REPORT

of

Committee on Thesis

The undersigned, acting as a Committee of the Graduate School, have read the accompanying thesis submitted by Hugh B. Wilcox for the degree of Master of Science They approve it as a thesis meeting the requirements of the Graduate School of the University of Minnesota, and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science

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May 29 1916

A study of accuracy in stellar photography
with special reference
to focal length and aperture

A Thesis submitted to the
Faculty of the Graduate School of the
University of Minnesota

by

Hugh B. Wilcox

In partial fulfillment of the requirements
for the Degree of
Master of Science

June 1916

Introduction

An examination of plates taken with the photographic objective of the Observatory of the University of Minnesota shows that perfect images are obtained only near the optical axis, and that upon leaving the center of the plate the eye encounters images which become increasingly " winged " until near the edge they are quite unmeasureable. For some time Prof. Leavenworth has sought to overcome this difficulty by reducing the aperture of the objective. His experience indicated that better results could be obtained; and it was with this fact in mind that he suggested for the object of my thesis, a more complete investigation into this matter.

Discussion of Problem

If with this arrangement, stars in all parts of the plate could be brought to yield results comparable with those at present obtainable only near the center of the plate, we might infer a considerable enhancement of the usefulness of our photographs. Let us enquire briefly into the merits of this assumption.

The objection might be raised at the outset that in order to procure this accuracy over the whole field the exposure time would be unduly increased. To this it may be answered that in ~~order~~ covering the whole field we increase the area about nine fold, with the very considerable advantage of having the whole field upon one plate instead of extending over nine plates. The number of plates would be further increased by the old method because over lapping of the plates would be necessary. Moreover there is a very great saving in expense and in time of development as well as in time consumed at the telescope in the various operations attendant upon exposing several plates instead of one. We should conclude, therefore, that this scheme would be more economical from every point of view, provided of course that the exposure time was not much increased over ten times its former value. The evidence upon this point will be examined in due time.

Let there be considered now an advantage which will, I think, be of more importance than any other connected with this work. Suppose for example that we are measuring from time to time the position of some object or objects so situated as to have a dearth of comparison stars in its neighborhood; and that the right ascension and declination of these objects is ~~was~~ required with the utmost accuracy. Capping down the telescope affords a very simple and effective means for dealing with this problem, for as already noted, an increase by nine time in the area increases in a like measure the probability of finding stars bright enough to be included in the catalogues. It is only necessary then to take a long exposure plate of the region, including not only the bright stars but the faint stars as well. Having once obtained the positions of the central faint stars with reference to the surrounding catalogue stars we are in possession of data which will make short exposurs with full aperture quite sufficient for all subsequent plates.

The aperture could not, of course, be cut down where rapidly moving faint objects were to be photographed, such as a faint comet or an asteroid. It is, however, often of the greatest importance as in the case of the Eros campaign to secure places for such an object with as much precision as possible. Again, the method outlined in the preceding paragraph will prove very practicable. It is of interest to note in this connection that Prof. Leavenworth is of the opinion that considerably more accurate results could have been obtained at Minneapolis during the Eros work, had the scheme of capping down the telescope occured to him at that time. It is significant that Hinks[1] found very serious discrepancies in measures made upon the Crossley Reflector plates and that upon investigation the errors were found to be directly traceable to optical distortion of images measured too far from the plate center. After this was recognized and corrected by confining the measures to more central stars the results were in very good agreement. Having considered now the advantages of good images over the whole plate, let us turn to the causes which give the distortion of out-lying images.

Causes of Optical Distortion

At first thought it might be supposed that the distortion would be due to the fact that the focal plane ~~in~~ of a corrected objective is sensibly spherical instead of plane and that as a consequence the only portion where perfect images could

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1 - A. R. Hinks, Mon. Not. 64. 8, p. 709

occur would be where the photographic plate was tangent to, or intersected by, the focal surface. It is easily seen that if the cone of rays is homogeneous and truly conical as it should be were this the only difficulty, the intersection of the cone with the plane of the plate would give an elliptical image. Now as a matter of fact the images are decidedly V shaped with the long axis of the V radial from the plate center. This together with the fact that we are using a long focal length in comparison to the full aperture ~~ua~~ (F - 13) makes it exceedingly probable that this is a minor cause of image distortion. At any rate the effect of capping down the objective would be to cause this difficulty to vanish, if indeed it exists, since the reduction of aperture gives the well known pin hole effect of a very flat field. This could be verified experimentally by taking extra-focal plates. The evidence pointing to another cause is so strong, however, that this was not thought necessary.

Since photographic objectives are corrected for chromatic aberration in the region of the spectrum best suited to the photographic plate, the discussion will be limited to the two most important aberrations, namely: spherical aberration and coma. Spharical aberration is defined as a spreading longitudinally of the ray cone due to different focal lengths belonging to different zones of the object glass. Of these the ones of most importance are the third and fifth order respectively. Coma is a V shaped blur in the image due to the rays of the image forming object striking the ~~o~~ lens obliquely. In the case of spherical aberration occuring with coma the images are rounded at the ends and take a sort of dumb-bell shape. An examination of the images obtained near the edge of the plate shows that coma very probably plays the chief part in the distortion. Coma varies directly as the square of the aperture while third and fifth order spherical aberrations vary as the square and the fourth power of the aperture respectively. From this law of variation we might expect a complete removal of the trouble provided we decreased the aperture sufficiently. This will be treated under the heading " Experimental Results." Let us now turn to an examination of the effect of cutting down the aperture upon resolving power.

Resolving Power

The resolving power of a telescope varies as is well known, directly with the aperture. It is shown in works on Optics that if an image is formed using light of wave length λ the angular separation of two images just resolvable is

$$\theta = a\lambda/R$$

where R is the radius of the objective and a is a constant depending upon the quality of image desired. The value of a is usually taken as 0.61 for telescopic work. Now $\theta = d/F$ where d is the linear separation between two images and F is the focal length. Substituting the value of θ in the above formula, we have

$$d/F = a\lambda/R$$

According to Nutting[1] two images must be about 20 μ apart to be separable upon a photographic plate. If we use a wave length of 0.5 μ and substitute for F its value of 136 inches we have 1 inch as the value of R and 2 inches for the value of the limiting aperture. The smallest opening used was just 2 inches.

