REMARKS ON EIGENVALUES OF HANKEL MATRICES

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Abstract. Two conditions for the n-tuples of eigenvalues of real symmetric $n \times n$ Hankel matrices are presented.

1. Introduction. K.R. Driessel posed the following problem: Characterize all n-tuples of real numbers which can serve as n-tuples of eigenvalues of some $n \times n$ real Hankel matrix. We shall present two necessary conditions for $n \times n$ matrices and show that the second condition is also sufficient for the case $n = 3$.

Let us recall that a Hankel matrix is a matrix H of the form $H = (h_{i+j})$, $i,j = 0,\ldots,n-1$, i.e.

$$H = \begin{pmatrix} h_0 & h_1 & h_2 & \ldots & h_{n-1} \\ h_1 & h_2 & \ldots & h_{n-1} & h_n \\ h_2 & & \ldots & & \\ \ldots & & \ldots & & \ldots \\ h_{n-1} & & \ldots & h_{2n-3} & h_{2n-2} \end{pmatrix}.$$

Thus it is always symmetric and in the case that it is real, its eigenvalues are real.

2. Results. Observe first that the identity matrix is for $n \geqq 3$ not Hankel so that not all n-tuples $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$ can be n-tuples of $n \times n$ real Hankel matrices. This also suggests that there could be a positive constant bound from below for the "relative variance" of a non-zero real $n \times n$ Hankel matrix, i.e. for the number $\sum_{i<j}(\lambda_i - \lambda_j)^2 / \sum \lambda_i^2$. The best such bound is given in the first theorem.

THEOREM 1. *An n-tuple of real numbers $\lambda_1, \ldots, \lambda_n$ can serve as the n-tuple of eigenvalues of an $n \times n$ real Hankel matrix only if*

$$\sum_{1 \leqq i < j \leqq n} (\lambda_i - \lambda_j)^2 \geqq K_n \sum_{i=1}^{n} \lambda_i^2, \tag{1}$$

where

$$K_n = 2\left(\frac{2}{3} + \frac{4}{5} + \frac{6}{7} + \cdots + \frac{n-2}{n-1}\right) \text{ if } n \text{ is even,}$$
$$K_n = 2\left(\frac{2}{3} + \frac{4}{5} + \cdots + \frac{n-3}{n-2} + \frac{n-1}{2n}\right) \text{ if } n \text{ is odd,} \tag{2}$$

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(thus $K_3 = \frac{2}{3}$).

The multiplicative constant K_n is the best possible for each $n \geqq 2$ and the equality in (1) is attained if and only if the Hankel matrix is a multiple of the $n \times n$ matrix

$$\widehat{H} = \begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & & \cdots \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots & & \\ \frac{1}{3} & 0 & \frac{1}{5} & & \cdots & & \\ 0 & \frac{1}{5} & & & \cdots & & \\ \frac{1}{5} & & & & \cdots & & \frac{1}{3} \\ \vdots & & & & \frac{1}{3} & & 0 \\ & & & \frac{1}{3} & 0 & & 1 \end{pmatrix},$$

i.e. $\widehat{H} = (\hat{h}_{i+j})$, $i, j = 0, \cdots, n-1$,

$$\hat{h}_t = 0 \text{ if } t \text{ is odd },\ \hat{h}_t = \hat{h}_{2n-2-t} = \frac{1}{t+1}$$

if t is even and $t \leqq n-1$.

The condition (1) is, however, not sufficient even for $n = 3$.

Proof. Let $H = (h_{i+j})$, $i, j = 0, \cdots, n-1$, be a real Hankel matrix, let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be its eigenvalues. Then

$$\sum_{i=1}^{n} \lambda_i = h_0 + h_2 + \cdots + h_{2n-2}; \tag{3}$$

since $\sum_i \lambda_i^2 = trHH^T$ is the sum of squares of all the entries of H, we have

$$\sum_{i=1}^{n} \lambda_i^2 = h_0^2 + 2h_1^2 + 3h_2^2 + \cdots (n-1)h_{n-2}^2 + nh_{n-1}^2 + \cdots + 2h_{2n-3}^2 + h_{2n-2}^2. \tag{4}$$

Denote by v_n, w_n respectively the vectors with n coordinates

$$v_n = (h_0, h_2\sqrt{3}, h_4\sqrt{5}, \cdots, h_{n-2}\sqrt{n-1}, h_n\sqrt{n-1}, h_{n+2}\sqrt{n-3}, \cdots, h_{2n-2}),$$
$$w_n = \left(1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}}, \cdots, \frac{1}{\sqrt{n-1}}, \frac{1}{\sqrt{n-1}}, \frac{1}{\sqrt{n-3}}, \cdots, 1\right)$$

if n is even,

$$v_n = (h_0, h_2\sqrt{3}, h_4\sqrt{5}, \cdots, h_{n-3}\sqrt{n-2}, h_{n-1}\sqrt{n}, h_{n+1}\sqrt{n-2}, \cdots, h_{2n-2}),$$
$$w_n = \left(1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}}, \cdots, \frac{1}{\sqrt{n-2}}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n-2}}, \cdots, 1\right)$$

if n is odd.

