Preservice Elementary Teachers' Understandings of the Connections Among Decimals, Fractions, and the Set of Rational Numbers: A Descriptive Case Study

A Dissertation
SUBMITTED TO THE FACULTY OF
UNIVERSITY OF MINNESOTA
BY

Christy Rae Pettis

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Dr. Kathleen Cramer, Advisor

Dr. Terrence Wyberg, Advisor

December, 2015

Acknowledgements

I would like to take this opportunity to thank the many people who supported me throughout this entire process. First, I would like to thank the members of my committee, Dr. Kathy Cramer, Dr. Terry Wyberg, Dr. Sashank Varma, and Dr. Tom Post. I am grateful to all of you for your feedback and your support. I would especially like to thank my advisors, Kathy and Terry. Kathy, I am forever grateful for the opportunities you gave me to put what I was learning into practice. I also am very appreciative of your consistent encouragement throughout my time at the U. Terry, I learned so much from you about both teaching and mathematics. I hope we can have many more conversations over good coffee. I am lucky to have had you both as mentors.

I would also like to thank my colleagues and friends at the STEM Education Center. These past few years were made infinitely more enjoyable because I got to share them with all of you. I especially wish to thank Aran, Bethann, and Christina. Your friendship and the conversations we had were truly a highlight of my time at the U. Here's to many more years of conversations that begin, "Okay, we're really going to get that done. But first, let me just show you this…"

I am deeply grateful to my family for their love, patience and understanding as I pursued this degree. Grace, Amelia, and William, you are all amazing. I am so lucky to get to be your mom. Finally, I want to thank my husband Matthew. There are not words to fully express how grateful I am to you. From practicing GRE words to staying up late to listen to my final presentation, you have been by my side, supporting me all the way. Talking and being with you has been the best part of…everything.

Dedication

This dissertation is dedicated to my family.

To my parents. My love of learning comes from the two of you.

To my children, Grace, Amelia, and William.

May you continue to love learning throughout your lives.

And finally, to my husband, Matt.

Thank you, thank you, for sharing this journey with me.

Abstract

The mathematical knowledge needed for teaching is a specialized form of mathematical knowledge, (Ball, Thames, & Phelps, 2008). One important area of mathematical understanding for elementary teachers is the area of number and numeration. Mathematically, the sets of whole and rational numbers and their corresponding notational systems are deeply interconnected. Ensuring that preservice elementary teachers understand the ways these sets of numbers and notations are connected, both mathematically and developmentally, is a critical component of teacher education coursework.

This study is a descriptive case study (Yin, 2014) documenting preservice elementary teachers' ways of understanding the relationships among fractions, decimals, and the sets of rational and irrational numbers. The unit of analysis was a single class of preservice elementary teachers participating in an eight-week instructional unit designed to support them in making explicit connections between concepts related to number and numeration. The broad agenda for this study is to support the development of curricula that may productively and efficiently develop preservice teachers' understandings of the connections among fractions, decimals, and the sets of rational and irrational numbers. This study extends prior work on bridging tools (Abrahamson & Wilensky, 2007) by documenting how two bridging tools were used to promote understanding of the connections between fraction and decimal notation.

Results from early in the unit indicate that preservice elementary teachers' initial understandings of the connections among fractions, decimals, and the set of rational

numbers were limited and often inaccurate. Limited understandings of decimal notation were also documented. Finally, the preservice teachers primarily used symbolic representations to explain the connection between fractions and decimals.

After the unit, the preservice teachers showed a more connected understanding of the relationships among fractions, decimals, and the set of rational numbers. The majority of preservice teachers demonstrated the ability to use multiple, non-symbolic representations in order to find and explain connections between fractions and decimals. Widespread understandings of decimal notation were documented, but these understandings were applied inconsistently. Together, the results suggest that a connected approach to curriculum design shows promise as a way to address multiple areas of preservice teachers' content understandings simultaneously.

Table of Contents

Acknowledgements i

Dedication ii

Abstract iii

List of Tables ix

List of Figures xii

Chapter 1 1

Statement of the Problem 3

Significance of the Study 9

Conceptual Models 14

Research Focus 17

Overview of the Chapters 18

Chapter 2 20

Theoretical Framework 22

Emergent Perspective 22

Mathematical Knowledge for Teaching 24

Understanding 26

The Development of Number Understanding 27

Models for Connection 29

Overview of Number and Notation for Elementary Teachers 32

The Real Number System 32

Knowledge of Number is Important for the Work of Teaching 33

Knowledge of Notation is Important for the Work of Teaching 35

Review of Literature Related to PSTs' Knowledge of Number and Numeration......................37
PSTs' Understandings of Number and Numeration 38
PSTs' Understanding of the Sets of Rational and Irrational Numbers 40
PSTs' Understandings Related to Notation and the Sets of Numbers 43
PSTs' Understanding of the Number Line..51
Summary 52
Review of Literature Related to Developing PSTs' Knowledge of Number and Numeration ..54
Challenges of Teaching Number and Numeration to PSTs 54
The Use of Alternate Numeration Systems 56
Instructional Approaches Related to the Connections Between Fractions and Decimals 60
Summary 64
Conclusion..65
Chapter 3..67
Research Design 67
Setting and Participants 69
Selection and Description of Interview Participants..69
Timeline 73
Key Instructional Tasks...73
Breaking Bread Activities...74
Number Line Activities...76
Data Sources and Collection Procedures 78
Introductory Questionnaire 79
Pre- and Posttest..80
Interview Protocols 85

Field Notes 91
Research Design and Data Analysis Procedures 91
Role of the Researcher 95
Trustworthiness 96
Conclusion 97
Chapter 4 98
Understandings Early in the Unit 101
Interview task: Definitions of Number and Types of Numbers 101
Pretest Question 1: Examples/Non-Examples of Numbers 106
Pretest Question 2: Is There a Number Between 0.66 and ⅔? 121
Pretest Question 4: Explain Why ⅓=0.333… 126
Interview 1: Number Sorting Task 134
Summary of Findings from Early in the Course 152
Relationship of Findings to Research Questions: Early in the Unit 158
Understandings After the Unit 162
Brief Summary of Relevant Activities from the Instructional Unit 164
Posttest Question 3: Are There Numbers Between .333 and ⅓? 165
Posttest Question 5: Find ⅙ as a Decimal 174
Posttest Question 6c: Locate 0.166… on a Number Line 189
Posttest Bonus Questions 2 and 3: Find ⅓ as a Decimal in Base Seven and Explain How It Relates to 2/9 in Base Ten 197
Summary of Findings From After the Unit 212
Relationship of Findings to Research Questions: After the Unit 217
Conclusion 221

Chapter 5 ... 222

Restatement of Purpose and Research Questions ... 224

Summary and Discussion of Results ... 226

Limitations ... 234

Implications and Recommendations for Future Research ... 235

Conclusion ... 244

References ... 245

Appendix A: Overview of Activities in Instructional Unit ... 256

Appendix B: Pretest ... 257

Appendix C: Posttest ... 261

Appendix D: Introductory Questionnaire ... 269

Appendix E: Description of Interview Participants ... 270

Appendix F: Interview Protocols ... 274

Appendix G: First Cycle Codes ... 277

List of Tables

Table 1. Timeline of the Study 73

Table 2. Data Sources Used in the Study as They Corresponded to Research Questions.79

Table 3. The Four Tasks Completed by Participants in Interview 1 86

Table 4. Summary of Ways Interviewees Defined Number and the Categories of Numbers They Listed During the First Interview 105

Table 5. Interviewees' Examples and Non-Examples of Rational and Irrational Numbers on the Pretest 108

Table 6. Summary of the Whole Class's Examples and Non-Examples of Rational and Irrational Numbers on Pretest Question 1 117

Table 7. Summary of Misunderstandings of Rational Numbers on Pretest Question 1 ..119

Table 8. Summary of Ways Interviewees Responded to Pretest Question 2 122

Table 9. Summary of the Whole Class's Responsed to Pretest Question 2 124

Table 10. Summary of Ways Interviewees Explained Why ⅓ = 0.33… on the Pretest..128

Table 11. Summary of Ways the Whole Class Explained Why ⅓ = 0.33… on the Pretest 131

Table 12. Definitions of the Subsets of the Real Numbers Provided to Preservice Teachers During the First Interview...135

Table 13. Summary of Ways PSTs Classified Terminating Decimals and Their Justifications for Doing So During the First Interview 136

Table 14. Summary of Ways Interviewees Classified Repeating Decimals and Their Justifications for Doing So During the First Interview 140

Table 15. Summary of Ways Interviewees Classified the Decimal Form of π and Fractions Involving π During the First Interview 147

Table 16. Summary of Themes from Early in the Unit, with Related Subthemes and Data Sources 159

Table 17. Interviewees' Explanation of How to Find a Number Between a Terminating Decimal and Related Fraction on the Pre- and Posttest 167

Table 18. Summary of Ways the Whole Class Responded to Posttest Question 3 170

Table 19. MKT2 Students' Work on Comparable Problems Relating a Fraction to a Repeating Decimal on the Pre- and Posttest 176

Table 20. MKT1 PSTs' Work on Comparable Problems Relating a Fraction to a Repeating Decimal on the Pre- and Posttest 178

Table 21. Summary of Ways the Whole Class Showed How to Find $\frac{1}{6}$ as a Decimal on the Posttest 187

Table 22. MKT2 Students' Work on Posttest Question 6c: Locate 0.166… on a Number Line 192

Table 23. MKT1 Students' Work on Posttest Question 6c: Locate 0.166… on a Number Line 193

Table 24. Summary of Ways the Whole Class Located the Repeating Decimal $0.1\overline{6}$ on a Number Line on the Posttest 196

Table 25. MKT2 Students' Posttest Work Showing how to Write the Fraction $\frac{1}{3}$ as a "Decimal" in Base 7 (Bonus Question 2) and Why $\frac{1}{3}$ in Base Seven and $\frac{2}{9}$ in Base Ten Have the Same Decimal Form (Bonus Question 3) 199

Table 26. MKT1 Students' Posttest Work Showing how to Write the Fraction ⅓ as a "Decimal" in Base 7 (Bonus Question 2) and Why ⅓ in Base Seven and $^2/_9$ in Base Ten Have the Same Decimal Form (Bonus Question 3) 204

Table 27. Summary of Ways the Whole Class Found ⅓ as a Decimal in Base 7 on the Posttest 209

Table 28. Summary of Ways the Whole Class Explained Why ⅓ in Base 7 is Similar to $^2/_9$ in Base 10 on Posttest 210

Table 29. Summary of Themes from After the Unit, with Related Research Questions and Data Sources 218

Table 30. Summary of Results by Data Source 227

Table 31. Summary of Findings, and Related Research Questions and Data Sources....229

List of Figures

Figure 1. The Lesh Translation model depicts ways of developing and assessing mathematical understandings by making connections within and between five representational areas. 14

Figure 2. The Number and Numeration System depicts the ways knowledge of number, notation, and (non-symbolic) representations are interconnected. 15

Figure 3. A 3x5 array as a bridging tool supporting understanding of the concept of the commutative property of multiplication, since the array can be viewed as 5 groups of 3 or 3 groups of 5. Adapted from Abrahamson and Wilensky (2007). 16

Figure 4. The development of the concept of number (Image from Sfard, 1991, p. 13). 28

Figure 5. The Lesh Translation Model, adapted from Lesh, Post, and Behr (1987).........29

Figure 6. The *Number and Numeration System* depicts the ways knowledge of number, notation, and (non-symbolic) representations are interconnected. 31

Figure 7. The Real Number System. 33

Figure 8. The relationship between the subsets of the real numbers and the three canonical notations. 36

Figure 9. The *Number and Numeration System* depicts the ways knowledge of number, notation, and (non-symbolic) representations are interconnected. 68

Figure 10. *Pretest questions 2, 4, 6, and 7.* 71

Figure 11. Pictures showing two ways three loaves of bread can be fairly shared by four people ("Grace," "Amelia," "William," and "Matt"), and their related symbolic notations. 75

Figure 12. Pictures showing how three loaves of bread can be fairly shared by four people (“Grace,” “Amelia,” “William,” and “Matt”), so that the resultant shares can be denoted using “decimal” notation in base ten and base five................................76

Figure 13. Two representative number line activities completed during the instructional unit. 77

Figure 14. Questions 1, 2, and 4 from the pretest. 82

Figure 15. Questions 3, 5, 6c, and Bonus Questions 2 and 3 from the posttest. 85

Figure 16. Formal definitions of numbers provided to participants between Tasks 1 and 2 during the first interview. 88

Figure 17. Examples of numbers provided to participants during the first interview, written in the same way they were written on the cards given to participants. 89

Figure 18. The major themes, supporting subthemes, and the connections between them that emerged from the Second Cycle of coding. 95

Figure 19. The *Number and Numeration System* depicts the ways knowledge of number, notation, and (non-symbolic) representations are interconnected. 100

Figure 20. Questions 1, 2, and 4 from the pretest. 101

Figure 21. “UT’s” explanation for why ⅓ =. 133

Figure 22. Themes, subthemes, and data sources related to PSTs’ nature of understanding the *Number and Numeration System* and use of representations early in the unit. .154

Figure 23. Posttest Questions 3, 5, 6c, and Bonus Questions 2 and 3. 164

Figure 24. An example of the *Breaking Bread* activity showing how three loaves of bread could be shared by four people so that the answer could be denoted by a fraction

(left) or decimal (right). 165

Figure 25. Soren and Eva use similar, but distinct, methods to locate the repeating decimal 0.16 on a number line. 194

Figure 26. Eva's work showing her attempts to write ⅓ as a "decimal" in base 7. 207

Figure 27. Nina and Korey's work showing their understanding of place value notation in base ten and base seven. 208

Figure 28. Themes, subthemes, and data sources related to PSTs' nature of understanding the *Number and Numeration System* and use of representations after the unit. 213

Figure 29. The *Number and Numeration System* depicts the ways knowledge of number, notation, and (non-symbolic) representations are interconnected. 226

Chapter 1

Introduction

"Mathematics has a remarkable beauty, power, and coherence, more than we could have ever expected. It is always changing, as we turn new corners and discover new delights and unexpected connections with old familiar grounds."
~William Thurston, mathematician

Making connections is at the heart of learning and doing mathematics. Learning how and why ideas are connected is integral to understanding mathematics. The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000b) includes "Connections" as one of the five process standards intended to highlight the processes by which mathematical knowledge is acquired. This document argues that instruction that emphasizes the interrelatedness of mathematical ideas is important because "mathematics is an integrated field of study. When students connect mathematical ideas, their understanding is deeper and more lasting, and they come to view mathematics as a coherent whole" (National Council of Teachers of Mathematics, 2000a, p. 4). High-achieving mathematics students focus on learning the big ideas of mathematics and how those ideas are connected, while low-achieving students are often those who focus on memorizing mathematics facts and procedures without connecting those facts and procedures to bigger concepts (Boaler, 2015).

Making connections is also at the heart of mathematical knowledge *for teaching*. The way teachers know mathematics can be expected to impact they way they teach mathematics. Teachers who know mathematics as a set of disconnected facts and procedures will be likely to pass these ways of knowing mathematics along to their students. In contrast, “if a teacher’s conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students” (Thompson, Carlson, & Silverman, 2007, pp. 416–417).

The importance of connections in mathematical knowledge for teaching also goes beyond teachers themselves having the type of connected understandings that are valuable to students. Teachers must also know how mathematical ideas develop, how to build on earlier learning in ways that support students in moving towards more complex ideas, and how current understandings will impact students’ reasoning when encountering new ideas (Ball et al., 2008; Ma, 2010; Silverman & Thompson, 2008). Essentially, teachers must have both their own connected understandings of mathematics and an understanding of how to unpack that knowledge in order to determine what a coherent and generative understanding of mathematics looks like for students at various developmental stages (Ball et al., 2008).

One important area of mathematical understanding that is developed during the elementary school years is that of *number*. During their first years of formal schooling, students are expected to learn what whole and (positive) rational numbers are, how they work, how they relate to real-life experiences with these quantities, and how they may be represented both symbolically and non-symbolically (National Council of Teachers of Mathematics, 2000b; National Governors Association, 2010). Mathematically, the sets of whole and rational numbers are deeply interconnected. Ensuring that preservice elementary teachers understand the ways these sets of numbers are connected, both mathematically and developmentally, is a critical component of teacher education coursework. In fact, the Conference Board of Mathematical Sciences (2012) advises that six of the recommended twelve semester hours of mathematics content instruction for preservice teachers be devoted developing their understanding of number and operations (p. 31).

Statement of the Problem

Mathematics content courses for preservice elementary teachers need to support them in developing the connected and coherent understandings of the whole and rational numbers necessary for the work of teaching. However, there are several challenges with accomplishing this. First, research suggests that, in general, preservice teachers lack a deep understanding of the various areas of number encountered in elementary school mathematics, including whole number and place value (Murawska, 2013; Thanheiser, 2009), fractions (Newton, 2008; Post, Harel, Behr, & Lesh, 1991; Tobias, 2012), and decimals (Chick, Baker, Pham, & Cheng, 2006; Kastberg & Morton, 2014). One challenge is finding ways to address all these areas in the limited time available during teacher education coursework.

A second challenge arises from the fact that preservice teachers have many years of experience working with whole and rational numbers. This is important because learning about topics perceived as "familiar" can actually be more challenging for preservice teachers than learning about new topics (Markovits & Sowder, 1990). Sinclair, Liljedahl, and Zazkis (2006) argue that preservice teachers' reluctance to revisit "familiar" topics is a psychological barrier which preservice teacher education course design must take into account. Course designers must also take into account the fact that these years of experience with numbers and notational systems mean that preservice teachers cannot be expected to engage in learning about number and notation in the same way that younger students would. This implies that coursework intended to deepen preservice elementary teachers' understanding of number and numeration cannot simply use the same activities that have been successfully used to deepen children's understanding of the same content and assume it will also work with preservice teachers. Instead, coursework for teachers take into account their prior learning experiences and how those will impact the learning of the material.

Finally, there are challenges inherent in learning about the whole and rational numbers as a connected, coherent system. Expanding the idea of number from the whole numbers to the rationals significantly alters what is meant, mathematically, by the concept of "number." The foundation of number, both historically and developmentally, is counting. Counting objects one-by-one, creating a correspondence between the verbal counting sequence, a set of objects, and the number symbols "1, 2, 3, 4, …," is how the notion of number itself is developed. Thus, whole numbers are intimately related to the idea of counts of discrete units.

The rational numbers generalize the concept of number to include *partitioned units*. Rational numbers arise out of sharing or measurement situations, where whole objects or units are equally partitioned into smaller sub-units. This shift from numbers always being able to be mapped to counts of discrete units to numbers being mapped to partitioned units has many ramifications. Thinking of a number as a count of objects, that there's always a "next biggest number," or that multiplication makes bigger are all very natural ideas when one's entire experience with number is with whole numbers. Yet none of these are true for the set of rational numbers. Thus the introduction of rational numbers forces the learner to fundamentally redefine what a number *is*.

The shift from the whole numbers to the rational numbers is further complicated by the fact that new notation is also needed in order to depict these new, more general forms of number. For rational numbers, two common forms of notation are used in our society, namely fractions and decimals. Each of these notations is similar to and different from the familiar whole number notation. Standard fraction notation makes use of whole numbers in two ways: the denominator is a whole number that indicates how a given unit has been partitioned, and numerator is a whole number that indicates a count of those partitioned pieces. Fraction notation can therefore be related to the whole number idea of counting, with the difference being that partitioned units are what is being counted. In this way, the whole numbers can be understood as a subset of the rationals by considering them as counts of units that have not been partitioned (i.e. 12/1 can be thought of as twelve iterations of a non-partitioned/whole unit. This is equivalent to the whole number notation 12, which denotes a count of twelve units). However, this connection between fraction and (Hindu-Arabic) whole number notation is not obvious. It is also complicated by the fact that while the numerator can be meaningfully mapped to whole number conceptions of counting, the denominator cannot, and yet both are written using whole number notation.

Decimal notation is the other common form of notation used for rational numbers.

Decimal notation is clearly an extension of whole numbers and is concerned both with partitioning and with counting groups of powers of ten. The powers of ten, however, may now include both whole number groups of powers of ten (the numerals to the left of the decimal point) and partitions by the powers of ten (the numerals to the right of the decimal point). Importantly, this feature allows decimals to approximate any quantity that can be measured to any desired degree of accuracy. All that is needed is to add up the number of groups and partitions of powers of ten that come within any specified margin of error. However, allowing base-ten positional notation to continue to the right of the decimal point introduces some cognitively complex ideas. First, unlike fraction notation, decimal notation looks very much like whole number notation. Yet, despite the visual similarity between the whole (Hindu-Arabic) numbers and decimals, they represent different sets of numbers. In fact, decimal notation can be used to not just one, but *two* new types of real numbers. Some decimals are related to rational numbers. Some are related to irrational numbers. To further complicate matters, decimals that represent rational numbers can look very similar to decimals that represent irrational numbers and vice versa (Markovits & Sowder, 1990). For instance, the terminating decimal 3.1415 represents a rational number, even though it looks very much like the decimal approximation for irrational number π. The infinite decimal 0.121121112… is irrational even though it is highly predictable and visually very similar to a rational, repeating decimal such as 0.112112112…. Furthermore, it is not always possible to tell just by looking at a decimal what kind of number it represents. For learners, this makes decimal notation very confusing.

Another point of confusion with decimal notation is the fact that infinite decimals can denote different kinds of numbers. Repeating infinite decimals, such as 0.333…, are rational, while non-repeating infinite decimals are not. Moreover, the fact that repeating decimals represent an infinite sum of partitions of powers of ten (e.g. $0.333... = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$) makes it difficult to see why all repeating decimals have an equivalent fraction form. A good understanding of place value can help learners see why terminating decimals relate to fractions (e.g. why 0.66 is 66/100), but it does little good in the case of repeating decimals. What fraction is equivalent to the repeating decimal $0.13\overline{24}$? How does one know that such a fraction exists? Knowing that division may be used to convert fractions to decimals and seeing that this division process sometimes results in repeating decimals, may help some students understand that a fraction equivalent to $0.13\overline{24}$ exists, but that same reasoning then makes it difficult to understand why there is not fraction equivalent for a number like π. It also does not help students to find the fraction equivalent because the division algorithm used to convert fractions to decimals is itself so opaque (not obviously related to partitioning by powers of ten).

Thus, the expansion of numbers from whole to rational numbers is complicated and difficult for learners both because it entails a reworking of what students think of as numbers, and also because it entails the introduction of two new forms of notation. However, it is also a fundamental component of elementary mathematics education and the foundation for much of the mathematics that students will learn in middle and high school and beyond. Thus, it is critical that elementary teachers understand this complicated system. They need to understand what whole, rational, and irrational numbers are, how they are related, and how they progressively expand and change the definition of number. They need to understand this clearly so that they can support students in navigating this challenging terrain. Elementary teachers also need to understand how whole number, fraction, and decimal notation work, how they may be used to denote the different types of quantities, and, equally as important, the limitations of each notation. This includes how, why, and when decimal notation may be used to denote rational quantities, and in turn, fraction and rational decimal equivalence. It also includes understanding the non-equivalence of fractions and non-rational decimals because rational and non-rational decimals can appear to be so similar.

Together the limited time available in preservice teacher education coursework, the fact that preservice teachers generally have a thin, procedural understanding of whole and rational numbers and yet perceive them to be familiar topics already learned, along with all the ways in which these connecting these concepts is inherently difficult present a formidable challenge for mathematics teacher educators. What learning is possible to do in the limited time available in content courses? If not all can be accomplished, what should be prioritized and why?

Many studies have addressed aspects of these concerns by documenting ways in which preservice teachers' mathematical knowledge for teaching needs support, by documenting the ramifications of having, or not having, adequate content knowledge on classroom teaching and/or student learning, or by documenting ways that a particular instructional sequence impacts teacher content understandings in targeted areas (Ball et al., 2008; Mewborn, 2001; Thanheiser et al., 2013). These are all necessary and useful for the design of curricula for mathematics content courses. However, as argued in the introductory section, if a *connected understanding* of mathematical concepts generally, and number and numeration specifically, is central to the work of teaching and learning mathematics meaningfully, then ensuring that preservice teachers are developing such a connected understanding should be central to conversations about curriculum design for content coursework. The fact that whole and rational numbers seem so different on the surface means that it cannot be taken for granted that coursework that effectively deepens preservice teachers' understandings of isolated areas within the number and numeration will also support the development of a connected understanding.

This study prioritizes *connection*. Deepening preservice teachers' understanding of both whole and rational numbers and their corresponding notations is acknowledged as important, but not more important than ensuring that the future teachers take from those learning experiences the idea that these seemingly different domains of number are deeply connected. As Fischbein, Jehiam, and Cohen (1995) argue, "If one intends to convey to the students the feeling of the structurality of mathematics, one has to emphasize, first of all, the coherent picture of the number system with its strict hierarchy" (p. 30). Where better to address the idea that mathematics is *not* comprised of disconnected facts and procedures than by helping the preservice teachers' understand how the facts and procedures they know for whole and rational numbers are connected?

Significance of the Study

This study seeks to add to the existing literature on preservice teachers' knowledge of number and numeration by focusing on preservice elementary teachers' understanding of rational number concepts from a *connected perspective*. In this study, a *connected perspective* means that supporting learners' in understanding the connections between mathematical concepts was considered in the design of all aspects of the unit, including the activities, the course sequence, and the assessments. In particular, during the instructional unit used in this study, a connected perspective meant that the study of whole and rational numbers was integrated together throughout the unit. The study of whole number notations (both modern and historical) was integrated with study of fraction and decimal notation. This integration was primarily accomplished by doing the same activities first with one type of number or notation and then shortly thereafter with another. For instance, in one set of activities, equal-sharing problems were used to model various fraction notations to quotients. This same activity was then used to model decimal notations as quotients.

The impetus for this study arose out of difficulties in finding research-based instructional sequences that intentionally supported preservice elementary teachers in making connections between and across the various aspects of number and numeration related to whole and rational numbers. A search of the literature revealed that few studies had intentionally looked at the ways preservice teachers understood, or made sense of, the *connections* between aspects of numbers and numeration relevant to elementary teaching (see Amato, 2005, 2006; Dubinsky, Arnon, & Weller, 2013; Fischbein, Deri, Nello, & Marino, 1985; LeSage, 2011; Sinclair, Liljedahl, & Zazkis, 2006 for notable exceptions). Those studies that addressed connections within number and numeration did so in limited ways. For instance, Amato (2005, 2006) documented the effects of an instructional sequence that used place value mats to connect whole and mixed number notations. Dubinsky and colleagues (2013) considered preservice teachers' understanding of the equivalence between $0.\overline{9}$ and 1. Fischbein and colleagues (1995) looked at preservice teachers' understanding of irrational numbers. LeSage (2011) considered preservice teachers' understanding of the connection between fractions and *terminating* decimals only. Sinclair and colleagues (2006) documented preservice teachers' ways of understanding repeating decimals when they used a web-based visual calculator, called the "Colour Calculator," to explore the patterns that resulted when various fractions were entered into the calculator as quotients. While all of this work is potentially useful in designing curricula that can support preservice teachers in developing a connected understanding of number and numeration, more information is needed about the nature of preservice teachers' understandings of the connected aspects of number and numeration if research-based instructional sequences that fully support this understanding are to be developed.

In his explanation of the stages of curriculum development, Clements (2007) explained that if initial cognitive models for a particular domain are not available *a priori*—as is the case with preservice teachers' understanding of number and numeration from a connected perspective—researchers should use grounded theory methods and/or clinical interviews "to examine students' knowledge of the content domain, including conceptions, strategies, intuitive ideas, and informal strategies used to solve problems" (p. 44). To do so, the researchers could "set up a situation or task to elicit pertinent concepts or processes" (p. 44). Thus, this study was designed as a descriptive case study

(Yin, 2014) in order to examine preservice teachers' conceptions, strategies, intuitive ideas, and informal strategies related to understanding the domain of number and numeration from a connected perspective. Interview tasks and tasks on a pre- and posttest were used to elicit their understandings of the connections between various aspects of number and numeration early in and after participating in an instructional sequence designed to support these understandings. In keeping with Clements' curriculum research framework (2007), this study therefore can support the initial phases of a curriculum for developing these types of connected understandings.

Data for this study came from a pre- and posttest completed on the first and last day of the 17-class unit on number and numeration within a mathematics content course for future elementary teachers, as well as from two sets of hour-long individual interviews with eight members of the class. The study was designed to document the understandings related to fractions and decimals, the relationship between them and their relationship to the set of rational numbers by this group of preservice teachers at two different points in their mathematical content coursework. The purpose of documenting their ways of thinking before and after the instructional unit was two-fold. First, knowledge of the initial understandings that preservice teachers bring to their mathematics education coursework supports the development of effective curricula that builds on and extends learners' current ways of knowing (Clements, 2007; Thanheiser et al., 2013). Second, noting ways that preservice teachers do, and do not, take up understandings developed during activities can help curriculum designers identify characteristics of potentially productive (or non-productive) tasks (Clements, 2008).

