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A DUAL SIMPLEX ALGORITHM FOR THE TWO-STEP TRANSPORTATION PROBLEM

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Abstract: *An algorithm is developd from the associated generator trees of basic dual solutions of a two-step transportation problem. This is completely appropiate for the "Balinski signature method" for the obtention of prime solutions of the two-step transportation problem*

Key words: transportation, 2-step, signature, pivot.

Introduction

Balinski [1] uses his signature method in order to prove Hirsch's [3] conjecture, which relates the diameter of the convex polyhedron of the dual problem to the standard transportation problem, this is the transportation problem without steps.

In general, *the diameter of a polyhedron P* is understood as the largest distance between a pair of extreme points, namely the largest number of edges in the shortest path that connects such pair of points. Hirsch's conjecture establishes that the diameter of a convex polyhedron defined by q semi-spaces of a p-dimensional space is at the most q-p.

In linear programming language this means that given r linearly independent equations (assuming the non-negativity of the variables) it is possible to change an admissible basic solution by another one without losing the feasibility in at the most r pivot steps.

However the conjecture is valid for an important class of polyhedrons, it is false for unbounded polyhedrons [4]. Nevertheless, Balinski [1], validates the conjecture for unbounded polyhedrons associated to the dual of transportation problems. Moreover, Balinski uses his special pivoting form in order to design an $O(n^3)$ order algorithm oriented towards the resolution of the assignation problem.

Our purpose in this paper is to use this idea of the polyhedron, which appears related to the dual of the 2-step transportation problem, introduced by Marchi and Tarazaga [7]. We will prove that this approach is valuable as well in order to establish a combinatorial-type characterization of the extreme points of the respective primal problem.

The paper has been organized by sections:

- In the first section we consider some preliminary questions
- In the second section a Balinski pivoting method is introduced as a principle to the problem of two-step transportation
- In section three, we describe the signature method to obtain the solutions
- In section four, we describe a simplex dual competitive method
- In the last section we state some conclusions

1- **Preliminaries**

A two-step transportation model as the one introduced in [7] is a transportation model which introduces a new variant in transportation theory. Besides studying a two-step model where the merchandise passes through a deposit without accumulation, in order to travel from a port to the destination.

For the sake of clearness we want to remind the model.

Let $i:1,\ldots,m$ be ports and $k:1,\ldots,p$ be destinations. The merchandise, which is assumed to be indistinguishable, goes from a port to a destination. However, the merchandise leaving a port goes through a deposit $j:1,\ldots,n$ and reaches a destination. Each port i has a capacity r_i and each destination k needs the amount t_R. We have the common condition

$$\sum_{i=1}^{m} r_i = \sum_{k=1}^{p} t_k = r$$

which is concerned with the fact that all the merchandise required is distributed. Such condition is natural in transportation problems. Thus, if $x_{ij}^1 \geq 0$ and $x_{jk}^2 \geq 0$ are the respective total amount transported from port i to the deposit j, and from there to destination k, then the two-step transportation problem can take the following expression:

$$\sum_{j=1}^{m} x_{ij}^1 = r_j \qquad i \in \{1,\ldots,m\} = I$$

$$\sum_{j=1}^{m} x_{jk}^2 = t_k \qquad k \in \{1,\ldots,p\} = K$$

$$\sum_{i=1}^{m} x_{ij}^i = \sum_{k=1}^{p} x_{jk}^2 = 0 \qquad j \in \{1,\ldots,n\} = J$$

with $x = \left(x^1, x^2\right) \geq 0$

The last equation expresses the fact that at each deposit all the incoming amounts go out.

The conditions above are concerned with the total transported amounts, but the complete transportation problem is related to a cost function

$$f(x) = c^1 x^1 + c^2 x^2 = min! \tag{1,1}$$

$$= \sum_{ij} c_{ij}^1 x_{ij}^1 + \sum_{jk} c_{jk}^2 x_{jk}^2 = min!$$

which is linear. The amounts c_{ij}^1 and c_{jk}^2 are the costs to carry the unit amount from i to j and from j to k, respectively.

