STRUCTURE-RANKS OF MATRICES

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Abstract. Structure-rank of a matrix, i.e. maximum order of a nonsingular submatrix all of whose entries are located in a given structure (subset of $M \times N$ where M, N are row- and column-index sets, respectively) were previously studied for the off-diagonal part. Here, results on more general ranks, e.g. off-block-diagonal, strictly upper-triangular and strictly upper-block-triangular are presented. In particular, it is shown that for a nonsingular matrix A, in each of the last mentioned three cases, the ranks of A and A^{-1} coincide.

1. Introduction. In [2], the notion of the off-diagonal rank of a square matrix was introduced as the maximum order of a nonsingular submatrix of A none of whose entries is a diagonal entry of A. In particular, it was shown that the off-diagonal ranks of a nonsingular matrix and its inverse coincide.

We intend to generalize this rank and obtain results of a similar nature. As usual, for an $m \times n$ matrix A, $M = \{1, 2, \cdots, m\}$, $N = \{1, 2, \cdots, n\}$, and $\alpha \subseteqq M$, $\beta \subseteqq N$, we denote by $A[\alpha|\beta]$ the submatrix of A with row-indices in α and column-indices in β.

Under a *structure* S in the class of all such $m \times n$ matrices we understand a subset of $M \times N$. We call then S-*rank* of A, denoted $r(S; A)$, the maximum order of a nonsingular submatrix of A all of whose entries are in S; in other words,

$$r(S; A) = \max\{\text{rank } A[\alpha|\beta] | \alpha \times \beta \subseteqq S\}. \tag{1}$$

Thus, for square $n \times n$ matrices and $S_\sigma = N \times N \setminus \{(1,1), (2,2), \cdots, (n,n)\}$, $r(S_\sigma; A)$ is the off-diagonal rank of A.

We shall also say that a structure S has *combinatorial rank* $cr(S)$ if

$$cr(S) = \min\{k | P \subseteqq M, Q \subseteqq N, S \subseteqq (P \times N) \cup (M \times Q), |P| + |Q| = k\}.$$

The following theorem ([1], Cor. 3) will be of crucial importance in the sequel:

THEOREM A. *Let A be a nonsingular $n \times n$ matrix and let α, β be subsets of N. Then*

$$\text{rank } A^{-1}[\alpha|\beta] = \text{rank } A[N \setminus \beta | N \setminus \alpha] + |\alpha| + |\beta| - n. \tag{2}$$

COROLLARY B. *For a nonsingular $n \times n$ matrix A and $\alpha \subseteqq N$,*

$$\text{rank } A^{-1}[\alpha | N \setminus \alpha] = \text{rank } A[\alpha | N \setminus \alpha]. \tag{3}$$

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2. Results. The following is immediate:

THEOREM 1. *Let S_1, S_2 be structures on $M \times N$ such that $S_1 \subseteqq S_2$. Then for each $|M| \times |N|$ matrix A,*

$$r(S_1; A) \leqq r(S_2; A) \leqq r(S_1; A) + cr(S_2 \setminus S_1).$$

Let us now generalize the theorem on the off-diagonal rank mentioned in the Introduction to off-block-diagonal ranks:

THEOREM 2. *Let $N = N_1 \cup N_2 \cup \cdots \cup N_p$ be a decomposition (i.e. $N_i \cap N_j = \emptyset$ for all $i, j, i \neq j$) of $N, |N| = n$. Let*

$$S = N \times N \setminus \cup_{i=1}^{p} (N_i \times N_i).$$

Then for every nonsingular $n \times n$ matrix A,

$$r(S; A^{-1}) = r(S; A). \tag{4}$$

Proof. Since (4) is symmetric *w.r.* to inversion, assume that

$$r(S; A^{-1}) \geqq r(S; A).$$

Let $r(S; A^{-1}) = \text{rank } A^{-1}[\alpha_0|\beta_0]$, $\alpha_0 \times \beta_0 \subseteqq S$, let $I = \{i | N_i \cap \alpha_0 \neq \emptyset\}$, and

$$\alpha_1 = \cup_{i \in I} N_i.$$

Since $\beta_0 \cap \alpha_1 = \emptyset$, we have $\beta_0 \subseteqq N \setminus \alpha_1$ so that

$$\alpha_0 \times \beta_0 \subseteqq \alpha_1 \times (N \setminus \alpha_1) \subseteqq S.$$

By Theorem 1 and Corollary B,

$$\begin{aligned} r(S; A^{-1}) &\leqq \text{rank } A^{-1}[\alpha_1 | N \setminus \alpha_1] \\ &= \text{rank } A[\alpha_1 | N \setminus \alpha_1] \\ &\leqq r(S; A). \end{aligned}$$

Thus (4) follows. □

In the next theorem, we prove a similar statement for strictly block-triangular structures.