The connections between fractions and decimals and their relationship to the set of rational numbers were targeted for several reasons. First, these numbers and notations are central to understanding rational numbers and also central to the work of teaching elementary mathematics. Second, there is a need for research that specifically targets preservice teachers' understandings of these concepts because their familiarity with these topics can impact how they engage in learning activities designed to deepen their understandings (Markovits & Sowder, 1990; Sinclair et al., 2006; Sowder et al., 1998). Third, there a need for more research that studied preservice teachers' understandings of "foundational fraction concepts" (Thanheiser et al., 2013, p. 22). Fourth, the explicit inclusion of decimals in the study also addresses the fact that there is a documented need for studies that "result in understandings of how [preservice teachers] develop decimal concepts including understanding notations and representations of decimal quantities" (Kastberg & Morton, 2014, p. 330), as well as for studies that "generate instructional activities and methods that mathematics teacher educators can implement with [preservice teachers] to create opportunities to understand decimal concepts and representation" (p. 330). Finally, the limited time available during teacher education is an important consideration in curriculum design for preservice teachers. This study's focus on the development of preservice teachers' understandings of fractions and decimals from a connected perspective addresses the need for research that can support the development of activities that *productively* and *efficiently* develop PSTs' understandings in these areas (Kastberg & Morton, 2014).

Conceptual Models

Two conceptual models were used to guide the development of the activities and assessments related to this study. These models were used because supporting the development of understanding the connections between two concepts or representations is central to both.

The first conceptual model used for this study was designed based on the translation model used by Lesh, Behr, and Post (1987), shown in *Figure 1*.

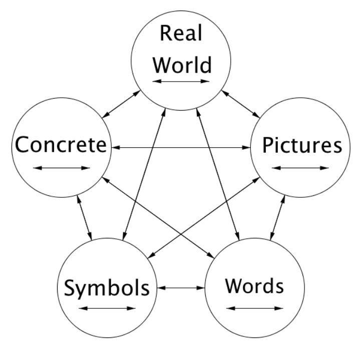


Figure 1. The Lesh Translation model depicts ways of developing and assessing mathematical understandings by making connections within and between five representational areas.

The Lesh Translation model depicts mathematical understanding as the ability to represent mathematical concepts in multiple representations and also make connections among those representations. Although the representations and translations depicted in the Lesh model are certainly relevant ways to represent concepts related to whole and rational numbers, a more specific translation model was created to explicitly show the representations and connections of interest to this study (*Figure 2*). In this study, this translation model is called the "Number and Numeration System."

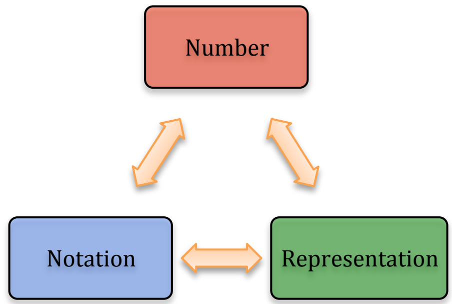


Figure 2. The Number and Numeration System depicts the ways knowledge of number, notation, and (non-symbolic) representations are interconnected.

The *Number and Numeration System* (*NNS*) conceptual model guided the overall design of the instructional sequence as well as the design of many of the activities and assessment tasks. Every lesson introduced during the unit was designed around the idea of making connections between or within representations depicted in the *NNS*. For example, whole number notation was introduced by asking students to learn about and make connections between four different historical number systems: the Ancient Egyptian, the traditional Chinese, the Mayan, and the Hindu-Arabic (modern). Fractional notation was later introduced with Egyptian fractions. Students were tasked with making connections between the ways the Ancient Egyptians wrote their whole numbers and their fractions, and then with connecting those to modern day ways of denoting whole numbers and fractions. A brief overview of the entire instructional unit is shown in Appendix A.

The second conceptual model that guided this study was the concept of *bridging tools*. Bridging tools are a concept proposed by Abrahamson and colleagues (Abrahamson & Wilensky, 2007; Abrahamson, 2004, 2006; Fuson & Abrahamson, 2005). The concept of a *bridging tool* makes the assumption that mathematical representations are *conceptual composites* (Abrahamson, 2006). This means that representations, by their nature, exist to coordinate two or more distinct ideas. Sometimes, ideas are connected in ways that are not easily apprehended by learners. In order to facilitate linking these ideas, *bridging tools* may be used. A bridging tool is intended be an intermediate and somewhat ambiguous representation that can be flexibly interpreted as clearly representing each of the separate ideas. Out of these different views of the same representation, a deeper understanding of the ways these different ideas are related to one another can then arise. This deeper understanding is the *target concept*, or the important mathematical idea that one hopes to support through the use of the bridging tool. For example, a 3 x 5 array may be viewed as three groups of five dots or five groups of three dots. Viewing the same picture in these two different ways supports the target concept of the commutative property of multiplication (*Figure 3*).

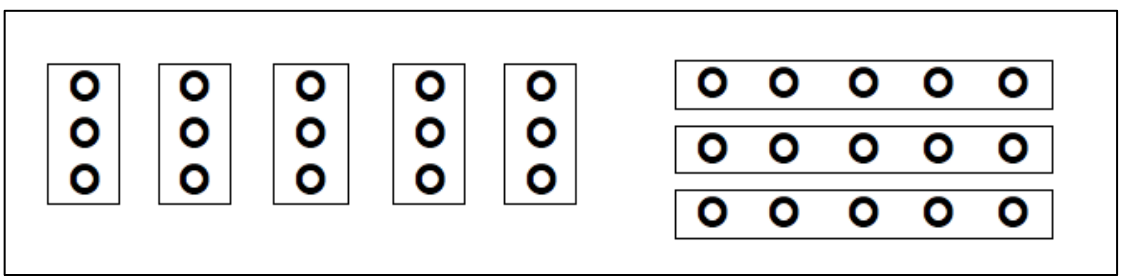


Figure 3. A 3x5 array as a bridging tool supporting understanding of the concept of the commutative property of multiplication, since the array can be viewed as 5 groups of 3 or 3 groups of 5. Adapted from Abrahamson and Wilensky (2007).

Abrahamson and colleagues posit that using bridging tools creates opportunities for learners to construct personal understandings of the target mathematical concept as they reconcile the differing interpretations of the concept highlighted by the bridging tool. They argue that focusing on creating bridging tools may be a more productive approach to the design of learning environments than focusing on directly creating understanding of mathematical concepts themselves. In particular, rather than simply giving learners formulas and definitions of a concept, creating bridging tools that require them to reconcile competing interpretations of a concept inherent in various representations may provide opportunities for developing a richer understanding of mathematical concepts than are possible when outright definitions and examples are used.

Two bridging tools were used in the instructional sequence related to this study. One was called the "Breaking Bread" tool because the activities all related to ways of sharing bread. This tool was used to support students in making connections between fractions and rational decimals in a way that also highlighted the central role that partitioning plays in both notations (and hence the reason fractions and terminating and repeating decimals are all rational numbers). The second bridging tool was a number line. The number line was used to highlight the equivalence of different notations, as well as the role that partitioning plays in locating all rational numbers on a number line.

Research Focus

There is a need for research-informed instructional sequences that may be used with preservice elementary teachers to deepen their understanding of the *interconnected* relationship between whole and rational number and notations so as to better prepare them for the work of teaching elementary mathematics. To support the development of such sequences, knowledge is needed of how preservice elementary teachers understand these connections and how such understanding may be developed during teacher education coursework. The purpose of this study is to document the nature of preservice teacher understanding of rational numbers and how decimals and fractions are *interconnected* before and after their participation in classroom experiences designed to support the development of these understandings from a connected perspective. In particular, the study will aim to reveal the representations and activities that support the development of an interconnected understanding of these numbers and notations in order to inform the development of tasks for use in mathematics content courses for preservice elementary teachers.

Two research questions and their related sub-questions guided this study:

1. What is the nature of preservice elementary teachers' (PSTs) understanding of the *Number and Numeration System* before and after participation in a unit designed to develop their understandings of this content?
    a. What is the nature of PSTs' understandings of the sets of numbers generally, and *rational numbers* specifically, early in the unit?
    b. What is the nature of PSTs' understandings of fractions, decimals, and the connections between them early in the unit?
    c. What is the nature of PSTs' understandings of fractions, decimals, and the connections between them after the unit?

2. What is the role of representations in the development of PSTs' understanding of the *Number and Numeration System*?
    a. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals before/early in the unit?
    b. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals after the unit?

Overview of the Chapters

This study is organized into five chapters. Chapter 1 presented the introduction to the study. Chapter 2 reviews the significant literature related to developing preservice teachers' mathematical knowledge for teaching, particularly in the area of rational numbers, and with a connected perspective. Chapter 3 describes the research methodology used the study, as well as the key components of the instructional sequence related to this study. The results are presented in Chapter 4. Chapter 5 concludes the study with an analysis of the findings, limitations, implications and recommendations for future research.

Chapter 2

Review of the Literature

"*We understand something when we see how it is related or connected to other things we know"* (Hiebert et al., 1997, p. 4).

The *Principles and Standards of School Mathematics* (National Council of Teachers of Mathematics, 2000b) emphasizes that one of the key goals of mathematics instruction is to ensure that all students understand numbers, including the ways of representing numbers, the relationships among them, and the real number system. For elementary students, the numbers they must understand include whole and rational quantities, although negative numbers and particular irrational quantities (such as π) are sometimes briefly introduced in the upper elementary grades. Children are typically taught two ways of representing these quantities, namely using base ten positional notation[1] first for whole and later for rational numbers, and fractional notation for rational numbers. In order to support students' learning about these different numbers and notational systems, teachers are expected to introduce and teach students to use a variety of physical and mental models, including concrete materials, pictures, and diagrams such as a number line (National Council of Teachers of Mathematics, 2000b; National Governors Association, 2010).

Ball and Bass (2003) argue that an important part of the mathematical work of teaching is "making connections across mathematical domains, helping students build links and coherence in their knowledge" (p. 12). For elementary teachers, this includes

[1] "Base ten positional" refers to both standard Hindu-Arabic numerals used for whole numbers, as well as the extension of this system to include the decimals.

helping students make connections between whole and rational numbers and between the different notational systems used to represent those quantities. Clearly, in order to effectively support students as they learn about whole and rational numbers, elementary teachers themselves must have a clear understanding of the nature of whole and rational *quantities*, the common *notational* systems used to symbolically denote these quantities, and common *non-symbolic representations* for each, as well as an understanding of the complex interrelationship between these concepts. However, this can be a difficult task, due in part to complexities inherent in the relationship between whole and rational numbers, in part to complexities in the notational systems, and in part to challenges that come with using a variety of representations for these ideas.

The purpose of this study was to document preservice teachers' understandings of fractions and decimals, the relationship between them and their relationship to the set of rational numbers in order to support the development of curricula appropriate for use in teacher education coursework. As will be further examined in the coming sections, preservice elementary teachers often have difficulties with many aspects number and numeration important for the work of teaching, including understanding the structural aspects of rational numbers (Dubinsky et al., 2013; Post et al., 1991; Tobias, 2012; Utley & Reeder, 2012), understanding fraction notation (Curtice, 2010; Siegler & Lortie-Forgues, 2015; Siegler, Thompson, & Schneider, 2011; Young & Zientek, 2011), understanding place value (McClain, 2003; Thanheiser, 2009), and using representations effectively to represent and reason about both whole and rational numbers (Mitchell, Charalambous, & Hill, 2013; Novick, 2004; Roche & Clarke, 2013).

In this chapter, an overview of the literature related to this study will be presented. First, the theoretical frameworks that guided this study will be introduced. Second, an overview of the knowledge of number and numeration that is necessary for the work of teaching will be presented. Third, the literature related to the PSTs' ways of understanding number and notation will be summarized. Finally, a summary of the literature related to developing these understandings in preservice teachers' coursework will be presented.

Theoretical Framework

Several theoretical perspectives were used together to guide this study. First, the emergent perspective (Cobb, Yackel, & Wood, 1992; Cobb & Yackel, 1996) was used to as an overarching principle defining what learning is and how it occurs. Second, the concept of *Mathematical Knowledge for Teaching* (Ball, Thames, & Phelps, 2008) was used to guide the nature of learning that is necessary for the work of teaching. Third, the study was guided by the theoretical perspective of understanding as "knowledge that is rich in relationships" (Hiebert & Lefevre, 1986). Fourth, Sfard's (1991) *object and processes* orientation for number is used to inform components of the study related specifically to how learning about number occurs. Finally, two models were used to guide the design of the tasks and assessments related to connection: a translation model (Cramer, 2003) and bridging tools (Abrahamson, 2006). Each of these will be briefly described in the sections that follow.

Emergent Perspective

To define what learning is and how it occurs, this study draws the emergent perspective which is a combination of a social interactionist and a constructivist theoretical framework (Cobb, Yackel, & Wood, 1992; Cobb & Yackel, 1996). Within this perspective, individual learning is considered a constructive process that occurs in concert with the individual's participation in and contributions to his/her learning community. Thus, the emergent perspective takes into consideration both individual learning and the environment in which it occurs.

An important feature of the emergent perspective is that while both the individual (psychological) and social (interactionist) perspectives are incorporated in this perspective, neither takes primacy over the other. Rather, depending on the nature of the study and research questions, "analyses conducted in line with this approach can give greater prominence to either the psychological or the interactionist perspective, depending on the issues and purposes at hand. In each case, one perspective comes to the fore against the background of the other" (Cobb & Yackel, 1996, p. 185). The emergent perspective therefore is a useful guide for research on student learning as it occurs over time in classrooms as it supports the study of both individual's activities and activities of the whole class as well as the consideration of how the individual contributed to group understanding and vice versa (Richardson, Berenson, & Staley, 2009). For this study, learning was viewed as process of knowledge construction for an individual that is reflexively influenced by and influencing the learning processes of the others in the course. Course activities were designed to take advantage of learning opportunities in both small and large group settings, and it is acknowledged that this group work certainly influenced individual student learning in ways that may be unapparent to the researcher (or even the individuals themselves).

Mathematical Knowledge for Teaching

Over two decades ago, Lee Shulman (1987) defined *pedagogical content knowledge* as knowledge that "represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8). The idea that teachers of mathematics need to blend their understanding of mathematical concepts with how those concepts come to be understood by students as well as with ways of teaching that will facilitate that student understanding has been a key component of numerous studies that have explored the nature of mathematical knowledge needed for teaching (Ball & Bass, 2003; Ball, Thames, & Phelps, 2008; Ball, 2002; Ball, Lubienski, & Mewborn, 2001; Shulman, 1987; Thanheiser et al., 2013).

Two common threads wind their way throughout much of the work on the nature of mathematical knowledge necessary for teaching, namely the idea that teachers need *deep* understanding of the mathematics they will teach, and that they need to understand how this knowledge is *connected* to fundamental mathematical concepts. The National Council of Teachers of Mathematics (NCTM) makes this clear the *Principles and Standards for School Mathematics* (2000). They state,

> Teachers need several different kinds of mathematical knowledge -- knowledge about the whole domain; deep, flexible knowledge about curriculum goals and about the important ideas that are central to their grade level; knowledge about the challenges students are likely to encounter in learning these ideas; knowledge

> about how the ideas can be represented to teach them effectively; and knowledge about how students' understanding can be assessed....Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise. (p. 17)

Similarly, Silverman and Thompson (2008) argue that teachers need "coherent and generative understandings of the big mathematical ideas that make up the curriculum" (p. 5) in order to support student learning that appreciates the way that mathematical concepts build on one another. Essentially, teacher need know about how the big ideas in mathematics carry through instructional sequences, how they build on earlier learning and toward new ideas, and how current understandings will impact students' reasoning when encountering new ideas (Ball et al., 2008; Ma, 2010; Silverman & Thompson, 2008).

However, mathematical knowledge for teaching goes beyond having a *deep* and *connected* understanding of the mathematics being taught, as it also involves knowing how to help students develop such an understanding. As Ball et al. (2001) argue, "It is not only what mathematics teachers know but also how they know it and what they are able to mobilize mathematically in the course of teaching" (p. 451). One of the challenges in developing mathematical knowledge for teaching is that *what mathematics teachers need to know* and *how they need to know it* can be at odds with one another. Typically, the goal of mathematical understanding is for it to become increasingly condensed and abstract. Teachers, on the other hand, need to move in the opposite direction and develop understanding of as many models, representations, and forms of a concept as possible so they can both offer and understand the multiple ways students might make connections between concrete representations and abstract concepts (Kajander, 2010), since when teachers do not have conceptual understanding, it can prove difficult or impossible for them to come up with good representations for students (Borko et al., 1992; Ma, 1999). Thus, helping teachers develop the mathematical knowledge necessary for teaching requires developing both the teachers' conceptual understanding of the mathematics they will teach as well as the ability to "unpack" that knowledge.

In this study, this need to both deepen PSTs' understanding of mathematical concepts and help them unpack that understanding was used to guide the design of course activities and assessments, and also influenced the evaluation and interpretation of the results. Parts of class activities and the assessments were designed to be opportunities for PSTs to learn about and unpack the big mathematical ideas that underpin and connect fractions, decimals, and whole numbers. In analyzing student work and interviews, attention was paid to the ways in which students were developing and attending to those big ideas.

Understanding

What is meant by "understanding" can vary greatly in mathematics education literature (Sierpinska, 1994). One commonly used definition of understanding is based upon Hiebert and Lefevre's (1986) definition of conceptual knowledge, or "knowledge that is rich in relationships. … a network in which the linking relationships are as prominent as the discrete pieces of information" (pp. 3-4). The connected nature of understanding has been stressed by others as well (Ma, 1999; Richardson et al., 2009; Simon, 2006). Another common usage for understanding is simply *knowing what to do and why*, which is in keeping with what Skemp (1976, 2006) called relational understanding (Simon, 2006). Richardson, Berenson, and Staley (2009) give what may be seen as a hybrid of these two common usages when they state that "Understanding and knowing mathematics are interchangeable and when students understand mathematics, they are able to see how things work, how things are related to each other, and why they work the way they do" (p. 188). This final definition succinctly captures the elements of what is meant by *deep understanding* in this study. The focus of this study is on how preservice elementary teachers develop a deep understanding of rational numbers necessary for the work of teaching, where "deep understanding" will be taken to mean knowledge of both how and why the fractions, decimals, and the set of rational numbers work, and also how they are connected to one another.

The Development of Number Understanding

This study focused specifically on learning about numbers. Sfard's (1991) notions how the concept of number develops was used to guide both the design of activities used during the study and the analysis of student work. Sfard (1991) states that there are two, complementary ways of thinking about mathematical concepts, namely as *objects* and as *processes*. Sfard argues that the development of the concept of number has been a continually, cyclic process of transitions from processes to abstract structure. The cycle begins with processes performed on accepted forms of number, which in turn result in new types of numbers. These new types of "numbers" are first regarded not as objects, or numbers, in their own right, but rather as the results of the processes. Over time, however, they become *reified*, and are seen as mathematical objects that may themselves be subject to processes.

For instance, the notion of natural numbers arises from the process of counting. At first, the quantity "five" is seen as the fifth step of a counting process: "one, two, three, four, five." This is the ordinal conception of numbers. In time, "five" became an entity in its own right, a *cardinal* number, which could be operated upon. The process of division leads to rational numbers. For instance, 3 divided by 4, is first a division process but in time became conceived of as a single quantity, a number, three-fourths (¾). Similarly, the process of subtraction led to the negative numbers: subtracting 5 from 3 is first viewed as a process that leads to the result of -2, or "owing 2." Over time, these "owed quantities" became accepted as numbers. This process is shown in *Figure 4*.

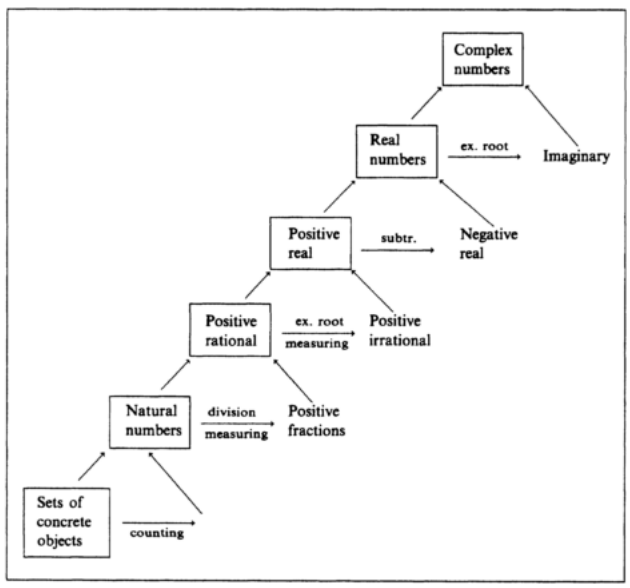


Fig. 3. Development of the concept of number.

Figure 4. The development of the concept of number (Image from Sfard, 1991, p. 13).

In this study, the notion that numbers move cyclically from a process orientation to an object orientation was used to guide the design of assessments and activities in the course as well as in the analysis and interpretation of the results. Portions of the assessments and interviews were designed to elicit whether students had a process or object orientation towards different number types and when written using different notations. Activities used during the course supported both process and object orientations for numbers written in both fraction and decimal notation.

Models for Connection

A translation model. Many mathematics education researchers have used multiple representations to develop and assess mathematical understanding (Bruner, 1971; Cramer, 2003; Dienes, 1960; Goldin & Steingold, 2001; Kinach, 2002b; Moore, Miller, Lesh, Stohlmann, & Kim, 2013; Post et al., 1991; Suh, Johnston, Jamieson, & Mills, 2008). Different representations of a mathematical concept typically emphasize different aspects of that concept, with the deeper meanings undergirding the concept "distributed across a variety of representations" (Moore et al., 2013, p. 146).

One model of how representations may be used in mathematics education to build student understanding is the Lesh Translation Model (LTM; *Figure 5*).

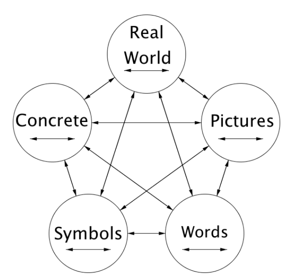


Figure 5. The Lesh Translation Model, adapted from Lesh, Post, and Behr (1987).

The key feature of the LTM is that it highlights that understanding is made up of both the ability to represent concepts in multiple ways as well as the ability to make connections (translations) between those representations (Cramer, 2003). Five different modes of representation are emphasized in the LTM, namely symbols, words, real-life contexts, pictures, and concrete manipulatives. The dual-direction arrows between the different modes indicate that translations between representations may and should occur in different directions and that these translations need not be synonymous (e.g. the ability to translate from a picture to symbols does not imply the ability to translate from symbols to a picture).

The LTM may be used to guide curricular development that supports the development of conceptual understanding of mathematical concepts by ensuring that students have opportunities to both represent concepts in multiple ways and to make connections between those representations. It also may be used to guide assessment and to measure understanding (Cramer, 2003; Lesh & Doerr, 2003). Although the representations and translations depicted in the Lesh model are certainly relevant ways to represent concepts related to whole and rational numbers, a more specific translation model was created to explicitly show the representations and connections of interest to this study (*Figure 2*). The version of the translation model used in this study is called the "Number and Numeration System." The *Number and Numeration System* (*NNS*) conceptual model guided the overall design of the instructional sequence as well as the design of many of the activities and assessment tasks. Every lesson introduced during the unit was designed around the idea of making connections between or within representations depicted in the *NNS*.

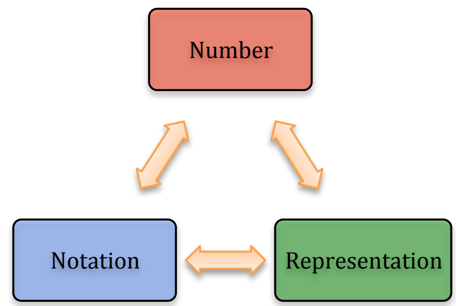


Figure 6. The *Number and Numeration System* depicts the ways knowledge of number, notation, and (non-symbolic) representations are interconnected.

Bridging tools. This study took place during an instructional sequence that was driven by the idea that activities that allow learners to meaningfully make connections between the representation and both fraction and base ten positional notation would support students in deepening their understandings of these notations as well as their understandings of the connections between these representations. One way to make connections between representations is to directly translate from one representation to the other. Sometimes, such direct translation is difficult, however, due to fact that the different aspects of a target concept that each representation makes transparent are not easily mapped to one another. In such situations, connections may be made between these representations by connecting both to a third representation that may meaningfully be connected to both representations and to the target concept. Abrahamson and colleagues (Abrahamson & Wilensky, 2007; Abrahamson, 2004, 2006; Fuson & Abrahamson, 2005) call such a representation a *bridging tool* and posit that using bridging tools creates opportunities for learners to construct personal understandings of the target mathematical concept as they reconcile the differing interpretations of the concept highlighted by the bridging tool. They argue that focusing on creating bridging tools may be a more productive approach to the design of learning environments than focusing on directly creating understanding of mathematical concepts themselves. In particular, rather than simply giving learners formulas and definitions of a concept, creating bridging tools that require them to reconcile competing interpretations of a concept inherent in various representations may provide opportunities for developing a richer understanding of mathematical concepts than are possible when outright definitions and examples are used. Two bridging tools were used in the unit related to this study, namely the "*Breaking Bread"* tool and the number line. Both are described in detail in Chapter 3.

Overview of Number and Notation for Elementary Teachers

In the previous section, the theoretical perspectives that were used to guide the design of the study and the instructional sequence were described. In this section, a brief overview of the subsets of the real numbers and how they relate to notation will be given. This will include how they relate to the work of teaching elementary mathematics. After this overview, the next section will review the literature related to preservice teachers' understandings of number and notation.

The Real Number System

The *real numbers* are all numbers that express measurable amounts and are often defined in terms of a number line that starts at zero and has unit intervals extending infinitely in both the left and right directions. The real numbers that are to the left of zero are known as the negative real numbers and those to the right are known as the positives. Every point on this continuous line is a real number and represents a directed, measurable distance from zero. A subset of the real numbers is the *rational numbers*. On the number line, the rationals are all those points that may be reached through a process of partitioning the unit into equal-length intervals and iterating some number of those intervals. The remaining points on the number line which cannot be reached through this process of partitioning and iterating are known as the *irrational* numbers. Every rational number is therefore also a real number, and every real number is either rational or irrational. A subset of the rational numbers is the *integers*. The integers are those rational numbers that are unit distances from zero, and thus may be positive or negative. Finally, a subset of the integers is the *whole numbers*. The whole numbers are those integers that are positive, or located at unit distances to the right of zero. Together, these nested subsets are called the *real number system (Figure 7*).

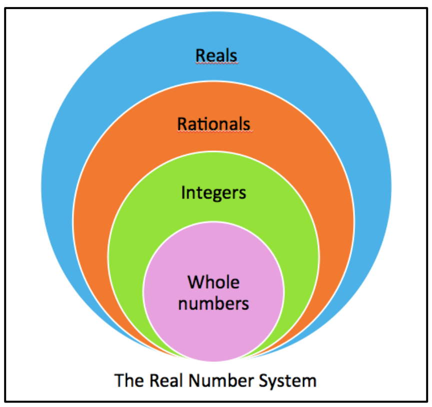


Figure 7. The Real Number System.

Knowledge of Number is Important for the Work of Teaching

In U.S. education, students are typically introduced to whole numbers in grades K-3, fractions and decimals in grades 4 and 5, negative numbers (integers) in grades 6 or 7, and irrational numbers in middle school (NCTM, 2000). This gradual introduction to more general forms of numbers mirrors the historical development of number and allows students to use counting as a way to build an understanding of number before considering more general forms. One of the challenges that comes with learning about numbers in this order, however, is that learning about each new, more inclusive set of numbers requires learners to radically reorganize their understanding of what is meant by number (Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004). In particular, when extending the notion of number from whole numbers (or integers) to rational numbers, learners must reconsider the notion that numbers should be identified with counting, despite the fact that counting is what gave rise to the conception of number in the first place (Stafylidou & Vosniadou, 2004). Extending from the rationals to the real requires learners to reconsider the idea that grouping and partitioning with respect to a given unit are sufficient processes that may be used to describe any measurable quantity (Yopp, Burroughs, & Lindaman, 2011). Students often generalize properties of sets of numbers by observing the results of operating with quantities. For instance, based on their work with whole numbers, many students, including preservice teachers, believe that addition and multiplication "make bigger" while subtraction and division "make smaller" (Greer, 1992; Siegler & Lortie-Forgues, 2015; Sowder et al., 1998), all generalizations that are not true for numbers in general. At the same time, learners must understand that not all aspects of number change when extending to new sets, including the fact that the algebraic properties such as the commutative property of addition are shared by whole and rational numbers.

