U. of M.

THE UNIVERSITY OF MINNESOTA

GRADUATE SCHOOL

Report

of

Committee on Examination

This is to certify that we the undersigned, as a committee of the Graduate School, have given Sally Elizabeth Carlson final oral examination for the degree of Master of Arts. We recommend that the degree of Master of Arts be conferred upon the candidate.

Minneapolis, Minnesota

June 5 1918

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Professor Bauer is on leave of absence in Government Thrift Stamp work and he could not attend the examination.

THE UNIVERSITY OF MINNESOTA

GRADUATE SCHOOL

Report

of

Committee on Thesis

The undersigned, acting as a Committee of the Graduate School, have read the accompanying thesis submitted by Elizabeth Carlson, for the degree of Master of Arts. They approve it as a thesis meeting the requirements of the Graduate School of the University of Minnesota, and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts.

Geo. N. Bauer
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June 5 1918

AN ANALYTIC GEOMETRY TREATMENT OF THE NATURE OF CONICS GENERATED BY PROJECTIVE RANGES AND PENCILS.

A Thesis submitted to the

Faculty of the Graduate School of the

University of Minnesota

by

ally

S. Elizabeth Carlson

In partial fulfillment of the requirements

for the degree of

Master of Arts

June

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OUTLINE.

I. Introduction and fundamental theorems.

1. Steiner obtained the nature of the conic generated by projective forms by means of synthetic geometry. #1.

2. The curve generated by projective forms is always a conic. #2.

II. The conic as the envelope of connectors of corresponding points.

1. Two classes of ranges.

a. Ranges without vanishing points. #3.

b. Ranges with vanishing points. #4.

2. Conics generated by similar ranges. #5.

a. In general, a parabola.

b. Degenerates into a point when:

(I) $\theta = 0$, ranges are parallel.

(II) $\cos\theta = \frac{d + l\sin\theta}{d}$

(III) $l = 0$ and $d = 0$

(IV) $l = 0$ and $\theta = \cos^{-1}\frac{1}{\alpha}$

1. Two projective pencils have a pair of corresponding rays, s and s', t and t', such that s is perpen-

dicular to t and s' is perpendicular to t'. #13.

2. For two projective pencils the relation $\tan(tr)\cdot\tan(s'r') =$ a constant is always true, and conversely. #14.

3. Conic generated when both centers are in the finite plane. #15.

a. When k is positive

I. Always a hyperbola.

b. When k is negative.

I. $0 > k > -1$.

(1) Parabola when $\phi + 2\Delta = 90$.

(2) Ellipse when $\phi + 2\Delta > 90$.

(3) Hyperbola when $\phi + 2\Delta < 90$.

II. $k < -1$.

(1) Parabola when $\phi + 90 = 2\Delta$

(2) Ellipse when $\phi + 90 > 2\Delta$

(3) Hyperbola when $\phi + 90 < 2\Delta$

4. One center at infinity and one in the finite plane. #16.

a. Hyperbola if b' is in finite plane.

b. Parabola if b' is the line at infinity.

c. Straight line if b and b' are both the same ray.

I. INTRODUCTION AND FUNDAMENTAL THEOREMS.

#1. A well known theorem in projective geometry is that conics are generated by projective pencils and ranges. The nature of the conic, as determined by the projective relation involved in its generation, has been obtained by Steiner by the use of synthetic methods.[1] The aim of this paper is to obtain the same and other results by the use of analytic geometry methods.

#2. The subject naturally divides itself into two parts: 1) conics generated by ranges; 2) conics generated by pencils. The generation of conics by projective ranges and pencils depends upon the two following theorems:- If two non-collinear ranges in the same plane are projective but not in perspective, the envelope of the connectors of pairs of corresponding points is a conic. If two non-concentric pencils in the same plane are projective but not in perspective, the locus of the intersection of corresponding rays is a conic. Steiner[2] and Cremona[3] give proofs of these theorems.

1. Steiner: Vorlesungen über Synthetische Geometrie. II.
2. Steiner: Vorlesungen über Synthetische Geometrie. II, #22.
3. Cremona: Elements of Projective Geometry. #150.

II. THE CONIC AS THE ENVELOPE OF CONNECTORS OF CORRESPONDING POINTS

#3. Projective ranges may be divided into two classes, ranges without vanishing points and ranges with vanishing points. If the ranges have no vanishing points, i.e. the points at infinity correspond to each other, the ranges are similar.[1] The ratio between corresponding segments of two similar ranges is constant; hence, if two corresponding points be chosen as origins, and the distances from the origins to corresponding points be denoted by ξ and ξ' we have the relation, $\xi' = \alpha\xi$. Conversely, if we have the relation $\xi' = \alpha\xi$ given, the two ranges are projective and have no vanishing points. This follows directly from the definition of similar ranges, and the theorem that two similar ranges are projective ranges whose points at infinity correspond to each other.[2]

#4. If the ranges have vanishing points, I and J', then the product IA x J'A' is constant.[3] If I and J be chosen as the fixed points from which ξ and ξ' are measured, the projective relation can be expressed by the equation $\xi' = \beta/\xi$. Conversely, it can be proved that if $\xi' = \beta/\xi$ the two ranges are projective. For if we take any four

1. Cremona: Elements of Projective Geometry. #100
2. Cremona: Elements of Projective Geometry. #99
3. Cremona: Elements of Projective Geometry. #74

sets of points determined by the equation $\xi' = \beta/\xi$ and call them ABCD and A'B'C'D', then the anharmonic ratio of ABCD is equal to the anharmonic ratio of A'B'C'D'. The following is a proof of this statement:- Let ξ_1, ξ_2, ξ_3, and ξ_4 be the distances of A, B, C, and D from the vanishing point I, and ξ_1', ξ_2', ξ_3', and ξ_4' the distances of A', B', C' and D' from J'. Then

$$(ABCD) = \frac{\xi_3 - \xi_1}{\xi_3 - \xi_2} \div \frac{\xi_4 - \xi_1}{\xi_4 - \xi_2} \quad \text{and}$$

$$(A'B'C'D') = \frac{\xi_3' - \xi_1'}{\xi_3' - \xi_2'} \div \frac{\xi_4' - \xi_1'}{\xi_4' - \xi_2'}$$

Since $\quad \xi_1' = \frac{\beta}{\xi_1}, \quad \xi_2' = \frac{\beta}{\xi_2'}, \quad etc.$

$$(A'B'C'D') = \frac{\frac{\beta}{\xi_3} - \frac{\beta}{\xi_1}}{\frac{\beta}{\xi_3} - \frac{\beta}{\xi_2}} \div \frac{\frac{\beta}{\xi_4} - \frac{\beta}{\xi_1}}{\frac{\beta}{\xi_4} - \frac{\beta}{\xi_2}}$$

$$= \frac{(\xi_1 - \xi_3)\xi_2}{(\xi_2 - \xi_3)\xi_1} \div \frac{(\xi_1 - \xi_4)\xi_2}{(\xi_2 - \xi_4)\xi_1}$$

$$= \frac{\xi_3 - \xi_1}{\xi_3 - \xi_2} \div \frac{\xi_4 - \xi_1}{\xi_4 - \xi_2}$$

$$= (ABCD)$$

From this it follows that the two ranges determined by $\xi' = \frac{\beta}{\xi}$ are projective, for if any four points of one range are equianharmonic with the four corresponding points of another range, the two ranges are projective.[1]

#5. Let us first consider the conics generated by similar ranges.

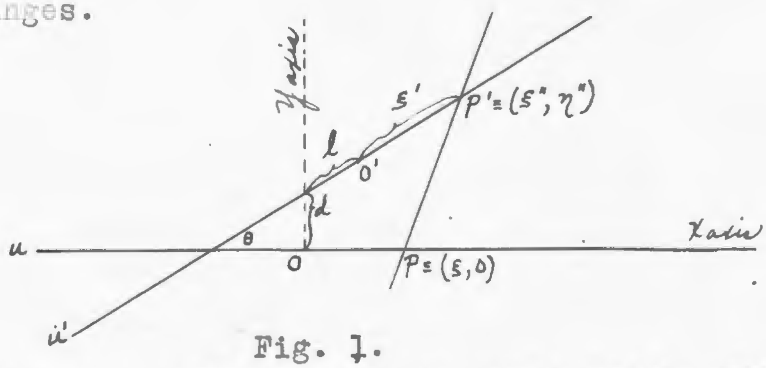

Fig. 1.

In figure 1, O and O' are the fixed points from which ξ and ξ' are measured. P and P' are any other two corresponding points determined by the relation $\xi' = \alpha\xi$. From the figure it is evident that the equation of PP' is

$$\frac{x-\xi}{\xi-\xi''} = \frac{y}{-\eta''} \quad \text{or} \quad \eta''(\xi - x) = y(\xi - \xi'').$$

Now $\xi'' = (l+\xi')c = lc + \alpha\xi c$ $\qquad (c = \cos\theta)$

and $\eta'' = d + ls + \alpha\xi s$ $\qquad (s = \sin\theta)$

Substituting these values for ξ'' and η'' in the equation for PP', we obtain a function of x, y, and ξ.

$$f(x, y, \xi) \equiv (d + ls + \alpha\xi s)(\xi - x) - y(\xi - ls - \alpha\xi c) = 0$$

1. Cremona: #78

$$f(x,y,\xi) \equiv d\xi + ls\xi + \alpha\xi^2 s - dx - lsx - \alpha\xi sx - y\xi + lcy + \alpha\xi cy = 0.$$

$$\frac{\partial f}{\partial \xi} = d + ls + 2\alpha\xi s - \alpha sx - y + \alpha cy = 0$$

$$\xi = \frac{\alpha sx - \alpha cy + y - d - ls}{2\alpha s}$$

Substituting this value of ξ in the expression for $f(x,y,\xi)$ we get as the equation of the envelope of the variable line, PP'

$$\alpha^2 s^2 x^2 + (1-\alpha c)^2 y^2 + 2\alpha s(1-\alpha c)xy + 2\alpha s(d+ls)x + (2\alpha cd - 2d - 2\alpha cls - 2ls)y + (d+ls)^2 = 0 \qquad (1)$$

(1) is an equation of the second degree and hence represents a conic. In this paper,

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

is taken as the general equation of the second degree. If the discriminant,

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

is equal to zero, the equation represents two straight lines. Otherwise, the equation represents an ellipse, a parabola, or a hyperbola according as h^2-ab is less than, is equal to, or is greater than zero.

Since h^2-ab in this case is equal to zero, (1) represents a parabola (Plate I). Therefore, two similar ranges generate a parabola.

The discriminant D of (1) is equal to $-4\alpha^2 s^2(\alpha cd - d - ls)^2$

$D=0$ if $\theta=0$; i,e, if the bases of the two ranges are parallel. In this case, the equation of the parabola reduces to the equation of two coincident straight lines.

$$[(1-\alpha)y - d]^2 = 0 \qquad \text{or} \qquad y = \frac{d}{1-\alpha} \tag{2}$$

Only the point $\left(\frac{l}{1-\alpha}, \frac{d}{1-\alpha}\right)$ is the envelope (Plate II). All the lines connecting corresponding points of the two ranges pass thru this point.

D=0 also if $(\alpha cd - d - ls) = 0$. Then (1) reduces to

$$\left[\alpha sx + (1-\alpha c)y + \frac{\alpha c ls}{\alpha c - 1}\right]^2 = 0. \tag{3}$$

All the connectors of corresponding points in this case are parallel to the line represented by (3); i.e. the point at infinity on (3) is the envelope (Plate III).

If d is equal to zero, the point O falls at the intersection of the two bases. Then (1) becomes

$$\alpha^2 s^2 x^2 + (1-\alpha c)^2 y^2 + 2\alpha s(1-\alpha c)xy + 2\alpha ls^2 x - 2ls(\alpha c+1)y + l^2 s^2 = 0. \qquad \text{(Plate IV)} \tag{4}$$

If l=0 when d=0, both O and O' fall at the point of intersection and the equation of the envelope becomes

$$\alpha sx + (1-\alpha c)y = 0 \tag{5}$$

with only the point at infinity as the actual envelope. (Plate V).

If l be chosen equal to zero when D is not zèro, the line OO' in figure 1 is perpendicular to the range u. In this case the equation of the parabola becomes

$$\alpha^2 s^2 x^2 + (1-\alpha c)^2 y^2 + 2\alpha s(1-\alpha c)xy + 2\alpha dsx - 2d(1-\alpha c)y + d^2 = 0. \qquad \text{(Plate VI)} \tag{6}$$

This equation breaks up into two linear factors when $(1-\alpha c)=0$; i.e. $\theta=\cos^{-1}\frac{1}{\alpha}$. (6) then reduces to

$$(\alpha^2-1)x^2+2dx\sqrt{\alpha^2-1}+d^2=0.$$

or
$$x=\frac{-d}{\sqrt{\alpha^2-1}}. \qquad (7)$$

In this case also, all the connectors of corresponding points are parallel to (7) and the point at infinity on (7) is the envelope (Plate VII).

#6. The nature of conics determined by ranges with vanishing points will be taken up next; first, parallel ranges and then non-parallel ranges.

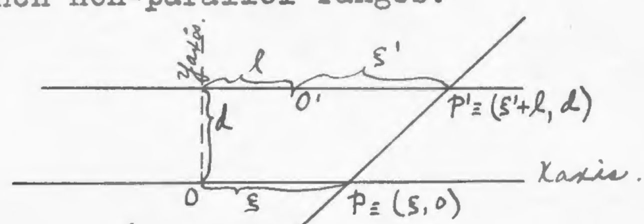

Fig. 2.

In figure 2, O and O' are the vanishing points, P and P' are any two corresponding points determined by the relation $\xi'=\beta/\xi$. From the figure it is evident that the equation of PP' is

$$\frac{x-\xi}{\xi-\xi'-l}=\frac{y}{-d}$$

or, by substituting $\frac{\beta}{\xi}$ for ξ', $d\xi-dx-y\xi+y\beta/\xi+ly=0.$

$$f(x,y,\xi)\equiv d\xi^2-dx\xi-y\xi^2+\beta y+ly\xi=0. \qquad (8)$$

$$\frac{\partial f}{\partial \xi}=2\xi(d-y)-dx+ly=0. \qquad (9)$$

$$\xi=\frac{dx-ly}{2(d-y)}$$

Eliminating ξ between (8) and (9), we obtain as the equation of the envelope of PP',

$$d^2x^2 - 2dlxy + y^2(l+4\beta) - 4\beta dy = 0. \tag{10}$$

The h^2- ab of (10) is $-4\beta d^2$; therefore if β is positive the conic is an ellipse (Plate VIII); and if β is negative it is a hyperbola (Plate IX). If β is positive the corresponding points move in opposite directions and if β is negative, in the same direction. The discriminant of (10) is $-4\beta^2 d^4$ which cannot be equal to zero if the ranges are non-collinear; i.e. (10) does not break up into two linear factors.

The center of the conic represented by (10) is at the point $\left(\frac{l}{2}, \frac{d}{2}\right)$. If β is negative, let $\beta = -k^2$; then the asymptotes of the hyperbola are

$$(l+2k)y = dx + kd.$$
$$(l-2k)y = dx - kd.$$

If l is 0, equation (10) reduces to

$$d^2x^2 + 4\beta y^2 - 4\beta dy = 0. \tag{11}$$

The center of this conic is at $\left(0, \frac{d}{2}\right)$. The lengths of the semi-axes are $\sqrt{\beta}$ and $\frac{d^2}{2}$. If $\beta = -k^2$, the asymptotes of the hyperbola are:

$$2ky - kd = \pm dx.$$

If $\beta = \frac{d^2}{4}$, the envelope is a circle (Plate X). If $\beta = -\frac{d^2}{4}$

the envelope is an equilateral hyperbola.

#7. When the bases of the ranges are not parallel, the nature of the envelope depends largely upon the position of the vanishing points.

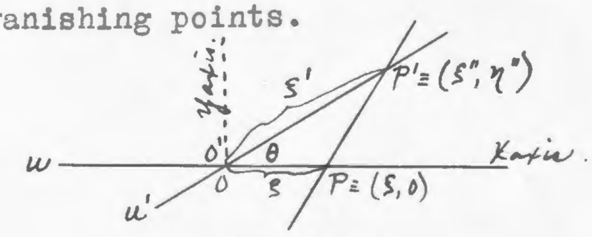

Fig. 3.

In figure 3 both vanishing points are at the intersection. The equation of PP' is

$$\eta''(\xi - x) = y(\xi - \xi'').$$

Now $\eta'' = \xi' \sin\theta = \beta/\xi\, s.$

and $\xi'' = \xi' \cos\theta = \beta/\xi\, c.$

Substituting these values for ξ'' and η'' in the equation for PP' and simplifying, we get the following function of x, y, and ξ:

$$f(x, y, \xi) \equiv \beta s \xi - \beta s x - y\xi^2 + \beta c y = 0. \qquad (12)$$

$$\frac{\partial f}{\partial \xi} = \beta s - 2y\xi = 0. \qquad (13)$$

$$\xi = \frac{\beta s}{2y}$$

Eliminating ξ between (12) and (13), we obtain the equation of the envelope.

$$4cy^2 - 4sxy + \beta s^2 = 0. \qquad \text{(Plate XI) (14)}$$

This equation always represents a hyperbola for $h^2 - ab = 4s^2$ which is always positive. The discriminant is equal to $-4\beta s^4$, which cannot equal zero, since the ranges are non-collinear. Therefore when both vanishing points are at the point of intersection of the two bases, the envelope of the connectors of corresponding points is always a hyperbola.

The equations of the asymptotes of the hyperbola are: $y=0$ and $y=x\tan\theta$; but $y=0$ is the equation of u, and $y=x\tan\phi$ is the equation of u'; hence the bases of the ranges are the asymptotes of the hyperbola and the intersection of the two bases is the center of the hyperbola. If $\theta = 90^{\circ}$, the ranges are perpendicular to each other and the hyperbola is equilateral (Plate XII).

