopic peridynamic equation is given by

$$\partial_t^2 u^H(x,t) = \int_{H_\gamma(x)} \lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(u^H(\hat{x},t)-u^H(x,t)\right)d\hat{x}, \tag{2.17}$$

supplemented with initial data

$$u^H(x,0) = \bar{F}h(x), \quad \partial_t u^H(x,0) = 0. \tag{2.18}$$

Here the macroscopic displacement u^H is the weak limit of the sequence of displacements u^ε. This is described by the following theorem.

Theorem 2.5. *Let u^ε be the solution of (1.10)-(1.12), where b, u^0, and v^0 are given by (2.16). Then as $\varepsilon \to 0$*

$$u^\varepsilon(x,t) \to u^H(x,t) \quad \textit{weakly in } L^s(\Omega\times(0,T))^3,$$

where $u^H \in C^2([0,T];\, L^s(\Omega)^3)$ is the unique solution of (2.17)-(2.18).

Moreover, assume that $h \in C(\bar{\Omega})^3$. Then u^H is in $C^2([0,T];\, C(\bar{\Omega})^3)$.

2.3.2 The Cell–Problem

The micro-level peridynamics is given by the following equations. For $j = 1, 2, 3$,

$$\begin{aligned}\partial_t^2 r^j(y,t) \;&=\; \int_{H_\delta(y)} \alpha(y,\hat{y})\frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|(\hat{y}-y)|^3}\left(r^j(\hat{y},t) - r^j(y,t)\right)d\hat{y} \\ &\quad -K\, r^j(y,t),\end{aligned} \tag{2.19}$$

supplemented with initial conditions

$$r^j(y,0) = F^j(y) - \bar{F}^j, \quad \partial_t r^j(y,0) = 0. \tag{2.20}$$

In (2.20), $F^j(y)$ and $\bar{F}^j$ denote the j^{th} columns of the matrices $F(y)$ and $\bar{F}$, respectively. The matrix K is given by (2.1).

2.3.3 Downscaling

The macroscopic displacement u^H together with the rescaled solution of the cell problem provide an approximation to the actual solution u^ε. This is expressed in the following theorem.

Theorem 2.6. *Let u^ε be the solution of (1.10)-(1.12), where b, u^0, and v^0 are given by (2.16). Assume that $h \in C(\bar{\Omega})^3$. Then for almost every $t \in (0,T)$,*

$$\lim_{\varepsilon\to 0}\left\| u^\varepsilon(x,t) - \left(u^H(x,t) + \sum_{j=1}^{3} r^j\left(\frac{x}{\varepsilon},t\right)h_j(x)\right)\right\|_{L^s(\Omega)^3} = 0, \tag{2.21}$$

where $r^j \in C^2([0,T]; L^s_{per}(Y)^3)$ is the unique solution of (2.19)-(2.20).

Moreover, assume that $\lambda = 0$ in equation (1.10). Then, for $t \in (0,T)$ and $h \in C^{0,\beta}(\bar{\Omega})^3$, the error in (2.21) is estimated by

$$\left\| u^\varepsilon(x,t) - \left(u^H(x,t) + \sum_{j=1}^{3} r^j\left(\frac{x}{\varepsilon}, t\right) h_j(x) \right) \right\|_{L^s(\Omega)^3} \leq M_3(t)\varepsilon^\beta, \tag{2.22}$$

where $M_3(t)$ is independent of ε. The function $M_3(t)$ is given explicitly in Section 5.2.3.

2.4 Fourth Case

In this section, the loading force and initial data are given by

$$\begin{aligned} b\left(x, \frac{x}{\varepsilon}, t\right) &= 0, && (2.23a)\\ u^0\left(x, \frac{x}{\varepsilon}\right) &= 0, && (2.23b)\\ v^0\left(x, \frac{x}{\varepsilon}\right) &= F\left(\frac{x}{\varepsilon}\right) h(x), && (2.23c) \end{aligned}$$

where $F \in L^s_{per}(Y)^{3\times3}$ and $h \in L^s(\Omega)^3$.

2.4.1 The Macroscopic Equation

The macroscopic peridynamic equation is given by

$$\partial_t^2 u^H(x,t) = \int_{H_\gamma(x)} \lambda(\hat{x}-x) \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3} \left(u^H(\hat{x},t) - u^H(x,t)\right) d\hat{x}, \tag{2.24}$$

supplemented with initial data

$$u^H(x,0) = 0, \quad \partial_t u^H(x,0) = \bar{F}h(x). \tag{2.25}$$

Here the macroscopic displacement u^H is the weak limit of the sequence of displacements u^ε. This is described by the following theorem.

Theorem 2.7. *Let u^ε be the solution of (1.10)-(1.12), where b, u^0, and v^0 are given by (2.23). Then as $\varepsilon \to 0$*

$$u^\varepsilon(x,t) \to u^H(x,t) \quad \text{weakly in } L^s(\Omega\times(0,T))^3,$$

where $u^H \in C^2([0,T]; L^s(\Omega)^3)$ is the unique solution of (2.24)-(2.25).

Moreover, assume that $h \in C(\bar{\Omega})^3$. Then u^H is in $C^2([0,T]; C(\bar{\Omega})^3)$.

2.4.2 The Cell–Problem

The micro-level peridynamics is given by the following equations. For $j = 1, 2, 3$,

$$\partial_t^2 r^j(y,t) = \int_{H_\delta(y)} \alpha(y,\hat{y}) \frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|(\hat{y}-y)|^3} \left(r^j(\hat{y},t) - r^j(y,t)\right) d\hat{y} \\ -K\, r^j(y,t), \tag{2.26}$$

supplemented with initial conditions

$$r^j(y,0) = 0, \quad \partial_t r^j(y,0) = F^j(y) - \bar{F}^j. \tag{2.27}$$

In (2.27), $F^j(y)$ and $\bar{F}^j$ denote the j^{th} columns of the matrices $F(y)$ and $\bar{F}$, respectively. The matrix K is given by (2.1).

2.4.3 Downscaling

The macroscopic displacement u^H together with the rescaled solution of the cell problem provide an approximation to the actual solution u^ε. This is expressed in the following theorem.

Theorem 2.8. *Let u^ε be the solution of (1.10)-(1.12), where b, u^0, and v^0 are given by (2.23). Assume that $h \in C(\bar{\Omega})^3$. Then for almost every $t \in (0,T)$,*

$$\lim_{\varepsilon\to 0} \left\| u^\varepsilon(x,t) - \left(u^H(x,t) + \sum_{j=1}^{3} r^j\left(\frac{x}{\varepsilon},t\right) h_j(x) \right) \right\|_{L^s(\Omega)^3} = 0, \tag{2.28}$$

where $r^j \in C^2([0,T]; L^s_{per}(Y)^3)$ is the unique solution of (2.26)-(2.27).

Moreover, assume that $\lambda = 0$ in equation (1.10). Then, for $t \in (0,T)$ and $h \in C^{0,\beta}(\bar{\Omega})^3$, the error in (2.28) is estimated by

$$\left\| u^\varepsilon(x,t) - \left(u^H(x,t) + \sum_{j=1}^{3} r^j\left(\frac{x}{\varepsilon},t\right) h_j(x) \right) \right\|_{L^s(\Omega)^3} \le M_4(t)\varepsilon^\beta, \tag{2.29}$$

where $M_4(t)$ is independent of ε. The function $M_4(t)$ is given explicitly in Section 5.2.3.

3 Existence and Uniqueness Results for the Peridynamic Equation

In this section, we make use of semigroup theory of operators to study the existence and uniqueness of (1.10)-(1.12). We begin by introducing the following operators. For $v \in L^s(\Omega)^3$, with $\frac{3}{2} < s < \infty$, let

$$A_{L,1}v(x) = \int_{H_\gamma(x)} \lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\, v(\hat{x})\, d\hat{x}, \tag{3.1}$$

$$A_{L,2}v(x) = \int_{H_\gamma(x)} \lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\, d\hat{x}\; v(x), \tag{3.2}$$

$$A^\varepsilon_{S,1}v(x) = \int_{H_{\varepsilon\delta}(x)} \frac{1}{\varepsilon^2}\,\alpha\left(\frac{x}{\varepsilon},\frac{\hat{x}}{\varepsilon}\right)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\, v(\hat{x})\, d\hat{x}, \tag{3.3}$$

$$A^\varepsilon_{S,2}v(x) = \int_{H_{\varepsilon\delta}(x)} \frac{1}{\varepsilon^2}\,\alpha\left(\frac{x}{\varepsilon},\frac{\hat{x}}{\varepsilon}\right)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\, d\hat{x}\; v(x). \tag{3.4}$$

Also we set

$$A_L = A_{L,1} - A_{L,2}, \tag{3.5}$$

$$A^\varepsilon_S = A^\varepsilon_{S,1} - A^\varepsilon_{S,2}, \tag{3.6}$$

$$A^\varepsilon = A_L + A^\varepsilon_S. \tag{3.7}$$

Then by making the identifications $u^\varepsilon(t) = u^\varepsilon(\cdot,t)$ and $b^\varepsilon(t) = b(\cdot,\frac{\cdot}{\varepsilon},t)$, we can write (1.10)-(1.12) as an operator equation in $L^s(\Omega)^3$

$$\begin{cases} \ddot{u}^\varepsilon(t) = A^\varepsilon u^\varepsilon(t) + b^\varepsilon(t), \quad t\in[0,T] \\ u^\varepsilon(0) = u^\varepsilon_0, \\ \dot{u}^\varepsilon(0) = v^\varepsilon_0. \end{cases} \tag{3.8}$$

or equivalently, as an inhomogeneous Abstract Cauchy Problem in $L^s(\Omega)^3 \times L^s(\Omega)^3$

$$\begin{cases} \dot{U}^\varepsilon(t) = \mathbb{A}^\varepsilon U^\varepsilon(t) + B^\varepsilon(t), \quad t\in[0,T] \\ U^\varepsilon(0) = U^\varepsilon_0. \end{cases} \tag{3.9}$$

where

$$U^\varepsilon(t) = \begin{pmatrix} u^\varepsilon(t) \\ \dot{u}^\varepsilon(t) \end{pmatrix},\ U^\varepsilon_0 = \begin{pmatrix} u^\varepsilon_0 \\ v^\varepsilon_0 \end{pmatrix},\ B^\varepsilon(t) = \begin{pmatrix} 0 \\ b^\varepsilon(t) \end{pmatrix},\ \text{and } \mathbb{A}^\varepsilon = \begin{pmatrix} 0 & I \\ A^\varepsilon & 0 \end{pmatrix}.$$

Here I denotes the identity map in $L^s(\Omega)^3$.

Proposition 3.1. *Let $\frac{3}{2} < s < \infty$ and assume that $b^\varepsilon \in C([0,T];\, L^s(\Omega)^3)$. Then*

(a) The operators A^ε and $\mathbb{A}^\varepsilon$ are linear and bounded on $L^s(\Omega)^3$ and $L^s(\Omega)^3 \times L^s(\Omega)^3$, respectively. Moreover, the bounds are uniform in ε.

(b) Equation (3.9) has a unique classical solution U^ε in $C^1([0,T];\, L^s(\Omega)^3 \times L^s(\Omega)^3)$ which is given by

$$U^\varepsilon(t) = e^{t\mathbb{A}^\varepsilon}U^\varepsilon_0 + \int_0^t e^{(t-\tau)\mathbb{A}^\varepsilon} B^\varepsilon(\tau)\, d\tau,\ \ t\in[0,T], \tag{3.10}$$

where

$$e^{t\mathbb{A}^\varepsilon} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \, (\mathbb{A}^\varepsilon)^n. \tag{3.11}$$

Moreover, equation (3.8) has a unique classical solution $u^\varepsilon \in C^2([0,T];\, L^s(\Omega)^3)$ *which is given by*

$$\begin{aligned} u^\varepsilon(t) &= \cosh\left(t\sqrt{A^\varepsilon}\right) u_0^\varepsilon + \sqrt{A^\varepsilon}^{\,-1} \sinh\left(t\sqrt{A^\varepsilon}\right) v_0^\varepsilon \\ &\quad + \sqrt{A^\varepsilon}^{\,-1} \int_0^t \sinh\left(t\sqrt{A^\varepsilon}\right) b^\varepsilon(\tau)\, d\tau \end{aligned} \tag{3.12a}$$

with the notation

$$\cosh\left(t\sqrt{A^\varepsilon}\right) := \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \, (A^\varepsilon)^n \tag{3.12b}$$

$$\sqrt{A^\varepsilon}^{\,-1} \sinh\left(t\sqrt{A^\varepsilon}\right) := \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \, (A^\varepsilon)^n \tag{3.12c}$$

(c) *The sequences* $(u^\varepsilon)_{\varepsilon>0}$*,* $(\dot{u}^\varepsilon)_{\varepsilon>0}$*, and* $(\ddot{u}^\varepsilon)_{\varepsilon>0}$ *are bounded in* $L^\infty([0,T];\, L^s(\Omega)^3)$*.*

Proof. Part (a). It is clear that the operators $A^\varepsilon_{S,1}$, $A^\varepsilon_{S,2}$, $A_{L,1}$, and $A_{L,2}$ are linear. So we begin the proof by showing that $A^\varepsilon_{S,1}$ and $A^\varepsilon_{S,2}$ are uniformly bounded on $L^s(\Omega)^3$ for $\frac{3}{2} < s < \infty$. Let $v \in L^s(\Omega)^3$. Then by the change of variables $\hat{x} = x + \varepsilon z$ in (3.3) we obtain

$$A^\varepsilon_{S,1} v(x) = \int_{H_\delta(0)} \alpha\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon} + z\right) \frac{z \otimes z}{|z|^3} \, v(x + \varepsilon z)\, dz. \tag{3.13}$$

Let $\alpha_{\max} = \max_{y,y' \in Y} \alpha(y,y')$. Then by taking the Euclidean norm in (3.13), we see that

$$\begin{aligned} |A^\varepsilon_{S,1} v(x)| &\leq \alpha_{\max} \int_{H_\delta(0)} \frac{1}{|z|} \, |v(x+\varepsilon z)|\, dz \\ &\leq \alpha_{\max} \left(\int_{H_\delta(0)} \frac{1}{|z|^{s'}}\, dz \right)^{1/s'} \left(\int_{H_\delta(0)} |v(x+\varepsilon z)|^s\, dz \right)^{1/s}, \end{aligned} \tag{3.14}$$

where Hölder's inequality was used in the second inequality, with $1/s + 1/s' = 1$ and $1 \leq s' < 3$. By changing the variable of integration back to $\hat{x}$ in the second integral, and then taking the limit as $\varepsilon \to 0$, we see that

$$\begin{aligned} \int_{H_\delta(0)} |v(x+\varepsilon z)|^s\, dz &= \frac{1}{\varepsilon^3} \int_{H_{\delta\varepsilon}(x)} |v(\hat{x})|^s\, d\hat{x} \\ &\to |H_\delta(x)|\, |v(x)|^s, \ \text{a.e. } x, \end{aligned} \tag{3.15}$$

where we have used Lebesgue's Differentiation Theorem to evaluate this limit. On the other hand, we observe that the first integral in (3.14) is finite because $s' < 3$. Therefore, it follows from (3.14) and (3.15) that

$$|A^\varepsilon_{S,1} v(x)| \leq M_1\, |v(x)|,$$

for some real number $M_1 > 0$ which is independent of ε. It follows that

$$\|A^{\varepsilon}_{S,1}v\|_{L^s(\Omega)^3} \quad \leq \quad M_1\,\|v\|_{L^s(\Omega)^3},$$

which shows that the operator $A^{\varepsilon}_{S,1}$ is uniformly bounded. Similarly, $A^{\varepsilon}_{S,2}$ can be written as

$$A^{\varepsilon}_{S,2}v(x) \quad = \quad \int_{H_\delta(0)} \alpha\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon} + z\right) \frac{z \otimes z}{|z|^3}\,dz\;\; v(x). \tag{3.16}$$

Thus

$$|A^{\varepsilon}_{S,2}v(x)| \quad \leq \quad \alpha_{\max} \int_{H_\delta(0)} \frac{1}{|z|}\,dz\;\; |v(x)|,$$

from which the boundedness of $A^{\varepsilon}_{S,2}$ immediately follows. Combining these results shows that A^{ε}_{S}, which is given by $A^{\varepsilon}_{S,1} - A^{\varepsilon}_{S,2}$, is a uniformly bounded operator on $L^s(\Omega)^3$.