The first fact that we would like to mention is that the feasible set of solutions of the matrices given by the previous equations, constitute a convex polyhedron. Such a system can be expressed as a linear system of equality $Ax = b$ where the matrix A is given by

1……….1	1……….1	. . . .	1……….1					↓ m ↑
				1……….1	1……….1	. . . .	1……….1	↓ p ↑
1 . . . 1	1 . . . 1	. . . . .	1 . . . 1	–1 . . . –1	–1 . . . –1	. . . . .	–1 . . . –1	↓ n ↑
$\xleftrightarrow{mn}$				$\xleftrightarrow{pn}$				

and $b = (r_j, t_k, 0)^t$. The rank of matrix A is $m + p + n - 1$.

Thus a two-step transportation model is completely determined by the matrix A of (m+n+p)x(mn+np) order and a vector $c = (c^1, c^2) \in R^{(mn+np)}$. As a linear program it consists of (1.1) and for $x = (x^1, x^2)$, with $x \in P_{mn+np}$, where

$$P_{mn+np} = \{x^1_{ij}, x^2_{jk} / \sum_j x^1_{ij} = r_i;\ \sum_j x^2_{jk} = t_k;\ \sum_j x^1_{ij} = \sum_j x^2_{jk};\ x^1_{ij}, x^2_{jk} \geq 0;\ i \in I, j \in J, k \in K\}$$

Such a two-step transportation problem can be formulated and solved as a linear program. If $\sum_i r_i = \sum_k t_k$, the problem is feasible [3] and at the extreme points of P_{mn+np}, the primal polyhedron, we must find the problem's optimum.

In this paper we will use arguments which are based in the following known questions. *I* represents the set of *m* port nodes, *J* the set of *n* deposit nodes and *K* the set of *p* destination points. (see fig. 1)

Then we have as a first result

Lemma 1:

Any generator tree T of edges (i,j) or (j,k), $i \in I$, $j \in J$, $k \in K$, contains exactly m+p+n-1 edges. It uniquely determines a basic solution x(T) in $\overline{P}_{mn+np}$ (P_{mn+np} without the non-negativity condition upon the variables) defined as:

$$x(T) = \{x \in \overline{P}_{mn+np} / x^1_{ij} = x^2_{jk} = 0 \text{ for } (i, j) \text{ or } (j, k) \notin T\}.$$

Furthermore, if $x(T) \geq 0$[1], x(T) is a feasible solution.

Proof. In order to prove that any generator tree T contains exactly m+p+n-1 edges, we will use the induction principle over the number n= # *J*, assuming that m= # *I* and p= # *K* are fixed.

In the fist place, assume that n=1. Let T be a generator tree with edges (i,j) or (j,k). Then T is convex and acyclic. Since # *I* x *J*= m and # *J* x *K* = p, necessarily T has (mx1)+(px1) edges, this is, it has m+p+1-1 edges.

Let us assume now that # *J*(T)= n+1 (n fixed). Consider a tree T´ with edges (i,j) or (j,k) in which # *J*(T´)=n, # *I*(T´)=m and #*K*(T´) = p. Design T in such a manner that

- $V(T) = V(T') \cup \{j_{n+1}\}$,
- $E(T) = E(T') \cup \{(i_m, j_{n+1})\}$.

This way the tree T ends up being a generator tree associated to the problem. By the induction principle T' has m+p+n-1 edges, therefore T has m+p+n = m+p+(n+1)-1 edges.

We complete the induction, in an analogous way, considering m variables, and maintaining n and p fixed. Finally we use p as the variable, with n and m fixed.

Using the same notation, let S be another generator associated tree to the problem and assume that # E(S)> m+p+n-1. Without loss of generality, let # E(S)=m+p+n. S contains T and

$$E(S)=E(T) \cup \{e_{m+p+n}\}**).$$

Therefore S contains a cycle, since T is a tree. This leads to an absurd.