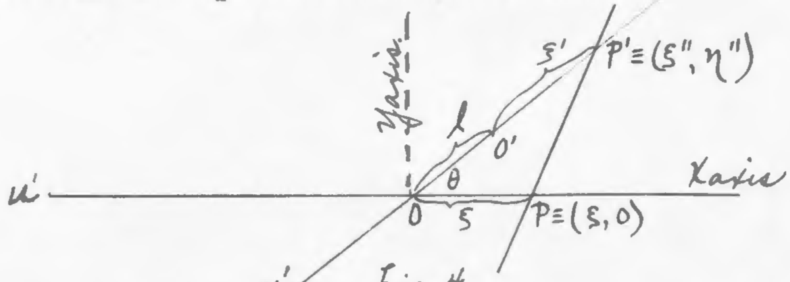

Fig. 4.

#8. In figure 4 only one of the vanishing points is at the intersection. The equation of PP' is

$$\eta''(\xi - x) = y(\xi - \xi'').$$

Now $\quad \xi'' = (l+\xi')c = lc + \frac{\beta}{\xi}c.$

and $\quad \eta'' = (l+\xi')s = ls + \frac{\beta}{\xi}s.$

Substituting these values for ξ'' and η'' in the equation for PP' and simplifying we get

$$f(x,y,\xi) \equiv \xi^2(ls - y) + \xi(\beta s - lsx + lcy) + \beta cy - \beta sx = 0.$$

$$\frac{\partial f}{\partial \xi} = 2\xi(ls - y) + (\beta s - lsx + lcy) = 0.$$

$$\xi = \frac{lsx - lcy - \beta s}{2(ls - y)}.$$

If this value of ξ is substituted in $f(x, y, \xi) = 0$, we obtain the equation of the envelope.

$$l^2s^2x^2 + (l^2c^2 + 4\beta c)y^2 - (2l^2sc + 4\beta s)xy + 2\beta ls^2x$$

$$- 2\beta lscy + \beta^2 s^4 = 0. \qquad \text{(Plate XIII)(15)}$$

This equation also always represents a hyperbola for $h^2 - ab$ is $4\beta^2 s^4$ which is always positive. (15) does not break up into two linear factors, because the discriminant equals $-4\beta^4 s^4$; hence we always have a hyperbola when one of the vanishing points is at the intersection of the two ranges.

The asymptotes of the hyperbola are:-
$sx - cy = 0$ (the equation of u') and $l^2sx - (4\beta + l^2c)y + 2\beta ls = 0.$
The center is at $(\frac{lc}{2}, \frac{ls}{2})$.

#9. When neither vanishing point is at the intersection of the ranges, the equation of the envelope of the connectors of corresponding points may represent either a hyperbola or an ellipse. The equation is obtained in the same way as in the previous cases. Equation of PP' in

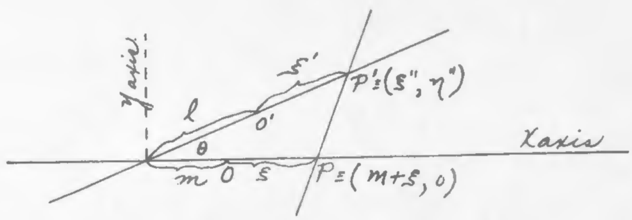

Fig 5.

figure 5 is $\eta''(m+\xi-x) - y(m+\xi-\xi'') = 0.$

$$\xi'' = (l+\xi')\cos\theta = lc + \frac{\beta}{\xi}c.$$

$$\eta'' = (l+\xi')\sin\theta = ls + \frac{\beta}{\xi}s.$$

Substituting these values for ξ'' and η'' in the equation for PP', we get

$$\left(ls + \frac{\beta}{\xi}s\right)(m+\xi-x) - y\left(m+\xi-lc-\frac{\beta}{\xi}c\right) = 0.$$

Simplifying this equation we obtain the following function of x, y and ξ :-

$$f(x,y,\xi) \equiv \xi^2(ls-y) + \xi(lms+\beta s-lsx-my+lcy) + \beta ms - \beta sx + \beta cy = 0.$$

$$\frac{\partial f}{\partial \xi} = 2\xi(ls-y) + (lms+\beta s-lsx-my+lcy) = 0.$$

$$\xi = \frac{-(lms+\beta s-lsx-my+lcy)}{2(ls-y)}$$

The equation of the envelope is:

$$l^2s^2x^2 + \left[(m-lc)^2+4\beta c\right]y^2 + \left[2ls(m-ls) - 4\beta s\right]xy + \left[2ls^2(\beta-lm)\right]x$$
$$+\left[2s(\beta-lm)(m-ls)\right]y + s^2(\beta-lm)^2 = 0. \qquad (16)$$

The h^2-ab of equation (16) is $4\beta^2s^2(\beta-lm)$. From this it is found that if βlm is negative, the envelope is a hyperbola (Plate XIV); if βlm is positive and $|\beta| > |lm|$, it is a hyperbola (Plate XV); if βlm is positive and $|\beta| < |lm|$, it is an ellipse (Plate XVI).

The discriminant of (16) is $D = 4\beta^2 s^4 (\beta - lm)^2$. $D = 0$ if $\beta = lm$. In this case (16) reduces to

$$y(m + ls) = lsx \qquad \text{(Plate XVII)} \qquad (17)$$

But only the point (m+lc, ls) is the envelope.

#10. In the case just discussed, Steiner determined the nature of the conic by the position of the line joining, the points of contact of the bases of the two ranges.[1] His result is that if the contact line, ie. the line joining these points of contact, lies between the vanishing points and the intersection of the two ranges, the envelope is an ellipse, otherwise it is a hyperbola.

The results obtained analytically can very easily be translated into Steiner's terms as follows: We shall choose as the positive direction on each line the direction from the intersection to the vanishing point on that line. βlm will then be positive or negative according as β is positive or negative. Then from the above, if β is negative, we have a hyperbola.

u (figure 6) will touch the conic at the point corresponding to I, thot of as belonging to u'.[2] The ξ' coordinate

1. Steiner: II, #26.
2. Steiner: II, #21.

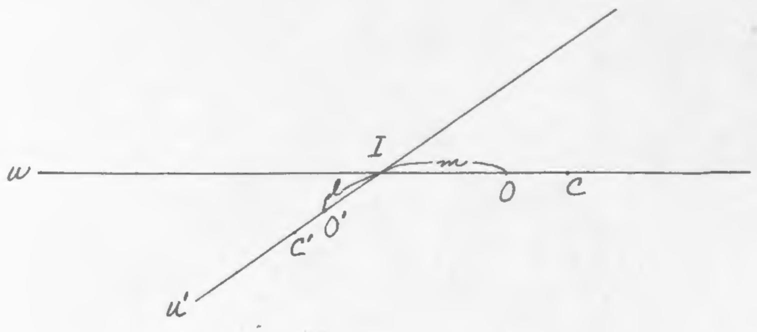

Fig. 6.

of I is -1. Therefore, from the relation $\xi' = \frac{\beta}{\xi}$, the ξ coordinate of the contact point, C, of u is $\xi_c = -\frac{\beta}{l}$. Similarly the ξ' coordinate of the contact point, C', of u' is $\xi'_{c'} = -\frac{\beta}{m}$. Since β is negative, both ξ_c and $\xi'_{c'}$ are positive; i.e., the contact point C will lie to the right of O, figure 6, and the contact point C' will lie below O' on u'. CC' falls outside of the triangle IOO', which is Steiner's condition for a hyperbola.

If β is positive and $\beta > lm$, the conic is also a hyperbola. In this case $\beta > lm$, and therefore $-\frac{\beta}{l} < -m$. But $\xi_c = -\frac{\beta}{l}$ and -m is the ξ coordinate of I; therefore C must lie to the left of I on u (figure 7). Similarly

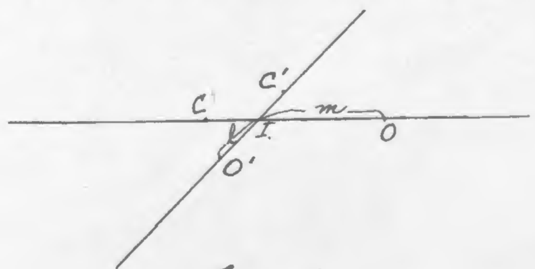

Fig. 7.

if $\beta > lm$, $-\frac{\beta}{m} < -l$; and since $\xi'_{c'} = -\frac{\beta}{m}$ and -1 is the ξ' coordinate of I, C'. must lie above I on u'. The line CC' again lies outside the triangle IOO', which is Steiner's condition for a hyperbola.

If β is positive and $\beta < lm$, the conic is an ellipse.

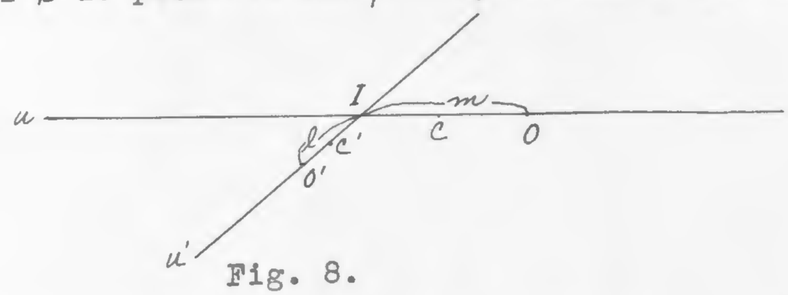

Fig. 8.

But if $\beta < lm$, we have $-\frac{\beta}{l} > -m$, and C lies to the right of I, (figure 8), but to the left of O, for $\frac{\beta}{l} < m$. Similarly C' lies between O' and I. Then we have the contact line lying between the vanishing points and the intersection, which is Steiner's condition for an ellipse.

#11. The results obtained thus far may be summarized as follows:- The envelope of connectors of corresponding points is:-

<u>a parabola</u>, if the ranges are similar, $\xi' = \alpha \xi$.

<u>an ellipse</u>, 1) when the ranges are parallel and β is positive in the equation $\xi = \frac{\beta}{\xi}$; 2) when neither vanishing point is at the point of intersection of two non-parallel ranges and βlm is positive with $\beta < lm$ (l and m are the distances of the vanishing points from the intersection).

a hyperbola, 1) when the ranges are parallel and β is negative; 2) when either one or both of the vanishing points are at the intersection of two non-parallel ranges; 3) when βlm is negative; 4) when βlm is positive and $|\beta| > |lm|$ Each of these conics may degenerate into a point under certain conditions.

III. THE CONIC AS THE LOCUS OF THE INTERSECTIONS OF CORRESPONDING RAYS.

#12. The study of the nature of the conic generated by projective pencils divides itself into three parts: 1) when the centers of both pencils are in the finite plane; 2) when one center is in the finite plane and one is at infinity; 3) when both centers are at infinity.

#13. If we have two pencils in perspective, B and B', there are two rays of B, s and t, perpendicular to each other, whose corresponding rays, s' and t' of B' are also perpendicular to each other. This theorem can be proved as follows: Let B and B' have their corresponding rays meet on CD. Draw the perpendicular bisector of BB'. With the point of

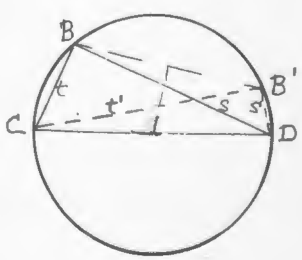

Fig. 9.

intersection of the bisector with CD as a center, describe a circle thru B and B'. CD becomes the diameter of the circle. Draw BC, BD, B'C, and B'D. BC is perpendicular to BD and B'C to B'D. BC and B'C are corresponding rays and BD and B'D are corresponding rays for they meet on CD. That BC, B'C and BD, B'D are the only rays of B and B' which satisfy these conditions can be proved as follows:

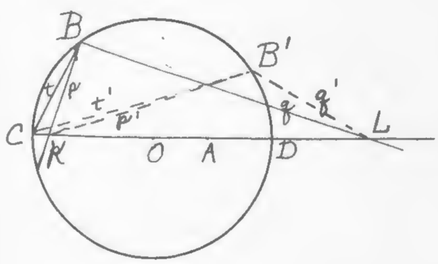

Fig. 10.

Let p and q be any two other rays of B which are perpendicular to each other. Let p' and q' be the corresponding rays of B'. p and p' meet in K on Cd, q and q' in L. Assume that p' and q' are perpendicular to each other. Then since KBL and KB'L are both right angles, KL must be the diameter of a circle passing thru B and B', and A, the middle point of KL, must be the center. But the center of any circle thru B and B' must lie on the perpendicular bisector of BB'. Since this bisector intersects CKDL in O, A must coincide with O, i.e. K and L must coincide with C and D respectively and p, p' q',q with s, s' t', t respectively. Hence B has only one pair of perpendicular rays such that the corresponding rays of B' are also perpendicular to each other.

This same theorem holds for two pencils which are projective but not in perspective.

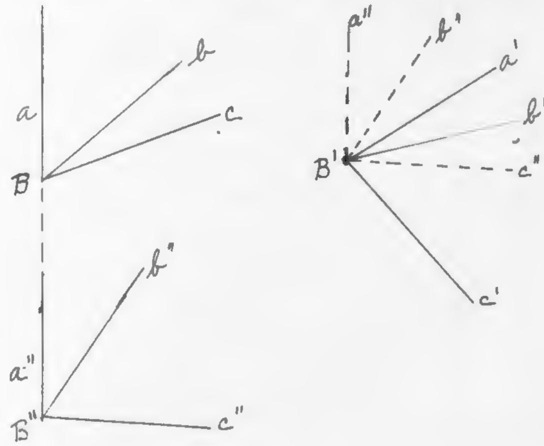

Fig. 11.

Let B and B' be two projective pencils. Rotate the pencil at B' about its vertex until a' is parallel to a. Translate B' so that its vertex lies on a of pencil B. Then B and B'' are in perspective for they have a self corresponding ray. By the previous theorem B and B" have each a pair of perpendicular rays whose corresponding rays are also perpendicular. If s" and t" of B" are perpendicular to each other, s' and t' of B' are also perpendicular to each other. Hence the theorem holds true for B and B', two pencils which are projective but not in perspective.

#14. If we have two projective pencils B and B' with s perpendicular to t and s' perpendicular to t', the product tan (tr)xtan (s'r') is constant, where r is any ray of B and r' is the corresponding ray of B'. This theorem may be proved as follows:-

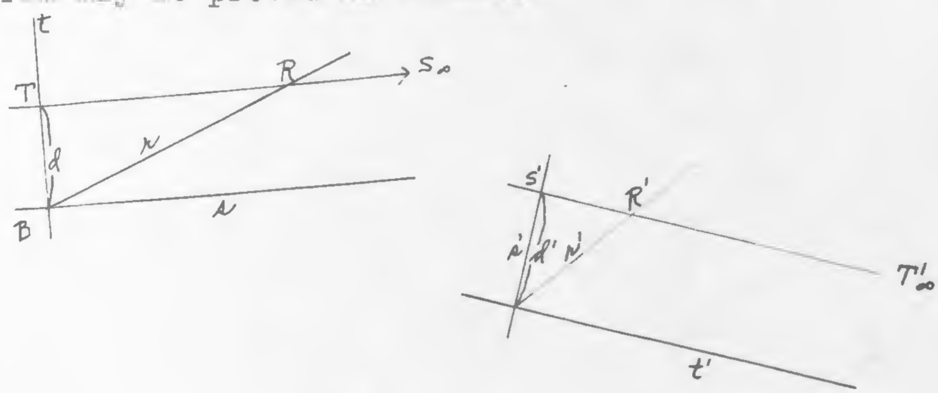

Fig. 12.

Cut B by a transversal parallel to s, cutting the rays in T, R, and S . Cut B' by a transversal parallel to t', intersecting the rays in T' ,R', S'. TRS and T' R'S' are projective ranges. Since S lies at infinity, S' is the vanishing point of the range T'R'S'. Similarly T is the vanishing point of the range TRS . From this we have the relation,TR·S'R' is constant.[1] Therefore,

$$\frac{TR}{d} \cdot \frac{S'R'}{d'} = \text{a constant.}$$

i.e. tan (tr)·tan (s'r')=a constant.

If the relation tan(tr)·tan(s'r)=a constant is given,

1. Cremona: #74.

it follows that the two pencils are projective. If $\tan(tr)\cdot\tan(s'r') =$ a constant, $TR\cdot S'R' =$ a constant. Since R and R' are any two corresponding points, this equation is the same as the equation $\xi' = \frac{\beta}{\xi}$. By #4 the two ranges TRS and T'R'S' will be projective, and hence the two pencils from which the ranges were derived are projective.

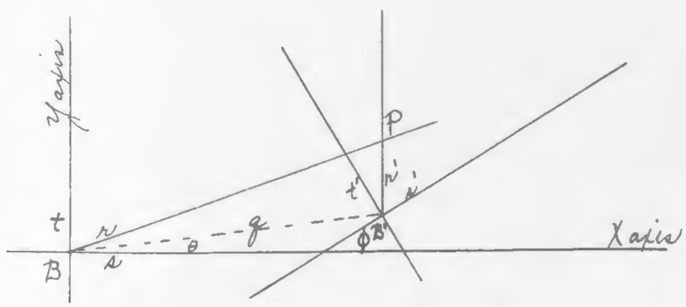

Fig. 13.

#15. In figure 13, B and B' are two projective pencils. s is perpendicular to t and s' to t'. ϕ is the angle between s and s'. B is chosen as the origin and B' is the point $(q\cos\theta, q\sin\theta)$. The equation of r is $y=mx$. m is the tangent of $[90-(rt)] - [90+(tr)]$. Therefore, $\tan(tr) = -\frac{1}{m}$ and since $\tan(tr)\cdot\tan(s'r') = k$, $\tan(s'r') = -km$. Therefore, the slope of $r' = \tan[\phi + (s'r')] = \frac{\tan\phi - km}{1 + km\tan\phi}$. The equation of r' will then be

$$y - q\sin\theta = \frac{\tan\phi - km}{1 + km\tan\phi}\,(x - q\cos\theta).$$

If we eliminate m between the equations for r and r', we obtain as the equation of the locus of P, the intersection of r and r',

$$x^2\tan\phi - y^2k\tan\phi - xy(k+1) + x(q\sin\theta - q\cos\theta\tan\phi)$$
$$+ y(kq\cos\theta + kq\sin\theta\tan\phi) = 0. \qquad (18)$$

The $h^2 - ab$ of equation (18) is $\left(\frac{k+1}{2}\right)^2 + k\tan^2\phi$. If k is positive, $h^2 - ab$ is positive, and the locus is a hyperbola. (Plate XXIII). When k is positive the two pencils are oppositely directed.. This gives us the same result that Steiner obtained by synthetic methods, namely that the locus of the intersection of the corresponding rays of two projective pencils is always a hyperbola when the pencils are oppositely directed.

