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IMA Preprint Series # 2215

(July 2008)

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GLOBAL SATURATION OF REGULARIZATION METHODS FOR INVERSE ILL-POSED PROBLEMS *

RUBEN D. SPIES† AND KARINA G. TEMPERINI‡

Abstract. In this article the concept of saturation of an arbitrary regularization method is formalized based upon the original idea of saturation for spectral regularization methods introduced by Neubauer [5]. Necessary and sufficient conditions for a regularization method to have global saturation are provided. It is shown that for a method to have global saturation the total error must be optimal in two senses, namely as optimal order of convergence over a certain set which at the same time, must be optimal (in a very precise sense) with respect to the error. Finally, two converse results are proved and the theory is applied to find sufficient conditions which ensure the existence of global saturation for spectral methods with classical qualification of finite positive order and for methods with maximal qualification.

Key words. Ill-posed, inverse problem, qualification, saturation.

AMS subject classifications. 47A52, 65J20

1. Introduction.

Let X, Y be infinite dimensional Hilbert spaces and $T : X \to Y$ a bounded linear operator such that $\mathcal{R}(T)$ is not closed. It is well known that under these conditions, the linear operator equation

$$Tx = y \tag{1.1}$$

is ill-posed, in the sense that $T^\dagger$, the Moore-Penrose generalized inverse of T, is not bounded [1]. The Moore-Penrose generalized inverse is strongly related to the least squares solutions of (1.1). In fact this equation has a least squares solution if and only if $y \in \mathcal{D}(T^\dagger) \doteq \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$. In that case, $x^\dagger \doteq T^\dagger y$ is the least squares solution of minimum norm and the set of all least-squares solutions of (1.1) is given by $x^\dagger + \mathcal{N}(T)$. If the problem is ill-posed then $x^\dagger$ does not depend continuously on the data y. Therefore, if instead of the exact data y, a noisy observation y^δ is available, with $\left\| y - y^\delta \right\| \leq \delta$, where $\delta > 0$ is small, then it is possible that $T^\dagger y^\delta$ does not even exist and if it does, it will not necessarily be a good approximation of $x^\dagger$. This instability becomes evident when trying to approximate $x^\dagger$ by traditional numerical methods and procedures. Thus, for instance, it is possible that the application of the standard least squares approximating procedure on an increasing sequence of finite-dimensional subspaces $\{X_n\}$ of X whose union is dense in X, result in a sequence $\{x_n\}$ of least squares solutions that does not converge to $x^\dagger$ (see [7]) or, even worst, that they diverge from $x^\dagger$ with speed arbitrarily large (see [8]).

Ill-posed problems must be first regularized if one wants to successfully attack the task of numerically approximating their solutions. Regularizing an ill-posed problem such as (1.1) essentially means approximating the operator $T^\dagger$ by a parametric family of bounded operators $\{R_\alpha\}$, where α is a regularization parameter. If $y \in \mathcal{D}(T^\dagger)$, then

*This work was supported by Consejo Nacional de Investigaciones Científicas y Técnicas, CONICET and by Universidad Nacional del Litoral, U.N.L. through project P.E. 236.

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the best approximate solution $x^\dagger$ of (1.1) can be written as $x^\dagger = \int_0^{\|T\|^2+} \frac{1}{\lambda}\, dE_\lambda T^* y$ where $\{E_\lambda\}$ is the spectral family associated to the operator T^*T (see [1]). From here that many regularization methods are based on spectral theory and consist in defining $R_\alpha \doteq \int_0^{\|T\|^2+} g_\alpha(\lambda)\, dE_\lambda T^*$ where $\{g_\alpha\}$ is a family of functions appropriately chosen such that for every $\lambda \in (0, \|T\|^2]$ there holds $\lim\limits_{\alpha\to 0^+} g_\alpha(\lambda) = \frac{1}{\lambda}$.

However, it is important to emphasize that no mathematical trick can make stable a problem that is intrinsically unstable. In any case there is always loss of information. All a regularization method can do is to recover the largest possible amount of information about the solution of the problem, maintaining stability. It is said that the art of applying regularization methods consist always in maintaining an adequate balance between accuracy and stability. In 1994, however, Neubauer ([5]) showed that certain spectral regularization methods "*saturate*", that is, they become unable to continue extracting additional information about the exact solution even upon increasing regularity assumptions on it. In this article, Neubauer introduced for the first time the idea of the concept of "*saturation*" of regularization methods. This idea referred to the best order of convergence that a method can achieve independently of the smoothness assumptions on the exact solution and on the selection of the parameter choice rule. Later on, in 1997, Neubauer ([6]) showed that this saturation phenomenon occurs in particular in the classical Tikhonov-Phillips method. The concept of saturation has however escaped rigorous formalization since then.

In 2001, Mathé and Pereverzev ([3]) used Hilbert scales to study the efficiency of approximating solutions based on observations with noise (stochastic or deterministic). In this context it is possible to quantify the degree of ill-posedness and to obtain general conditions on projection methods so that they attain optimal order of convergence. These concepts were later extended by the same authors ([4]) who studied the optimal convergence problem in variable Hilbert scales. In this article they showed that there is a close relationship between the optimal convergence of a method and the *"a-priori"* regularity (in terms of source sets) for spectral methods possessing qualification of finite order.

In 2004, Mathé ([2]) proposed general definitions of the concepts of qualification and saturation for spectral regularization methods. However, the concept of saturation defined by Mathé is not applicable to general (non-spectral) regularization methods and it is not fully compatible with the original idea of saturation proposed by Neubauer in [5]. In particular, for instance, the definition of saturation given in [2] does not imply uniqueness and therefore, neither a best global order of convergence.

In this article a general theory of global saturation for arbitrary regularization methods (not necessarily spectral) is developed. This concept involves two aspects: on one hand (just like in Neubauer's original idea) the best global order of convergence of the method, and on the other hand, the characterization of the source set on which such a best global order of convergence is achieved. Also, necessary and sufficient conditions are found for a regularization method to have global saturation. In particular, it is shown that for a method to have saturation, it is necessary that the total error be optimal in two senses, namely as optimal order of convergence over a certain set and at the same time, this set must satisfy a certain optimality condition with respect to the error. Moreover, an explicit form for the global saturation is given in terms of the family of regularization operators and the operator associated to the problem. Lastly, sufficient conditions are provided for spectral methods with qualification of positive finite order and for spectral methods with maximal qualification to have global saturation.

The organization of the paper is as follows. In Section 2 convergence bounds for regularization methods are defined and an appropriate framework for their comparison is developed. In Section 3 the concept of global saturation is introduced, its relation with the total error and with convergence bounds is shown and necessary and sufficient conditions for the existence of global saturation are provided. In Section 4, a few converse results are proved which, together with the results of Section 3, are used to derive sufficient conditions for the existence of global saturation for certain spectral regularization methods.

2. Upper bounds of convergence for regularization methods. In sequel and for convenience of notation, unless otherwise specified, we shall assume that all subsets of the Hilbert space X under consideration are not empty and they do not contain $x = 0$. Also, without loss of generality we will assume that the operator T is invertible. Given $M \subset X$, we will denote with $\mathcal{F}_M$ the set of the following functions: we will say that $\psi \in \mathcal{F}_M$ if there exists $a = a(\psi) > 0$ such that ψ is defined in $M \times (0, a)$, with values in $(0, \infty)$ and it satisfies the following conditions:

1. $\lim\limits_{\delta\to 0^+} \psi(x, \delta) = 0$ for all $x \in M$, and
2. ψ is continuous and increasing as a function of δ in $(0, a)$ for each fixed $x \in M$.

Definition 2.1. *Let $M \subset X$ and $\psi, \tilde{\psi} \in \mathcal{F}_M$.*

i) We say that "ψ precedes $\tilde{\psi}$ on M", and we denote it $\psi \overset{M}{\preceq} \tilde{\psi}$, if there exist a constant $r > 0$ and $p : M \to (0, \infty)$ such that $\psi(x, \delta) \le p(x)\tilde{\psi}(x, \delta)$ for all $x \in M$ and for every $\delta \in (0, r)$.

ii) We say that "$\psi, \tilde{\psi}$ are equivalent on M", and we denote it $\psi \overset{M}{\approx} \tilde{\psi}$, if $\psi \overset{M}{\preceq} \tilde{\psi}$ and $\tilde{\psi} \overset{M}{\preceq} \psi$.

iii) We say that "ψ strictly precedes $\tilde{\psi}$ on M" and we denote it $\psi \overset{M}{\prec} \tilde{\psi}$ if $\psi \overset{M}{\preceq} \tilde{\psi}$ and $\limsup\limits_{\delta\to 0^+} \frac{\psi(x,\delta)}{\tilde{\psi}(x,\delta)} = 0$ for every $x \in M$.

The following observations follow immediately from these definitions.

• Given that $\psi, \tilde{\psi} > 0$, in *iii)* the condition $\limsup\limits_{\delta\to 0^+} \frac{\psi(x,\delta)}{\tilde{\psi}(x,\delta)} = 0$ is equivalent to $\lim\limits_{\delta\to 0^+} \frac{\psi(x,\delta)}{\tilde{\psi}(x,\delta)} = 0$, i.e., $\psi(x, \delta) = o(\tilde{\psi}(x, \delta))$ for $\delta \to 0^+$.

• The relation "$\overset{M}{\preceq}$" introduces a partial ordering in $\mathcal{F}_M$ and "$\overset{M}{\approx}$" is an equivalence relation in $\mathcal{F}_M$.

• If $\psi \overset{M}{\preceq} (\overset{M}{\approx}, \overset{M}{\prec})\, \tilde{\psi}$ then $\psi \overset{\tilde{M}}{\preceq} (\overset{\tilde{M}}{\approx}, \overset{\tilde{M}}{\prec})\, \tilde{\psi}$ for every $\tilde{M} \subset M$.

With $\npreceq$, $\nprec$ and $\not\approx$ we will denote the negation of the relations $\preceq$, $\prec$ and $\approx$, respectively.

Lemma 2.2. *Let $M \subset X$ and $\psi, \tilde{\psi} \in \mathcal{F}_M$. If $\psi \overset{M}{\prec} \tilde{\psi}$ then $\tilde{\psi} \overset{\tilde{M}}{\npreceq} \psi$ for every $\tilde{M} \subset M$.*

Proof. For the contrareciprocal. Suppose there exists $\tilde{M} \subset M$ such that $\tilde{\psi} \overset{\tilde{M}}{\preceq} \psi$. Let $x_0 \in \tilde{M}$, then $\tilde{\psi} \overset{\{x_0\}}{\preceq} \psi$, that is, there exist constants $0 < p < \infty$ and $r > 0$ such that $\sup\limits_{\delta\in(0,r)} \frac{\tilde\psi(x_0,\delta)}{\psi(x_0,\delta)} \le p < \infty$. Then,

$$\begin{aligned}\limsup_{\delta\to0^+}\frac{\psi(x_0,\delta)}{\tilde\psi(x_0,\delta)} &\ge \liminf_{\delta\to0^+}\frac{\psi(x_0,\delta)}{\tilde\psi(x_0,\delta)} \ge \inf_{\delta\in(0,r)}\frac{\psi(x_0,\delta)}{\tilde\psi(x_0,\delta)}\\ &= \left(\sup_{\delta\in(0,r)}\frac{\tilde\psi(x_0,\delta)}{\psi(x_0,\delta)}\right)^{-1} \ge \frac{1}{p} > 0.\end{aligned}$$

Therefore, $\psi \overset{\{x_0\}}{\not\prec} \tilde\psi$, from which we deduce that $\psi \overset{M}{\not\prec} \tilde\psi$, since $x_0 \in M$. □

Definition 2.3. *Let $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ be a family of regularization operators for the problem $Tx = y$. We define the "total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ at $x \in X$ for a noise level δ" as*

$$\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta) \doteq \inf_{\alpha\in(0,\alpha_0)} \sup_{y^\delta\in\overline{B_\delta(Tx)}} \left\|R_\alpha y^\delta - x\right\|,$$

where $\overline{B_\delta(Tx)} \doteq \{y \in Y : \|Tx - y\| \le \delta\}$.

Note that $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}$ is the error in the sense of the largest possible discrepancy that can be obtained for an observation within the noise level δ, with any choice of the regularization parameter α.

Remark 2.4. *Let $a > 0$, $M \subset X$ and $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} : M \times (0,a) \to (0,\infty)$ be the total error. Then $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \in \mathcal{F}_M$. In fact, for each $x \in M$, $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta)$ is increasing as a function of δ, and given that $\{R_\alpha\}$ is a family of regularization operators, it follows that $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta)$ is continuous as a function of δ for each fixed $x \in M$ and $\lim\limits_{\delta\to0^+}\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta) = 0$ for every $x \in M$.*

Definition 2.5. *Let $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ be a family of regularization operators for the problem $Tx = y$, $M \subset X$ and $\psi \in \mathcal{F}_M$.*

i) We say that ψ is an "upper bound of convergence for the total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on M" if $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \overset{M}{\preceq} \psi$.

ii) We say that ψ is a "strict upper bound of convergence for the total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on M" if $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \overset{M}{\prec} \psi$.

iii) We say that ψ is an "optimal upper bound of convergence for the total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on M" if $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \overset{M}{\preceq} \psi$ and

$$\limsup_{\delta\to0^+}\frac{\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta)}{\psi(x,\delta)} > 0 \quad \textit{for every} \quad x \in M,$$

or equivalently, if $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta) \ne o(\psi(x,\delta))$ when $\delta \to 0^+$.

We will denote with $\mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$, $\mathcal{U}^{\text{est}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ and $\mathcal{U}^{\text{opt}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ the set of all functions $\psi \in \mathcal{F}_M$ that are, respectively, upper bounds, strict upper bounds and optimal upper bounds of convergence for the total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on M. In view of Remark 2.4, it is clear that $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \in \mathcal{U}^{\text{opt}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ for every $M \subset X$.

The observations below follow immediately from the previous definitions.