Next we show that the linear operator $A_L = A_{L,1} - A_{L,2}$ is bounded on $L^s(\Omega)^3$. Let $\lambda_{\max} = \max_{\xi \in H_\gamma(0)} \lambda(\xi)$. Then by taking the Euclidean norm in (3.1), we see that

$$\begin{aligned} |A_{L,1}v(x)| \quad &\leq \quad \lambda_{\max} \int_{H_\gamma(x)} \frac{1}{|\hat{x} - x|} |v(\hat{x})|\,d\hat{x} \\ &\leq \quad \lambda_{\max} \left(\int_{H_\gamma(x)} \frac{1}{|\hat{x} - x|^{s'}}\,d\hat{x} \right)^{1/s'} \left(\int_{H_\gamma(x)} |v(\hat{x})|^s\,d\hat{x} \right)^{1/s}, \end{aligned} \tag{3.17}$$

where Hölder's inequality was used in the second inequality, with $1/s + 1/s' = 1$ and $1 \leq s' < 3$. By the change of variables $\xi = \hat{x} - x$, it is easy to see that the first integral in (3.17) is independent of x and finite because $s' < 3$. Therefore from (3.17) we obtain

$$\|A_{L,1}v\|_{L^s(\Omega)^3} \leq \lambda_{\max} \left(\int_{H_\gamma(0)} \frac{1}{|z|^{s'}}\,dz \right)^{1/s'} \|v\|_{L^s(\Omega)^3}.$$

This shows that $A_{L,1}$ is bounded on $L^s(\Omega)^3$. The boundedness of $A_{L,2}$, which is given by (3.2), is clear. Therefore A_L is bounded on $L^s(\Omega)^3$.

Since $A^{\varepsilon} = A_L + A^{\varepsilon}_{S}$, we conclude that

$$\|A^{\varepsilon}v\|_{L^s(\Omega)^3} \leq M\,\|v\|_{L^s(\Omega)^3}, \tag{3.18}$$

for some real number $M > 0$ which is independent of ε.

The operator $\mathbb{A}^{\varepsilon}$ is clearly linear, thus it remains to show that this operator is uniformly bounded on $L^s(\Omega)^3 \times L^s(\Omega)^3$. To see this, we let $(v, w) \in L^s(\Omega)^3 \times L^s(\Omega)^3$. The norm in this Banach space is given by

$$\|(v, w)\|_{L^s(\Omega)^3 \times L^s(\Omega)^3} = \|v\|_{L^s(\Omega)^3} + \|w\|_{L^s(\Omega)^3}.$$

We note that

$$\mathbb{A}^{\varepsilon} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & I \\ A^{\varepsilon} & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} w \\ A^{\varepsilon}v \end{pmatrix}.$$

Thus by taking the norm, we obtain

$$\begin{aligned}\|\mathbb{A}^\varepsilon(v,w)\|_{L^s(\Omega)^3\times L^s(\Omega)^3} &= \|w\|_{L^s(\Omega)^3} + \|A^\varepsilon v\|_{L^s(\Omega)^3} \\ &\leq \|w\|_{L^s(\Omega)^3} + \|A^\varepsilon\|\,\|v\|_{L^s(\Omega)^3}. \end{aligned} \tag{3.19}$$

From (3.19) and since we may assume that $M > 1$ in (3.18), it follows that

$$\|\mathbb{A}^\varepsilon(v,w)\|_{L^s(\Omega)^3\times L^s(\Omega)^3} \leq M\|(v,w)\|_{L^s(\Omega)^3\times L^s(\Omega)^3}, \tag{3.20}$$

completing the argument.

Part (b). We have seen from Part (a) that $\mathbb{A}^\varepsilon$ is a bounded linear operator on the Banach space $L^s(\Omega)^3 \times L^s(\Omega)^3$. Also, since b^ε is in $C([0,T];\, L^s(\Omega)^3)$ by assumption, it follows that $B^\varepsilon = (0, b^\varepsilon)$ is in $C([0,T];\, L^s(\Omega)^3 \times L^s(\Omega)^3)$. From these facts, it follows from the theory of semigroups that[1]

1. The operator $\mathbb{A}^\varepsilon$ generates a uniformly continuous semigroup $\{e^{t\mathbb{A}^\varepsilon}\}_{t\geq 0}$ on $L^s(\Omega)^3 \times L^s(\Omega)^3$, where $e^{t\mathbb{A}^\varepsilon}$ is given by (3.11).
2. The inhomogeneous Abstract Cauchy Problem (3.9) has a unique classical solution $U^\varepsilon \in C^1([0,T];\, L^s(\Omega)^3 \times L^s(\Omega)^3)$ which is given by (3.10).

It immediately follows from (2) that the second order inhomogeneous Abstract Cauchy Problem (3.8) has a unique classical solution $u^\varepsilon \in C^2([0,T];\, L^s(\Omega)^3)$. It remains to show that u^ε is given explicitly by (3.12). To see this, we begin by the following observations which can be easily shown using mathematical induction. For $n = 0, 1, 2, \ldots$, we have

$$\begin{pmatrix} 0 & I \\ A^\varepsilon & 0 \end{pmatrix}^{2n} = \begin{pmatrix} (A^\varepsilon)^n & 0 \\ 0 & (A^\varepsilon)^n \end{pmatrix} \tag{3.21}$$

$$\begin{pmatrix} 0 & I \\ A^\varepsilon & 0 \end{pmatrix}^{2n+1} = \begin{pmatrix} 0 & (A^\varepsilon)^n \\ (A^\varepsilon)^{n+1} & 0 \end{pmatrix} \tag{3.22}$$

From (3.11) and by using these two equations we see that

$$e^{t\mathbb{A}^\varepsilon} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} 0 & I \\ A^\varepsilon & 0 \end{pmatrix}^n = \begin{pmatrix} \displaystyle\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\,(A^\varepsilon)^n & \displaystyle\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,(A^\varepsilon)^n \\ \displaystyle\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,(A^\varepsilon)^{n+1} & \displaystyle\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\,(A^\varepsilon)^n \end{pmatrix} \tag{3.23}$$

Equation (3.12) follows from equations (3.10) and (3.23), and the fact that

$$U^\varepsilon = \begin{pmatrix} u^\varepsilon \\ \dot{u}^\varepsilon \end{pmatrix}.$$

[1]see for example [20, 12].

Part (c). We recall that

$$\begin{aligned} u_0^\varepsilon(x) &:= u^0(x, \frac{x}{\varepsilon}) \\ v_0^\varepsilon(x) &:= v^0(x, \frac{x}{\varepsilon}) \end{aligned}$$

Also by assumption $u^0(x,y)$, $v^0(x,y)$ are in $L^s(\Omega; L^s_{per}(Y)^3)$. Therefore we see that

$$\begin{aligned} \|u_0^\varepsilon\|_{L^s(\Omega)^3} &\leq \|u^0\|_{L^s(\Omega; L^s_{per}(Y)^3)} := \left(\int_\Omega \int_Y |u^0(x,y)|^s\, dydx\right)^{1/s}, \\ \|v_0^\varepsilon\|_{L^s(\Omega)^3} &\leq \|v^0\|_{L^s(\Omega; L^s_{per}(Y)^3)} := \left(\int_\Omega \int_Y |v^0(x,y)|^s\, dydx\right)^{1/s}. \end{aligned}$$

Thus u_0^ε and v_0^ε are uniformly bounded in $L^s(\Omega)^3$, which implies that U_0^ε is uniformly bounded in $L^s(\Omega)^3 \times L^s(\Omega)^3$. Similarly we can show that for $t \in [0,T]$, $b^\varepsilon(t)$ is uniformly bounded in $L^s(\Omega)^3$. Since $b^\varepsilon(t)$ is continuous in t, it follows that b^ε is uniformly bounded in $C([0,T];\, L^s(\Omega)^3)$, which implies that B^ε is uniformly bounded in $C([0,T];\, L^s(\Omega)^3 \times L^s(\Omega)^3)$.

Next we note that

$$\begin{aligned} \|e^{t\mathbb{A}^\varepsilon}\| &\leq e^{t\|\mathbb{A}^\varepsilon\|} \\ &\leq e^{tM}, \end{aligned} \tag{3.24}$$

where in the last inequality we have used the fact that $\mathbb{A}^\varepsilon$ is uniformly bounded. Taking the norm in both sides of (3.10) and by using (3.24), we obtain

$$\|U^\varepsilon(t)\|_{L^s(\Omega)^3 \times L^s(\Omega)^3} \leq M_1 e^{tM} + \int_0^t e^{(t-\tau)M} M_2\, d\tau, \tag{3.25}$$

for some positive numbers M_1, M_2, and M. This implies that U^ε is uniformly bounded in $L^\infty([0,T];\, L^s(\Omega)^3 \times L^s(\Omega)^3)$. Therefore the sequences $(u^\varepsilon)_{\varepsilon>0}$ and $(\dot{u}^\varepsilon)_{\varepsilon>0}$ are bounded in $L^\infty([0,T];\, L^s(\Omega)^3)$. Finally, it follows from equation (3.8) that the sequence $(\ddot{u}^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty([0,T];\, L^s(\Omega)^3)$, completing the proof. □

4 Two-Scale Convergence and the Two-Scale Limit Equation

The aim of this section is to identify the two-scale limit of the peridynamic initial-value problem (1.10)-(1.12).

4.1 Two-Scale Convergence

We begin by defining two-scale convergence and recalling some results from two-scale convergence. In the subsequent discussion, we will often refer to the following function spaces

$$\begin{aligned} \mathcal{K} &= \{\psi \in C_c^\infty(\mathbb{R}^3 \times Y),\ \psi(x,y) \text{ is } Y\text{-periodic in } y\}, \\ \mathcal{J} &= \{\psi \in C_c^\infty(\mathbb{R}^3 \times Y \times \mathbb{R}^+),\ \psi(x,y,t) \text{ is } Y\text{-periodic in } y\}, \\ \mathcal{Q} &= \{w \in C^2([0,T];\, L^s(\Omega \times Y)^3),\ w(x,y,t) \text{ is } Y\text{-periodic in } y, \text{ and } 3/2 < s < \infty\}. \end{aligned}$$

Let p and p' be two real numbers such that $1 < p < \infty$ and $1/p + 1/p' = 1$.

Definition 4.1 (Two-scale convergence [18, 1]). *A sequence* (v^ε) *of functions in* $L^p(\Omega)$*, is said to two-scale converge to a limit* $v \in L^p(\Omega \times Y)$ *if, as* $\varepsilon \to 0$

$$\int_\Omega v^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right)\, dx \to \int_{\Omega\times Y} v(x,y)\psi(x,y)\, dxdy \tag{4.1}$$

for all $\psi \in L^{p'}(\Omega; C_{per}(Y))$*. We will often use* $v^\varepsilon \xrightarrow{2} v$ *to denote that* (v^ε) *two-scale converges to* v*.*

If the sequence (v^ε) is bounded in $L^p(\Omega)$ then $L^{p'}(\Omega; C_{per}(Y))$ can be replaced by $\mathcal{K}$ in Definition (4.1) (see [19]).

The following are well-known results on two-scale convergence, which can be found in [19].

Proposition 4.2. *If* (v^ε) *converges to* v *in* $L^p(\Omega)$ *then* (v^ε) *two-scale converges to* $\tilde{v}(x,y) = v(x)$*.*

Proposition 4.3. *If* $\psi \in \mathcal{K}$ *then* $\psi(x, \frac{x}{\varepsilon})$ *two-scale converges to* $\psi(x,y)$*.*

Proposition 4.4. *Let* (v^ε) *be a sequence in* $L^p(\Omega)$ *which two-scale converges to* $v \in L^p(\Omega \times Y)$*. Then*

$$\int_\Omega v^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right)\, dx \to \int_{\Omega\times Y} v(x,y)\psi(x,y)\, dxdy,$$

for every ψ *of the form* $\psi(x,y) = \psi_1(x)\psi_2(y)$*, where* $\psi_1 \in L^{rp'}(\Omega)$ *and* $\psi_2 \in L^{r'p'}_{per}(Y)$*, with* $1 \le r \le \infty$ *and* $1/r + 1/r' = 1$*.*

Proposition 4.5. *Let* (v^ε) *be a sequence in* $L^p(\Omega)$ *which two-scale converges to* $v \in L^p(\Omega \times Y)$*. Then as* $\varepsilon \to 0$

$$v^\varepsilon \to \int_Y v(x,y)\, dy \quad \text{weakly in } L^p(\Omega).$$

Definition 4.1 is motivated by the following compactness result of Nguetseng, see [18].

Theorem 4.6. *Let* (v^ε) *be a bounded sequence in* $L^p(\Omega)$*. Then there exists a subsequence and a function* $v \in L^p(\Omega \times Y)$ *such that the subsequence two-scale converges to* v*.*

For the time-dependent problems studied in this work, we slightly modify the above two-scale convergence definition and results to allow for homogenization with a parameter, see [5, 8]. Here the parameter is denoted by t.

Definition 4.7. *A sequence* (v^ε) *of functions in* $L^p(\Omega \times (0,T))$*, is said to two-scale converge to a limit* $v \in L^p(\Omega \times Y \times (0,T))$ *if, as* $\varepsilon \to 0$

$$\int_{\Omega\times(0,T)} v^\varepsilon(x,t)\psi\left(x, \frac{x}{\varepsilon}, t\right)\, dxdt \to \int_{\Omega\times Y\times(0,T)} v(x,y,t)\psi(x,y,t)\, dxdydt \tag{4.2}$$

for all $\psi \in \mathcal{J}$*.*

Theorem 4.8. *Let (v^ε) be a bounded sequence in $L^p(\Omega \times (0,T))$. Then there exists a subsequence and a function $v \in L^p(\Omega \times Y \times (0,T))$ such that the subsequence two-scale converges to v.*

The proof of this result is essentially the same as the proof of Theorem 4.6. A slight variation of Theorem 4.8 can be found in [8] and [5].

The following is a direct consequence of Definition 4.7 and the definition of weak convergence.

Proposition 4.9. *Let (v^ε) be a bounded sequence in $L^p(\Omega \times (0,T))$ that two-scale converges to $v \in L^p(\Omega \times Y \times (0,T))$. Then as $\varepsilon \to 0$*

$$v^\varepsilon \to \int_Y v(x,y,t)\,dy \quad \textit{weakly in } L^p(\Omega \times (0,T)).$$

Finally, we state the following well-known result on the weak limit of oscillatory periodic functions, which can be found in [6].

Proposition 4.10. *Let $h \in L^q(\Omega)$ be a Y-periodic function, where $1 \le q \le \infty$. Set $h^\varepsilon(x) = h(\frac{x}{\varepsilon})$ for $x \in \Omega$. Then as $\varepsilon \to 0$,*

$$h^\varepsilon \to \bar{h} = \int_Y h(y)\,dy \quad \textit{weakly in } L^q(\Omega), \tag{4.3}$$

if $1 \le q < \infty$, and

$$h^\varepsilon \to \bar{h} \quad \textit{weakly-}* \textit{ in } L^\infty(\Omega), \tag{4.4}$$

if $q = \infty$.

4.2 The Two-Scale Limit Equation

In this section, we use two-scale convergence to identify the limit of (1.10)-(1.12). We observe that the loading force and initial data given by equations (2.2), (2.9), (2.16), or (2.23), satisfy the following

$$b\left(x, \frac{x}{\varepsilon}, t\right) \;\stackrel{2}{\rightharpoonup}\; b(x,y,t), \tag{4.5a}$$

$$u^0\left(x, \frac{x}{\varepsilon}\right) \;\stackrel{2}{\rightharpoonup}\; u^0(x,y), \tag{4.5b}$$

$$v^0\left(x, \frac{x}{\varepsilon}\right) \;\stackrel{2}{\rightharpoonup}\; v^0(x,y). \tag{4.5c}$$

We note that from Proposition 3.1(c) and Theorem 4.8 it follows that, up to some subsequences, $u^\varepsilon \stackrel{2}{\rightharpoonup} u$, $\dot{u}^\varepsilon \stackrel{2}{\rightharpoonup} u^*$, and $\ddot{u}^\varepsilon \stackrel{2}{\rightharpoonup} u^{**}$, where u, u^*, and u^{**} are in $L^s([0,T];\, L^s(\Omega \times Y)^3)$. We shall see later that $u(x,y,t)$ is uniquely determined by an initial value problem. Therefore u is independent of the subsequence, and the whole sequence (u^ε) two-scale converges to u.