THEOREM 3. *Let* $N = N_1 \cup N_2 \cup \cdots \cup N_p$ *be a decomposition of* N, *let*

$$S_t = \cup_{(i,j),i<j}(N_i \times N_j).$$

Then, for any nonsingular $n \times n$ *matrix* A,

$$r(S_t; A^{-1}) = r(S_t; A). \tag{5}$$

Proof. We shall first prove a lemma.

LEMMA. *In our notation,*

$$r(S_t; A) = \max_{q=1,2,\cdots,p} \text{rank } A\left[\cup_{i=1}^q N_i | N \setminus \cup_{i=1}^q N_i\right]. \tag{6}$$

Proof. Denote the right-hand side in (6) by R, set

$$\sigma_q = \cup_{i=1}^q N_i \times \left(N \setminus \cup_{i=1}^q N_i\right), q = 1, \cdots, p.$$

Since $\sigma_q \subsetneqq S_t$ for each q, it follows that

$$r(S_t; A) \geqq r(\sigma_q; A) \text{ for each } q$$

so that

$$r(S_t; A) \geqq R.$$

Let

$$r(S_t; A) = \text{rank } A[\alpha_0|\beta_0], \alpha_0 \times \beta_0 \subsetneqq S_t.$$

Define $I = \{i|\alpha_0 \cap N_i \neq \emptyset\}$, $J = \{j|\beta_0 \cap N_j \neq \emptyset\}$, $\alpha_1 = \cup_{i\in I} N_i$. Since $\alpha_0 \times \beta_0 \subsetneqq S_t$, every index in I is smaller than every index in J. We have thus

$$\alpha_0 \times \beta_0 \subsetneqq \alpha_1 \times (N \setminus \alpha_1)$$

so that

$$\begin{aligned} r(S_t; A) &\leqq \text{rank } A[\alpha_1|N \setminus \alpha_1] \\ &\leqq R. \end{aligned}$$

To return to the proof of Theorem 3, (5) follows immediately since by Corollary B, the right hand sides of (6) for A and A^{-1} are equal. □

We can thus speak about *strictly upper-triangular rank* of A in the case that

$$S_u = \cup_{1 \leqq j < k \leqq n}(j, k), \tag{7}$$

about the *strictly upper-block-triangular rank* of A if

$$S_{ub} = \cup_{1 \leqq j < k \leqq p}(N_j \times N_k) \tag{8}$$

and all indices in N_k are smaller than all indices in N_{k+1}, $1 \leqq k \leqq p-1$.

We have thus

COROLLARY 4. *If A is a nonsingular $n \times n$ matrix then the strictly upper-triangular ranks of A and A^{-1} coincide. The same is true for strictly upper-block-triangular ranks.*

In the next theorem, we admit to include to S_u in (7) the diagonal entries as well, thus obtaining the *weakly upper-triangular rank* or, S_w-rank.

THEOREM 5. *The weakly upper-triangular ranks of A and A^{-1} differ at most by one:*

$$|r(S_w; A) - r(S_w; A^{-1})| \leqq 1. \tag{9}$$

Proof. As in the proof of the Lemma, one shows easily that

$$r(S_w; A) = \max_{k=1,\cdots,n} \text{rank } A[\{1,\ldots,k\}|\{k, k+1,\cdots,n\}]. \tag{10}$$

By Theorem A,

$$\text{rank } A[\{1,\cdots,k\}|\{k,\cdots,n\}] = \text{rank } A^{-1}[\{1,\cdots,k-1\}|\{k+1,\cdots,n\}] + 1$$

(and just 1 on the right-hand side if $k = 1$, or $k = n$). Hence

$$\begin{aligned} \text{rank } A^{-1}[\{1,\cdots,k\}|\{k,\cdots,n\}] &\leqq \text{rank } A^{-1}[\{1,\cdots,k-1\}|\{k+1,\cdots,n\}] + 2 \\ &= \text{rank } A[\{1,\cdots,k\}|\{k,\cdots,n\}] + 1. \end{aligned}$$

Thus also the maxima differ at most by one which, by (10), yields (9).

Example. For

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

so that equality is attained in (9).

In the following theorem, relation of the ranks to the LU-decomposition is cleared.

THEOREM 6. *Let, for an $n \times n$ matrix A and for a decomposition $N = N_1 \cup N_2 \cup \cdots \cup N_p$, all principal submatrices*

$$A\left[\cup_{j=1}^k N_j | \cup_{j=1}^k N_j\right], k = 1, \cdots, p$$

be nonsingular so that a block LU-decomposition of A exists, where L has zero blocks in all $N_j \times N_k$ positions for $j < k$ and U has zero blocks in all $N_j \times N_k$ positions for $j > k$.