- If $\psi \in \mathcal{F}_M$, then $\psi \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ if (and only if) $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta) = O(\psi(x,\delta))$ as $\delta \to 0^+$ for every $x \in M$. Moreover, $\mathcal{U}^{\text{est}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ and $\mathcal{U}^{\text{opt}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ are disjoint subsets of $\mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$, although their union is not all of $\mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ (except when M consists of just one element).

- If $\tilde{M} \subset M$, then $\mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}) \subset \mathcal{U}_{\tilde{M}}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$, $\mathcal{U}^{\text{opt}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}) \subset \mathcal{U}^{\text{opt}}_{\tilde{M}}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ and $\mathcal{U}^{\text{est}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}) \subset \mathcal{U}^{\text{est}}_{\tilde{M}}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$.
- If $\psi \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$, $\tilde{\psi} \in \mathcal{F}_M$ and $\psi \overset{M}{\preceq} \tilde{\psi}$, then $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$.
- If $\psi \in \mathcal{U}^{\text{opt}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$, $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ and $\tilde{\psi} \overset{M}{\preceq} \psi$, then $\tilde{\psi} \in \mathcal{U}^{\text{opt}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$.
- If $\psi \in \mathcal{U}^{\text{est}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$, $\tilde{\psi} \in \mathcal{F}_M$ and $\psi \overset{M}{\preceq} \tilde{\psi}$, then $\tilde{\psi} \in \mathcal{U}^{\text{est}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$.

Definition 2.6. *Let $\psi, \tilde{\psi} \in \mathcal{F}_M$. We say that "$\psi$ and $\tilde{\psi}$ are comparable on M" if they verify $\psi \overset{M}{\preceq} \tilde{\psi}$ or $\tilde{\psi} \overset{M}{\preceq} \psi$ (or both).*

Definition 2.7. *Let $\mathcal{A} \subset \mathcal{F}_M$ and $\psi^* \in \mathcal{A}$. We say that "ψ^* is a minimal element of $\left(\mathcal{A}, \overset{M}{\preceq}\right)$" if $\psi^* \overset{M}{\preceq} \psi$ for every $\psi \in \mathcal{A}$ comparable with ψ^* on M. Equivalently, ψ^* is minimal element of $\left(\mathcal{A}, \overset{M}{\preceq}\right)$ if for every $\psi \in \mathcal{A}$, the condition $\psi \overset{M}{\preceq} \psi^*$ implies $\psi^* \overset{M}{\preceq} \psi$.*

Lemma 2.8. *Let $\mathcal{A} \subset \mathcal{F}_M, \psi, \psi^* \in \mathcal{A}$ and ψ, ψ^* be comparable on M. If there exists $M_0 \subset M$ such that $\psi \overset{M_0}{\prec} \psi^*$ then ψ^* is not a minimal element of $\left(\mathcal{A}, \overset{M}{\preceq}\right)$.*

Proof. Let $\mathcal{A} \subset \mathcal{F}_M$ and $\psi, \psi^* \in \mathcal{A}$ be comparable on M. Let us suppose that there exists $M_0 \subset M$ such that $\psi \overset{M_0}{\prec} \psi^*$, then it follows from Lemma 2.2 that $\psi^* \overset{M_0}{\npreceq} \psi$. Thus $\psi^* \overset{M}{\npreceq} \psi$ and since $\psi, \psi^* \in \mathcal{A}$ are comparable on M, it follows from Definition 2.7 that ψ^* cannot be a minimal element of $\left(\mathcal{A}, \overset{M}{\preceq}\right)$. ∎

Corollary 2.9. *If $\psi^* \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ and there exist $x_0 \in M$ and $\psi_0 \in \mathcal{U}_{\{x_0\}}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ such that $\psi_0 \overset{\{x_0\}}{\prec} \psi^*$ then ψ^* is not a minimal element of $\left(\mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}), \overset{M}{\preceq}\right)$.*

Proof. This corollary is an immediate consequence of the previous lemma with $\mathcal{A} = \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$, $M_0 = \{x_0\}$ and

$$\psi(x,\delta) \doteq \begin{cases} \psi_0(x_0,\delta), & \text{if } x = x_0 \\ \psi^*(x,\delta), & \text{if } x \neq x_0. \end{cases}$$

Note that this function ψ so defined is in $\mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ and it is comparable with ψ^* on M (moreover $\psi \overset{M}{\preceq} \psi^*$). ∎

Next we will show that the optimal upper bounds of convergence for the total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on M are characterized by being minimal elements of the partially ordered set $\left(\mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}), \overset{M}{\preceq}\right)$. More precisely, we have the following result.

Theorem 2.10. *Let $\psi \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$. Then $\psi \in \mathcal{U}^{\text{opt}}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ if and only if ψ is a minimal element of $\left(\mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}), \overset{M}{\preceq}\right)$.*

Proof. Let $\psi \in \mathcal{U}_M^{\mathrm{opt}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ and suppose that ψ is not a minimal element of $\left(\mathcal{U}_M(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}), \overset{M}{\preceq}\right)$. Then there exists $\psi_c \in \mathcal{U}_M(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ comparable with ψ on M for which it is not true that $\psi \overset{M}{\preceq} \psi_c$. Then, there exists $x_0 \in M$ such that

$$\limsup_{\delta\to 0^+} \frac{\psi(x_0,\delta)}{\psi_c(x_0,\delta)} = \infty. \tag{2.1}$$

Now, since $\psi \in \mathcal{U}_M^{\mathrm{opt}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ and $x_0 \in M$, we have that

$$\limsup_{\delta\to 0^+} \frac{\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}(x_0,\delta)}{\psi(x_0,\delta)} > 0. \tag{2.2}$$

Since

$$\limsup_{\delta\to 0^+} \frac{\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}(x_0,\delta)}{\psi_c(x_0,\delta)} = \limsup_{\delta\to 0^+} \frac{\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}(x_0,\delta)}{\psi(x_0,\delta)} \frac{\psi(x_0,\delta)}{\psi_c(x_0,\delta)},$$

using (2.1) and (2.2) we deduce that $\limsup\limits_{\delta\to 0^+} \frac{\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}(x_0,\delta)}{\psi_c(x_0,\delta)} = \infty$, which implies that $\psi_c \notin \mathcal{U}_{\{x_0\}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$. This contradicts the fact that $\psi_c \in \mathcal{U}_M(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$. Therefore, ψ is a minimal element of $\left(\mathcal{U}_M(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}), \overset{M}{\preceq}\right)$.

Conversely, assume that $\psi \in \mathcal{U}_M(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ and $\psi \notin \mathcal{U}_M^{\mathrm{opt}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$. Then there exists $x_0 \in M$ such that $\psi \in \mathcal{U}^{\mathrm{est}}_{\{x_0\}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$, which implies that $\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}} \overset{\{x_0\}}{\prec} \psi$. By virtue of Lemma 2.8 we deduce that ψ is not a minimal element of $\left(\mathcal{U}_M(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}), \overset{M}{\preceq}\right)$. ∎

Note that ψ is a minimal element of $\left(\mathcal{U}_M(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}), \overset{M}{\preceq}\right)$ if and only if it is minimal of $\left(\mathcal{U}_{M^*}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}), \overset{M^*}{\preceq}\right)$ for every $M^* \subset M$.

Corollary 2.11. *Let $\mathcal{U}_M^{\mathrm{opt}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ and $\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}$ be as before*

i) *If $\psi \in \mathcal{U}_M^{\mathrm{opt}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ then $\psi \overset{M}{\approx} \mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}$.*

ii) *If $\psi, \tilde{\psi} \in \mathcal{U}_M^{\mathrm{opt}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ then $\psi \overset{M}{\approx} \tilde{\psi}$.*

Proof.

i) If $\psi \in \mathcal{U}_M^{\mathrm{opt}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ then $\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}} \overset{M}{\preceq} \psi$, from which it follows that $\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}$ and ψ are comparable on M. Then, since $\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}} \in \mathcal{U}_M(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ and by Theorem 2.10 ψ is a minimal element of $\left(\mathcal{U}_M(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}), \overset{M}{\preceq}\right)$, we have that $\psi \overset{M}{\preceq} \mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}$. Therefore, $\psi \overset{M}{\approx} \mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}$ as we wanted to prove.

ii) This is an immediate consequence of **i)** and the properties of the equivalence relation "$\overset{M}{\approx}$", because every $\psi \in \mathcal{U}_M^{\mathrm{opt}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ is equivalent to $\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}$ on M. ∎

Note that if ψ is an optimal upper bound of convergence on M for the total error of a regularization method, then at every point of M ψ tends to zero, as the noise level tends to zero, exactly with the same speed with which the total error does.

In order to introduce the concept of saturation in the next section, we will further need some definitions and tools that will allow us to compare convergence bounds on different sets of X.

DEFINITION 2.12. *Let $M, \tilde{M} \subset X$, $\psi \in \mathcal{F}_M$ and $\tilde{\psi} \in \mathcal{F}_{\tilde{M}}$.*

i) We say that "ψ on M precedes $\tilde{\psi}$ on $\tilde{M}$", and we denote it with $\psi \overset{M,\tilde{M}}{\preceq} \tilde{\psi}$, if there exist a constant $d > 0$ and $k : M \times \tilde{M} \to (0, \infty)$ such that $\psi(x, \delta) \leq k(x, \tilde{x})\, \tilde{\psi}(\tilde{x}, \delta)$ for every $x \in M$, for every $\tilde{x} \in \tilde{M}$ and for every $\delta \in (0, d)$.

ii) We say that "ψ on M is equivalent to $\tilde{\psi}$ on $\tilde{M}$", and we denote it with $\psi \overset{M,\tilde{M}}{\approx} \tilde{\psi}$, if $\psi \overset{M,\tilde{M}}{\preceq} \tilde{\psi}$ and $\tilde{\psi} \overset{\tilde{M},M}{\preceq} \psi$.

iii) We say that "ψ on M strictly precedes $\tilde{\psi}$ on $\tilde{M}$", and we denote it with $\psi \overset{M,\tilde{M}}{\prec} \tilde{\psi}$, if $\psi \overset{M,\tilde{M}}{\preceq} \tilde{\psi}$ and $\limsup_{\delta \to 0^+} \frac{\psi(x,\delta)}{\tilde{\psi}(\tilde{x},\delta)} = 0$ for every $x \in M$ and for every $\tilde{x} \in \tilde{M}$.

REMARK 2.13. *In a certain sense, the previous definitions generalize (although they are slightly stronger than) the relations introduced in Definition 2.1. Note for instance that if $\psi \overset{M,M}{\prec} \tilde{\psi}$ then $\psi \overset{M}{\prec} \tilde{\psi}$, although the converse, in general, is not true.*

It follows immediately from Definition 2.12 that if $\psi \overset{M,N}{\preceq} \tilde{\psi}$ then $\psi \overset{\tilde{M},\tilde{N}}{\preceq} \tilde{\psi}$ for every $\tilde{M} \subset M$ and for every $\tilde{N} \subset N$. The same happens for the relations "$\overset{M,N}{\approx}$" and "$\overset{M,N}{\prec}$".

Next, we will extend the notion of "comparability" given in Definition 2.6, to the case of different subsets of X.

DEFINITION 2.14. *Let $M, \tilde{M} \subset X$, $\psi \in \mathcal{F}_M$ and $\tilde{\psi} \in \mathcal{F}_{\tilde{M}}$.*

i) We say that "ψ on M is comparable with $\tilde{\psi}$ on $\tilde{M}$" if $\psi \overset{M,\tilde{M}}{\preceq} \tilde{\psi}$ or $\tilde{\psi} \overset{\tilde{M},M}{\preceq} \psi$.

ii) We say that "ψ is invariant over M" if $\psi \overset{M,M}{\approx} \psi$.

REMARK 2.15. *It is immediate that the condition $\psi \overset{M,M}{\approx} \psi$ is equivalent to $\psi \overset{M,M}{\preceq} \psi$.*

Roughly speaking, this last notion of "invariance" establishes that if ψ is invariant over M then the orders of convergence of ψ as a function of δ when $\delta \to 0^+$, in any two points of M, are equivalent.

The following result is related to a certain transitivity property of the invariance relation.

LEMMA 2.16. *Let $M \subset X$, $\psi, \tilde{\psi} \in \mathcal{F}_M$ be such that $\tilde{\psi} \overset{M}{\approx} \psi$ and $\psi \overset{M,M}{\approx} \psi$. Then:*

i) *$\tilde{\psi} \overset{M,M}{\approx} \psi$ and*

ii) *$\tilde{\psi} \overset{M,M}{\approx} \tilde{\psi}$ (i.e. $\tilde{\psi}$ is also invariant over M).*

Proof.

Let $M \subset X$, $\psi, \tilde{\psi} \in \mathcal{F}_M$ and $x, \tilde{x} \in M$. Suppose that $\psi \overset{M,M}{\approx} \psi$ and $\tilde{\psi} \overset{M}{\approx} \psi$.

i) Since $\tilde{\psi} \overset{M}{\approx} \psi$, there exist positive constants d, k_x and $k_{\tilde{x}}$ such that for every $\delta \in (0, d)$,

$$\tilde{\psi}(x, \delta) \leq k_x \psi(x, \delta) \quad \text{and} \quad \psi(\tilde{x}, \delta) \leq k_{\tilde{x}} \tilde{\psi}(\tilde{x}, \delta). \tag{2.3}$$

From the invariance of ψ over M it follows that there exist positive constants d^* y $k^*_{x,\tilde{x}}$ such that $\psi(x, \delta) \leq k^*_{x,\tilde{x}} \psi(\tilde{x}, \delta)$ for every $\delta \in (0, d^*)$, which together with (2.3) implies that for every $\delta \in (0, \min\{d, d^*\})$,

$$\tilde{\psi}(x,\delta) \le k_x \psi(x,\delta) \le k_x k^*_{x,\tilde{x}} \psi(\tilde{x},\delta) \quad \text{and} \quad \psi(x,\delta) \le k^*_{x,\tilde{x}} \psi(\tilde{x},\delta) \le k^*_{x,\tilde{x}} k_{\tilde{x}} \tilde{\psi}(\tilde{x},\delta). \tag{2.4}$$

Then, $\tilde{\psi} \overset{M,M}{\preceq} \psi$ and $\psi \overset{M,M}{\preceq} \tilde{\psi}$, that is, $\tilde{\psi} \overset{M,M}{\approx} \psi$.

ii) From the first inequality in (2.4) and from the second inequality in (2.3) it follows immediately that $\tilde{\psi} \overset{M,M}{\preceq} \tilde{\psi}$ and therefore, $\tilde{\psi}$ is invariant over M. ◻

The following result is analogous to Lemma 2.2 for this case of comparison of convergence bounds on different sets.