In order to identify the two-scale limit of (1.10), we multiply both sides by a test function $\psi(x, \frac{x}{\varepsilon}, t)$, where $\psi(x, y, t)$ is Y-periodic in y and is such that $\psi \in C_c^\infty(\mathbb{R}^3 \times Y \times \mathbb{R})^3$, and integrate on $\Omega \times \mathbb{R}^+$

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^+} \partial_t^2 u^\varepsilon(x,t) \cdot \psi\left(x, \frac{x}{\varepsilon}, t\right) dxdt \\ &= \int_{\Omega\times\mathbb{R}^+} \left((A_L + A_S^\varepsilon)u^\varepsilon(x,t) + b\left(x, \frac{x}{\varepsilon}, t\right)\right) \cdot \psi\left(x, \frac{x}{\varepsilon}, t\right) dxdt \end{aligned}$$

After integrating by parts twice, we obtain

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^+} u^\varepsilon(x,t) \cdot \partial_t^2 \psi\left(x, \frac{x}{\varepsilon}, t\right) dxdt - \int_\Omega \partial_t u^\varepsilon(x,0) \cdot \psi\left(x, \frac{x}{\varepsilon}, 0\right) dx \\ &+ \int_\Omega u^\varepsilon(x,0) \cdot \partial_t \psi\left(x, \frac{x}{\varepsilon}, 0\right) dx \\ &= \int_{\Omega\times\mathbb{R}^+} \left((A_L + A_S^\varepsilon)u^\varepsilon(x,t) + b\left(x, \frac{x}{\varepsilon}, t\right)\right) \cdot \psi\left(x, \frac{x}{\varepsilon}, t\right) dxdt \end{aligned}$$

By letting $\varepsilon \to 0$ we obtain

$$\begin{aligned} &\int_{\Omega\times Y\times\mathbb{R}^+} u(x,y,t) \cdot \partial_t^2 \psi(x,y,t)\, dxdydt - \int_{\Omega\times Y} v^0(x,y) \cdot \psi(x,y,0)\, dxdy \\ &+ \int_{\Omega\times Y} u^0(x,y) \cdot \partial_t \psi(x,y,0)\, dxdy \\ &= \lim_{\varepsilon\to 0} \int_{\Omega\times\mathbb{R}^+} (A_L + A_S^\varepsilon)u^\varepsilon(x,t) \cdot \psi\left(x, \frac{x}{\varepsilon}, t\right) dxdt \\ &+ \int_{\Omega\times Y\times\mathbb{R}^+} b(x,y,t) \cdot \psi(x,y,t)\, dxdydt \end{aligned} \tag{4.6}$$

For $i = 1, 2, 3$, we extend $u_i(x, y, t)$ by periodicity from $\Omega \times Y \times (0, T)$ to $\Omega \times \mathbb{R}^3 \times (0, T)$. We will use the following lemma to compute the limit on the right hand side of (4.6).

Lemma 4.11. *Let w be in $L^s(\Omega; L^s_{per}(Y)^3)$ and define*

$$\begin{aligned} B_L w(x,y) &= \int_{H_\gamma(x)} \lambda(\hat{x} - x) \frac{(\hat{x} - x) \otimes (\hat{x} - x)}{|\hat{x} - x|^3} \left(\int_Y w(\hat{x}, y')\, dy' - w(x,y)\right) d\hat{x}, \\ B_S w(x,y) &= \int_{H_\delta(y)} \alpha(y, \hat{y}) \frac{(\hat{y} - y) \otimes (\hat{y} - y)}{|\hat{y} - y|^3} \left(w(x, \hat{y}) - w(x,y)\right) d\hat{y}. \end{aligned}$$

Then as $\varepsilon \to 0$,

(a) $A_L u^\varepsilon(x,t) \overset{2}{\rightharpoonup} B_L u(x,y,t)$.
Moreover, the operator B_L is linear and bounded on $L^s(\Omega; L^s_{per}(Y)^3)$.

(b) $A_S^\varepsilon u^\varepsilon(x,t) \overset{2}{\rightharpoonup} B_S u(x,y,t)$.
Moreover, the operator B_S is linear and bounded on $L^s(\Omega; L^s_{per}(Y)^3)$.

The proof of this lemma is provided at the end of this section.

Using Lemma (4.11) and Lebesgue's dominated convergence theorem, it follows that

$$\lim_{\varepsilon\to 0}\int_{\Omega\times\mathbb{R}^+}(A_L+A_S^\varepsilon)u^\varepsilon(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)dxdt$$
$$=\int_{\Omega\times Y\times\mathbb{R}^+}(B_L+B_S)u(x,y,t)\cdot\psi(x,y,t)\,dxdydt.$$

Thus (4.6) becomes

$$\int_{\Omega\times Y\times\mathbb{R}^+}u(x,y,t)\cdot\partial_t^2\psi(x,y,t)\,dxdydt-\int_{\Omega\times Y}v^0(x,y)\cdot\psi(x,y,0)\,dxdy$$
$$+\int_{\Omega\times Y}u^0(x,y)\cdot\partial_t\psi(x,y,0)\,dxdy$$
$$=\int_{\Omega\times Y\times\mathbb{R}^+}((B_L+B_S)u(x,y,t)+b(x,y,t))\cdot\psi(x,y,t)\,dxdydt \tag{4.7}$$

We shall see from Lemma 4.13, provided before the end of this section, that u has two classical partial derivatives with respect to t, for almost every t, and the initial conditions supplementing (4.7) are given by

$$u(x,y,0)=u^0(x,y),\;\;\partial_t u(x,y,0)=v^0(x,y). \tag{4.8}$$

Thus by integrating by parts twice, equation (4.7) becomes

$$\int_{\Omega\times Y\times\mathbb{R}^+}\partial_t^2u(x,y,t)\cdot\psi(x,y,t)\,dxdydt$$
$$=\int_{\Omega\times Y\times\mathbb{R}^+}((B_L+B_S)u(x,y,t)+b(x,y,t))\cdot\psi(x,y,t)\,dxdydt \tag{4.9}$$

Since this is true for any function $\psi\in C_c^\infty(\mathbb{R}^3\times Y\times\mathbb{R})^3$ for which $\psi(x,y,t)$ is Y-periodic in y, we obtain that for almost every x,y, and t

$$\partial_t^2u(x,y,t)=Bu(x,y,t)+b(x,y,t), \tag{4.10}$$

where $B=B_L+B_S$. It follows from Lemma 4.11 that B is a bounded linear operator on $L^s(\Omega;\,L^s_{per}(Y)^3)$. Therefore, the initial value problem given by (4.10) and (4.8), interpreted as a second-order inhomogeneous abstract Cauchy problem defined on $L^s(\Omega;\,L^s_{per}(Y)^3)$, has a unique solution $u\in\mathcal{Q}$.

The following summarizes the results of this section.

Theorem 4.12. *Let (u^ε) be the sequence of solutions of (1.10)-(1.12). Then $u^\varepsilon\xrightarrow{2}u$ where $u\in\mathcal{Q}$ is the unique solution of*

$$\begin{aligned}\partial_t^2u(x,y,t) &= \int_{H_\gamma(x)}\lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(\int_Y u(\hat{x},y',t)\,dy'-u(x,y,t)\right)d\hat{x}\\ &\quad+\int_{H_\delta(y)}\alpha(y,\hat{y})\frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|\hat{y}-y|^3}\left(u(x,\hat{y},t)-u(x,y,t)\right)d\hat{y}\\ &\quad+b(x,y,t),\end{aligned} \tag{4.11}$$

supplemented with initial conditions

$$u(x,y,0) = u^0(x,y), \tag{4.12}$$
$$\partial_t u(x,y,0) = v^0(x,y). \tag{4.13}$$

Lemma 4.13. *Let $t \in [0,T]$ and define*

$$g(x,y,t) = \int_0^t \int_0^\tau u^{**}(x,y,l)\, dl d\tau + t u^*(x,y,0) + u(x,y,0). \tag{4.14}$$

Then g is in $L^s(\Omega \times Y \times (0,T))^3$, twice differentiable with respect to t almost everywhere, and satisfies

(a) For almost every x, y, and t, $g(x,y,t) = u(x,y,t)$, $\partial_t g(x,y,t) = u^*(x,y,t)$, and $\partial_t^2 g(x,y,t) = u^{**}(x,y,t)$.

(b) For almost every x and y

$$g(x,y,0) = u(x,y,0) = u^0(x,y),$$
$$\partial_t g(x,y,0) = u^*(x,y,0) = v^0(x,y).$$

Proof. Part (a). Let $\psi_1(x,y)$ be in $C_c^\infty(\Omega\times Y)^3$ and Y-periodic in y, and let ϕ be in $C_c^\infty(\mathbb{R}^+)$. Then by using integration by parts, we see that

$$\int_{\Omega\times\mathbb{R}^+} \partial_t u^\varepsilon(x,t) \cdot \psi_1\left(x, \frac{x}{\varepsilon}\right) \phi(t)\, dxdt = -\int_{\Omega\times\mathbb{R}^+} u^\varepsilon(x,t) \cdot \psi_1\left(x, \frac{x}{\varepsilon}\right) \dot{\phi}(t)\, dxdt.$$

Sending ε to 0 and using the fact that, up to a subsequence, $\partial_t u^\varepsilon \stackrel{2}{\rightharpoonup} u^*$, we obtain

$$\int_{\Omega\times Y\times\mathbb{R}^+} u^*(x,y,t) \cdot \psi_1(x,y)\, \phi(t)\, dxdydt$$
$$= -\int_{\Omega\times Y\times\mathbb{R}^+} u(x,y,t) \cdot \psi_1(x,y)\, \dot{\phi}(t)\, dxdydt.$$

Since this holds for every ψ_1 we conclude that

$$\int_{\mathbb{R}^+} u^*(x,y,t)\phi(t)\, dt = -\int_{\mathbb{R}^+} u(x,y,t)\dot{\phi}(t)\, dt, \tag{4.15}$$

for almost every x and y and for every $\phi \in C_c^\infty(\mathbb{R}^+)$. Similarly, by using the fact that, up to a subsequence, $\partial_t^2 u^\varepsilon \stackrel{2}{\rightharpoonup} u^{**}$, we see that

$$\int_{\mathbb{R}^+} u^{**}(x,y,t)\phi(t)\, dt = \int_{\mathbb{R}^+} u(x,y,t)\ddot{\phi}(t)\, dt, \tag{4.16}$$

for almost every x and y and for every $\phi \in C_c^\infty(\mathbb{R}^+)$. We note that from (4.14) it is easy to see that g is twice differentiable in t almost everywhere and satisfies

$$\partial_t g(x,y,t) = \int_0^t u^{**}(x,y,\tau)\, d\tau + u^*(x,y,0), \tag{4.17}$$
$$\partial_t^2 g(x,y,t) = u^{**}(x,y,t). \tag{4.18}$$

We will use these facts together with (4.15) and (4.16) to show that $\partial_t g = u^*$ almost everywhere and $g = u$ almost everywhere.

For $\phi \in C_c^\infty(\mathbb{R}^+)$, we have

$$\begin{aligned} \int_{\mathbb{R}^+} \partial_t g(x,y,t)\dot{\phi}(t)\,dt &= -\int_{\mathbb{R}^+} \partial_t^2 g(x,y,t)\phi(t)\,dt \\ &= -\int_{\mathbb{R}^+} u^{**}(x,y,t)\phi(t)\,dt \\ &= -\int_{\mathbb{R}^+} u(x,y,t)\ddot{\phi}(t)\,dt \\ &= \int_{\mathbb{R}^+} u^*(x,y,t)\dot{\phi}(t)\,dt \end{aligned}$$

where (4.18) and (4.16) were used in the second and third steps, respectively. Thus we obtain

$$\int_{\mathbb{R}^+} \left(\partial_t g(x,y,t) - u^*(x,y,t)\right)\dot{\phi}(t)\,dt = 0, \tag{4.19}$$

for every $\phi \in C_c^\infty(\mathbb{R}^+)$. Since $\partial_t g(x,y,0) = u^*(x,y,0)$, we conclude from (4.19) that $\partial_t g(x,y,t) = u^*(x,y,t)$ almost everywhere.

We also have

$$\begin{aligned} \int_{\mathbb{R}^+} g(x,y,t)\dot{\phi}(t)\,dt &= -\int_{\mathbb{R}^+} \partial_t g(x,y,t)\phi(t)\,dt \\ &= -\int_{\mathbb{R}^+} u^*(x,y,t)\phi(t)\,dt \\ &= \int_{\mathbb{R}^+} u(x,y,t)\dot{\phi}(t)\,dt \end{aligned}$$

where the fact that $\partial_t g(x,y,t) = u^*(x,y,t)$ almost everywhere was used in the second step and (4.15) was used in the third step. Thus we see that

$$\int_{\mathbb{R}^+} \left(g(x,y,t) - u(x,y,t)\right)\dot{\phi}(t)\,dt = 0, \tag{4.20}$$

for every $\phi \in C_c^\infty(\mathbb{R}^+)$. Since $g(x,y,0) = u(x,y,0)$, we conclude from (4.20) that $g(x,y,t) = u(x,y,t)$ almost everywhere, completing the proof of Part (a).

Part (b). Let $\psi(x,y,t)$ be in $C_c^\infty(\Omega \times Y \times \mathbb{R})^3$ and Y-periodic in y. Then by using integration by parts, we see that

$$\begin{aligned} \int_{\Omega\times\mathbb{R}^+} \partial_t u^\varepsilon(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt &= -\int_{\Omega\times\mathbb{R}^+} u^\varepsilon(x,t)\cdot\partial_t\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt \\ &\quad -\int_{\Omega} u^\varepsilon(x,0)\cdot\psi\left(x,\frac{x}{\varepsilon},0\right)\,dx. \end{aligned}$$

Sending ε to 0, we obtain

$$\begin{aligned} \int_{\Omega\times Y\times\mathbb{R}^+} u^*(x,y,t)\cdot\psi\left(x,y,t\right)\,dxdydt &= -\int_{\Omega\times Y\times\mathbb{R}^+} u(x,y,t)\cdot\partial_t\psi\left(x,y,t\right)\,dxdydt \\ &\quad -\int_{\Omega\times Y} u^0(x,y)\cdot\psi\left(x,y,0\right)\,dxdy. \end{aligned} \tag{4.21}$$

On the other hand, using Part (a), we see that

$$\begin{aligned}\int_{\Omega\times Y\times\mathbb{R}^+} u^*(x,y,t)\cdot\psi\,(x,y,t)\;dxdydt &= \int_{\Omega\times Y\times\mathbb{R}^+}\partial_t g(x,y,t)\cdot\psi\,(x,y,t)\;dxdydt\\ &= -\int_{\Omega\times Y\times\mathbb{R}^+} g(x,y,t)\cdot\partial_t\psi\,(x,y,t)\;dxdydt\\ &\quad -\int_{\Omega\times Y} g(x,y,0)\cdot\psi\,(x,y,0)\;dxdy\\ &= -\int_{\Omega\times Y\times\mathbb{R}^+} u(x,y,t)\cdot\partial_t\psi\,(x,y,t)\;dxdydt\\ &\quad -\int_{\Omega\times Y} u(x,y,0)\cdot\psi\,(x,y,0)\;dxdy.\end{aligned} \tag{4.22}$$

From (4.21) and (4.22) we obtain that

$$\int_{\Omega\times Y}\big(u^0(x,y)-u(x,y,0)\big)\cdot\psi\,(x,y,0)\;dxdy=0,$$

for every ψ. Therefore

$$u(x,y,0)=u^0(x,y),$$

almost everywhere. Similarly we can show that

$$\partial_t u(x,y,0)=v^0(x,y),$$

almost everywhere, completing the proof of Part (b). □

Proof of Lemma 4.11. Part (a). Since $A_L = A_{L,1} - A_{L,2}$, we will compute the two-scale limits of $A_{L,1}u^\varepsilon$ and $A_{L,2}u^\varepsilon$, then combine them to show that as $\varepsilon\to 0$,

$$A_L u^\varepsilon(x,t)\overset{2}{\rightharpoonup} B_L u(x,y,t). \tag{4.23}$$

Let $\psi\in C_c^\infty(\mathbb{R}^3\times Y)^3$ such that $\psi(x,y)$ is Y-periodic in y, and $\phi\in C_c^\infty(\mathbb{R}^+)$. Then from the definition of $A_{L,1}$, equation (3.1), we see that

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}^+} A_{L,1}u^\varepsilon(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon}\right)\phi(t)\,dxdt\\ &=\int_{\Omega\times\mathbb{R}^+}\int_{H_\gamma(x)}\lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\;u^\varepsilon(\hat{x},t)\,d\hat{x}\cdot\psi\left(x,\frac{x}{\varepsilon}\right)\phi(t)\,dxdt,\end{aligned} \tag{4.24}$$

Since $u^\varepsilon(x,t)\overset{2}{\rightharpoonup} u(x,y,t)$, we obtain using Proposition 4.9 that, as $\varepsilon\to 0$,

$$u^\varepsilon\to\int_Y u(x,y,t)\,dy\;\text{ weakly in } L^s(\Omega\times(0,T))^3. \tag{4.25}$$