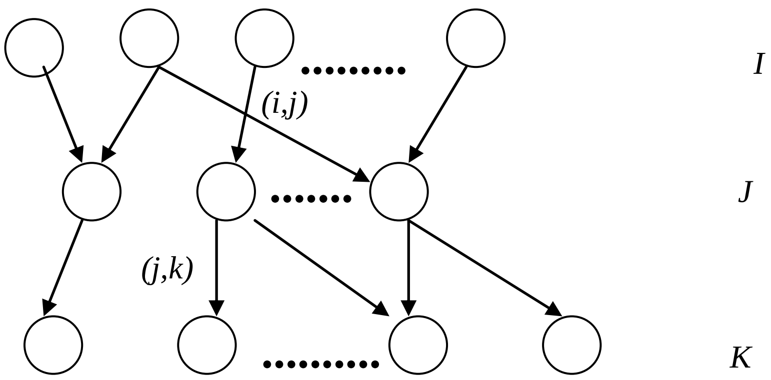

Figure 1: generator tree T of the tripartite graphic.

Having this result, it is also true since T has exactly m+p+n-1 edges and x(T) is derfined as

$$x(T) = \{x \in \overline{P}^{mn+mp} \,/\, x^1_{ij} = x^2_{jk} = 0 \text{ for } (i, j) \text{ or } (j, k) \notin T\},$$

that it has m+p+n-1 non-negative components. Consequently it is possible to assign a basic solution to the two-step transportation problem (see [3]) Q.E.D.

In this context, the *dual* of the two-step transportaation problem is given by

$$(1.2)\quad \textit{máx } ru + 0v + tz$$

for $(u, v, z) \in D_{m+n+p}$ *, In this case*

$$D_{m+n+p} = \{u_i, v_j, z_k \,/\, u_1 = 0;\ u_i + v_j \le c^1_{ij};\ z_k - v_j \le c^2_{jk};\ i \in I, j \in J, k \in K\}$$

The condition $u_1 = 0$ is arbitrary. This is established only with the purpose of obtaining unique values for the dual variables.

As a direct consequence of the definition and the number of edges of T we have:

Lemma 2:

A genearator tree T (constructed as in Lemma 1) determines a UNIQUELY feasible dual solution, u(T), v(T), z(T) of D_{m+n+p}*, defined by:*

$$\{u_i, v_j, z_k \,/\, u_1 = 0;\ u_i + v_j = c^1_{ij};\ z_k - v_j = c^2_{jk};\ (i,j) \text{ or } (j,k) \in T\}$$

We will now call *w(T) to* $w=(w^1,w^2)$ defined by:

$$w^1_{ij} = c^1_{ij} - u_i - v_j \quad ; \quad w^2_{jk}= c^2_{jk} - z_k + v_j$$

and it should be noticed that w is uniquely determined by each *u(T), v(T), z(T).*

It is easy to see the next result:

Lemma 4:

For any T, x(T) and w(T) are orthogonal and fulfill the slackness property:

$$x^1_{ij}(T).\ w^1_{ij}(T) = 0\ ;\ x^2_{jk}(T)\ .\ w^2_{jk}(T) = 0 \qquad \text{for any } i,j,k$$

The proof is immediate and by this reason we will skip it. Remember that if w(T) is basic, it satisfies $c^1_{ij}= u_i + v_j$ y $c^2_{jk} = z_k - v_j$.

q.e.d

Maintaning the same notation it is possible to prove:

Lemma 5:

If x(T) and w(T) are feasible, then x(T) is a solution for (1.1) and w(T) is a solution for (1.2).

The proof is trivial, due to Lemma1

q.e.d

2- **Balinski-Guzner Pivot Method**

A generator tree T, in linear programming language is simply a base. Given a generator tree T, pivoting basically consists in deleting an edge of T and adding another one which is not an edge of T. This has the purpose of obtaining a new generator tree T' and the respective values of the prime variables and duals *x´= x(T´)* and *w´= w(T´).*

The extreme points of P^{mn+np} are in correpondence with a *tripartite generator graph* such as those in fig. 1.

Extending this notion, a graph G=(V,E) is said to be tripartite if V can be written as an disjoint union of three non-empty sets. In this I,J,K, in such a manner that if e is an edge of E, e one and only one of the following forms: e=(i, j), $i \in I$, $j \in J$ or e=(j, k) with $j \in J$ and $k \in K$.

Consider in the first place, the values of x. $T \cup \{(g,h)\}$(Fig. 2) contains only one cycle to which (g,h) and (l,t) belong.