When the two pencils are equally directed, k is negative. Let $k = -j^2$. The $h^2 - ab$ of equation (18) then becomes $\left(\frac{1-j^2}{2}\right)^2 - j^2\tan^2\phi$. Let j (a constant which can have any value from $-\infty$ to $+\infty$) $= \tan\Delta$ (which can have any value from $-\infty$ to $+\infty$). Then this value of $h^2 - ab$ becomes

$$\left(\frac{1-\tan^2\Delta}{2}\right)^2 - \tan^2\Delta\tan^2\phi$$

$$= \tan^2\Delta\left[\left(\frac{1-\tan^2\Delta}{2\tan\Delta}\right)^2 - \tan^2\phi\right]$$

$$= \tan^2\Delta\left[\cot^2 2\Delta - \tan^2\phi\right].$$

The conic is an ellipse, parabola, or hyperbola according as this is less than, equal to, or greater than zero.

But the angle ϕ can always be chosen as the acute angle between s and s' or t and t'. The angle Δ can also be chosen acute, for as Δ varies from 0 to 90°, $k=-j^2=-\tan^2\Delta$ will have all possible negative values from 0 to $-\infty$. Therefore, if $\cot 2\Delta = \tan\phi$, (18) represents a parabola; if $\cot 2\Delta > \tan\phi$, a hyperbola; if $\cot 2\Delta < \tan\phi$; an ellipse. If k lies between 0 and -1, $\tan\Delta$ is less than 1 and 2Δ is acute. The condition that (18) represent a parabola will then be $\tan(90-2\Delta)=\tan\phi$ or $90-2\Delta=\phi$ or $\phi+2\Delta=90$ (Plate XIX). For a hyperbola, the condition is $\phi+2\Delta<90$ (Plate XX), and for an ellipse, $\phi+2\Delta>90$ (Plate XXI).

If k is less than -1, 2Δ will be obtuse. The conditions determining the nature of the conic can then be written as follows:

$\phi+90=2\Delta$ --parabola

$\phi+90<2\Delta$ --hyperbola

$\phi+90>2\Delta$ --ellipse

If $k=-1$, (18) becomes

$$x^2\tan\phi+y^2\tan\phi+x(q\sin\theta-q\cos\theta\tan\phi)+y(q\cos\theta+q\sin\theta\tan\phi)=0. \quad (19)$$

Which is the equation of a circle (Plate XXII). When $k=-1$, $\tan(tr)\cdot\tan(s'r')=-1$; i.e. the tangent of one angle is the negative reciprocal of the other, or r and r' are always perpendicular to each other. Then, k being negative, the two pencils are directly equal. Therefore, when the two pencils are directly equal, a circle is generated.

If $k = 1$, (18) reduces to

$$x^2\tan\phi - y^2\tan\phi - 2xy + x(q\sin\theta - q\cos\theta\tan\phi) + y(q\cos\theta + q\sin\theta\tan\phi) = 0 \qquad (20)$$

which is the equation of an equilateral hyperbola (Plate XXIII) In this case the two pencils are equal but oppositely directed.

The discriminant of (18) is D

$$D = \frac{kq^2}{2} \sec^2\phi\left[\tan\phi(\cos^2\theta - k\sin^2\theta) - \sin\theta\cos\theta(1+k)\right].$$

$$D = 0 \text{ if } \tan\phi = \frac{\sin\theta\cos\theta(1+k)}{\cos^2\theta - k\sin^2\theta}.$$

In this case equation (18) reduces to

$$\left[x\sin\theta - y\cos\theta\right]\left[x\cos\theta(1+k) + k(1+k)y\sin\theta - kq\right] = 0 \qquad (21)$$

which represents two straight lines, one of which is the line joining the centers of the two pencils (Plate XXIV). The two pencils are in perspective, the line joining the two centers being a self corresponding ray, and all the other corresponding rays intersect on the other straight line of the locus.

If $\phi = 90$, the coefficients of x and y in (18) become infinite. The equation for this case can be obtained from figure 14.

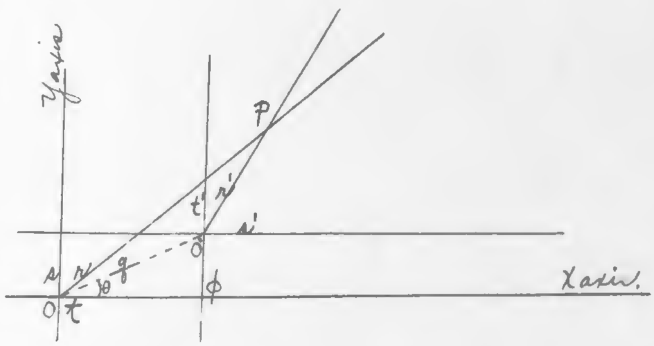

Fig. 14.

The equation of r is $y = mx$. m is the tan(tr); therefore, $\frac{k}{m}$ is the tan(s'r'). The equation of r' is therefore $(y-q\sin\theta) = \frac{k}{m}(x-q\cos\theta)$. Eliminating m between the equations for r and r', we obtain as the equation of the locus of the intersection of r and r':-

$$kx^2+y^2+yq\sin\theta-xkq\cos\theta = 0 \tag{22}$$

The h^2-ab of (22) is k. Therefore, if k is positive, the locus is a hyperbola (Plate XXIII); if k is negative, an ellipse (Plate XXV).

If ϕ is 0, equation (18) reduces to

$$-xy(k+1)+xq\sin\theta+ykq\cos\theta = 0 \tag{23}$$

which represents a hyperbola unless $k = -1$ or $\theta = 0$. When $k = -1$, the line joining the centers of the two pencils is a self corresponding ray and all the other corresponding rays are parallel to each other (Plate XXVI).

If $\theta = 0$ when ϕ is 0, the locus consists of the two straight lines: $y = o$ and $-x(k+1) - kq\cos\theta = o$ (Plate XXVII) (24)

#16. When the center of one of the pencils is at infinity and the center of the other is in the finite plane, the locus of the intersections of corresponding rays is either a hyperbola, a parabola, or a straight line.

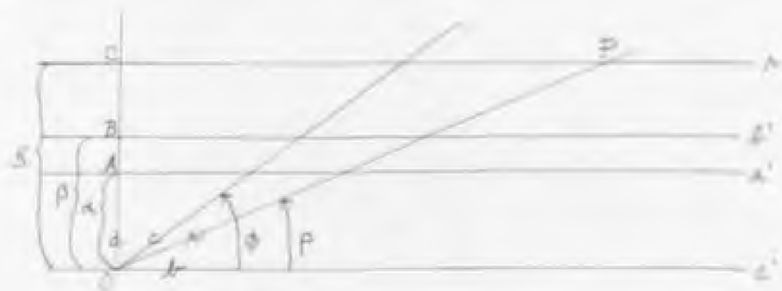

Fig. 15.

If the projective pencils are as in figure 15 with a, b, c corresponding to a', b', c' respectively, the locus is a hyperbola. The proof is as follows: Let P be the intersection of any two corresponding rays r and r', whose equations are evidently $y = x\tan\rho$ and $y = s$. Since the pencils O and O' are projective, the anharmonic ratio (a'b'c'r') = (abcr)

But (OABC) = (a'b'c'r')

Therefore (OABC) = (abcr)

$$\frac{\beta}{\beta-\alpha} \div \frac{\xi}{\xi-\alpha} = \frac{\sin\phi}{1} \div \frac{\sin(\phi-\rho)}{\sin(90-\rho)}$$

$$\frac{\beta(\xi-\alpha)}{\xi(\beta-\alpha)} = \frac{\sin\phi\cos\rho}{\sin\phi\cos\rho-\cos\phi\sin\rho}$$

$$\frac{\beta\xi-\alpha\beta}{\beta\xi-\alpha\xi} = \frac{1}{1-\cot\phi\tan\rho}$$

$$\beta\xi-\alpha\beta-\beta\xi\cot\phi\tan\rho+\alpha\beta\cot\phi\tan\rho = \beta\xi-\alpha\xi$$

$$\tan\rho = \frac{\alpha(\beta-\xi)}{\beta\cot\phi(\alpha-\xi)} = \frac{\alpha(\beta-\xi)\tan\phi}{\beta(\alpha-\xi)}$$

Eliminating ρ and ξ between this equation and the equations for r and r', we get as the equation of the locus of P,

$$y = x\frac{\alpha(\beta-y)}{\beta(\alpha-y)}\tan\phi.$$

$$xy\alpha\tan\phi-\beta y^2-x\alpha\beta\tan\phi+\alpha\beta y=0 \quad \text{(Plate XXVIII)} \qquad (25)$$

The h^2-ab of (25) is $\frac{\alpha^2}{4}\tan^2\phi$, which is always positive; therefore (25) always represents a hyperbola. The discriminant cannot be equal to zero; hence (25) can not break up into linear factors.

If b' of figure 15 is the line at infinity, the figure becomes figure 16 and the locus is a parabola. The proof is as follows:

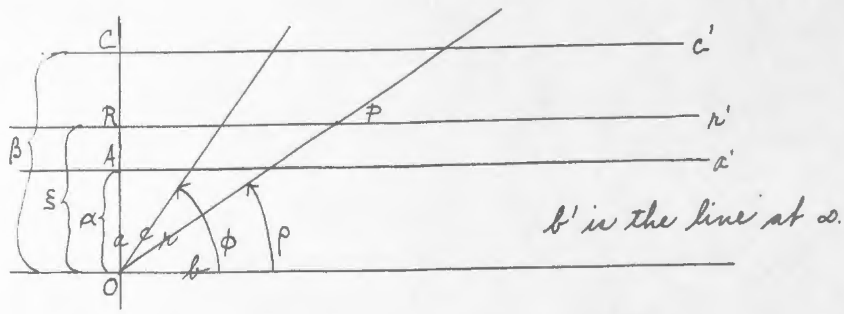

Fig. 16.

Let P be the intersection of any two corresponding rays r and r' whose equations are evidently $y = x\tan\rho$ and $y = \xi$.

Since O and O' are projective,

$(a'r'c'b') = (arcb)$

But $(ARCB) = (a'r'c'b')$

Therefore $(ARCB) = (arcb)$

$$\frac{AC}{RC} = \frac{\sin ac}{\sin rc} \div \frac{\sin ab}{\sin rb}$$

$$\frac{\beta-\alpha}{\beta-\xi} = \frac{\sin(90-\phi)}{\sin(\rho-\phi)} \div \frac{\sin 90}{\sin\rho}$$

$$\frac{\beta-\alpha}{\beta-\xi} = \frac{\cos\phi\sin\rho}{\sin\rho\cos\phi - \cos\rho\sin\phi} = \frac{1}{1-\tan\phi\cot\rho}$$

$$\beta-\alpha- \beta\tan\phi\cot\rho + \alpha\tan\phi\cot\rho$$

$$\frac{1}{\tan\rho}(\alpha\tan\phi - \beta\tan\phi) = \alpha-\xi$$

$$\tan\frac{(\alpha-\beta)}{(\alpha-\xi)} = \tan\rho$$

Eliminating ρ and ξ between this equation and the equation for r and r' we obtain the equation for the locus of P,

$$\alpha y - y^2 = x(\alpha - \beta) \tan \phi \qquad (26)$$

which represents a parabola (Plate XXIX).

If the line joining the two centers is a self corresponding ray, the two pencils are in perspective, and the locus of the intersection of corresponding rays is a straight line. The equation of the locus can be found as follows:

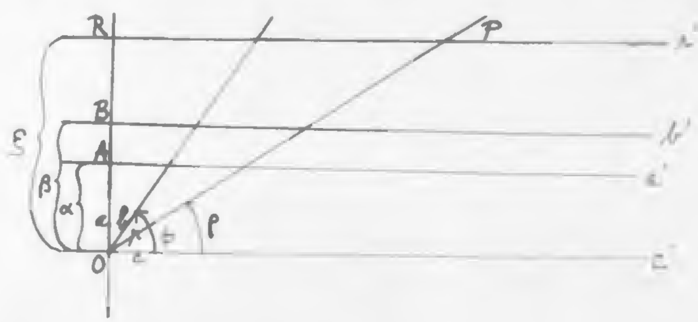

Fig. 17.

$$(OABR) = (c'a'b'r') = (cabr)$$

$$\frac{\beta}{\beta-\alpha} \div \frac{\xi}{\xi-\alpha} = \frac{\sin cb}{\sin ab} \div \frac{\sin cr}{\sin ar}$$

$$\frac{\beta(\xi-\alpha)}{\xi(\beta-\alpha)} = \frac{\sin\phi}{\sin(90-\phi)} \div \frac{\sin\rho}{\sin(90-\rho)}$$

$$\frac{\beta\xi-\alpha\beta}{\beta\xi-\alpha\xi} = \frac{\tan\phi}{\tan\rho}$$

$$\tan\rho = \frac{\tan\phi\,(\beta\xi-\alpha\xi)}{(\beta\xi-\alpha\beta)}$$

The equation of r' is $y=\xi$, and the equation of r is $y=x\tan\rho$. Eliminating ρ and ξ from the last three equations, we get

$$\beta y^2 - \alpha\beta y - xy(\beta-\alpha)\tan\phi = 0 \qquad (27)$$

or $y=0$ and $\beta y - x(\beta-\alpha)\tan\phi - \alpha\beta = 0$

$y=0$ is the self corresponding ray, and the other line is the locus of the intersections of corresponding rays. (Plate XXX).

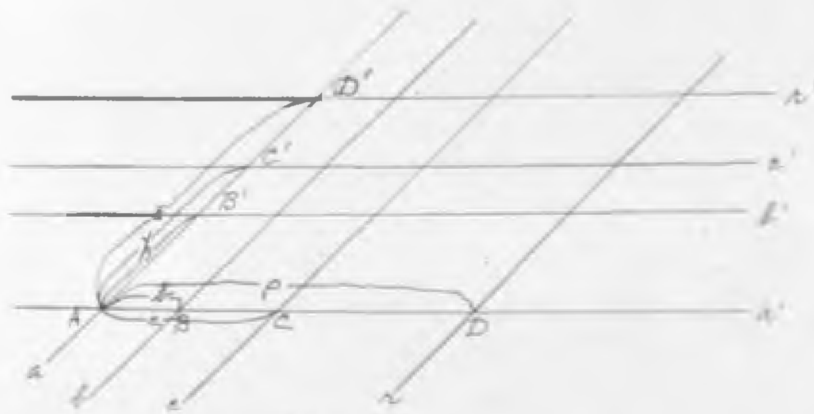

Fig. 18.

§17. In figure 18, both pencils have their centers at infinity. Using oblique coordinates, with a and a' as axes, $y=\xi$ is the equation of r', and $x=\rho$ is the equation of r. Since the two pencils are projective,

(ABCD) = (AB'C'D')

$$\frac{c}{c-b} \div \frac{\rho}{\rho-b} = \frac{\gamma}{\gamma-\beta} \div \frac{\xi}{\xi-\beta}$$

Let $\dfrac{c}{c-b} = k$ and $\dfrac{\gamma}{\gamma-\beta} = l$. (k and l are constants).

$$\frac{k(\rho-b)}{\rho} = \frac{l(\xi-\beta)}{\xi}$$

$k\rho\xi - kb\xi = l\xi\rho - l\beta\rho.$

Now $y = \xi$ and $x = \rho$; therefore the equation for the locus of P is

$$(k-1)xy - kby + l\beta x = 0 \tag{28}$$

which represents a hyperbola (Plate XXXI).

When k=1, (28) reduces to

$$l\beta x - kby = 0 \tag{29}$$

which is the equation of a straight line. When k = 1,

$$\frac{c}{c-b} = \frac{\gamma}{\gamma-\beta}$$

or $\frac{\beta}{\gamma} = \frac{b}{c}.$

When $\frac{\beta}{\gamma} = \frac{b}{c}$, the locus is a straight line, otherwise a hyperbola (Plate XXXII).

#18. The results obtained for projective pencils may be summarized as follows:-

The conic generated is:-

<u>a parabola</u>, 1) if $\phi + 90 = 2\Delta$ or $\phi + 2\Delta = 90$, k being negative; 2) when O' is at infinity and b which is the line OO' corresponds to the line at infinity.

<u>an ellipse,</u> if $\phi + 2\Delta > 90$ or $\phi + 90 > 2\Delta$, k being negative.

<u>a hyperbola,</u> 1) when k is positive; 2) when $\phi + 2\Delta < 90$ or $\phi + 90 < 2\Delta$, k being negative; 3) when one center is at infinity, b' lying in the finite plane; 4) when both

centers are at infinity.

Each of the conics may degenerate into one or two straight lines depending upon certain other relations already given.