It follows from (4.25) that, for fixed x,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^+}\int_{H_\gamma(x)}\lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\,u^\varepsilon(\hat{x},t)\phi(t)\,d\hat{x}dt$$
$$=\int_{\mathbb{R}^+}\int_{H_\gamma(x)}\lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(\int_Y u(\hat{x},y',t)\,dy'\right)\phi(t)\,d\hat{x}dt. \tag{4.26}$$

We note that by replacing $v(x)$ with $u^\varepsilon(x,t)$ in (3.17), we obtain

$$\left|\int_{H_\gamma(x)}\lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\,u^\varepsilon(\hat{x},t)\,d\hat{x}\right|$$
$$\leq \lambda_{\max}\left(\int_{H_\delta(x)}\frac{1}{|\hat{x}-x|^{s'}}\,d\hat{x}\right)^{1/s'}\left(\int_{H_\delta(x)}|u^\varepsilon(\hat{x},t)|^s\,d\hat{x}\right)^{1/s}$$
$$\leq \lambda_{\max}\left(\int_{H_\delta(x)}\frac{1}{|\hat{x}-x|^{s'}}\,d\hat{x}\right)^{1/s'}\|u^\varepsilon\|_{L^\infty([0,T];\,L^s(\Omega)^3)}. \tag{4.27}$$

From Proposition 3.1, $\|u^\varepsilon\|_{L^\infty([0,T];\,L^s(\Omega)^3)}$ is bounded. Thus from (4.26), and (4.27) and by using Lebesgue's dominated convergence theorem, we conclude that the convergence of the sequence of functions in (4.26) is not only point-wise in x convergence but also strong in $L^s(\Omega)^3$. Therefore we can use Proposition 4.2 and (4.26) to evaluate the limit of (4.24) as $\varepsilon\to 0$. We find that

$$\lim_{\varepsilon\to 0}\int_{\Omega\times\mathbb{R}^+}A_{L,1}u^\varepsilon(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon}\right)\phi(t)\,dxdt$$
$$=\int_{\Omega\times\mathbb{R}^+}\int_{H_\gamma(x)}\lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\left(\int_Y u(\hat{x},y',t)\,dy'\right)d\hat{x}\cdot\psi\left(x,\frac{x}{\varepsilon}\right)\phi(t)\,dxdt, \tag{4.28}$$

Next we evaluate the two-scale limit of $A_{L,2}u^\varepsilon$. We recall from (3.2) that

$$A_{L,2}u^\varepsilon(x,t)=\int_{H_\gamma(x)}\lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\,d\hat{x}\;u^\varepsilon(x,t), \tag{4.29}$$

from which immediately follows that as $\varepsilon\to 0$,

$$A_{L,2}u^\varepsilon \xrightarrow{2}\int_{H_\gamma(x)}\lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\,d\hat{x}\;u(x,y,t). \tag{4.30}$$

Combining equations (4.28) and (4.30), the result (4.23) follows.

The fact that the two operators B_L and B_S are linear and bounded on the Banach space $L^s(\Omega;\,L^s_{per}(Y))$ can be shown by arguments similar to those used in the proof of Proposition 3.1.

Part (b). Since $A^\varepsilon_S=A^\varepsilon_{S,1}-A^\varepsilon_{S,2}$, we will compute the two-scale limits of $A^\varepsilon_{S,1}u^\varepsilon$ and $A^\varepsilon_{S,2}u^\varepsilon$, then combine them to show that as $\varepsilon\to 0$,

$$A^\varepsilon_S u^\varepsilon(x,t)\xrightarrow{2} B_S u(x,y,t). \tag{4.31}$$

Let $\psi(x,y,t) = \psi_2(x)\psi_1(y)\phi(t)$, where $\psi_2 \in C_c^\infty(\mathbb{R}^3)$, $\psi_1 \in C_{per}^\infty(Y)^3$, and $\phi \in C_c^\infty(\mathbb{R}^+)$. Then by using (3.13), replacing $v(x)$ with $u^\varepsilon(x,t)$, we see that

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}^+} A^\varepsilon_{S,1} u^\varepsilon(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt\\ &= \int_{\Omega\times\mathbb{R}^+}\int_{H_\delta(0)} \alpha\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z\right)\frac{z\otimes z}{|z|^3}\,u^\varepsilon(x+\varepsilon z,t)\,dz\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt.\end{aligned} \tag{4.32}$$

We recall that $\alpha\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z\right)$ is defined by equation (1.5). Without loss of generality, we may assume that $\alpha\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z\right)$ is given by

$$\alpha\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z\right) = \chi_{\mathrm{f}}\left(\frac{x}{\varepsilon}\right)\chi_{\mathrm{f}}\left(\frac{x}{\varepsilon}+z\right).$$

Thus after a change in the order of integration in the right hand side of equation (4.32), we see that

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}^+} A^\varepsilon_{S,1} u^\varepsilon(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt\\ &= \int_{H_\delta(0)}\frac{1}{|z|^3}\int_{\Omega\times\mathbb{R}^+}\chi_{\mathrm{f}}\left(\frac{x}{\varepsilon}\right)\chi_{\mathrm{f}}\left(\frac{x}{\varepsilon}+z\right)u^\varepsilon(x+\varepsilon z,t)\cdot z\;\psi_1\left(\frac{x}{\varepsilon}\right)\cdot z\;\psi_2(x)\phi(t)\,dxdtdz.\end{aligned} \tag{4.33}$$

Now we focus on evaluating the limit as $\varepsilon \to 0$ of the inner integral in (4.33). By the change of variables $r = x + \varepsilon z$ we obtain

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}^+}\chi_{\mathrm{f}}\left(\frac{x}{\varepsilon}\right)\chi_{\mathrm{f}}\left(\frac{x}{\varepsilon}+z\right)u^\varepsilon(x+\varepsilon z,t)\cdot z\;\psi_1\left(\frac{x}{\varepsilon}\right)\cdot z\;\psi_2(x)\phi(t)\,dxdt\\ &= \int_{\mathbb{R}^3\times\mathbb{R}^+}\chi_\Omega(r-\varepsilon z)\,\chi_{\mathrm{f}}\left(\frac{r}{\varepsilon}-z\right)\chi_{\mathrm{f}}\left(\frac{r}{\varepsilon}\right)u^\varepsilon(r,t)\cdot z\;\psi_1\left(\frac{r}{\varepsilon}-z\right)\cdot z\;\psi_2(r-\varepsilon z)\phi(t)\,drdt \qquad (4.34)\\ &:= a^\varepsilon(z),\end{aligned}$$

where χ_Ω denotes the indicator function of Ω. We will show that for $z \in H_\delta(0)$,

$$\lim_{\varepsilon\to 0} a^\varepsilon(z) = \int_{\Omega\times Y\times\mathbb{R}^+}\chi_{\mathrm{f}}(y-z)\,\chi_{\mathrm{f}}(y)\,u(r,y,t)\cdot z\;\psi_1(y-z)\cdot z\;\psi_2(r)\phi(t)\,drdydt. \tag{4.35}$$

To see this, we approximate χ_Ω by smooth functions ζ_n such that as $n\to\infty$, $\zeta_n(r)\to\chi_\Omega(r)$ almost everywhere and $\zeta_n \to \chi_\Omega$ in $L^{s'}_{loc}(\Omega)$, with $1/s + 1/s' = 1$. Then by adding and subtracting $\zeta_n(r-\varepsilon z)$ to and from $\chi_\Omega(r-\varepsilon z)$ in (4.34), we obtain that

$$a^\varepsilon(z) = a_1^{n,\varepsilon}(z) + a_2^{n,\varepsilon}(z), \tag{4.36}$$

where,

$$a_1^{n,\varepsilon}(z) := \int_{\mathbb{R}^3\times\mathbb{R}^+} (\chi_\Omega(r-\varepsilon z) - \zeta_n(r-\varepsilon z)) \times \chi_{\rm f}\left(\frac{r}{\varepsilon}-z\right)\chi_{\rm f}\left(\frac{r}{\varepsilon}\right) u^\varepsilon(r,t)\cdot z\ \psi_1\left(\frac{r}{\varepsilon}-z\right)\cdot z\ \psi_2(r-\varepsilon z)\phi(t)\,drdt, \tag{4.37}$$

$$a_2^{n,\varepsilon}(z) := \int_{\mathbb{R}^3\times\mathbb{R}^+} \zeta_n(r-\varepsilon z) \times \chi_{\rm f}\left(\frac{r}{\varepsilon}-z\right)\chi_{\rm f}\left(\frac{r}{\varepsilon}\right) u^\varepsilon(r,t)\cdot z\ \psi_1\left(\frac{r}{\varepsilon}-z\right)\cdot z\ \psi_2(r-\varepsilon z)\phi(t)\,drdt. \tag{4.38}$$

From (4.37) and by using Hölder's inequality, we see that

$$|a_1^{n,\varepsilon}(z)| \leq \left(\int_{\mathbb{R}^3} |\chi_\Omega(r-\varepsilon z) - \zeta_n(r-\varepsilon z)|^{s'}\,dr\right)^{1/s'} \times \int_{\mathbb{R}^+}\left(\int_{\mathbb{R}^3} \chi_{\rm f}\left(\frac{r}{\varepsilon}-z\right)\chi_{\rm f}\left(\frac{r}{\varepsilon}\right)\left|u^\varepsilon(r,t)\cdot z\ \psi_1\left(\frac{r}{\varepsilon}-z\right)\cdot z\ \psi_2(r-\varepsilon z)\right|^s dr\right)^{1/s} \times \phi(t)\,dt. \tag{4.39}$$

We note that the second term on the right hand side of (4.39) is bounded above uniformly in ε. This follows from Hölder's inequality applied to the inner integral and the fact that $(u^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty_{loc}(\mathbb{R}^+;\, L^s(\Omega)^3)$. On the other hand, by the change of variables $r' = r-\varepsilon z$, the first term on the right hand side of (4.39) becomes

$$\left(\int_{\mathbb{R}^3} |\chi_\Omega(r') - \zeta_n(r')|^{s'}\,dr'\right)^{1/s'},$$

which goes to zero as $n\to\infty$. From these two facts and (4.39), we conclude that for all $\varepsilon>0$ and $z\in H_\delta(0)$,

$$\lim_{n\to\infty} a_1^{n,\varepsilon}(z) = 0. \tag{4.40}$$

Now for fixed n, since ζ_n and ψ_2 are smooth functions, we see that as $\varepsilon\to 0$, $\zeta_n(r-\varepsilon z)\psi_2(r-\varepsilon z)\to\zeta_n(r)\psi_2(r)$ uniformly. Therefore, we see from (4.38) that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} a_2^{n,\varepsilon}(z)\\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\times\mathbb{R}^+} \zeta_n(r)\ \chi_{\rm f}\left(\frac{r}{\varepsilon}-z\right)\chi_{\rm f}\left(\frac{r}{\varepsilon}\right) u^\varepsilon(r,t)\cdot z\ \psi_1\left(\frac{r}{\varepsilon}-z\right)\cdot z\ \psi_2(r)\phi(t)\,drdt\\
&= \int_{\mathbb{R}^3\times Y\times\mathbb{R}^+} \zeta_n(r)\chi_{\rm f}(y-z)\,\chi_{\rm f}(y)\,u(r,y,t)\cdot z\ \psi_1(y-z)\cdot z\ \psi_2(r)\phi(t)\,drdydt,
\end{aligned} \tag{4.41}$$

where in the last step the fact that $(u^\varepsilon)_{\varepsilon>0}$ two-scale converges to $u(r,y,t)$ was used. By taking the limit as $n \to \infty$ in (4.41), we obtain

$$\begin{aligned} &\lim_{n\to\infty}\lim_{\varepsilon\to 0} a_2^{n,\varepsilon}(z) \\ &= \int_{\Omega\times Y\times\mathbb{R}^+} \chi_{\mathrm{f}}(y-z)\,\chi_{\mathrm{f}}(y)\,u(r,y,t)\cdot z\;\psi_1(y-z)\cdot z\;\psi_2(r)\phi(t)\,drdydt. \end{aligned} \tag{4.42}$$

From (4.40) and (4.42) and since

$$\lim_{\varepsilon\to 0} a^\varepsilon(z) = \lim_{n\to\infty}\lim_{\varepsilon\to 0}(a_1^{n,\varepsilon}(z) + a_2^{n,\varepsilon}(z)),$$

equation (4.35) follows.

From (4.33) and (4.35), and by using Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\Omega\times\mathbb{R}^+} A^\varepsilon_{S,1}u^\varepsilon(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt \\ &= \int_{H_\delta(0)}\frac{1}{|z|^3}\int_{\Omega\times Y\times\mathbb{R}^+} \chi_{\mathrm{f}}(y-z)\,\chi_{\mathrm{f}}(y)\,u(r,y,t)\cdot z\;\psi_1(y-z)\cdot z\;\psi_2(r)\phi(t)\,drdydtdz \\ &= \int_{\Omega\times\mathbb{R}^+}\int_{H_\delta(0)}\frac{1}{|z|^3}\int_Y \chi_{\mathrm{f}}(y-z)\,\chi_{\mathrm{f}}(y)\,u(r,y,t)\cdot z\;\psi_1(y-z)\cdot z\,dydz\;\psi_2(r)\phi(t)drdt, \end{aligned} \tag{4.43}$$

where we have changed the order of integration in the last step. After shifting the domain of integration in the inner integral of the right hand side of equation (4.43), we obtain

$$\begin{aligned} &\int_Y \chi_{\mathrm{f}}(y-z)\,\chi_{\mathrm{f}}(y)\,u(r,y,t)\cdot z\;\psi_1(y-z)\cdot z\,dy \\ &= \int_{Y-z} \chi_{\mathrm{f}}(y)\,\chi_{\mathrm{f}}(y+z)\,u(r,y+z,t)\cdot z\;\psi_1(y)\cdot z\,dy \\ &= \int_Y \chi_{\mathrm{f}}(y)\,\chi_{\mathrm{f}}(y+z)\,u(r,y+z,t)\cdot z\;\psi_1(y)\cdot z\,dy, \end{aligned} \tag{4.44}$$

where in the last step the fact that the integrand is Y-periodic in y was used. Substituting (4.44) in equation (4.43), then by changing the order of integration we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\Omega\times\mathbb{R}^+} A^\varepsilon_{S,1}u^\varepsilon(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt \\ &= \int_{\Omega\times\mathbb{R}^+}\int_Y\int_{H_\delta(0)} \chi_{\mathrm{f}}(y)\,\chi_{\mathrm{f}}(y+z)\,\frac{z\otimes z}{|z|^3}u(r,y+z,t)dz\cdot\psi_1(y)dy\;\psi_2(r)\phi(t)drdt \\ &= \int_{\Omega\times Y\times\mathbb{R}^+}\int_{H_\delta(y)} \chi_{\mathrm{f}}(y)\chi_{\mathrm{f}}(\hat{y})\,\frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|\hat{y}-y|^3}\;u(r,\hat{y},t)d\hat{y}\cdot\psi(r,y,t)\,drdydt. \end{aligned} \tag{4.45}$$

In the last equality the change of variables $\hat{y} = y + z$ was used.