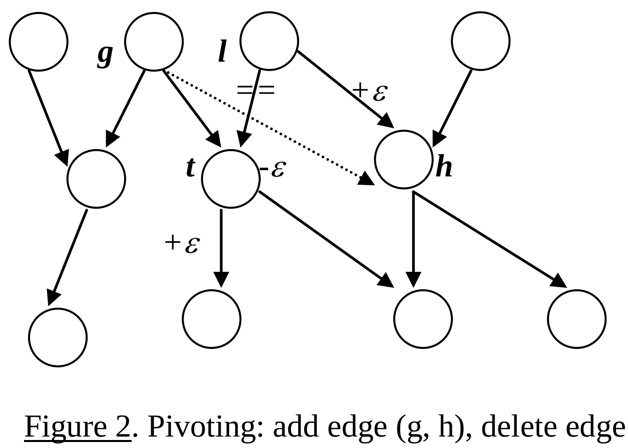

Figure 2. Pivoting: add edge (g, h), delete edge (l, t), change the values of x.

The edge (l, t) is "less" and for each pair of edges in the cycle, one is "less" and the other one is more. Doing*) $x^{*}_{lt} = \varepsilon$ [3] (see), the current values of x´(T) are obtained defining $x'^{*}_{\nu\beta} = x_{\nu\beta} - \varepsilon$, if (ν, β) is "less" or $x'^{*}_{\nu\beta} = x_{\nu\beta} + \varepsilon$, if (ν,β) is "more".

On the other hand, a triple (u,v,z) is an extreme point of D_{m+n+p} if and only if there exists a generator tree T of such tripartite graph for which

$$0 = c^1_{ij} - u_i - v_j\,;\ 0 = c^2_{jk} - z_k + v_j \qquad \text{for } (i,j) \text{ and } (j,k) \text{ in } T$$

$$0 \le c^1_{ij} - u_i - v_j\,;\ 0 \le c^2_{jk} - z_k + v_j \qquad \text{for } (i,j) \text{ and } (j,k) \text{ not in } T$$

Under these conditions we say that T is a suitable dual. Given a tree T for the extreme point (u,v,z), a new tree T´, corresponding now to the extreme point (u´,v´,z´), might be obtained (Figura 3) *pivoting* upon the edge (l,t), the valences of *l and t at least of less/minus two, in the following way*:

- Notice that T-{(l,t)} consists of two conex components, each of them in a tree, in a form such that if the node l belongs to one of them, the node t belongs to the other;
- Call T^l to the subtree of T tht contains node l and T^t to the subtree that contains node t;
- compute:

$$\delta_1 = \text{mín}\{c^1_{ij} - u_i - v_j,\ i \in T^t,\ j \in T^l\}$$

$$\delta_2 = \text{mín}\{c^2_{jk} - z_k + v_j,\ j \in T^t,\ k \in T^l\}$$

$$(2.1)\quad \delta = \text{mín}\,\{\delta_1, \delta_2\};$$

- call (g,h) to the edge at which δ is obtained;
- make $T' = T^l \cup T^t \cup (g,h)$;
- if port node $1 \in T^l$, define

$$(2.2)\begin{cases} u'_i = u_i + \delta & i \in T^t \\ u'_i = u_i & \text{other case} \end{cases}$$

$$(2.3)\begin{cases} v'_j = v'_j - \delta & j \in T^t \\ v'_j = v_j & \text{other case} \end{cases}$$

$$(2.4)\begin{cases} z'_k = z_k - \delta & k \in T^t \\ z'_k = z_k & \text{other case} \end{cases};$$

- if port node $1 \in T^t$ define

$$(2.6)\ \begin{cases} z'_k = z_k + \delta & k \in T^l \\ z'_k = z_k & \text{other case} \end{cases}$$

$$(2.7)\quad \begin{cases} v'_j = v_j + \delta & j \in T^l \\ v'_j = v_j & \text{other case} \end{cases}$$

$$(2.8)\quad \begin{cases} u'_i = u_i - \delta & i \in T^l \\ u'_i = u_i & \text{other case} \end{cases}$$

Proposition 2.1

(u′,v′,z′) is an extreme point associated to the tree T'.