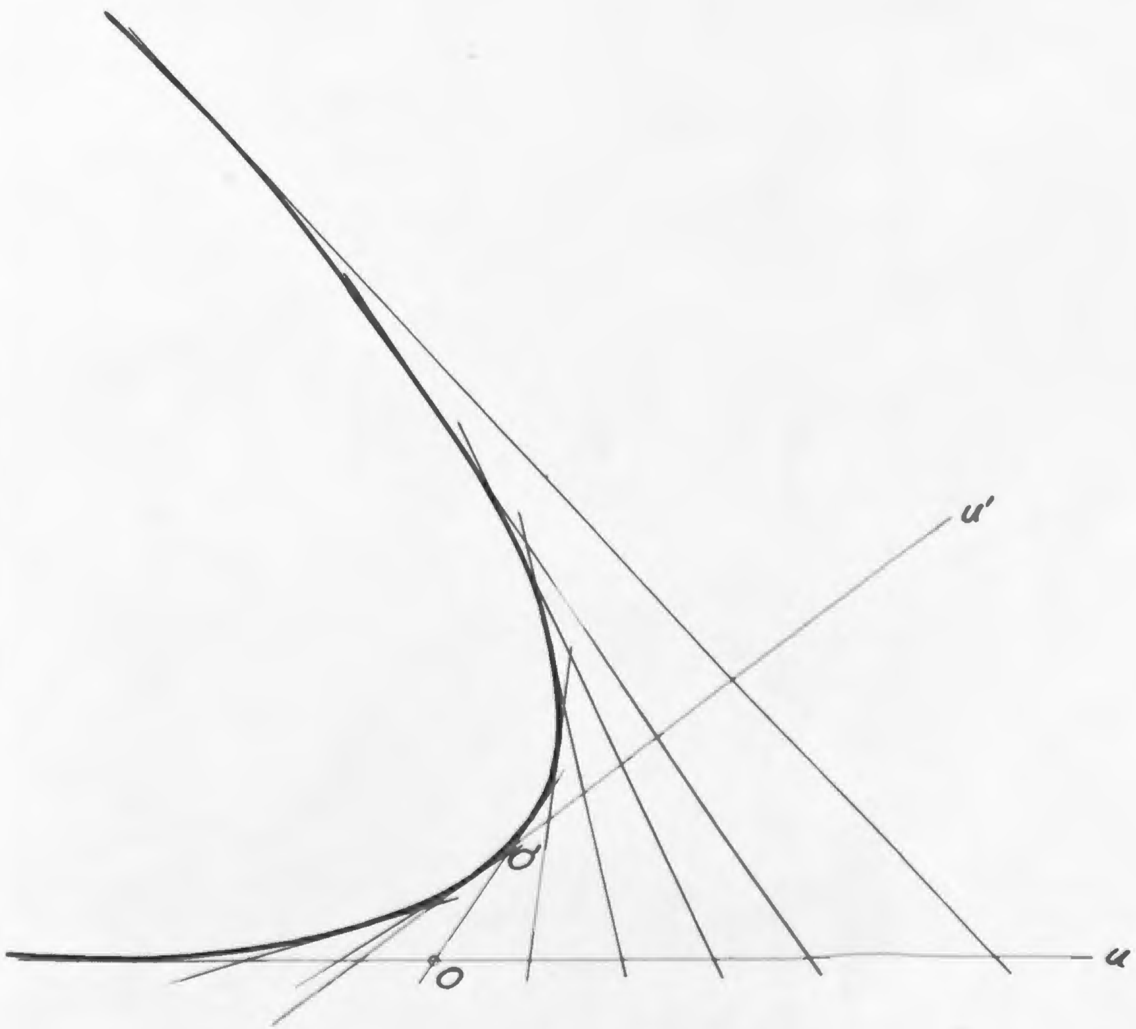

PLATE I

$\alpha = \frac{1}{2}$

$d = .3$

$l = .5$

$\theta = \tan^{-1}\frac{3}{4}$

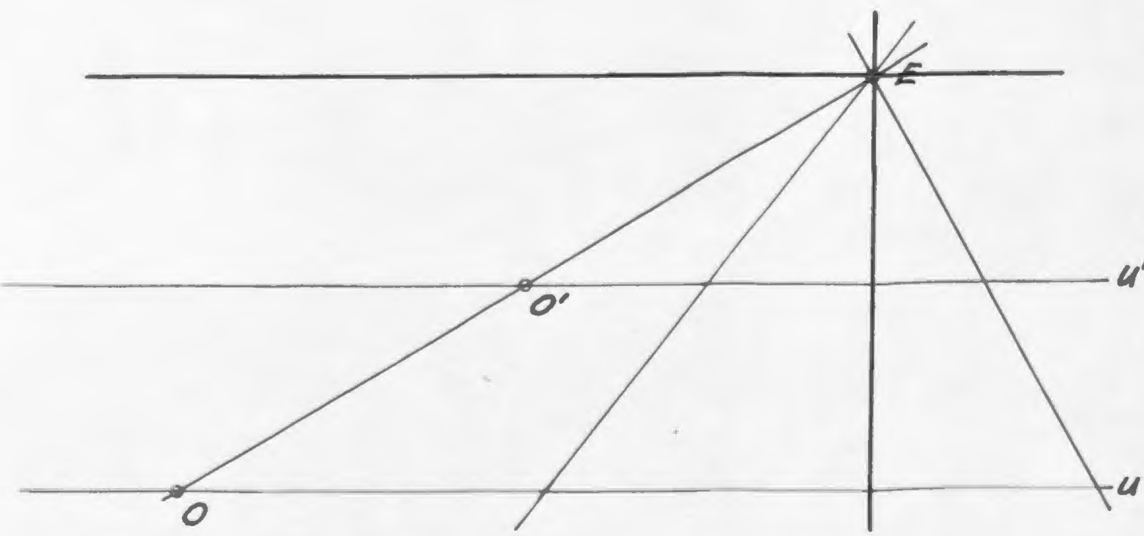

PLATE II

$\alpha = \frac{1}{2}$

$d = .3$

$l = .5$

$\theta = 0°$

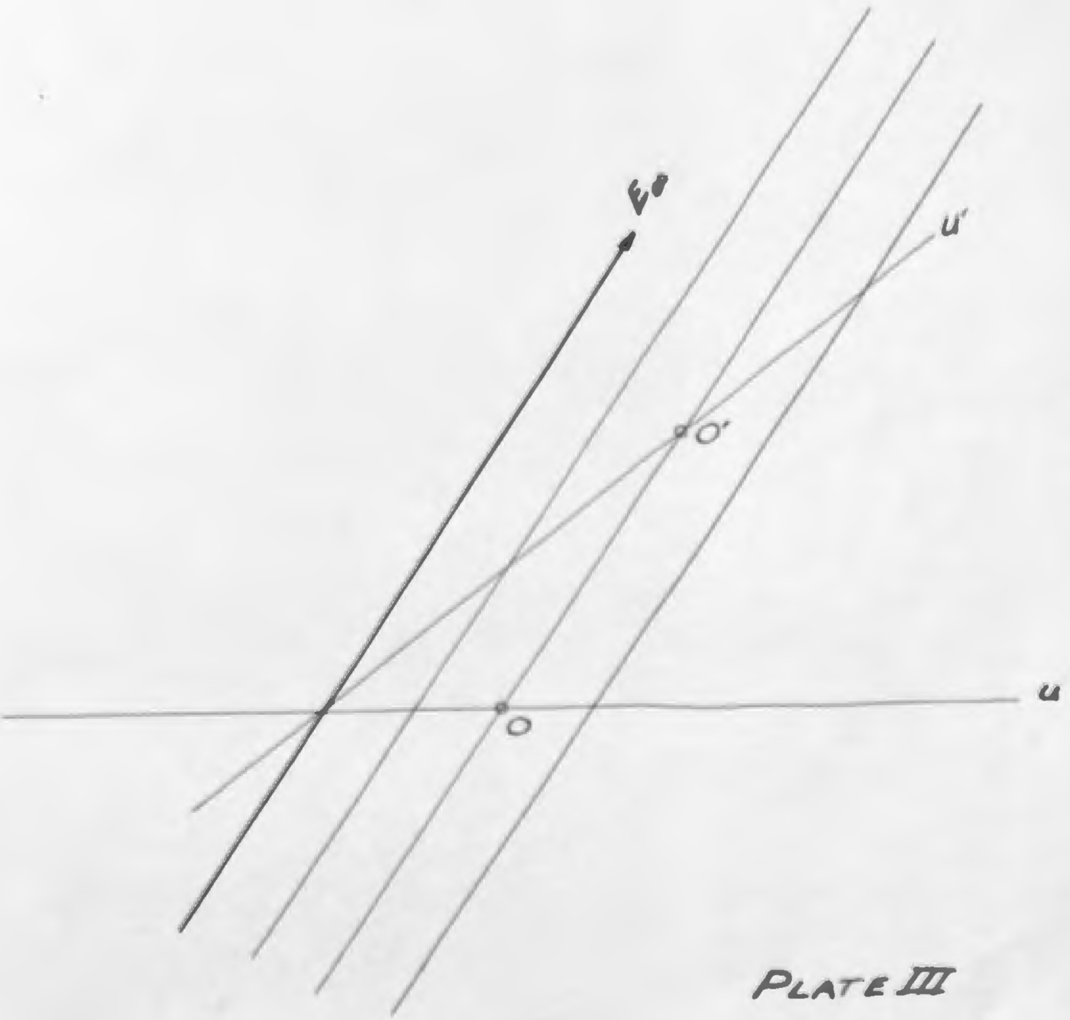

PLATE III

$\alpha = \frac{5}{2}$

$d = .3$

$\lambda = .5$

$\theta = \tan^{-1}\frac{3}{4}$

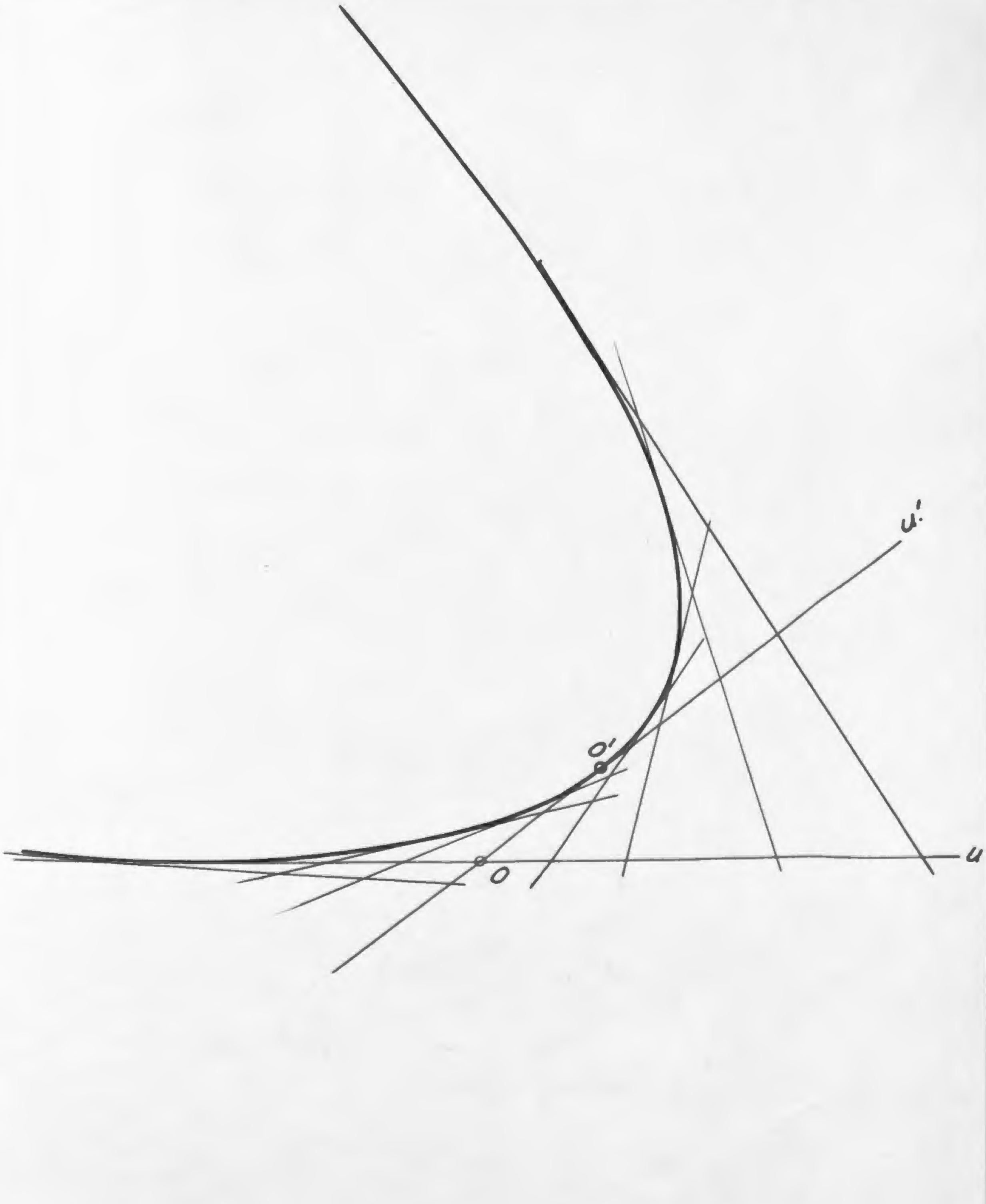

PLATE IV

$\alpha = \frac{1}{2}$

$d = 0$

$\eta = .5$

$\theta = \tan^{-1}\frac{3}{4}$

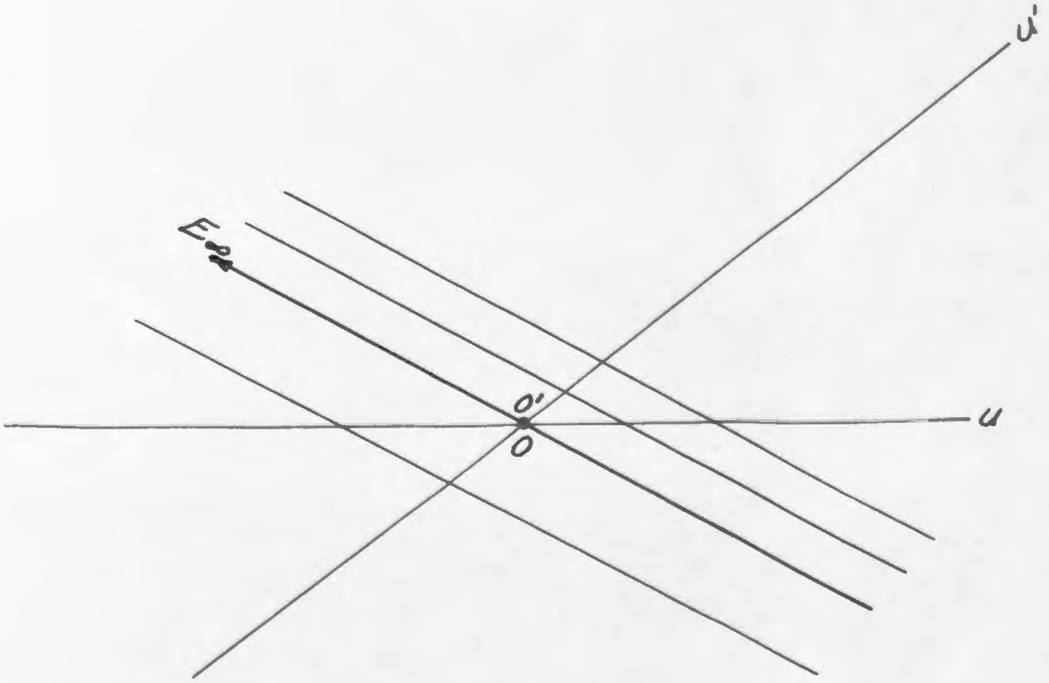

PLATE V