Next we evaluate the two-scale limit of $A^{\varepsilon}_{S,2}u^{\varepsilon}$. Let ψ be a test function in $\mathcal{J}$. Then by using (3.16), replacing $v(x)$ with $u^{\varepsilon}(x,t)$, we obtain

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}^+} A^{\varepsilon}_{S,2}u^{\varepsilon}(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt\\&=\int_{\Omega\times\mathbb{R}^+}\int_{H_\delta(0)}\alpha\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z\right)\frac{z\otimes z}{|z|^3}\,dz\;u^{\varepsilon}(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt.\end{aligned}\tag{4.46}$$

The right hand side of (4.46), after changing the order of integration, is equal to

$$\int_{H_\delta(0)}\frac{1}{|z|^3}\int_{\Omega\times\mathbb{R}^+}\alpha\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z\right)u^{\varepsilon}(x,t)\cdot z\;\psi\left(x,\frac{x}{\varepsilon},t\right)\cdot z\,dxdtdz.\tag{4.47}$$

Using the fact that $(u^{\varepsilon})_{\varepsilon>0}$ two-scale converges to $u(x,y,t)$, we see that for $z\in H_\delta(0)$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega\times\mathbb{R}^+}\alpha\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z\right)u^{\varepsilon}(x,t)\cdot z\;\psi\left(x,\frac{x}{\varepsilon},t\right)\cdot z\,dxdt\\&=\int_{\Omega\times Y\times\mathbb{R}^+}\alpha\left(y,y+z\right)u(x,y,t)\cdot z\;\psi\left(x,y,t\right)\cdot z\,dxdydt.\end{aligned}\tag{4.48}$$

From (4.46), (4.47) and (4.48), and by using Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega\times\mathbb{R}^+} A^{\varepsilon}_{S,2}u^{\varepsilon}(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt\\&=\int_{H_\delta(0)}\frac{1}{|z|^3}\int_{\Omega\times Y\times\mathbb{R}^+}\alpha\left(y,y+z\right)u(x,y,t)\cdot z\;\psi\left(x,y,t\right)\cdot z\,dxdydtdz\end{aligned}\tag{4.49}$$

By changing the order of integration and then using the change of variables $\hat{y}=y+z$, we conclude that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega\times\mathbb{R}^+} A^{\varepsilon}_{S,2}u^{\varepsilon}(x,t)\cdot\psi\left(x,\frac{x}{\varepsilon},t\right)\,dxdt\\&=\int_{\Omega\times Y\times\mathbb{R}^+}\int_{H_\delta(y)}\alpha\left(y,\hat{y}\right)\frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|\hat{y}-y|^3}\,d\hat{y}\;u(x,y,t)\cdot\psi\left(x,y,t\right)\,dxdydt.\end{aligned}\tag{4.50}$$

Equation (4.31) follows from combining (4.45) and (4.50), completing the proof. □

5 The Macroscopic Equation and Downscaling

The aim of this section is to justify the main results of Section 2.

5.1 Derivation of the Macroscopic Equation

We begin this section with the following observation. Let ϕ be a function in $L^s_{per}(Y)^3$. Then

$$\int_Y \int_{H_\delta(y)} \alpha(y,\hat{y}) \frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|(\hat{y}-y)|^3} \left(\phi(\hat{y}) - \phi(y)\right) d\hat{y}\, dy = 0. \tag{5.1}$$

To see this, we note that using Fubini's theorem and the assumption that ϕ is Y-periodic, the double integral in (5.1) can be written as

$$\begin{aligned} &\int_Y \int_{H_\delta(\hat{y})} \alpha(y,\hat{y}) \frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|(\hat{y}-y)|^3} \left(\phi(\hat{y}) - \phi(y)\right) dy\, d\hat{y} \\ &= -\int_Y \int_{H_\delta(\hat{y})} \alpha(\hat{y},y) \frac{(y-\hat{y})\otimes(y-\hat{y})}{|(y-\hat{y})|^3} \left(\phi(y) - \phi(\hat{y})\right) dy\, d\hat{y}, \end{aligned} \tag{5.2}$$

where in the last equality we have used the fact $\alpha(y,\hat{y}) = \alpha(\hat{y},y)$. Comparing the double integral in (5.1) with (5.2) the result follows.

Now let

$$u^H(x,t) = \int_Y u(x,y,t)\, dy.$$

Then from Proposition 4.9, we have that $u^H(x,t)$ is the weak limit of $u^\varepsilon(x,t)$ in $L^p(\Omega \times (0,T))^3$. To identify the equation that u^H solves, we integrate (4.11) over Y to obtain

$$\begin{aligned} \partial_t^2 u^H(x,t) \;=\; & \int_{H_\gamma(x)} \lambda(\hat{x}-x) \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3} (u^H(\hat{x},t) - u^H(x,t))\, d\hat{x} \\ & + \int_Y \int_{H_\delta(y)} \alpha(y,\hat{y}) \frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|\hat{y}-y|^3} \left(u(x,\hat{y},t) - u(x,y,t)\right) d\hat{y} dy \\ & + \int_Y b(x,y,t)\, dy. \end{aligned} \tag{5.3}$$

Using (5.1), the second integral on right hand side of (5.3) is equal to zero for all $x \in \Omega$ and $t \in (0,T)$. Thus u^H solves

$$\partial_t^2 u^H(x,t) = \int_{H_\gamma(x)} \lambda(\hat{x}-x) \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3} \left(u^H(\hat{x},t) - u^H(x,t)\right) d\hat{x} + \int_Y b(x,y,t)\, dy, \tag{5.4}$$

supplemented with initial data

$$u^H(x,0) = \int_Y u^0(x,y)\, dy, \quad \partial_t u^H(x,0) = \int_Y v^0(x,y)\, dy. \tag{5.5}$$

The initial value problem (5.4)-(5.5) can be written as the following operator equation in $L^s(\Omega)^3$

$$\left\{ \begin{array}{rcll} \ddot{u}^H(t) &=& A_L u^H(t) + \bar{b}(t), & t \in [0,T] \\ u^H(0) &=& \bar{u}^0, & \\ \dot{u}^H(0) &=& \bar{v}^0. & \end{array} \right. \tag{5.6}$$

where

$$\begin{aligned}\bar{b}(x,t) &= \int_Y b(x,y,t)\,dy,\\ \bar{u}^0(x) &= \int_Y u^0(x,y)\,dy, \text{ and}\\ \bar{v}^0(x) &= \int_Y v^0(x,y)\,dy.\end{aligned}$$

We have seen from the proof of Proposition 3.1 that A_L is a bounded linear operator on $L^s(\Omega)^3$, thus $u^H \in C^2([0,T]; L^s(\Omega)^3)$ is the unique solution of 5.6.

To complete the proof of Theorems 2.1, 2.3, 2.5, and 2.7, we show that u^H is in $C^2([0,T]; C(\bar{\Omega})^3)$, when the initial data $\bar{u}^0$ and $\bar{v}^0$ are in $C(\bar{\Omega})^3$, and the loading force $\bar{b}$ is in $C([0,T]; C(\bar{\Omega})^3)$. In fact, it suffices to show that the linear operator A_L is bounded on the Banach space of continuous functions $C(\bar{\Omega})^3$ equipped with the uniform norm. So we let $v \in C(\bar{\Omega})^3$ and denote the uniform norm on $C(\bar{\Omega})^3$ by $\|\cdot\|_{C(\bar{\Omega})^3}$. Then, we recall from (3.5) that $A_L = A_{L,1} + A_{L,2}$, where $A_{L,1}$ and $A_{L,2}$ can be written as

$$A_{L,1}v(x) = \int_{H_\gamma(0)} \lambda(\xi)\frac{\xi\otimes\xi}{|\xi|^3}\, v(x+\xi)\,d\xi, \tag{5.7}$$

$$A_{L,2}v(x) = \int_{H_\gamma(0)} \lambda(\xi)\frac{\xi\otimes\xi}{|\xi|^3}\, d\xi\, v(x), \tag{5.8}$$

respectively. Taking the norm in (5.7) we see that

$$\begin{aligned}\|A_{L,1}v\|_{C(\bar{\Omega})^3} &= \max_{x\in\bar{\Omega}} \left| \int_{H_\gamma(0)} \lambda(\xi)\frac{\xi\otimes\xi}{|\xi|^3}\, v(x+\xi)\,d\xi \right|\\ &\le \left(\max_{\xi\in H_\gamma(0)} \lambda(\xi)\right) \max_{x\in\bar{\Omega}} \int_{H_\gamma(0)} \frac{1}{|\xi|}\,|v(x+\xi)|\,d\xi\\ &\le \left(\max_{\xi\in H_\gamma(0)} \lambda(\xi)\right) \int_{H_\gamma(0)} \frac{1}{|\xi|}\,d\xi\, \|v\|_{C(\bar{\Omega})^3}.\end{aligned}$$

Thus $A_{L,1}$ is bounded on $C(\bar{\Omega})^3$. It is clear that $A_{L,2}$ is also bounded on $C(\bar{\Omega})^3$, and therefore A_L is bounded completing the argument.

5.2 Justifying the Downscaling Step

In this section we prove Theorems 2.2, 2.4, 2.6, and 2.8. We begin by showing that for fixed $t \in (0,T)$,

$$\lim_{\varepsilon\to 0} \left\| u^\varepsilon(x,t) - u\left(x, \frac{x}{\varepsilon}, t\right)\right\|_{L^s(\Omega)^3} = 0.$$

By shifting the domains of integration, equation (4.11) can be written as follows

$$\begin{aligned}\partial_t^2 u(x,y,t) &= \int_{H_\gamma(0)} \lambda(\xi)\frac{\xi\otimes\xi}{|\xi|^3}\left(\int_Y u(x+\xi,y',t)\,dy' - u(x,y,t)\right)d\xi \\ &+\int_{H_\delta(0)} \alpha(y,y+z)\frac{z\otimes z}{|z|^3}\left(u(x,y+z,t)-u(x,y,t)\right)dz \\ &+b(x,y,t).\end{aligned} \tag{5.9}$$

Since $u(x,y,t)$ is in $\mathcal{Q}$ and solves (5.9) with initial conditions (4.12) and (4.13), then $u(x,\frac{x}{\varepsilon},t)$ is in $C^2([0,T];\,L^s(\Omega)^3)$ and solves

$$\begin{aligned}\partial_t^2 u\left(x,\frac{x}{\varepsilon},t\right) &= \int_{H_\gamma(0)} \lambda(\xi)\frac{\xi\otimes\xi}{|\xi|^3}\left(\int_Y u(x+\xi,y',t)\,dy' - u\left(x,\frac{x}{\varepsilon},t\right)\right)d\xi \\ &+\int_{H_\delta(0)} \alpha\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z\right)\frac{z\otimes z}{|z|^3}\left(u\left(x,\frac{x}{\varepsilon}+z,t\right)-u\left(x,\frac{x}{\varepsilon},t\right)\right)dz \\ &+b\left(x,\frac{x}{\varepsilon},t\right),\end{aligned} \tag{5.10}$$

supplemented with initial conditions

$$u(x,y,0) = u^0\left(x,\frac{x}{\varepsilon}\right), \tag{5.11}$$

$$\partial_t u(x,y,0) = v^0\left(x,\frac{x}{\varepsilon}\right). \tag{5.12}$$

We let $e^\varepsilon(x,t)=u^\varepsilon(x,t)-u(x,\frac{x}{\varepsilon},t)$. Then by subtracting (5.10) from (1.10), we find that $e^\varepsilon\in C^2([0,T];\,L^s(\Omega)^3)$ solves

$$\partial_t^2 e^\varepsilon(x,t) = A^\varepsilon e^\varepsilon(x,t)+d^\varepsilon(x,t), \tag{5.13}$$

$$e^\varepsilon(x,0) = 0, \tag{5.14}$$

$$\partial_t e^\varepsilon(x,0) = 0. \tag{5.15}$$

where A^ε is given by (3.7) and $d^\varepsilon(x,t)$ is given by

$$d^\varepsilon(x,t) = d_L^\varepsilon(x,t)+d_S^\varepsilon(x,t), \tag{5.16}$$

$$d_L^\varepsilon(x,t) = \int_{H_\gamma(0)} \lambda(\xi)\frac{\xi\otimes\xi}{|\xi|^3}\left(u\left(x+\xi,\frac{x+\xi}{\varepsilon},t\right)-\int_Y u(x+\xi,y',t)\,dy'\right)d\xi, \tag{5.17}$$

$$d_S^\varepsilon(x,t) = \int_{H_\delta(0)} \alpha\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z\right)\frac{z\otimes z}{|z|^3}\left(u\left(x+\varepsilon z,\frac{x}{\varepsilon}+z,t\right)-u\left(x,\frac{x}{\varepsilon}+z,t\right)\right)dz. \tag{5.18}$$

Since A^ε is bounded, the solution of (5.13)-(5.15) is explicitly given by

$$e^\varepsilon(x,t)=\int_0^t\sum_{n=0}^\infty \frac{(t-\tau)^{2n+1}}{(2n+1)!}\,(A^\varepsilon)^n d^\varepsilon(x,\tau)\,d\tau.$$

Thus

$$
\begin{aligned}
\|e^{\varepsilon}(\cdot,t)\|_{L^s(\Omega)^3} &\le \int_0^t \sum_{n=0}^{\infty} \frac{(t-\tau)^{2n+1}}{(2n+1)!} \, \|(A^{\varepsilon})^n\| \, \|d^{\varepsilon}(\cdot,\tau)\|_{L^s(\Omega)^3} \, d\tau \\
&\le \int_0^t \frac{1}{\sqrt{M}} \sinh\left(\sqrt{M}(t-\tau)\right) \|d^{\varepsilon}(\cdot,\tau)\|_{L^s(\Omega)^3} \, d\tau \qquad (5.19)
\end{aligned}
$$

where in the second inequality we have used the fact that A^{ε} is bounded above by an $M > 0$ independent of ε.

In the following sections we will show that for $t \in (0,T)$,

$$\lim_{\varepsilon\to 0} \|d^{\varepsilon}(\cdot,t)\|_{L^s(\Omega)^3} = 0, \qquad (5.20)$$

for each of the four cases of initial and loading conditions that has been introduced in Section 2. On the other hand, from (5.16)-(5.18) and the fact that u is continuous on $[0,T]$, it follows that $d^{\varepsilon}(\cdot,\tau)$ is continuous on $[0,t]$ for $t \le T$. Thus, from equations (5.19) and (5.20), and Lebesgue's convergence theorem, we see that

$$\lim_{\varepsilon\to 0} \|e^{\varepsilon}(\cdot,t)\|_{L^s(\Omega)^3} = 0,$$

from which the result follows.

In order to prove (5.20), we will make use of the following observation: *The solution of each cell-problem of Section 2 has zero average over the unit cell.* To see this, we integrate equation (2.5) over Y to obtain

$$\ddot{\bar{r}}(t) = \int_Y \int_{H_\delta(y)} \alpha(y,\hat{y}) \frac{(\hat{y}-y)\otimes(\hat{y}-y)}{|(\hat{y}-y)|^3} \left(r(\hat{y},t) - r(y,t)\right) d\hat{y}\, dy - K\, \bar{r}(t), \qquad (5.21)$$

supplemented with initial conditions

$$\bar{r}(0) = 0, \quad \dot{\bar{r}}(0) = 0. \qquad (5.22)$$

Using (5.1), the integral on the right hand side of (5.21) is equal to zero for all $t \in (0,T)$. Thus $\bar{r}$ solves

$$\ddot{\bar{r}}(t) = -K\, \bar{r}(t), \qquad (5.23)$$

supplemented with zero initial conditions. Obviously the solution of (5.23) is given by

$$\int_Y r(y,t)\, dy = \bar{r}(t) = 0, \qquad (5.24)$$

for all $t \in (0,T)$. Similarly we can show that

$$\int_Y r^j(y,t)\, dy = \bar{r}^j(t) = 0, \qquad (5.25)$$

for all $t \in (0,T)$, where r^j is the solution of (2.12), (2.19), or (2.26).

5.2.1 First Case

In this section we complete the proof of Theorem 2.2 by showing that equation (5.20) holds true when b, u^0, and v^0 are given by (2.2). We also prove the error estimate (2.8).

Using the fact that $r(y,t)$, the solution of the cell problem (2.5)-(2.6), has zero average over Y, and by linearity, it is easy to check that $u^H(x,t)+r(y,t)$ solves (4.11)-(4.13), where u^H is the solution of (2.3)-(2.4). Thus by uniqueness we conclude that

$$u(x,y,t) = u^H(x,t) + r(y,t). \tag{5.26}$$

Using this representation of $u(x,y,t)$ and from equations (5.17) and (5.18), we see that $d^\varepsilon_L(x,t)$ and $d^\varepsilon_S(x,t)$ are now given by

$$d^\varepsilon_L(x,t) = \int_{H_\gamma(0)} \lambda(\xi)\frac{\xi\otimes\xi}{|\xi|^3}\, r\left(\frac{x+\xi}{\varepsilon},t\right)\, d\xi, \tag{5.27}$$

$$d^\varepsilon_S(x,t) = \int_{H_\delta(0)} \alpha\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z\right)\frac{z\otimes z}{|z|^3}\left(u^H(x+\varepsilon z,t)-u^H(x,t)\right)\, dz, \tag{5.28}$$

respectively.