The proof is immediate from the definitions of δ and of u´, v´, z´.

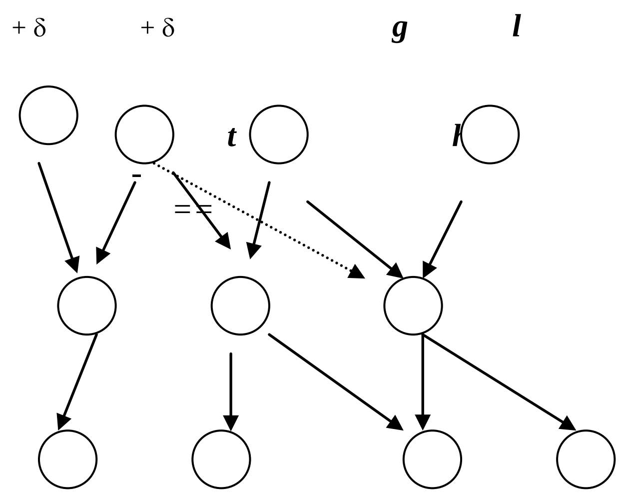

Figure 3. Pivoting: delete (l, t), add (g, h), change the values of u, v, z.

Briefly: a primal simplex method begins with a feasible primal base T, that is with an x(T) which is an extreme point of P_{mn+np} and one continues pivoting in such a way that one deletes an edge (l,t) from T nd adds an edge (g,h), for which the respective dual restrictions are violated. A *simplex dual method* begins with a feasible dual basis T and the corresponding extreme point w(T) of D_{m+n+p} and continues pivoting, deleting in each step an edge, and adding an edge (g,h) which makes δ.

Both, the primal and the dual simplex methods solve the two-step transportation problem in a finite number of steps (if one avoids to incorporate an edge already deleted previously). In the following section we describe a dual simplex method that does not take into account the degeneration and does not make any difference between primal and dual changes. This method is based on a characterization of the extreme points D_{m+n+p}, and it is called signature

3- **Signatures**

Given a two step transportation problem and a generator tree T, as that of fig. 1, it is possible to associate three vertices to T. From now on, *signature*, which gives the valence of their nodes.

We understand by *strong signature of T* a vector $a=(a_i) \in R^m$, where for each i,

$$a_i = \# \{j \in V(T) / (i,j) \in E(T)\}.$$

Analogously we call *deposit signature of T* to a vector $b=(b_j) \in R^n$, where for each j

$$b_j = \# (\{i \in V(T) / (i, j) \in E(T)\} \cup \{k \in V(T)/ (j, k) \in E(T)\}),$$

Moreover it is possible to define *the destination signature of T* as a vector c= $(c_k) \in R^p$ where, for each k,

$$c_k = \# \{j \in V(T)/ (j,k) \in E(T)\}.$$

As an easy consequence of the previous definitons

Proposition 3.1

For each i, j, k, a_i, b_j and $c_k \geq 1$, it is valid that:

$$\sum_i a_i + \sum_k c_k = \sum_j b_j = m + p + n - 1$$

In order to verify it, it is enough to take into consideration the associated graph characteristics of the problem and to observe the definiton of P_{mn+pn} and Lemma 1.

q.e.d.

It turns out that if (T) is a feasible solution, w(T)={u(T), v(T), z(T)}, and the equations $u_1=0$, $c^1_{ij} = u_i + v_j$, $c^2_{jk} = z_k - v_j$ form a maximal linear independent set if T is a tree. Reciprocally, exactly an extreme point corresponds to any generaror tree T(*).

Furthermore, starting from the notion of signature, it is possible to generalize, Balinski's theorem [1], which according to our context, has the following form:

Theorem 3.2:

Two different trees T and T', with the same signature correspond to a same extreme point.

Proof: We will provide a constructive proof. We will find a tree T" associated to the same extreme point of T', but with more edges in common with T than T'.