$\alpha = \frac{1}{2}$

$d = 0$

$l = 0$

$\theta = \tan^{-1}\frac{3}{4}$

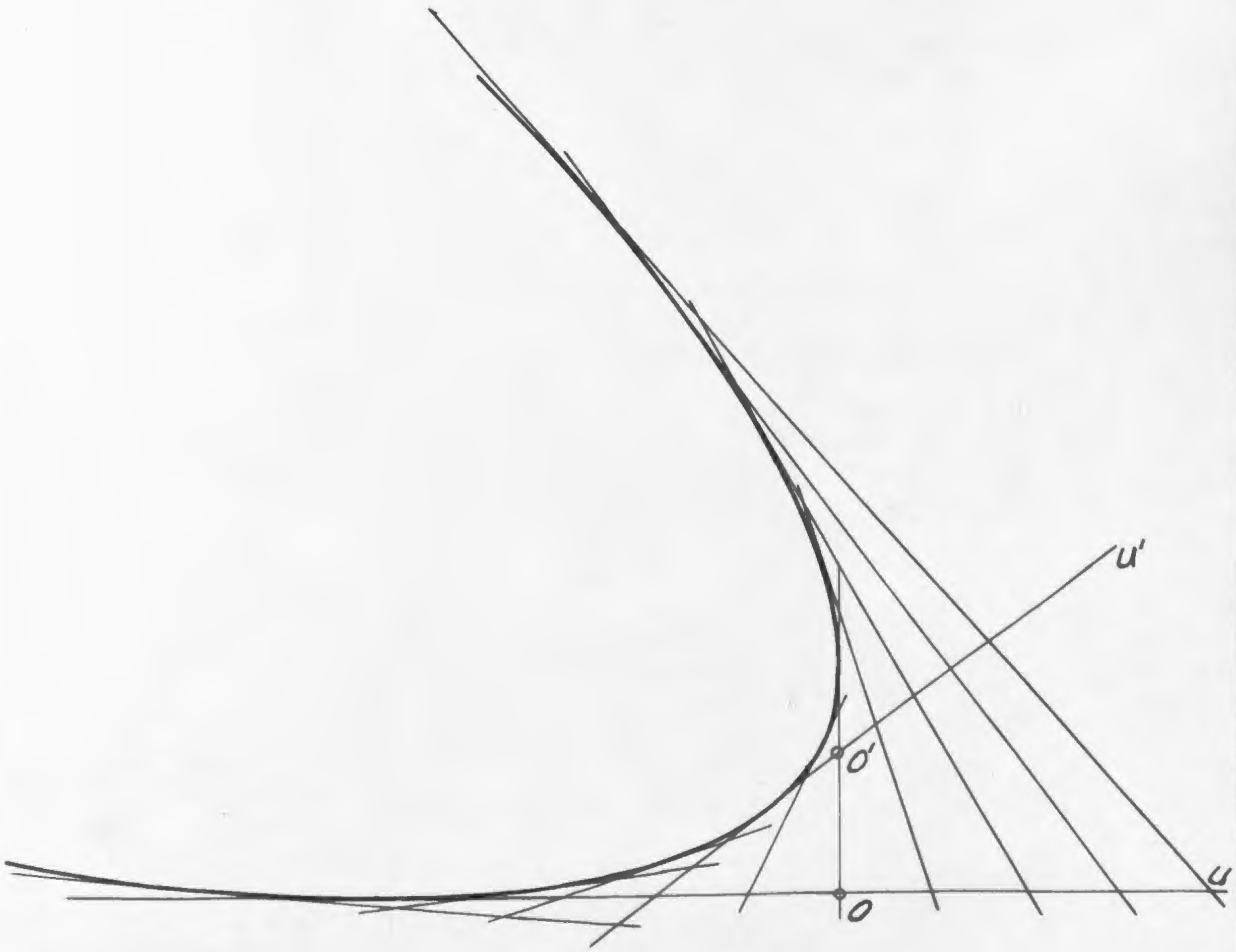

PLATE VI

$\alpha = \frac{1}{2}$

$d = .3$

$\gamma = 0$

$\theta = \tan^{-1}\frac{3}{4}$

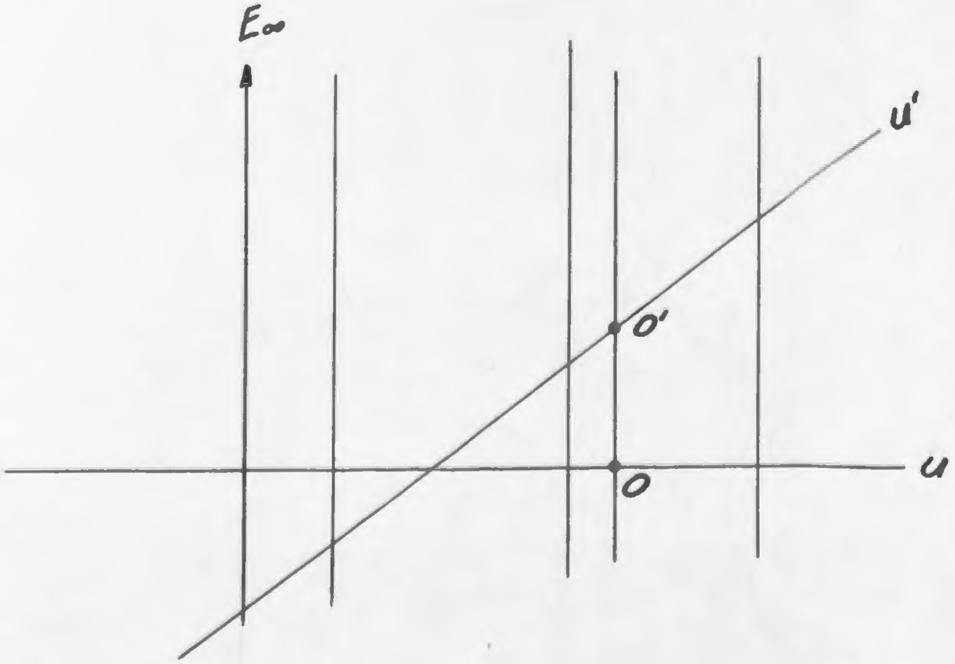

PLATE VII

$\alpha = \frac{5}{4}$

$d = .3$

$\eta = 0$

$\theta = \tan^{-1}\frac{3}{4}$

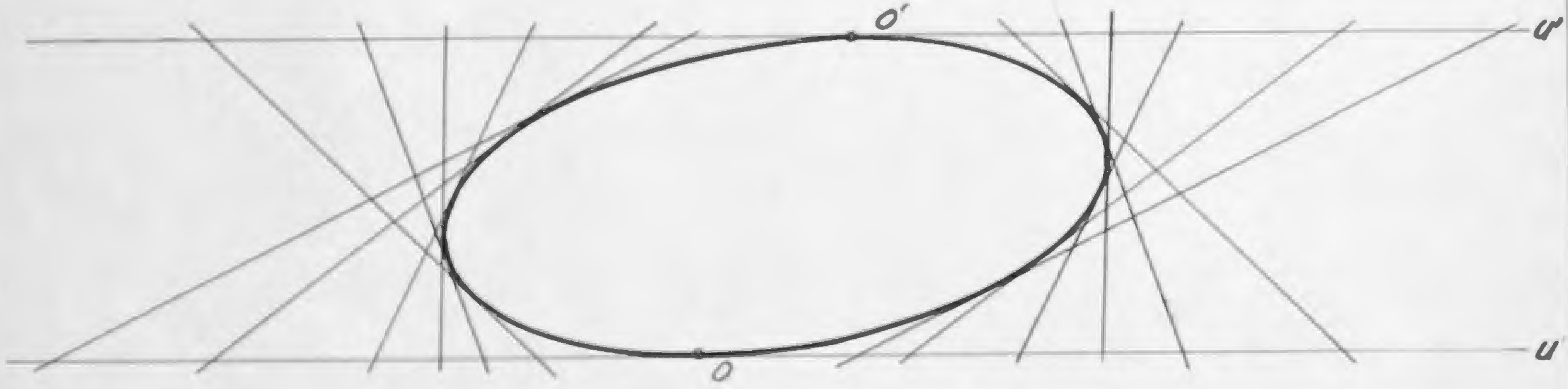

PLATE VIII

$\beta = +1$

$\lambda = .5$

$d = 1$

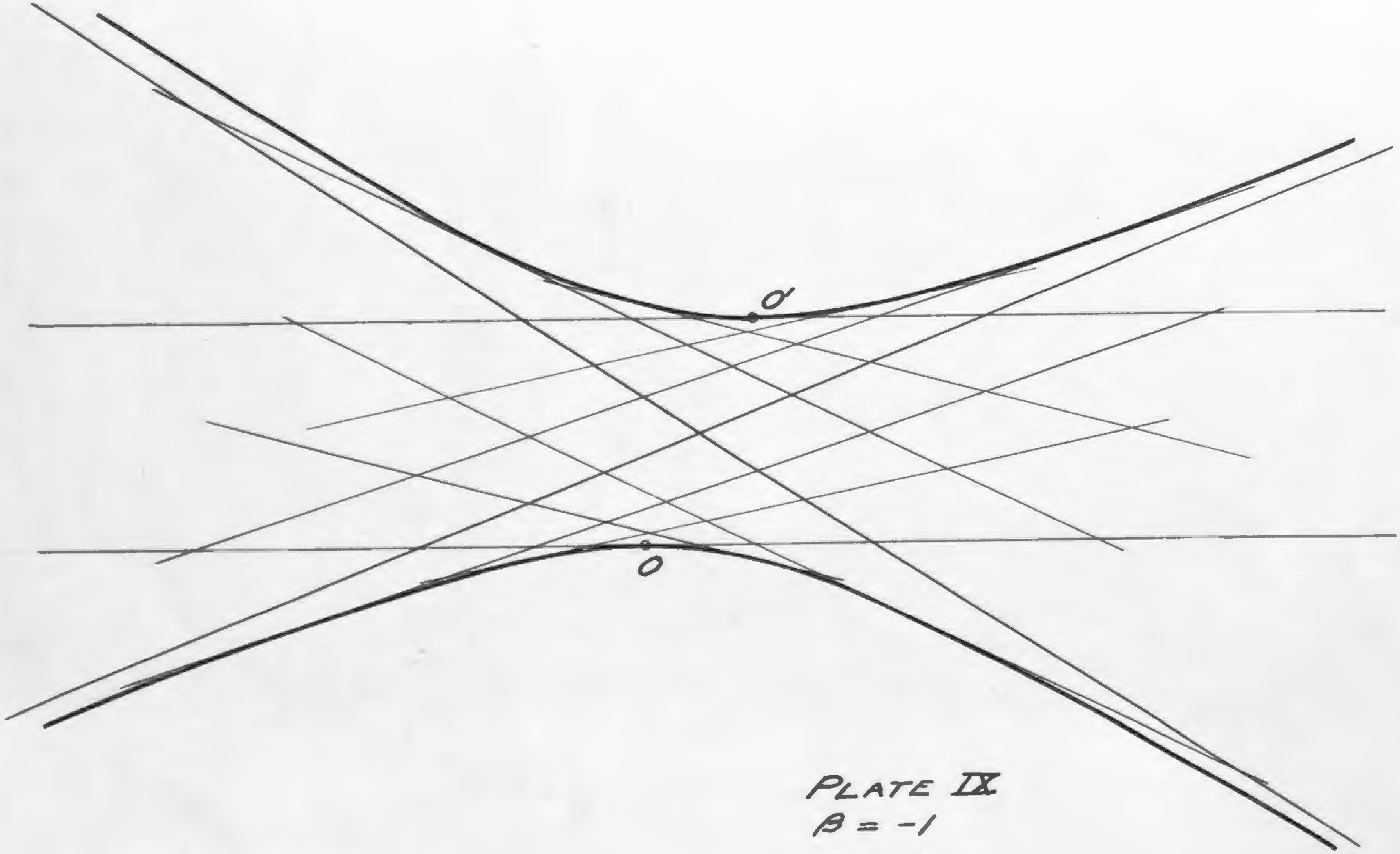

PLATE IX

$\beta = -1$

$\gamma = .5$

$d = 1$

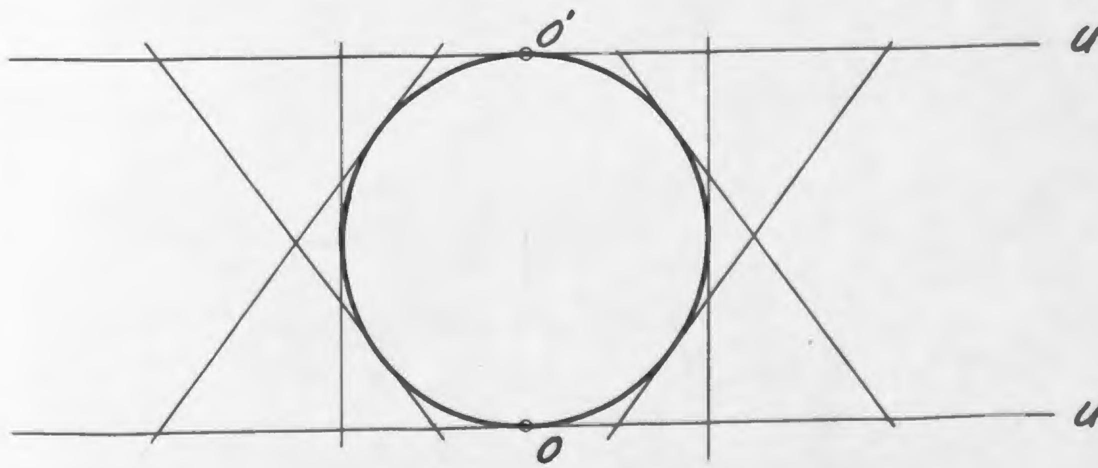

PLATE X.