Changing variables of integration, equation (5.27) becomes

$$d^\varepsilon_L(x,t) = \int_{H_\gamma(x)} \lambda(\hat{x}-x)\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^3}\, r\left(\frac{\hat{x}}{\varepsilon},t\right)\, d\hat{x}. \tag{5.29}$$

Since $r(y,t)$ is Y-periodic in y and from Proposition 4.10, we see that for fixed t, as $\varepsilon\to 0$

$$r\left(\frac{\hat{x}}{\varepsilon},t\right) \to \int_Y r(y,t)\, dt = 0 \quad \text{weakly in } L^s(\Omega)^3.$$

Thus from (5.29) we obtain that

$$\lim_{\varepsilon\to 0} d^\varepsilon_L(x,t) = 0,$$

for $x\in\Omega$ and $t\in(0,T)$. It follows from Lebesgue's convergence theorem that

$$\lim_{\varepsilon\to 0} \|d^\varepsilon_L(\cdot,t)\|_{L^s(\Omega)^3} = 0, \tag{5.30}$$

for $t\in(0,T)$. On the other hand, by taking the Euclidean norm of $d^\varepsilon_S(x,t)$ in (5.28), we obtain

$$|d^\varepsilon_S(x,t)| \le \alpha_{\max}\int_{H_\delta(0)} \frac{1}{|z|}\left|u^H(x+\varepsilon z,t)-u^H(x,t)\right|\, dz, \tag{5.31}$$

where $\alpha_{\max} = \max_{y,y'\in Y}\alpha(y,y')$. Since $u^H\in C^2([0,T];C(\bar{\Omega})^3)$ (see Section 5.1), it follows that for $x\in\Omega$ and $t\in(0,T)$

$$\lim_{\varepsilon\to 0} |d^\varepsilon_S(x,t)| = 0. \tag{5.32}$$

Thus using Lebesgue's convergence theorem, we obtain

$$\lim_{\varepsilon\to 0}\|d_S^\varepsilon(\cdot,t)\|_{L^s(\Omega)^3}=0, \tag{5.33}$$

for $t\in(0,T)$. Equation (5.20) follows from equations (5.30) and (5.33).

Now we prove the error estimate (2.8). By setting $\lambda=0$ in equation (2.3), we see that its solution u^H is given explicitly by

$$u^H(x,t)=u_0(x)+t\,v_0(x)+\int_0^t(t-\tau)l(x,\tau)\,d\tau. \tag{5.34}$$

By assumption u_0, v_0, and $l(\cdot,t)$ are in $C^{0,\beta}(\bar\Omega)$. Thus for $z\in H_\delta(0)$, we see from (5.34) that

$$\begin{aligned}|u^H(x+\varepsilon z,t)-u^H(x,t)| &\le C|\varepsilon z|^\beta+t\,C|\varepsilon z|^\beta+\int_0^t(t-\tau)C|\varepsilon z|^\beta\,d\tau\\ &= C\left(1+t+\frac{t^2}{2}\right)|z|^\beta\varepsilon^\beta,\end{aligned} \tag{5.35}$$

for some $C>0$. We use this bound in inequality (5.31) to obtain

$$|d_S^\varepsilon(x,t)|\le C\left(1+t+\frac{t^2}{2}\right)\alpha_{\max}\int_{H_\delta(0)}|z|^{\beta-1}\,dz\,\varepsilon^\beta. \tag{5.36}$$

Since $\lambda=0$ we see from (5.16)-(5.18) that $d^\varepsilon=d_S^\varepsilon$. Therefore from (5.36), after a simple calculation, we obtain

$$\|d^\varepsilon(\cdot,t)\|_{L^s(\Omega)^3}\le 4\pi C\alpha_{\max}|\Omega|^{1/s}\frac{\delta^{\beta+2}}{\beta+2}\left(1+t+\frac{t^2}{2}\right)\varepsilon^\beta. \tag{5.37}$$

By using (5.37) to bound $\|d^\varepsilon(\cdot,\tau)\|_{L^s(\Omega)^3}$ in (5.19), the error estimate (2.8) follows.

5.2.2 Second Case

In this section we complete the proof of Theorem 2.4 by showing that equation (5.20) holds true when b, u^0, and v^0 are given by (2.9). We also prove the error estimate (2.15).

Using the fact that $r(y,t)$, the solution of the cell problem (2.12)-(2.13), has zero average over Y, and by linearity, it is easy to check that $u^H(x,t)+\sum_{j=1}^3 r^j(y,t)h_j(x)$ solves (4.11)-(4.13), where u^H is the solution of (2.10)-(2.11). Thus by uniqueness we conclude that

$$u(x,y,t)=u^H(x,t)+\sum_{j=1}^3 r^j(y,t)h_j(x). \tag{5.38}$$

Using this representation of $u(x,y,t)$ and from equations (5.17) and (5.18), we see that $d_L^\varepsilon(x,t)$ is now given by

$$d_L^\varepsilon(x,t)=\int_{H_\gamma(0)}\lambda(\xi)\frac{\xi\otimes\xi}{|\xi|^3}\,\sum_{j=1}^3 r^j\left(\frac{x+\xi}{\varepsilon},t\right)h_j(x+\xi)\,d\xi, \tag{5.39}$$

and $d_S^\varepsilon(x,t)$ can be written as

$$d_S^\varepsilon(x,t) = d_{S,1}^\varepsilon(x,t) + d_{S,2}^\varepsilon(x,t), \tag{5.40}$$

where,

$$d_{S,1}^\varepsilon(x,t) = \int_{H_\delta(0)} \alpha\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon} + z\right) \frac{z \otimes z}{|z|^3} \left(u^H(x+\varepsilon z, t) - u^H(x,t)\right) dz, \tag{5.41}$$

$$d_{S,2}^\varepsilon(x,t) = \int_{H_\delta(0)} \alpha\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon} + z\right) \frac{z \otimes z}{|z|^3} \sum_{j=1}^{3} r^j\left(\frac{x}{\varepsilon} + z, t\right) \left(h_j(x+\varepsilon z) - h_j(x)\right) dz. \tag{5.42}$$

Applying the methods developed in Section 5.2.1 for (5.30) and (5.33), we can show that for $t \in (0,T)$,

$$\lim_{\varepsilon \to 0} \|d_L^\varepsilon(\cdot,t)\|_{L^s(\Omega)^3} = 0, \tag{5.43}$$

and

$$\lim_{\varepsilon \to 0} \|d_{S,1}^\varepsilon(\cdot,t)\|_{L^s(\Omega)^3} = 0. \tag{5.44}$$

It remains to show that for $t \in (0,T)$,

$$\lim_{\varepsilon \to 0} \|d_{S,2}^\varepsilon(\cdot,t)\|_{L^s(\Omega)^3} = 0. \tag{5.45}$$

From equation (5.42), we see that

$$|d_{S,2}^\varepsilon(x,t)| \le \alpha_{\max} \int_{H_\delta(0)} \frac{1}{|z|} \sum_{j=1}^{3} \left| r^j\left(\frac{x}{\varepsilon} + z, t\right)\right| |h_j(x+\varepsilon z) - h_j(x)| \, dz, \tag{5.46}$$

where $\alpha_{\max} = \max_{y,y' \in Y} \alpha(y,y')$. Since $\frac{3}{2} < s < \infty$, we can choose s', with $\frac{3}{2} < s' < \infty$, and s'', with $1 \le s'' < 3$, such that $1/s + 1/s' + 1/s'' = 1$. By Hölder's inequality we obtain

$$\begin{aligned} |d_{S,2}^\varepsilon(x,t)| \le\ & \alpha_{\max} \left(\int_{H_\delta(0)} \frac{1}{|z|^{s''}} \, dz\right)^{1/s''} \sum_{j=1}^{3} \left(\int_{H_\delta(0)} \left| r^j\left(\frac{x}{\varepsilon} + z, t\right)\right|^{s'} dz\right)^{1/s'} \\ & \times \left(\int_{H_\delta(0)} |h_j(x+\varepsilon z) - h_j(x)|^s \, dz\right)^{1/s}. \end{aligned} \tag{5.47}$$

It is easy to see that

$$\left(\int_{H_\delta(0)} \left| r^j\left(\frac{x}{\varepsilon} + z, t\right)\right|^{s'} dz\right)^{1/s'} \le \|r^j(\cdot,t)\|_{L^{s'}(\Omega)^3}. \tag{5.48}$$

Thus from (5.47) and (5.48), and by using the triangle inequality in L^s, we obtain

$$\begin{aligned} \|d_{S,2}^\varepsilon(\cdot,t)\| \le\ & \alpha_{\max} \left(\int_{H_\delta(0)} \frac{1}{|z|^{s''}} \, dz\right)^{1/s''} \sum_{j=1}^{3} \|r^j(\cdot,t)\|_{L^{s'}(\Omega)^3} \\ & \times \left(\int_\Omega \int_{H_\delta(0)} |h_j(x+\varepsilon z) - h_j(x)|^s \, dz \, dx\right)^{1/s}. \end{aligned} \tag{5.49}$$

Since h_j is continuous on $\bar{\Omega}$, we obtain from Lebesgue's convergence theorem that

$$\lim_{\varepsilon\to 0}\int_\Omega\int_{H_\delta(0)}|h_j(x+\varepsilon z)-h_j(x)|^s\,dz\,dx=0. \tag{5.50}$$

Equation (5.45) follows from (5.49) and (5.50). This shows that (5.20) holds true for this case.

Now we prove the error estimate (2.15). By setting $\lambda=0$ in equation (2.10), we see that its solution u^H is given explicitly by

$$u^H(x,t)=\int_0^t(t-\tau)\bar{F}(\tau)\,d\tau\,h(x). \tag{5.51}$$

By assumption h is in $C^{0,\beta}(\bar{\Omega})$. Thus for $z\in H_\delta(0)$, we see from (5.51) that

$$|u^H(x+\varepsilon z,t)-u^H(x,t)| \;\le\; C|\varepsilon z|^\beta\int_0^t(t-\tau)\bar{F}(\tau)\,d\tau, \tag{5.52}$$

for some $C>0$. Taking the Euclidean norm in both sides of (5.41) and using the bound (5.52), we see that

$$|d^\varepsilon_{S,1}(x,t)|\le C\alpha_{\max}\int_0^t(t-\tau)\bar{F}(\tau)\,d\tau\int_{H_\delta(0)}|z|^{\beta-1}\,dz\,\varepsilon^\beta, \tag{5.53}$$

and it follows that

$$\|d^\varepsilon_{S,1}(\cdot,t)\|_{L^s(\Omega)^3}\le 4\pi C\alpha_{\max}|\Omega|^{1/s}\frac{\delta^{\beta+2}}{\beta+2}\left(\int_0^t(t-\tau)\bar{F}(\tau)\,d\tau\right)\varepsilon^\beta. \tag{5.54}$$

On the other hand from (5.49), after a straight forward calculation, we obtain

$$\|d^\varepsilon_{S,2}(\cdot,t)\|_{L^s(\Omega)^3}\le C\alpha_{\max}\left(4\pi\frac{\delta^{3-s''}}{3-s''}\right)^{1/s''}\left(4\pi|\Omega|\,\frac{\delta^{s\beta+3}}{s\beta+3}\right)^{1/s}\sum_{j=1}^3\|r^j(\cdot,t)\|_{L^{s'}(\Omega)^3}\,\varepsilon^\beta. \tag{5.55}$$

Since $\lambda=0$ we see that $d^\varepsilon=d^\varepsilon_{S,1}+d^\varepsilon_{S,2}$. Therefore by combining (5.54) and (5.55) to bound $\|d^\varepsilon(\cdot,\tau)\|_{L^s(\Omega)^3}$ in (5.19), the error estimate (2.15) follows.

5.2.3 Third and Fourth Cases

Arguments similar to those presented in Section 5.2.2 show that equation (5.20) holds true when the loading and initial conditions are given by (2.16) or (2.23). Also, the proofs of the error estimates (2.22) and (2.29) are similar to the proof of (2.15) provided in Section 5.2.2. For completeness, we explicitly provide the functions $M_3(t)$ and $M_4(t)$ of Theorems 2.6 and 2.6, respectively. The function $M_3(t)$ is given by

$$M_3(t)=\int_0^t\frac{1}{\sqrt{M}}\sinh\left(\sqrt{M}(t-\tau)\right)\,f_3(\tau)\,d\tau$$

where

$$f_3(t) = C\alpha_{\max}|\Omega|^{1/s}\left(4\pi|\bar{F}|\frac{\delta^{\beta+2}}{\beta+2} + \left(4\pi\frac{\delta^{3-s''}}{3-s''}\right)^{1/s''}\left(4\pi\frac{\delta^{s\beta+3}}{s\beta+3}\right)^{1/s}\sum_{j=1}^{3}\|r^j(\cdot,t)\|_{L^{s'}(\Omega)^3}\right)$$

and r^j solves (2.19)-(2.20).

The function $M_4(t)$ is given by

$$M_4(t) = \int_0^t \frac{1}{\sqrt{M}}\sinh\left(\sqrt{M}(t-\tau)\right) f_4(\tau)\,d\tau$$

where

$$f_4(t) = C\alpha_{\max}|\Omega|^{1/s}\left(4\pi|\bar{F}|\frac{\delta^{\beta+2}}{\beta+2}\,t + \left(4\pi\frac{\delta^{3-s''}}{3-s''}\right)^{1/s''}\left(4\pi\frac{\delta^{s\beta+3}}{s\beta+3}\right)^{1/s}\sum_{j=1}^{3}\|r^j(\cdot,t)\|_{L^{s'}(\Omega)^3}\right)$$

and r^j solves (2.26)-(2.27).

This completes the proofs of Theorems 2.6 and 2.8.

6 Fluctuating Long-Range Bond Model

In this section, we present a new multiscale analysis method for computing the deformation of fiber-reinforced composites modeled by the peridynamic formulation. This is done for the Fluctuating Long-Range Bond model described in Section 1.2. The method provides a computationally inexpensive multiscale numerical method. This is described by Theorem 6.1. A homogenization result for this model is expressed in Theorem 6.2.

We begin by recalling the peridynamic equation of motion for this model. By expanding α_L^ε in equation (1.15), then collecting the $\chi_{\mathrm{f}}^\varepsilon$ terms, we obtain

$$\begin{aligned}\partial_t^2 u^\varepsilon(x,t) &= \chi_{\mathrm{f}}^\varepsilon(x)\int_{I_\delta^n(x)}(C_{\mathrm{f}}-\varepsilon C_{\mathrm{m}})\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2}(u^\varepsilon(\hat{x},t)-u^\varepsilon(x,t))\,dl_{\hat{x}}\\ &+ \int_{H_\delta(x)}\varepsilon C_{\mathrm{m}}\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2}(u^\varepsilon(\hat{x},t)-u^\varepsilon(x,t))\,d\hat{x}, \qquad (6.1)\end{aligned}$$

where the first integral in (6.1) is a line integral over the set

$$I_\delta^n(x) = \{\hat{x}\in H_\delta(x) \text{ such that } \hat{x}-x \text{ is parallel to } n\}.$$

The initial conditions supplementing this equation are given by

$$\begin{aligned} u^\varepsilon(x,0) &= u^0(x), \qquad (6.2)\\ \partial_t u^\varepsilon(x,0) &= v^0(x). \qquad (6.3)\end{aligned}$$

The well-posedness of equation (6.1)-(6.3) is provided in Section 6.1 (Proposition 6.4).

Theorem 6.1 (Downscaling). *Let $u^\varepsilon \in C^2([0,T];\, L^p(\Omega)^3)$ be the solution of (6.1)-(6.3), where $1 \le p < \infty$. Then for $t \in [0,T]$,*

$$\lim_{\varepsilon\to 0} \left\| u^\varepsilon(x,t) - \left(\chi_f^\varepsilon(x) w(x,t) + u^0(x) + t v^0(x)\right) \right\|_{L^p(\Omega)^3} = 0, \tag{6.4}$$

where $w \in C^2([0,T];\, L^p(\Omega)^3)$ is the solution of

$$\begin{aligned} \partial_t^2 w(x,t) &= \int_{I_\delta^n(x)} C_f \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2} (w(\hat{x},t) - w(x,t))\, dl_{\hat{x}} \\ &+ \int_{I_\delta^n(x)} C_f \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2} (u^0(\hat{x}) + t v^0(\hat{x}) - (u^0(x) + t v^0(x)))\, dl_{\hat{x}} \end{aligned} \tag{6.5}$$

supplemented with the initial conditions

$$w(x,0) = 0, \tag{6.6}$$
$$\partial w(x,0) = 0. \tag{6.7}$$

Moreover, for $t \in [0,T]$ the error in (6.4) is estimated by

$$\left\| u^\varepsilon(x,t) - \left(\chi_f^\varepsilon(x) w(x,t) + u^0(x) + t v^0(x)\right) \right\|_{L^p(\Omega)^3} \le \varepsilon\, M_5(t), \tag{6.8}$$

where

$$M_5(t) = \left(\|u^0\|_{L^p(\Omega)^3} \cosh \sqrt{M} t + \|v^0\|_{L^p(\Omega)^3} \frac{1}{\sqrt{M}} \sinh \sqrt{M} t \right),$$

and where M is a positive constant.

Theorem 6.1 is proved in Section 6.2.