Experimental Methods

Several photographs were taken of the Pleiades cluster each one of which bore twelve exposurs. The exposurs were repeated in the reverse order so that the mean time of exposure for any one aperture would be the same as the time for any other aperture. This arrangement was adopted to compensate for any atmospheric disturbance which might be taking place in a fairly uniform way, such as a gradual thickening up of the sky. The following table gives the exposure number, aperture and exposure time.

Exp.	Aperture in.	Exp. Time m	s
(1 - 12)	10.50	1	0
(2 - 11)	9.25	1	10.0
(3 - 10)	7.87	1	37.0
(4 - 9)	6.00	2	47.0
(5 - 8)	4.00	6	15.0
(6 - 7)	2.00	25	0

From these the best No. 575 was selected for measurement. The exposure times were purposely made inversely proportional to the square of the aperture so that the amount of light would be the same for each exposure. The exposurs are all correct with the exception of No.(1 - 12) which was made 6 seconds too long, amounting to an over exposure of about ten per cent. Alcyone was used as a guiding star. The telescope was moved a small amount between each exposure in order to displace the star image on the plate; a movable reticle on the guiding telescope made this very easy. Seed 30 plates were used throughout.

Measurement and Reduction of the Plate

Twenty-four stars covering an area of 2° by 1°5 were selected from the work of Elkins and Smith.[1] These stars had been observed by Bessel at Königsburg with the heliometer and have their places fixed therefore with very great precision. The average of two measures (Epoch 1885) was taken for the definitive places for all stars with the exception of two which have measureable proper motion. Elkin's[2] values for proper motion were used to reduce Smith's places to the revised places of Elkin. The average was then taken. My places were ~~then~~ reduced to 1885 using first the proper motion of Elkin and second the proper motions obtained by R. Trumpler. The latter motions were obtained from a discussion of nearly all the available material.

The scale coordinates in millimeters of the stars were obtained from measures parallel to the right ascension and declination axes, the plate being reversed (turned through 180°) and remeasured along each axis. These readings, after

[1] Elkin , Trans. Ast. Obs. Yale, Vol.1 (1887 - 1901)

[2] " " " " " " " " p. 357

having been corrected for error of run and scale error, were averaged giving the scale coordinates of the center of the plate Using the coordinates of the center of the plate as an origin the coordinates of the stars were obtained by subtracting the center coordinates from the scale coordinates of ~~the~~ the star. Since two exposures were made with each cap, the average of the coordinates (using the same cap) were used in the final solution. Fifteen comparison stars were used to obtain the constants of the plate. Prof. Leavenworth's tables and formulae were employed, the solutions being affected by the method of Least Squares. The right ascension and declination was next computed for each star from the measured coordinates, and the residuals for all the stars were computed. An inspection of the residuals obtained by comparing the computed places with those of the catalogue shows that they run so nearly equal to those obtained in checking the least square solutions as to make a computation of the star places unnecessary in any but the smallest aperture exposures. In other words, the residuals obtained from the least square solutions furnish a sufficient criterion as to the probable error in each case, without going to the additional work of computing the separate right ascensions and declinations.

Star Places Used

These are for the epoch 1885 and are the average of the results of the first and second Yale trianglulations. The star 2J is taken from the work of Jacoby[L]. For proper motion stars see page

Star	Right Asc. 1885			Declination 1885		
	h	m	s	°	′	″
1	3	36	53.032	24	0	33.62
2J	3	36	53.241	23	17	2.07
6	3	38	2.816	23	45	2.76
8	3	38	16.595	23	20	26.05
9	3	38	18.037	24	28	38.22
10	3	38	21.777	24	6	19.40
13	3	38	36.521	23	40	25.82
19	3	38	51.066	23	55	40.37
27	3	$3\overset{9}{8}$	36.904	23	53	45.60
29	3	39	49.321	23	44	41.08
30	3	40	8.241	24	9	44.61
31	3	40	11.561	24	27	44.20
42	3	40	34.783	23	33	28.27
44	3	40	38.835	23	55	54.50
51	3	41	32.341	23	4	0.30
52	3	41	39.133	23	59	28.35
56	3	42	20.691	23	47	2.86
57 See below.						
58	3	42	25.565	24	2	37.19
62	3	42	54.069	23	21	37.87
68	3	43	36.352	24	8	42.38
69	3	44	2.095	23	36	45.92
57	3	42	22.338	23	32	3.10

[L] Jacoby, Contributions of Columbia Observatory, No. 3, p 23

Table of mean coordinates, given in millimeters and in terms of time and arc.

		Star 1				
Exp.	Av. X	Av. Y	X'		Y'	
			m	s	'	"
(1 - 12)	48.2614	12.9984	- 3	30.420	+ 12	57.30
(2 - 11)	48.0739	13.9313	- 3	29.602	+ 13	53.09
(3 - 10)	47.7045	14.7266	- 3	27.991	+ 14.	40.65
(4 - 9)	47.2576	15.6097	- 3	26.043	+ 15	33.46
(5 - 8)	46.8415	16.3955	- 3	24.229	+ 16	20.45
(6 - 7)	46.3660	17.3654	- 3	22.156	+ 17	16.12
		Star 2 J				
(1 - 12)	48.3469	30.6512	- 3	30.792	- 30	32.94
(2 - 11)	48.1566	29.7238	- 3	29.963	- 29	37.48
(3 - 10)	47.7875	28.9221	- 3	28.353	- 28	49.54
(4 - 9)	47.3408	28.0434	- 3	26.406	-27	56.99
(5 - 8)	46.9284	27.2382	- 3	24.608	- 27	8.84
(6 - 7)	46.4450	26.2932	- 3	22.500	- 26	1233
		Star 6				
(1 - 12)	32.3686	2.5995	- 2	21.127	- 2	35.45
(2 - 11)	32.1577	1.6586	- 2	20.208	- 1	39.18
(3 - 10)	31.7651	0.8627	-2	18.496	- 0	51.59
(4 - 9)	31.3053	0.1418	- 2	16.491	+ 0	8.48
(5 - 8)	30.8944	0.8182	- 2	14.700	+ 0	48.93
(6 - 7)	30.4125	1.7486	- 2	12.598	+ 1	44.57