$\beta = \frac{1}{4}$

$d = 1$

$l = 0$

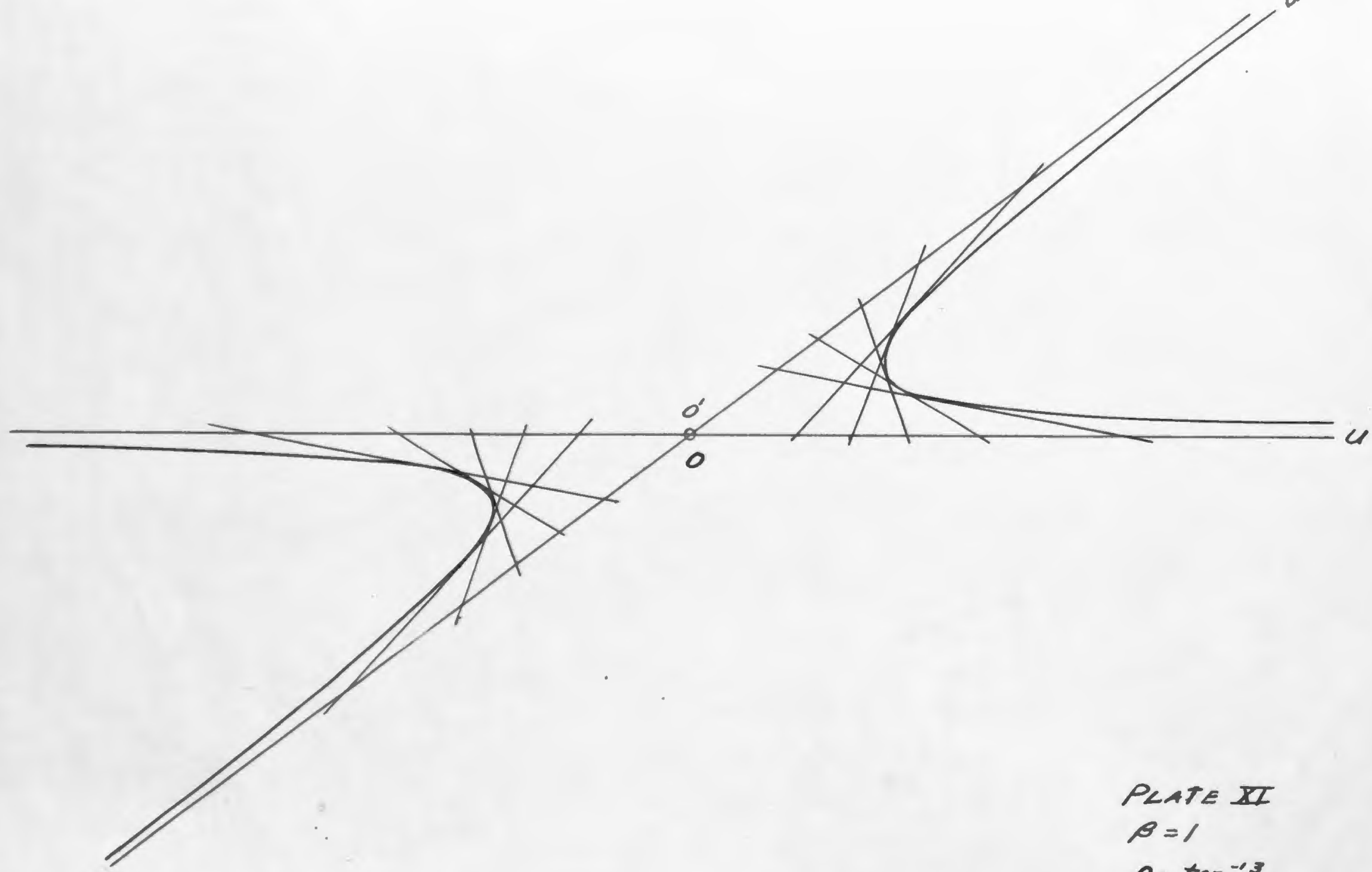

PLATE XI

$\beta = 1$

$\theta = \tan^{-1}\frac{3}{4}$

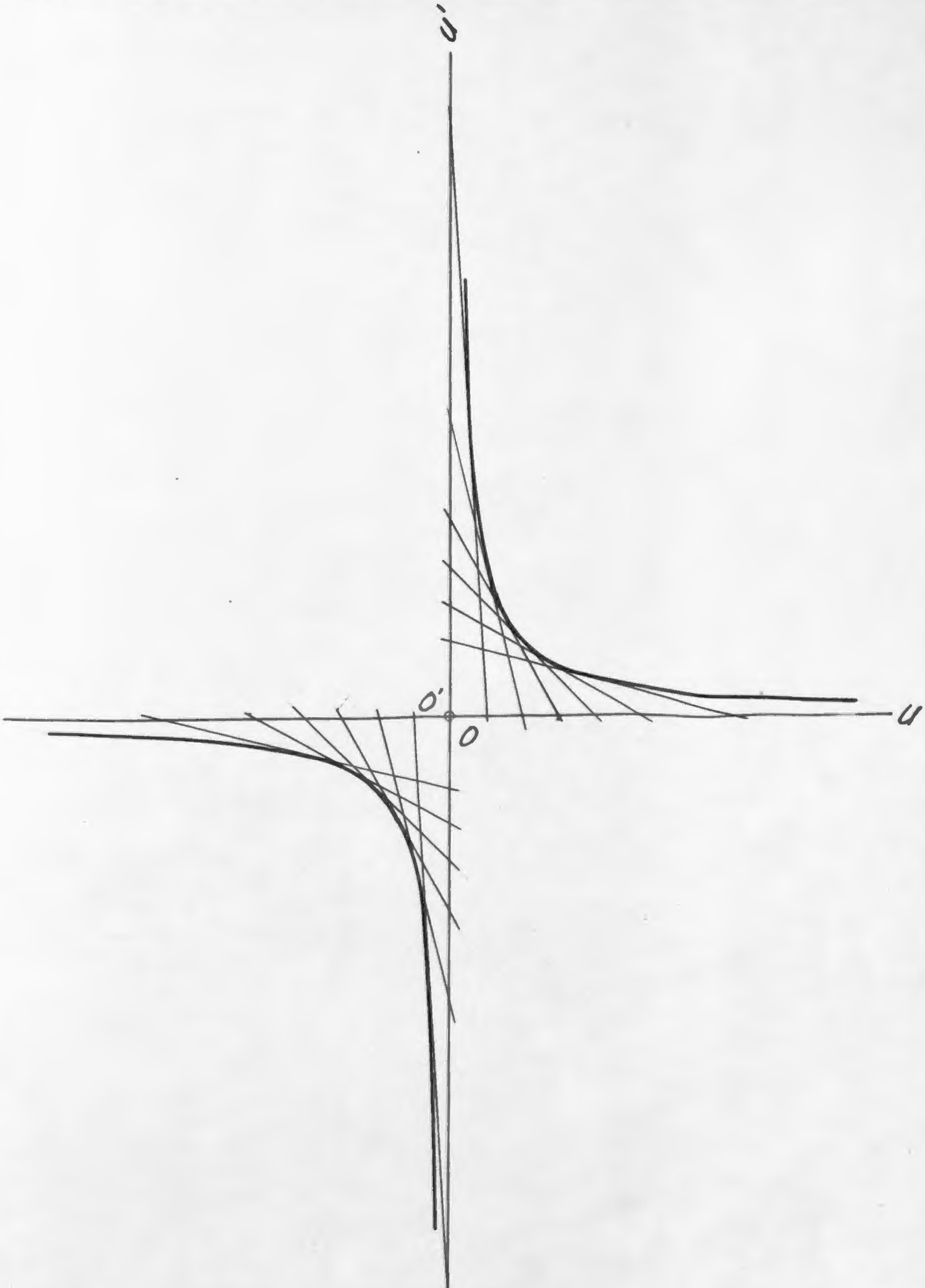

PLATE XII
$\beta = 1$
$\theta = 90°$

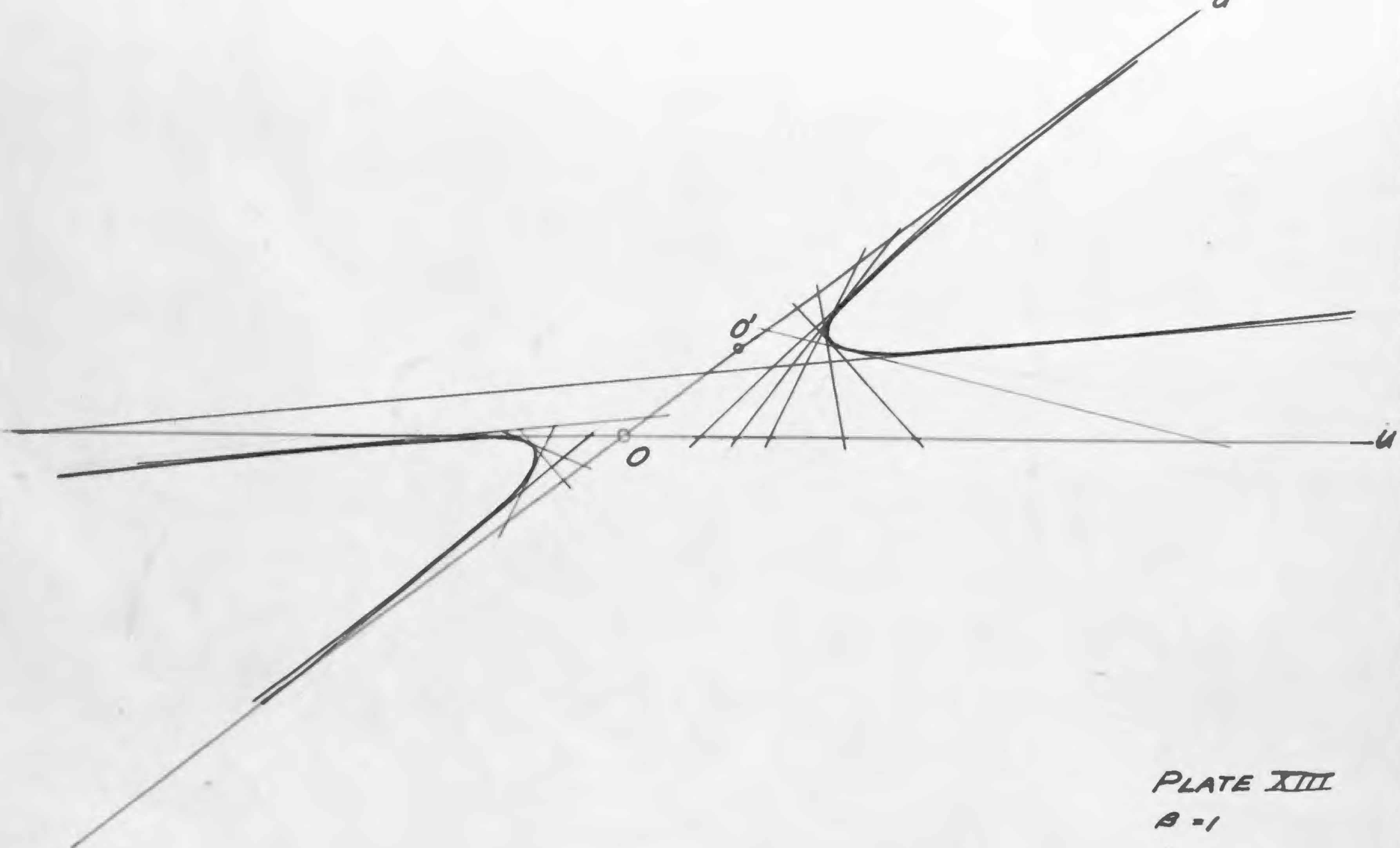

PLATE XIII

$\beta = 1$

$\gamma = 1$

$\theta = \tan^{-1} \frac{3}{4}$

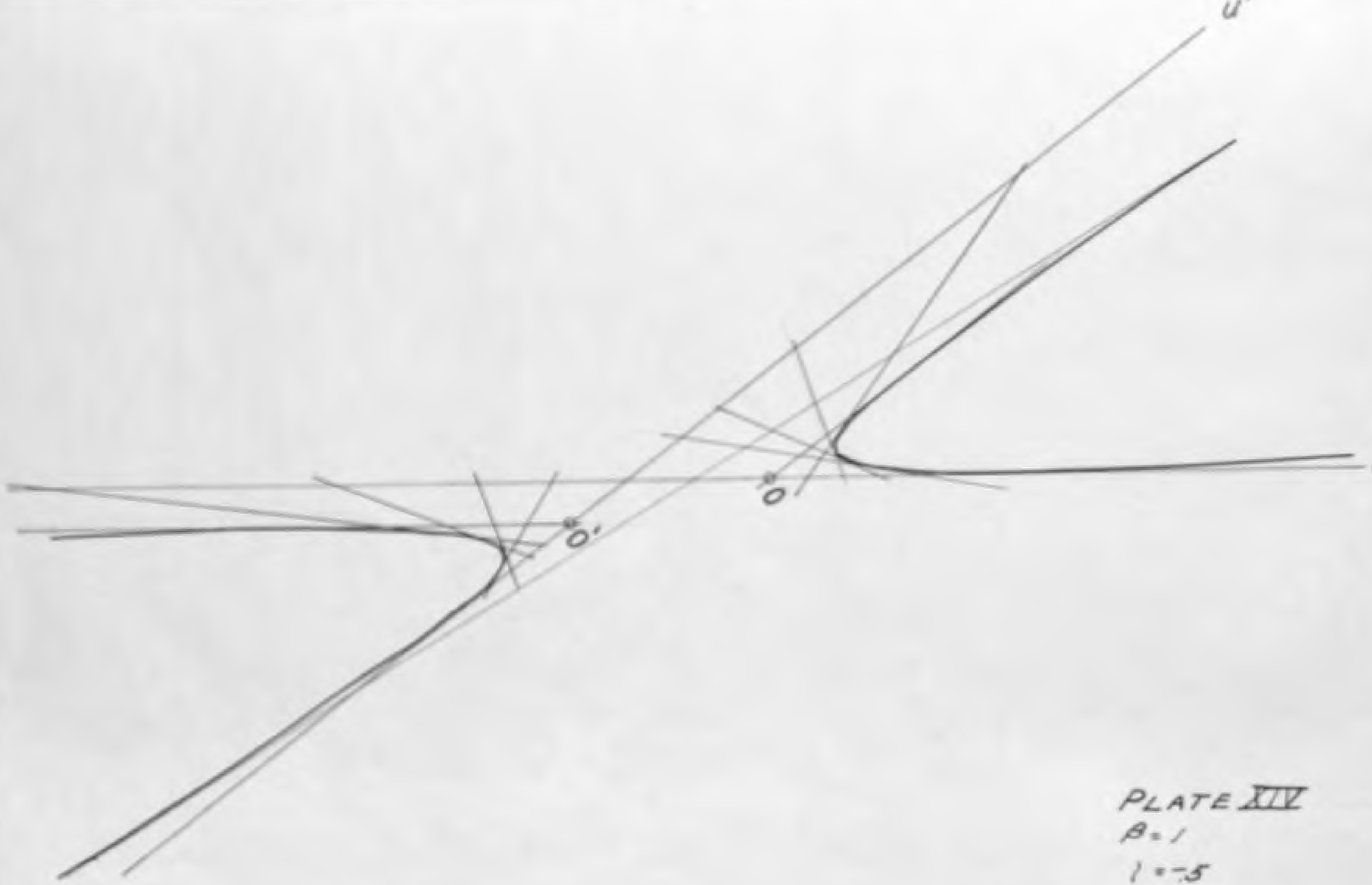

PLATE XIV

$\beta = 1$

$l = -.5$

$m = 1$

$\theta = \tan^{-1}\frac{3}{4}$

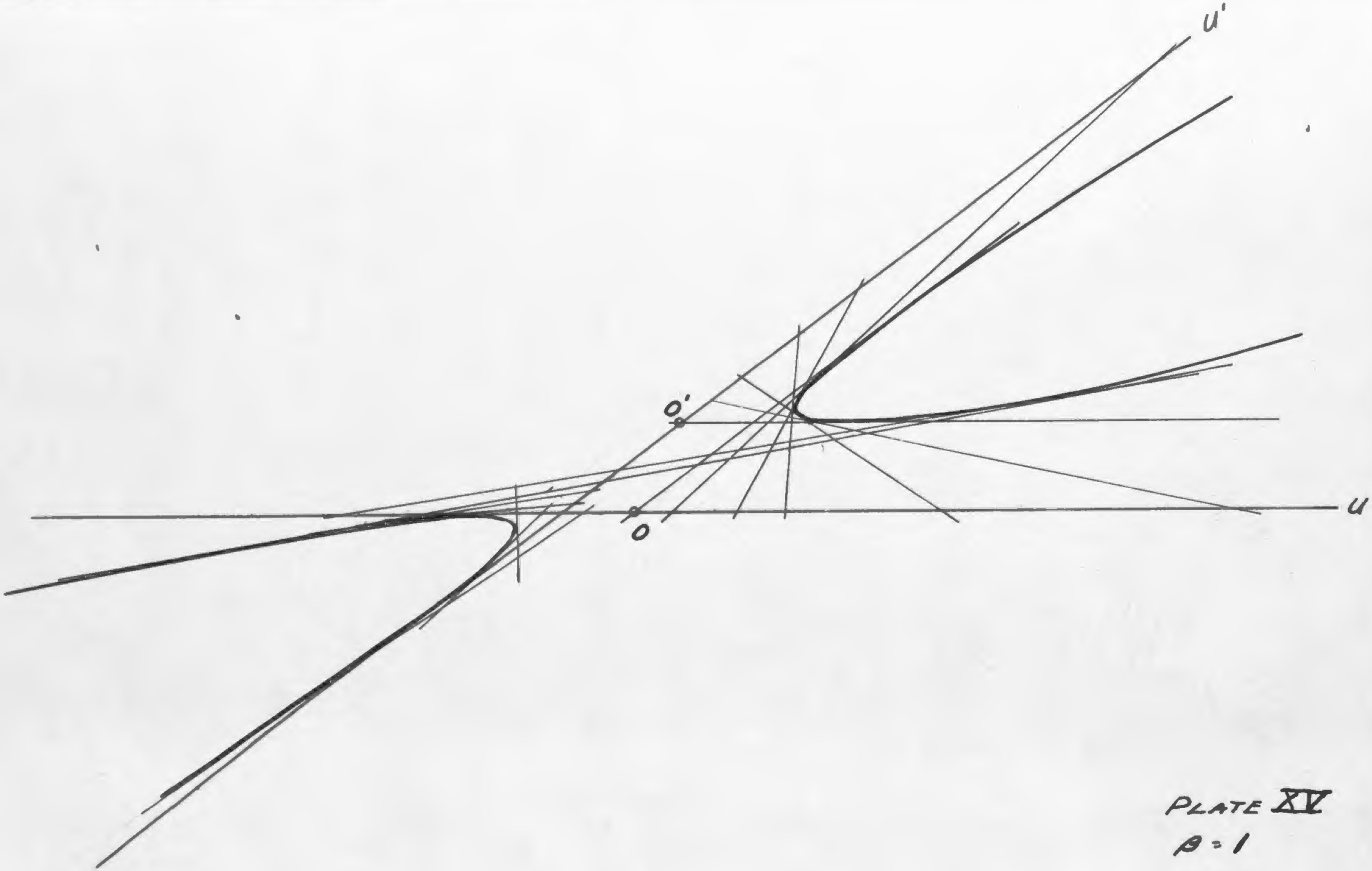

PLATE XV

$\beta = 1$

$l = 1$

$m = .5$

$\theta = \tan^{-1}\frac{3}{4}$

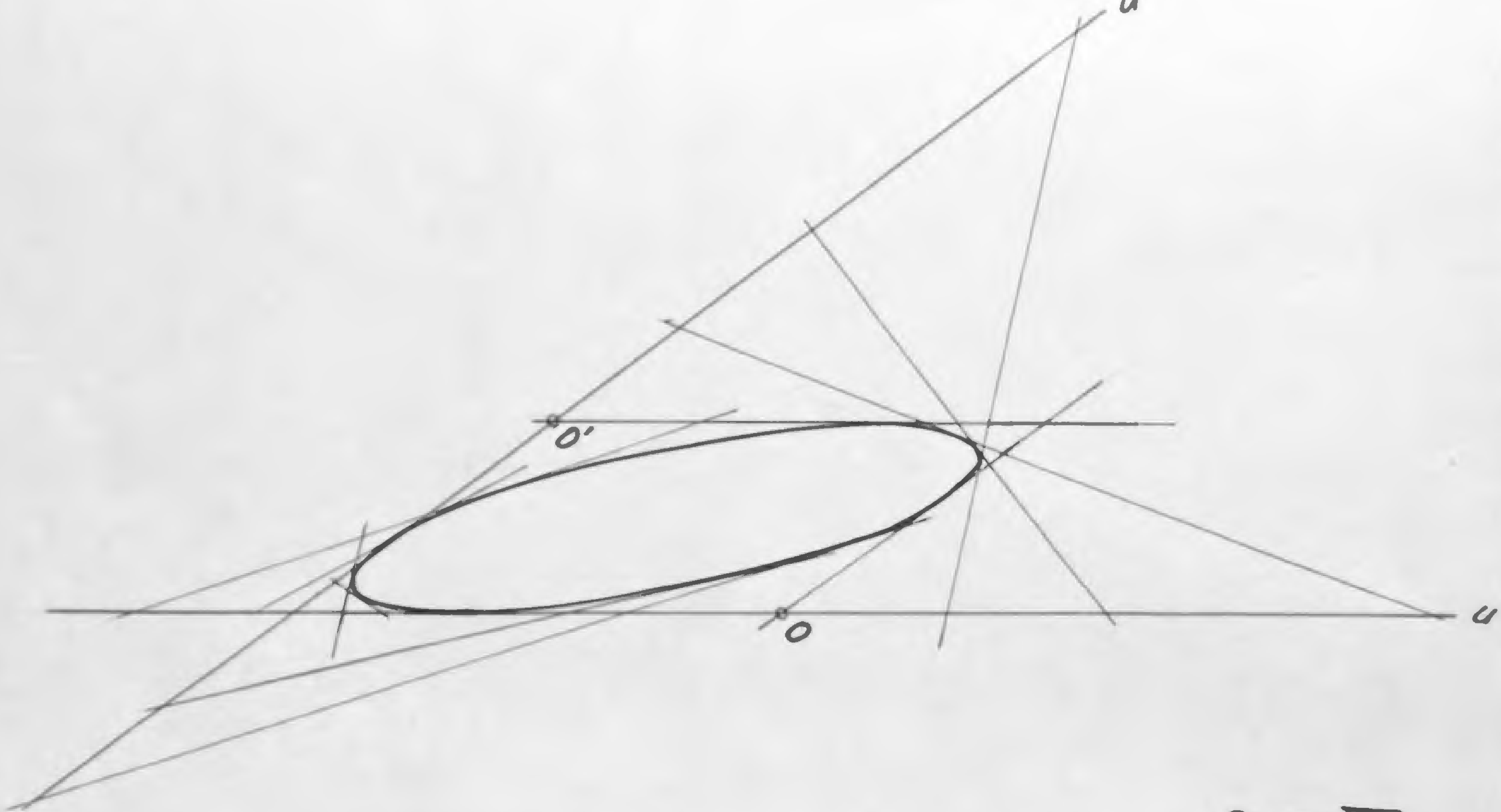

PLATE XVI

$\beta = 1$

$l = 1$

$m = 1.5$

$\theta = \tan^{-1}\frac{3}{4}$

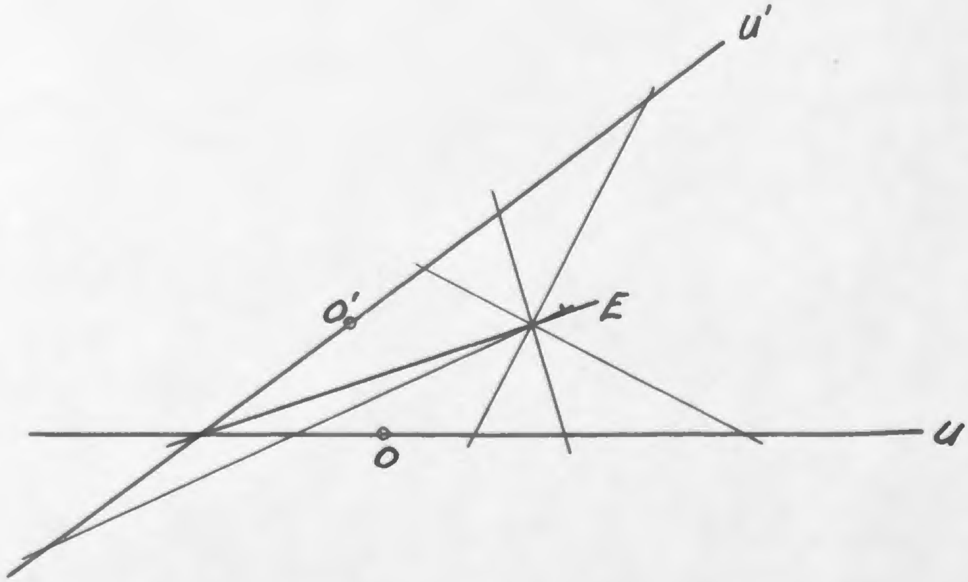

PLATE XVII

$\beta = 1$

$l = 1$

$m = 1$

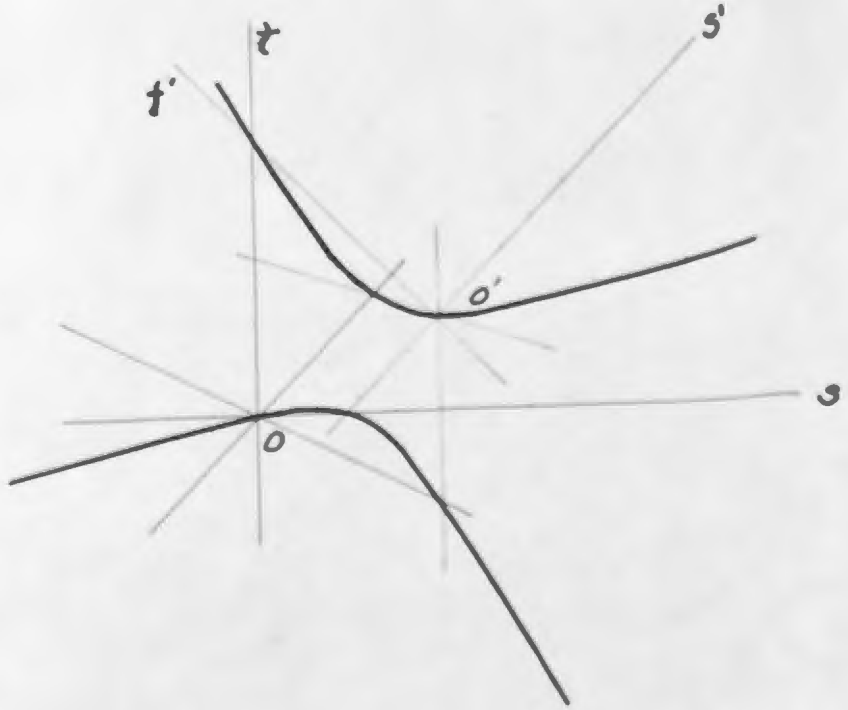

PLATE XVIII

$K = 2$