For elementary teachers, understanding that whole numbers, rational numbers, and real numbers are a *nested system*, rather than separate number domains, is necessary if teachers are to support students in building a coherent knowledge of this system. After all, "[i]f one intends to convey to the students the feeling of the structurality of mathematics, one has to emphasize, first of all, the coherent picture of the number system with its strict hierarchy" (Fischbein et al., 1995, p. 30). They also must understand that the fact that we build up this system over time with students means that students' conceptions of numbers will need to be changed and adapted in order to incorporate new, more general forms of numbers, and that part of the work of teaching will be supporting students as they adapt their understandings of number. Clearly, teachers will only be able to do this if they themselves understand the real number system in this way. Without such knowledge, teachers will struggle to "make connections across time, as mathematical ideas develop and extend" (Ball & Bass, 2003, p. 11).

Knowledge of Notation is Important for the Work of Teaching

There are two essential components of any numeration system, the quantities that are being represented and the written symbols and notational system being used to represent those quantities (Hiebert, 1992). In addition to the fact that students are introduced to increasingly general forms of *number* during the elementary years, these new number forms also come with new forms of *notation.* The Hindu-Arabic, standard fraction, and decimal notations are the three canonical ways that real numbers are expressed in our society. Each notation is particularly well-suited to notate different subsets of the real numbers: the Hindu-Arabic denote the whole numbers and integers, the fractions to denote the rational numbers, and the decimals denote both rational and irrational numbers. As the whole numbers, integers, and rational numbers are nested subsets of the real numbers, this means that there are non-symmetric mappings between notations and different subsets of the real numbers. The relationships between these sets of numbers and notation are shown in *Figure 8*.

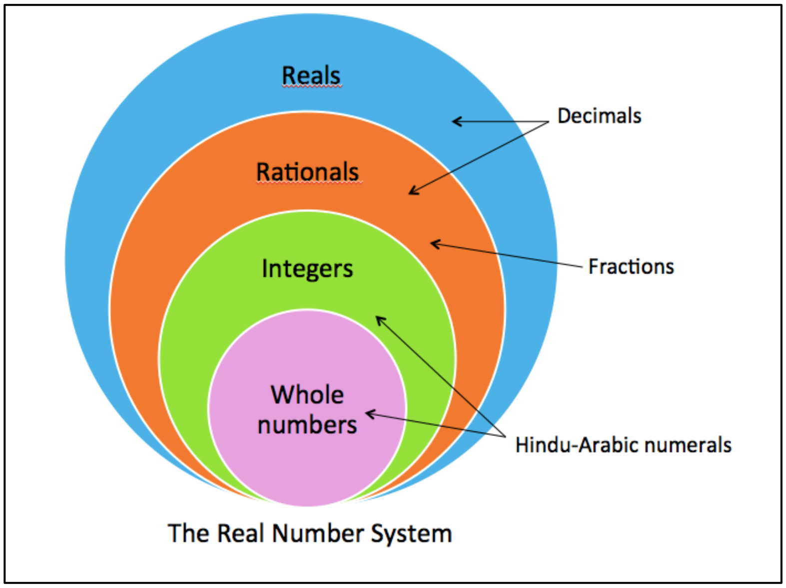


Figure 8. The relationship between the subsets of the real numbers and the three canonical notations.

The use of these different notational systems for nested subsets means that we sometimes use “different symbols to represent the same idea and similar-looking symbols to represent different ideas" (Markovits & Sowder, 1990, p. 5). For instance, $^{12}/_{2}$ and 6 both represent the same quantity, despite the fact that they look very different. In contrast, the terminating decimal 3.1415 represents a quantity that falls in the domain of *rational* numbers, while the similar-looking non-terminating, non-repeating decimal 3.1415… approximates the *irrational* number pi. Furthermore, the fact that decimals may be used to denote all real numbers, while rational numbers cannot means that there is a non-symmetric relationship between fraction and decimal notation, wherein all fractions have a decimal representation but not all decimals have a fractional notation. This makes supporting students in developing an understanding of notation and its relationship to the various sets of numbers challenging, but important, work for elementary teaching (O'Connor, 2001).

A second challenge related to teaching about notation is that, by design, notational systems for numbers convey information in a very condensed, efficient form. However, this efficiency comes with a cost as notational systems can conceal as much about number as they reveal (Hiebert, 1988; Zazkis & Gadowsky, 2001). Moreover, much of the power of mathematics comes from the fact that symbols can be used and manipulated without regard to the referents that were used to generate and give meaning to the symbols in the first place (Hiebert, 1988). Thus, when teaching about how notation works, care must be taken to ensure that learners are able to make meaningful connections between aspects of the numbers being represented and the features of the notational system, a process which "involves building bridges between symbols and referents and crossing over them mentally many times " (Hiebert, 1988, p. 336).

Review of Literature Related to PSTs' Knowledge of Number and Numeration

In the previous sections, the theoretical perspectives that were used to guide the design of the study and the instructional sequence were described. Then, a brief overview of the subsets of the real numbers and how they relate to notation was given. This section will present a review of the literature related to preservice teachers' understandings of number and notation from a connected perspective. Recall that, for this study, the *connected perspective* meant that supporting learners' in understanding the connections between mathematical concepts was considered in the design of all aspects of the unit, including the activities, the course sequence, and the assessments. This literature review will therefore give an overview of studies that documented preservice and in-service teachers' understandings of the connections between fractions and decimals and their relationship to the sets of rational and irrational numbers. Note that interconnected nature of these understandings means that there is a great deal of overlap in the categories presented and several studies could have fit into more than one section. Following this section, a review of the literature related to interventions used in preservice teachers' coursework will be presented.

PSTs' Understandings of Number and Numeration

Many research studies have found that preservice elementary teachers' understanding of whole and rational numbers to be thin, fragmented, and largely procedural- and rule-based (Ball, 1990; Cramer & Lesh, 1988; Fischbein et al., 1995; Ma, 1999; Newton, 2008; Siegler & Lortie-Forgues, 2015; Simon, 1993; Timmerman, 2010; Tobias, 2012). As knowledge of number comes from learning about what happens as actions are performed on that number (Sfard, 1991), it is perhaps not surprising that much of the literature that relates to PSTs' understanding of number examines that understanding as it is operationalized through computation. One of the major themes found by Thanheiser and colleagues (2013) in their meta-analysis of studies of preservice teachers' mathematical knowledge for teaching was that "PSTs tended to use procedures, algorithms, and memorized rules to address problem situations. For areas of fractions, whole numbers, decimals, algebra, and measurement, PSTs struggled when asked to explain why the algorithms work" (p. 13). Similarly, a recent, large-scale international study of preservice teacher knowledge found that, in the United States, PSTs' understandings of algorithms is very thin (Tatto & Senk, 2011). About half of the 950 U.S. preservice elementary teachers included in the study could "apply whole number arithmetic in simple problem-solving situations, [but] they tended to overgeneralize and had difficulty solving abstract problems and those requiring multiple steps" (Tatto & Senk, 2011, p. 128).

Numerous studies have found that many preservice elementary teachers struggle to make connections between procedural understandings and other representations (see Newton, 2008; Thanheiser et al., 2013 for an overview). For instance, Simon (1993) found that the majority preservice elementary teachers in his study struggled with creating story problems for division problems, even when they could solve the problems. In particular, he asked the teachers to create three different story problems for a division problem involving remainder that would yield different answers depending upon the context (i.e. 51 ÷ 4 could be 12, 13, or 12¾). PSTs generally did not connect the context to the division problem in meaningful ways and seemed to believe that directions about rounding were most important for deciding which of the three answers was correct. Sowder et al. (1998) observed that, for many of the PSTs in their teacher education courses, their "mental images of fractions may be limited to fraction symbols and their language limited to procedural terms for algorithms" (Sowder et al., 1998, p. 145). Amato (2005, 2006) found that the PSTs in her study were able to draw part-whole diagrams of fractions but were unable to use those diagrams as tools for reasoning about fraction addition. For instance, to represent the addition sentence ½ + ¼, several PSTs drew three different part-whole diagrams, one for each addend and one for the result. Importantly, these PSTs did not relate the diagrams in any visible way to one another or to the addition sentence but rather appeared to have solved the diagram using a written algorithm and then simply represented each fraction pictorially.

Finally, in a recent study, Siegler and Lortie-Forgues (2015) found that whole number knowledge influences PSTs' thinking about rational numbers, even in cases where no computational errors are made. They found that preservice elementary teachers frequently made errors when asked to predict the magnitude of the answer of a multiplication and division problem involving fractions despite being able to accurately solve fraction multiplication and division problems when given the opportunity to perform the computation. In particular, they made predications that were in keeping with the notions that "multiplication makes bigger and division makes smaller." These PSTs were also able to accurately place fractions and decimals along a number line, indicating that they understood the magnitudes of the individual fractions involved in the calculations. These results indicate that the PSTs' understandings of multiplication, division, whole and rational numbers were not well-connected.

PSTs' Understanding of the Sets of Rational and Irrational Numbers

Rational numbers are commonly defined in two different ways: as the set of all numbers that may be written in the form ${}^{a}/_{b}$ ($b \neq 0$) and as the set of terminating and repeating decimals. Irrationals are commonly defined either as "not rational," or as the set of non-terminating, non-repeating decimals. Making sense of the definition of rational numbers thus requires an understanding of the relationship between fractions and repeating decimals and of the relationship between terminating and repeating decimals (i.e. seeing terminating decimals as a particular type of repeating decimal, namely one that ends in a series of zeroes). Making sense of the definition of irrational numbers requires understanding what it means to be not rational and an understanding of the difference between repeating and non-repeating decimals. The studies that address preservice or inservice teachers' understandings of these definitions, though limited in number, suggest that teachers do not understand these definitions or relationships well.

After investigating high school students and preservice teachers' understandings of rational and irrational numbers, Fischbein, Jehiam, and Cohen (1995) concluded that both groups "have a totally confused idea about the concepts of rational and irrational numbers and the relationships between them" (1995, p. 38). They found that many of the PSTs in their study did not have an understanding of the relationship between the sets of rational and irrational numbers and that these two sets of numbers together make up the number line. For instance, 44% of the PSTs believed that every point on the real number line corresponded to a rational number (p. 37), while 20% disagreed that every irrational number had a corresponding point on a number line. Their investigation revealed that many of the preservice teachers in their study did not fully understand what rational numbers are and how they are related to the real numbers. For instance, 24% of the PSTs were unable to identify -22/7 as a real number but did identify it as a rational number (p. 32), indicating a lack of understanding of the fact that the rationals are a subset of the reals. Similarly, while nearly all PSTs were able to identify 0.055….as a rational number (97%), only 83% also identified it as a real number. They had less success with the rational number 34.2727…, however, with only 59% identifying it as rational. About 30% of the PSTs were unable to identify $\sqrt{16}$ as a rational number, but more interestingly, only 3% identified it as an irrational number instead. This implies that many of the PSTs identified it as neither rational nor irrational.

Research by Fischer (2014) suggests that some of this confusion may be due to the fact that both fractions and decimals are used in different ways to define and describe rational numbers. He found that middle school mathematics teachers gave *definitions* of rational numbers that contrasted sharply with their *descriptions* of rational numbers. For instance, some defined rational numbers as "not irrational" which they in turn associated with infinite decimals, and then described rational numbers in terms of fraction notation (i.e. numbers written in the form $^{a}/_{b}$). He found that the "majority of participants viewed rational numbers as a collection of different 'types' of numbers that were mutually exclusive" (p. 119). Another source of confusion is the fact that fractions and terminating decimals are commonly used in place of both irrational numbers and non-terminating rational numbers in computation, such as using 3.14 or $^{22}/_{7}$ to represent the irrational number π or 0.67 to represent the rational number ⅔ (Arcavi, Bruckheimer, & Ben-zvi, 1987).

Finally, Widjaja, Stacey, and Steinle (2008) found the ways in which PSTs in their study understood decimals were related to their understanding of the density of rational numbers. They determined that there were four different types of incorrect strategies used by PSTs when asked to identify the number of decimals between two given (non-equivalent) decimals, such as 3.14 and 3.15. First, some PSTs thought there were no numbers between the two because "14 and 15 are consecutive numbers" (p. 125). Others thought that there were a finite number between the two, such as 3.141, 3.142, 3.143, … , 3.149. For problems such as finding decimals between 0.799 and 0.80, some PSTs used a "rounding strategy" (p. 125) to say that 0.799 rounds to 0.80 so they are the same. Finally, some PSTs subtracted the two decimals and interpreted the result as naming the number of decimals between the two rather than the distance between them. They also found that PSTs tended to want to work with decimals of the same digit-length, a strategy that inhibited their ability to perceive that there are infinitely many decimals between any two given decimals, which in turn impacted their understanding of the density principle of rational numbers. They note that mathematics textbooks and teachers often suggest working with decimals that are the same length in order to make problems easier to understand, largely because working with decimals that are the same length typically allows students to work with the decimals as if they were whole numbers.

PSTs' Understandings Related to Notation and the Sets of Numbers

The relationship between decimals and fractions. Several studies have documented preservice teachers' misunderstandings of the relationship between fractions and decimals. For instance, preservice teachers in a study by Sinclair, Liljedahl, and Zazkis (2006) participated in an investigation of the connections between fractions and decimals in which they input a fraction as a quotient into an online "Colour Calculator." The calculator displayed the decimal to 100 places and also as a colored grid where each digit was assigned a particular color. Many preservice teachers expressed surprise at "seeing" a fraction and decimal representation as representing the same number as they participated in the investigation. The authors argue that this is likely because "fractions and decimals *look* very different and it therefore requires some cognitive work to see them as both representing the same number. Prior to using the Colour Calculator, the participants were perhaps more influenced by the *form* of these different representations than by the quantities they denote" (p. 189, emphasis in original). They go on to argue that believing that "¼ can be represented 0.25" is different than believing that ¼ and 0.25 are equal (p. 189). Studies that ask PSTs to work with sets of numbers that include both fractions and decimals also suggest that some PSTs see them as distinct (Fischer, 2014; Markovits & Sowder, 1990; Tsao, 2005). Both Markovits and Sowder (1990) and Tsao (2005) found that some preservice teachers separated fractions from decimals when ordering, indicating that they think of these as separate entities, not necessarily related. Tsao (2005) found that this disconnection between fractions and decimals was most pronounced with elementary PSTs he had classified as "low ability."

Zazkis and Sirotic (2010) found that prospective secondary teachers had difficulties with how different infinite decimal representations relate to rational and irrational numbers and to fraction representations. They also showed a preference for working with decimal representations in order to determine irrationality. For instance, when asked if the fraction 53/83 is rational or irrational, many of the PSTs either converted it to a decimal using a calculator or discussed doing so. As the period of the decimal equivalent to 53/83 is long, several then concluded that the decimal did not repeat and so 53/83 is irrational. Others concluded it was rational because the decimal terminated (at the end of the calculator display). Still others argued that it could be rational or irrational: “I cannot tell whether the digits will repeat because too few digits are shown. They might repeat and they might not” (p. 211). PSTs also overgeneralized the idea of a “repeating pattern” in decimal representations to mean any pattern, and thus incorrectly categorized decimals such as 0.121221222… as rational. Notably, this study was with secondary preservice teachers who typically have more mathematics background than their elementary counterparts. The authors conclude that the connection between the definition of rational numbers as being any number that can be written in the form $^a/_b$ ($b \neq 0$) and as terminating or repeating decimals is a “missing link” in students’ understanding. That is, students do not have a good understanding of the equivalence of these two definitions and thus do not understand why every division of two whole numbers yields a repeating (or terminating) decimal and vice versa. They argue that understanding this link is important for teachers and students alike:

> Explicit attention to the link between two representations, where both exist, would reinforce students’ understanding of rational numbers and serve well in preparing them for the encounter with the counterpart – the irrationals. Simply put, we suggest that directing students’ explicit attention to representations and to mathematical connections that render the two representations equivalent is helpful in acquiring a more profound understanding of number. (p. 26)

PSTs’ attention to surface features of notation. One reason why PSTs may have such difficulties making sense of the relationship between notations and the sets of numbers being denoted is that many PSTs seem to have a superficial understanding of the notations themselves, particularly with regards to decimal notation (Chick, Baker, Pham, & Cheng, 2006; Hiebert, 1988; Kastberg & Morton, 2014; Khoury & Zazkis, 1994; Sinclair et al., 2006; Thanheiser et al., 2013; Zazkis & Gadowsky, 2001; Zazkis & Sirotic, 2010). In particular, several studies noted that many preservice and inservice teachers seem to regard the names of the positions in base ten positional notation more as labels than as meaningfully denoting quantities. For instance, Chick, Baker, Pham, and Cheng (2006) found that inservice teachers in their study, “spoke accurately about place value, yet struggled in other areas, such as number line representations. Most identified place value as a key underlying principle for decimal understanding, but very few articulated the fundamental factor-of-10 iteration that underpins place value” (p. 2-303). LeSage (2011) noted that PSTs’ superficial understandings of place value sometimes led to the belief that the place immediately to the right of the decimal point should be called the “oneths” position.

Khoury and Zazkis (1994) used alternate bases as a way to explore PSTs’ understanding of place value structure as it relates to non-integer numbers. They used the alternate bases as a way to uncover PSTs’ conceptions of place value structure that may have been masked by their procedural knowledge of base 10 language and symbols if they had investigated their understanding using standard decimals. When they asked PSTs to make sense of decimals written in alternate bases (e.g. 0.243_{FIVE}[2]), they found that PSTs did not consistently maintain the 5-to-1 multiplicative relationship between place values in base five, suggesting a fragile understanding of the 10-to-1 relationship in base 10. For instance, some students identified the three positions immediately to the right of the decimal point in base five as “1/5, 1/50, and 1/500” (p. 195), a pattern that suggests they were attending to the “10” in “1/10, 1/100, 1/1000” for same positions in base ten rather than to the notion of partitioning by powers of the base. They also

[2] This means that the number 0.243 should be interpreted as being written in base-5, and therefore is equivalent to the number $^{2}/_{5} + {}^{4}/_{25} + {}^{3}/_{125}$.

investigated PSTs' understanding of the relationship between fractions and positional notation and found that PSTs' understanding of fraction-decimal equivalence was often based on surface features rather than understandings of the properties of quantities being depicted. For instance, several PSTs believed that one-half in base three must not be the same as one-half in base five. Preservice teachers also had difficulty comparing "decimals" that looked the same but were written using different bases, such as 0.2_{THREE} and 0.2_{FIVE} (Khoury & Zazkis, 1994).

A superficial understanding of place value notation, including the notion that decimals are composites of multiples of units, was noted by Kastberg and Morton (2014) as a likely source of difficulties that PSTs in their study had with comparing and ordering decimals. Stacey et al. (2001) examined preservice teachers' understanding of decimal numeration and their knowledge of student errors and found that 1 in 5 made errors that showed a lack of understanding of fundamental aspects of decimal numeration. A common source of errors was comparing decimal numbers with zero, with some PSTs appearing to believe that decimals are smaller than zero, a finding supported by research by Widjaja, Stacey, and Steinle (2011).

Having a thin understanding of fraction notation is also associated with difficulties in making sense of fractions as rational numbers and with their relationship to decimals (Domoney, 2001; Khoury & Zazkis, 1994; Newton, 2008). Domoney (2001) found that the preservice teachers in his study had strong part-whole images of fractions that dominated their thinking, a finding in keeping with research on younger students (Moseley, 2005). Furthermore, while they were aware of the fact that fractions are considered numbers, in problem situations, they reverted to their part-whole imagery and did not appear to make connections to other forms of representation that would support a more robust understanding of fractions. He argues that this suggests that “within the classroom…when responses have to be made with no time for reflection, it is likely that they will fall back exclusively upon their favoured [sic] images, for example comparing two-thirds and three-quarters with pictures but without noting their positions on a number line, or translating them into decimal notation by dividing” (p. 17). Khoury and Zazkis (1994) noted similar difficulties with PSTs’ understanding of fractions as depicting a particular numerical quantity with regard to a given unit that must remain invariant when described by different notations. For example, they found that only 26% of the PSTs were able to correctly conclude that “one-half” in base three and “one half” in base five must refer to the same quantity.

PSTs’ understandings of infinite decimals as numbers. A small group of studies investigated preservice teachers’ understandings of infinite decimals as numbers. In particular, two sets of studies investigated preservice teachers’ conceptions of the repeating decimal 0.999… and the equality 0.999…=1 (Burroughs & Yopp, 2010; Dubinsky et al., 2013; Yopp et al., 2011). Much of the research on students’ conceptions of the equality 0.999…=1 comes from literature related to limits and calculus and suggests that students’ difficulties are due to difficulties with the notion of limits. However, Burroughs and Yopp (2010) concluded that preservice teachers’ difficulties with the equality are likely not rooted in misconceptions about calculus concepts but rather deep-seated misconceptions about repeating decimals stemming from work that primarily occurred in the elementary grades (p. 39). In particular, PSTs seem to understand infinite decimals more as a *process* than as a *number*.

Dubinsky and colleagues (2013) argued that the lack of understanding of this equality also reveals limitations in learners' understanding infinite decimals as numbers. These researchers used APOS Theory (*Actions, Processes, Objects,* and *Schemas*) to argue that the distinction between *actions*, *processes,* and *objects* is important for problems involving infinity. Actions are so-named because they require actual execution of each step and so must, by necessity, be finite. Processes, on the other hand, allow one to imagine a series of actions continuing indefinitely. This allows learners to imagine, for example, that the process of division of 1 by 3 could result in a notation composed of an infinite number of threes: 0.3333…. Encapsulating this infinite process into an object corresponds to conceiving of this notation, 0.333… as representing a *number*, or something that *is* rather than something that *one does* (p. 236). However, they also found that the progression from *process* to *object* for infinite processes may be particularly difficult for learners. They did find that PSTs who displayed evidence of an object orientation, or at least the ability to imagine all the steps of the process as occurring at once, were more likely to believe the equality 0.999…=1.

Burroughs and Yopp (2011) found that several PSTs in their study did seem to hold process-oriented conceptions of infinite decimals, and others held action orientations (conceptions not yet at the process level). PSTs with an object-oriented view of infinite decimals spoke about the process of creating decimal notation using long division but did not speak of the resulting decimals as numbers themselves. For instance, one student, “Sally,” described the decimal $0.\overline{3}$ as being derived from the process of dividing 1 by 3 but being “basically an unobtainable decimal because it repeats forever” (p. 36). Other students used imagery that suggested that they thought of the infinite decimal as process of moving towards one, as along an asymptote, getting very close to one but not quite reaching it. The researchers argued that students need support in learning that infinite decimals are numbers, as objects rather than as a process that is related to division. They suggested that work with repeating decimals and number lines could support the development of such an understanding. They also recommended giving learners opportunities to compare various notational forms for representing particular numbers, including repeating decimals. Finally, they recommended that elementary students be given opportunities to develop object conceptions of infinite decimals in order to better support their work in middle and high school mathematics. Clearly, giving elementary students such opportunities necessitates their teachers having such a conception themselves.

Evidence of a process orientation towards infinite decimals was also found by Fischer (2014) who investigated inservice middle school teachers’ understandings of repeating decimals. Two of the participants held what appeared to be process-oriented understandings of repeating decimals. One participant believed that 0.333… could not be placed on a number line because the division process could never be completed. The other defined a rational number as “a number in some state of division” (p. 121). Moreover, several of the teachers in the study drew sharp distinctions between terminating and repeating decimals, believing for instance, that terminating decimals could not be interpreted as repeating decimals where the repeating portion was simply the digit "0" (e.g. $0.25 = 0.25\overline{0}$). Those that made clear distinctions between repeating and terminating decimals demonstrated less flexibility in their understandings about rational numbers. These results are particularly notable since these were all practicing teachers who taught about repeating decimals as part of their curriculum.

PSTs' Understanding of the Number Line

One representation that may be used to support student understanding of the relationships between whole, rational, and real numbers and the equivalence (or non-equivalence) of different notations is a number line. A compelling feature of the number line is that it can be meaningfully used to represent fractions, decimals, and whole number since all three express magnitudes that can be located on the real number line (Siegler & Lortie-Forgues, 2015). It can therefore be a tool for reasoning about the magnitude of numbers expressed in all three notations, a tool for comparing numbers expressed in any of the notations, and a tool for making sense of the equivalence of different notations. Moreover, since points on a number line represent distinct numbers, it can also be a tool for reasoning about numbers as objects rather than processes (Kinach, 2002a; Saxe et al., 2007; Shaughnessy, 2009; Siegler et al., 2011).

Research on PSTs' understandings of the number line and their ability to use it as a tool for reasoning about and making sense of notation is mixed. Siegler and Lortie-Forgues (2015) found that PSTs were very successful at estimating the location of positive fractions and whole numbers on a number line and could use it as a tool for thinking about the magnitude of positive numbers. Widjaja, Stacey, and Steinle (2011) found that PSTs could successfully locate positive decimals on a number line, but some had difficulties with negative decimals. Others have found that preservice and inservice teachers have difficulty using number lines when working with negative numbers (Kinach, 2002b; Mitchell et al., 2013; Steiner, 2009). Finally, locating infinite decimals on number lines can also be problematic for some PSTs, as they associate the infinite decimal with an incomplete *process* rather than a *number* (Burroughs & Yopp, 2010; Fischbein et al., 1995; Yopp et al., 2011).

Preservice teachers' may also have a limited understanding of the features of number lines and how they relate to the properties of the subsets of the real numbers. Doritou and Gray (2009) found that preservice and inservice elementary teachers in their study had limited conceptual understanding of the number line. Few of the teachers appeared to recognize how the structural features of a number line, particularly its continuity and density, related to the number system. Rather than seeing the number line as an abstract representation of the real number system, they instead associated the as "a series of discrete representations of particular elements of the number system. The notion that it evolved from a unit that could be repeated and partitioned was less important than the notion that actions could be carried out with it" (pp. 1742-1743). Clearly, if teachers themselves do not understand the number line as a representation of the number system, they cannot use it effectively as a model to develop that understanding in children (Doritou & Gray, 2009; Mitchell et al., 2013; Timmerman, 2010).

Summary

In summary, the body of research on PSTs' understanding of number and notation suggests that their knowledge is fragmented and superficial in many ways. They generally have some knowledge of the rules and algorithms that apply to whole and rational numbers (Tatto & Senk, 2011), but these are applied inconsistently and with little connection to the deeper principles that tie together the concepts that underlie number and numeration. Studies investigating preservice teachers' understandings of notation, particularly decimal notation, have revealed that preservice teachers' have misunderstandings about notation and about the properties of rational, irrational, and real numbers are deeply intertwined and problematic (Chick, Baker, Pham, & Cheng, 2006; Hiebert, 1988; Kastberg & Morton, 2014; Khoury & Zazkis, 1994; Sinclair et al., 2006; Thanheiser et al., 2013; Zazkis & Gadowsky, 2001; Zazkis & Sirotic, 2010). Many PSTs appear to have an incomplete understanding of the relationship between numeration and the number systems, as well as limited understandings about the differences between rational and irrational numbers (Fischbein et al., 1995; Sirotic & Zazkis, 2006; Widjaja et al., 2008). Difficulties seem to stem partly from the ways in which the different notations and numbers are commonly taught and used (Arcavi et al., 1987; Fischer, 2014), partly from misunderstandings about the hierarchical structure of the real number system (Fischbein et al., 1995), and partly from difficulties related to infinite decimal representations (Burroughs & Yopp, 2010; Dubinsky et al., 2013; Yopp et al., 2011). Finally, the number line representation is not well-understood by many PSTs, in part due to challenges with the number line itself and in part due to challenges related to their other (mis)understandings about number and notations.

Review of Literature Related to Developing PSTs' Knowledge of Number and Numeration

The previous section presented an overview of studies that documented preservice and in-service teachers' understandings of the connections between fractions and decimals and their relationship to the sets of rational and irrational numbers. In this section, the literature related to developing these understandings in teacher education coursework is reviewed. First, some challenges related to deepening PSTs' understandings of number and numeration are summarized. The next section presents a summary of ways alternate numeration systems have been used to deepen PSTs' understanding of number and notation. Finally, a summary of the literature related specifically to supporting preservice teachers in understanding the connections between fractions and decimals is presented.

Challenges of Teaching Number and Numeration to PSTs

Preservice teachers having procedurally-focused understandings of number and numeration (Tatto & Senk, 2011) poses a challenge for preservice teacher education in several ways. First, knowledge of an algorithm or procedure can interfere with students' ability and desire to get a deeper understanding of the underlying concepts (Hiebert & Carpenter, 1992; Sinclair, Liljedahl, & Zazkis, 2006). Markovits and Sowder (1990) found that that pre-service teachers in their study were more successful when learning new mathematical content than when re-learning topics they considered to be "familiar." This is likely due to the fact that their knowledge of these topics does not seem to them to be problematic. As Sowder et al. (1998) conclude, "Our experience suggests that one critical aspect of the teachers' knowledge of rational numbers is that *they do not realize that they lack the understanding* of rational numbers necessary to teach this topic in a meaningful way" (Sowder et al., 1998, p. 145, emphasis added). A study by Stacey et al. (2001) suggests that this lack of awareness of their own (mis)understandings may be very problematic in the classroom as a 43% of the PSTs in their study failed to report that students would have difficulty with problems of the type that they got wrong themselves, "indicating that quite a sizeable proportion of preservice elementary school teachers may not suspect they are making errors" (p. 222). In other words, if rule-based learning was the norm in their own education and it allows them to solve some problems involving whole and rational numbers, then work that focuses on deepening that understanding may not seem necessary or important to preservice teachers. Sinclair, Liljedahl, and Zazkis (2006) argue that PSTs may be content with their superficial understandings and conclude that preservice teachers' reluctance to revisit "familiar" topics are psychological barriers that should be a consideration in preservice teacher education course design.