Star 8

(1 - 12)	29.1415	27.2808	- 2	7.057	- 27	11.39
(2 - 11)	28.9535	26.3496	- 2	6.237	- 26	15.71
(3 - 10)	28.5851	25.5502	- 2	4.631	- 25	27.90
(4 - 9)	28.1432	24.6598	- 2	2.704	- 24	34.66
(5 - 8)	27.7319	23.8535	- 2	0.911	- 23	46.44
(6 - 7)	27.2580	22.9148	- 1	58.845	- 22	50.30

Star 9

(1 - 12)	28.8361	41.2046	- 2	5.725	+ 41	4.03
(2 - 11)	28.6343	42.0914	- 2	4.846	+ 41	57.07
(3 - 10)	28.2438	42.8869	- 2	3.143	+ 42	44.64
(4 - 9)	27.7858	43.7437	- 2	11.146	+ 43	35.87
(5 - 8)	27.3750	44.5138	- 1	59.355	+ 44	21.92
(6 - 7)	26.8993	45.4332	- 1	57.281	+ 45	16.91

Star 10

(1 - 12)	27.9948	18.7886	- 2	2.057	+ 18	43.56
(2 - 11)	27.7892		- 2	1.161		
(3 - 10)	27.3990	20.4936	- 1	59.460	+ 20	25.52
(4 - 9)	26.9426	21.3600	- 1	57.470	+ 21	17.33
(5 - 8)	26.5296	22.1429	- 1	55.670	+ 22	4.15
(6 - 7)	26.0554	23.0702	- 1	53.601	+ 22	59.60

Star 13

(1 - 12)	24.5878	7.2319	- 1	47.203	- 7	12.47
(2 - 11)	24.3880	6.2955	- 1	46.332	- 6	16.47
(3 - 10)	24.0109	5.5007	- 1	44.687	- 5	28.94
(4 - 9)	23.5574	4.6147	- 1	42.710	- 4	35.96
(5 - 8)	23.1519		-1	40.942		
(6 - 7)	22.6781	2.8800	- 1	38.876	- 2	52.22

Star 19

(1 - 12)	21.2342	8.0677	- 1	32.581	+ 8	2.45
(2 - 11)	21.0393	8.9983	- 1	31.731	+ 8	58.10
(3 - 10)	20.6712	9.7924	- 1	30.126	+ 9	45.59
(4 - 9)	20.2263	10.6714	- 1	28.187	+ 10	38.15
(5 - 8)	19.8215	11.4675	- 1	26.422	+ 11	25/76
(6 - 7)	19.3525	12.3970	- 1	24.377	+ 12	21.34

Star ~~20~~ 27

(1 - 12)	10.7281	6.1666	- 0	46.774	+ 6	8.76
(2 - 11)	10.5401	7.0924	- 0	45.955	+ 7	4.12
(3 - 10)	10.1633	7.8867	- 0	44.312	+ 7	51.62
(4 - 9)	9.7232	8.7661	- 0	42.393	+ 8	44.21
(5 - 8)	9.3191	9.5573	- 0	40.631	+ 9	31.53
(6 - 7)	8.8498	10.4896	- 0	38.585	+ 10	27.28

Star 29

(1 - 12)	7.8503	2.9465	- 0	34.227	- 2	56.20
(2 - 11)	7.6616	2.0117	- 0	33.405	- 2	0.30
(3 - 10)	7.2937	1.2$\overset{1}{8}$97	- 0	31.800	- 1	12.94
(4 - 9)	6.8523	0.3337	- 0	29.876	- 0	19.96
(5 - 8)	6.4559	0.4645	- 0	28.148	+ 0	27.78
(6 - 7)	5.9860	1.4008	- 0	26.099	+ 1	23.77

Star 30

(1 - 12)	3.5826	22.2388	- 0	15.620	+ 22	9.88
(2 - 11)	3.3934	23.1575	- 0	14.795	+ 23	4.82
(3 - 10)	3.0232	23.9391	- 0	13.181	+ 23	51.56
(4 - 9)	2.5866	24.8007	- 0	11.277	+ 24	43.08
(5 - 8)	2.1875	25.5971	- 0	9.537	+ 25	30.71
(6 - 7)	1.7208	26.5219	- 0	7.503	+ 26	26.01

Star 31

(1 - 12)	2.8663	40.2429	- 0	12.497	+ 40	6.52
(2 - 11)	2.6780	41.1761	- 0	11.676	+ 41	2.33
(3 - 10)	2.3117	41.9622	- 0	10.079	+ 41	49.34
(4 - 9)	1.8743	42.8485	- 0	8.172	+ 42	42.34
(5 - 8)	1.4772	43.6272	- 0	6.440	+ 43	28.91
(6 - 7)	1.0119	44.5510	- 0	4.412	+ 44	24.15

Star 42

(1 - 12)	2.6277	14.1818	+ 0	11.457	- 14	8.07
(2 - 11)	2.8163	13.2429	+ 0	12.279	- 13	11.92
(3 - 10)	3.1855	12.4399	+ 0	13.889	- 12	23.91
(4 - 9)	3.6147	11.5473	+ 0	15.760	- 11	30.53
(5 - 8)	4.0175	10.7431	+ 0	17.516	- 10	42.44
(6 - 7)	4.4804	9.8091	+ 0	19.534	- 9	46.58

Star 44

(1 - 12)	3.4793	8.3575	+ 0	15.170	+ 8	19.78
(2 - 11)	3.6638	9.2838	+ 0	15.974	+ 9	15.17
(3 - 10)	4.0320	10.0743	+ 0	17.580	+ 10	2.44
(4 - 9)	4.4636	10.9538	+ 0	19.461	+ 10	55.04
(5 - 8)	4.8610	11.7482	+ 0	21.194	+ 11	42.54
(6 - 7)	5.3238	12.6780	+ 0	23.212	+ 12	38.14

Star 47

(1 - 12)	7.0256	36.4594	+ 0	30.632	- 36	20.27
(2 - 11)	7.2152	35.5195	+ 0	31.458	- 35	24.07
(3 - 10)	7.5771	34.7242	+ 0	33.036	- 34	36.51
(4 - 9)	Rej. Meas. wrong star.					
(5 - 8)	8.4061	33.0165	+ 0	36.651	- 32	54.39
(6 - 7)	8.8750	32.0750	+ 0	38.685	- 31	58.08