$\phi = 45°$

$\theta = \tan^{-1}\frac{1}{2}$

$q = \sqrt{5}$

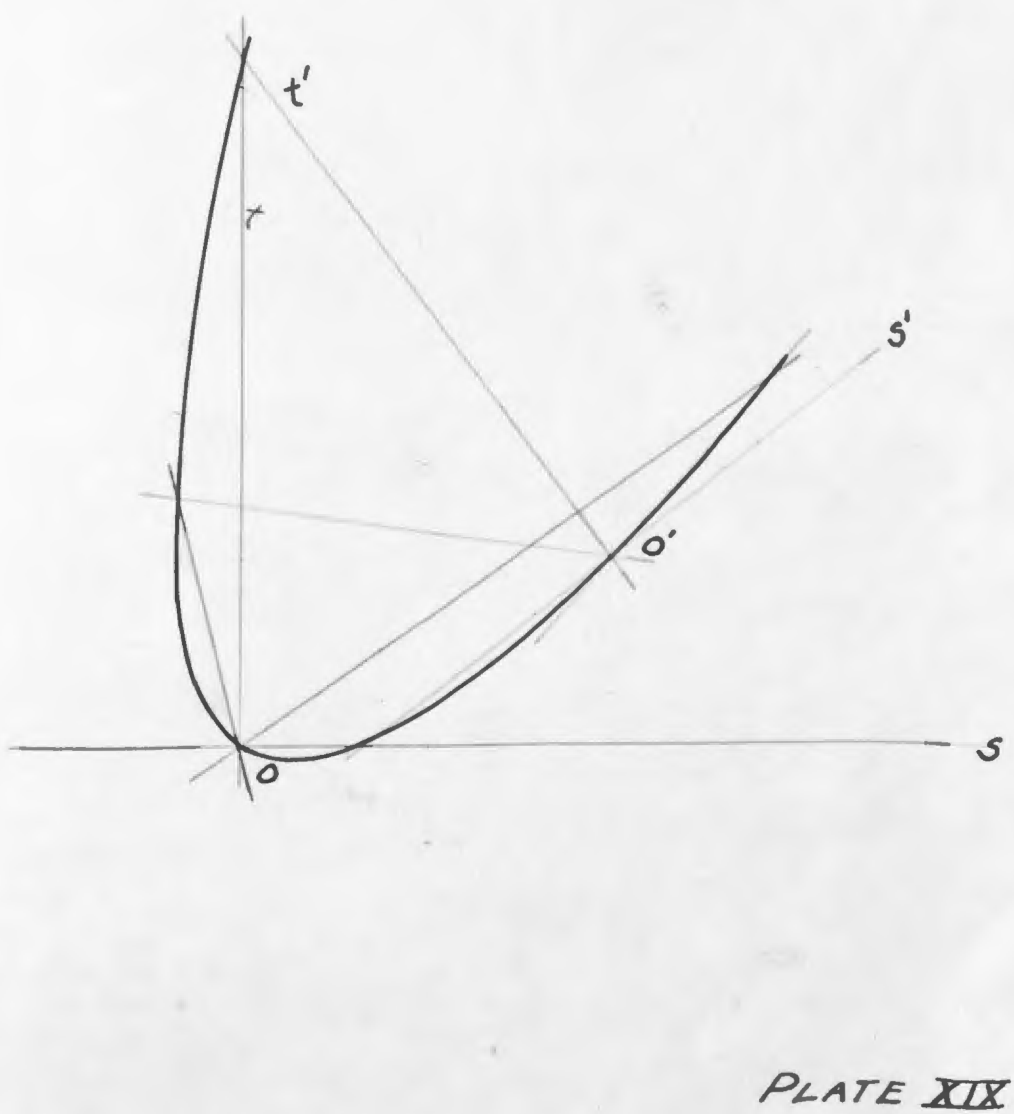

PLATE XIX

$k = -\frac{1}{4}$

$\phi = \tan^{-1}\frac{3}{4}$

$\theta = \tan^{-1}\frac{1}{2}$

$q = \sqrt{5}$

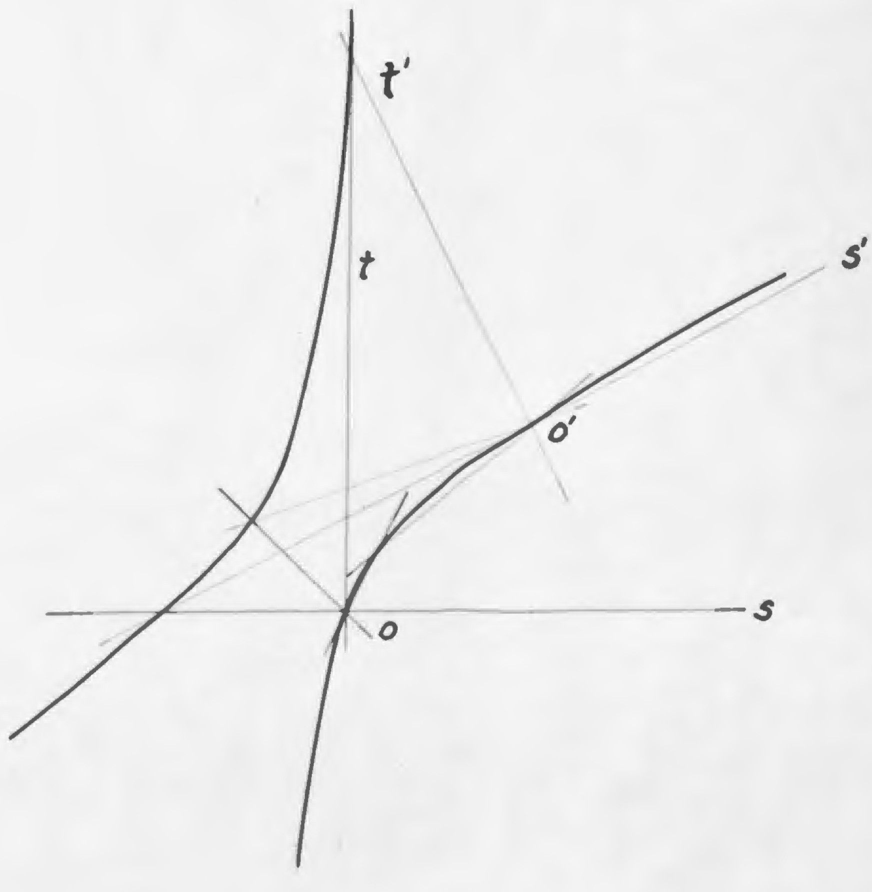

PLATE XX

$k = -\frac{1}{8}$

$\phi = \tan^{-1}\frac{1}{2}$

$\theta = 45°$

$q = \sqrt{2}$

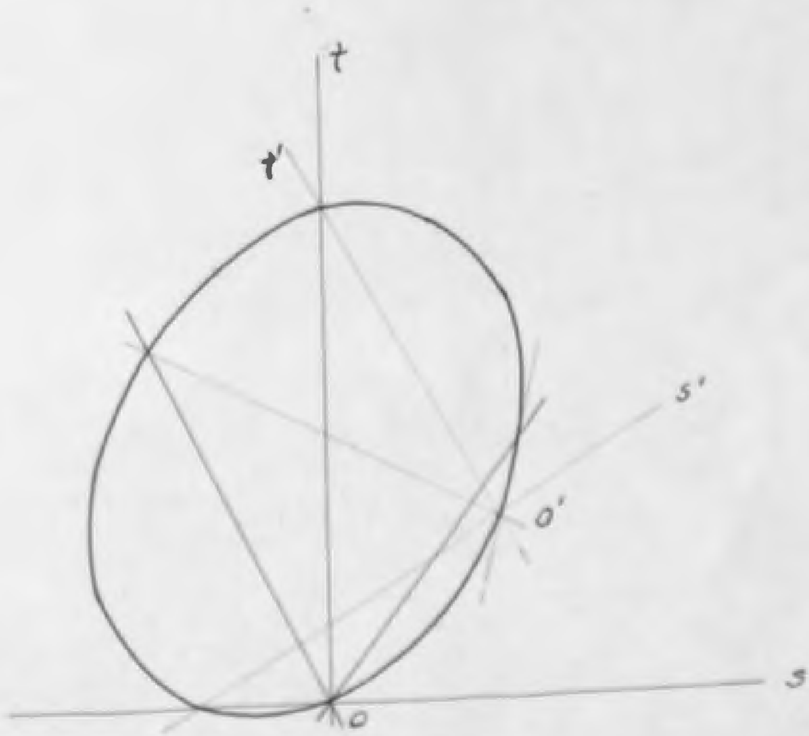

PLATE XXI

$K = -\frac{3}{4}$

$\phi = 30°$

$\theta = 45°$

$q = \sqrt{2}$

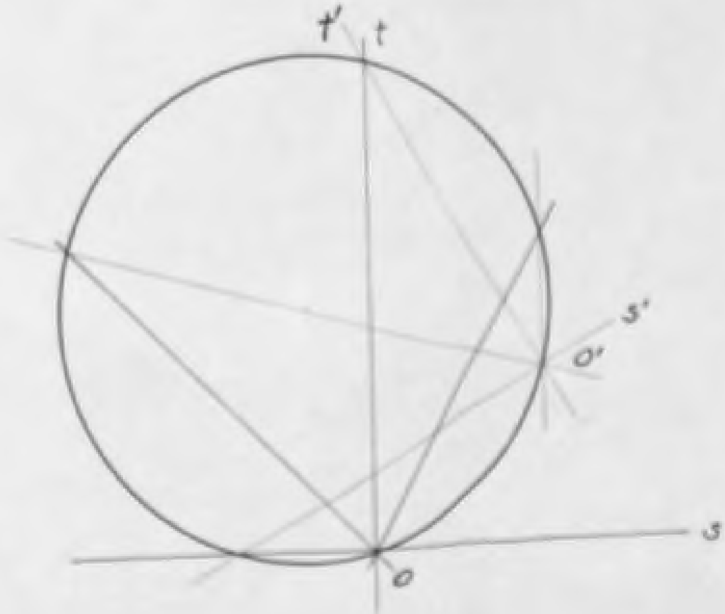

PLATE XXII

$k = -1$

$\phi = 30°$

$\theta = 45°$

$q = \sqrt{2}$

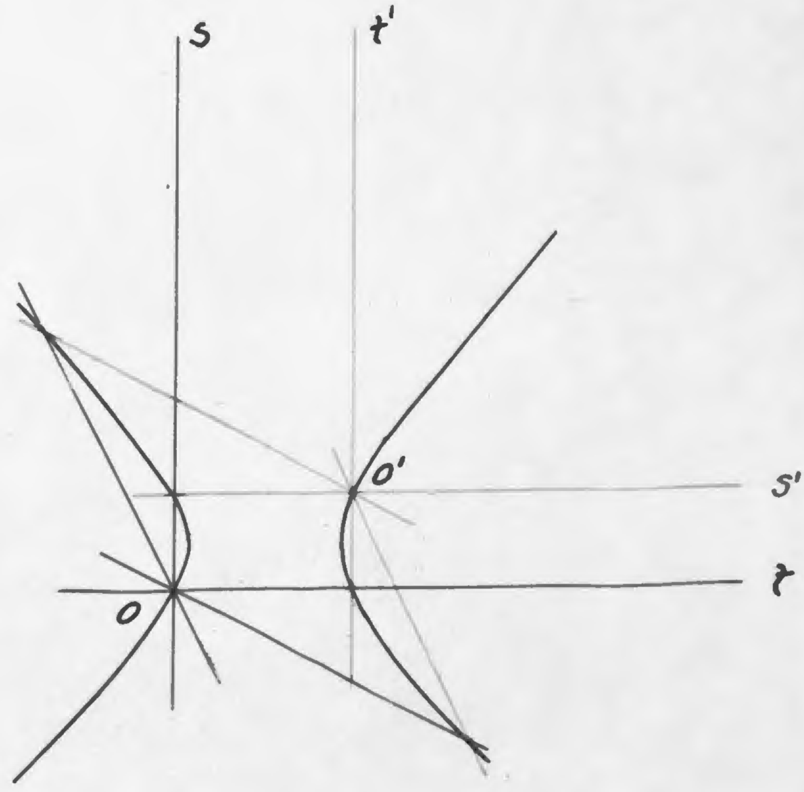

PLATE XXIII
$k = 1$
$\phi = 90°$
$\theta = \tan^{-1}\frac{1}{2}$
$q = \sqrt{5}$

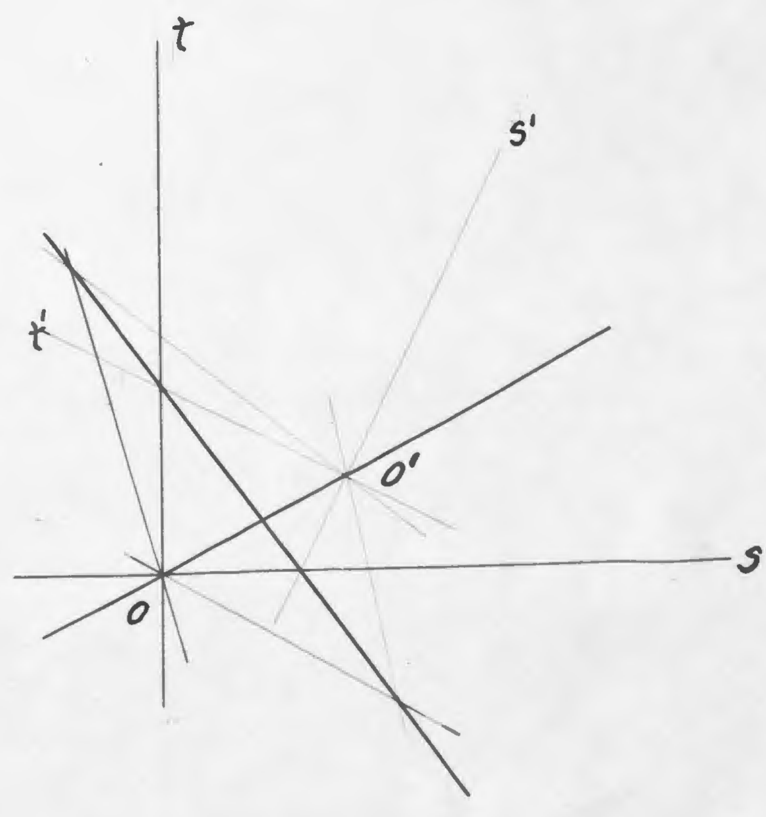

PLATE XXIV
$k = 2$
$\phi = \tan^{-1} 3$
$\theta = \tan^{-1} \frac{1}{2}$
$q = \sqrt{5}$