The fact that many preservice elementary teachers can be expected to have had largely procedurally-focused experiences in mathematics creates an additional challenge for teacher education coursework in that they will not have learned how to learn mathematics using the types of models and strategies that are the cornerstone of curricula that emphasizes building conceptual understanding. This means that in addition needing to learn more about the mathematical content, they also need to learn how to learn mathematics using models and through group-based problem solving activities. They also must learn about why such methods are necessary and appropriate ways of learning mathematics, and about the big mathematical ideas that underlie the concepts they are learning. Lawson and Stienstra (2008) emphasize that a focus on developing preservice teachers' content knowledge without such a focus will likely mean that the preservice teachers miss out on much of the important pedagogical learning that could be available to them because they will not know how to engage in the activities in ways that will deepen their understanding.

Coursework that is intended to deepen understanding of fractions, decimals, and the set of rational numbers is clearly re-treading familiar ground for preservice teachers. It should be expected, therefore, that PSTs may have difficulty fully engaging in activities related to these areas and may rely on or use procedural understanding to solve problems, even when those problems are designed to require or deepen conceptual understanding. Understanding ways in which activities may be designed in order to maximize PSTs' willingness and ability to engage fully in activities that are designed to deepen their understanding of "familiar" topics is therefore important for design of mathematics coursework for preservice teachers.

The Use of Alternate Numeration Systems

Two factors may be identified as important considerations for content coursework design that will deepen preservice teachers understanding of number and numeration in a way that is useful for the work of teaching: (1) the need to make the content interesting and challenging for preservice teachers in a way that acknowledges and builds on their prior knowledge of this content (Sowder et al., 1998), and (2) the need to incorporate opportunities to make connections between different representations and notations

(Kastberg & Morton, 2014; Mitchell et al., 2013).

One approach that addresses both of these needs is the use of units wherein PSTs work with numeration systems different than our own. These include working with historical number systems (Nataraj & Thomas, 2009; Radin, 2007; Thanheiser, 2014), invented number systems (Hopkins & Cady, 2007) and using positional notation with alternative bases (Andreasen, 2006; Khoury & Zazkis, 1994; McClain, 2003; Murawska, 2013; Roy, 2008; Wheeldon, 2008; Yackel, Underwood, & Elias, 2007). These approaches allow PSTs to represent quantities using numeration systems different from our own with a goal of allowing PSTs a window into the structural features of our own numeration system that can be difficult to appreciate in the absence of other examples. These features include the *symbols* we use (e.g. “2” is not the only shape that can denote a quantity of “two”), the fact that the *base* we use for grouping is ten, the fact that we use the principle of *position* (or *place value*) to denote the size of a given group. Work with alternate bases and numeration systems has also been used as a way to “reveal PSTs’ conceptions of the underlying place value structure of numbers that would [be] masked by procedural use of base ten language and symbols” (Fasteen, Melhuish, & Thanheiser, 2015, p. 87).

Varying the symbols as well as possibly the base or how numbers are structured are all approaches that can support PSTs in moving beyond their rule-based understanding of number. Hopkins and Cady (2007) invented a numeration system in base five that also used different symbols than our own, which they called the “Ordpa system.” They found that using the Ordpa system helped support their inservice and preservice teachers in deepening their understanding of place value, particularly the concept of zero as both a placeholder and a number and the fact that a number's position determines its value. One challenge they identified was with teachers' initial reluctance to use manipulatives to make sense of mathematical concepts themselves. At first, the teachers did not use the cubes as "tools for thinking" (Diezmann & English, 2001) but rather only to display the answers, and appeared to believe that needing to use the cubes was a sign of weakness. They had to be encouraged by the instructors to use the cubes more productively, and it was only after doing so that the deeper insights were made.

Another challenge related to the use of alternate bases or numeration systems as a context for deepening PSTs' understanding of our number system is the fact that PSTs sometimes continue to rely on their rule-based understandings of numbers written in base ten positional notation to make sense of these new quantities. For instance, Fasteen et al. (2015) designed a set of activities in which PSTs examined patterns with multiplication by five in base five. Investigations included opportunities to explore multiplication in base five numerically and with manipulatives, particularly Dienes blocks in base five (analogous to base ten blocks). While they found that the activities ultimately supported PSTs' in deepening their understanding of the structural properties of our base ten positional number system and how those relate to multiplication, the PSTs' initial approach to investigating the patterns was very shallow and rule-based. It was only after pressing from the instructor to examine and explain why the patterns they were observing were happening and how they related to known patterns in base ten multiplication and their work modeling multiplication using base five blocks that students were able to gain deeper insights.

Similarly, McClain (2003) reports that she changed the instructional sequence she was using to not include using base eight notation because PSTs were relying solely on the notation to try to make sense of operations in base eight. As a result, she found that they were creating rules for computation rather than working to develop a conceptual understanding of the notation and place value structure, and that the activities actually served to reinforce the idea that understanding numbers is about memorizing rules.

Several studies reported that PSTs often converted the alternate base problems into standard base ten notation, did the work in base ten, then converted back to the other numeration system, thereby undermining the purpose of the activity (Fasteen et al., 2015; Khoury & Zazkis, 1994; McClain, 2003; Morris, Hiebert, & Spitzer, 2009). Finding ways to ensure that PSTs actually engage in the activities in alternate numeration systems and that they appreciate the importance of doing so is thus an important consideration when using such approaches to deepen or assess their understanding.

While the use of alternate numeration systems is a common approach to developing PSTs' understanding of whole numbers notation, it has not been commonly applied to developing their understanding of decimal or fraction notation. However, Zazkis and colleagues (Khoury & Zazkis, 1994; Zazkis & Whitkanack, 1993) did use alternate bases as a way to explore PSTs' understanding of place value structure as it relates to non-integer numbers. It is important to note that the authors reported that they used these investigations only as a tool for assessment rather than as a way to deepen PSTs' understanding of fractions or the relationship between fractions and decimals, and so it is not clear how activities could be developed that would support PSTs in making sense of the relationships between place value structure and non-integer numbers. Zazkis and Whitknack (1993) suggest that such activities could be productively used in classrooms for that purpose but no study citing that work documented using that approach with young students or preservice teachers.

Instructional Approaches Related to the Connections Between Fractions and Decimals

Very few studies directly address ways to support preservice teachers in explicitly making connections between fractions and decimals. However, the small body of work that exists suggests that using multiple representations is can be a productive approach to deepening PSTs' understanding of these notations.

LeSage (2011) studied preservice elementary teachers' understanding of decimals before and after participation in a computer-based learning module intended to deepen their understanding of decimals and their connection to fractions. The module followed the learning sequence recommended by Moss and Case (1999) and built on student knowledge of rational numbers by starting with percentage representations, then connecting percentage to decimals, and finally connecting decimal to fraction representations. One component of the sequence involved using geoboards to construct visual representations of decimal quantities that terminated in one or two places. This supported PSTs in differentiating between 0.2 as "two-tenths" and 0.20 as "twenty-hundredths" while also allowing them to understand why the two quantities are equal. Another component was related to supporting PSTs' in their ability to accurately use and understand the importance of decimal terminology (e.g. “tenths”). LeSage reports that the series of activities did improve the PSTs’ conceptual understanding of terminating decimals. This work is in keeping with recent research with younger students that suggests that use of a hundreds grid to visually represent decimals along with supportive language use can help students develop a deeper understanding of decimals (Cramer et al., 2015).

It is important to note that the activities reported in LeSage (2011) were related to developing a deeper understanding of *terminating* decimals and their relationship to fractions with denominators with powers of ten and would not necessarily support a deeper understanding of repeating decimals or the relationship between fractions and repeating decimals. For instance, learners were told that “the word ‘*point*’ should not be used when naming a decimal…and that the digit furthest to the right dictates the name of a decimal (e.g., 3.04 is read ‘three and four hundredths’)” (p. 415, emphasis in original). However, these naming conventions may only be usefully applied to decimals that have a “last digit.”

A study by Sinclair, Liljedahl, and Zazkis (2006) was designed to support PSTs’ understanding of both repeating and terminating decimals and their relationship to fractions. It was also designed to support PSTs in differentiating between repeating and non-repeating infinite decimals. This study also used a grid to represent decimals but not as an area model. Rather, they used an online calculator they designed, called the Colour Calculator [sic], to support preservice elementary teachers in investigating fractions and decimals. The Colour Calculator has a standard simple calculator interface (including π and square root keys) to input calculations and outputs both a number and a grid that is colored based upon the digits to the right of the decimal point. Terminating decimal grids simply stop after the number of places in which the decimal terminates, while repeating decimals are shown up to 100 places. Each digit receives its own color, which allows patterns in repeating decimals and the lack of patterns in irrational numbers to become visible, particularly since 100 digits to the right of the decimal point are displayed. In other words, the grid is not colored to show the fraction represented but rather to help reveal patterns in repeating decimal notation.

After completing investigations in this environment related to terminating, repeating, and non-repeating decimals, PSTs in the study had a much deeper understanding of the relationship between fractions and repeating and terminating decimals. They also distinguished between repeating decimals and non-repeating decimals as being different types of numbers. Several aspects of the investigation appeared to be useful. First, having the fraction, decimal, and colored grid all together as a single display seemed to help promote PSTs in making connections. The colored grid and the number of digits displayed both helped make patterns in repeating decimals apparent, and helped PSTs to literally see the difference between rational and irrational decimals. Moreover, the authors found that investigating the relationship between fractions and decimals in this new environment was enjoyable and interesting for the PSTs, which helped them to engage more deeply in the investigations and to "re-examine and enrich their understandings of the mathematical concepts previously encountered" (p. 200). Finally, seeing the infinite decimal representations depicted on a grid as well as by a series of digits appeared to support the PSTs in thinking of these decimals, and their corresponding fractions in the case of repeating decimals, as *objects* rather than as processes.

One possible concern with the Colour Calculator sequence is the fact that a hundreds grid is commonly used to visually represent the magnitude of a decimal fraction and to demonstrate fraction-decimal equivalence (e.g. for the decimal 0.25, 25-hundredths, or ¼, of a hundreds grid would be shaded). It is not clear whether PSTs would find the different uses of the hundreds grid confusing. Furthermore, although Sinclair et al. (2011) report that seeing the fraction as a quotient and the decimal to 100-places as both symbols and a pattern supported PSTs in believing that these two representations were equal, the depth of this connection is unclear.

Finally, Amato (2005, 2006) used a place value board to represent whole numbers, fractions, and decimals. First, she taught her PSTs to use a place value board to represent whole numbers, decimals, and mixed number fractions and supported her PSTs in making explicit connections between the concrete representations and the different symbolic notations. She then helped teachers use the board to model addition and subtraction with the different number types and to connect these to the standard written algorithms related to each representation. She found that the PSTs' conceptual understanding of both the notations and the algorithms improved after use with the place value board. More importantly, she also found that their understandings of the connections between these representations was improved as a result of using the place value board with all three notations and algorithms.

All of the above results indicate that PSTs' understanding of fractions, decimals, whole numbers, and the relationships between them may be deepened when instruction is designed to engage them tasks that build on the knowledge they bring about these numbers and notations and support them in explicitly making connections between them. However, each study is very limited in scope, and addresses only one specific area of learning about number and numeration. It is not clear if addressing learning in this piecemeal fashion would ultimately support PSTs in developing a deep understanding the many ways in which number and notation are interconnected.

Summary

In summary, several challenges related to developing PSTs' understanding of number and numeration during their teacher education coursework have been identified. Coursework related to number and numeration is re-treading familiar ground for preservice teachers. This familiarity may result in PSTs finding it difficult to fully engage in activities designed to deepen their conceptual understandings (Markovits & Sowder, 1990). Work requiring preservice teachers to work in alternative bases and/or alternative number systems has been used as a way to address this concern (Fasteen et al., 2015; Hopkins & Cady, 2007; McClain, 2003). However, even in these situations, some PSTs relied on or used their procedural knowledge of standard, base ten numeration to solve problems, and therefore failed to engage in the aspects of activities that could productively deepen their understandings (Fasteen et al., 2015; Khoury & Zazkis, 1994; McClain, 2003; Morris et al., 2009). Studies that have specifically addressed ways to support PSTs in deepening their understanding of the relationship between fractions and decimals suggests that using multiple representations can be a productive approach to deepening PSTs' understanding of these notations and the connections between them (Amato, 2005, 2006; Sinclair et al., 2006). However, these studies were limited in scope.

Conclusion

In summary, studies have found that many preservice elementary teachers' understanding of fractions, decimals and whole numbers is fragmented and procedurally-driven (Thanheiser et al., 2013). The connection between fractions and decimals is weak, and their overall understanding of the relationships between number and numeration for whole and rational numbers lacks coherence (Dubinsky et al., 2013; Fischbein et al., 1995). Work with alternate numeration systems and connecting the notations to multiple representations have been shown to help deepen PSTs' understandings of number and notation (Hopkins & Cady, 2007; McClain, 2003), including some work that related specifically to developing understandings of rational numbers through alternative bases (Khoury & Zazkis, 1994). Work with representations that allow PSTs to investigate the differences between repeating and non-repeating decimals and the connection between fractions and terminating or repeating decimals has been helpful as well (Sinclair et al., 2006). Additionally, work with non-symbolic representations of repeating decimals may support PSTs in developing an understanding of infinite decimals as numbers rather than processes (Sinclair et al., 2006). Notably, across these studies, there little attention paid to how coursework might explicitly support PSTs in making connections between fractions and base ten positional notation and in understanding how these notations relate to the real number system.

The present study adds to the body of literature presented in this chapter by documenting preservice elementary teachers' understanding of the relationships between fractions and decimals and the set of rational numbers as the PSTs participated in an instructional unit designed to support these understandings. The study also documents how the preservice teachers' use of representations during the course supported and/or constrained the development of their understanding of the relationships between number and numeration. This study also extended prior work on bridging tools (Abrahamson & Wilensky, 2007; Abrahamson, 2006; Fuson & Abrahamson, 2005) by documenting how bridging tools might be used specifically to promote connections between fraction and decimal notation and to develop an understanding of both as representations of numbers.

This chapter presented an overview of the theoretical frameworks and the real number system that guided the study. Next, the literature related to the PSTs' ways of understanding number and notation was summarized, followed by a summary of the literature related to developing these understandings in preservice teachers' coursework. The research methodology that guided this study is presented in the following chapter.

Chapter 3

Method

This study was a descriptive case study (Yin, 2014) of the ways in which one class of preservice elementary teachers developed their understandings of number and numeration in the social context of a classroom community. This qualitative research study documented the ways in which individual preservice elementary teachers in the first semester of an undergraduate mathematics content course understood and represented fractions and decimals, the relationship between them, and between those notations and the sets of rational and irrational numbers. The primary goal of this study was to document the understandings and use of representations by a group of preservice teachers at two different points in their mathematical content coursework in order to support the design of curricula that promotes the development of preservice elementary teachers' mathematical understandings from a *connected* perspective. Recall that in this study, a *connected perspective* meant that supporting learners' in understanding the connections between mathematical concepts was considered in the design of all aspects of the unit, including the activities, the course sequence, and the assessments. In this chapter, the research design, participants, intervention, data collection and analysis, and credibility and transferability of the study will be described.

Research Design

This research was designed as a descriptive case study (Yin, 2014) in order to contribute to the understandings of how individual preservice elementary teachers develop an understanding of number and numeration as it relates to the set of rational numbers within a collective classroom setting. A case study design was deemed appropriate for this study as the purpose of a case study is to gather systematic, comprehensive, and in-depth information about the case of interest (Patton, 2002). The unit of analysis in this study was a single class of preservice elementary teachers participating in an instructional unit designed to support them in making explicit connections between concepts related to number and numeration.

Two research questions and their related sub-questions guided this study:

1. What is the nature of preservice elementary teachers' (PSTs) understanding of the *Number and Numeration System* (*Figure 2*) before and after participation in a unit designed to develop their understandings of this content?
    a. What is the nature of PSTs understandings of the sets of numbers generally, and *rational numbers* specifically, early in the unit?
    b. What is the nature of PSTs understandings of fractions, decimals, and the connections between them early in the unit?
    c. What is the nature of PSTs understandings of fractions, decimals, and the connections between them after the unit?

2. What is the role of representations in the development of PSTs' understanding of the *Number and Numeration System*?
    a. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals before/early in the unit?
    b. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals after the unit?

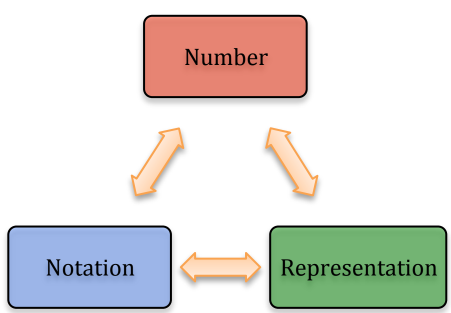


Figure 9. The *Number and Numeration System* depicts the ways knowledge of number,

notation, and (non-symbolic) representations are interconnected.

Setting and Participants

Participants in the study were thirty-two undergraduate preservice elementary teachers enrolled in a semester-long mathematics content course at a large, Midwestern university. This course was the first of two required mathematics content courses designed to deepen prospective elementary teachers' content knowledge of mathematics and to provide a model of reform-based teaching. The course instructor was a tenured professor with extensive experience teaching the mathematics content course and knowledgeable about research in the areas of elementary number and operations. The researcher participated as co-instructor of the instructional sequence and as participant-observer in small- and large-group discussions. The researcher was also an experienced teacher of the course content, having taught the course for the five semesters prior to the intervention.

The course met twice a week for 75 minutes. The number and numeration unit took place during the first seventeen (of 30) class sessions. All students in the course completed the same course assignments. A total of thirty-three students were enrolled in the course; one student opted out of the study and that student's work was excluded from analysis.

Selection and Description of Interview Participants

In order to examine individuals with different levels of initial content knowledge, eight preservice teachers in the course were selected as participants for the interview component of the study. For the interview component, a purposive sampling strategy (Merriam, 2009) was employed to identify students with relatively weak and strong levels of initial knowledge of the *Number and Numeration System* compared to their classmates. As differences in students' initial understandings of the material may be expected to influence their understandings during the course, it was considered desirable to have a range of mathematical understanding in the interview group.

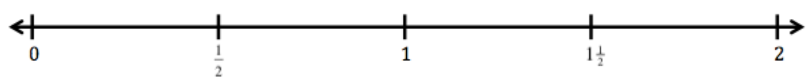
In order to identify the interview participants, two pieces of data were collected on the first day of the course: a pretest and an introductory questionnaire. Both are described in more detail in the "Data Sources" section below. Both the pretest and introductory questionnaire were given to all thirty-three students enrolled in a mathematics content course for preservice teachers on the first day of the course. Students were informed of the study prior to taking the pretest. Students who were interested in participating in the interview portion were asked to indicate their interest at the same time. Thirty-two of the thirty-three students opted-in to the whole class study. Work from the student who opted out of the study was not collected by the researcher, but that student did participate in all the same activities as the rest of the students. Twenty-eight students indicated a willingness to participate in the interview study as well.

Responses to the pretest Questions 2, 4, 6, and 7 were used to identify students who showed evidence of differing levels of initial content knowledge of fractions and decimals on the pretest. These particular pretest questions were used because they directly related to this study. Pretest Questions 2 and 4 both asked students to make connections between fractions and decimals. Pretest Question 6 asked them to place a point on a number line, one of the key representational tools used in the instructional sequence. Question 7 (parts a, b, and c) asked students to solve then create a story and picture related to three different number sentences. Figure 10 shows these questions. The complete pretest may be found in Appendix B.

2. Are there any numbers between 0.66 and $\frac{2}{3}$? If so, give an example and briefly explain how you know it is between 0.66 and $\frac{2}{3}$. If not, explain why not.

4. Tell how you would help a student understand why $\frac{1}{3} = 0.333\ldots$ when written as a decimal.

6. Accurately locate $\frac{3}{5}$ on the given number line **without using a ruler** or other measuring device. Briefly explain or show how you located the number.



7. For each open number sentence, (a) solve, (b) write a story problem that matches the sentence *exactly as written*, and (c) show how to solve by drawing pictures.

a. $\frac{2}{5} \times 1\frac{1}{4}$

b. $1\frac{1}{4} \div \frac{3}{4}$

c. $^{-}3 - {}^{-}5$

Figure 10. *Pretest questions 2, 4, 6, and 7.*

To identify interviewees, responses to pretest Questions 2, 4, 6, and 7 were scored as correct, partially correct, incorrect, and incorrect with clear misunderstandings. Students who answered all five questions fully or partially correctly were categorized as having relatively strong initial content knowledge. Six students were in this group. Among those six students, four also provided at least one correct story or picture for a number sentence in Question 7. Three of these students offered a correct story for fraction multiplication or division, and one student gave a correct story for the integer subtraction sentence. As the ways students could make connections between representations was of particular interest to this study, these four students were selected for participation in the interview. All four students asked to participate in the interview study agreed to do so. All four students in this group reported having taken calculus. All four reported being good at math. This group consisted of three women and one man. Three were in their sophomore year of college, one was a junior. One student was studying abroad from an Asian country and spoke English as an additional language.

Students who had three or more misunderstandings on the five selected pretest questions were considered to have relatively weak initial content knowledge. Eleven of the thirty-two students fell into this category based on their pretest responses. From this group of eleven, eight stated they would be willing to participate in the interview study as well as the whole class study. The goal was to select students who represented a range of misunderstandings as well as a range of mathematical backgrounds. One student in this group reported taking mathematics courses at the calculus level or higher and was selected for participation in the study. Three more students were selected for participation in the study based on misunderstandings shown about fractions in their responses to pretest Question 7. Two agreed to participate in the study. One of these students never responded to scheduling requests for an interview, and was dropped from the interview portion of the study only. Another student was selected as a replacement based on the fact that she had the most incorrect responses from the remaining group, and agreed to participate in the study. All four students in this group were women. Three were in their sophomore year of college. One student was older than typical undergraduates at the university and was a junior.

The four students who were selected because they displayed relatively weak initial content knowledge on the pretest will be referred to in this study as being in the *Mathematical Knowledge for Teaching 1 (MKT1)* group. The four students who were selected because they showed relatively strong initial content knowledge on the pretest will be referred to as being in the *Mathematical Knowledge for Teaching 2 (MKT2)* group. Eva, Nina, Willa, and Korey are in the MKT1 group, and Soren, Jo, Andie, and Mei are in the MKT2 group. A longer description of each of the eight interview participants can be found in Appendix E.

Timeline

Table 1 shows the timeline of the study.

Table 1
Timeline of the Study

Description	Time frame
Instructional sequence for number and numeration in mathematics content course along with supporting activities. Items for pre- and posttest piloted.	Fall, 2014
Pre-test	January, 2015
Interview 1	January-February, 2015
Instructional sequence	January-March, 2015
Posttest	March, 2015
Interview 2	March-April, 2015

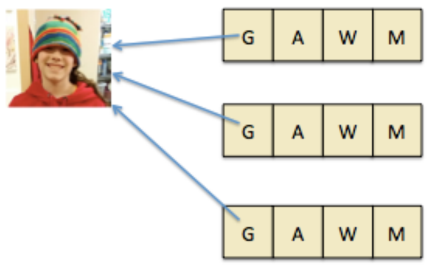
Key Instructional Tasks

Two sets of instructional tasks that occurred during the instructional unit were directly related to this study. These are described in more detail in this section. An overview of the complete unit may be found in Appendix A.

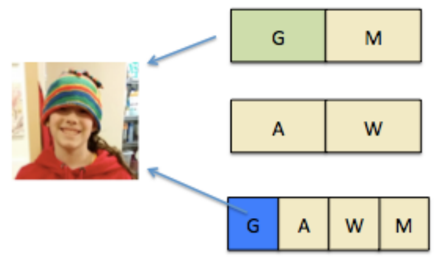
Breaking Bread Activities

Portions of two class periods were spent creating and interpreting pictures related to the context "How can X loaves of bread be shared fairly among Y people?" These activities were called the *Breaking Bread* activities. The first day of *Breaking Bread* activities took place on day 7 of the 17-day unit, and the second day was on day 12. The *Breaking Bread* activities were designed by the researcher and piloted over two previous semesters of the same course. These activities were modeled after the equi-partitioning activities found in Empson and Levi (2011). The context of sharing loaves of bread was chosen for two reasons. First, loaves of bread are naturally modeled by rectangles, a shape that can be used to model fair sharing situations that result in quantities that can be denoted using both fraction and decimal notation. Second, similar problems are found on the Rhind/Ahmes papyrus, an historical document related to the mathematics of ancient Egypt. Historical numbers systems, including the Ancient Egyptian, were studied early in the unit as a way of building meaning for our whole number numeration system. Using this context allowed for the connection to be made between the ways the Ancient Egyptians denoted whole numbers and the ways they denoted fractions.

On the first day of the *Breaking Bread* activities, students were encouraged to fairly share the X loaves of bread in multiple ways and then to connect their pictures to the correct fraction notation. For instance, for the problem "How can three loaves of bread be shared fairly by four people?," if a student drew a picture depicting each loaf cut in fourths with each person receiving one-fourth from each loaf (see the left hand picture in *Figure 11*), they were encouraged to symbolically represent one person's share as "¼ + ¼ + ¼" or "3 x ¼." Similarly, if they instead showed two loaves cut in half and the third cut in fourths with each person receiving one-half and one-fourth of a loaf, they were encouraged to represent one person's shares symbolically as "½ + ¼" (see the right hand picture of *Figure 11*).



Grace's share is ¼ +¼ +¼ of a loaf



Grace's share is ½ + ¼ of a loaf

Figure 11. Pictures showing two ways three loaves of bread can be fairly shared by four people ("Grace," "Amelia," "William," and "Matt"), and their related symbolic notations.

On the second day of the *Breaking Bread* activities, students were encouraged to share the bread in such a way that the answer could be written using decimal notation. In other words, they had to share the bread so that the pieces that were created were always fractional pieces that could be described by powers of ten. This was referred to in the class as "decimal law." For example, in the case of three loaves of bread shared by four people, this would result in each person receiving seven-tenths and five-hundredths, or 0.75, of a loaf of bread (see left hand picture in *Figure 12*). Approximately the last ten minutes of this class was spent looking examples of how this context could be used to find "decimals" in bases other than ten by always partitioning loaves into say, powers of five, rather than powers of ten in order to write the answer in base five "decimal" notation (see right hand picture in *Figure 12*). This was the only time spent extending the idea of decimal notation to non-base ten contexts. The limited class time spent generalizing the *Breaking Bread* activities is notable as two questions that could be solved using these ideas were included as “bonus” questions on the posttest.

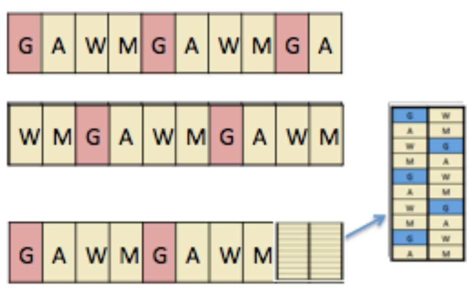
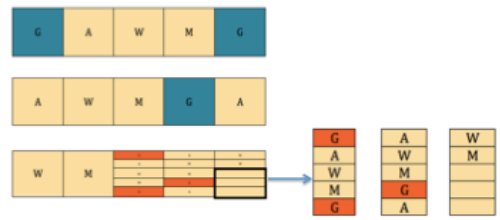
Base 10	Base 5
	
Grace’s share is 7/10 + 5/100 or .75 of a loaf	Grace’s share is 3/5 + 3/25 + …, or $0.33\ldots_{\text{FIVE}}$ of a loaf

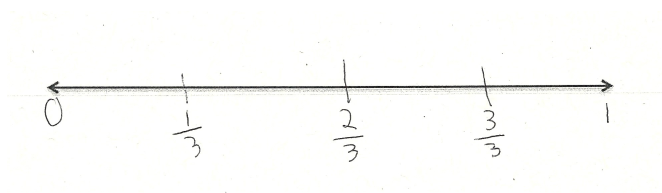
Figure 12. Pictures showing how three loaves of bread can be fairly shared by four people (“Grace,” “Amelia,” “William,” and “Matt”), so that the resultant shares can be denoted using “decimal” notation in base ten and base five.