LEMMA 2.17. *Let $M, N \subset X$, $\psi \in \mathcal{F}_M$ and $\tilde{\psi} \in \mathcal{F}_N$. If $\psi \overset{M,N}{\prec} \tilde{\psi}$ then $\forall\, \tilde{M} \subset M$, $\forall\, \tilde{N} \subset N$ we have that $\tilde{\psi} \overset{\tilde{N},\tilde{M}}{\npreceq} \psi$.*

Proof. By the contrareciprocal. Suppose that there exist $\tilde{M} \subset M$ and $\tilde{N} \subset N$ such that $\tilde{\psi} \overset{\tilde{N},\tilde{M}}{\preceq} \psi$. Then there exist a constant $d > 0$ and $k : \tilde{N} \times \tilde{M} \to (0,\infty)$ such that $\tilde{\psi}(\tilde{x},\delta) \le k(\tilde{x},x)\,\psi(x,\delta)$ for every $\tilde{x} \in \tilde{N}$, $x \in \tilde{M}$ and $\delta \in (0,d)$. Let $x_0 \in \tilde{M}$ and $\tilde{x}_0 \in \tilde{N}$, then $\tilde{\psi}(\tilde{x}_0,\delta) \le k(\tilde{x}_0,x_0)\,\psi(x_0,\delta)$ for every $\delta \in (0,d)$. Thus, $\sup\limits_{\delta \in (0,d)} \frac{\tilde{\psi}(\tilde{x}_0,\delta)}{\psi(x_0,\delta)} \le k(\tilde{x}_0,x_0) < \infty$. Then,

$$\begin{aligned}
\limsup_{\delta\to 0^+} \frac{\psi(x_0,\delta)}{\tilde{\psi}(\tilde{x}_0,\delta)} &\ge \liminf_{\delta\to 0^+} \frac{\psi(x_0,\delta)}{\tilde{\psi}(\tilde{x}_0,\delta)} \ge \inf_{\delta\in(0,d)} \frac{\psi(x_0,\delta)}{\tilde{\psi}(\tilde{x}_0,\delta)} \\
&= \left(\sup_{\delta\in(0,d)} \frac{\tilde{\psi}(\tilde{x}_0,\delta)}{\psi(x_0,\delta)} \right)^{-1} \ge \frac{1}{k(\tilde{x}_0,x_0)} > 0.
\end{aligned}$$

Therefore, $\psi \overset{\{x_0\},\{\tilde{x}_0\}}{\nprec} \tilde{\psi}$, from which it follows that $\psi \overset{M,N}{\nprec} \tilde{\psi}$, since $x_0 \in M$ and $\tilde{x_0} \in N$. ◻

3. Global Saturation. We will now proceed to formalize the concept of global saturation.

DEFINITION 3.1. *Let $M_S \subset X$ and $\psi_S \in \mathcal{U}_{M_S}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$. We say that ψ_S is a "global saturation function of $\{R_\alpha\}$ over M_S" if ψ_S satisfies the following three conditions:*

S1. For every $x^* \in X$, $x^* \neq 0$, $x \in M_S$, $\limsup\limits_{\delta\to 0^+} \frac{\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta)}{\psi_S(x,\delta)} > 0$.

S2. ψ_S is invariant over M_S.

S3. There is no upper bound of convergence for the total error of $\{R_\alpha\}$ that is a proper extension of ψ_S (in the variable x) and satisfies S1 and S2, that is, there exist no $\tilde{M} \supsetneq M_S$ and $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ such that $\tilde{\psi}$ satisfies S1 and S2 with $M_S = \tilde{M}$ and $\psi_S = \tilde{\psi}$.

REMARK 3.2. *Note that condition S1 implies that for every $M \subset X$ and for every $\psi \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$, $\limsup\limits_{\delta\to 0^+} \frac{\psi(x^*,\delta)}{\psi_S(x,\delta)} > 0$ for every $x^* \in M$, $x \in M_S$ (this is an immediate consequence of S1 and the fact that $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \overset{M}{\preceq} \psi \ \forall\ \psi \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$). Therefore, it cannot happen that $\psi \overset{M,M_S}{\prec} \psi_S$. On the other hand, if $\psi \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ then it is not necessarily true that $\psi_S \overset{M_S,M}{\preceq} \psi$ even if ψ on M is comparable to ψ_S on M_S, because in this case it can happen that $\liminf\limits_{\delta\to 0^+} \frac{\psi(x,\delta)}{\psi_S(x_S,\delta)} = 0$ for some $x \in M$ and some $x_S \in M_S$*

(which implies that $\psi_S \overset{M_S,M}{\not\preceq} \psi$*), and still have* $\limsup_{\delta\to0^+} \frac{\psi(x,\delta)}{\psi_S(x_S,\delta)} > 0$*. However, if* ψ *on* M *is comparable with* ψ_S *on* M_S *and there exists* $\lim_{\delta\to0^+} \frac{\psi(x,\delta)}{\psi_S(x_S,\delta)}$ *for every* $x \in M$ *and for every* $x_S \in M_S$*, then it is in fact true that* $\psi_S \overset{M_S,M}{\preceq} \psi$*.*

Note also that condition S1 can be replaced by

$$\limsup_{\delta\to0^+} \frac{\psi(x^*,\delta)}{\psi_S(x,\delta)} > 0 \quad \forall\, \psi \in \mathcal{U}_{\{x^*\}}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}), \forall\, x^* \in X, x^* \neq 0, x \in M_S.$$

This notion of global saturation essentially establishes that in no point $x^* \in X$, $x^* \neq 0$, can exist an upper bound of convergence for the total error of the regularization method that is "strictly better" than the saturation function ψ_S in any point of M_S.

Next we will show that a saturation function is always an optimal upper bound of convergence.

Lemma 3.3. *Let* $\psi_S \in \mathcal{U}_{M_S}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$*. If* ψ_S *satisfies the condition S1 on* M_S*, then* $\psi_S \in \mathcal{U}^{\mathrm{opt}}_{M_S}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$*. In particular, if* ψ_S *is a saturation function of* $\{R_\alpha\}$ *on* M_S *then* $\psi_S \in \mathcal{U}^{\mathrm{opt}}_{M_S}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$*.*

Proof. The condition *S1* implies in particular that $\limsup_{\delta\to0^+} \frac{\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}(x,\delta)}{\psi_S(x,\delta)} > 0$ for every $x \in M_S$. Since also by definition $\psi_S \in \mathcal{U}_{M_S}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$ it follows that $\psi_S \in \mathcal{U}^{\mathrm{opt}}_{M_S}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$, as we wanted to show. ∎

An immediate corollary of this lemma is the equivalence between the saturation function and the total error.

Corollary 3.4. *If* ψ_S *is a saturation function of* $\{R_\alpha\}$ *on* M_S *then* $\psi_S \overset{M_S}{\approx} \mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}$*. Moreover, we have the stronger equivalence* $\psi_S \overset{M_S,M_S}{\approx} \mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}$*.*

Proof. The first part of the corollary is an immediate consequence of the previous lemma and of Corollary 2.11 **i)**. The second part follows from the first and the fact that $\psi_S \overset{M_S,M_S}{\approx} \psi_S$ via Lemma 2.16 **i)**. ∎

Remark 3.5. *A consequence of the first part of this corollary and of Lemma 2.16 ii) is that if* ψ_S *is a saturation function of* $\{R_\alpha\}$ *on* M_S*, then* $\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}} \overset{M_S,M_S}{\approx} \mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}$*, that is, the total error must be invariant over* M_S*. We will shed more light on this matter in Theorem 3.8.*

Definition 3.6. *Let* $M \subset X$ *and* $\psi \in \mathcal{U}_X(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$*. We say that "*$M$ *is optimal for* ψ*", and we denote it with* $M \in \mathcal{O}(\psi)$*, if the following condition hold:*

C2. For every $x \in M$*,* $x_c \in M^c$ *neither* $\psi \overset{\{x_c\},\{x\}}{\prec} \psi$ *nor* $\psi \overset{\{x_c\},\{x\}}{\approx} \psi$*.*

That a set M be optimal for ψ essentially means that at any point of the complement of M, the order of convergence of ψ as a function of δ for $\delta \to 0^+$ cannot be better nor even equivalent to the order of convergence of ψ at any point of M; that is, at any point outside of M, the order of convergence of ψ must be worst than itself at any point of M. However, we will see next that this optimality condition imposes a very precise restriction. As we shall see later on (Theorem 3.8), it is precisely this restriction on the total error, together with its invariance, what allows us to characterize the regularization methods that have saturation.

Let $\psi \in \mathcal{U}_X(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$, $M \subset X$ and consider the following conditions:

C1. $\psi \overset{M,M^c}{\prec} \psi$.

C3. $\psi \overset{M^c,M}{\not\prec} \psi$ and $\psi \overset{M^c,M}{\not\approx} \psi$.

Note that condition *C2* (of optimal set) is strictly stronger than condition *C3*, and strictly weaker than condition *C1*. In fact, if M is optimal for ψ then for every $x \in M$, $x_c \in M^c$ we have that $\psi \overset{\{x_c\},\{x\}}{\not\prec} \psi$ and $\psi \overset{\{x_c\},\{x\}}{\not\approx} \psi$, from which it follows immediately that $\psi \overset{M^c,M}{\not\prec} \psi$ and $\psi \overset{M^c,M}{\not\approx} \psi$, that is, *C3* holds. However, for condition *C3* to hold it is sufficient that there exist $x \in M$ and $x_c \in M^c$ such that $\psi \overset{\{x_c\},\{x\}}{\not\prec} \psi$ and $\psi \overset{\{x_c\},\{x\}}{\not\approx} \psi$, which obviously does not imply condition *C2*. On the other hand if *C1* holds, then it follows from Lemma 2.17 that for every $x \in M$, $x_c \in M^c$, there holds $\psi \overset{\{x_c\},\{x\}}{\not\preceq} \psi$ and therefore, $\psi \overset{\{x_c\},\{x\}}{\not\prec} \psi$ and $\psi \overset{\{x_c\},\{x\}}{\not\approx} \psi$ for every $x \in M$, $x_c \in M^c$, that is, condition *C2* holds. However, *C2* does not imply *C1* since it can happen that M be optimal for ψ and that there exist $x \in M$ and $x_c \in M^c$ such that $\psi \overset{\{x\},\{x_c\}}{\not\preceq} \psi$ and therefore, $\psi \overset{M,M^c}{\not\prec} \psi$.

In order to be able to characterize the regularization methods that have saturation, we will previously need the following result.

Lemma 3.7. *Suppose that $\{R_\alpha\}$ has saturation function on $M \subset X$ and for every $x \in M$, $x_c \in M^c$ there holds $\mathcal{E}^{\rm tot}_{\{R_\alpha\}} \overset{\{x_c\},\{x\}}{\not\prec} \mathcal{E}^{\rm tot}_{\{R_\alpha\}}$. Then $\mathcal{E}^{\rm tot}_{\{R_\alpha\}} \overset{\{x_c\},\{x\}}{\not\approx} \mathcal{E}^{\rm tot}_{\{R_\alpha\}}$ for every $x \in M$, $x_c \in M^c$.*

Proof. Since $\{R_\alpha\}$ has saturation function on M, it follows from Remark 3.5 that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}}$ is invariant over M. Suppose that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}} \overset{\{x_c\},\{x\}}{\not\prec} \mathcal{E}^{\rm tot}_{\{R_\alpha\}}$ for every $x \in M$, $x_c \in M^c$ and that there exist $\tilde{x} \in M$, $\tilde{x}_c \in M^c$ such that

$$\mathcal{E}^{\rm tot}_{\{R_\alpha\}} \overset{\{\tilde{x}_c\},\{\tilde{x}\}}{\approx} \mathcal{E}^{\rm tot}_{\{R_\alpha\}}. \tag{3.1}$$

Then,

$$\limsup_{\delta\to 0^+} \frac{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(\tilde{x},\delta)}{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(\tilde{x}_c,\delta)} > 0. \tag{3.2}$$

We define $\tilde{M} \doteq M \cup \{\tilde{x}_c\}$ and

$$\tilde{\psi}(x,\delta) \doteq \begin{cases} \psi(x,\delta), & \text{if } x \in M \\ \mathcal{E}^{\rm tot}_{\{R_\alpha\}}(x,\delta), & \text{if } x = \tilde{x}_c, \end{cases}$$

where ψ is a saturation function of $\{R_\alpha\}$ on M. Next we will show that $\tilde{\psi}$ is saturation function on $\tilde{M}$. Clearly, $\tilde{\psi}$ is upper bound of convergence for the total error on $\tilde{M}$, i.e., $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}(\mathcal{E}^{\rm tot}_{\{R_\alpha\}})$ and since ψ is saturation on M, it follows that $\tilde{\psi}(x,\delta)$ satisfies condition *S1* for all $x \in M$.

We will now check that $\tilde{\psi}(\tilde{x}_c,\delta)$ also satisfies *S1*. Since $\tilde{x} \in M$ it follows that

$$\limsup_{\delta\to 0^+} \frac{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(x^*,\delta)}{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(\tilde{x},\delta)} > 0 \quad \forall\, x^* \in X,\, x^* \neq 0. \tag{3.3}$$

If $x^* \in M$, this inequality follows from the fact that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}}$ is invariant over M and if $x^* \in M^c$, it is a consequence of the fact that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}} \overset{\{x^*\},\{\tilde{x}\}}{\not\preceq} \mathcal{E}^{\rm tot}_{\{R_\alpha\}}$.