Star 51

(1 - 12)	16.0403	43.7570	+ 1	9.936	- 43	36.67
(2 - 11)	16.2123	42.7933	+ 1	10.686	- 42	39.04
(3 - 10)	16.5730	41.9688	+ 1	12.258	- 41	49.73
(4 - 9)						
(5 - 8)	17.3879	40.2300	+ 1	15.811	- 40	5.75
(6 - 7)	17.8496	39.2828	+ 1	17.824	- 39	9.11

Star 52

(1 - 12)	17.3047	12.0017	+ 1	15.448	+ 11	57.70
(2 - 11)	17.4816	12.9250	+ 1	16.220	+ 12	52.91
(3 - 10)	17.8408	13.7166	+ 1	17.786	+ 13	40.25
(4 - 9)	18.2677	14.5950	+ 1	19.647	+ 14	32.78
(5 - 8)	18.6534	15.3828	+ 1	21.329	+ 15	19.89
(6 - 7)	19.1156	16.3167	+ 1	23.344	+ 16	15.74

Star 56

(1 - 12)	26.9338	0.4303	+ 1	57.431	- 0	25.73
(2 - 11)	27.1025	0.5026	+ 1	58.167	+ 0	30.05
(3 - 10)	27.4578	1.2994	+ 1	59.716	+ 1	17.70
(4 - 9))	27.8750	2.1905	+ 2	1.535	+ 2	10.99
(5 - 8)	28.2538		+ 2	3.187		
(6 - 7)	28.7146	3.9193	+ 2	5.196	+ 3	54 .37

Star 57

(1 - 12)	27.3781	15.4694	+ 1	59.368	- 15	25.07
(2 - 11)	27.5616	14.5409	+ 2	0.168	- 14	29.55
(3 - 10)	27.9206	13.7439	+ 2	1.734	- 13	41.88
(4 - 9)	28.3367	12.8514	+ 2	3.545	- 12	48.51
(5 - 8)	28.7252	12.0457	+ 2	5.242	- 12	0.33
(6 - 7)	29.1828	11.1063	+ 2	7.237	- 11	4.16

		Star 58				
(1 - 12)	27.9232	15.2051	+ 2	1.745	+ 15	9.26
(2 - 11)	28.1028	16.1373	+ 2	2.528	+ 16	5.01
(3 - 10)	28.4602	16.9326	+ 2	4.086	+ 16	52.57
(4 - 9)	28.8792	17.8118	+ 2	5.913	+ 17	45.15
(5 - 8)	29.2655	18.5982	+ 2	7.598	+ 18	32.17
(6 - 7)	29.7216	19.5279	+ 2	9.586	+ 19	27.77
		Star 62				
(1 - 12)	34.7961	25.9162	+ 2	31.711	- 25	49.79
(2 - 11)	34.9649	24.9639	+ 2	32.447	- 24	52.84
(3 - 10)	35.3129	24.1581	+ 2	33.964	- 24	4.65
(4 - 9)	35.7218	23.2541	+ 2	35.747	- 23	10.59
(5 - 8)	36.0974	22.4421	+ 2	37.385	- 22	22.04
(6 - 7)	36.5546	21.4981	+ 2	39.378	- 21	25.59
		Star 64				
(1 - 12)	36.7638	4.5976	+ 2	40.290	+ 4	34.94
(2 - 11)	36.9521	5.5315	+ 2	41.111	+ 5	30.78
(3 - 10)	37.3097	6.3274	+ 2	42.670	+ 6	18.38
(4 - 9)	37.7381	7.2101	+ 2	44.538	+ 7	11.16
(5 - 8)	38.1250	8.0107	+ 2	46.225	+ 7	59.04
(6 - 7)	38.5815	8.9455	+ 2	48.215	+ 8	54.94
		Star 68				
(1 - 12)		21.4293			+ 21	21.47
(2 - 11)		22.3601			+ 22	17.13
(3 - 10)		23.1582			+ 23	4.86
(4 - 9)	45.0301	24.0473	+ 3	16.331	+ 23	58.03
(5 - 8)	45.4090	24.8372	+ 3	17.983	+ 24	45.26
(6 - 7)	45.8624	25.7674	+ 3	19.960	+ 25	40.89

Star 69

(1 - 12)	50.3082	10.5626	+ 3	39.344	- 10	31.64
(2 - 11)	50.4321	9.6292	+ 3	39.884	- 9	35.83
(3 - 10)	50.8261	8.8308	+ 3	41.602	- 8	48.08
(4 - 9)	51.2341	7.9415	+ 3	43.381	- 7	54.90
(5 - 8)	51.6044	7.1345	+ 3	44.995	- 7	6.64
(6 - 7)	52.0551	6.1976	+ 3	46.960	- 6	10.62

del

Observation~~al~~ Equations

These equations are:

$$a\,z_1 + b\,z_2 + c\,z_3 + M = 0$$

$$a\,z_1' + b\,z_2' + c\,z_3' + M' = 0$$

Since the coefficients a, b, and c are common to both equations, we shall in order to avoid repetition, combine the two into the single equation

$$a\,Z_1 + b\,Z_2 + c\,Z_3 + M \;....\; + M' = 0$$

where the capital Zs take the place of either z or z' depending upon which M is used in the solution. The right hand member is understood to be zero in the following equations.

$a = 1$

$b = X / 52.1$

$c = Y / 45.4$

where X and Y are the measured coordinates in millimeters.

Star	Equations Exp. (6 - 7)
1	$Z_1 - 0.891\,Z_2 + 0.381\,Z_3 + 0.225 \;...+\; 10.88$
2 J	$Z_1 - 0.892\,Z_2 - 0.579\,Z_3 - 0.346 \;+\; 7.44$
8	$Z_1 - 0.524\,Z_2 - 0.504\,Z_3 - 0.308 \;+\; 4.14$
9	$Z_1 - 0.517\,Z_2 + 1.000\,Z_3 + 0.597 \;+\; 9.36$
19	$Z_1 - 0.372\,Z_2 + 0.273\,Z_3 + 0.176 \;+\; 5.51$
27	$Z_1 - 0.170\,Z_2 + 0.231\,Z_3 + 0.141 \;+\; 3.65$
29	$Z_1 - 0.115\,Z_2 + 0.031\,Z_3 + 0.024 \;+\; 2.48$
300	$Z_1 - 0.033\,Z_2 + 0.584\,Z_3 + 0.368 \;+\; 3.67$