Number Line Activities

Number line activities took place during three class periods in the middle of the instructional unit (days 5, 6, and 7). Number line activities were co-designed by the instructor for the course and the researcher and piloted during the previous semester of the same course. The first day of number line activities were based on work from the Rational Number Project (Cramer, Wyberg, & Levitt, 2009). These included looking at student work on number lines and discussing common errors made by students. The second and third day included activities based on the “non-routine” number lines developed by Saxe and colleagues (Saxe et al., 2007; Saxe, Diakow, & Gearhart, 2012).

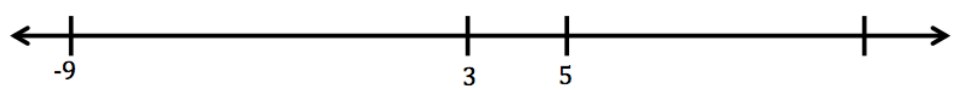
An example of a number line activity involving student errors and one involving a “non-routine” number line are shown in *Figure 13*.

Common error made by elementary students



Task: Describe the error made by the student. What does the student know? What does the student not know?

A “non-routine” number line problem



Task: Are the numbers shown on the number line correct? How do you know? What number do you believe should go at the far right tick mark? Why?

Figure 13. Two representative number line activities completed during the instructional unit.

All days of number line instruction emphasized the idea that non-equivalent quantities lie at different points on a number line while equivalent quantities lie at the same point. The role of partitioning with regard to locating rational quantities was emphasized in activities on the second and third day of number line activities. The number line was chosen as the second “bridging tool” used in the instructional sequence because it embodies important structural features of real, rational, and integer quantities within the same representation (National Mathematics Advisory Panel, 2008; Saxe, Diakow, & Gearhart, 2012). It also supports reasoning about fraction and rational decimal equivalence, as well as the non-equivalence of fractions and non-rational decimals. Finally, as a linear representation for all real numbers, it also supports the notions of numbers as *measures*, or (directed) distances from zero, an important unifying property for the sets of whole, integer, rational, and irrational numbers.

Data Sources and Collection Procedures

Data used in this study was collected from several sources, including a pre- and posttest, two sets of individual interviews, an introductory questionnaire, and field notes taken by the researcher during the implementation of the instructional sequence. These instruments are described in the following sections. Table 2 shows how the data sources related to the two research questions guiding the study.

Table 2

Data Sources Used in the Study as They Corresponded to the Research Questions

Research question	Subquestion	Data sources
1. What is the nature of preservice elementary teachers' (PSTs') understanding of the *Number and Numeration System* before and after participation in a unit designed to develop their understandings of this content?	a. What is the nature of PSTs' understandings of the sets of numbers generally, and *rational numbers* specifically, early in the unit?	Interview 1: Def'n of numbers Interview 1: Def'n of rational/irrational numbers Interview 1: Number sort Pretest Q1 Pretest Q2
	b. What is the nature of PSTs' understandings of fractions, decimals, and the connections between them early in the unit?	Interview 1: Number sort Pretest Q2 Pretest Q4
	c. What is the nature of PSTs' understandings of fractions, decimals, and the connections between them after the unit?	Interview 2 Posttest Q3 Posttest Q5 Posttest Q6c Posttest Bonus Q2 Posttest Bonus Q3
2. What is the role of representations in the development of PSTs' understanding of the *Number and Numeration System*?	a. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals before/early in the unit?	Pretest Q4 Interview 1:Number sort
	b. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals after the unit?	Posttest Q5 Posttest Q6c Posttest Bonus Q2 Posttest Bonus Q3 Interview 2

Note. Def'n= Definition

Introductory Questionnaire

An introductory questionnaire was administered on the first day of class to all students enrolled in the course to gather information about their mathematical background. This included information about the highest level of mathematics taken, their feelings towards mathematics, and their perceptions of their mathematical abilities. A copy of the questionnaire is included in Appendix D.

Pre- and Posttest

This study was designed as a descriptive case study (Yin, 2014) to document the ways that preservice teachers understood the *connections* between fractions and decimals, and between those notations and the sets of rational numbers. Thus, the pre- and posttest questions were designed by the researcher to elicit preservice teachers' understandings of the connections between the various areas of the *Number and Numeration System*. Two principles from the literature guided the question design. First, several problems were set within the context of teaching. This was done to promote an understanding that the mathematical knowledge necessary for teaching is specialized. It was also done to elicit explanations from the preservice teachers that they deemed adequate for the work of teaching elementary students. Other tests designed to measure mathematical knowledge *for teaching* have used the context of teaching to elicit such understandings (Rowan et al., 2001). Second, to elicit students' understandings of the connections between mathematical ideas, questions asking them to translate between different concepts were designed. These questions drew on prior work with a translation model (Cramer, 2003). These questions included pretest Questions 1, 2, and 4 and posttest Questions 3, 5, 6, and bonus questions 2 and 3. These questions are described in more detail in the following sections.

The questions used on the pre- and posttest of interest to this study were piloted on the final exam for the same course the previous semester. The instructor for the course, a tenured professor of mathematics education with many years' experience teaching the course, also reviewed and gave feedback on the tests. Changes were made based on the results of the pilot study and the instructor's feedback. In particular, for the questions relevant to this study, the categories of "non-examples" and "unsure" were added to pretest Question 1, and posttest Bonus Questions 2 and 3 were moved from the main test to the "bonus" section.

Pretest. The pretest was given on the first day of the course. Students were given approximately twenty-five minutes to complete the pretest. They were told that the pretest would not count as a graded assignment. They were encouraged to do their best work and told that the questions were designed based on the types of knowledge they would need for teaching. Three items on the pretest were used as data sources for this study, namely pretest Questions 1, 2, and 4 (*Figure 14*). Pretest Question 1 asked students to translate between the sets of numbers and notations by giving the students the sets of numbers and asking them to write examples and non-examples of numbers (notations) for each set. Question 2 asked students to translate between a fraction and a terminating decimal notation. Question 4 asked students to translate between a fraction and a repeating decimal notation, and was set in the context of teaching.

1. Give some examples and non-examples for each kind of number. If you are unsure whether a number is a given type, put it in the "unsure" category.

Whole Numbers	NOT whole numbers	Unsure

Rational Numbers	NOT rational numbers	Unsure

Integers	NOT integers	Unsure

Irrational Numbers	NOT irrational numbers	Unsure

2. Are there any numbers between 0.66 and $\frac{2}{3}$? If so, give an example and briefly explain how you know it is between 0.66 and $\frac{2}{3}$. If not, explain why not.

4. Tell how you would help a student understand why $\frac{1}{3} = 0.333...$ when written as a decimal.

Figure 14. Questions 1, 2, and 4 from the pretest.

In addition to pretest Questions 1, 2, and 4, the three parts of pretest Question 7 were used for the selection of the interview participants. The second research question guiding this study asked about the role of representations in preservice teachers' ways of understanding. In order to document some of the ways the preservice teachers used non-symbolic representations as tools for thinking about rational numbers at the start of the course, three questions asked students to write a story, draw a picture, and solve a given number sentence. One number sentence involved fraction multiplication, one involved fraction division, and the third involved integer subtraction. Fraction multiplication and division and integer subtraction were chosen as these areas have been documented in other research as being difficult for preservice teachers and/or students (Ball, 1990a; Olanoff, 2011; Post et al., 1991). These questions thus created an opportunity to identify students with comparatively higher levels of initial content knowledge with regard to non-symbolic representations with operations involving non-whole numbers. In particular, one measure that was used to identify students as being in the “relatively high initial content knowledge” group (MKT2) was whether they were able to create at least one correct story or picture for these problems. Similarly, these problems were also used to identify students for the “relatively weak initial content knowledge” group (MKT2) if the students’ responses to these questions showed misconceptions. The complete pretest may be found in Appendix B.

Posttest. The posttest was given on the last day of the seventeen-day unit. It served as the final exam for the number and numeration component of the course. This test was not timed, but was designed to take approximately one hour to complete. Students were told in advance that they could stay and finish the test after the regular class period ended if they desired to do so. The majority of participants finished within the allotted one-hour-and-fifteen minutes of class time, and all others finished within an additional thirty minutes. Three participants were absent on the day of the posttest and took it at another time, within one week from the missed class. These tests were proctored by the instructor for the course.

Five items on the posttest were used as data sources in this study, namely posttest Questions 3, 5, 6c, and Bonus Questions 2, and 3 (*Figure 15*). These questions were designed to ask students to make similar translations between representations as on the pretest. In particular, posttest Question 3 asked students to translate between a fraction and a terminating decimal and was related to pretest Question 2. Question 5 was set in the context of teaching. In this question, students were given a fraction ($^1/_6$) and asked to connect it to its decimal representation (the repeating decimal $0.1\overline{6}$). This question was related to pretest Question 4. Question 6c utilized the same repeating decimal ($0.1\overline{6}$) as in Question 5, and asked students to place it on a number line. As the decimal was repeating, this necessitated connecting the decimal back to its fraction representation. Two additional questions were added as "bonus" questions at the end of the test that asked students to translate between a fraction and a non-base-ten decimal. These questions were included as bonus questions rather than on the regular test because the idea of translating between fractions and non-base-ten decimals was only been briefly addressed for approximately ten minutes of one class period. The use of bases other than ten has been used in other work with preservice teachers to increase their understanding of the structure of the base ten numbers (Fasteen et al., 2015; Safi, 2009; Zazkis & Whitkanack, 1993). The complete posttest may be found in Appendix C.

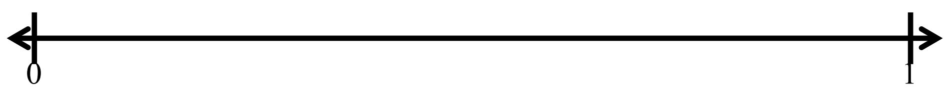
3.	Are there any rational numbers **between** $\frac{1}{3}$ and 0.333? If yes, give an example and justify how you know it is between $\frac{1}{3}$ and 0.333. If not, explain why not.
5.	Show/explain how you could help a student find the decimal representation of the fraction ⅙ **without** using the standard division algorithm or a calculator.
6c.	Accurately locate each value on the given number line. When more than one number is given, position and label both numbers on the same number line. Briefly explain or show how you positioned each point. c. 0.1666... 
B2.	Write $\frac{1}{3}$ as a decimal in base-7. Show/explain how you got your answer.
B3.	Explain why $\frac{1}{3}$ written in base-7 decimals is similar to $\frac{2}{9}$ in base-10 decimals.

Figure 15. Questions 3, 5, 6c, and Bonus Questions 2 and 3 from the posttest.

Interview Protocols

Two sets of semi-structured interviews with the eight interview participants were conducted with a purpose of collecting more in-depth information about their understanding of number and numeration than was possible through written instruments alone. Each interview was conducted with individual interview participants, lasted between 60 and 80 minutes, and was videotaped. Notes taken during the interview as well as student work produced were also kept as part of the case study record.

The first interview took place during the first two weeks of the course. The main objective of this interview was to find out how participants classified whole and rational numbers written in different notations (whole number, fractions, and decimals), how they understood fraction and decimal notation (including terminating and repeating decimals), and what they knew about the connections between fractions and decimals.

The first interview was organized around four tasks (Table 3). The format for the interview was semi-structured and the course of each interview was guided by participant responses. The main variation between interviews was the types of numbers participants were asked to classify, with these being adapted based upon the way they classified other numbers. The general goal was to find which numbers and notations were generally well-understood by the participant and which ones the individual preservice teachers found difficult or troubling. Unusual responses and/or those that represented misconceptions were tested for consistency by asking the preservice teacher to categorize more than one number of that given type. Due to time constraints, not all participants completed Task 3. The basic series of questions that guided Interview 1 are shown in Appendix F.

Table 3

The Four Tasks Completed by Participants in Interview 1

Task	Description
1	Define "number" then create a list of types or categories of numbers
2	Sort numbers into provided categories (whole, integer, rational, irrational, real, imaginary, unsure/not a number). Justify classifications when prompted by interviewer.
3	Show the relationship between the different categories of numbers.
4	Describe addition, subtraction, multiplication, and division for whole numbers and fractions.

Tasks 1 and 2 are the two tasks from the first interview relevant to this study, so they will be described in more detail. Data from Tasks 3 and 4 were not used in this study. Task 1 was designed to assess participants' initial conceptions of numbers. First, all participants were asked for their definition of number in order to allow them to describe numbers in their own words. Second, all participants were asked to create a list of different kinds of numbers. This was done in order to gain a more complete picture of the kinds of numbers that participants thought about under the umbrella term "number." By asking participants to describe what they meant by different kinds of numbers as they listed them (e.g. if a student wrote "rational numbers," they were asked, "What do you mean by rational numbers?"), this second question also created more opportunities for participants to describe their initial conceptions of numbers.

After Task 1 but prior to beginning Task 2, all participants were given the same set of formal definitions of numbers (*Figure 16*). Participants were provided with these definitions as it was clear from the pretests that the students did not have a common understanding of these sets of numbers at the start of the course and how participants reasoned about why various numbers fell into the given categories based on common definitions was of interest in this study. In addition to providing written definitions, each definition was discussed with the participants. They were asked if they understood the definition, and any questions they had were answered.

Whole numbers (sometimes called the ***counting*** or ***natural*** numbers): The positive integers, {1, 2, 3, ...}.
Integers: A number that can be written without a fractional part. *or* The positive and negative whole numbers and 0.
Rational Numbers: All numbers that can be written in the form $\frac{a}{b}$ where *a* and *b* are integers and $b\neq0$.
Real Numbers: A number that can represent a quantity along a continuous line.
Irrational Numbers: Real numbers that are *not* rational, or cannot be written in the form $\frac{a}{b}$ where *a* and *b* are integers and $b\neq0$.

Figure 16. Formal definitions of numbers provided to participants between Tasks 1 and 2 during the first interview.

Task 2 was designed to assess how participants reasoned about numbers presented in various notational forms as belonging to different sets of numbers. In Task 2, participants were given a variety of numbers written as numerals on small cards (*Figure 17*) and asked to sort them into the categories. Categories included whole numbers, integers, rational numbers, irrational numbers, real numbers, and imaginary numbers. An "unsure" or "not a number" category was added if students needed such a category during the course of their interview. As participants categorized numbers, they were frequently asked to justify their thinking about the categorization. They were not asked to justify every categorization in order to allow them to categorize a greater variety of numbers. They were particularly asked to justify whole or integer numbers that were not obviously written in standard form, and numbers they classified as rational or irrational. They were also asked about numbers they classified as real if they did not also classify the number as rational or irrational and/or if they had given non-standard reasons for classifying numbers as real earlier in the interview. Every participant classified at least one of each of the following: a whole number in non-standard form (e.g. $^{18}/_{3}$), a common fraction, a mixed number, a terminating decimal, a repeating decimal, the decimal form of pi, a non-standard fraction (e.g. $^{\pi}/_{2}$, $^{3}/_{-5}$), and a fraction with a denominator of zero. Based on their classifications and justifications, participants were asked to categorize additional numbers in order to test the consistency of their responses or to challenge justifications they had given. For instance, after one student stated that $^{\pi}/_{2}$ was irrational because no rational number could have π in it, she was asked to classify the number $^{\pi}/_{2\pi}$. One purpose of this activity was to document the ways that participants classified numbers in different notations. A second purpose was to document how participants responded when presented with notations that challenged the participant's stated ways of categorizing numbers. A third purpose was to assess when and if participants drew on class inclusion to reason about categorizing numbers (e.g. a participant could reason that if a number was rational, it also had to be real).

3.14	$\frac{\pi}{3}$	14.00	$\frac{2}{3}$
$\frac{18}{3}$	$-\frac{1}{2}$	72.3	7
$\frac{40}{41}$	$892\frac{5}{13}$	0.6215	$\frac{67}{94}$
0.66…	3.1415…	0.000003400	$0.\overline{583}$

Figure 17. Examples of numbers provided to participants during the first interview, written in the same way they were written on the cards given to participants.

The second interview was completed during the latter half of the course following the completion of the posttest for the number and numeration unit. The time between the posttest and the interview ranged from seven to twenty-six days, with all but two participants completing the second interview between seven and sixteen days following the posttest. Interview 2 was semi-structured and individualized based upon PSTs' responses to interview questions and their performance on the posttest.

The second interview was not designed nor intended to be the same as the first interview. As described in the previous section, the tasks in the first interview were designed to elicit the PSTs' initial understandings of connected aspects of number and numeration as knowledge of these initial conceptions is important for curriculum design. In contrast, the second interview was designed to support a deeper, richer exploration of the PSTs' understandings developed during the instructional sequence, particularly with regard to the connections among fractions, decimals, and the set of rational numbers. In this interview, the eight participants were given an opportunity to describe what they had learned from the instructional sequence in their own words as students' perceptions of what they have learned is one aspect of the nature of their understanding (Borko et al., 1992). Each individual's posttest was reviewed prior to the interview. Participants were asked to elaborate on and /or solve problems from the posttest where their understandings were unclear from the written work. Thus, Interview 2 was used as an opportunity to test conjectures made about the students' understandings as a result of observations made during the course and to verify with the students the accuracy of these conjectures. As such, the second interview served as a form of *member checking,* which is a way to increase the trustworthiness of the results (Patton, 2002). The basic series of questions that guided Interview 2 are shown in Appendix F.

Field Notes

As the researcher was a co-instructor in the instructional sequence and therefore actively interacting with students throughout the class period, field notes were taken during the class sessions were brief "jottings" about the development of student understandings that stood out and notes about what activities and representations were or were not supportive for different students. More extensive field notes were completed as soon as possible after each class. This is common in research where the researcher is a participant-observer (Patton, 2002). Meetings with the co-instructor related to the instructional sequence were documented in the field notes as well.

Research Design and Data Analysis Procedures

A case study design was used where the case of interest was the single class of preservice elementary teachers in the initial mathematics content course designed for prospective teachers. A case study design was used in this study in order to gain deeper insights into preservice elementary teachers' understandings of the connections between the various aspects of number and numeration related to rational numbers. As described previously, the researcher designed the pre- and posttest based on a review of the literature and with feedback from the instructor in the course. The researcher also designed and conducted all interviews. The researcher and instructor for the course acted as co-designers and co-instructors for the instructional unit. The analysis for this research was conducted using the transcriptions of the interviews, the pre- and posttests completed by the students, and field notes taken by the researcher during the implementation of the instructional unit.

To support the analysis of the data for this study, the software program MAXQDA for Mac was used. MAXQDA is a software problem designed to assist in the organization and analysis of qualitative data. The pre- and posttest data were all deidentified, scanned, and imported in to the MAXQDA database. The interviews were transcribed using F5 software and imported into the MAXQDA database as well. The use of F5 and MAXQDA software together allowed the researcher to easily view relevant excerpts of the video while coding the transcripts. The MAXQDA software also allowed for direct coding of the students' written work. This was important because the types of non-symbolic representations the students used, the ways they used them, and the ways they showed connections between representations was of interest to this study. These data could not easily be captured in text format alone.

The stages of the data analysis were guided by the 7-step method for analyzing video data described by Powell, Francisco, and Maher (2003). They developed the model for studying the development of mathematical thinking using video data and involves seven interacting, non-linear phases: (1) Viewing attentively the video data; (2) Describing the video data; (3) Identifying critical events; (4) Transcribing; (5) Coding; (6) Constructing the storyline; and (7) Composing the narrative (2003, p. 413). They note that this model is specifically meant to guide the phases of data analysis rather than act as a prescription for the analysis itself. These stages were used with both the transcripts generated from the video data and the written artifacts from the pre- and posttests.

Coding and analysis of the data proceeded in two phases using First Cycle and

Second Cycle coding methods (Saldana, 2013). In the first phase of the data analysis used a combination of descriptive coding and open coding methods (Saldana, 2013). First, a database of descriptive codes that summarized the areas of number and numeration of interest to this study was created in MAXQDA. These included "Notation," "Number," and "Representation," with sub-codes that referred to the specific aspect of each these areas of the *NNS*. For this study, *Notations* included fraction, decimal, and whole number notation. *Number* included the subdomains of the real numbers, including whole numbers, rational numbers, irrational numbers, real numbers. *Representations* included non-symbolic representations of numbers including pictures, words, stories (real world context), and concrete manipulatives. All data was coded using these descriptive codes in order to "fracture" the data in a way that would support coding for themes within and between these areas. Due to the fact that this study was designed to investigate preservice teachers' understandings of the connections within and between these areas of number and numeration, the majority of data segments were simultaneously coded in more than one category. The overlapping codes were used to identify areas of connection (or disconnection) in students' understandings.

After all data from the pretest, posttest, and interviews were coded using descriptive codes, the data were sorted based on the categorical codes. These sorted data were then coded using the constant comparative method (Strauss & Corbin, 1998). All data for the pretest and first interview were coded prior to coding the data for the posttest and second interview. This resulted in six major categories of codes: Understanding, representation, notation, number, intentional connection, and critical event. Each major category had several nested subcategories. The nested first cycle codes are shown in Appendix G.

Saldana (2013) states that the "primary goal during Second Cycle coding is to develop a sense of categorical, thematic, conceptual, and/or theoretical organization from your First Cycle codes" (p. 207). Pattern coding was used at this stage to reassemble the data into dominant categories and related subcategories. A challenge arose in collapsing the initial codes into non-overlapping categories based on the interconnected nature of the understandings being documented. After several cycles of review of the data corpus, eight larger themes were identified, along with several supporting subthemes. However, the subthemes could not be linearly aligned with the larger themes. Instead, different aspects of the subthemes aligned with different aspects of the larger themes. In order to capture the complexity of these relationships, two diagrams depicting the ways the themes related to the subthemes before and after the unit were created. The interconnections between subthemes and larger patterns was deemed appropriate and desirable given the study's stated purpose of documenting understanding from a *connected* perspective. *Figure 18* shows the codes and sub-codes that emerged from the second cycle of coding.

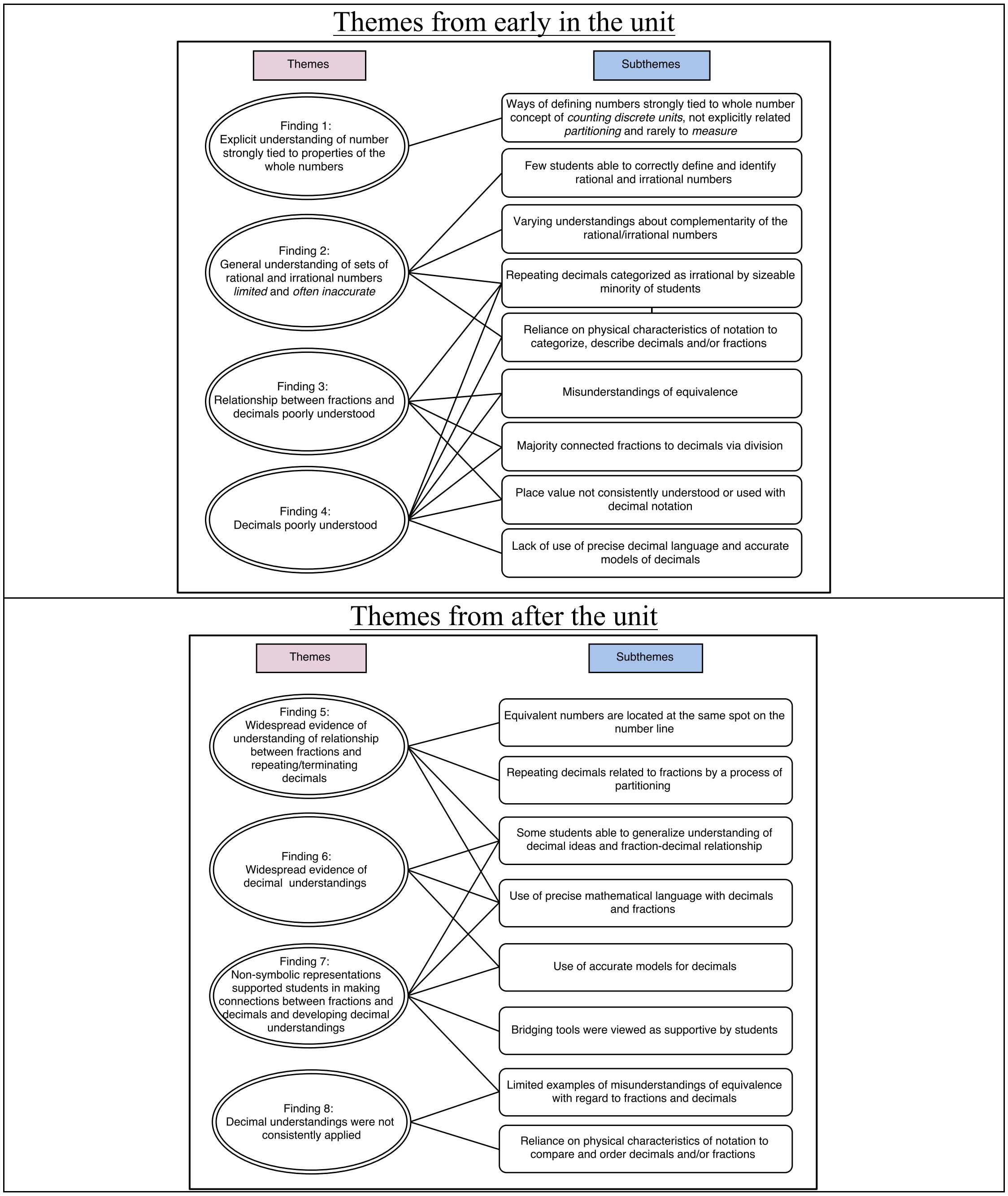


Figure 18. The major themes, supporting subthemes, and the connections between them that emerged from the Second Cycle of coding.

Role of the Researcher

In this study, the researcher was the curriculum designer and co-instructor in the course. As researcher-teacher, the investigator was also a member of the classroom and therefore was an active contributor to the mathematical practices that develop. Additionally, design and implementation of all activities were necessarily influenced by the researcher's knowledge of the students and classroom. Finally, it is to be expected that student interactions with the instructors could have been influenced by their roles as evaluators of student performance. Although the other course instructor was responsible for all grading for the course, students may still have perceived the researcher as being in an evaluative position due to the co-instruction.

Trustworthiness

Several methods were used to enhance the credibility of the study. First, the fact that multiple written records of the data analysis process along with the systematic process by which the data analysis proceeded together enhanced the credibility of the study as this record makes it possible to backtrack through the data to justify the claims and analysis. Second, the researcher's prolonged engagement with the participants of the study as researcher-teacher also enhances the credibility of the study (Lincoln & Guba, 1985) as first-hand experiences are a critical source of insight when attempting to account for students' understanding (Cobb & Whitenack, 1996).

Third, multiple forms of data were collected on the individual and class levels throughout the study, including videotape and transcript data of interviews, artifacts of written classwork, videotape and transcript data of class sessions, notes of meetings between the researcher and co-instructor of the course, field notes, and reflective journal kept by the researcher. The study also included multiple methods including interviews, observations, and written documents. Such forms of triangulation are another way that credibility may be enhanced in qualitative studies (Merriam, 2009).

The results of this study are not generalizable in the sense of generalizing to other populations due to the small number of cases and purposive sampling strategy used. However, as this study seeks to support the development of models of *connected* understanding for which no such models exist a priori, the use such strategy is appropriate as it supported the rich, thick description necessary for initial model development (Merriam, 2009). Thus the results of this study may be considered transferable to the extent that the findings may be used and tested in future teaching experiments, including those conducted by other researchers and/or in other locations. This is common in descriptive, qualitative studies wherein “generalization is accomplished by means of an explanatory framework rather than by means of a representative sample” (Cobb, 2003, p. 4).

Conclusion

This chapter summarized the research design, participants, intervention, data collection and analysis, and credibility and transferability of the study. Results from the study are described in the next chapter.

Chapter 4

Results

This chapter contains a description of the data collected during an eight-week instructional unit on number and numeration for preservice elementary teachers. The study was set in a mathematics content course for preservice elementary teachers as a large, Midwestern university. Data was collected from pre- and posttests designed by the researcher, from written artifacts produced during the instructional unit including classwork and homework, from field notes taken by the researcher, and from one-on-one interviews conducted with eight students from the course. A purposive selection strategy (Merriam, 2009) was used to select interview participants with relatively weak and relatively strong initial content understandings. The four interview participants who were selected because they displayed multiple misunderstandings on the pretest will be referred to as being in the *Mathematical Knowledge for Teaching 1 (MKT1)* group. The four students who were selected because they gave mostly correct responses on the pretest will be referred to as being in the *Mathematical Knowledge for Teaching 2 (MKT2)* group. Eva,[3] Nina, Willa, and Korey are in the MKT1 group, and Soren, Jo, Andie, and Mei are in the MKT2 group.