Then, for every $x^* \in X$, $x^* \neq 0$ we have that

$$\limsup_{\delta\to 0^+} \frac{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(x^*,\delta)}{\tilde{\psi}(\tilde{x}_c,\delta)} = \limsup_{\delta\to 0^+} \frac{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(x^*,\delta)}{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(\tilde{x},\delta)} \frac{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(\tilde{x},\delta)}{\tilde{\psi}(\tilde{x}_c,\delta)} > 0$$

by virtue of (3.2) y (3.3). Thus, $\tilde{\psi}(x,\delta)$ satisfies *S1* for every $x \in \tilde{M}$.

We will now check that $\tilde{\psi}$ satisfies *S2* on $\tilde{M}$. Since ψ is saturation function of $\{R_\alpha\}$ on M, we have that $\tilde{\psi}$ is invariant over M. It remains to prove that $\tilde{\psi} \overset{\{\tilde{x}_c\},M}{\approx} \tilde{\psi}$, i.e. that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}} \overset{\{\tilde{x}_c\},M}{\approx} \psi$. But this is an immediate consequence of (3.1), of Corollary 3.4 which implies that $\psi \overset{M}{\approx} \mathcal{E}^{\rm tot}_{\{R_\alpha\}}$ and from the fact that ψ is invariant over M.

Thus, we have shown that $\tilde{\psi}$ is a proper extension of ψ satisfying *S1* and *S2* on $\tilde{M}$, which then implies that ψ does not satisfy condition *S3*. This contradicts the fact that ψ is saturation function of $\{R_\alpha\}$ on M. Therefore, for every $x \in M$, $x_c \in M^c$ we have that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}} \overset{\{x_c\},\{x\}}{\not\approx} \mathcal{E}^{\rm tot}_{\{R_\alpha\}}$, as we wanted to prove. ∎

THEOREM 3.8. *(Necessary and sufficient condition for the existence of saturation.) A regularization method $\{R_\alpha\}$ has saturation function if and only if there exists $M \subset X$ $(M \neq \{0\}, M \neq \emptyset)$ such that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}}$ is invariant over M and M is optimal for $\mathcal{E}^{\rm tot}_{\{R_\alpha\}}$. In such case, $\mathcal{E}^{\rm tot}_{M}(x,\delta) \doteq \mathcal{E}^{\rm tot}_{\{R_\alpha\}}(x,\delta)$ for $x \in M$ and $\delta > 0$ is saturation function of $\{R_\alpha\}$ on M.*

Proof. Suppose that $\{R_\alpha\}$ has saturation function ψ on M. Then it follows from Remark 3.5 that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}}$ is invariant over M.

Let us now check that M is optimal for $\mathcal{E}^{\rm tot}_{\{R_\alpha\}}$. Let $x \in M$ and $x_c \in M^c$. We will first show that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}} \overset{\{x_c\},\{x\}}{\not\preceq} \mathcal{E}^{\rm tot}_{\{R_\alpha\}}$. Since $\psi \in \mathcal{U}_M(\mathcal{E}^{\rm tot}_{\{R_\alpha\}})$ and $x \in M$, there exist positive constants d and k_x such that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(x,\delta) \le k_x\psi(x,\delta)$ for every $\delta \in (0,d)$. Then

$$\limsup_{\delta\to 0^+} \frac{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(x_c,\delta)}{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(x,\delta)} \ge \limsup_{\delta\to 0^+} \frac{\mathcal{E}^{\rm tot}_{\{R_\alpha\}}(x_c,\delta)}{k_x\psi(x,\delta)} > 0,$$

where the last inequality follows from the fact that ψ satisfies condition *S1* on M. Therefore $\forall\, x \in M$, $\forall\, x_c \in M^c$, $\mathcal{E}^{\rm tot}_{\{R_\alpha\}} \overset{\{x_c\},\{x\}}{\not\preceq} \mathcal{E}^{\rm tot}_{\{R_\alpha\}}$. This condition together with the fact that $\{R_\alpha\}$ has saturation function ψ on M implies, moreover, that $\forall\, x \in M$, $\forall\, x_c \in M^c$, $\mathcal{E}^{\rm tot}_{\{R_\alpha\}} \overset{\{x_c\},\{x\}}{\not\approx} \mathcal{E}^{\rm tot}_{\{R_\alpha\}}$, what follows by virtue of Lemma 3.7. We have thus shown that M is optimal for $\mathcal{E}^{\rm tot}_{\{R_\alpha\}}$.

Conversely, suppose that there exists $M \subset X$ $(M \neq \{0\}, M \neq \emptyset)$ such that $\mathcal{E}^{\rm tot}_{\{R_\alpha\}}$ is invariant over M and M is optimal for $\mathcal{E}^{\rm tot}_{\{R_\alpha\}}$ and define $\mathcal{E}^{\rm tot}_{M}(x,\delta) \doteq \mathcal{E}^{\rm tot}_{\{R_\alpha\}}(x,\delta)$ for $x \in M$ and $\delta > 0$. We will show that $\mathcal{E}^{\rm tot}_{M}$ is saturation function of $\{R_\alpha\}$ on M. Clearly, $\mathcal{E}^{\rm tot}_{M} \in \mathcal{U}_M(\mathcal{E}^{\rm tot}_{\{R_\alpha\}})$ and since by hypothesis $\mathcal{E}^{\rm tot}_{M}$ is invariant over M, it only remains to prove that $\mathcal{E}^{\rm tot}_{M}$ satisfies conditions *S1* and *S3*.

To prove *S1*, let $x^* \in X$, $x^* \neq 0$ and $x \in M$. If $x^* \in M$, then the invariance of $\mathcal{E}_M^{\rm tot}$ over M implies that $\mathcal{E}_M^{\rm tot} \overset{\{x^*\},\{x\}}{\approx} \mathcal{E}_M^{\rm tot}$ and therefore

$$\limsup_{\delta\to 0^+} \frac{\mathcal{E}_{\{R_\alpha\}}^{\rm tot}(x^*,\delta)}{\mathcal{E}_M^{\rm tot}(x,\delta)} = \limsup_{\delta\to 0^+} \frac{\mathcal{E}_M^{\rm tot}(x^*,\delta)}{\mathcal{E}_M^{\rm tot}(x,\delta)} > 0. \tag{3.4}$$

On the other hand, if $x^* \in M^c$, the previous limit is also positive due to the fact that $\mathcal{E}_{\{R_\alpha\}}^{\rm tot} \overset{\{x^*\},\{x\}}{\not\prec} \mathcal{E}_M^{\rm tot}$ (condition *C2*) because M is optimal for $\mathcal{E}_{\{R_\alpha\}}^{\rm tot}$. Then, $\mathcal{E}_M^{\rm tot}$ satisfies condition *S1*.

Finally, suppose that $\mathcal{E}_M^{\rm tot}$ does not satisfy condition *S3*, i.e. there exist $\tilde{M} \supsetneqq M$ and $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}(\mathcal{E}_{\{R_\alpha\}}^{\rm tot})$ such that $\tilde{\psi}$ is a proper extension of $\mathcal{E}_M^{\rm tot}$ satisfying conditions *S1* and *S2* on $\tilde{M}$. Let $\tilde{x} \in \tilde{M} \setminus M$, then the invariance of $\tilde{\psi}$ over $\tilde{M}$ implies that $\tilde{\psi} \overset{\{\tilde{x}\},M}{\approx} \tilde{\psi}$ and since $\tilde{\psi}$ coincides with $\mathcal{E}_M^{\rm tot}$ on M,

$$\tilde{\psi} \overset{\{\tilde{x}\},M}{\approx} \mathcal{E}_M^{\rm tot}. \tag{3.5}$$

Since $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}(\mathcal{E}_{\{R_\alpha\}}^{\rm tot})$ satisfies *S1* on $\tilde{M}$, Lemma 3.3 implies that $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}^{opt}(\mathcal{E}_{\{R_\alpha\}}^{\rm tot})$. Then, by virtue of Corollary 2.11.i we have that $\mathcal{E}_{\{R_\alpha\}}^{\rm tot} \overset{\tilde{M}}{\approx} \tilde{\psi}$. In particular, $\mathcal{E}_{\{R_\alpha\}}^{\rm tot} \overset{\{\tilde{x}\}}{\approx} \tilde{\psi}$, which, together with (3.5) imply that $\mathcal{E}_{\{R_\alpha\}}^{\rm tot} \overset{\{\tilde{x}\},M}{\approx} \mathcal{E}_M^{\rm tot}$, that is, $\mathcal{E}_{\{R_\alpha\}}^{\rm tot} \overset{\{\tilde{x}\},M}{\approx} \mathcal{E}_{\{R_\alpha\}}^{\rm tot}$. Given that $\tilde{x} \in M^c$, this equivalence contradicts the fact that M is optimal for $\mathcal{E}_{\{R_\alpha\}}^{\rm tot}$. Therefore, $\mathcal{E}_M^{\rm tot}$ must satisfy condition *S3* and, as a consequence, it is saturation function of $\{R_\alpha\}$ on M. ∎

REMARK 3.9. *From the previous theorem we conclude that a saturation function of a regularization method is an optimal upper bound of convergence for the total error, invariant and without a proper extension.*

Note that a saturation function must be optimal in two senses. In fact, if ψ is saturation function on M, then M is optimal for ψ and ψ is optimal (upper bound) for the total error of $\{R_\alpha\}$ on M. Moreover, M and ψ (modulus M, M equivalence) are uniquely determined. In fact, if the domain M is changed, this is no longer optimal for ψ and if the function ψ is changed, even at a single point of M, in such a way that ψ is not invariant on M, then it is no longer an optimal upper bound. Suppose that at the point $x \in M$, we redefine $\psi(x,\delta) \doteq \tilde{\psi}(x,\delta)$, where $\tilde{\psi} \in \mathcal{F}_M$. If $\tilde{\psi} \overset{\{x\}}{\prec} \psi$, then ψ is no longer an upper bound for the total error of $\{R_\alpha\}$ on M and if $\psi \overset{\{x\}}{\prec} \tilde{\psi}$ then ψ is upper bound but it is not optimal.

4. Saturation for spectral regularization methods. The objective of this section is to apply the theory previously developed to the case of spectral regularization methods. Further, we show that this theory is consistent with previously existing results about optimal convergence of spectral regularization methods.

Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family associated to the linear selfadjoint operator T^*T and $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ a parametric family of functions $g_\alpha : [0, \|T\|^2] \to \mathbb{R}$ for $\alpha \in (0,\alpha_0)$, and consider the following hypotheses:

H1. For every $\alpha \in (0,\alpha_0)$ the function g_α is piecewise continuous on $[0, \|T\|^2]$.

H2. There exists a constant $C > 0$ (independent of α) such that $|\lambda g_\alpha(\lambda)| \leq C$ for every $\lambda \in [0, \|T\|^2]$.

H3. For every $\lambda \in (0, \|T\|^2]$, there exists $\lim_{\alpha \to 0^+} g_\alpha(\lambda) = \frac{1}{\lambda}$.

H4. $G_\alpha \doteq \|g_\alpha(\cdot)\|_\infty = O\left(\frac{1}{\sqrt{\alpha}}\right)$ for $\alpha \to 0^+$.

If $\{g_\alpha\}_{\alpha \in (0,\alpha_0)}$ satisfies hypotheses *H1-H3*, then (see [1], Theorem 4.1) the family $\{R_\alpha\}_{\alpha \in (0,\alpha_0)}$, with

$$R_\alpha \doteq \int g_\alpha(\lambda)\, dE_\lambda\, T^* = g_\alpha(T^*T)T^*, \tag{4.1}$$

is a family of regularization operators for $T^\dagger$. In this case we will say that $\{R_\alpha\}_{\alpha \in (0,\alpha_0)}$ is a family of spectral regularization operators for $Tx = y$, since each one of its elements is defined in terms of an integral with respect to the spectral family associated to the operator T^*T.

Next, we introduce the classical definition of qualification for family of spectral regularization operators.

DEFINITION 4.1. *Let $\{R_\alpha\}_{\alpha \in (0,\alpha_0)}$ be the family of spectral regularization operators for $Tx = y$ generated by the family of functions $\{g_\alpha\}_{\alpha \in (0,\alpha_0)}$, and let us denote with $\mathcal{I}(g_\alpha)$ the set*

$$\mathcal{I}(g_\alpha) \doteq \{\mu \geq 0 : \exists\, k > 0 \text{ and } \lambda^\mu\, |1 - \lambda g_\alpha(\lambda)| \leq k\, \alpha^\mu\ \forall\, \lambda \in [0, \|T\|^2],\ \forall\, \alpha \in (0, \alpha_0)\}.$$

We define the order of the classical qualification of $\{R_\alpha\}_{\alpha \in (0,\alpha_0)}$ as $\mu_0 \doteq \sup_{\mu \in \mathcal{I}(g_\alpha)} \mu$ and we say that $\{R_\alpha\}_{\alpha \in (0,\alpha_0)}$ has classical qualification of order μ_0.