Star	Equation
31	$Z_1 - 0.019\,Z_2 + 0.982\,Z_3 + 0.597 \;..+ 5.23$
42	$Z_1 + 0.086\,Z_2 - 0.216\,Z_3 - 0.127 \quad - 0.05$
44	$Z_1 + 0.102\,Z_2 + 0.279\,Z_3 + 0.186 \quad + 1.50$
52	$Z_1 + 0.367\,Z_2 + 0.359\,Z_3 + 0.216 \quad - 0.91$
57	$Z_1 + 0.560\,Z_2 - 0.245\,Z_3 - 0.115 \quad - 4.48$
58	$Z_1 + 0.571\,Z_2 + 0.430\,Z_3 + 0.259 \quad - 2.09$
69	$Z_1 + 1.000\,Z_2 - 0.137\,Z_3 - 0.065 \quad - 8.13$

The following normal equations were derived from the preceding observational equations:

$$\begin{array}{rrrrl}
15.000\,Z_1 & -\;0.847\,Z_2 & +\;2.869\,Z_3 & +\;1.827 & ..\;+ 38.200 \\
-\;0.847\,Z_1 & +\;4.105\,Z_2 & -\;0.145\,Z_3 & -\;0.048 & \quad - 38.530 \\
2.869\,Z_1 & -\;0.145\,Z_2 & +\;3.685\,Z_3 & +\;2.226 & \quad + 18.228
\end{array}$$

The Least Square solution of these equations gave the following values for Z_1, Z_2 and Z_3:

$Z_1 \quad z_1 = -\,0\overset{s}{.}0079 \qquad z_1' = -\,1\overset{''}{.}364$

$Z_2 \quad z_2 = -\,0.0111 \qquad z_2' = +\,8.982$

$Z_3 \quad z_3 = -\,0.5987 \qquad z_3' = -\,3.532$

If we let $Z_1 = a$, $Z_2 = b$, and $Z_3 = c$, there is obtained:

$a = -\,0\overset{s}{.}0079 \qquad a' = -\,1\overset{''}{.}364$

$b = -\,0.000212 \qquad b' = +\,0.172$

$c = -\,0.0132 \qquad c' = -\,0.0778$

Obsevational Equations

Exp.(5 - 8)

Star	Equation
1	$Z_1 \not+ 0.908 Z_2 + 0.3\ 69 Z_3 + 0.180 \,..\, +0 \quad 11.71$
2J	$Z_1 - 0.910 Z_2 - 0.612 Z_3 - 0.369 + 9.11$
8	$Z_1 - 0.537 Z_2 - 0.536 Z_3 - 0.363 + 5.40$
9	$Z_1 - 0.531 Z_2 + 1.000 Z_3 + 0.570 + 9.45$
19	$Z_{11} - 0.384 Z_2 + 0.258 Z_3 + 0.114 + 6.16$
27	$Z_1 - 0.181 Z_2 + 0.215 Z_3 + 0.084 + 4.42$
29	$Z_1 - 00.125 Z_2 + 0.010 Z_3 - 0.029 + 3.48$
30	$Z_1 - 0.043 Z_2 + 0.575 Z_3 + 0.308 + 3.97$
31	$Z_1 - 0.029 Z_2 + 0.981 Z_{33} + 0.535 + 5.45$
42	$Z_1 + 0.078 Z_2 - 0.241 Z_3 - 0.209 + 0.79$
44	$Z_1 + 0.094 Z_2 + 0.264 Z_3 + 0.109 + 2.07$
52	$Z_1 + 0.362 Z_2 + 0.346 Z_3 + 0.145 - 0.13$
57	$Z_1 + 0.557 Z_2 -0.271 Z_3 - 0.208 - 3.43$
58	$Z_1 + 0.567 Z_2 + 0.418 Z_3 + 0.167 - 1.62$
69	$Z_1 + 1.000 Z_2 -0\ .161 Z_3 -.0.176 - 7.32$

Normal Equations

$$15.000 Z_1 - 0.990 Z_2 + 2.615 Z_3 + 0.858 + 49.510$$

$$-0.990 Z_1 + 4.198 Z_2 - 0.157 Z_3 - 0.171 - 40.712$$

$$2.615 Z_1 - 0.157 Z_2 + 3.726 Z_3 + 2.089 + 17.247$$

$z_1 = +0.0482$ $\quad z_1' = -2.238$

$z_2 = +0.0299$ $\quad z_2' = +9.070$

$z_3 = -0.5930$ $\quad z_3' = -2.669$

Observational Equations

Exp. (4 - 9)

Star	Equation	
1	$z_1 - 0.925\,z_2 + 0.357\,z_3 + 0.276$	... + 7.75
2J	$z_1 - 0.927\,z_2 - 0.642\,z_3 - 0.299$	+ 6.30
8	$z_1 - 0.551\,z_2 - 0.564\,z_3 - 0.288$	+ 2.67
9	$z_1 - 0.544\,z_2 + 1.000\,z_3 + 0.658$	+ 4.47
19	$z_1 - 0.396\,z_2 + 0.244\,z_3 + 0.173$	+ 2.73
27	$z_1 - 0.190\,z_2 + 0.201\,z_3 + 0.144$	+ 0.68
29	$z_1 - 0.134\,z_2 - 0.007\,z_3 - 0.004$	+ 0.15
30	$z_1 - 0.051\,z_2 + 0.568\,z_3 + 0.352$	+ 0.51
31	$z_1 - 0.039\,z_2 + 0.981\,z_3 + 0.575$	+ 0.93
42	$z_1 + 0.071\,z_2 - 0.264\,z_3 - 0.155$	- 2.22
44	$z_1 + 0.087\,z_2 + 0.251\,z_3 + 0.148$	- 1.53
52	$z_1 + 0.358\,z_2 + 0.334\,z_3 + 0.139$	- 4.19
57	$z_1 + 0.555\,z_2 - 0.294\,z_3 - 0.202$	- 6.43
58	$z_1 + 0.565\,z_2 + 0.408\,z_3 + 0.168$	- 5.79
69	$z_1 + 1.000\,z_2 - 0.182\,z_3 - 0.241$	- 10.30

Normal Equations

$$15.000\,z_1 - 1.121\,z_2 + 2.391\,z_3 + 1.444 \quad - 4.270$$

$$-1.121\,z_1 + 4.299\,z_2 - 0.163\,z_3 - 0.518 \quad - 37.135$$

$$2.391\,z_1 - 0.163\,z_2 + 3.770\,z_3 + 2.240 \quad + 3.896$$

$z_1 = + 0.0057$ $\qquad z_1' = + 1.1707$

$z_2 = + 0.0995$ $\qquad z_2' = + 8.891$

$- 0.5935$

$z_3 = - 0.$[illegible] $\qquad z_3' = - 1.392$

Observational Equations

Exp. (3 - 10)

The right hand member of all equations = 0.