Then, in the notation (8),

$$r(S_{ub}; U) = r(S_{ub}; A). \tag{11}$$

In other words, U has the same strictly upper-block-triangular rank as A (and as A^{-1} by Theorem 3).

Proof. Since $A = LU$,

$$A\left[\cup_{i=1}^{k} N_i \middle| N \setminus \cup_{i=1}^{k} N_i\right] = L\left[\cup_{i=1}^{k} N_i \middle| \cup_{i=1}^{k} N_i\right] U\left[\cup_{i=1}^{k} N_i \middle| N \setminus \cup_{i=1}^{k} N_i\right]$$

which implies

$$\text{rank } A\left[\cup_{i=1}^{k} N_i \middle| N \setminus \cup_{i=1}^{k} N_i\right] = \text{rank } U\left[\cup_{i=1}^{k} N_i \middle| N \setminus \cup_{i=1}^{k} N_i\right],$$

$k = 1, \cdots, p$.

By (6), (11) follows.

3. Applications. In this concluding section, we shall first generalize the structure-rank as follows:

We assume that in $M \times N$, two structures S_1, S_2 are given such that $S_1 \subset S_2$, $S_1 \neq S_2$. If A is an $m \times n$ matrix, the *difference* $S_2 \ominus S_1$*-rank* of A is the maximum order of a nonsingular submatrix $A[\alpha|\beta]$ for which

$$S_1 \subseteqq \alpha \times \beta \subseteqq S_2.$$

We denote it as $r(S_2 \ominus S_1; A)$ so that

$$r(S_2 \ominus S_1; A) = \max\{\text{rank } A[\alpha|\beta] | \alpha \subseteqq M, \beta \subseteqq N, S_1 \subseteqq \alpha \times \beta \subseteqq S_2\}.$$

In this notation, we prove:

THEOREM 7. *Let $N = N_1 \cup \cdots \cup N_p$, $p \geqq 2$, be a decomposition of N. Let A be an $n \times n$ matrix for which $A[N_1|N_1]$ is invertible. Then, for*

$$S_1 = N_1 \times N_1, S_2 = N \times N \setminus \cup_{i=1}^{p}(N_i \times N_i),$$

and

$$\hat{S} = \cup_{i=2}^{p} N_i \times \cup_{i=2}^{p} N_i \setminus \cup_{i=2}^{p}(N_i \times N_i)$$
$$(\text{ in } (N \setminus N_1) \times (N \setminus N_1)),$$

$$r(S_2 \ominus S_1; A) = r\left(\hat{S}; A^{-1}\left[\cup_{i=2}^{p} N_i \middle| \cup_{i=2}^{p} N_i\right]\right) + |N_1|. \tag{12}$$

Proof. For notational convenience, we can assume that N_1 consists of the first n_1 indices, N_2 of the next n_2 consecutive indices $, \cdots ,$ N_p of the last n_p indices and $\sum_{k=1}^{p} n_k = n$. We can then write A in the block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} is $n_1 \times n_1$, $A_{22}(n_2 + \cdots + n_p) \times (n_2 + \cdots + n_p)$. Let the inverse conformally partitioned be denoted as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

As is well known [3], the Schur complement of A_{11} in A is

$$A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

and satisfies

$$(A/A_{11})^{-1} = B_{22}.$$

By Theorem 2 on off-block-diagonal ranks,

$$r(\hat{S}; B_{22}) = r(\hat{S}; A/A_{11})$$

which is, however, equal to

$$r(S_2 \ominus S_1; A) - n_1$$

since

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix} \begin{pmatrix} I_1 & A_{11}^{-1}A_{12} \\ 0 & I_2 \end{pmatrix}.$$

This proves (12) in the original notation. □

Let us add an observation on orthogonal and unitary matrices.

THEOREM 8. *Let C be a unitary or orthogonal matrix. Assign to every $(p \times q)$-submatrix C_0 of C the number $k(C_0) = \frac{1}{2}(p + q) - \text{rank } C_0$. Then, whenever C_1 and C_2 are complementary submatrices of C (the rows as well as the columns are complementary) then $k(C_1) = k(C_2)$.*

Proof. Follows easily from (2) and the fact that $C^{-1} = C^*$ (or C^T).

COROLLARY 9. *In a unitary or orthogonal $n \times n$ matrix, the ranks of complementary $(p \times q)$-submatrices with $p + q = n$ coincide.*

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