REMARK 4.2. *Note that by virtue of H2, $0 \in \mathcal{I}(g_\alpha)$ and the order μ_0 of the classical qualification of a regularization method is always nonnegative (it can even be equal to 0 or $+\infty$).*

4.1. Spectral methods with classical qualification of finite positive order. LEMMA 4.3. *Suppose that $\{g_\alpha\}_{\alpha \in (0,\alpha_0)}$ satisfies the hypotheses H1-H4. If the family of regularization operators $\{R_\alpha\}_{\alpha \in (0,\alpha_0)}$, with R_α defined as in (4.1), has classical qualification of order μ_0, $0 < \mu_0 < +\infty$, then $\psi_{\mu_0}(x, \delta) \doteq \delta^{\frac{2\mu_0}{2\mu_0+1}}$, for $x \in X_{\mu_0} \doteq \mathcal{R}((T^*T)^{\mu_0}) \setminus \{0\}$ and $\delta > 0$, is upper bound of convergence for the total error of $\{R_\alpha\}_{\alpha \in (0,\alpha_0)}$ in X_{μ_0}, that is, $\psi_{\mu_0} \in \mathcal{U}_{X_{\mu_0}}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$.*

Proof. Since $\{g_\alpha\}$ satisfies hypothesis *H4*, we have that $G_\alpha = O\left(\frac{1}{\sqrt{\alpha}}\right)$ when $\alpha \to 0^+$, and then $G_\alpha = o(\frac{1}{\alpha})$ when $\alpha \to 0^+$. From this and from the fact that $\{g_\alpha\}$ satisfies hypothesis *H1-H3* and $\{R_\alpha\}$ has classical qualification of order μ_0, $0 < \mu_0 < +\infty$, it follows that (see [1], Corollary 4.4 and Remark 4.5) there exists an *a-priori* parameter choice rule $\alpha^* : \mathbb{R}^+ \to (0, \alpha_0)$ such that the regularization method (R_α, α^*) is of optimal order on X_{μ_0}, that is, for every $x \in X_{\mu_0}$ there exists $k(x) > 0$ such that for every $\delta > 0$,

$$\sup_{y^\delta \in \overline{B_\delta(Tx)}} \left\| R_{\alpha^*(\delta)} y^\delta - x \right\| \leq k(x)\, \delta^{\frac{2\mu_0}{2\mu_0+1}}.$$

Then, for every $x \in X_{\mu_0}$ and for every $\delta > 0$

$$\inf_{\alpha \in (0,\alpha_0)} \sup_{y^\delta \in \overline{B_\delta(Tx)}} \left\| R_\alpha y^\delta - x \right\| \leq k(x)\, \delta^{\frac{2\mu_0}{2\mu_0+1}},$$

that is, $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \overset{X_{\mu_0}}{\preceq} \psi_{\mu_0}$, which implies that $\psi_{\mu_0} \in \mathcal{U}_{X_{\mu_0}}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$. ☐

THEOREM 4.4. *(Saturation for families of spectral regularization operators with classical qualification of finite positive order.)*

Suppose that $\{g_\alpha\}_{\alpha>0}$ satisfies hypotheses H1-H4. Suppose further that:

i) *The spectrum of T^*T has $\lambda = 0$ as accumulation point.*

ii) *There exist positive constants $\lambda_1 \leq \|T\|^2$ and γ_1, γ_2 and $c_1 > 1$ such that*

a) $0 \leq r_\alpha(\lambda) \leq 1$, $\alpha > 0$, $0 \leq \lambda \leq \lambda_1$;

b) $r_\alpha(\lambda) \geq \gamma_1$, $0 \leq \lambda < \alpha \leq \lambda_1$;

c) $|r_\alpha(\lambda)|$ *is monotone increasing with respect to α for $\lambda \in (0, \|T\|^2]$;*

d) $g_\alpha(c_1\alpha) \geq \frac{\gamma_2}{\alpha}$, $0 < c_1\alpha \leq \lambda_1$ *and*

e) $g_\alpha(\lambda) \geq g_\alpha(\tilde{\lambda})$, *for* $0 < \alpha \leq \lambda \leq \tilde{\lambda} \leq \lambda_1$.

iii) *The family of regularization operators $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ defined by (4.1), where $\alpha_0 \doteq \min\{\lambda_1, \frac{\lambda_1}{c}\}$, has classical qualification of order μ_0, $0 < \mu_0 < +\infty$.*

iv) *There exist constants $\gamma, c > 0$ such that*

$$\left(\frac{\lambda}{\alpha}\right)^{\mu_0} |r_\alpha(\lambda)| \geq \gamma, \quad \text{for every } 0 < c\alpha \leq \lambda \leq \|T\|^2, \tag{4.2}$$

where $r_\alpha(\lambda) \doteq 1 - \lambda g_\alpha(\lambda)$.

Then $\psi_{\mu_0}(x,\delta) \doteq \delta^{\frac{2\mu_0}{2\mu_0+1}}$ for $x \in X_{\mu_0} \doteq \mathcal{R}((T^*T)^{\mu_0}) \setminus \{0\}$ and $\delta > 0$, is saturation function of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on X_{μ_0}.

Note that the hypothesis ***i)*** is trivially satisfied if T is compact. To prove this theorem we will need two previous lemmas.

LEMMA 4.5. *Suppose that $\{g_\alpha\}_{\alpha>0}$ satisfies hypotheses H1-H4 and that hypotheses **ii.b)**, **ii.c)**, **iii)** and **vi)** of Theorem 4.4 also hold. Then for $\alpha \in (0, \alpha_0)$ the operator $r_\alpha(T^*T)$ is invertible, where $\alpha_0 \doteq \min\{\lambda_1, \frac{\lambda_1}{c}\}$.*

Proof. We will prove that for every $\alpha \in (0, \alpha_0)$ and for every $x \in X$, the function $r_\alpha^{-2}(\lambda)$ is integrable with respect to the measure $d\|E_\lambda x\|^2$. Let $\alpha \in (0, \alpha_0)$ be arbitrary but fixed. Since $\alpha_0 \leq \lambda_1$, it follows from hypothesis ***ii.b)*** that $r_\alpha(\lambda) \geq \gamma_1 > 0$ for every $\lambda \in [0, \alpha)$. Then

$$\int_0^\alpha \frac{1}{r_\alpha^2(\lambda)}\, d\|E_\lambda x\|^2 \leq \frac{\|x\|^2}{\gamma_1^2} < +\infty. \tag{4.3}$$

It remains to prove that $\int_\alpha^{\|T\|^2+} \frac{1}{r_\alpha^2(\lambda)}\, d\|E_\lambda x\|^2 < +\infty$. For that we shall consider two cases.

Case I: $c \leq 1$. In this case, for every $\lambda \in [\alpha, \|T\|^2]$ we have that $\lambda \geq \alpha \geq c\,\alpha > 0$ and from (4.2) it follows that $|r_\alpha(\lambda)| \geq \gamma\left(\frac{\alpha}{\lambda}\right)^{\mu_0}$ for every $\lambda \in [\alpha, \|T\|^2]$. Therefore

$$\int_\alpha^{\|T\|^2+} \frac{1}{r_\alpha^2(\lambda)}\, d\|E_\lambda x\|^2 \leq \int_\alpha^{\|T\|^2+} \frac{\lambda^{2\mu_0}}{(\alpha^{\mu_0}\gamma)^2}\, d\|E_\lambda x\|^2 \leq \frac{\|(T^*T)^{\mu_0} x\|^2}{(\alpha^{\mu_0}\gamma)^2} < +\infty.$$

Case II: $c > 1$. In this case we write

$$\int_\alpha^{\|T\|^2+} \frac{1}{r_\alpha^2(\lambda)}\, d\|E_\lambda x\|^2 = \int_\alpha^{c\,\alpha} \frac{1}{r_\alpha^2(\lambda)}\, d\|E_\lambda x\|^2 + \int_{c\,\alpha}^{\|T\|^2+} \frac{1}{r_\alpha^2(\lambda)}\, d\|E_\lambda x\|^2. \tag{4.4}$$

Like in the previous case, by virtue of (4.2), the second integral on the RHS of (4.4) is bounded above by $\frac{\|(T^*T)^{\mu_0}x\|^2}{(\alpha^{\mu_0}\gamma)^2} < +\infty$. For the first integral on the RHS of (4.4), by virtue of hypothesis ***ii.c)*** we have that

$$r_\alpha^2(\lambda) \geq r_{\alpha/c}^2(\lambda), \tag{4.5}$$

because $\frac{\alpha}{c} < \alpha$. On the other hand, again by using (4.2), and given that $0 < c(\frac{\alpha}{c}) \leq \lambda$ we have that

$$\left(\frac{\lambda}{\alpha/c}\right)^{2\mu_0} r_{\alpha/c}^2(\lambda) \geq \gamma^2. \tag{4.6}$$

From (4.5) and (4.6) we conclude that for every $\lambda \in [\alpha, c\,\alpha]$, $r_\alpha^2(\lambda) \geq \gamma^2 \left(\frac{\alpha}{c\,\lambda}\right)^{2\mu_0}$. Thus, for the first integral on the RHS of (4.4) we have the estimate

$$\int_\alpha^{c\,\alpha} \frac{1}{r_\alpha^2(\lambda)}\, d\,\|E_\lambda x\|^2 \leq \int_\alpha^{c\,\alpha} \frac{c^{2\mu_0}}{\alpha^{2\mu_0}\gamma^2}\lambda^{2\mu_0}\, d\,\|E_\lambda x\|^2 \leq \frac{c^{2\mu_0}}{\alpha^{2\mu_0}\gamma^2}\,\|(T^*T)^{\mu_0}x\|^2 < \infty.$$

Therefore, $r_\alpha(T^*T)$ is an invertible operator. □

LEMMA 4.6. *Suppose that $\{g_\alpha\}_{\alpha>0}$ satisfies the hypotheses H1-H4 and suppose further that hypotheses **ii.b)**, **ii.c)**, **iii)** and **vi)** of Theorem 4.4 hold. Let $\varphi : [0, \|T\|^2] \to \mathbb{R}^+$ be continuous, strictly increasing with $\varphi(0) = 0$. If for some $x^* \in X$, $x^* \neq 0$ we have that $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*, \delta) = o(\varphi(\delta))$ for $\delta \to 0^+$, then there exists an a priori parameter choice rule $\tilde{\alpha}(\delta)$ such that*

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \left\|R_{\tilde{\alpha}(\delta)}y^\delta - x^*\right\| = o(\varphi(\delta)) \quad \textit{for } \delta \to 0^+.$$

The same remains true if we replace $o(\varphi(\delta))$ by $O(\varphi(\delta))$.

Proof. Suppose that there exists $x^* \in X$, $x^* \neq 0$ such that $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*, \delta) = o(\varphi(\delta))$ for $\delta \to 0^+$. Then by definition of $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}$,

$$\lim_{\delta\to 0^+} \frac{\inf\limits_{\alpha\in(0,\alpha_0)} \sup\limits_{y^\delta \in \overline{B_\delta(Tx^*)}} \left\|R_\alpha y^\delta - x^*\right\|}{\varphi(\delta)} = \lim_{\delta\to 0^+} \inf_{\alpha\in(0,\alpha_0)} \frac{\sup\limits_{y^\delta \in \overline{B_\delta(Tx^*)}} \left\|R_\alpha y^\delta - x^*\right\|}{\varphi(\delta)} = 0. \tag{4.7}$$

For the sake of simplify we shall denote with:

$$f(\alpha, \delta) \doteq \frac{\sup\limits_{y^\delta \in \overline{B_\delta(Tx^*)}} \left\|R_\alpha y^\delta - x^*\right\|}{\varphi(\delta)} \quad \text{and} \quad h(\delta) \doteq \inf_{\alpha\in(0,\alpha_0)} f(\alpha, \delta).$$

Then $h(\delta) > 0$ for every $\delta \in (0, \infty)$ and (4.7) can be simply written as $\lim\limits_{\delta\to 0^+} h(\delta) = 0$. Next, for $n \in \mathbb{N}$ we define

$$\delta_n \doteq \sup\left\{\delta > 0 : h(\delta) \leq \frac{1}{n}\right\}.$$

Clearly, $\delta_n \downarrow 0$ and $h(\delta) = \inf\limits_{\alpha\in(0,\alpha_0)} f(\alpha, \delta) \leq \frac{1}{n}$ for every $\delta \in (0, \delta_n]$ for every $n \in \mathbb{N}$. Then, there exists $\alpha_n = \alpha_n(\delta_n) \in (0, \alpha_0)$ such that

$$f(\alpha_n, \delta) \leq \frac{2}{n} \quad \forall\, \delta \in (0, \delta_n],\ \forall\, n \in \mathbb{N}. \tag{4.8}$$

We then define $\alpha(\delta) \doteq \alpha_n$ for all $\delta \in (\delta_{n+1}, \delta_n]$ for every $n \in \mathbb{N}$. From this, from the fact that $\delta_n \downarrow 0$ and from (4.8) we deduce that $\lim_{\delta\to 0^+} f(\alpha(\delta),\delta) = \lim_{n\to+\infty} f(\alpha_n,\delta_n) = 0$.

We cannot guarantee the existence of the limit of $\alpha(\delta)$ for $\delta \to 0^+$. However, we will see next that $\alpha(\delta)$ can be replaced by a function $\tilde{\alpha} : \mathbb{R}^+ \to (0, \alpha_0)$ such that $\lim_{\delta\to 0^+} \tilde{\alpha}(\delta) = 0$ (i.e, such that $\tilde{\alpha}(\delta)$ is an admissible parameter choice rule) maintaining $\lim_{\delta\to 0^+} f(\tilde{\alpha}(\delta),\delta) = 0$. In fact, since $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,\alpha_0)$ is a bounded sequence of real numbers, it contains a convergent subsequence $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$, con $\alpha_{n_k} \to \alpha^*$ for $k \to +\infty$, $\alpha^* \in [0, \alpha_0]$. We define $\tilde{\alpha}(\delta) \doteq \alpha_{n_k}$ for all $\delta \in (\delta_{n_{k+1}}, \delta_{n_k}]$, for every $k \in \mathbb{N}$. Then,

$$\lim_{\delta\to 0^+} \tilde{\alpha}(\delta) = \lim_{k\to+\infty} \alpha_{n_k} = \alpha^*. \tag{4.9}$$

Since $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ and $\{\delta_{n_k}\}_{k\in\mathbb{N}}$ are subsequences of $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$, $\lim_{\delta\to 0^+} f(\tilde{\alpha}(\delta),\delta) = \lim_{k\to+\infty} f(\alpha_{n_k}, \delta_{n_k}) = 0$. Then, by definition of f,

$$\lim_{\delta\to 0^+} \frac{\sup\limits_{y^\delta \in B_\delta(Tx^*)} \left\| R_{\tilde{\alpha}(\delta)} y^\delta - x^* \right\|}{\varphi(\delta)} = 0,$$

that is,

$$\sup_{y^\delta \in B_\delta(Tx^*)} \left\| R_{\tilde{\alpha}(\delta)} y^\delta - x^* \right\| = o(\varphi(\delta)), \quad \text{as} \quad \delta \to 0^+. \tag{4.10}$$