Star	Equation
1	$Z_1 - 0.939\,Z_2 + 0.343\,Z_3 + 0.302 \ldots + 7.72$
2J	$Z_1 - 0.941\,Z_2 - 0.674\,Z_3 - 0.284 \quad + 5.98$
8	$Z_1 - 0.563\,Z_2 - 0.596\,Z_3 - 0.281 \quad + 2.90$
9	$Z_1 - 0.556\,Z_2 + 1.000\,Z_3 + 0.753 \quad + 2.75$
19	$Z_1 - 0.407\,Z_2 + 0.228\,Z_3 + 0.205 \quad + 2.32$
27	$Z_1 - 0.200\,Z_2 + 0.184\,Z_3 + 0.160 \quad + 0.28$
29	$Z_1 - 0.144\,Z_2 - 0.029\,Z_3 + 0.016 \quad + 0.12$
30	$Z_1 - 0.059\,Z_2 + 0.558\,Z_3 + 0.360 \quad - 1.00$
31	$Z_1 - 0.046\,Z_2 + 0.978\,Z_3 + 0.592 \quad + 0.90$
42	$Z_1 + 0.063\,Z_2 - 0.290\,Z_3 - 0.184 \quad - 1.88$
44	$Z_1 + 0.079\,Z_2 + 0.235\,Z_3 + 0.133 \quad - 1.97$
52	$Z_1 + 0.351\,Z_2 + 0.320\,Z_3 + 0.112 \quad - 4.77$
57	$Z_1 + 0.550\,Z_2 - 0.320\,Z_3 - 0.279 \quad - 6.19$
58	$Z_1 + 0.561\,Z_2 + 0.395\,Z_3 + 0.114 \quad - 6.33$
69	$Z_1 + 1.000\,Z_2 - 0.206\,Z_3 - 0.339 \quad - 10.35$

Normal Equations

$$15.000\,Z_1 - 1.251\,Z_2 + 2.126\,Z_3 + 1.380 \ldots\ldots - 9.520$$

$$-1.251\,Z_1 + 4.375\,Z_2 - 0.160\,Z_3 - 0.833 \qquad -36.290$$

$$2.126\,Z_1 - 0.160\,Z_2 + 3.822\,Z_3 + 2.395 \qquad + 0.708$$

$$z_1' = +0^{s}.0109 \qquad z_1' = +1''.448$$

$$z_2' = +0^{s}.171 \qquad z_2' = +8''.684$$

$$z_3' = -0^{s}.625 \qquad z_3' = -0''.626$$

Observational Equations

Exp. (2 - 11)

The right hand member of all equations = 0.

Star	Equation
1	$Z_1 - 0.954\ Z_2 + 0.331\ Z_3 + 0.297 .. + 7.57$
2J	$Z_1 - 0.955\ Z_2 - 0.706\ Z_3 - 0.302 + 6.25$
8	$Z_1 - 0.574\ Z_2 - 0.626\ Z_3 - 0.294 + 2.99$
9	$Z_1 - 0.568\ Z_2 + 1.000\ Z_3 + 0.852 + 2.59$
19	$Z_1 - 0.418\ Z_2 + 0.214\ Z_3 + 0.202 + 2.07$
27	$Z_1 - 0.209\ Z_2 + 0.168\ Z_3 + 0.201 + 0.01$
29	$Z_1 - 0.152\ Z_2 - 0.048\ Z_3 + 0.018 - 0.29$
30	$Z_1 - 0.067\ Z_2 + 0.550\ Z_3 + 0.377 - 2.05$
31	$Z_1 - 0.053\ Z_2 + 0.978\ Z_3 + 0.595 + 0.11$
42	$Z_1 + 0.056\ Z_2 - 0.315\ Z_3 - 0.176 - 2.68$
44	$Z_1 + 0.073\ Z_2 + 0.221\ Z_3 + 0.142 - 2.52$
52	$Z_1 + 0.347\ Z_2 + 0.307\ Z_3 + 0.088 - 5.28$
57	$Z_1 + 0.547\ Z_2 - 0.346\ Z_3 - 0.306 - 6.39$
58	$Z_1 + 0.558\ Z_2 + 0.383\ Z_3 + 0.088 - 6.65$
69	$Z_1 + 1.000\ Z_2 - 0.229\ Z_3 - 0.201 - 10.55$

Normal Equations

$$15.000\ Z_1 - 1.369\ Z_2 + 1.882\ Z_3 + 1.581 - 14.820$$

$$-1.369\ Z_1 + 4.461\ Z_2 - 0.157\ Z_3 - 0.783 - 36.990$$

$$1.882\ Z_1 - 0.157\ Z_2 + 3.894\ Z_3 + 2.511 - 1.001$$

$z_1 = -\ 0.0124 \qquad z_1' = +\ 1.829$

$z_2 = +\ 0.150 \qquad z_2' = +\ 8.841$

$z_3 = -\ 0.633 \qquad z_3' = -\ 0.271$

Observational Equations

Exp. (1 - 12)

The right hand member of all equations = 0.