Let T and T´ be as required by the conditions. Since T and T´ are generator trees of a same graph, then V(T) = V(T´). Since T ≠ T´, necessarily E(T) ≠ E(T´). Then, without loss of generality, we can conclude that there exists at least $i_1 \in I$ such that

$$(i_1, j_1) \in T'-T.$$

Make a path *Π* in T, whose first edge is (i_1, j_1). We assign the sign "-" to it, and proceed in the following way:

3.2 if an edge (i_ν, j_ν) "minus" has been added to *Π*, choose

$$(i_{\nu+1}, j_\nu) \text{ or } (j_\nu, k_{\nu+1})$$

as the new edge, so that in T, it belongs to the unique path that joins i_1 con j_ν[2]. In all cases we assign "+" to the new edge;

3.3 if a edge $(i_{\nu+1}, j_\nu)$ "plus" has bee added to *Π*, choose as the new edge, if possible

$$(i_{\nu+1}, j_{\nu+1}) \text{ or } (j_{\nu+1}, k_{\nu+1})$$

an edge that belongs to T´-T. Otherwise, choose an edge frome T´$\cap$ T that belongs to the unique path in T that connects i $_{\nu+1}$ with i $_{\nu}$ (respectively z $_{\nu+1}$). In any of the two cases, assign "-" to it.

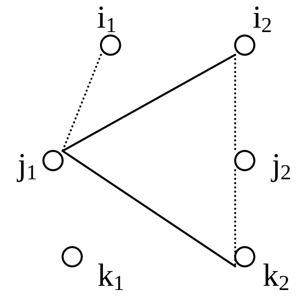

Figure 4. “plus” edges: continuous line; “minus” edges: dotted line

Because of the wat they were assembled, all the “minus” edges belong to T´, meanwhile the "+"edges belong to T. Repeating this procedure, a cycle X appears.
Let u(T), v(T), z(T), u(T´), v(T´), and z(T´) be the extreme points associated to T and T´ respectively. For the edges (i, j) and (j, k) of the cycle, we observe that, since they are edges of T,

3.4 $u_i + v_j = c^1 ij$ for (i,j) "plus" ; $z_k - v_j = c^2{}_{jk}$ for (j, k) "plus"
3.5 $u_i + v_j \leq c^1 ij$ for (i,j) "minus" ; $z_k - v_j \leq c^2{}_{jk}$ for (j, k) "minus".

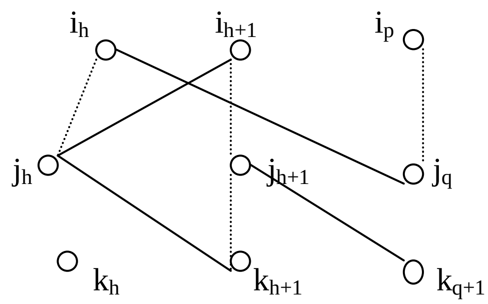

Figure 5. How a cycle might appear

Reasoning in an analogous way, but now upon T´,

3.6 $u'_i + v'_j \leq c^1 ij$ for (i,j) "plus" ; $z'_k - v'_j \leq c^2{}_{jk}$ for (j, k) "plus"
3.7 $u'_i + v'_j = c^1 ij$ for (i,j) "minus" ; $z'_k - v'_j = c^2{}_{jk}$ for (j, k) "minus"

Adding in 3.7, and applying succesively 3.6, 3.4 y 3.5,

3.8 $$\sum_{"minus"} c^1_{ij} = \sum_{"minus"} (u'_i + v'_j) \leq \sum_{"plus"} c^1_{ij} = \sum_{"plus"} (u_i + v_j) \leq \sum_{"minus"} c^1_{ij}$$

3.9 $$\sum_{"minus"} c^2_{ij} = \sum_{"minus"} (z'_i - v'_j) \leq \sum_{"plus"} c^2_{ij} = \sum_{"plus"} (z_i - v_j) \leq \sum_{"minus"} c^2_{ij}$$

Therefore, for any (i, j) or (j, k) in *X*,

$$3.10 \qquad \begin{cases} u_i + v_j = u'_i + v'_j \\ z_k - v_j = z'_k - v'_j \end{cases}$$