This study was guided by two research questions and their related sub-questions:

1. What is the nature of preservice elementary teachers' (PSTs) understanding of the *Number and Numeration System* before and after participation in a unit designed to develop their understandings of this content?
    a. What is the nature of PSTs understandings of the sets of numbers generally, and *rational numbers* specifically, early in the unit?
    b. What is the nature of PSTs understandings of fractions, decimals, and

---

[3] All names are pseudonyms.

the connections between them early in the unit?

c. What is the nature of PSTs understandings of fractions, decimals, and the connections between them after the unit?

2. What is the role of representations in the development of PSTs' understanding of the *Number and Numeration System*?
   a. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals before/early in the unit?
   b. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals after the unit?

In order to answer these questions, data from the interviews as well as from the pre- and posttest will be presented. The purpose of this study was to document the nature of preservice teacher understanding of the ways in which decimals and fractions and the set of rational numbers are *interconnected* before and after their participation in classroom experiences designed to support the development of these understandings from a connected perspective. Recall that two sets of activities related to developing preservice teachers' understandings of the connections between fractions and decimals were included in the unit. These are described in more detail in Chapter 3 as the *Breaking Bread* and number line activities. Questions related to the preservice teachers' understandings of the connections between fractions and decimals were included on the pre- and posttest, and data from these questions are presented herein.

As this study was interested with the development of preservice teachers' understandings of number and numeration from a connected perspective (*Figure 19*), data related to the preservice teachers' understandings of the concept of number as well as their understandings of the subsets of the real number system were included on the pretest and in the first interview. However, these understandings were only documented early in the unit as the instructional sequence did not include activities directly related to developing preservice teachers’ understandings in these areas.

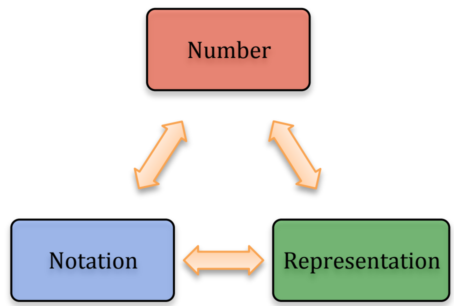


Figure 19. The *Number and Numeration System* depicts the ways knowledge of number, notation, and (non-symbolic) representations are interconnected.

The data in this chapter are organized according to participant responses to a selected set of questions from the first interview as well as the pre- and posttests. In all cases, interviewees’ responses are presented first, and information about how the whole class responded is presented afterwards if available. Data are organized in two sections based on whether they are related to understandings early in the unit or after the unit. Due to the fact that the preservice teachers’ understandings of the concept of number and the sets of rational and irrational numbers were only documented early in the unit, there is an asymmetry to these sections. The first section includes data from the first interview and pretest related to preservice teachers’ understandings of the concept of “number” and subsets of the real numbers, as well as data from these sources related to the preservice teachers’ understandings of fractions, decimals, and the connections between them. The second section includes data from the second interview and posttest related to the preservice teachers’ understandings of understandings of fractions, decimals, and the connections between them after their participation in a unit designed to support the development of these understandings. At the end of each section, the themes that emerged in that section are summarized. The ways that these themes relate to the research questions guiding the study are also addressed.

Understandings Early in the Unit

The following sections present the data the first set of individual interviews along with the responses to pretest Questions 1, 2, and 4 as the data from these questions relate directly to the research questions guiding this study. *Figure 20* shows pretest Questions 1, 2, and 4. The interview tasks are described with data to which they relate.

1. Give some examples and non-examples for each kind of number. If you are unsure whether a number is a given type, put it in the "unsure" category.

Whole Numbers	NOT whole numbers	Unsure

Rational Numbers	NOT rational numbers	Unsure

Integers	NOT integers	Unsure

Irrational Numbers	NOT irrational numbers	Unsure

2. Are there any numbers between 0.66 and ⅔? If so, give an example and briefly explain how you know it is between 0.66 and ⅔ . If not, explain why not.

4. Tell how you would help a student understand why ⅓ =0.333… when written as a decimal.

Figure 20. Questions 1, 2, and 4 from the pretest.

Interview task: Definitions of Number and Types of Numbers

Early in the first interview, after asking participants to describe their mathematical backgrounds, feelings about mathematics, and their ability levels, every student was asked to define the term "number." Students who gave a definition without elaboration were prompted to explain their thinking, and asked, "Is there anything else a number could be?" in order to get as complete an idea as possible of their initial intuitions about the defining characteristics of numbers. After they wrote down and/or discussed their definition, they were then asked to create a list of all the kinds of numbers they could.

The primary purpose of asking students to define "number" was to document the types of properties that they could explicitly identify as being important to the concept of number. The purpose of asking them to list kinds of numbers was two-fold. First, this supported the documentation of ways in which the preservice teachers' explicit definitions of number were related to the types of numbers listed. Second, it was expected that students might find the task of defining numbers to be challenging and would find it easier to give examples of numbers. These examples could therefore provide more holistic understanding of the preservice teachers' initial intuitions of what numbers are.

When asked to define *number*, all eight interviewees indicated that they found the question difficult, but had an easier time giving examples of numbers. As shown in Table 4, all eight interviewees gave remarkably similar initial definitions of number. In particular, every student defined a number as a way to represent a quantity and/or as a way to keep track of a count of discrete objects. For seven of the eight, this was their initial definition of number, and it was the follow-up statement made by Mei. All eight students either stated directly that numbers are used to count or gave an example that involved counting a discrete quantity (e.g. "two chairs," "three ducks," "a quantity of markers"). In other words, the idea that *numbers are related to counting* was evident in all eight preservice teachers' definitions of number and was the initial intuition upon which they drew to define number. All eight students also tried to expand on the idea of *numbers as counts* in some way. Only two of the eight candidates, Soren and Andie, expanded their definition to include the notion that number can be a *measure*. Andie made a distinction between what she called a number as a "value," by which she meant “something on a number line,” and a number as “quantity,” by which she meant something “physical, like I have three pens in front of me.” Soren gave examples of real world situations where numbers denote *measurements* rather than *counts*, such as a “quantity of coffee” or “quantity of heat energy,” and explicitly named this as “measurement.” No other student included this notion of *number as measurement* in their definition of number, but all eight included sets of numbers such as fractions, decimals, rational and irrational numbers, that are all more meaningfully interpreted as measures than as counts.

One student, Willa, also specifically included the idea that a “quantity” could represent more than just a count of discrete, real objects. In particular, she argued that negative numbers can be thought of as a kind of quantity by thinking of them as debt, so “like if you have negative three, you’re three dollars in debt. So the quantity is you owe three dollars.” She also noted that she thought of fractions and decimals in a similar manner, stating, “I think about it is mostly in terms of money. Because if you even have like decimals or fractions/ say you have a third and you have like thirty-three cents or whatever.” Although she was not able to clearly articulate what a general notion of “number as quantity” entailed, she did make it clear that she had an intuitive understanding that the whole numbers, integers, and rational numbers are related by a more generalized notion of number than just counting discrete objects. What she did not do was clearly articulate how the negative integers are the same as *and different from* the whole numbers and how the rational numbers (fractions and decimals) were the same as *and different* from the integers. Thus, she had a supportive intuition about these sets of numbers at the start of the course, but did not yet express the understanding necessary to support her future work teaching these concepts to children.

The remaining five students did not define numbers in ways related to any of the other subdomains of the real number system. Six of the eight students (all but Eva and Korey) did note that they could think of either specific numbers or categories of numbers that did not fit their definition (and/or the idea of a *number as a count*). Soren and Willa both noted that imaginary numbers were in this category. Nina and Andie both noted that the trigonometric functions (e.g. sine, cosine, tangent) seemed to be a different type of number. Similarly, Willa also noted that a number like “3x” is a different kind of number because it represents “what you need to do.” Jo noted that pi is “such a different kind of quantity” than a counting number but justified this based on the fact that “it’s not complete.” Thus, she did not categorize pi as an example of a *measure*, although that is, in fact, an important characteristic, but instead noted that its *notation* was different because “it’s not complete.” Mei noted that numbers can be “road numbers, house numbers, things like that,” and was the only interviewee to reference the idea that numbers can act as *labels*.

Eva and Korey did not try to extend their definition of number to include numbers beyond counting numbers. Eva simply stated, “It’s a tool,” and gave counting as her only example of using a number as a tool. Korey added the idea that “numbers represent money” to her initial idea that numbers are used to count objects, but did not offer a substantive explanation of how numbers that “represent money” are different from or similar to numbers that count objects.

Table 4

Summary of Ways Interviewees Defined "Number" and the Categories of Numbers They Listed During the First Interview

Student	Definition of number	Categories of numbers (in order stated/listed)
Soren	"*A representation of quantity*…So just like the *quantity of markers*. Or like volume, the quantity of coffee. Or I guess, that's not really right, in terms of like temperature. The quantity of heat energy. …. Well, I mean, I guess, *measurement*." Also, a number can be "a concept, rather than, *you know, like imaginary numbers that aren't really like either of these things*. Just kind of like an idea."	Negative numbers, positive numbers, rational numbers, irrational, whole, integers, imaginary
Jo	"I would say it's the *symbol of a quantity*….There's so many different kinds. Because some numbers, you can, like whole numbers, *I can count*. Like I have *five things*. But then think of like, *pi is a number*, but I wouldn't be able to say that that's like a/ I mean it is a symbol of a quantity, *but that's like such a different kind of quantity*, I guess, because it's not complete."	Whole numbers, integers ("positive and negative, the counting numbers again"), rational, irrational, imaginary, fractions
Andie	"I would say a number is like *a symbol to represent a quantity* maybe, or value.?...I guess when I think of value I think of *something on a number line*. And when I think of quantity, I think of physical, like I have *three pens* in front of me….I'm not sure where like sine and cosine and tangent would fall."	Whole numbers, fractions or decimals, irrational, imaginary ("are irrational and imaginary the same thing?"), negative numbers, sine or cosine
Mei	A number is "kind of like for defining the number of stuff…. *Like how many it is of it*." Numbers can also be "like road numbers, house numbers, things like that."	Positive numbers, negative, rational, irrational, whole numbers, the opposite of a whole number (e.g. 1.2, ⅓)
Eva	"Like for *how many chairs are in this room*, if I wanted to tell someone upstairs how many chairs there are. Instead of saying, 'I don't know. Can you just come down here and count them?*' I could count them* and there is a way to categori/, or to explain, instead of/ It's a tool?"	Whole numbers, negative numbers, rational, irrational, fractions, decimals, symbols
Nina	"A symbol that *represents an amount of something*….Like if there's *two chairs*, you'd be like there's two. So it's like you're grouping them. It represents an amount so when you look at something you're able to put a word to what you're seeing.…Well, *it represents money*." A different kind of number would be "if you put words with the numbers. Like sine of three pi, that is different….Yeah, it will give you a number as an answer, but you have to figure out what it is. Whereas these kind of just tell you."	1, 2, 3, 4, pi ("just a different symbol"), sine 3π, fractions, whole, integer
Willa	"A number is used to *represent a quantity*…Something like if you have negative three, you're three dollars in debt. So the quantity is you owe three dollars. And the way *I think about it is mostly in terms of money*. Because if you even have like decimals or fractions/ say you have a third and you have like thirty-three cents or whatever. I feel like it just can work with most anything….It could be/ it's like a representative of anything. Like if you have *three ducks*, three/ But it can also represent what you need to do. So if you have like three x, you would multiply it by three, whatever x is…. So it really depends on you're using the number for….I suppose when you take *imaginary numbers, it would be completely different than this* [pointing to definition]."	Real numbers, "whole numbers like pi," imaginary, repeating decimals, negative numbers
Korey	"A number represents something, anything, like a object or like there's one of something or there's two of something, so maybe *it represents like an object*….*Numbers are used to count*, we need numbers to count things. To find out how many things you have. *You need to know numbers for money*, I would say, they represent money. I'm always telling my daughter numbers represent money so you need to learn about numbers"	1, 2, 3, …, 5, 10, 15,…,skip counts by tens or twenties, "the in between kind of numbers" (e.g. 1.5, 2.5), fractions, numbers in the thousands, negatives

Summary of findings related to the definitions of numbers. The primary theme to emerge from the data related to the preservice teachers' understanding of number early in the unit was that their *general understanding of number was strongly tied to conceptions of whole numbers.* In particular, the idea that numbers are related to *counts of discrete units* was expressed by all eight interviewees. Moreover, although six of the eight students noted in some way that this notion of *numbers as counts* was incomplete, only two of these students (Andie and Soren) extended their definition to include the conception of *number as measure*, a critical understanding related to understanding the set of real numbers. Notably, *no student described numbers as being related to partitioning*, a critical conception related to understanding the rational numbers (Mack, 1993).

Pretest Question 1: Examples/Non-Examples of Numbers

On Question 1 of the pretest, students were asked to list examples and non-examples of whole numbers, integers, rational, and irrational numbers. The purpose of this question was to document the types of numbers and notations that students associated with the various subdomains of the real numbers at the start of the course. With regards to this study, there was particular interest in documenting when and how students included (or excluded) fractions and decimals from the set of rational and irrational numbers.

In the following sections, a summary of the types of examples and non-examples that the preservice teachers gave of rational and irrational numbers as part of their response to Question 1 will be presented (see pretest in Appendix B). Relevant excerpts from the first interviews as well as a summary of the whole class' response will be included in order to give a more complete picture of the preservice teachers' understanding of rational and irrational numbers at the start of the course.

Interviewees' responses to pretest Question 1. This section will present a summary of each of the eight interviewees' understanding of rational and irrational numbers based on the examples and non-examples they provided on the pretest as well as ways they talked about these sets of numbers during their individual interviews. Table 5 shows the examples and non-examples of rational and irrational numbers provided by the eight interviewees. All numbers are presented exactly as written by the students. As shown in Table 5, three of the eight interviewees, Soren, Jo, and Willa, provided correct examples and non-examples of rational and irrational numbers on the pretest. However, as will be shown in the following sections, the interviews with Jo and Willa revealed that the two women had misconceptions about the sets of rational and/or irrational numbers at the start of the course that were not documented in their response to this question on the pretest.

Table 5

Interviewees' Examples and Non-Examples of Rational and Irrational Numbers on the Pretest

Group	Student	Rational			Irrational		
		Correct?	Examples	Non-examples	Correct?	Examples	Non-examples
MKT2	Soren	Yes	$\frac{4}{3}, \frac{1}{2}, \frac{6}{4}, 3$	*e, π*	Yes	*e, π*	2, -4, 3, 0
	Jo	Yes	$6.5, \frac{3}{4}, 8$	π, *e*	Yes	π	7, 3.2
	Andie	Yes	-3, 6, 10.2	$2i, -3\sqrt{i}$	No	$\mathbf{3}\boldsymbol{i}, \boldsymbol{\sqrt{i}}, \mathbf{.3\overline{3}}$	6
	Mei	No	0, 1, 2	$\mathbf{\frac{1}{3}, \frac{1}{7}}$	No	$\mathbf{\frac{1}{3}, \frac{1}{7}}$	0, 1, 2
MKT1	Eva	No	.83, 3.7, 8, $\frac{3}{4}$	$\mathbf{\frac{1}{3}, .\overline{6}666}$	No	$\mathbf{\frac{1}{3}, .\overline{1}111}$	$\frac{1}{2}, .6, 40.7$
	Nina	Yes	2, 4, 5	π	No	$\mathbf{-.3}, -\sqrt{2}$	2, 4, 5
	Willa	Yes	4.5, 7	π	Yes	π	6.2
	Korey	No	2, 4, 6, 8 "not sure yet"	**1, 3, 7, 9** "not sure yet"	No	**1, 3, 7, 9** "not sure yet"	2, 4, 6, 8 "not sure yet"

Note. Bolded numbers indicate numbers that are not correct. "Correct" means that all examples and non-examples for both categories are correct. All numbers are typed exactly as the student wrote them.

Soren was one of the three interviewees who gave correct examples and non-examples of rational and irrational numbers on the pretest. Notably, he included in his list common fractions related to both terminating and non-terminating decimals as examples of rational numbers (i.e. $\frac{4}{3}$ and $\frac{1}{2}$). During his interview, when asked to define rational numbers, Soren responded, "numbers that can be written as fractions." This definition is both in keeping with his pretest responses and very similar to the standard textbook definition of rational numbers being "any number that can be written in the form a over b, where a and b are integers and b is not zero."

During his interview, Soren defined irrational numbers as "numbers that can't be written as fractions." He also knew that both *e* and π were considered irrational numbers and that this meant that they could not be written as a fraction. However, he could not explain why this was the case and knew little about the two numbers. When asked what he meant by *e*, he responded, “What is *e*? I have no clue what *e* is.” When asked what he knew about pi other than its decimal expansion, he responded, “I mean it’s everywhere with circles. Two pi *r* is circumference.” Thus, although Soren was not entirely clear about why his two known examples of irrational numbers were irrational, he was clear that rational numbers can be written as fractions while irrational numbers cannot. He was the only interviewee to give this response. He was also only interviewee to demonstrate a correct understanding of rational and irrational numbers at the start of the course on both the pretest and during the interview.

Like Soren, Jo gave correct examples and non-examples of rational and irrational numbers on the pretest. Notably, Jo included a whole number, fraction with a commonly known decimal equivalent, and a terminating decimal in her list of rational numbers. This list showed that she had an understanding that whole numbers, fractions, and terminating decimals can be part of the same set of numbers at the start of the course. However, she did not include repeating decimals nor their fraction equivalents anywhere in her lists.

The significance of her exclusion of repeating decimals and their fraction equivalents from her list became apparent early in the first interview when she was asked to describe rational and irrational numbers.

> J: “So, rational come to an actual end, whereas irrational go on. So pi would be irrational because it keeps going on forever and gets unpredictable as it keeps going.”
> I: “What about point three repeating? Three three three three three?”
> J: “Um. (…) I think that’s still rational, if I remember correctly, because it’s predictable, so you know it’s always going to be three forever.”
> I: “So rational numbers come to an end, but that doesn’t come to an end.”
> J: “So it comes to an end or has a predictable pattern?”

I: “Okay.”
J: “But then I know we can predict the next numbers of pi so now I’m second guessing that.”
I: “Or what if it was a number like this: one two, one one two, one one one two, one one one one two. That’s predicable. What do you think, is that rational or irrational?”
J: “Uh (…) Let’s go (…) I don’t know.”

A short while later, when Jo was asked about fractions, she then “reversed” her decision to include ⅓ in the set of rationals.

I: “What are fractions?”
J: “They (…) are like one (…) whole number over/ one integer over another. And then that can be simplified to a rational or irrational number.”
I: “So could fractions be rational or irrational?”
J: “Yes.”
I: “Can you think of a rational fraction?”
J: “Three-fourths.”
I: “Can you think of an irrational fraction?”
J: “I want to say one-third is irrational. But now I’m changing what I said before.”

Clearly, Jo was trying to use properties of the decimal notation, such as whether it terminated or was “predictable,” to classify numbers as rational or irrational, but she was struggling with the limitations of this approach. She was, however, clear that there were inconsistencies in her definitions. Note that, unlike Soren who defined rational numbers based on whether or not they could be written as a fraction, Jo’s definitions were focused on the decimal form of the number. Furthermore, Jo’s difficulties were specific to *non-terminating* decimals and their related fractions. Finally, Jo’s statement that fractions such as ¾ and ⅓ can be considered rational and irrational, respectively, demonstrates that she did not have a clear understanding that all fractions[4] are rational numbers. Thus

[4] Unless specified, “fractions” will mean numbers written in the form a/b where a and b are integers and b is not equal to zero.

although Jo's examples and non-examples of rational and irrational numbers were correct on the pretest, her interview quickly made it clear that she did not have a good understanding of these groups of numbers.

As shown in Table 5, Andie gave correct examples and non-examples of rational numbers on the pretest. However, she included the repeating decimal $0.\overline{3}$ as an example of an irrational numbers. Thus, like with Jo, the fact that Andie included only integers (-3 and 6) and a terminating decimal (10.2) as her examples of rational numbers was significant. During her interview, Andie verified that she considered repeating decimals to be irrational.

> Gosh. I remember talking about irrational numbers in calculus. So are irrational, I can't remember, is an irrational number *a number that doesn't end*? So like it would be like *three point three three three repeating*? (Andie, interview 1, emphasis added)

Note that Andie, like Jo, characterized an irrational number in terms of its decimal expansion being non-terminating. She also made it clear that she was uncertain about her response. Unlike Jo, Andie did not try to further delineate between non-terminating, repeating decimals like $0.\overline{3}$ and non-terminating, non-repeating decimals like π. She did, however, wonder if there was a difference between imaginary and irrational numbers, which is in keeping with the fact that she also listed imaginary numbers as both examples of irrational numbers and non-examples of rational numbers on her pretest. Thus, Andie's responses on the pretest and during the interview indicate that she certainly did not have clear understanding of characteristics of rational and irrational numbers at the start of the course, and that she was attending to characteristics of the decimal notation itself as she tried to reason about these sets of numbers. The extent of her understanding (or misunderstanding) is unclear since she only gave the one example ($0.\overline{3}$) with little other explanation during the interview.

Mei gave correct examples of rational numbers on the pretest, but her list was limited to the whole numbers 0, 1, and 2. She then listed the fractions $\frac{1}{3}$ and $\frac{1}{7}$ as not rational (and irrational), so her answer to this question was considered incorrect. However, it was unclear whether she meant to exclude fractions in general from the set of rational numbers or if she was thinking more specifically in some way about characteristics of those fractions. She also did not list any decimals in those categories. Furthermore, it should be recalled that Mei was studying abroad from an Asian country and spoke English as an additional language.

Her interview did not help to clarify her thinking, mostly because she found it difficult to describe her thinking about these numbers and terms in English. The following exchange took place early in the interview, immediately after she had listed "opposites of whole numbers" as a type of number.

> I: "What's a whole number?"
> M: "It's like one and two."
> I: "Okay, so what would be an example of something that was an opposite of a whole number?"
> M: "Like one point two, or one third, or something like that."
> I: "Okay. What would you say are rational numbers?"
> M: "Rational number is like one and two and something like that."
> I: "Okay. And how about irrational numbers? What would you say those are?"
> M: "It would be like one-third and one-seventh."
> I: "Okay. Would that help you with the opposites?"
> M: "I don't know what the opposite word is to say that."

Given the fact that Mei did not speak English as a first language and her difficulties in trying to articulate what she meant by "opposite of a whole number," it is not possible to draw conclusions about Mei's understanding of the sets of rational and irrational numbers at the start of the course based on her responses to this pretest question or the interview.

Like Soren, Jo, Andie, and Mei, Eva also listed correct examples of rational numbers on her pretest. Notably, she included a whole number (8), terminating decimals (3.2 and .83), and a fraction (¾) in her list, indicating that she did understand that the different notations could be related to one another and to the set of rational numbers. However, she then listed both the fraction ⅓ and the repeating decimal $0.\overline{6}$ as non-examples of rational numbers, and ⅓ and $0.\overline{1}$ as examples of irrational numbers. Thus, Eva's responses on the pretest indicated that she was distinguishing between terminating and repeating decimals (and their fraction equivalents) when she characterized rational and irrational numbers.

The fact that she was attending to whether or not the decimal notation terminated was verified during the first interview. When asked what rational numbers were early in the interview, she replied, "Numbers that terminate…. As opposed to *an irrational number that does not terminate*. Like the square root of three, I think. A number that continues on and does not have an end" (emphasis added).

However, shortly after making that statement, Eva went on to argue that repeating decimals "terminate."

E: "So if it were, if you're taking one third, or two thirds and divided it, it would terminate. Or have that bar over…."
I: "Okay. So you said the three with a bar over it. What does that mean?"
E: "Repeating."
I: "Repeating. So is that a number that terminates or not?"
E: "I never thought it was, but I think, I think it terminates. You can say it terminates? That one I don't remember."

Eva's argument that repeating decimals *can be thought of as terminating* and so are rational, is a clear shift from her pretest responses. It is not clear why she made that shift. It is also unclear here what Eva meant by the idea that a repeating decimal can be thought of as terminating. Regardless, it is clear that Eva did not have a clear understanding of the relationship between repeating decimals and rational numbers at the start of the unit.

Like Mei, Nina, also listed whole numbers only (i.e. 2, 4, 5) as examples of rational numbers, and she listed "π" as a non-example of a rational number. Interestingly, she did not list π as irrational, but rather listed two negative numbers: -.3, and $-\sqrt{2}$. Unlike the other interviewees, Nina did not give an explanation for why she included the terminating decimal "-.3" as irrational during her interview, but its inclusion in her list does indicate that she certainly did not have a fully correct understanding of rational (and irrational) numbers at the start of the course.

In fact, she indicated that she could remember little about these sets of numbers. Early in the interview, when she was asked about fractions, she responded,

> Gosh that part of the test, I did not remember how to solve that. I knew what whole numbers were, that was it. When I think of fractions, I always just think of division. Or I always think of them (incomprehensible) to decimals. It's a lot easier for me. (Nina, interview 1)

She did not elaborate further and she did not describe what she thought rational and irrational numbers were prior to me giving her the definitions. Thus, it is not possible to tell what kinds of decimals Nina meant to include in the set of irrationals; it is even possible that she meant only to include the *specific* decimal "-.3." What is clear is that in any case, she certainly did not have a correct understanding of the set of rationals and irrationals at the start of the unit.

Willa was one of the three interviewees who gave correct examples and non-examples of rational and irrational numbers on the pretest. In particular, she listed "4.5" and "7" as examples of rational numbers, and "π" as a non-example of rational and example of an irrational number. Like Jo, Willa's non-inclusion of repeating decimals or their fraction equivalents proved significant during her interview. The following exchange occurred early in the first interview, while she was talking about different types of numbers she knew.

W: "And then real numbers. It's kind of the same as whole. But I don't remember if real includes negative numbers. I feel like it's just anything without *i*. But it's also like a whole number. I feel like pi is an imagin/[5] No. I feel like, yeah. Because if you have *like three point one four that's a whole number*. But then if you have pi, it goes into a separate category because it's repeating." (emphasis added)

I: "So both four and three point one four would be whole numbers? Can you tell me a little bit more about what whole numbers are?"

W: "Or maybe that's a real number." [Seemed to be referring to pi, but not completely clear].

I: "What makes a number a whole number?"

W: "Well, a whole number, it's/ If *it has a set quantity*. Because *if it repeats, it's not whole* because it's never ending and you can't cut it off" (emphasis added)

I: "But ones that end, like this one [pointing to examples on paper]. Four ends. Three point one four ends. Then it?"

W: "Like you can put something in front of yourself to represent that number."

As demonstrated in the above exchange, Willa was reasoning *directly from the decimal notation* to explain why a number was rational or not. She described whole numbers (which notably terminating decimals) as numbers that have "a set quantity," where "you can put something in front of yourself to represent that number." In contrast,

[5] The "/" is used throughout the interviews to indicate partially-completed sentences. When interviewees began a new sentence partway through a word, the partial word is written to indicate where the interviewee left off.

repeating decimals were not "whole" because their decimal notation did not end. Thus, although Willa's examples and non-examples of rational and irrational numbers were correct on the pretest, her interview quickly made it clear that she did not have a good understanding of these groups of numbers.

Unlike the other seven interviewees, Korey listed only whole numbers as her examples and non-examples of rational and irrational numbers. In fact, she listed the even numbers 2, 4, 6, and 8 as examples of rational numbers and the odd numbers 1, 3, 5, and 7 as non-examples (and vice-versa for the irrationals). During her interview, she verified that she had intentionally split the sets of numbers in this way. Early in the interview, she was asked what she remembered about the rational and irrational numbers. She responded,

> I remember learning them. Um (…) I feel like a rational number/ I put even numbers because I was thinking it had something to do with you had to split them apart or something. Like they have to divide into each other. Or be able to half, like you can half them. And irrational numbers you can't. That's what I was thinking. (Korey, interview 1)

Korey's idea that rational numbers "had something to do with you had to split them apart or something" is a beneficial intuition about the nature of rational numbers. In fact, Moss and Case (1999) describe "a numerical structure for 'splitting' or 'doubling'" (p. 125) as one of the two primitive psychological units for rational numbers. The fact that Korey splits in *half* in particular, is in keeping with Confrey and Smith's (1994) proposal that "doubling and halving are the most primitive of units in a splitting world" (p. 44). Thus, although her response is clearly incorrect, Korey's reasons for thinking about rational numbers as even numbers in in keeping with important fundamental properties of the set of rational numbers.

Whole class response to pretest Question 1. Table 6 shows a summary of the whole class' responses to question one on the pretest where they gave examples and non-examples of rational and irrational numbers.

Table 6

*Summary of the Whole Class's Examples and Non-Examples of Rational and Irrational Numbers on Pretest Question 1 (*N=*32)*

	Rational			Irrational		
		Sample response			Sample response	
Type of response	Number	Examples	Non-examples	Number	Examples	Non-examples
Correct examples and non-examples	12	½, 4, .25	$\sqrt{-5}$	10	π	100
Incorrect non-example, correct example	11	$\frac{2}{3}, \frac{1}{10}, \frac{7}{9}$	1, 3, 5	0		
Incorrect example, correct non-example,	0			12	$4.\overline{6666}, \pi$	5, 10
Incorrect examples and non-examples	0			0		
Other	3	$\sqrt{64}$	*blank*	2	*blank*	$\sqrt{i}$
Blank	6			8		

Note. All numbers are typed exactly as the student wrote them.