Let us observe that by (2), (3) and (4) for the inner products

$$(v_n, w_n) = \sum_{i=1}^{n} \lambda_i,$$
$$(w_n, w_n) = n - K_n,$$
$$(v_n, v_n) \leqq \sum_{i=1}^{n} \lambda_i^2,$$

with equality if and only if $h_{2k+1} = 0$ for $k = 0, \cdots, n-1$.

We have now by the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{1 \leqq i < j \leqq n} (\lambda_i - \lambda_j)^2 &= n \sum \lambda_i^2 - \left(\sum \lambda_i\right)^2 \\ &= n \sum \lambda_i^2 - (v_n, w_n)^2 \\ &\geqq n \sum \lambda_i^2 - (w_n, w_n)(v_n, v_n) \\ &\geqq n \sum \lambda_i^2 - (n - K_n) \sum \lambda_i^2 \\ &= K_n \sum \lambda_i^2. \end{aligned}$$

The equality conditions are easily established.

An example of a triple satisfying (1) but not corresponding to any 3×3 Hankel matrix will be given in Remark A.

THEOREM 2. *An n-tuple $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$ of real numbers can serve as the n-tuple of eigenvalues of a Hankel $n \times n$ matrix only if both conditions*

$$\begin{aligned} &\lambda_1 - \lambda_{n-1} - 2\lambda_n \geqq 0, \\ &2\lambda_1 + \lambda_2 - \lambda_n \geqq 0 \end{aligned} \tag{5}$$

are fulfilled.

For $n = 3$, these conditions are also sufficient and a Hankel matrix with the eigenvalues $\lambda_1 \geqq \lambda_2 \geqq \lambda_3$ satisfying (5) is

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \beta \\ \gamma & \beta & \alpha \end{pmatrix}, \tag{6}$$

where

$$\begin{aligned} \alpha &= \frac{1}{3}(\lambda_1 + 2\lambda_2 + \lambda_3), \\ \beta &= \sqrt{\frac{1}{18}(2\lambda_1 + \lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 - 2\lambda_3)}, \\ \gamma &= \frac{1}{3}(\lambda_1 - \lambda_2 + \lambda_3). \end{aligned}$$

Proof. Let H be an $n \times n$ real Hankel matrix with the eigenvalues $\lambda_1 \geqq \cdots \geqq \lambda_n$. Let $Z = (z_{ik})$ be the symmetric matrix defined by

$$z_{1,2s-1} = z_{2s-1,1} = 1, \quad z_{ss} = -2 \text{ for } s = \left[\frac{n+1}{2}\right],$$

$z_{ik} = 0$ otherwise.

Thus for the eigenvalues $\beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_n$ of Z,

$$\beta_1 = 1, \beta_2 = \cdots = \beta_{n-2} = 0, \beta_{n-1} = -1, \beta_n = -2.$$

We shall now apply the well known fact (cf. [1]) that for our symmetric matrices

$$\sum_{i=1}^{n} \lambda_i \beta_{n+1-i} \leqq trHZ \leqq \sum_{i=1}^{n} \lambda_i \beta_i.$$

Since $trHZ = 0$, we obtain

$$-2\lambda_1 - \lambda_2 + \lambda_n \leqq 0 \leqq \lambda_1 - \lambda_{n-1} - 2\lambda_n,$$

i.e. (5).

Since (6) has an eigenvalue $\alpha - \gamma = \lambda_2$ corresponding to the eigenvector $(1, 0, -1)^T$, the remaining two are easily seen to be λ_1 and λ_3.

Remark A. That the condition (1) is indeed a consequence of (5) in the case $n = 3$, follows immediately from the identity

$$(\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2 - \frac{2}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \equiv$$
$$\equiv \frac{2}{27}(7(\lambda_1 - \lambda_2 - 2\lambda_3)(2\lambda_1 + \lambda_2 - \lambda_3) + (2\lambda_1 - 5\lambda_2 + 2\lambda_3)^2).$$

It follows that (1) can be strengthened to

$$(\lambda_1-\lambda_2)^2+(\lambda_1-\lambda_3)^2+(\lambda_2-\lambda_3)^2 \geqq \frac{2}{3}(\lambda_1^2+\lambda_2^2+\lambda_3^2)+\frac{2}{27}(2\lambda_1-5\lambda_2+2\lambda_3)^2 (\text{ for } \lambda_1 \geqq \lambda_2 \geqq \lambda_3),$$

which is then, for $n = 3$ and $\lambda_1 \geqq \lambda_2 \geqq \lambda_3$, a necessary and sufficient condition. (The triple 2.9,1,1 shows that (1) does not suffice, (5) is not satisfied.)

Remark B. Since the inverse of a nonsingular Hankel matrix is a Bézout matrix [2], (1) and (5) yield conditions for the spectrum of a Bézout matrix if we set $\lambda_i = \frac{1}{\mu_i}$.

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