Since $X \not\subset T'$, X contains a “plus” edge of T-T´. Without loss of generality, assume that such edge $(i_\nu, j_{\nu-1}) \in \Pi\text{-}X$. Then, $(i_\nu, j_{\nu-1})$ is the precedecesor of a “minus” edge (i_ν, j_ν) or $(j_{\nu-1}, k_\nu)$ which consequently belongs to T´-T. The valence of the node i_ν is larger or equal to two, in T as well as in T´, because of the signature hypothesis. Besides $T' \cup \{(i_\nu, j_{\nu-1})\}^3$ contains a cycle which includes (i_ν, j_ν) or any other edge of T´, incidental to i_ν, but that does not belong to *X*. If (i_ν, j_ν) belongs to X, then we delete (i_ν, j_ν). Assign the name T´´ to the constructed edge this way. If this not were the case, delete the edge of T´ incidental to i_ν, in order to obtain a new tree associated to the same extreme point and signature as T´, but with an edge "+". From here it has one more edge in common with T.
Repeating the procedure, it is possible to find a tree T´´ in which all the "+" edges that have a succesor in $T \cap T'$, where there exists an edge "+" $(i_\lambda, j_\beta) \in$ T-T´. Since $(i_\lambda, j_\beta) \in$ T-T´, i_λ must have an edge in T´-T and consequently, by construction, $(i_h, j_h) \notin t$. Hence, adding (i_λ, j_β) a tree T´´is obtained, with one more edge in common with T.

q.e.d.

It is interesting as well to use the notion of signature in order to caractherize a certain class of feasible dual solutions.

Theorem3.3:

Given the vector $(n,1,.....,1) \in R^m$, it is always possible to build a tree T, with that vector as port signature and feasible dual if we define:

3.11 $\quad u_1 = 0 \;;\quad v_j = c^1_{ij} \;;\quad u_i = \text{mín}_j\,(c^1_{ij} - c^1_{1j}) \;;\quad z_k = \text{mín}_j\,(c^2_{jk} + c^1_{1j})$

and T is constructed in the following fashion:

$$3.12 \begin{cases} an \;\; edge \;\; (1,j) & \text{for all } j \text{ in } J \\ an \;\; edge \;\; (i,j) & \text{if } j \text{ reaches } \text{mín}_j \text{, any } i \\ an \;\; edge \;\; (j,k) & \text{if } j \text{ reaches } \text{mín}_j \text{, any } k \end{cases}$$

Proof. The solution 3.11 is dual feasible. Observe that:

3.12. $u_i + v_j = \text{mín}_j\,(c^1_{ij} - c^1_{1j}) \leq (c^1_{ij} - c^1_{1j}) + c^1_{1j} \leq c^1_{ij}$

3.13. $z_k - v_j = \text{mín}_j\,(c^2_{jk} + c^1_{1j}) \leq c^2_{jk}$

3.14. Let j_1 y j_2 such that

$$\text{mín}_j\,(c^1_{ij} - c^1_{1j}) = c^1_{ij_1} - c^1_{1j_1} \quad \text{and} \quad \text{mín}_j\,(c^2_{jk} + c^1_{1j}) = c^2_{j_2k} + c^2_{1j_2}$$

Then the remaining dual conditions are valid.

3.15. Let T be constructed as required in 3.12. T has n+(m-1)+p edges, and does not have isolated nodes; therefore it does not have cycles. Then T is a tree.

---

q.e.d.

In the specific case of two-step transportation problems, where r_i and t_k are all rational, this is to say

$$r_i = \frac{p_i}{q_i} \quad ; \quad t_k = \frac{\bar{p}_k}{\bar{q}_k} \qquad \text{for any } i, k$$

it is possible to obtain important results in this context.