It remains to be shown that $\alpha^* = 0$. If $\alpha^* > 0$, then it follows from (4.9) that there exists $\delta_0 > 0$ such that $\tilde{\alpha}(\delta) > \frac{\alpha^*}{2}$ for all $\delta \in (0, \delta_0)$. Hypothesis ***ii.c)*** of Theorem 4.4 implies then that for every $\delta \in (0,\delta_0)$, $\left| r_{\tilde{\alpha}(\delta)}(\lambda) \right| \geq \left| r_{\frac{\alpha^*}{2}}(\lambda) \right|$ for all $\lambda \in (0, \|T\|^2]$. Then, for every $\delta \in (0, \delta_0)$,

$$\begin{aligned}
\left\| r_{\tilde{\alpha}(\delta)}(T^*T)x^* \right\|^2 &= \int_0^{\|T\|^2+} r^2_{\tilde{\alpha}(\delta)}(\lambda)\, d\left\| E_\lambda x^* \right\|^2 \\
&\geq \int_0^{\|T\|^2+} r^2_{\frac{\alpha^*}{2}}(\lambda)\, d\left\| E_\lambda x^* \right\|^2 \\
&= \left\| r_{\frac{\alpha^*}{2}}(T^*T)x^* \right\|^2.
\end{aligned}$$

Now, for all $\delta \in (0, \delta_0)$,

$$\begin{aligned}
\sup_{y^\delta \in B_\delta(Tx^*)} \left\| R_{\tilde{\alpha}(\delta)} y^\delta - x^* \right\| &\geq \left\| R_{\tilde{\alpha}(\delta)} Tx^* - x^* \right\| = \left\| (I - g_{\tilde{\alpha}(\delta)}(T^*T)T^*T)x^* \right\| \\
&= \left\| r_{\tilde{\alpha}(\delta)}(T^*T)x^* \right\| \geq \left\| r_{\frac{\alpha^*}{2}}(T^*T)x^* \right\|.
\end{aligned}$$

Taking limit for $\delta \to 0^+$, it follows that

$$\lim_{\delta\to 0^+} \sup_{y^\delta \in B_\delta(Tx^*)} \left\| R_{\tilde{\alpha}(\delta)} y^\delta - x^* \right\| \geq \left\| r_{\frac{\alpha^*}{2}}(T^*T)x^* \right\|,$$

which, together with (4.10) implies that $\left\| r_{\frac{\alpha^*}{2}}(T^*T)x^* \right\| = 0$. But since $\frac{\alpha^*}{2} < \alpha_0$, it follows from Lemma 4.5 that $r_{\frac{\alpha^*}{2}}(T^*T)$ is invertible and therefore $x^* = 0$, which is a contradiction since x^* was not zero. Hence, α^* must be equal to zero, as wanted.

We proceed now to prove the second part of the Lemma. Suppose that there exists $x^* \in X$, $x^* \neq 0$ such that $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta) = O(\varphi(\delta))$ as $\delta \to 0^+$. Then there exist positive constants k and d such that $\inf\limits_{\alpha\in(0,\alpha_0)} f(\alpha,\delta) \le k$ for every $\delta \in (0,d)$. Let $\{\delta_n\}_{n\in\mathbb{N}} \subset (0,d)$ be such that $\delta_n \downarrow 0$ and $\alpha_n = \alpha_n(\delta_n) \in (0,\alpha_0)$ such that

$$f(\alpha_n,\delta) \le k + \delta_n,\ \forall\,\delta \in (0,d),\ \forall\, n \in \mathbb{N}.$$

We define (just like we did it previously for the case "o") $\alpha(\delta) \doteq \alpha_n$ for all $\delta \in (\delta_{n+1},\delta_n]$ for every $n \in \mathbb{N}$. Since $\delta_n \downarrow 0$ it follows that $f(\alpha(\delta),\delta) \le k + \delta_1$ for every $\delta \in (0,d)$ and therefore

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \left\| R_{\alpha(\delta)} y^\delta - x^* \right\| = O(\varphi(\delta)) \quad \text{as } \delta \to 0^+. \tag{4.11}$$

Exactly in the same way as we proceeded before in the first part of the proof, by defining the function $\tilde\alpha(\delta)$ (from a convergent subsequence of $\{\alpha_n\}_{n\in\mathbb{N}}$), equation (4.11) is proved with $\tilde\alpha(\delta)$ in place of $\alpha(\delta)$. Finally, and also by proceeding in an analogous way, it is shown that $\tilde\alpha(\delta)$ converges to zero as $\delta \to 0^+$, i.e. that $\tilde\alpha(\delta)$ is an admissible parameter choice rule. Since the steps are the same we do not give details here. ◻

We are now in condition of proving Theorem 4.4.

Proof of Theorem 4.4. We will show that $\psi_{\mu_0}(x,\delta) \doteq \delta^{\frac{2\mu_0}{2\mu_0+1}}$ for $x \in X_{\mu_0}$ and $\delta > 0$, is saturation function of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on X_{μ_0}.

By virtue of Lemma 4.3 we have that $\psi_{\mu_0} \in \mathcal{U}_{X_{\mu_0}}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$. Next we will show that ψ_{μ_0} satisfies condition *S1* (see Definition 3.1) of saturation on X_{μ_0}. Suppose that is not true, i.e., suppose that there exist $x^* \in X$, $x^* \neq 0$ and $x \in X_{\mu_0}$ such that $\limsup\limits_{\delta\to 0^+} \frac{\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta)}{\psi(x,\delta)} = 0$. Then $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta) = o\left(\delta^{\frac{2\mu_0}{2\mu_0+1}}\right)$ as $\delta \to 0^+$ and from Lemma 4.6 it follows that there exists an *a-priori* admissible parameter choice rule $\alpha(\delta)$ such that

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \left\| R_{\alpha(\delta)} y^\delta - x^* \right\| = o(\delta^{\frac{2\mu_0}{2\mu_0+1}}) \quad \text{ for } \delta \to 0^+.$$

Note that hypothesis *H4* implies that there exists a finite positive constant β such that $\sqrt{\lambda}\,|g_\alpha(\lambda)| \le \frac{\beta}{\sqrt{\alpha}}$, for every $\alpha \in (0,\alpha_0)$ and for every $\lambda \in [0,\|T\|^2]$. Since $\{g_\alpha\}$ satisfies the hypotheses *H1-H4* and ***i)-iv)*** hold, it follows from Theorem 3.1 of [5] that $x^* = 0$, which contradicts the fact that x^* was different from zero. Hence, ψ_{μ_0} satisfies condition *S1* on X_{μ_0}.

Since ψ_{μ_0} does not depend on x, we further have that ψ_{μ_0} is (trivially) invariant over X_{μ_0}, i.e., it satisfies condition *S2*.

It only remains to prove that ψ_{μ_0} satisfies condition *S3*, that is, that the set X_{μ_0} is optimal for ψ_{μ_0}. Suppose not. Then there must exist $M \supsetneqq X_{\mu_0}$ and $\tilde\psi \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ such that $\tilde\psi\,|_{X_{\mu_0}} = \psi_{\mu_0}$ and $\tilde\psi$ satisfies *S1* and *S2* on M. Let $x^* \in M \setminus X_{\mu_0}$, $x^* \neq 0$. Since $\tilde\psi \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ we have that

$$\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \overset{\{x^*\}}{\preceq} \tilde\psi. \tag{4.12}$$

Since $\tilde\psi$ is invariant over M, we have that $\tilde\psi \overset{\{x^*\},X_{\mu_0}}{\preceq} \tilde\psi$ and since $\tilde\psi$ coincides with ψ_{μ_0} on X_{μ_0}, it follows that $\tilde\psi \overset{\{x^*\},X_{\mu_0}}{\preceq} \psi_{\mu_0}$. This, together with (4.12) implies that

$\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}} \overset{\{x^*\},X_{\mu_0}}{\preceq} \psi_{\mu_0}$ and therefore $\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}(x^*,\delta) = O\left(\delta^{\frac{2\mu_0}{2\mu_0+1}}\right)$ as $\delta \to 0^+$. Lemma 4.6 then implies that there exists an *a-priori* admissible parameter choice rule $\alpha(\delta)$ such that

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \left\| R_{\alpha(\delta)} y^\delta - x^* \right\| = O\left(\delta^{\frac{2\mu_0}{2\mu_0+1}}\right) \quad \text{as } \delta \to 0^+.$$

Since $\mu_0 < +\infty$ it follows that $x^* \in \mathcal{R}((T^*T)^{\mu_0})$ (see [5], Corollary 2.6) and since $x^* \neq 0$, we have that $x^* \in X_{\mu_0}$ which contradicts that $x^* \in M \setminus X_{\mu_0}$. Thus, ψ_{μ_0} satisfies condition *S3* and ψ_{μ_0} is saturation function of $\{R_\alpha\}$ on X_{μ_0}, as we wanted to prove.

4.2. Spectral Methods with maximal qualification. The concept of classical qualification is a special case of a more general definition of qualification introduced by Mathé and Pereverzev ([2], [4]).

DEFINITION 4.7. *Let $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ be a family of spectral regularization operators for $Tx = y$ generated by the family of functions $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ and let $r_\alpha(\lambda) \doteq 1 - \lambda g_\alpha(\lambda)$. We say that a function $\rho : (0, \|T\|^2] \to \mathbb{R}^+$ is qualification of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ if ρ is increasing and there exists a constant $\gamma > 0$ such that*

$$\sup_{\lambda\in(0,\|T\|^2]} |r_\alpha(\lambda)|\ \rho(\lambda) \leq \gamma\, \rho(\alpha) \quad \textit{for every } \alpha \in (0, \alpha_0).$$

If, moreover, for every $\lambda \in (0, \|T\|^2]$ there exists a constant $c \doteq c(\lambda) > 0$ such that

$$\inf_{\alpha\in(0,\alpha_0)} \frac{|r_\alpha(\lambda)|}{\rho(\alpha)} \geq c$$

then we say that ρ is a maximal qualification $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$.

Note then that the classical qualification of order μ corresponds to the case in which the functions ρ are restricted to monomials $\rho(t) = t^\mu$ for $0 \leq \mu < +\infty$.

These two definitions of qualification are related. For instance, if a spectral regularization method $\{R_\alpha\}$ possesses classical qualification of order $\mu_0 < \infty$, then any increasing function $\tilde\rho : (0, \|T\|^2] \to \mathbb{R}^+$ satisfying $\alpha^{\mu_0} \leq k\,\tilde\rho(\alpha)$ for α in a neighborhood of $\alpha = 0$ and for some constant $k > 0$, is also qualification of $\{R_\alpha\}$. Also, if α^{μ_0} y $\tilde\rho(\alpha)$ are two maximal qualifications then they are necessarily equivalent in the sense that there exist constants $k, \tilde k > 0$ such that $k\,\alpha^{\mu_0} \leq \tilde\rho(\alpha) \leq \tilde k\,\alpha^{\mu_0}$. On the other hand, if a spectral regularization method $\{R_\alpha\}$ has classical qualification of infinite order, then not necessarily has maximal qualification.

Next, we will show that under certain general hypotheses, it is also possible to characterize the saturation of spectral regularization methods possessing maximal qualification. For that we will previously need the following definition.

DEFINITION 4.8. *Let $\rho : (0, a] \to (0, +\infty)$ be a continuous non-decreasing function such that $\lim_{t\to 0^+} \rho(t) = 0$ and $\beta \in \mathbb{R}$, $\beta \geq 0$. We say that ρ is of local upper type β if there exists a positive constant d such that $\rho(t) \leq d\,(\frac{1}{s})^\beta \rho(s\,t)$ for every $s \in (0, 1]$, $t \in (0, a]$.*

A function of finite upper type is also said to satisfy a Δ_2 condition.

THEOREM 4.9. *(Saturation for families of spectral regularization operators with maximal qualification.)*

Let T be a compact linear operator. Suppose that $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ satisfies hypotheses H1-H4 and let $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ be defined by (4.1). Suppose further that the following hypotheses are satisfied:

i) *There exist $\{\tilde\lambda_k\}_{k=1}^\infty \subset \sigma_p(TT^*)$ and $c \ge 1$ such that $\tilde\lambda_k \downarrow 0$ and $\frac{\tilde\lambda_k}{\tilde\lambda_{k+1}} \le c$ for every $k \in \mathbb{N}$.*

ii) *There exist positive constants $\lambda_1 \le \|T\|^2$, γ_1, γ_2 and $c_1 > 1$ such that*

a) $0 \le r_\alpha(\lambda) \le 1$, $\alpha > 0$, $0 \le \lambda \le \lambda_1$;

b) $r_\alpha(\lambda) \ge \gamma_1$, $0 \le \lambda < \alpha \le \lambda_1$;

c) *$|r_\alpha(\lambda)|$ is monotone increasing with respect to α for $\lambda \in (0, \|T\|^2]$;*

d) *$g_\alpha(c_1\alpha) \ge \frac{\gamma_2}{\alpha}$, $0 < c_1\alpha \le \lambda_1$ and*

e) *$g_\alpha(\lambda) \ge g_\alpha(\tilde\lambda)$, for $0 < \alpha \le \lambda \le \tilde\lambda \le \lambda_1$.*

iii) *There exists $\rho : (0, \|T\|^2] \to (0, +\infty)$, strictly increasing and of local upper type β, for some $\beta \ge 0$, such that ρ is maximal qualification of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ and satisfies:*

a) *there exist positive constants a and k such that*

$$\frac{\rho(\lambda)\,|r_\alpha(\lambda)|}{\rho(\alpha)} \ge a, \quad \text{for all } \alpha,\ \lambda \text{ such that } 0 < k\,\alpha \le \lambda \le \|T\|^2 .$$

iv) *There exists a constant $b > 0$ such that*

$$\sup_{\lambda\in(0,\|T\|^2]} \sqrt{\lambda}\, |g_\alpha(\lambda)| \ge \frac{b}{\sqrt{\alpha}} \text{ for every } \alpha \in (0, \alpha_0).$$

v) *For every $\alpha \in (0, \alpha_0)$ the function $\lambda \to |r_\alpha(\lambda)|^2$, $\lambda \in (0, \|T\|^2]$ is convex.*

Let $\Theta(t) \doteq \sqrt{t}\rho(t)$ for $t \in (0, \|T\|^2]$. Then $\psi(x, \delta) \doteq (\rho \circ \Theta^{-1})(\delta)$ for $x \in X^\rho \doteq \mathcal{R}(\rho(T^*T)) \setminus \{0\}$ and $\delta \in (0, \Theta(\alpha_0))$, is saturation function of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on X^ρ.