The fact that only twelve of the 32 students (38%) provided correct examples and non-examples of rational numbers and demonstrates that rational numbers certainly were not well understood as a set of numbers by this set of students at the start of the unit. Furthermore, of the twelve students who gave correct examples and non-examples of rational numbers, only two of these, including Soren, included fractions with repeating decimal equivalents in their examples. This was significant because the most common way that students in the class showed a misunderstanding of the sets of rational numbers on the pretest was by either excluding repeating decimals or their fraction equivalents from the set of rationals and/or including them in the set of irrationals (see Table 7). A total of six students made this error on the pretest. Combined with the fact that Willa and Jo both revealed a similar understanding of repeating decimals being irrational during their interviews, this means that at minimum eight of 32 (25%) of the class started the course believing that repeating decimals and/or their fraction equivalents were not rational and/or irrational.

Similarly, only ten of 32 students (31%) gave correct examples and non-examples of irrational numbers. Of these, seven included π as their sole example of an irrational number. One additional student, Soren, included π and *e*. The remaining two wrote "8.4328…[6]" and "$\sqrt{7}$." This does not suggest a robust understanding of irrational numbers. Certainly these students may have known more about irrational numbers but chose to only list π as their example. However, even if this were the case, it would still be only ten of 32 students with a correct understanding of irrationality at the start of the course. Moreover, recall that Soren, who was chosen as an interview participant because of his strong mathematical background and understandings, could explain neither *e* nor π when asked. (i.e. "What is *e*? I have no clue what *e* is" and pi is "everywhere with circles. Two pi r is circumference"). Overall, the irrational numbers were also not well understood as a set of numbers by this set of students at the start of the unit.

Finally, an uncommon, but notable, misunderstanding was the idea that rational numbers are even numbers. Three students out of 32 (9%) gave a response in keeping

---

[6] This student is being given the benefit of the doubt and assuming she means that the decimal does not terminate or repeat. She did write repeating decimals using the bar over the numbers as non-examples of rational numbers, suggesting that she would use the bar notation if she meant to signify that she meant the decimal to be repeating.

with this belief. As argued in the section above, Korey's statement that rational numbers have "something to do with you had to split them apart or something" is actually in keeping with the notion that "a numerical structure for 'splitting' or 'doubling'" (Moss & Case, 1999, p. 125) is one of the two primitive psychological units for rational numbers.

Table 7

Summary of Misunderstandings of Rational Numbers on Pretest Question 1 (N=15)

Type of response	Number[a]	Responses[b]	Note
Listed repeating decimals or their fraction equivalent as non-examples of rational	5	$\frac{1}{3}, \frac{1}{7}$ (Mei) $\frac{1}{3}, .\overline{6}666$ (Eva) π, $3.\overline{3}$ $7.\overline{6}, 11.\overline{34}$ 1⅓	3 of these students also listed repeating decimals or the equivalent fractions as examples of irrational numbers.
Listed repeating decimals or their fraction equivalent as examples of irrational	4	$3i$, $\sqrt{i}$, $.3\overline{3}$ (Andie) $\frac{1}{3}, \frac{1}{7}$ (Mei) $\frac{1}{3}, .\overline{1}111$ (Eva) $4.\overline{6}$, π	3 of these students also listed repeating decimals or the equivalent fractions as examples of irrational numbers.
Listed whole numbers as non-examples of rational	5	∞, -8 -1, 11, 13 1, 3, 5 1, 3, 5 1, 3, 5, 7 (Korey)	3 of these students also listed odd numbers as examples of irrational numbers.
Listed whole numbers as examples of irrational	3	∞, -8 -1, 11, 13 1, 3, 5, 7 (Korey)	All these students also listed these same numbers as non-examples of rational numbers.
Listed a terminating decimal as non-example of rational	1	1.4, π	This student listed ⅔ and $^4/_6$ as examples of rational numbers.
Listed a terminating decimal as example of irrational	1	$-.3, -\sqrt{2}$ (Nina)	Nina listed whole numbers only as her examples of rational numbers.
Listed improper fractions as example of irrational	2	$\frac{10}{9}, \frac{3}{2}, \frac{24}{8}$ $\frac{7}{2}$	Both students also listed proper fractions as examples of rational numbers.

Note. All numbers are typed exactly as the student wrote them. All numbers the student wrote are included in example, even if correct.
[a]Total is more than 15 since some students counted in multiple categories.
[b]If a response was given an interview participant, their name is shown in parentheses. All other responses were from other students participating in the study.

While excluding repeating decimals from the set of rationals (or including them in the irrationals) was the most common way that students demonstrated a misunderstanding of the set of rational numbers on the pretest, the next most common was excluding whole numbers. As shown in Table 7, a total of five students listed whole numbers as non-examples of rational numbers, three of whom also listed whole numbers as examples of irrational numbers. Notably, three of the five students, including the interview participant Korey, wrote *even numbers* as examples of rational numbers and *odd numbers* as non-examples. As only Korey was interviewed, it is not possible to state with certainty that the other two students intended to suggest that rational numbers were even and irrational odd.

Summary of findings from pretest Question 1. The primary theme to emerge from students' responses to pretest Question 1 was that their *general understanding of the sets of rational and irrational numbers may be characterized as limited and often inaccurate.* This theme included three related sub-themes. First, only twelve of thirty-two students were able to provide correct examples and non-examples of rational numbers on the pretest. Furthermore, only one of the eight interviewees (Soren) was able to provide both correct examples and non-examples of rational and irrational numbers on the pretest and an accurate definition of both sets of numbers during the interview. Thus, *few students were able to correctly define and give examples of rational and irrational numbers at the start of the course.* Second, *repeating decimals were categorized as irrational* by many students. One-quarter of the students explicitly named repeating decimals as irrational and/or not rational either in their response to pretest Question 1 or during the first interview. Only two students explicitly included repeating and terminating decimals or their fraction equivalents in their examples of rational numbers. Third, four of the eight interviewees (Jo, Andie, Eva, and Willa) *relied on the physical characteristic of a given number's notation to categorize it as rational or irrational* during the portion of the first interview reported above. In particular, all four considered whether or not a decimal terminated to be a defining and distinguishing characteristic of the set of rational numbers at least once during the interview.

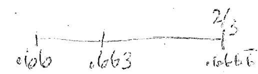
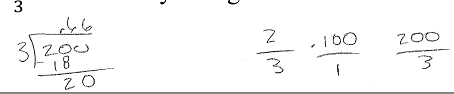
Pretest Question 2: Is There a Number Between 0.66 and ⅔?

Question 2 asked: "Are there any numbers between 0.66 and ⅔? If so, give and example and briefly explain how you know it is between 0.66 and ⅔. If not, explain why not." The purpose of this question was to see if and in what ways preservice teachers distinguished the two numbers since they are not equivalent but are often used interchangeably in practice. In the following sections, a summary of the ways the eight interviewees responded to this question will be presented first. Then a summary the ways the whole class responded will be presented in order to give a more complete picture of the nature of the preservice teachers' understanding of the relationship between 0.66 and ⅔ at the start of the unit.

Interviewees' response to pretest Question 2. Table 8 shows the eight interviewees' responses to pretest Question 2.

Table 8

Summary of Ways Interviewees Responded to Pretest Question 2

Group	Student	Are there numbers between .66 and ⅔?	Reason
MKT2	Soren	Yes	"0.662 is between 0.66 and ⅔ because ⅔ =$0.\overline{66}$."
	Jo	Yes	"⅔ is really a number that continues after .66 when made into a decimal. Therefore .663 would be between .66 & ⅔ drawn on a number line." 
	Andie	Yes	$\frac{1}{3} = .3\overline{3}$ $\frac{2}{3} = .6\overline{6}$ "Yes because $\frac{2}{3} = .6\overline{6}$, therefore .66 ends with a 6 in the 100ths place, while $\frac{2}{3}$ or $.6\overline{6}$ continues infinitely."
	Mei	Yes	"Yes. For example, 0.665 is between 0.66 and ⅔. Because ⅔ is irrational number, it's 0.66666…. There're numbers that are larger than 0.66 but smaller than 0.66666…. One example is 0.665."
MKT1	Eva	No	"⅔ = .66 no numbers exist between them."
	Nina	No	"No, ⅔ is equivalent to $0.66\overline{6}$. I know this because $\frac{2}{3} \cdot 100$ then by doing 200 ÷ 3 = $.66\overline{6}$." 
	Willa	Yes	"Yes, 0.661 would be between them since ⅔ is $0.\overline{66}$."
	Korey	Yes	"Yes, 0.665. I know this because ⅔ is $.66\overline{6}$ and decimal places can go on for awhile."

Notes. Spelling, grammar, and punctuation are preserved. Numbers are represented exactly as each student wrote them. If the student included a drawing or similar work, it is included.

As shown in Table 8, six of the eight interviewees (all but Eva and Nina) correctly identified that there are numbers between 0.66 and ⅔ and five of the six (all but Andie) correctly named a number between 0.66 and ⅔. Soren, Willa, and Mei all noted that ⅔ is equivalent to $0.\overline{6}$ but offered no justification for how they knew their chosen number was between 0.66 and ⅔ or $0.\overline{6}$. Jo, Korey, and Andie all noted that the decimal $0.\overline{6}$ "continues" after the terminating decimal 0.66, but did not otherwise justify why that meant that their chosen numbers were between 0.66 and $0.\overline{6}$. Note that Andie was the only interviewee to mention place value explicitly in her response. However, also note that she did not use place value to explain why there are numbers greater than 0.66 and less that $0.\overline{6}$ (in fact, she did not even name a number between 0.66 and ⅔), but instead simply stated that ".66 ends with a 6 in the 100ths place."

Two of the interviewees, Eva and Nina, both stated that there were *not* any numbers between 0.66 and ⅔. Eva argued that 0.66 and ⅔ were equivalent, so "no numbers exist between them." In contrast, Nina stated that "⅔ is equivalent to $0.66\overline{6}$," but then stated that there were *not* numbers between the two. One possible explanation for Nina's response is that she misread "0.66" as "$0.\overline{66}$." Another possible explanation for Nina's response is that she treated 0.66 and "$0.66\overline{6}$" as equivalent. Her discussion of 0.66 and $0.\overline{6}$ during the first interview lends credence to the latter explanation. The following exchange occurred about halfway through the first interview when she was asked to categorize a variety of numbers as whole, integers, rational, irrational, and/or real, and to explain her thinking.

> I: "Negative two-thirds. Where would you put that on a number line?"
> N: "So that's negative point six. Back this way, it would be right here."
> I: "Okay. How did you know it was negative point six?"
> N: "I just kind of have that memorized now from doing it so many. You can just divide two into three. To figure it out, you can find the higher common denominator. And multiply by the common denominator (incomprehensible) to get this one."
> I: "What about this one, zero point six repeating?"
> N: "Um (…) Well, technically, I'm pretty sure two-thirds is zero point six repeating."
> I: "Okay."
> N: "So it would be the same part. Well I guess/ Well I guess it would be. This could be written as a fraction, so it would be on here."
> I: "And what fraction would you say?"
> N: "Two-thirds."
> I: "Exactly two-thirds?"
> N: "(…) Um, yeah?"
> I: "Okay. What's making you doubt it?"

N: "I'm just trying to think, I haven't done it in a calculator in so long, I'm trying to remember if it comes out as point six six six. Or if it's just like, I'm just used to saying two-thirds is point six."

As this exchange shows, Nina was not clear about whether two-thirds was the same as 0.6, 0.666, or $0.\overline{6}$. She also located the decimal -0.6 at the point -⅔ on a number line, which is one way to show that two numbers are equivalent. Importantly, neither Nina nor Eva drew on place value to distinguish between these different numbers.

Whole class response to pretest Question 2**.** To situate the interviewees' responses to question two on the pretest within the larger scope of the class, Table 9 summarizes the responses of the class to this question.

Table 9

Summary of the Whole Class's Response to Pretest Question 2 (*N*=32)

Type of response	Number	Example
Yes because $\frac{2}{3} = 0.\overline{6}$	14	"Yes, there are numbers between 0.66 & ⅔. This is because ⅔ is equal to $0.\overline{66}$, so the number 6 is infinite. This means the number 0.666 or 0.6666 is in between 0.66 and ⅔."
Yes (misconception about ⅔ as a decimal)	5	"Yes, 0.666. ⅔ has many '6'es before reaching a final 7 in its decimal 0.666667."
No because $\frac{2}{3} = 0.66$	7	"No. ⅔ is exactly .66."
No because $\frac{2}{3} = 0.\overline{66}$	3	"$0.\overline{66}$ is equivalent to ⅔. I know this because 2÷3 =$0.\overline{66}$. Fractions are essentially division problems. There are no numbers between 0.66 and ⅔ except $0.\overline{66}$."
Unsure	2	"⅔ = .66 would $.6666\overline{6}$ be between the two? I am unsure."
Blank	1	

Notes. Spelling, grammar, and punctuation are preserved. Numbers are represented exactly as each student wrote them. If the student included a drawing or similar work, it is included.

As can be seen in Table 9, only fourteen of the 32 students in the class (approximately 44%) responded that there were numbers between ⅔ and 0.66 *and* correctly named ⅔ as a decimal. Soren, Jo, Andie, Mei, Willa, and Korey's responses are included in this group of fourteen. As with these six interviewees, the other eight who responded that there were numbers between 0.66 and ⅔ all noted that ⅔ is equal to $0.\overline{6}$.

However, none of these students then gave a substantive explanation for why this meant that their chosen number was between 0.66 and ⅔. Instead, students either simply stated that their number was between the two or they stated that $0.\overline{6}$ has an infinite number of sixes with no explanation for why this was important. An additional five students in the class provided a similar response, but misrepresented what ⅔ was as a decimal. Typically this involved stating that ⅔ was equal to a *terminating* decimal such as 0.66666 or 0.666667 (see the example in Table 9 for "Yes (misconception about ⅔ as a decimal)" for a typical response). This was not represented in among the interviewees.

A total of ten students in the course stated that there were *no* numbers between ⅔ and 0.66. Seven of these students (22%), including Eva, responded that there were no numbers between 0.66 and ⅔ because the two were equal. Three students (9%), including Nina, stated that ⅔ = $0.\overline{66}$ but that there were no numbers between 0.66 and ⅔. Two additional students used a combination of Eva and Nina's arguments, stating that 0.66 and ⅔ were the same and then wondering if $0.\overline{6}$ might be between the two (see the example Table 9 for "No because ⅔ = $0.\overline{66}$"). One student left the problem unanswered. In other words, Nina and Eva were not alone in their misconceptions for this problem.

Summary of findings from pretest Question 2. In this section, the ways interviewees and students in the class responded to a question asking them to find numbers between 0.66 and ⅔ and justify their response were summarized. Two primary themes emerged from students' responses to pretest Question 2. First, *the relationship between fractions and decimals was poorly understood* by a majority of students. Eighteen of the 32 students in the course (56%) gave a partially or completely incorrect response to this question. Second, *decimals were poorly understood* by many students as well. One subtheme related to both of these primary themes was a *misunderstandings of equivalence*, with some students stating that $0.\overline{6}$ was equivalent to 0.66, 0.66667, or a other terminating decimal, and others stating that 0.66 was equivalent to ⅔.

Two related subthemes overlapped in this case as *place value was not used to describe the decimal* meaningfully by any student and thus there was a *lack of use of precise decimal language* throughout. The lack of use of decimal language and/or place value was particularly problematic for that group of students that treated non-equivalent numbers as equivalent. Furthermore, the ability to use place value to compare and order decimals is required of fourth grade students in the Minnesota State Academic Standards in Mathematics (2007) so the preservice teachers responses to this question showed a more limited understanding than will be necessary for teaching elementary students to do similar tasks.

Finally, in lieu of using place value to meaningfully compare and order the numbers in this problem, many students instead *relied on the physical characteristics of the decimal notation* to justify the ordering of the numbers. In particular, the fact that the $0.\overline{6}$ has an infinite (or many) sixes in a row was the most common justification given for why a number was between 0.66 and ⅔ (or $0.\overline{6}$).

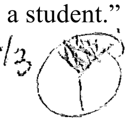
Pretest Question 4: Explain Why ⅓=0.333…

Pretest Question 4 asked: "Tell how you would help a student understand why ⅓ = 0.333… when written as a decimal." The purpose of this question was to see how preservice teachers described the relationship between a fraction and a terminating decimal. The context, “Tell how you would help a student understand,” was used in order to prompt students to give the clearest description this relationship possible. In the following two sections, a summary of the ways the eight interviewees responded to this question will be presented first, drawing on data from the interviews as necessary to better explain their responses. Then a summary of the ways the whole class responded will be presented in order to give a more complete picture of the nature of the preservice teachers’ understanding of the relationship between ⅓ and $0.\overline{3}$ early in the course.

Interviewees’ responses to pretest Question 4. Table 10 shows how the eight interviewees explained why ⅓ is equal to $0.\overline{3}$ on the pretest.

Table 10

Summary of Ways Interviewees Explained Why ⅓ = 0.333… on the Pretest

Group	Student	Pretest (Question 4)
MKT2	Soren	"$\frac{1}{3}=\frac{3}{9}$ $\frac{1}{9}=0.\overline{111}$ $\frac{2}{9}=0.\overline{222}$ $\frac{3}{9}=0.\overline{333}$ $\frac{4}{9}=0.\overline{444}$"
	Jo	"⅓ shows how much of a whole is present. One 'whole' in terms of decimals is 1.0. I could show a pie chart. I could also use percentages or division to show this to a student." .33 = 1/3 
	Andie	"I would explain that when we divide 1 by 3 the answer is not whole, & the 3 repeats infinitely forever."
	Mei	"I would let them to use the math '1÷3' and let them try to solve the math. Once they see they will always get 3 after the decimal, we would make the predication that there will always be 3 after the decimal."
MKT1	Eva	$\frac{1}{3}$ = 0.333… when written $\frac{3}{3}$ = 1 pie "$\frac{3}{3}$ = 1 pie. If you think about the fraction as a pie, there are 3 slices of pie total. We need to figure out how to think about how 1 slice of pie from 3 and your numbers should be less than 1."
	Nina	$\frac{1}{3}\cdot\frac{100}{1}=\frac{100}{3}$ $3\overline{)100}$ = .333 9 10 9 "I would help the student by showing them to multiply by 100 and then in you divide 3 by 100 you get .333."
	Willa	"I would explain that 1 cannot be divided by 3 evenly, so adding it 3 times written as 0.333… would get it closest to 1."
	Korey	"To find the decimal of ⅓ you would divide the bottom number (denominator) into the top number (numerator) and you would get the decimal .333."

Note. Minor corrections were made to spelling, grammar, and punctuation to improve readability. Numbers are represented exactly as each student wrote them. If the student included a drawing or similar work, it is included.

As shown in Table 10, six of the eight interviewees (all but Soren and Willa) relied wholly or in part on the idea that one divided by three is equal to $0.\overline{3}$. Of these, Andie, Mei, Nina, and Korey relied solely on division as the way to show the equivalence of ⅓ and $0.\overline{3}$, although Nina changed the problem to be 100÷3 rather than 1÷3[7]. While it is certainly possible to show that dividing one by three results in the repeating decimal $0.\overline{3}$ using the standard long division algorithm, the problem is that the division algorithm is efficient because it "hides" the place-value ideas upon which it relies (Battista, 2012, p. 4). Thus, just using the division algorithm to show that ⅓ = $0.\overline{3}$ hides the fact that 0.333… is equivalent to the infinite sum $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\cdots$. It also offers no insight into *why* this infinite sum should be equal to one-third.

Two of the interviewees, Jo and Eva, offered additional strategies in addition to the division algorithm. In particular, both used the idea that a "pie" (Eva) or circle (Jo) could be split in three parts to show the fraction ⅓. Jo even added an illustration of a circle with ⅓ shaded (see Table 10). However, neither woman then explained how this would show that ⅓ = $0.\overline{3}$. Jo simply wrote next to her illustration ".33=⅓," despite the fact that her circle showed the fraction only. Jo also offered that "One 'whole' in terms of decimals is 1.0," and that she could "use percentages" to show this equivalence to a student, but it is unclear how she saw those two ideas as illustrating why ⅓ equals $0.\overline{3}$. Eva referenced the idea that a pie can be divided into three slices, but gave no explanation for how this related to the decimal $0.\overline{3}$ or to the fact that this repeating decimal is equal to one-third.

The remaining two interviewees, Willa and Soren, took a different approach to

---

[7] Note that Nina did the same thing when converting ⅔ to a decimal; that is, she instead solved 200 ÷ 3. See Table 8.

explaining the equivalence between ⅓ and $0.\overline{3}$. In particular, both utilized the idea that three thirds are equal to one. Willa's primary argument was that adding $0.\overline{3}$ three times would get an answer that is "closest to," rather than *equal* to one. Willa also referenced division but only to note, "1 cannot be divided by 3 evenly."

Soren gave a partial response to this question. First, he noted that "⅓ = $^3/_9$," and then he listed the decimal equivalents to $^1/_9$, $^2/_9$, $^3/_9$, and $^4/_9$. Clearly, if Soren had continued his pattern, he would have ended with the line "$^9/_9 = 0.\overline{999}$." The idea that $0.\overline{9}$ equals one is a difficult concept for many mathematics students to accept (Dubinsky et al., 2013; Richman, 1999), so it was possible that he had abandoned his pattern because he was troubled by the idea that $1=0.\overline{9}$. He verified that this was indeed the case during the first interview when he was asked about where he thought $0.\overline{9}$ should go on a number line. The following conversation took place during the sorting activity, shortly after he explained that $0.\overline{6}$ would be located at the same point as ⅔ on the number line.

I: "What about this one, point nine repeating. Where would you put that?"
S: "Um (…) Oh, it's (…) oh man. (…) Now, I'm reconsidering."
I: "What's making you, what's tripping you up?"
S: "Well, I mean, I just, well I guess not many of the other ones. So if point nine nine repeated infinitely it would equal one. Hypothetically. Theoretically. So it could be one is a whole number and an integer. Point nine nine isn't I would say, but they're the same thing. So, okay. But I don't know if that's a real number necessarily. Because, well yeah it's a real number."
I: "Where would it go on the number line?"
S: "Really really really close to one. Or at one, depending on/"
I: "Where does point six repeating go on a number line?"
S: "Um (…) at two-thirds."
I: "Okay, where does point nine repeating go on a number line?"
S: "So it would go a one by that logic."
I: "Okay."
S: "Um."
I: "What's the problem?"

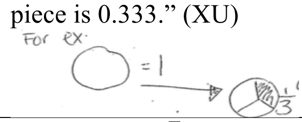
S: “Well, it’s not one.”
I: “Why not?”
S: “It’s infinitely close to one. But it’s not one.”

As this conversation demonstrates, Soren saw the logic in the argument that $1=0.\overline{9}$, but also believed that the two were not actually equal. Soren’s struggles with this equivalence demonstrate the difficulties with the approach of trying to use repeated addition to argue that since three thirds equal one, three groups of $0.\overline{3}$ are also equal to one.

Whole class response to pretest Question 4. To situate the interviewees’ responses to question four on the pretest within the larger scope of the class, Table 11 summarizes the responses of the class to this question.

Table 11

Summary of Ways the Whole Class Explained why ⅓ = 0.333… on the Pretest (*N*=32)

Type of response	Number (%)	Example[a]
Division (1 ÷ 3)	9 (28%)	“I would explain that when you are converting a fraction to a decimal, you divide the top number by the bottom. 1÷3=.33.” (EL)
Division (100 ÷ 3)	2 (6%)	“’That’s just how it is.’ Kidding, I would try to use a visual, for example, to show that when you split 100 into 3 parts, there’s a little leftover that you need to divide evenly.” (OD)
Division and picture or story (all circle models)	7 (22%)	“Explain that the 1 on top is the numerator and shade one part and explain that we are splitting it into 3 (denominator) equally and one piece is 0.333.” (XU) 
Three thirds equals 1 whole (repeated addition)	7 (22%)	“The 3 in $0.33\overline{3}$ repeats to represent ⅓ because ⅓ +⅓+⅓=1, so if the 3 didn’t repeat and we just added 0.333+0.333+0.333=0.999, 0.999 does not equal 1. The 3 repeats in order to get the decimal form of ⅓ as close to 1 as possible.” (UT)
Other	3 (9%)	“I would show him or her by making the denominator 100 and showing what happens to the numerator.” (ET)
Blank/ “Don’t know”	4 (13%)	“I’m not sure how to explain this yet.” (RE)

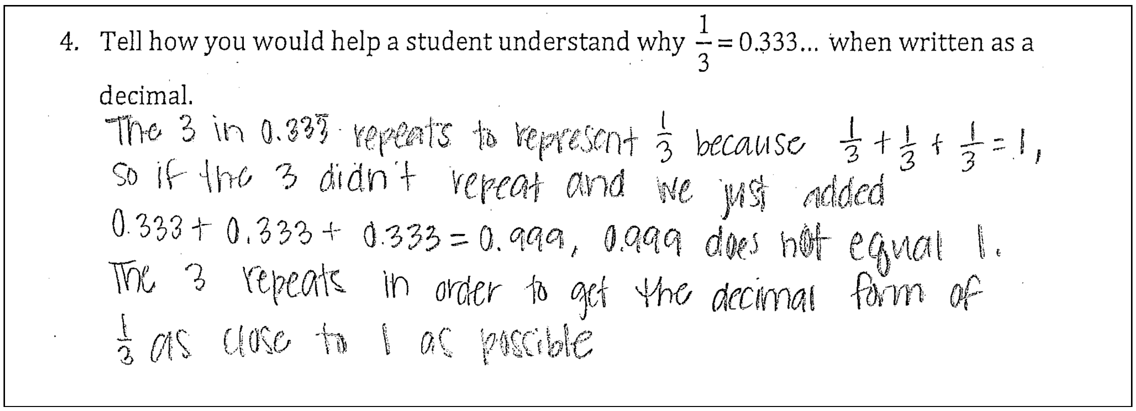
Note. Minor corrections to spelling, grammar, and/or punctuation were made to improve readability. All numbers are presented exactly as written by the students.
[a]Student initials (pseudonyms) are shown in parentheses.

As shown in Table 11, a total of eleven people in the class, including Andie, Mei, Korey, and Nina, relied solely on division to explain why ⅓ = 0.333…. Two of these, including Nina, first converted the problem to 100÷3. Seven people, including Jo and Eva, relied primarily on division but also included a picture or story to illustrate. In all cases, the picture or related to a circle model and showed only the fraction ⅓, not why ⅓ equals $0.\overline{3}$ (see Jo and Eva's pictures in Table 10 and OD's picture in Table 11). Together, this means that 18 of the 32 students (56%) used division as their only viable method to describe the equivalence of ⅓ and $0.\overline{3}$.

The next most common strategy used to show why ⅓ is equal to $0.\overline{3}$ was the repeated addition strategy used by Soren and Willa. A total of seven students used some version of the idea that since three groups of one-third equal one, three groups of $0.\overline{3}$ would also equal one. This line of reasoning would lead to the true, but generally misunderstood (Dubinsky et al., 2013), statement that $0.\overline{9}=1$. However, no student in this group directly claimed that $0.\overline{9}=1$. Four students simply stated that three groups of $0.\overline{3}$ would equal one, rather than $0.\overline{9}$. Two students, including Willa, argued that it the total would be close, but not equal, to one (see Figure 21 for an example). The remaining student was Soren, who abandoned his argument before writing "$\frac{9}{9} = 0.\overline{999}$." As described in the previous section, he did not believe $0.\overline{9}=1$ to be a true statement of equivalence.

Finally, three students from the class suggested unclear methods for showing that $\frac{1}{3}=0.\overline{3}$. For instance, student "OE" wrote, "I would explain it to them that we notate

⅓=.333… because the 3’s continue on.” An additional four students left the problem blank or wrote, “I don’t know.” Thus, the majority of students (56%) suggested using division to show that ⅓ equals $0.\overline{3}$. A substantial minority (22%) suggested using repeated addition of ⅓ and $0.\overline{3}$, with no explanation of how they would respond to the implied idea that $0.\overline{9} = 1$. The remaining 22% offered no strategy for explaining this equivalence.



4. Tell how you would help a student understand why $\frac{1}{3} = 0.333...$ when written as a decimal.

The 3 in $0.33\overline{3}$ repeats to represent $\frac{1}{3}$ because $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, so if the 3 didn't repeat and we just added 0.333 + 0.333 + 0.333 = 0.999, 0.999 does not equal 1. The 3 repeats in order to get the decimal form of $\frac{1}{3}$ as close to 1 as possible

Figure 21. “UT’s” explanation for why ⅓ $= 0.\overline{3}$.