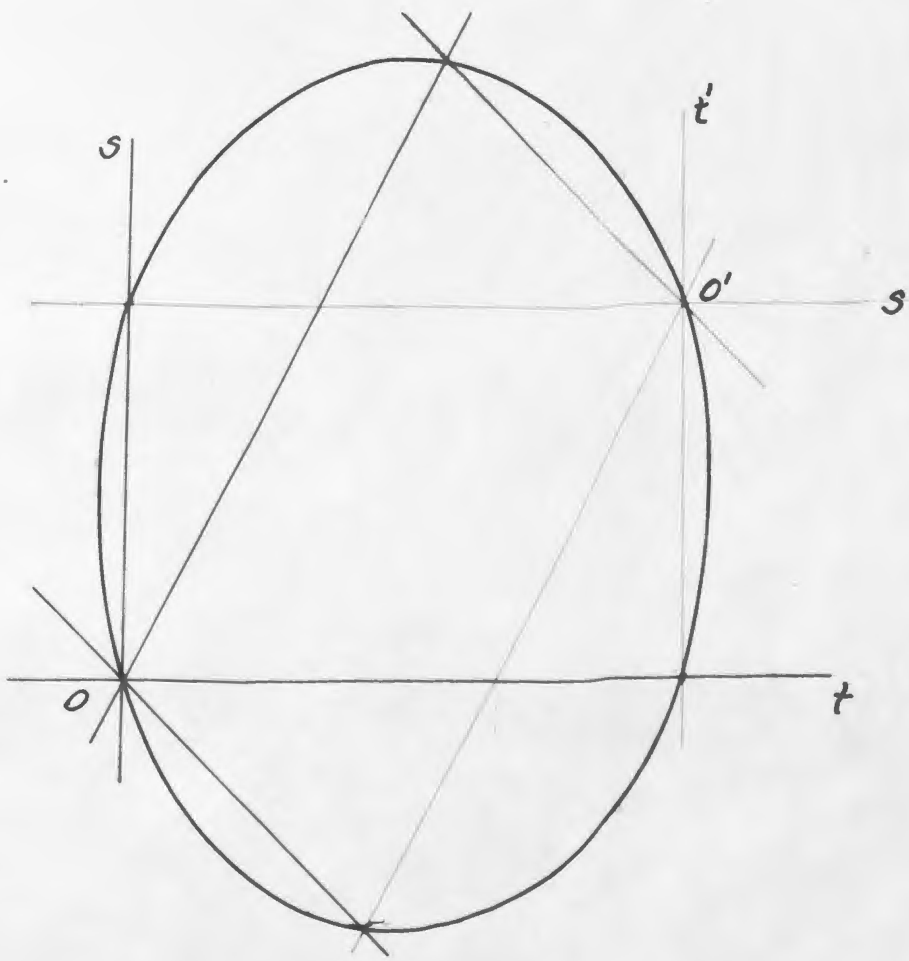

PLATE XXV

$k = -2$

$\phi = 90°$

$\theta = \tan^{-1}\frac{2}{3}$

$q = \sqrt{13}$

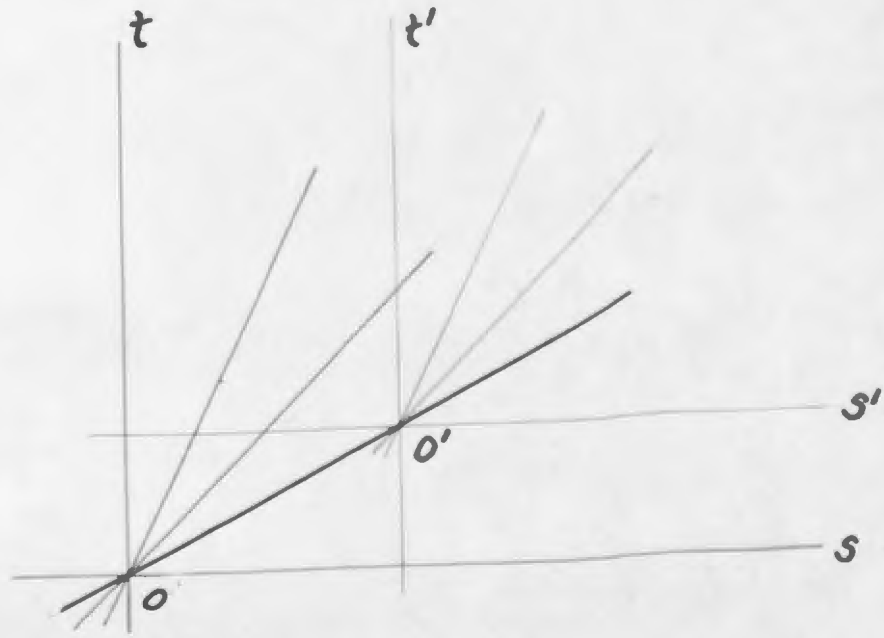

PLATE XXVI

$k = -1$

$\phi = 0$

$\theta = \tan^{-1}\frac{1}{2}$

$q = \sqrt{3}$

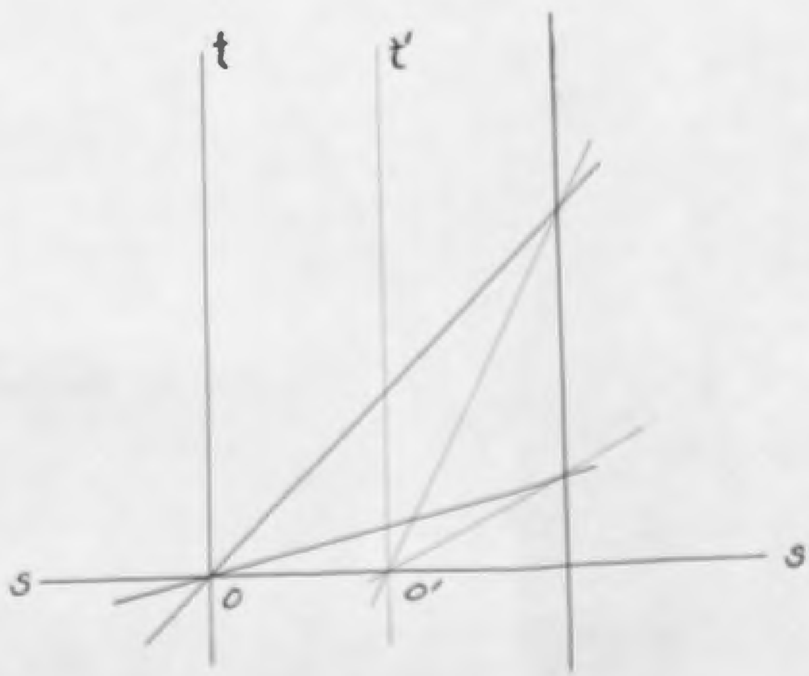

PLATE XXVII

$k = -2$

$\phi = 0$

$\theta = 0$

$q = 1$

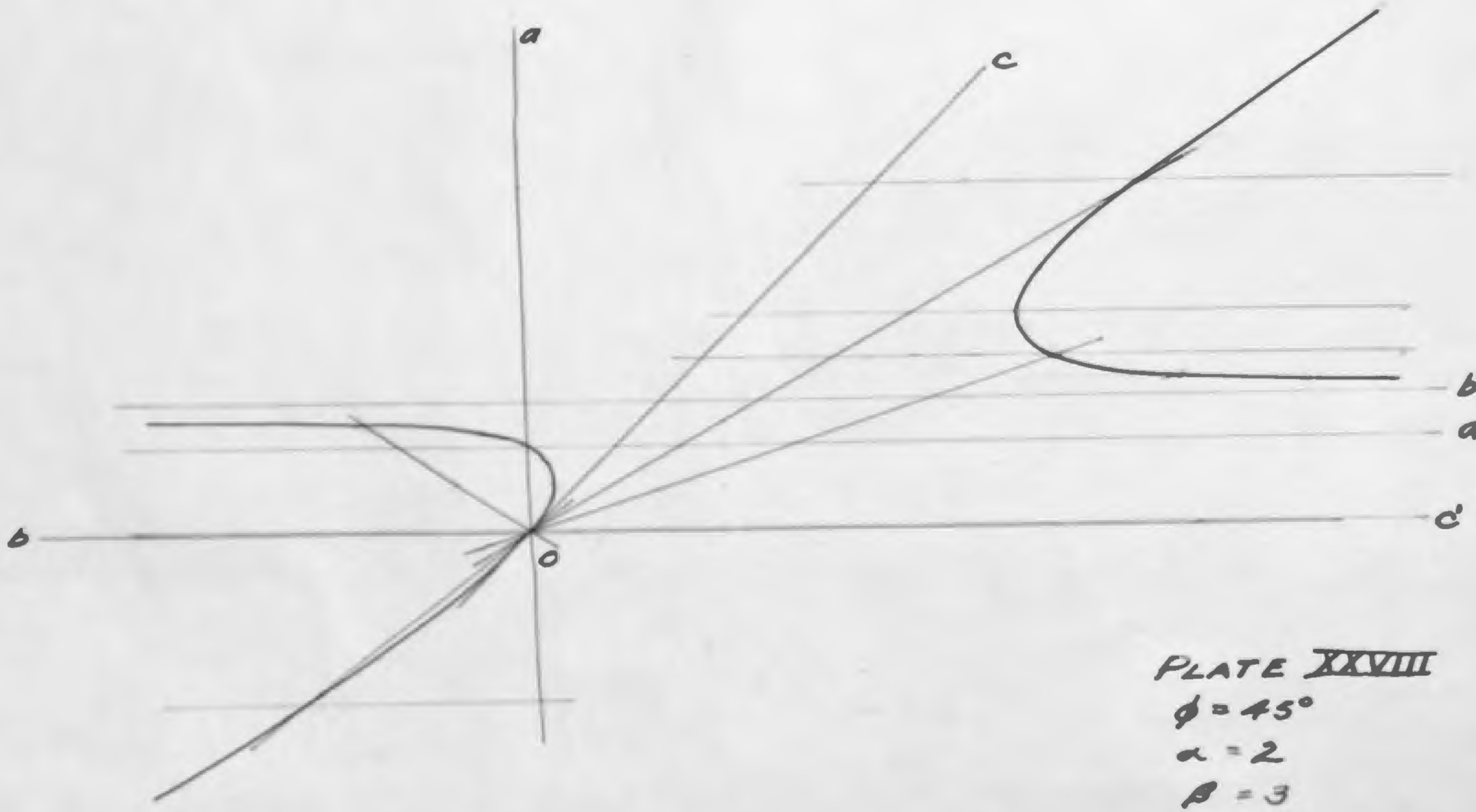

PLATE XXVIII

$\phi = 45°$

$\alpha = 2$

$\beta = 3$

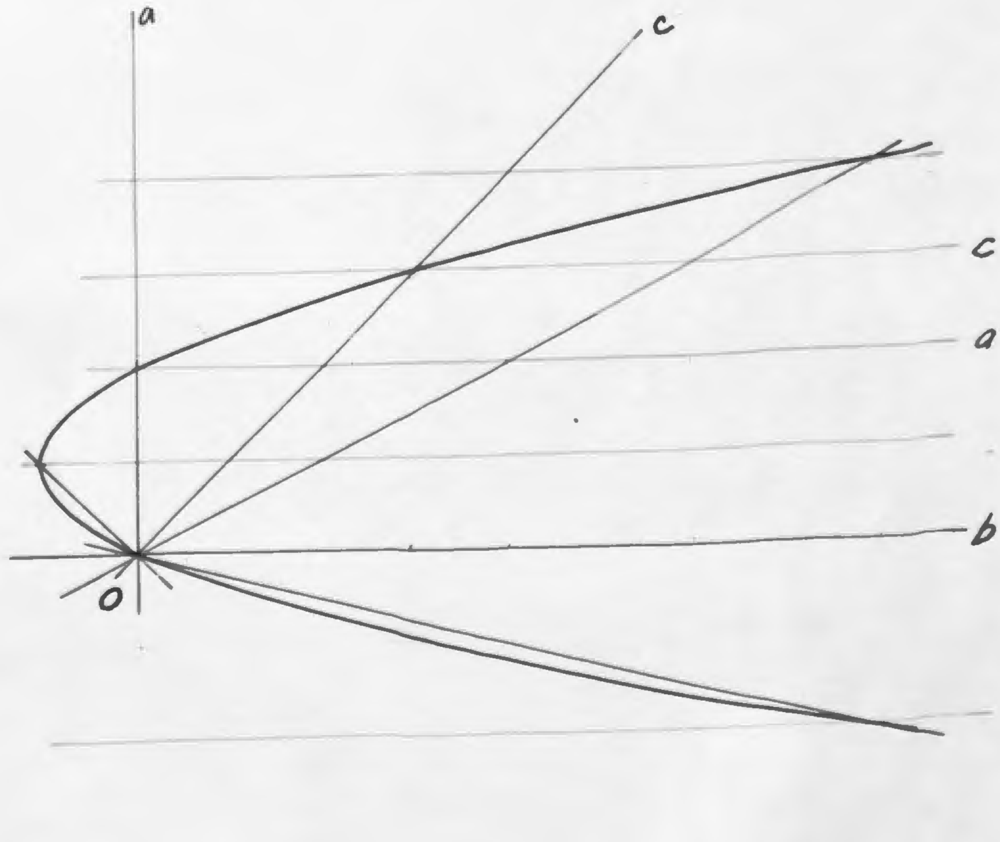

PLATE XXIX

$\phi = 45°$

$\alpha = 2$

$\beta = 3$

b' is line at ∞

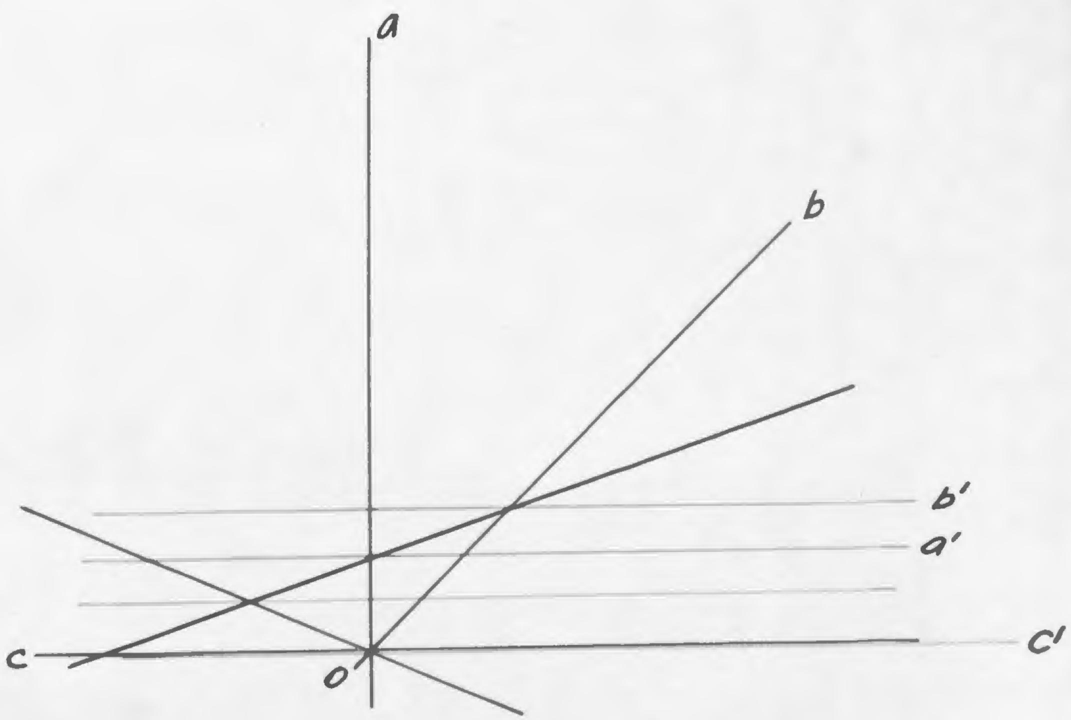

PLATE XXX

$\phi = 45^\circ$

$\alpha = 2$

$\beta = 3$

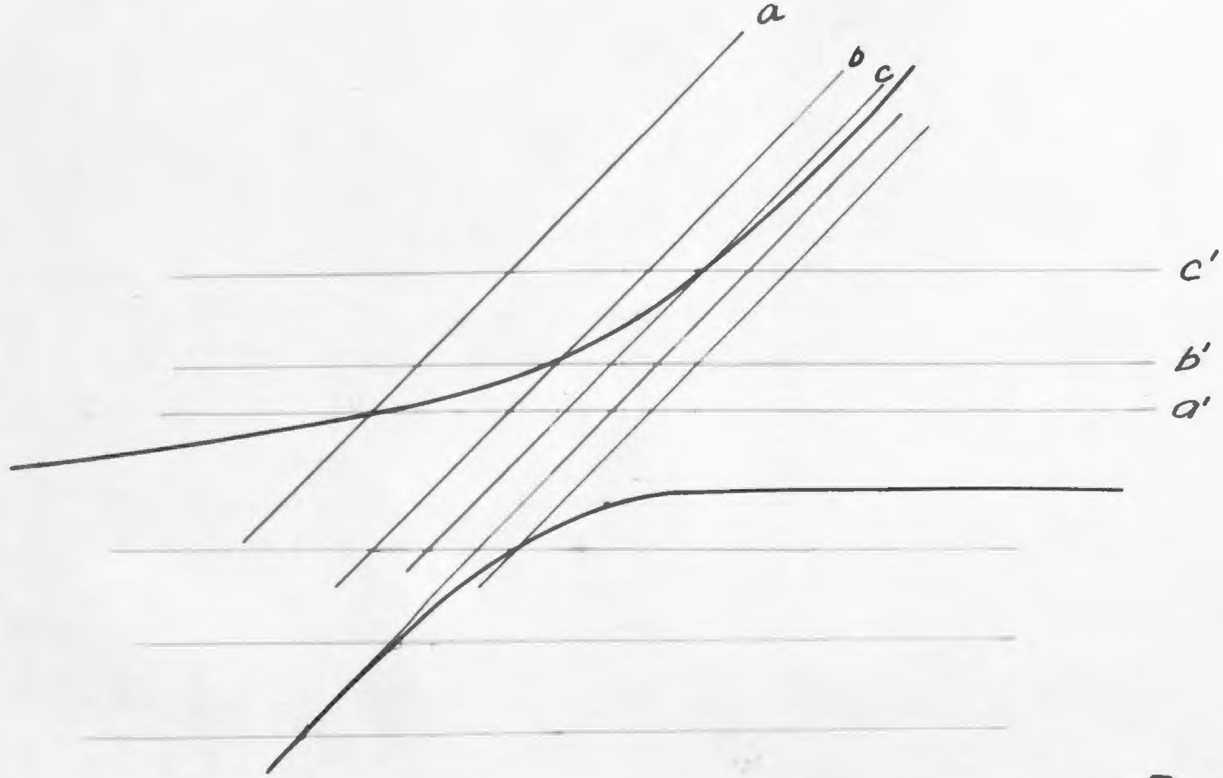

PLATE XXXI

$b = 3$

$c = 4$

$\beta = \sqrt{2}$

$\gamma = 3\sqrt{2}$

PLATE XXXII

$b = 1$

$c = 2$

$\beta = \sqrt{2}$

$\gamma = 2\sqrt{2}$