In order to prove this theorem we will previously need two converse results that we establish in the following two Lemmas.

Lemma 4.10. *Let $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ be a family of spectral regularization operators for $Tx = y$ and $\rho : (0, \|T\|^2] \to \mathbb{R}^+$ a strictly increasing continuous function satisfying hypothesis **iii.a)** of Theorem 4.9. If for some $x \in X$, $\|R_\alpha Tx - x\| = O(\rho(\alpha))$ for $\alpha \to 0^+$, then $x \in \mathcal{R}(\rho(T^*T))$.*

Proof. From hypothesis ***iii.a)*** of Theorem 4.9 it follows that

$$\|R_\alpha Tx - x\|^2 = \int_0^{\|T\|^2+} r_\alpha^2(\lambda)\, d\,\|E_\lambda x\|^2 \ge a^2\,\rho^2(\alpha) \int_{k\,\alpha}^{\|T\|^2+} \rho^{-2}(\lambda)\, d\,\|E_\lambda x\|^2 . \tag{4.13}$$

Since $\|R_\alpha Tx - x\| = O(\rho(\alpha))$ for $\alpha \to 0^+$, it then follows that there are constants $C > 0$ and α^*, $0 < \alpha^* \le \alpha_0$ such that

$$\int_{k\,\alpha}^{\|T\|^2+} \rho^{-2}(\lambda)\, d\,\|E_\lambda x\|^2 \le \frac{\|R_\alpha Tx - x\|^2}{a^2\rho^2(\alpha)} \le \frac{C^2}{a^2} \quad \text{for every } \alpha \in (0, \alpha^*).$$

Taking limit for $\alpha \to 0^+$ it follows that $\int_0^{\|T\|^2+} \rho^{-2}(\lambda)\, d\,\|E_\lambda x\|^2 < +\infty$, from which we deduce that $w \doteq \int_0^{\|T\|^2+} \rho^{-1}(\lambda)\, dE_\lambda x \in X$. Then,

$$\rho(T^*T)w = \int_0^{\|T\|^2+} \rho(\lambda)\rho^{-1}(\lambda)\, dE_\lambda x = x$$

and therefore $x \in \mathcal{R}(\rho(T^*T))$. ∎

LEMMA 4.11. *Under the same hypotheses of Theorem 4.9, if for some $x \in X$ we have that*

$$\sup_{y^\delta \in \overline{B_\delta(Tx)}} \inf_{\alpha \in (0,\alpha_0)} \left\| R_\alpha y^\delta - x \right\| = O(\rho(\Theta^{-1}(\delta))) \quad \text{when } \delta \to 0^+, \tag{4.14}$$

then $x \in \mathcal{R}(\rho(T^*T))$.

Proof. Without loss of generality we suppose that $\alpha_0 \doteq \min\{\frac{\lambda_1}{c_1}, \frac{\lambda_1}{k}\}$ and that $x \neq 0$ (if $x = 0$ the result is trivial).

Let $\bar\lambda \in \sigma_p(TT^*)$ be such that $0 < c_1\,\bar\lambda \leq \lambda_1$ (the compactness of T guarantees the existence of such $\bar\lambda$), and define

$$\bar\delta \doteq \frac{\bar\lambda^{1/2}}{\gamma_2} \left\| R_{\bar\lambda} Tx - x \right\|.$$

Then, clearly the equation

$$\left\| R_\alpha Tx - x \right\|^2 = \frac{(\gamma_2\,\bar\delta)^2}{\alpha} \tag{4.15}$$

has $\alpha = \bar\lambda$ as a solution. Moreover, from the hypothesis ***ii.c)*** and given that $x \neq 0$, it follows that $\alpha = \bar\lambda$ is the unique solution of (4.15). Note also that $\bar\delta \to 0^+$ if and only if $\bar\lambda \to 0^+$.

Now, given $\delta > 0$ and $\bar\lambda \in \sigma_p(TT^*)$, $c_1\bar\lambda \leq \lambda_1$, we define

$$\bar y^{\,\delta} \doteq Tx - \delta G_{\bar\lambda} z, \quad \forall\, \delta > 0, \tag{4.16}$$

with

$$z \doteq \begin{cases} Tx \left\| G_{\bar\lambda} Tx \right\|^{-1}, & \text{if } G_{\bar\lambda} Tx \neq 0 \\ \text{arbitrary with } \left\| G_{\bar\lambda} z \right\| = 1, & \text{in other case} \end{cases}$$

where $G_{\bar\lambda} \doteq F_{c_1\,\bar\lambda} - F_{\bar\lambda}$ and $\{F_\lambda\}$ is the spectral family associated to TT^*. Since $\bar\lambda \in \sigma_p(TT^*)$ it follows that $G_{\bar\lambda}$ is not the null operator and therefore the definition makes sense. Note that $\left\| \bar y^{\,\delta} - Tx \right\| = \delta$, which implies that $\bar y^{\,\delta} \in \overline{B_\delta(Tx)}$.

Now, from (4.1), (4.16) and from the fact that $g_\alpha(T^*T)T^* = T^* g_\alpha(TT^*)$ it follows that for every $\alpha \in (0, \alpha_0)$ and $\delta > 0$,

$$\begin{aligned}
\left\langle R_\alpha Tx - x\ , R_\alpha(\bar y^{\,\delta} - Tx) \right\rangle &= \langle g_\alpha(T^*T)T^*Tx - x, -g_\alpha(T^*T)T^*\,\delta G_{\bar\lambda} z\rangle \\
&= \delta\,\langle g_\alpha(T^*T)T^*Tx - x, -T^* g_\alpha(TT^*) G_{\bar\lambda} z\rangle \\
&= \delta\,\langle T g_\alpha(T^*T)T^*Tx - Tx, -g_\alpha(TT^*) G_{\bar\lambda} z\rangle \\
&= \delta\,\langle (TT^* g_\alpha(TT^*) - I)Tx, -g_\alpha(TT^*) G_{\bar\lambda} z\rangle \\
&= \delta\,\langle -r_\alpha(TT^*)Tx, -g_\alpha(TT^*) G_{\bar\lambda} z\rangle \\
&= \delta \int_0^{\|T\|^2+} r_\alpha(\lambda) g_\alpha(\lambda)\, d\,\langle F_\lambda Tx, G_{\bar\lambda} z\rangle\,.
\end{aligned} \tag{4.17}$$

Since $c_1\bar\lambda \leq \lambda_1$, it follows from hypothesis ***ii.a)*** that both $g_\alpha(\lambda)$ and $r_\alpha(\lambda)$ are nonnegative for all $\lambda \in [0, c_1\bar\lambda]$. On the other hand, from the definitions of $G_{\bar\lambda}$ and z it follows immediately that the function $h(\lambda) \doteq \langle F_\lambda Tx, G_{\bar\lambda} z\rangle$ is real and non-decreasing and therefore

$$\int_0^{c_1\bar\lambda+} r_\alpha(\lambda) g_\alpha(\lambda)\, d\,\langle F_\lambda Tx, G_{\bar\lambda} z\rangle \geq 0. \tag{4.18}$$

On the other hand, since $h(\lambda) = \langle Tx, F_\lambda G_{\bar\lambda} z\rangle$ and $F_\lambda G_{\bar\lambda} = G_{\bar\lambda}$ for every $\lambda \geq c_1\bar\lambda$, it follows that $h(\lambda)$ is constant for every $\lambda \geq c_1\bar\lambda$ and therefore

$$\int_{c_1\bar\lambda+}^{\|T\|^2+} r_\alpha(\lambda) g_\alpha(\lambda)\, d\,\langle F_\lambda Tx, G_{\bar\lambda} z\rangle = 0. \tag{4.19}$$

From (4.18) and (4.19) we have that

$$\int_0^{\|T\|^2+} r_\alpha(\lambda) g_\alpha(\lambda)\, d\,\langle F_\lambda Tx, G_{\bar\lambda} z\rangle \geq 0,$$

which, by virtue of (4.17), implies that

$$\left\langle R_\alpha Tx - x, R_\alpha(\bar y^{\,\delta} - Tx)\right\rangle \geq 0. \tag{4.20}$$

By using once again (4.1) and (4.16) together with (4.20) it then follows that for every $\alpha \in (0, \alpha_0)$, for every $\bar\lambda \in \sigma_p(TT^*)$ such that $c_1\bar\lambda \leq \lambda_1$ and for every $\delta > 0$,

$$\begin{aligned}
\left\|R_\alpha \bar y^{\,\delta} - x\right\|^2 &= \|R_\alpha Tx - x\|^2 + \left\|R_\alpha(\bar y^{\,\delta} - Tx)\right\|^2 + 2\left\langle R_\alpha Tx - x, R_\alpha(\bar y^{\,\delta} - Tx)\right\rangle \\
&= \|R_\alpha Tx - x\|^2 + \delta^2 \left\|g_\alpha(T^*T)T^* G_{\bar\lambda} z\right\|^2 + 2\left\langle R_\alpha Tx - x, R_\alpha(\bar y^{\,\delta} - Tx)\right\rangle \\
&\geq \|R_\alpha Tx - x\|^2 + \delta^2 \left\|g_\alpha(T^*T)T^* G_{\bar\lambda} z\right\|^2 \\
&= \|R_\alpha Tx - x\|^2 + \delta^2 \int_0^{+\infty} \lambda\, g_\alpha^2(\lambda)\, d\,\|F_\lambda G_{\bar\lambda} z\|^2 \\
&\geq \|R_\alpha Tx - x\|^2 + \delta^2 \int_{\bar\lambda}^{c_1\,\bar\lambda} \lambda\, g_\alpha^2(\lambda)\, d\,\|F_\lambda G_{\bar\lambda} z\|^2 .
\end{aligned} \tag{4.21}$$

We now consider two different possible cases.

<u>Case I</u>: $\alpha \leq \bar\lambda$. Since $c_1\bar\lambda \leq \lambda_1$ and $c_1 > 1$, it follows from hypothesis ***ii.e)*** that

$$g_\alpha(\lambda) \geq g_\alpha(c_1\bar\lambda) \geq g_\alpha(\lambda_1) \quad \text{for every} \quad \lambda \in [\bar\lambda, c_1\,\bar\lambda]. \tag{4.22}$$

On the other hand, from hypothesis ***ii.a)*** it follows that $r_\alpha(\lambda_1) \leq 1$, which implies that $\lambda_1\, g_\alpha(\lambda_1) \geq 0$ and therefore, $g_\alpha(\lambda_1) \geq 0$. It then follows from (4.22) that $g_\alpha^2(\lambda) \geq g_\alpha^2(c_1\,\bar\lambda)$ for every $\lambda \in [\bar\lambda, c_1\,\bar\lambda]$. Then,

$$\begin{aligned}
\int_{\bar\lambda}^{c_1\,\bar\lambda} \lambda\, g_\alpha^2(\lambda)\, d\,\|F_\lambda G_{\bar\lambda} z\|^2 &\geq \bar\lambda\, g_\alpha^2(c_1\,\bar\lambda) \int_{\bar\lambda}^{c_1\,\bar\lambda} d\,\|F_\lambda G_{\bar\lambda} z\|^2 \\
&= \bar\lambda\, g_\alpha^2(c_1\,\bar\lambda),
\end{aligned} \tag{4.23}$$

where the last equality follows from the fact that $\int_{\bar\lambda}^{c_1\,\bar\lambda} d\,\|F_\lambda G_{\bar\lambda} z\|^2 = 1$, which is a consequence of the fact that $\int_{\bar\lambda}^{c_1\,\bar\lambda} d\,\|F_\lambda G_{\bar\lambda} z\|^2 = \left\|F_{c_1\,\bar\lambda} G_{\bar\lambda} z\right\|^2 - \|F_{\bar\lambda} G_{\bar\lambda} z\|^2$, from the definition of $G_{\bar\lambda}$, from the fact that $F_\lambda F_\mu = F_{\min\{\lambda,\mu\}}$ for every $\lambda, \mu \in \mathbb{R}$ and the fact that $\|G_{\bar\lambda} z\| = 1$.

At the same time, the hypotheses ***ii.a)*** and ***ii.c)*** imply that $g_\alpha(\lambda)$ is monotone decreasing with respect to α for every $\lambda \in [0, \lambda_1]$. Since $\alpha \leq \bar\lambda$ and $c_1\,\bar\lambda \leq \lambda_1$, we then have that

$$g_\alpha(c_1\,\bar\lambda) \geq g_{\bar\lambda}(c_1\,\bar\lambda), \tag{4.24}$$

and from hypothesis ***ii.d)*** that

$$g_{\bar\lambda}(c_1\,\bar\lambda) \geq \gamma_2/\bar\lambda > 0. \tag{4.25}$$

From (4.24) and (4.25) we conclude that

$$g^2_\alpha(c_1\,\bar\lambda) \geq \left(\gamma_2/\bar\lambda\right)^2. \tag{4.26}$$

Substituting (4.26) into (4.23) we obtain

$$\int_{\bar\lambda}^{c_1\,\bar\lambda} \lambda\, g^2_\alpha(\lambda)\, d\left\|F_\lambda G_{\bar\lambda} z\right\|^2 \geq \gamma_2^2/\bar\lambda,$$

which, together with (4.21) imply that if $\alpha \leq \bar\lambda$, then $\left\|R_\alpha \bar y^{\,\delta} - x\right\|^2 \geq (\gamma_2\,\delta)^2/\bar\lambda$.