The macroscopic peridynamic equation for this model is given by

$$\begin{aligned} \partial_t^2 u^H(x,t) &= \int_{I_\delta^n(x)} C_{\mathrm{f}} \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2} (u^H(\hat{x},t) - u^H(x,t))\, dl_{\hat{x}} \\ &+ (\theta_f - 1) \int_{I_\delta^n(x)} C_{\mathrm{f}} \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2} (u^0(\hat{x}) + t v^0(\hat{x}) - (u^0(x) + t v^0(x)))\, dl_{\hat{x}}, \end{aligned} \tag{6.9}$$

supplemented with initial conditions

$$u^H(x,0) = u^0(x), \tag{6.10}$$
$$\partial_t u^H(x,0) = v^0(x). \tag{6.11}$$

Here the macroscopic displacement u^H is the weak limit of the sequence of displacements u^ε. This is described by the following theorem.

Theorem 6.2 (Homogenization). *Let* $u^\varepsilon \in C^2([0,T];\, L^p(\Omega)^3)$ *be the solution of (6.1)-(6.3), where* $1 \le p < \infty$. *Then for* $t \in [0,T]$, *as* $\varepsilon \to 0$,

$$u^\varepsilon(\cdot,t) \to u^H(\cdot,t) \quad \textit{weakly in } L^p(\Omega)^3,$$

where $u^H \in C^2([0,T];\, L^p(\Omega)^3)$ *is the solution of (6.9)-(6.11). Equivalently,* u^H *can be computed as follows*

$$u^H(x,t) = \theta_f w(x,t) + u^0(x) + t v^0(x), \tag{6.12}$$

where w *solves (6.5)-(6.7).*

Theorem 6.2 is proved in Section 6.2.

Remark 6. We observe that the macroscopic peridynamic equation (6.9) has a nonzero loading force, although the original peridynamic equation (6.1) has no loading force. The physical interpretation for this phenomenon is not well-understood up to this point.

6.1 Existence and Uniqueness Results

Without loss of generality, we may choose the fiber direction to be parallel to the x_1-axis. So let $n = (1,0,0)$. We note that the matrix multiplying $(u^\varepsilon(\hat{x},t) - u^\varepsilon(x,t))$ in the first integral of (6.1) is now given by

$$\frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $\hat{x}_1 \neq x_1$. Thus equation (6.1), after shifting the domain of integration in the first integral, becomes

$$\begin{aligned} \partial_t^2 u^\varepsilon(x,t) &= (C_{\mathrm{f}} - \varepsilon C_{\mathrm{m}})\, \chi_{\mathrm{f}}^\varepsilon(x) \int_{-\delta}^{\delta} (u_1^\varepsilon(x+(l,0,0),t) - u_1^\varepsilon(x,t))\, dl \\ &+ \int_{H_\delta(x)} \varepsilon C_{\mathrm{m}} \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2} (u^\varepsilon(\hat{x},t) - u^\varepsilon(x,t))\, d\hat{x}. \end{aligned} \tag{6.13}$$

Let $v = (v_1, v_2, v_3) \in L^p(\Omega)^3$ with $1 \le p < \infty$. Then we define the following operators

$$A_{\mathrm{f}} v(x) = C_{\mathrm{f}} \int_{-\delta}^{\delta} (v_1(x+(l,0,0)) - v_1(x))\, dl, \tag{6.14}$$

$$A_{\mathrm{f}}^\varepsilon v(x) = \chi_{\mathrm{f}}^\varepsilon(x)\, A_{\mathrm{f}} v(x), \tag{6.15}$$

$$A_{\mathrm{m}} v(x) = \int_{H_\delta(x)} C_{\mathrm{m}} \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2} (v(\hat{x}) - v(x))\, d\hat{x}, \tag{6.16}$$

$$A^\varepsilon = A_{\mathrm{f}}^\varepsilon + \varepsilon \left(A_{\mathrm{m}} - \frac{C_{\mathrm{m}}}{C_{\mathrm{f}}} A_{\mathrm{f}}^\varepsilon \right). \tag{6.17}$$

The initial value problem (6.1)-(6.3) can be written as the following operator equation in $L^p(\Omega)^3$

$$\begin{cases} \ddot{u}^\varepsilon(t) = A^\varepsilon u^\varepsilon(t), \; t \in [0,T] \\ u^\varepsilon(0) = u^0, \\ \dot{u}^\varepsilon(0) = v^0. \end{cases} \tag{6.18}$$

Existence and uniqueness of solution of (6.18) is given by the following proposition.

Proposition 6.4. *Let* $1 \leq p < \infty$*. Then*

(a) The operator A^ε *is linear and uniformly bounded on* $L^p(\Omega)^3$*.*

(b) Equation (6.18) has a unique classical solution $u^\varepsilon \in C^2([0,T];\, L^p(\Omega)^3)$ *which is given by*

$$u^\varepsilon(t) \;=\; \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\,(A^\varepsilon)^n u^0 + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,(A^\varepsilon)^n v^0. \tag{6.19}$$

Proof. Part (a). First, we show that the linear operator A_{m} is bounded on $L^p(\Omega)^3$. Let $v \in L^p(\Omega)^3$. Then from (6.16), A_{m} can be written as

$$A_{\mathrm{m}} = C_{\mathrm{m}}(A_{\mathrm{m},1} - A_{\mathrm{m},2}),$$

where

$$A_{\mathrm{m},1}v(x) \;=\; \int_{H_\delta(x)} \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2} v(\hat{x})\,d\hat{x}, \tag{6.20}$$

$$A_{\mathrm{m},2}v(x) \;=\; \int_{H_\delta(x)} \frac{(\hat{x}-x)\otimes(\hat{x}-x)}{|\hat{x}-x|^2}\,d\hat{x}\,v(x). \tag{6.21}$$

From equation (6.20) we see that

$$\begin{aligned} \|A_{\mathrm{m},1}v\|^p_{L^p(\Omega)^3} &\leq \int_\Omega \left(\int_{H_\gamma(x)} |v(\hat{x})|\,d\hat{x} \right)^p dx \\ &\leq |\Omega|\,\|v\|^p_{L^p(\Omega)^3}, \end{aligned} \tag{6.22}$$

where the fact that $\|v\|_{L^1(\Omega)^3} \leq \|v\|_{L^p(\Omega)^3}$ was used in the last step. This shows that $A_{\mathrm{m},1}$ is bounded on $L^p(\Omega)^3$. The boundedness of $A_{\mathrm{m},2}$ is clear. Therefore A_{m} is bounded on $L^p(\Omega)^3$.

Next we note that A_{f} is bounded on $L^p(\Omega)^3$, which is a consequence of Lemma 6.5 given at the end of this section. Thus it follows from (6.15) that $A^\varepsilon_{\mathrm{f}}$ is uniformly bounded on $L^p(\Omega)^3$.

Combining these results with equation (6.17), it follows that A^ε is uniformly bounded on $L^p(\Omega)^3$, completing the proof of Part (a).

The proof of Part (b) is similar to the proof of Part (b) of Proposition 3.1.

□

Lemma 6.5. *Let* v *be in* $L^p(\Omega)^3$*, where* $1 \leq p < \infty$*, and define*

$$\check{v}(x) = \int_{-\delta}^{\delta} v(x + (l,0,0))\,dl.$$

Then $\check{v}$ *is in* $L^p(\Omega)^3$ *and*

$$\|\check{v}\|_{L^p(\Omega)^3} \;\leq\; 2\gamma\,\|v\|_{L^p(\Omega)^3}. \tag{6.23}$$

Proof. From the definition of $\check{v}$ it is easy to see that

$$\int_\Omega |\check{v}(x)|^p\,dx \;\leq\; \int_\Omega \left(\int_{-\delta}^{\delta} |v(x_1+l,x_2,x_3)|\,dl \right)^p dx_1dx_2dx_3. \tag{6.24}$$

Using Hölder's inequality in the inner integral with $v \in L^p(\Omega)^3$ and $1 \in L^{p'}(\Omega)^3$, where $1/p + 1/p' = 1$, we obtain

$$\begin{aligned} \int_\Omega |\check{v}(x)|^p\,dx \;&\leq\; (2\delta)^{p/p'} \int_\Omega \int_{-\delta}^{\delta} |v(x_1+l,x_2,x_3)|^p\,dl\,dx_1dx_2dx_3 \\ &=\; (2\delta)^{p/p'} \int_{-\delta}^{\delta} \int_\Omega |v(x_1+l,x_2,x_3)|^p\,dx_1dx_2dx_3dl, \end{aligned} \tag{6.25}$$

by Fubini's theorem. We extend v to $\mathbb{R}^3$ by setting $v = 0$ outside Ω. Then by the change of variables $\hat{x}_1 = x_1 + l$ in the inner integral of (6.25), we obtain

$$\int_\Omega |v(x_1+l,x_2,x_3)|^p\,dx_1 \;\leq\; \int_\Omega |v(x_1,x_2,x_3)|^p\,dx_1.$$

Using this estimate in (6.25), we conclude that

$$\int_\Omega |\check{v}(x)|^p\,dx \;\leq\; (2\delta)^{p/p'}(2\delta) \int_\Omega |v(x)|^p\,dx, \tag{6.26}$$

and (6.23) follows, completing the proof. □

6.2 Multiscale Analysis Using the Semigroups Approach

The aim of this section is to prove Theorems 6.1 and 6.2. Our approach is summarized by the following steps:

1. Compute the two-scale limit $u(x,y,t)$ of the sequence (u^ε) using the explicit representation of u^ε, equation (6.19). We show that for fixed $t \in [0,T]$, as $\varepsilon \to 0$,

$$u^\varepsilon(x,t) \xrightarrow{2} u(x,y,t), \tag{6.27}$$

   where u is given by

$$\begin{aligned} u(x,y,t) \;&=\; u^0(x) + tv^0(x) + \chi_{\mathrm{f}}(y) \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\,(A_{\mathrm{f}})^n u^0(x) \\ &\quad + \chi_{\mathrm{f}}(y) \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,(A_{\mathrm{f}})^n v^0(x). \end{aligned} \tag{6.28}$$

2. Compute $\partial_t^2 u$ in (6.28) then use it to identify the two-scale limit equation. We find that $u \in C^2([0,T]; L^p(\Omega)^3)$ uniquely solves

$$\left\{ \begin{aligned} \partial_t^2 u(x,y,t) \;&=\; \tilde{A}_{\mathrm{f}} u(x,y,t) + b(x,y,t), \\ u(x,y,0) \;&=\; u^0(x), \\ \partial u(x,y,0) \;&=\; v^0(x), \end{aligned} \right. \tag{6.29}$$

where b is given by

$$b(x,y,t) = (\chi_{\rm f}(y) - 1)A_{\rm f}(u^0 + tv^0)(x).$$

Here the operator $\tilde{A}_{\rm f}$ is defined as follows. For $\tilde{v} \in L^p(\Omega \times Y)^3$,

$$\tilde{A}_{\rm f}\tilde{v}(x,y) = C_{\rm f} \int_{-\delta}^{\delta} (\tilde{v}_1(x + (l,0,0), y) - \tilde{v}_1(x,y))\, dl. \tag{6.30}$$

3. The macroscopic equation is found by integrating (6.29) over Y. We find that the macroscopic displacement u^H solves

$$\begin{cases} \partial_t^2 u^H(x,t) &=& A_{\rm f} u^H(x,t) + \bar{b}(x,t), \\ u^H(x,0) &=& u^0(x), \\ \partial u^H(x,0) &=& v^0(x), \end{cases} \tag{6.31}$$

where $\bar{b}$ is given by

$$\bar{b}(x,t) = (\theta_f - 1)A_{\rm f}(u^0 + tv^0)(x).$$

Here for fixed $t \in [0,T]$, as $\varepsilon \to 0$,

$$u^\varepsilon(\cdot,t) \to u^H(\cdot,t) \;\text{ weakly in }\; L^p(\Omega)^3. \tag{6.32}$$

4. The two-scale limit u can also be computed by the following method. This method is numerically inexpensive.

$$u(x,y,t) = \chi_{\rm f}(y)w(x,t) + u^0(x) + tv^0(x), \tag{6.33}$$

where $w \in C^2([0,T];\, L^p(\Omega)^3)$ solves

$$\begin{cases} \partial_t^2 w(x,t) &=& A_{\rm f} w(x,t) + A_{\rm f}(u^0 + tv^0)(x), \\ w(x,0) &=& 0, \\ \partial w(x,0) &=& 0. \end{cases} \tag{6.34}$$

It follows from integrating (6.33) over Y that u^H can also be computed by

$$u^H(x,t) = \theta_f w(x,t) + u^0(x) + tv^0(x). \tag{6.35}$$

5. Extend u by periodicity from $\Omega \times Y \times (0,T)$ to $\Omega \times \mathbb{R}^3 \times (0,T)$. Then we use the explicit representations of u^ε and u, equations (6.19) and (6.28), respectively, to show that for fixed $t \in [0,T]$,

$$\lim_{\varepsilon \to 0} \left\| u^\varepsilon(x,t) - u\left(x, \frac{x}{\varepsilon}, t\right) \right\|_{L^p(\Omega)^3} = 0. \tag{6.36}$$

Now we justify Steps (1)-(5).

Proof of Step (1). Let $v \in L^p(\Omega)^3$, where $1 \le p < \infty$. Then we first show that

$$(A_{\rm f}^\varepsilon)^n v(x) = \chi_{\rm f}^\varepsilon(x)(A_{\rm f})^n v(x) \;\text{ for all } n \in \mathbb{N}. \tag{6.37}$$

The proof is by induction on n. The formula (6.37) holds for $n=1$ by the definition of $A_{\mathrm{f}}^{\varepsilon}$. Assume that it holds for $n=k$. Then for $n=k+1$,

$$\begin{aligned}(A_{\mathrm{f}}^{\varepsilon})^{k+1}v(x) &= \chi_{\mathrm{f}}^{\varepsilon}(x)C_{\mathrm{f}}\int_{-\delta}^{\delta}\left((A_{\mathrm{f}}^{\varepsilon})^{k}v_1(x+(l,0,0))-(A_{\mathrm{f}}^{\varepsilon})^{k}v_1(x)\right)\,dl\\ &= \chi_{\mathrm{f}}^{\varepsilon}(x)C_{\mathrm{f}}\int_{-\delta}^{\delta}\big(\chi_{\mathrm{f}}^{\varepsilon}(x+(l,0,0))(A_{\mathrm{f}})^{k}v_1(x+(l,0,0))\\ &\qquad -\chi_{\mathrm{f}}^{\varepsilon}(x)(A_{\mathrm{f}})^{k}v_1(x)\big)\,dl. \end{aligned} \tag{6.38}$$

Note that since x lies in a fiber if and only if $x+(l,0,0)$ lies in the same fiber, then $\chi_{\mathrm{f}}^{\varepsilon}(x+(l,0,0))=\chi_{\mathrm{f}}^{\varepsilon}(x)$. On the other hand $(\chi_{\mathrm{f}}^{\varepsilon})^2=\chi_{\mathrm{f}}^{\varepsilon}$, thus (6.38) becomes

$$\begin{aligned}(A_{\mathrm{f}}^{\varepsilon})^{k+1}v(x) &= \chi_{\mathrm{f}}^{\varepsilon}(x)\left(C_{\mathrm{f}}\int_{-\delta}^{\delta}\left((A_{\mathrm{f}})^{k}v_1(x+(l,0,0))-(A_{\mathrm{f}})^{k}v_1(x)\right)\,dl\right)\\ &= \chi_{\mathrm{f}}^{\varepsilon}(x)(A_{\mathrm{f}})^{k+1}v(x).\end{aligned}$$

Therefore (6.37) follows. Since $(A_{\mathrm{f}})^n v\in L^p(\Omega)$, it follows from Propositions 4.3 and 4.4 of Section 4.1 that

$$\chi_{\mathrm{f}}^{\varepsilon}(x)(A_{\mathrm{f}})^n v(x)\xrightarrow{2}\chi_{\mathrm{f}}(y)(A_{\mathrm{f}})^n v(x). \tag{6.39}$$

Next we show that

$$(A^{\varepsilon})^n v(x)\xrightarrow{2}\chi_{\mathrm{f}}(y)(A_{\mathrm{f}})^n v(x). \tag{6.40}$$

To see this, we note that from (6.17), the operator $(A^{\varepsilon})^n$, $n\in\mathbb{N}$, can be written in the following form

$$(A^{\varepsilon})^n=(A_{\mathrm{f}}^{\varepsilon})^n+\varepsilon D_n^{\varepsilon}, \tag{6.41}$$

where the operator D_n^{ε} is bounded on $L^p(\Omega)^3$ and satisfies

$$\|D_n^{\varepsilon}\|<M^n \tag{6.42}$$

for some $M>0$ independent of ε. It follows that for fixed $n\in\mathbb{N}$,

$$\lim_{\varepsilon\to 0}\varepsilon D_n^{\varepsilon}v=0,\quad \text{in } L^p(\Omega)^3, \tag{6.43}$$

and thus by Proposition 4.2, the sequence $(\varepsilon D_n^{\varepsilon}v)_{\varepsilon>0}$ two-scale converges to 0. Therefore the result follows by combining (6.41), (6.39), and (6.37).