Certainly, in such cases the problem can be "normalized". Observing that:

$$\frac{r_i}{\tau} \text{ y } \frac{t_k}{\tau} \in \mathrm{N}$$

and only considering

$$\tau = \frac{1}{.m.c.m\{q_1, \ldots, q_m, \bar{q}_1, \ldots, \bar{q}_m\}}.$$

Furthermore, if we define $\bar{x} = \left(\bar{x}^1; \bar{x}^2\right) \geq 0$, problem [4] can be reformulated in a completely equivalent maner, as follows:

3.16 $\displaystyle\sum_{j=1}^{n} \frac{1}{\tau}\, \bar{x}^1_{\bar{i}j} = 1$ $\qquad$ *for* $\bar{i} = 1, \ldots, \displaystyle\sum_{i=1}^{m} \frac{r_i}{\tau} = \overline{m}$

3.17 $\displaystyle\sum_{j=1}^{n} \frac{1}{\tau}\, \bar{x}^2_{j\bar{k}} = 1$ $\qquad$ *for* $\bar{j} = 1, \ldots, \displaystyle\sum_{k=1}^{p} \frac{t_k}{\tau} = \overline{p}$

3.18 $\displaystyle\sum_{\bar{i}=1}^{\overline{m}} \bar{x}^1_{\bar{i}j} - \sum_{\bar{k}=1}^{\overline{p}} \bar{x}^2_{j\bar{k}} = 0$ $\qquad$ *for* $j = 1, \ldots, n$

In the specific case where $\overline{m} = \overline{n} = \overline{p}$, notice:

Lemma:

If T is a feasible dual generator tree for 3.16/18 and whose deposit signature contains exactly a two, then it is an optimal primal basis and x(T) defined by:

$\bar{x}^1_{\bar{i}j} = \bar{x}^2_{j\bar{k}} = 1$ *for each node j of valence 2 such that* $(\bar{i}, \bar{j}) \in T$;

$\bar{x}^1_{\tilde{i}j} = \bar{x}^2_{j\tilde{k}}$ *for* $\begin{cases} \bar{i} \neq \tilde{i} \\ \bar{k} \neq \tilde{k} \end{cases}$ *, where* $(\tilde{i}, j)\, y\, (j, \tilde{k})$ *are the only edges in T incidental to*

j which does not belong in the path that connects $\bar{i}$ *with* $\tilde{i}$ and $\tilde{k}$ *with* $\bar{k}$ *;*

$\bar{x}^{1}_{\tilde{i}j} = \bar{x}^{2}_{j\tilde{k}} = 0 \quad other \;\; cases$

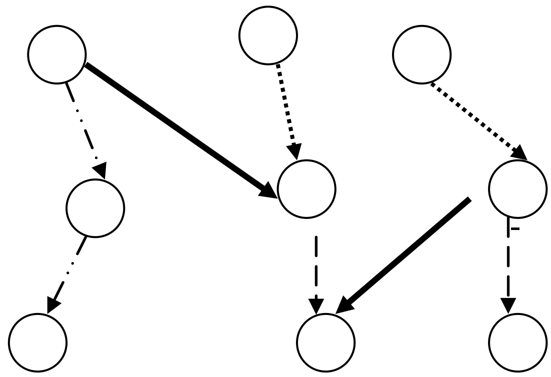

Figure 5: Optimal primal solution, according to the Lemma, for a problem with three ports, three deposit and three destinations.

Proof: It is enough to observe that the hypothesis implies that the deposit signature of T can only consist of a 2 and 3 for the remaining. Any basis of deposit signature (2,3,...,3) is feasible primal. The Glover procedure [2] for pivoting, maintaining the complete signature, completes the proof.

4- **The dual simplex method for the two-step tansportation problem**

In the case of equal ports, deposits and destinations, the problem of finding an optimal solution to the two-step problem is reduced to find a feasible basis of the deposit signature (2,3,...,3).

As it is known, a base T with port signature (n,1,..,1)- (op. cit. pg. 9)- if *w(T) and x(T) ≥ 0, the desired solution has been found*. In the opposite case the process will consist in pivoting, following the Balinski-Guzner rule, upon the edges (j, k) of T until the deposit signature is found.

If m ≠ n or n ≠ p or m ≠ p, it is possible to add fictitious nodes and assign:

- capacity, maintaining the feasibility, in the case that such points represent ports or destinations,
- "great" costs, concerning the remaining costs of the problem, to the edges incidental to them.

5- **Conclusion**

We have considered the application of a form of pivoting of Balinski's type. This determines feasible dual solutions for two-step transportation problems. At the same time, some results of combinational nature related the notion of signature have been generalized.

From the obtained results and efficiency and simplicity of the proposed methods it is possible to develop a simple computational interpretation.

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