Case II: $\alpha > \bar\lambda$. In this case, it follows from hypothesis ***ii.c)*** that $r^2_\alpha(\lambda) \geq r^2_{\bar\lambda}(\lambda)$ for every $\lambda \in (0, \|T\|^2]$. Then,

$$\|R_\alpha Tx - x\|^2 = \int_0^{+\infty} r^2_\alpha(\lambda)\, d\,\|E_\lambda x\|^2 \geq \int_0^{+\infty} r^2_{\bar\lambda}(\lambda)\, d\,\|E_\lambda x\|^2 = \|R_{\bar\lambda} Tx - x\|^2,$$

which, together with (4.21) imply that $\left\|R_\alpha \bar y^{\,\delta} - x\right\|^2 \geq \|R_{\bar\lambda} Tx - x\|^2$.

Summarizing the results obtained in cases I and II, we can write:

$$\begin{aligned} \left\|R_\alpha \bar y^{\,\delta} - x\right\|^2 &\geq \begin{cases} \|R_{\bar\lambda} Tx - x\|^2, & \text{if } \alpha > \bar\lambda \\ (\gamma_2\,\delta)^2/\bar\lambda, & \text{if } \alpha \leq \bar\lambda. \end{cases} \\ &\geq \min\{\|R_{\bar\lambda} Tx - x\|^2, (\gamma_2\,\delta)^2/\bar\lambda\}, \end{aligned} \tag{4.27}$$

which is valid for every $\alpha \in (0, \alpha_0)$, $\bar\lambda \in \sigma_p(TT^*)$ such that $c_1\bar\lambda \leq \lambda_1$ and for every $\delta > 0$. Then

$$\begin{aligned} \min\left\{\|R_{\bar\lambda} Tx - x\|, \gamma_2\,\delta/\sqrt{\bar\lambda}\right\} &= \left(\min\{\|R_{\bar\lambda} Tx - x\|^2, (\gamma_2\,\delta)^2/\bar\lambda\}\right)^{1/2} \\ &\leq \inf_{\alpha\in(0,\alpha_0)} \left\|R_\alpha \bar y^{\,\delta} - x\right\| \qquad \text{(by (4.27))} \\ &\leq \sup_{y^\delta \in \overline{B_\delta(Tx)}} \inf_{\alpha\in(0,\alpha_0)} \left\|R_\alpha y^\delta - x\right\| \qquad (\text{since } \bar y^{\,\delta} \in \overline{B_\delta(Tx)}) \\ &= O(\rho(\Theta^{-1}(\delta))) \quad \text{for} \quad \delta \to 0^+ \quad \text{(by hypothesis)}. \end{aligned}$$

Now, given that $\bar\lambda = \alpha(\bar\delta)$ solves equation (4.15), from the previous inequality we have that

$$\left\|R_{\alpha(\bar\delta)} Tx - x\right\| = \gamma_2\,\bar\delta/\sqrt{\bar\lambda} = O(\rho(\Theta^{-1}(\bar\delta))) \quad \text{for} \quad \bar\delta \to 0^+, \tag{4.28}$$

which implies that

$$\frac{\bar\delta}{\rho(\Theta^{-1}(\bar\delta))} = O\left(\sqrt{\alpha(\bar\delta)}\right) \quad \text{for} \quad \bar\delta \to 0^+. \tag{4.29}$$

Since $\delta = \Theta(\Theta^{-1}(\delta))$ it follows from the definition of Θ that $\delta = \sqrt{\Theta^{-1}(\delta)}\rho(\Theta^{-1}(\delta))$. Then, it follows from (4.29) that $\sqrt{\Theta^{-1}(\bar{\delta})} = O(\sqrt{\alpha(\bar{\delta})})$ for $\bar{\delta} \to 0^+$. From this and (4.28) we then deduce that:

$$\left\| R_{\alpha(\bar{\delta})}Tx - x \right\| = O(\rho(\alpha(\bar{\delta}))) \quad \text{for } \bar{\delta} \to 0^+ \ \forall \ \alpha(\bar{\delta}) \in \sigma_p(TT^*) : c_1\, \alpha(\bar{\delta}) \le \lambda_1. \quad (4.30)$$

Now, let $\alpha \in \mathbb{R}^+$ such that $\alpha \le \max_{k\in\mathbb{N}}\{\tilde{\lambda}_k : \tilde{\lambda}_k \le \frac{\lambda_1}{c_1}\}$. Then, there exist $\tilde{\lambda}_k$, $\tilde{\lambda}_{k+1} \in \sigma_p(TT^*)$ such that $\frac{\tilde{\lambda}_k}{\tilde{\lambda}_{k+1}} \le c$, $\tilde{\lambda}_{k+1} < \alpha \le \tilde{\lambda}_k$ and $(\tilde{\lambda}_{k+1}, \tilde{\lambda}_k) \cap \sigma_p(TT^*) = \emptyset$.

From hypothesis ***ii.c)*** and the fact that $\tilde{\lambda}_k \in \sigma_p(TT^*)$ and $\tilde{\lambda}_k \le \frac{\lambda_1}{c_1}$ it follows that

$$\begin{aligned} \|R_\alpha Tx - x\|^2 &= \int_0^{+\infty} r_\alpha^2(\lambda)\, d\,\|E_\lambda x\|^2 \\ &\le \int_0^{+\infty} r_{\tilde{\lambda}_k}^2(\lambda)\, d\,\|E_\lambda x\|^2 \\ &= \left\| R_{\tilde{\lambda}_k} Tx - x \right\|^2 \\ &= O(\rho^2(\tilde{\lambda}_k)), \quad \text{(by virtue of (4.30))}. \end{aligned} \quad (4.31)$$

From hypothesis ***i)*** we have that $\tilde{\lambda}_k \le c\,\tilde{\lambda}_{k+1}$ and since ρ is strictly increasing and non-negative it follows that (for all k big enough, more precisely for all k such that $c\,\tilde{\lambda}_{k+1} \le \|T\|^2$)

$$\rho^2(\tilde{\lambda}_k) \le \rho^2(c\,\tilde{\lambda}_{k+1}). \quad (4.32)$$

Given that $c \ge 1$ and ρ is of local upper type β for some $\beta \ge 0$ (hypothesis ***iii)***), there exists a positive constant d such that

$$\rho(c\,\tilde{\lambda}_{k+1}) \le d\, c^\beta \rho\left(\frac{1}{c}\, c\,\tilde{\lambda}_{k+1}\right) \le d\, c^\beta \rho(\tilde{\lambda}_{k+1}). \quad (4.33)$$

From (4.31), (4.32), (4.33) and from the fact that $\rho^2(\tilde{\lambda}_{k+1}) < \rho^2(\alpha)$ it follows that $\|R_\alpha Tx - x\| = O(\rho(\alpha))$ for $\alpha \to 0^+$. Therefore, Lemma 4.10 now implies that $x \in \mathcal{R}(\rho(T^*T))$. This concludes the proof of the Lemma. □

From the definition of qualification it follows that

$$\|R_\alpha Tx - x\|^2 \le \gamma^2\, \rho^2(\alpha) \int_0^{+\infty} \rho^{-2}(\lambda)\, d\,\|E_\lambda x\|^2 .$$

Therefore, in the previous Lemma, the hypotheses ***i)*** and that ρ be of local upper type β for some $\beta \ge 0$ can be substituted by the requirement that $\rho(T^*T)$ be invertible, or equivalently, that $\rho^{-2}(\lambda)$ be integrable with respect to the measure $d\,\|E_\lambda x\|^2$ for every $x \in X$.

Proof of Theorem 4.9. Without loss of generality we suppose that $\alpha_0 \doteq \min\{\frac{\lambda_1}{c_1}, \frac{\lambda_1}{k}\}$. First we will prove that $\psi(x,\delta) \doteq (\rho \circ \Theta^{-1})(\delta)$ for $x \in X^\rho$ and $\delta \in (0, \Theta(\alpha_0))$, is an upper bound of convergence for the total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ in X^ρ, that is, we will show that $\psi \in \mathcal{U}_{X^\rho}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$. For every $r \ge 0$ we define the source sets

$X^{\rho,r} \doteq \{x \in X : x = \rho(T^*T)\xi,\ \|\xi\| \le r\}$. Let $x \in X^\rho$, then there exists $r \ge 1$ such that $x \in X^{\rho,r}$. Since Θ is continuous and strictly increasing in $(0, \alpha_0)$, there exists a unique $\tilde{\alpha} \in (0, \alpha_0)$ such that $x \in X^{\rho,r}$ and $\Theta(\tilde{\alpha}) = \frac{\delta}{r}$. Therefore,

$$\begin{aligned}\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta) &= \inf_{\alpha\in(0,\alpha_0)} \sup_{y^\delta \in \overline{B_\delta(Tx)}} \left\|R_\alpha\, y^\delta - x\right\| \\ &\le \sup_{y^\delta \in \overline{B_\delta(Tx)}} \left\|R_{\tilde{\alpha}}\, y^\delta - x\right\| \\ &\le \sup_{x\in X^{\rho,r}} \sup_{y^\delta \in \overline{B_\delta(Tx)}} \left\|R_{\tilde{\alpha}}\, y^\delta - x\right\|. \end{aligned} \tag{4.34}$$

On the other hand, from hypotheses *H1-H4*, the fact that the function ρ is qualification of $\{R_\alpha\}$, the fact that ρ trivially *covers* ρ with constant equals to unity (see [4], Definition 2) and given that $\Theta(\tilde{\alpha}) = \frac{\delta}{r}$, it follows by virtue of Theorem 2 in [4], that there exists a positive constant K, independent of δ such that

$$\sup_{x\in X^{\rho,r}} \sup_{y^\delta \in \overline{B_\delta(Tx)}} \left\|R_{\tilde{\alpha}}\, y^\delta - x\right\| \le K\,\rho\left(\Theta^{-1}\left(\frac{\delta}{r}\right)\right), \text{ for } 0 < \delta \le r\,\Theta(\|T\|^2). \tag{4.35}$$

From (4.34) and (4.35) it follows that for every $\delta \in (0, \Theta(\alpha_0))$,

$$\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta) \le K\,\rho\left(\Theta^{-1}\left(\frac{\delta}{r}\right)\right) \le K\,\rho(\Theta^{-1}(\delta)) = K\,\psi(x,\delta),$$

where the last inequality follows from the fact that $r \ge 1$ and both ρ and Θ^{-1} are increasing functions. This proves that $\psi \in \mathcal{U}_{X^\rho}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$.

Next we will see that ψ satisfies condition *S1* of saturation on X^ρ. From hypotheses *H1-H4*, ***iv)*** and ***v)*** and the fact that ρ is maximal qualification of $\{R_\alpha\}$ it follows by virtue of Theorem 2.3 and Definition 2.2 in [2], that for every $x^* \in X$, $x^* \ne 0$ and $x \in X^\rho$ there exist positive constants $a \doteq a(x, x^*)$ and $d = d(x, x^*)$ such that

$$\frac{\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta)}{\psi(x,\delta)} \ge a \quad \forall\, \delta \in (0, d).$$

Then, $\limsup\limits_{\delta\to0^+} \frac{\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta)}{\psi(x,\delta)} > 0$ for every $x^* \in X$, $x^* \ne 0$ and $x \in X^\rho$, that is, ψ satisfies condition *S1* on X^ρ.

Also, since ψ does not depend on x, it is invariant over X^ρ, i.e., ψ satisfies condition *S2* of saturation.

It remains to prove that ψ satisfies condition *S3*. Suppose not. Then, there exist $M \supsetneqq X^\rho$ and $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ such that $\tilde{\psi}\,|_{X^\rho} = \psi$ and $\tilde{\psi}$ satisfies *S1* and *S2* over M. Let $x^* \in M \setminus X^\rho$, $x^* \ne 0$. Since $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ we have that

$$\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \overset{\{x^*\}}{\preceq} \tilde{\psi}. \tag{4.36}$$

Since $\tilde{\psi}$ is invariant over M and $X^\rho \subset M$, it follows that $\tilde{\psi} \overset{\{x^*\},X^\rho}{\preceq} \tilde{\psi}$ and since $\tilde{\psi}$ coincides with ψ on X^ρ, it follows that $\tilde{\psi} \overset{\{x^*\},X^\rho}{\preceq} \psi$. This together with (4.36) imply that $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \overset{\{x^*\},X^\rho}{\preceq} \psi$ and therefore $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta) = O(\rho(\Theta^{-1}(\delta)))$ for $\delta \to 0^+$.

Lemma 4.6 then implies that there exists an *a-priori* admissible parameter choice rule $\tilde{\alpha} : \mathbb{R}^+ \to (0, \alpha_0)$ such that

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \left\| R_{\tilde{\alpha}(\delta)} y^\delta - x^* \right\| = O(\rho(\Theta^{-1}(\delta))) \quad \text{for } \delta \to 0^+.$$

Then,

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \inf_{\alpha \in (0,\alpha_0)} \left\| R_\alpha y^\delta - x^* \right\| = O(\rho(\Theta^{-1}(\delta))) \quad \text{for } \delta \to 0^+.$$

Finally, Lemma 4.11 implies that $x^* \in \mathcal{R}(\rho(T^*T))$ and since $x^* \neq 0$, we have that $x^* \in X^\rho$, which contradicts the fact that $x^* \in M \setminus X^\rho$. Hence, ψ satisfies condition *S3* and therefore, ψ is saturation function of $\{R_\alpha\}$ on X^ρ. □

Note that both Lemma 4.5 and Lemma 4.6 remain true if hypotheses ***iii)*** and ***iv)*** of Theorem 4.4 are replaced by the requirement that there exists $\rho : (0, \|T\|^2] \to (0, \infty)$ that is qualification of $\{R_\alpha\}_{\alpha \in (0,\alpha_0)}$ and satisfies hypothesis ***iii.a)*** of Theorem 4.9.

5. Conclusions. In this article we have developed a general theory of global saturation for arbitrary regularization methods for inverse ill-posed problems. This concept of saturation formalizes the best global order of convergence that a method can achieve independently of the smoothness assumptions on the exact solution and on the selection of the parameter choice rule. Necessary and sufficient conditions for a methods to have global saturation have been provided. It was shown that for a method to have saturation the total error must be optimal in two senses, namely as optimal order of convergence over a certain set which at the same time, must be optimal with respect to the error. Finally, we have proved two converse results and applied the theory to derive sufficient conditions for the existence of global saturation for spectral methods with classical qualification of finite positive order and for methods with maximal qualification.

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