Star	Equation
1	$Z_1 - 0.959\, Z_2 + 0.316\, Z_3 + 0.291 \,..\, + 8.26$
2J	$Z_1 - 0.961\, Z_2 - 0.744\, Z_3 - 0.301 \quad +- \; 6.58$
8	$Z_1 - 0.579\, Z_2 - 0.662\, Z_3 - 0.292 \quad + \; 3.63$
9	$Z_1 - 0.573\, Z_2 + 1.000\, Z_3 + 0.921 \quad + \; 0.50$
19	$Z_1 - 0.422\, Z_2 + 0.196\, Z_3 + 0.242 \quad + \; 2.58$
27	$Z_1 - 0.213\, Z_2 + 0.150\, Z_3 + 0.215 \quad + \; 0.22$
29	$Z_1 - 0.156\, Z_2 - 0.072\, Z_3 + 0.036 \quad + \; 0.45$
30	$Z_1 - 0.071\, Z_2 + 0.540\, Z_3 + 0.404 \quad - \; 2.27$
31	$Z_1 - 0.057\, Z_2 + 0.977\, Z_3 + 0.619 \quad + \; 0.75$
42	$Z_1 + 0.052\, Z_2 - 0.344\, Z_3 - 0.154 \quad - \; 0.71$
44	$Z_1 + 0.069\, Z_2 + 0.203\, Z_3 + 0.148 \quad - \; 2.30$
52	$Z_1 + 0.342\, Z_2 + 0.291\, Z_3 + 0.070 \quad - \; 5.26$
57	$Z_1 + 0.544\, Z_2 - 0.375\, Z_3 - 0.292 \quad - \; 6.08$
58	$Z_1 + 0.555\, Z_2 + 0.369\, Z_3 + 0.087 \quad - \; 7.12$
69	$Z_1 + 1.000\, Z_2 - 0.256\, Z_3 - 0.436 \quad - \; 9.99$

Normal Equations

$$15.000\, Z_1 - 1.429\, Z_2 + 1.589\, Z_3 + 1.558 \quad - 10.760$$

$$- 1.429\, Z_1 + 4.490\, Z_2 - 0.135\, Z_3 - 1.088 \quad - 36.965$$

$$+ 1.589\, Z_1 - 0.135\, Z_2 + 3.990\, Z_3 + 2.685 \quad - 3.719$$

$z_1 = - 0.0132 \qquad z_1' = + 1.481$

$z_2 = + 0.2183 \qquad z_2' = + 8.725$

$z_3 = - 0.6601 \qquad z_3' = + 0..638$

Table of Residuals

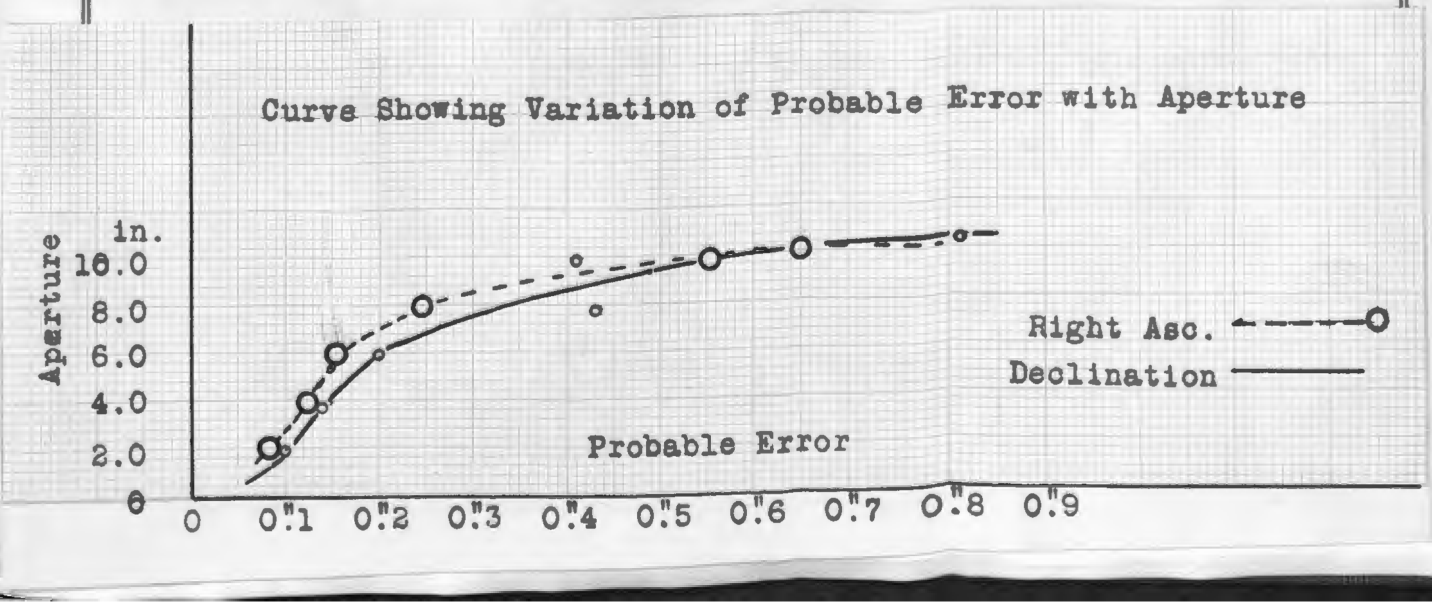
for

Comparison Stars

Right Asc.

Star	Exp.	(6 - 7)	(5 - 8)	(4 - 9)	(3 - 10)	(2 - 11)	(1 - 12)
1		-0.001	-0.018	-0.022	-0.051	-0.068	-0.131
2		+0.003	+0.015	-0.004	-0.013	-0.010	-0.032
8		-0.006	-0.013	-0.002	+0.006	+0.004	+0.005
9		-0.004	+0.009	+0.017	+0.044	+0.122	+0.123
19		+0.009	-0.002	-0.006	+0.005	-0.008	+0.008
27		-0.004	-0.000	+0.012	+0.022	+0.052	+0.057
29		+0.001	+0.021	-0.008	+0.011	+0.013	+0.037
30		+0.010	+0.014	+0.016	+0.012	+0.007	+0.020
31		+0.001	+0.000	-0.005	-0.016	-0.044	-0.051
42		-0.007	-0.016	+0.015	+0.019	+0.019	+0.071
44		+0.010	+0.004	+0.014	+0.010	+0.001	+0.016
52		-0.011	-0.001	-0.017	-0.017	-0.066	-0.061
57		+0.018	+0.017	+0.033	+0.027	-0.017	+0.060
58		-0.013	-0.016	-0.012	-0.026	-0.082	-0.049
69		-0.003	-0.004	-0.028	-0.028	+0.082	-0.062

See page 25 for probable error tables.