Now we recall from (6.19) that $u^{\varepsilon}(x,t)$ is given by

$$u^{\varepsilon}(x,t) = u^0(x)+tv^0(x)+\sum_{n=1}^{\infty}\frac{t^{2n}}{(2n)!}\,(A^{\varepsilon})^n u^0(x)+\sum_{n=1}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\,(A^{\varepsilon})^n v^0(x). \tag{6.44}$$

Using (6.40), we will show in Section 6.2.1 that for $\psi \in \mathcal{K}$,

$$\lim_{\varepsilon\to 0}\int_\Omega \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\,(A^\varepsilon)^n u^0(x)\cdot\psi\left(x,\frac{x}{\varepsilon}\right)\,dx$$

$$= \int_\Omega\int_Y \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\,\chi_{\mathrm{f}}(y)(A_{\mathrm{f}})^n u^0(x)\cdot\psi\left(x,y\right)\,dydx, \tag{6.45}$$

$$\lim_{\varepsilon\to 0}\int_\Omega \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,(A^\varepsilon)^n v^0(x)\cdot\psi\left(x,\frac{x}{\varepsilon}\right)\,dx$$

$$= \int_\Omega\int_Y \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,\chi_{\mathrm{f}}(y)(A_{\mathrm{f}})^n v^0(x)\cdot\psi\left(x,y\right)\,dydx. \tag{6.46}$$

It follows from (6.45) and (6.46) that for fixed $t \in [0,T]$, as $\varepsilon \to 0$,
$u^\varepsilon(x,t) \stackrel{2}{\rightharpoonup} u(x,y,t)$, where u is given by (6.28).

Proof of Step (2). We can see from (6.28) that $u \in C^2([0,T];\, L^p(\Omega\times Y)^3)$. Then by taking the second time derivative of both sides (6.28), we obtain

$$\begin{aligned}
\partial_t^2 u(x,y,t) &= \chi_{\mathrm{f}}(y)\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\,(A_{\mathrm{f}})^{n+1}u^0(x) + \chi_{\mathrm{f}}(y)\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\,(A_{\mathrm{f}})^{n+1}v^0(x)\\
&= \chi_{\mathrm{f}}(y)A_{\mathrm{f}}(u^0+tv^0)(x)\\
&\quad +\chi_{\mathrm{f}}(y)A_{\mathrm{f}}\sum_{n=1}^{\infty}\frac{t^{2n}}{(2n)!}\,(A_{\mathrm{f}})^{n}u^0(x) + \chi_{\mathrm{f}}(y)A_{\mathrm{f}}\sum_{n=1}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\,(A_{\mathrm{f}})^{n}v^0(x)
\end{aligned} \tag{6.47}$$

From (6.28) and the definition of $\tilde{A}_{\mathrm{f}}$, given by (6.30), we see that

$$\begin{aligned}
\tilde{A}_{\mathrm{f}}u(x,y,t) - A_{\mathrm{f}}(u^0+tv^0)(x) &= \chi_{\mathrm{f}}(y)A_{\mathrm{f}}\sum_{n=1}^{\infty}\frac{t^{2n}}{(2n)!}\,(A_{\mathrm{f}})^{n}u^0(x)\\
&\quad + \chi_{\mathrm{f}}(y)A_{\mathrm{f}}\sum_{n=1}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\,(A_{\mathrm{f}})^{n}v^0(x)
\end{aligned} \tag{6.48}$$

Thus from (6.47) and (6.48) we obtain that

$$\partial_t^2 u(x,y,t) = \tilde{A}_{\mathrm{f}}u(x,y,t) + (\chi_{\mathrm{f}}(y)-1)A_{\mathrm{f}}(u^0+tv^0)(x), \tag{6.49}$$

and hence (6.29) follows. The linear operator $\tilde{A}_{\mathrm{f}}$ is bounded on $L^p(\Omega\times Y)^3$. Thus u is the unique solution of (6.29).

Proof of Step (3). From (6.27) and Proposition 4.9, we obtain that for fixed
$t \in [0,T]$, as $\varepsilon \to 0$,

$$u^\varepsilon(\cdot,t) \to \int_Y u(\cdot,y,t)\,dy \ \text{ weakly in } \ L^p(\Omega)^3.$$

By definition $u^H(x,t) = \int_Y u(x,y,t)\,dy$, thus (6.32) follows. It is clear that (6.31) follows from integrating (6.29) over Y.

Proof of Step (4). Define

$$w(x,t) = \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\,(A_{\mathrm{f}})^n u^0(x) + \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,(A_{\mathrm{f}})^n v^0(x). \tag{6.50}$$

Combining this equation with (6.28) gives (6.33). On the other hand, equation (6.50) implies that $w \in C^2([0,T];\, L^p(\Omega)^3)$. Thus by taking the second time derivative of both sides of (6.50) gives

$$\begin{aligned}
\partial_t^2 w(x,t) &= \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\,(A_{\mathrm{f}})^{n+1} u^0(x) + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,(A_{\mathrm{f}})^{n+1} v^0(x) \\
&= A_{\mathrm{f}}(u^0 + tv^0)(x) + A_{\mathrm{f}} \left(\sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\,(A_{\mathrm{f}})^n u^0 + \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,(A_{\mathrm{f}})^n v^0 \right)(x) \\
&= A_{\mathrm{f}}(u^0 + tv^0)(x) + A_{\mathrm{f}} w(x,t).
\end{aligned} \tag{6.51}$$

Note that from (6.50) it is easy to see that $w(x,0) = 0$ and $\partial_t w(x,0) = 0$. Combining this fact with (6.51), equation (6.34) follows. The fact that A_{f} is linear and bounded on $L^p(\Omega)^3$ implies that w is the unique solution of (6.34).

Proof of Step (5). Extend χ_{f} from Y to $\mathbb{R}^3$ by periodicity. Then by making the substitution $y = \frac{x}{\varepsilon}$ in (6.28), we obtain

$$\begin{aligned}
u\left(x, \frac{x}{\varepsilon}, t\right) &= u^0(x) + tv^0(x) + \chi_{\mathrm{f}}^{\varepsilon}(x) \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\,(A_{\mathrm{f}})^n u^0(x) \\
&\quad + \chi_{\mathrm{f}}^{\varepsilon}(x) \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,(A_{\mathrm{f}})^n v^0(x) \\
&= u^0(x) + tv^0(x) + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\,(A_{\mathrm{f}}^{\varepsilon})^n u^0(x) \\
&\quad + \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\,(A_{\mathrm{f}}^{\varepsilon})^n v^0(x),
\end{aligned} \tag{6.52}$$

where in the last equality we have used equation (6.37).

Now we compute the difference $u^\varepsilon(x,t) - u(x, \frac{x}{\varepsilon}, t)$ using equations (6.19) and (6.52). We see that

$$\begin{aligned} u^\varepsilon(x,t) - u\left(x, \frac{x}{\varepsilon}, t\right) &= \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \; ((A^\varepsilon)^n - (A^\varepsilon_{\mathrm{f}})^n)\, u^0(x) \\ &\quad + \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \; ((A^\varepsilon)^n - (A^\varepsilon_{\mathrm{f}})^n)\, v^0(x) \\ &= \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \; (\varepsilon D^\varepsilon_n)\, u^0(x) + \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \; (\varepsilon D^\varepsilon_n)\, v^0(x), \end{aligned} \tag{6.53}$$

where in the last equality we have used equation (6.41). By taking the L^p norm in (6.53) and by using (6.42), we see that

$$\begin{aligned} \left\| u^\varepsilon(x,t) - u\left(x, \frac{x}{\varepsilon}, t\right) \right\|_{L^p(\Omega)^3} &\leq \varepsilon \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \; M^n \|u^0\|_{L^p(\Omega)^3} \\ &\quad + \; \varepsilon \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \; M^n \|v^0\|_{L^p(\Omega)^3} \\ &= \varepsilon \left(\|u^0\|_{L^p(\Omega)^3} \cosh \sqrt{M} t + \|v^0\|_{L^p(\Omega)^3} \frac{1}{\sqrt{M}} \sinh \sqrt{M} t \right) \end{aligned}$$

thus (6.36) follows, completing the proof.

6.2.1 Proof of (6.45) and (6.46)

In this section we prove (6.45). Equation (6.46) can be derived similarly.

We begin by the following observation

$$\sum_{n=1}^{\infty} \int_\Omega \left| \frac{t^{2n}}{(2n)!} \; (A^\varepsilon)^n u^0(x) \cdot \psi\left(x, \frac{x}{\varepsilon}\right) \right| \, dx < \infty. \tag{6.54}$$

To see this, we use Cauchy-Schwarz inequality to obtain

$$\int_\Omega \left| (A^\varepsilon)^n u^0(x) \cdot \psi\left(x, \frac{x}{\varepsilon}\right) \right| \; dx \leq \left\| (A^\varepsilon)^n u^0 \right\|_{L^2(\Omega)^3} \; \left\| \psi\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)^3}. \tag{6.55}$$

From Part (a) of Proposition 6.4, the operator A^ε is uniformly bounded on $L^2(\Omega)^3$. Also, it is easy to see that

$$\left\| \psi\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)^3} \leq \|\psi\|_{L^2(\Omega; C_{per}(Y)^3)} := \left(\int_\Omega \sup_{y \in Y} |\psi(x,y)|^2 \, dx \right)^{1/2}.$$

We use these two facts in (6.55) to obtain

$$\int_\Omega \left| (A^\varepsilon)^n u^0(x) \cdot \psi\left(x, \frac{x}{\varepsilon}\right) \right| \; dx \leq M^n \; \left\| u^0 \right\|_{L^2(\Omega)^3} \; \|\psi\|_{L^2(\Omega; C_{per}(Y)^3)}, \tag{6.56}$$

for some $M > 0$. Therefore

$$\sum_{n=1}^{\infty} \int_{\Omega} \left| \frac{t^{2n}}{(2n)!} \; (A^{\varepsilon})^n u^0(x) \cdot \psi\left(x, \frac{x}{\varepsilon}\right) \right| \, dx \leq \left\| u^0 \right\|_{L^2(\Omega)^3} \, \|\psi\|_{L^2(\Omega; C_{per}(Y)^3)} \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} M^n,$$

from which (6.54) follows.

Now from (6.54) and by using Lebesgue's dominated convergence theorem, it is straightforward to show that

$$\int_{\Omega} \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \; (A^{\varepsilon})^n u^0(x) \cdot \psi\left(x, \frac{x}{\varepsilon}\right) \, dx = \sum_{n=1}^{\infty} \int_{\Omega} \frac{t^{2n}}{(2n)!} \; (A^{\varepsilon})^n u^0(x) \cdot \psi\left(x, \frac{x}{\varepsilon}\right) \, dx. \tag{6.57}$$

For $n \in \mathbb{N}$, we define

$$S_{N,\varepsilon} = \sum_{n=1}^{N} \frac{t^{2n}}{(2n)!} \int_{\Omega} (A^{\varepsilon})^n u^0(x) \cdot \psi\left(x, \frac{x}{\varepsilon}\right) \, dx.$$

Then using (6.40) we see that

$$\lim_{\varepsilon \to 0} S_{N,\varepsilon} = \sum_{n=1}^{N} \frac{t^{2n}}{(2n)!} \int_{\Omega \times Y} \chi_{\mathrm{f}}(y) (A_{\mathrm{f}})^n u^0(x) \cdot \psi\,(x, y) \; dxdy, \tag{6.58}$$

and hence

$$\lim_{N \to \infty} \lim_{\varepsilon \to 0} S_{N,\varepsilon} = \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \int_{\Omega \times Y} \chi_{\mathrm{f}}(y) (A_{\mathrm{f}})^n u^0(x) \cdot \psi\,(x, y) \; dxdy. \tag{6.59}$$

Below we will show that the order of the limits in (6.59) can be interchanged, i.e.,

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} S_{N,\varepsilon} = \lim_{N \to \infty} \lim_{\varepsilon \to 0} S_{N,\varepsilon}. \tag{6.60}$$

Combining this with (6.57) we obtain

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\Omega} \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \; (A^{\varepsilon})^n u^0(x) \cdot \psi\left(x, \frac{x}{\varepsilon}\right) \, dx \\ = \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \int_{\Omega \times Y} \chi_{\mathrm{f}}(y) (A_{\mathrm{f}})^n u^0(x) \cdot \psi\,(x, y) \; dxdy. \end{aligned} \tag{6.61}$$

Applying arguments similar to those used in obtaining (6.57), we can show that

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{\Omega \times Y} \frac{t^{2n}}{(2n)!} \chi_{\mathrm{f}}(y) (A_{\mathrm{f}})^n u^0(x) \cdot \psi\,(x, y) \; dxdy \\ = \int_{\Omega \times Y} \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \chi_{\mathrm{f}}(y) (A_{\mathrm{f}})^n u^0(x) \cdot \psi\,(x, y) \; dxdy. \end{aligned} \tag{6.62}$$

From (6.61) and (6.62), the result (6.45) follows.

To complete the proof, it remains to justify (6.60). It is sufficient to show the double sequence $(S_{N,\varepsilon})$ is Cauchy. So assume that $N, L \in \mathbb{N}$ such that $N \geq L$. Then

$$\begin{aligned}
|S_{N,\varepsilon} - S_{L,\varepsilon}| &= \left| \sum_{n=L+1}^{N} \frac{t^{2n}}{(2n)!} \int_\Omega (A^\varepsilon)^n u^0(x) \cdot \psi\left(x, \frac{x}{\varepsilon}\right) dx \right| \\
&\leq \sum_{n=L+1}^{N} \frac{t^{2n}}{(2n)!} \int_\Omega \left| (A^\varepsilon)^n u^0(x) \cdot \psi\left(x, \frac{x}{\varepsilon}\right) \right| dx \\
&\leq \left\| u^0 \right\|_{L^2(\Omega)^3} \|\psi\|_{L^2(\Omega; C_{per}(Y)^3)} \sum_{n=L+1}^{N} \frac{t^{2n}}{(2n)!} M^n, \qquad (6.63)
\end{aligned}$$

where (6.56) was used in the last step. We note that the term $\sum_{n=L+1}^{N} \frac{t^{2n}}{(2n)!} M^n$ in (6.63) can be made arbitrarily small by choosing large values of N and L. We conclude that for given $\zeta > 0$, there exists a positive integer $K(\zeta)$ such that for $N, L > K(\zeta)$ and all $\varepsilon > 0$,

$$|S_{N,\varepsilon} - S_{L,\varepsilon}| < \zeta. \qquad (6.64)$$

From (6.58) and (6.64), and by using Lemma 6.6 below, it follows that the double sequence $(S_{N,\varepsilon})$ is Cauchy.

Lemma 6.6. *Let $(a_{n,k})$ be a double sequence in $\mathbb{R}^d$, $d \in \mathbb{N}$, such that*

(a) For each $n \in \mathbb{N}$,

$$\lim_{k \to \infty} a_{n,k} = \bar{a}_n.$$

(b) Given $\zeta > 0$, there exists a positive integer $N = N(\zeta)$ such that for $n, l > N$ and all $k \in \mathbb{N}$,

$$|a_{n,k} - a_{l,k}| < \zeta. \qquad (6.65)$$

Then the double sequence $(a_{n,k})$ is Cauchy, and hence convergent.

Proof. Let $\zeta > 0$ and assume that $N \in \mathbb{N}$ satisfies Part (b). Then consider the sequence $(a_{N,k})_{k \in \mathbb{N}}$. It follows from Part (a) that this sequence is convergent, and hence Cauchy. Thus there exists a positive integer $K = K(N, \zeta)$ such that for $k, m > K$,

$$|a_{N,k} - a_{N,m}| < \zeta. \qquad (6.66)$$

Let $J = \max\{N, K\}$. Then from (6.65) and (6.66) we obtain that for $n, l, k, m > J$,

$$\begin{aligned}
|a_{n,k} - a_{l,m}| &\leq |a_{n,k} - a_{N,k}| + |a_{N,k} - a_{N,m}| + |a_{N,m} - a_{l,m}| \\
&\leq 3\zeta,
\end{aligned}$$

and therefore the double sequence $(a_{n,k})$ is Cauchy. □

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