Curve Showing Variation of Probable Error with Aperture

Residuals

Declination

Star	(6 - 7)	(5 - 8)	(4 - 9)	(3 - 10)	(2 - 11)	(1 - 12)
1	+ 0.17	+ 0.25	+0.20	+ 0.81	+ 0.88	+ 1.58
2	+ 0.11	+ 0.25	+0.12	- 0.32	- 0.17	- 0.80
8	- 0.15	- 0.28	-0.28	- 0.17	- 0.08	- 0.36
9	- 0.17	- 0.28	-0.59	- 1.26	- 0.87	- 2.38
19	- 0.15	- 0.25	+0.04	+ 0.10	+ 0.15	+ 0.50
27	- 0.06	- 0.03	-0.10	- 0.13	- 0.06	- 0.07
29	- 0.02	+ 0.14	+0.13	+ 0.30	+ 0.21	+ 0.53
30	- 0.05	- 0.20	+0.44	- 0.41	- 0.96	- 1.06
31	+ 0.23	+ 0.33	+0.39	+ 1.34	+ 1.21	+ 2.35
42	+ 0.12	- 0.10	-0.05	+ 0.30	- 0.27	+ 1.00
44	+ 0.07	- 0.02	+0.06	+ 0.02	- 0.11	- 0.09
52	- 0.24	- 0.01	-0.30	- 0.47	- 0.46	- 0.61
57	+ 0.05	+ 0.10	+0.08	+ 0.23	+ 0.36	- 0.10
58	+ 0.16	+ 0.16	-0.17	- 0.26	+ 0.01	- 0.56
69	- 0.03	- 0.05	+0.01	- 0.09	+ 0.18	+ 0.05

Probable Errors

Exposure	(6 - 7)	(5 - 8)	(4 - 9)	(3 - 10)	(2 - 11)	(1 - 12)
	"	"	"	"	"	"
Declination	0.103	0.145	0.193	0.431	0.412	0.822
Right Asc.	0.086	0.125	0.155	0.249	0.554	0.653

Comparison of Yale Triangulation with Photographic Positions

Exposure (6 - 7)

Star	Mpls. - Y.		Star	Mpls. - Y.		Star	Mpls. - Y	
1	+0^{s}.002	+0$''$.17	27	-0^{s}.004	-0$''$.05	52	-0^{s}.006	-0$''$.18
2	+0.006	+0.16	29	+0.010	-0.02	56	-0.030	-0.05
6	+0.017	+0.02	30	+0.011	-0.05	57	+0.016	+0.04
8	+0.004	-0.09	31	-0.001	+0.24	58	-0.012	+0.12
9	-0.010	-0.16	42	-0.007	+0.12	62	+0.001	+0.07
10	-0.016	-0.16	44	+0.008	+0.06	64	See p. m. star	
13	-0.006	0.00	47	See p.m. star		68	-0.010	+0.49
19	+0.008	-0.16	51	+0.013	+0.26	69	-0.005	-0.05

The photographic positions and the Second Yale places for the proper motion stars, numbers 47 and 64 have been reduced to 1885 by means of the proper motions given by Elkin[1] and Trumpler.[2] The following comparison gives, as above, the difference between Minneapolis and Yale.

Star	Mpls. - Y. Proper Motion by Elkin		Mpls. - Y. Proper Motion by Trumpler	
64	- 0^{s}.007	- 0$''$.03	- 0^{s}.008	- 0$''$.22
47	- 0.025	- 0.13	+ 0.005	0.00

On page 27 will be found a table giving the places of these stars for 1885 as obtained by Bessel, Elkin, Smith and myself.

[1] Elkin, Trans. Ast. Obs. Yale. Vol. I, p. 357

[2] Trumpler, A. N. NO. 4790, p. 218.

Proper Motion

The proper motion has been derived from the following places by the method of least squares.

Place of Observation	Epoch	Star	Right Asc.			Declination		
			°	'	"	°	'	"
K.	1840	47	55	13	25.71	23	11	12.45
Y_1	1885				23.70			13.49
Y_2	1901				22.95			13.45
Mpls.	1915				22.72			13.86
K.	1840	64	55	45	60.98	23	51	56.68
Y_1	1885				59.27			58.13
Y_2	1901				59.60			59.26
Mpls.	1915				58.98			59.61
K	1840	68	55	54	4.73 [illegible]	24	8	42.04
Y_1	1885				5.20			41.97
Y_2	1901				5.36			42.80
Mpls.	1915				5.43			43.09

Proper Motion Values

Star		Right Asc.	Declination
47	Elkin	-0.045	+ 0.018
	Trumpler	0.030 ± .005	0.024 ± .005
	Mpls.	0.046 ± .001	0.020 ± .002
64	Elkin	-0.026	+ 0.041
	Trumpler	0.027 ± .010	0.032 ± .005
	Mpls.	0.028 ± .005	0.044 ± .005
68	Elkin		
	Trumpler	+0.012 ± .004	+ 0.004 ± .004
	Mpls.	0.011 ± .0003	0.014 ± .006

Results

An inspection of the probable error curve page 24 shows that we may expect a probable error of about $0\overset{''}{.}14$ for a four inch aperture with a lengthening of exposure by about six times. It was shown earlier in the paper that we should be justified in capping down the objective provided that the exposure time was not increased over ten fold. A ten fold exposure would correspond to a three inch aperture which in turn, as is shown by the curve, yields a probable error of about $0\overset{''}{.}10$. Since $0\overset{''}{.}10$ has been shown by previous work at this observatory to be about the minimum error attainable at the central region of the plate it seems justifiable to conclude that the advantages outlined at the beginning of this paper are fully realized.

A comparison of the places obtained in this work with those of Bessel, Elkin and Smith yield values of proper motion which agree well with those of Elkin and Trumpler.

A phase of the subject which might profitably have been taken up is the variation of image size with the aperture. If the first images, Exp. (1 - 12) be excluded by reason of over exposure, there is still a gradual diminution in image diameter with a decrease in aperture. This appears to take place very slowly until the last aperture is reached where the change is more abrupt. However in no case was the change considerable enough to destroy an image just at the limit of visibility. At the edge ~~this f~~ of the plate this tendency was not so marked, doubtless due to the fact that in the small aperture images all of the light was concentrated in a point and not spread out in " wings."

In conclusion I wish to thank Prof. Leavenworth for the kindly interest and valuable advice which he has at all times been so willing to give.