Summary of findings for pretest Question 4. This section presented as summary of the ways interviewees’ and students in the class responded to a question asking them to explain why ⅓ is equal to the repeating decimal $0.\overline{3}$. Two themes emerged from this data. First, *the relationship between fractions and decimals was poorly understood* by the students. No student offered a clear explanation of the equivalence of ⅓ and $0.\overline{3}$. The overwhelming majority of students stated that they would *use the division algorithm to explain this equivalence* to children. The division algorithm is opaque in that it hides the role of the powers of ten in the decimal representation of one-third.

Second, *decimals were poorly understood.* In particular, there was *a lack of use of* *accurate models for decimals*. One of the indicators of decimal understanding is the ability to connect decimal symbols with pictorial representations (Cramer et al., 2015), but in this problem, no preservice teacher represented, or even attempted to represent, the decimal $0.\overline{3}$ in any non-symbolic way. In fact, the few students who attempted to use a non-symbolic representation to show this equivalence all chose a circle model and represented only the fraction ⅓. Clearly, preservice teachers must understand how choose appropriate representations for decimals if they are to teach this topic meaningfully to elementary students.

Interview 1: Number Sorting Task

The previous sections presented a summary of the ways the preservice teachers in the course responded to questions that asked them to reason about or explain the relationship between 0.66 and ⅔ (pretest question 2) and $0.\overline{3}$ and ⅓ (pretest question 4). This section will present as summary of the ways the eight interviewees categorized a variety of decimals and fractions as rational or irrational during the first interview. The number categorizing activity, herein called the *number sort*, took place during the first interview after students had defined and listed categories of number, as described in the previous section. At the start of this activity, each interviewee was provided with the definitions of whole numbers, integers, rational, irrational, real, and imaginary numbers shown in Table 12. Each definition was briefly discussed. Each interviewee was given an opportunity to ask questions about the definitions and provided clarification as necessary.

Although interviewees were asked to categorize a variety of whole numbers, integers, rational, and irrational numbers during the interview, in this section, only on the ways they categorized terminating and repeating decimals, as well as numbers involving π will be reported. Note that the provided definition of rational numbers focused on the fact that rational numbers can be written using *fraction notation*; that is, in the form $^a/_b$ (where *a* and *b* are integers, *b*≠0). Thus, categorizing various decimals as rational (or not) using this definition created an opportunity for students to both talk about their understandings of the set of rational numbers and their understandings of the fraction-decimal relationship. Since the number π is commonly used as an example of an irrational number and has a well-known decimal approximation, categorizing numbers involving π, including decimals (e.g. 3.1415…) and fractions (e.g. π/2) created an opportunity for students to talk about their understandings of fraction and decimal notation in relation to the set of rational and irrational numbers.

Table 12

Definitions of the Subsets of the Real Numbers Provided to Preservice Teachers During the First Interview

Number set	Definition
Whole numbers (or ***counting*** or ***natural*** numbers)	The positive integers, {1, 2, 3, …}.
Integer	A number that can be written without a fractional part. *or* The positive and negative whole numbers and 0.
Rational numbers	All numbers that can be written in the form $\frac{a}{b}$ where *a* and *b* are integers and *b*≠0.
Irrational numbers	Real numbers that are *not* rational, or cannot be written in the form $\frac{a}{b}$ where *a* and *b* are integers and *b*≠0.
Real number	A number that can represent a quantity along a continuous line

Number sort: Terminating decimals. For this aspect of the number sorting task, the interviewees were a given card with terminating decimal written on it and asked to place the decimal in all the categories of numbers into which the student believed it belonged. As they placed the number in different categories, students were asked to justify their response. As can be seen in Table 13, terminating decimals were overwhelmingly categorized as rational by participants. Only Korey failed to classify all terminating decimals as rational. In particular, she classified 72.3 as real only (not rational or irrational) because she "just felt like it went there," and 0.000003400 as irrational "because it's a lot of zeros in front of it."

Table 13

Summary of Ways PSTs Classified Terminating Decimals and Their Justifications for Doing So During the First Interview

Group	Student	Classification	Typical Reason
MKT2	Soren	Rational	Converted to fraction using place value for all
	Jo	Rational	Converted to fraction using place value for all
	Andie	Rational	Converted to fraction using place value for all
	Mei	Rational	Converted to fraction using place value for all
MKT1	Eva	Rational	"Because it [0.6215] terminates"
	Nina	Rational	"Well, this [72.3] could be written as a fraction. I don't know what it is."
	Willa	Rational	"Because it could be seventy two and then one third."
	Korey	Usually rational	"I don't know what the fraction is [for 1.25], but I know a decimal is a fraction."
		Just real	"Ooh. I don't know. I just felt like it [72.3] went there."
		Irrational	"Because it's a lot of zeroes in front of it [0.000003400]."

Note. Minor grammatical changes made to improve readability.

Although interviewees generally categorized terminating decimals as rational, only the four students in the MKT2 group did so because they converted the terminating decimal to a fraction. These four students consistently justified the placement of a terminating decimal in the rational category by explaining how to convert the decimal to a fraction using place value. For instance, given the decimal 0.66, the four MKT2 students responded as follows:

> Rational number because it's sixty-six over a hundred. (Soren)

> I guess I don't know what the fraction form of point six six is. Well, it must be/ it would be sixty-six over a hundred so it would be [rational]. (Andie)

> It would be sixty-six over a hundred. (Jo)

> That can be written as the sixty-six over one hundred. (Mei)

In contrast, the four students in the MKT1 group never used place value to convert terminating decimals to fractions. Instead, they relied on a variety of other reasons to justify how they knew that terminating decimals were rational. Eva consistently stated that numbers that terminated were rational and used that as her justification for placing terminating decimals in the rational category. For instance, she after she categorized 0.6215 as rational, the following exchange took place.

> I: "How do you know it's a rational number?"
> E: "Because it ends. It terminates. And b is not zero, so I know that it could be written as a fraction where both a and b are integers."
> I: "What fraction could it be, do you know?"
> E: "Oh, I have no idea."
> I: "Do you have any idea how you might figure it out?"
> E: "Um (…) Not at this moment, no."
> I: "So how you do know it could be?"
> E: "Because it terminates."

As the above exchange illustrates, Eva did not use place value to convert terminating decimals to fractions. She also justified the placement of terminating decimals in the rational category based on the decimal notation, rather than the fact that the decimal could be written as a fraction.

Similarly, Nina and Korey also both stated that they did not know what the fraction was for the terminating decimals they were presented in their interview (even common ones like "0.3"). However, both also stated that they believed that terminating decimals could be written as a fractions because fractions can be converted to decimal using division (with the above noted exceptions for Korey).

Willa, on the other hand, sometimes converted the terminating decimal to a fraction, but she did so incorrectly in most cases. For instance, she consistently converted 0.3 to ⅓ because "it's just like cut off from point three three repeating." She did, however, correctly convert 1.25 to 1¼. For less common decimals, Willa stated that she knew that the decimal could be converted to a fraction, but she was never able to correctly describe how to do so. For example, the following exchange occurred after she was asked to categorize the terminating decimal 3.14.

> W: "I'm assuming, but I'm not exactly sure. I feel like if it's a decimal, you can write it as a fraction (incomprehensible)."
> I: "Do you know how to or are you not sure?"
> W: "You have to/ I think you'd have to figure out how many times point one four goes into three. If it would be like three and then like whatever point one four would tell me."

In other words, although Willa was not able to use place value to convert terminating decimals to fractions, she did believe that (terminating) decimals could be written as fractions. Note that her method of trying to convert the decimal to a fraction involved division ("how many times point one four goes into three"), which also suggests that Willa had a poor understanding of how the division algorithm works.

*Summary of findings related to ways interviewees categorized terminating decimals***.** The primary theme to emerge from the ways the interviewees categorized terminating decimals was that half of the interviewees showed a *lack of decimal understanding*. In particular, *place value was not understood* by the four students in the MKT1 group, as all consistently stated that they did not know how to translate terminating decimals to fractions.

Number sort: Repeating decimals. For this aspect of the number sorting task, the interviewees were a given card with repeating decimal written on it and asked to place the decimal in all the categories of numbers into which the student believed it belonged. As they placed the number in different categories, students were asked to justify their response. For repeating decimals, there were again differences between how the MKT2 and MKT1 groups categorized the decimals and justified their categorization. Three of the four students in the MKT2 group consistently classified all repeating decimals as rational, whereas only one student in the MKT1 group (Nina) consistently did so. Table 14 shows a summary of the ways that the interviewees categorized and justified their categorizations of common and uncommon repeating decimals during the first interview.

Table 14

Summary of Ways Interviewees Classified Repeating Decimals and Their Justifications for Doing So During the First Interview

Group	Student	Common repeating (e.g. $0.\overline{6}$)	Uncommon repeating (e.g. $.\overline{583}$)
MKT2	Soren	Rational *"That's two-thirds so it's rational"*	Rational *"Typically repeating numbers are fractions represented as decimals."*
	Jo	Rational "*Because it's predictable*" *"Two over three is point six six repeating"*	Rational *"I just feel like this is going to be a fraction."*
	Andie	Irrational *"Is an irrational number a number that doesn't end?"* Then rational *"I just I knew that point six six was two thirds"*	Unsure *"I don't know how you'd figure what fraction is that makes this, or if there is a fraction that gives you this number."*
	Mei	Rational *"It would be two-thirds"*	Rational *"I think that could be written in the rational number, but I don't know exactly how to write that"*
MKT1	Eva	Irrational *Wrote "$0.\overline{6}666$" and "$0.\overline{1}111$" an examples of "not rational" and "irrational" number on pretest* Then rational *"I think it terminates. You can say it terminates?"*	Rational *"I think rational numbers can be written as repeating but I don't/."*
	Nina	Rational *"I'm pretty sure two-thirds is zero point six repeating"*	Rational *"Because when you divide a over b, you'll come to a decimal."*
	Willa	Irrational *"Because you can't represent it as a fraction because it doesn't have a set quantity. I mean, I suppose it depends on who you're talking to. Because some people think of two thirds as point six six repeating, and then some people cut it off."*	Irrational *Same problem as* $0.\overline{6}$
	Korey	Irrational *" Because you can't make it into a decimal. (...) Or sorry a fraction."*	NA

Note. NA="Not Applicable" as the preservice teacher was not asked to categorize a decimal of this type during the interview.

The one student in the MKT2 group that did not consistently classify repeating decimals as rational was Andie. Initially, she thought that an irrational number was a “number that does not end” and gave “point three three repeating” as her example. After being given the definition of rational number, she then switched to classifying the two common repeating decimals ($0.\overline{3}$ and $0.\overline{6}$) as rational, in both cases because she knew the equivalent fraction. She did not, however, then classify *all* repeating decimals as rational. Instead, she declined to categorize $0.\overline{583}$ as rational or irrational, arguing ,“I don’t know how you’d figure what fraction is that makes this, or if there is a fraction that gives you this number.” She did, however, know that $0.\overline{583}$ would be rational if there was a related, equivalent fraction, and irrational if not.

In contrast, the other three MKT2 students all classified $0.\overline{583}$ as rational despite knowing neither the equivalent fraction nor how to find it. Instead, they all simply felt confident that repeating decimals could be represented as a fraction even if they did not know how to find the fraction.

> I feel like I’ve seen that somewhere, when I did calculating. Maybe like doing the a over b, something like that. (Mei)
>
> I just feel like this is going to be a fraction. (Jo)
>
> Yeah, typically repeating numbers are fractions represented as decimals, I think…Don’t know why. (Soren)

Note that Mei, Jo, and Soren’s reasons for categorizing repeating decimals as rational based on a “feeling” that repeating decimals are rational is similar to the reasons given by MKT1 group when categorizing terminating decimals as rational. In other words, while the MKT2 group’s place value understanding was useful in helping them find an equivalent fraction in the case of terminating decimals, it was insufficient in the case of repeating decimals and they had no alternative way to make sense of why a given decimal should be rational.

Given the MKT1 groups' limited understanding of the relationship between terminating decimals and fractions, it is not surprising that they also had difficulties categorizing repeating decimals. Two of the students in the MKT1 group categorized repeating decimals as rational, while the other two categorized them as irrational.

Both Eva and Nina consistently classified repeating decimals as rational, but their reasons for doing so revealed that each had misconceptions about decimals. Nina, for instance, said that $0.\overline{583}$ was rational because "when you divide a over b, you'll come to a decimal." However, she applied this same reasoning to *all* decimals, including the irrational number pi.

I: "Okay. How about this. Three point one four one five, it goes on. Pi."
N: "Well, that could be (…) a fraction."
I: "Okay. How so?"
N: "If you divide two numbers to get the decimal."
I: "How do you know you can do that?"
N: "(…) Well they probably somewhere along the line, they probably got that number multiple times and that's why they made it pi."

In other words, Nina classified repeating decimals as rational not because she remembered that repeating decimals could be written as fractions, but rather because she believed that *all* decimals came from a fraction by the process of division.

As described in previous sections, Eva listed the repeating decimals $0.\overline{6}666$ and $0.\overline{1}111$ as examples of "non-rational" and "irrational" numbers, respectively. Then, early in her interview, she argued that the repeating decimal $0.\overline{3}$ could be considered rational because, "you can say it terminates." For the remainder of the interview, Eva then simply stated that she thought that "rational numbers can be written as repeating" without further elaboration. This included the repeating decimals $0.\overline{6}$ and $0.\overline{583}$. It is not clear why she shifted from stating that repeating decimals were not rational/irrational on the pretest to saying that they were rational during the interview. However, her explanation that $0.\overline{3}$ is rational because "you can say it terminates" along with her changing categorizations indicate a lack of understanding of both decimal notation and the relationship between fractions and repeating decimals. Thus, although Eva and Nina correctly categorized repeating decimals as rational, their reasons for doing so revealed misunderstanding about the relationship between fractions and decimals and misunderstandings about decimal notation.

In contrast to the other six interviewees, Willa and Korey classified repeating decimals as *irrational* even after being given the definition of rational numbers. For instance, Willa argued the repeating decimal $0.\overline{6}$ was irrational because "you can't represent it as a fraction because it doesn't have a set quantity." This was in keeping with her definition of whole numbers (which she said included numbers like 3.14) early in the interview as a number that "has a set quantity." Interestingly, Willa qualified her argument that the decimal $0.\overline{6}$ cannot be represented by a fraction by saying, "I mean, I suppose it depends on who you're talking to. Because some people think of two thirds as point six six repeating, and then some people cut it off." However, when asked directly if two-thirds actually equal to $0.\overline{6}$, she stated, "No." The fact that Willa did not believe that ⅔ and $0.\overline{6}$ were equivalent meant that working with these numbers during the sorting activity did not challenge her self-made definition of rational numbers as numbers that have a "set quantity." If anything, it validated this as a useful way of defining the set of numbers since it allowed her to classify ⅔ and $0.\overline{6}$ in different sets. Notably, Willa also stated she would use the same reasoning to categorize $0.\overline{583}$ as irrational, so the idea that sets of numbers could be defined based on whether or not the decimal terminated was a stable conception for her throughout the interview.

Korey also classified $0.\overline{6}$ as irrational based on the fact that the decimal "continues," although she was not confident in her answer.

> K: "Um (…) I want to say irrational."
> I: "Okay. How come?"
> K: "Because you cannot make it into a decimal. Yes you can, because that's why it continues. No you can't. Wait hold on. No you can't. So it's irrational because you can't make it into a decimal. (…) Or sorry a fraction."
> I: "Okay, so you can't make it into a fraction?"
> K: "Yeah."
> I: "Okay. Because it continues forever?"
> K: "Yeah. But I feel like that's wrong somehow. I feel like I'm wrong."

Korey was not asked to classify any other repeating decimals during her interview. Recall that on the pretest, Korey had listed even whole numbers as "rational" and odd numbers as "irrational" examples. After being given the definition of rational numbers, however, Korey did shift to talking about rational numbers in terms of fractions. However, like Willa, Korey did not state that $0.\overline{6}$ was equivalent to ⅔ but instead stated she did not know what fraction it could be. Interestingly, she did seem to know this on question two of her pretest, writing "⅔ is $.66\overline{6}$," and it is not clear why she did not draw on this knowledge during the interview. However, the fact that she did not suggests at least that the equivalence of ⅔ and $0.\overline{6}$ was not a stable understanding for Korey early in the course. Importantly, during the interview, when she did not draw on this equivalence, she instead reasoned about the rationality of the repeating decimal based on the fact that the decimal notation "continues." In other words, at least for the repeating decimals they were presented during the interview, both Willa and Korey relied on the surface features of the decimal notation as their primary reason for stating that repeating decimals could not be rational.

*Summary of findings related to interviewees' categorization of repeating decimals***.** Three related themes emerged from the data related to the ways the preservice teachers categorized repeating decimals during the interview. The first theme was that *repeating decimals were not well understood* by the students. The second theme was that the *preservice teachers' understanding of the sets of rational (and irrational) numbers was often inaccurate*. The third theme was that *the relationship between fractions and decimals was poorly understood*. These three themes were related in this case because the given definition of rational numbers stated that rational numbers were numbers that could be written in (standard) fraction form. Thus this activity essentially asked students to draw on their understanding of the relationship between fractions and repeating decimals as well as their understanding of the set of rational numbers.

Several subthemes related to these larger themes also emerged from the data from this activity. First, although five of the eight students categorized all repeating decimals as rational during the number sorting activity, none of the students could correctly explain why repeating decimals had equivalent fraction forms. Instead, Soren, Jo, Mei, and Eva all simply stated that it was something that they believed to be true (i.e. a "rule without reason" (Skemp, 1976)). Nina stated it was true because all decimals come from fractions "somehow." Willa consistently stated that ⅔ and $0.\overline{6}$ were not exactly equivalent, and then added that it "depends on who you talk to." Willa's response is another example of the subtheme of *misunderstandings related to equivalence.* Korey was uncertain about how to classify $0.\overline{6}$ and did not note that it was equal to ⅔, even when she was trying to decide if it was rational or irrational. Two additional subthemes to emerge from this data were that some of the preservice teachers *relied on the physical characteristic of the notation to categorize repeating decimal,* with two of them *categorizing repeating decimals as irrational.* In particular, Korey and Willa used the idea that the decimal "continues" (Korey) or is "not a set quantity" to classify repeating decimals as irrational even after being given the definition of a rational number. Eva used the idea that "you can say it terminates" to classify the repeating decimal $0.\overline{3}$ as rational, which also refers to decimal notation itself.

Number sort: Numbers involving π. During the number sorting activity, interviewees were asked to classify numbers involving pi in two different ways. Recall that during the number sorting task, the interviewees were a given card with number written on it and asked to place the number into all the categories of numbers which the student believed it belonged. For the numbers involving pi, all interviewees were asked to categorize pi written as a non-terminating, non-repeating decimal, "3.1415…." Six of the students were also asked to classify at least one fraction involving π, such as $^{\pi}/_{3}$. Eva and Willa were not asked to classify a fraction involving pi due to time constraints. Table 15 shows how the interviewees classified numbers involving pi during the first interview.

Table 15

Summary of Ways Interviewees Classified the Decimal Form of π and Fractions Involving π During the First Interview

Group	Student	3.1415…	Fraction involving π (e.g. $\frac{\pi}{3}$; should be irrational)
MKT2	Soren	Irrational *"If it were a fraction it would repeat, and pi doesn't repeat"*	Irrational *No explanation offered*
	Jo	Irrational *"Because it keeps going on forever and gets unpredictable as it keeps going*	Irrational (π/3) *"Because pi is in there and I feel like there's no way that pi over anything could be rational."* Maybe irrational (π/$\sqrt{3}$) *"I'm just wondering if the square root does something."*
	Andie	Unsure *"I don't know what the fraction is for pi, or if there is a fraction"*	Irrational "*It still wouldn't be rational because pi's not a whole number, or an integer. So then it would be irrational. Right? Yeah, it would be irrational."* Then unsure *If there is a fraction for pi, then $\frac{\pi}{3}$ would be rational*
	Mei	Irrational *" I was taught that pi can't be written in the form a over b"*	Irrational *"[T]hey are not integers, so it's irrational"*
MKT1	Eva	Irrational *"It does not terminate"*	NA
	Nina	Rational *Pi "could be a fraction… If you divide two numbers to get the decimal"*	Real only *"So it wouldn't be irrational because they're not integers. So none of these besides putting it on a number line? Because … it has to be an integer, and it can't be in the integers."* Then irrational *"So it could be an irrational number then because it's not rational."*
	Willa	Irrational *Pi "goes into a separate category because it's repeating"*	NA
	Korey	Irrational *No explanation offered*	Rational and irrational (unsure) *"Since you wrote it as a fraction, it could be a rational number. But pi is always irrational. So then it would go into the irrational too"*

Note. Minor edits to grammar and spelling were made to improve readability. NA = Not Asked.

As can be seen in Table 15, all the students except Nina and Andie classified the decimal form of pi as irrational. As described in the previous section, Nina classified the decimal form of pi as *rational* because she believed that all decimals came from a fraction “somehow.” Andie was unsure if she should categorize pi as rational because she did not know if there was a fraction form for pi or not. Mei was the only student who both classified pi as irrational and stated that she knew that it did not have a fraction form.

Soren, Jo, Eva, and Willa all focused instead on the nature of the decimal. Soren focused on the fact that the decimal was non-terminating and non-repeating, a common metric used to describe irrational numbers. He did not offer an explanation for why these decimals were irrational. Jo focused on the fact that the decimal was both non-terminating and *unpredictable*. Such a description excludes irrational-but-predictable decimals such as 0.121121112…, a decimal that Jo declined to categorize as rational or irrational earlier in the interview. Eva focused on the fact that the decimal did not *terminate*, which is in keeping with her definition of rational numbers as “numbers that terminate.” This definition, of course, fails to exclude repeating decimals from the set of rationals. Recall that Eva had, in fact, given the repeating decimal “$0.\overline{6}666$” as an example of an irrational number on the pretest, but then switched to stating that such decimals were rational during the interview. While it is unclear why she made such a switch, it is clear that she was focused on the fact that irrational numbers had decimal form that were infinite. Finally, Willa stated that pi would “go in a separate category because it is *repeating*.” It is unclear what she meant by the word “repeating.” Thus, of the students who focused on the decimal form of pi in order to justify classifying it as irrational, only Soren gave a justification that accurately reflected irrational numbers.

Notably, none of the students gave any explanation for why a non-terminating, non-repeating decimal could not be written as a fraction.

Further limitations in some students' understanding of pi specifically, and irrational numbers more generally, were revealed when they were asked to categorize a fraction that involved pi. Four of the six students who were asked to categorize fractions involving pi expressed uncertainty about whether or not at least one fraction involving pi would be an irrational number. Only Soren and Mei categorized both pi as a decimal and a fraction involving pi as irrational.

Jo stated that the fraction $\pi/3$ was irrational because "pi is in there and I feel like there's no way that pi over anything could be rational." She then stated that $\pi/2\pi$ was rational "because the pis would cancel." But when asked about $\pi/\sqrt{3}$, she expressed uncertainty.

> J: "Mmm (…) Not as confident about this, but we'll still go irrational."
> I: "Why are you not as confident?"
> J: "I'm just wondering if the square root does something. I'm thinking of how it would affect the value of pi. [Appears to be trying to calculate the value of the square root of three, mouthing numbers]. Yeah, I think it's just because I'm unsure of the value of the square root of three."
> I: "Where would you put just the square root of three as a number?"
> J: "That would be real. [Very quietly] And I'm not sure if it's rational or irrational."

She also was unsure if the square root of three was rational or irrational. In other words, Jo knew that pi was irrational, but had little other knowledge about irrational numbers.

Recall that Andie was unsure whether the decimal form of pi, 3.1415…, was rational or irrational because she was not sure if there was a fraction related to pi. However, when given the number $\pi/3$, she immediately classified it as irrational, stating

"It still wouldn't be rational because pi's not a whole number, or an integer. So then it would be irrational. Right? Yeah, it would be irrational." In order to further test her thinking about the necessity of the numerator and denominator being integers in a rational number, she was then asked about the fraction $^{1.5}/_3$. She quickly realized that the latter fraction was equivalent to one-half, which then led her to question the rationality of $^{\pi}/_3$ as well.

> Yeah, but now I'm questioning going back with some of these that I was saying weren't rational. I can't remember what else I said. Well pi over three, is like well, well that's going back to I don't know if pi is rational or not. Because if pi is rational [trailed off]. (Andie, interview 1)

Thus Andie was able to quickly recognize that the same reasons she used to determine that $^{1.5}/_3$ was rational would be applicable to the fraction $^{\pi}/_3$ if pi was rational. Yet her initial inclination was to categorize $^{\pi}/_3$ as irrational because "pi's not a whole number, or an integer." In other words, she first reasoned about the irrationality of $^{\pi}/_3$ based on the characteristics of the fraction notation itself, rather than based on deeper understandings of the sets of rational and irrational numbers.

Like Andie, Nina also classified π as rational and a fraction involving π as irrational. In Nina's case, the difference was even more striking because the fraction she was given was $^{\pi}/_1$. She was asked to categorize $^{\pi}/_1$ immediately after she stated that pi as a decimal was rational.

> I: "So pi over one? Would let you get here [in the rational category] or wouldn't?"
> N: "It wouldn't, because it's not a whole number."
> I: "So where would you put it?"
> N: "I think it would have to go [reading definition of irrationals]. So it wouldn't be irrational because they're not integers. So none of these besides putting it on a number line? Because it's not, yeah because it has to

be an integer, and it can't be in the integers."

I: "This one's real numbers that are NOT rational. So just numbers that you can't write as this integer over integer."

N: "Oh, sorry, I got that wrong. So it could be an irrational number then because it's not rational."

Unlike Andie, who used the fact that she knew that $^{1.5}/_{3}$ was rational to reason that if pi was rational, then $^{\pi}/_{3}$ would be rational as well, the fact that Nina believed π to be rational did not lead her to conclude that $^{\pi}/_{1}$ would also be rational. Instead, in her categorization of $^{\pi}/_{1}$, Nina revealed a lack of understanding that every real number must be either rational or irrational, as well as a general lack of understanding of the set of irrational numbers.

Korey also revealed a lack of understanding that every real number must be either rational or irrational when asked to categorize a fraction involving pi. In particular, Korey stated that while she knew π was irrational, π/3 could be both rational *and* irrational since it was written in the form of a fraction. This revealed further limitations in Korey's understanding of the definition of rational numbers, in particular, since she interpreted this the definition to mean any number written in fraction form.

Summary of findings related to interviewees' categorization of numbers involving pi. In this section, data on the ways the eight interviewees categorized numbers involving pi were presented. Categorizing numbers involving pi revealed limitations in the preservice teachers' understandings of the sets of rational and irrational numbers. Thus, the primary theme to emerge from this data was that the preservice teachers' *general understandings of the sets of rational and irrational numbers may be characterized as limited and often inaccurate*. Despite the fact that pi is a well-known example of an irrational number, not all the interviewees' were certain it was irrational. In some cases, their justifications for categorizing fractions involving pi in some cases revealed *misunderstandings about the complementarity of the sets of rational and irrational numbers*. Nina first categorized $^{\pi}/_{1}$ as neither rational nor irrational but real (because it was on a number line). Korey thought that perhaps $^{\pi}/_{3}$ could be both rational and irrational since it was written in the form of a fraction.

An additional subtheme that emerged from this data was that three of the six students (Andie, Nina, and Korey) who were asked to categorize a fraction involving pi *relied on the physical characteristic of the fraction notation* to justify their categorization, at least initially. Andie and Nina utilized the fact that it was not written in the form $^{a}/_{b}$ where *a* and *b* were integers, to justify their initial categorization of the fraction as non-rational. Korey used the fact that $^{\pi}/_{3}$ was written in fraction form to justify placing it in the rational category.

Finally, Nina's categorization of the decimal form of pi as rational was another example of a way that *the relationship between fractions and decimals was poorly understood*. The fact that she justified this categorization by stating that the number pi must have come from a fraction "somehow" because all decimals come from fractions was another example of the subtheme that *fractions are connected to decimals by division.*

Summary of Findings from Early in the Course

In the previous sections, data from the pretest and first interview were presented that documented the nature of eight preservice elementary teachers' understanding of number, the sets of rational and irrational numbers, fraction and decimal notation, and the relationships between them. Data documenting how all thirty-two members of the class responded to the pretest questions was also presented in order to give a more complete picture of these eight teachers' understandings within the context of their classroom learning community. In particular, the whole class data showed that the responses given by the eight interviewees on the pretest were typically similar to responses given by other members of the same class.

This study focused on preservice elementary teachers' understanding of fractions, decimals, the sets of rational and irrational numbers, and the relationships between them within the larger frame of number and numeration. The data from the above sections came from a pretest given on the first day and from interviews conducted in the first two weeks of the course. Four themes emerged from this data that relate to the nature of the preservice teachers' understanding of these areas of number and numeration early in the unit. In some cases, supporting "subthemes" emerged as well. *Figure 22* summarizes these findings as well as the data sources to which they relate. This section will conclude with a statement of how these findings relate to the research questions guiding this study.

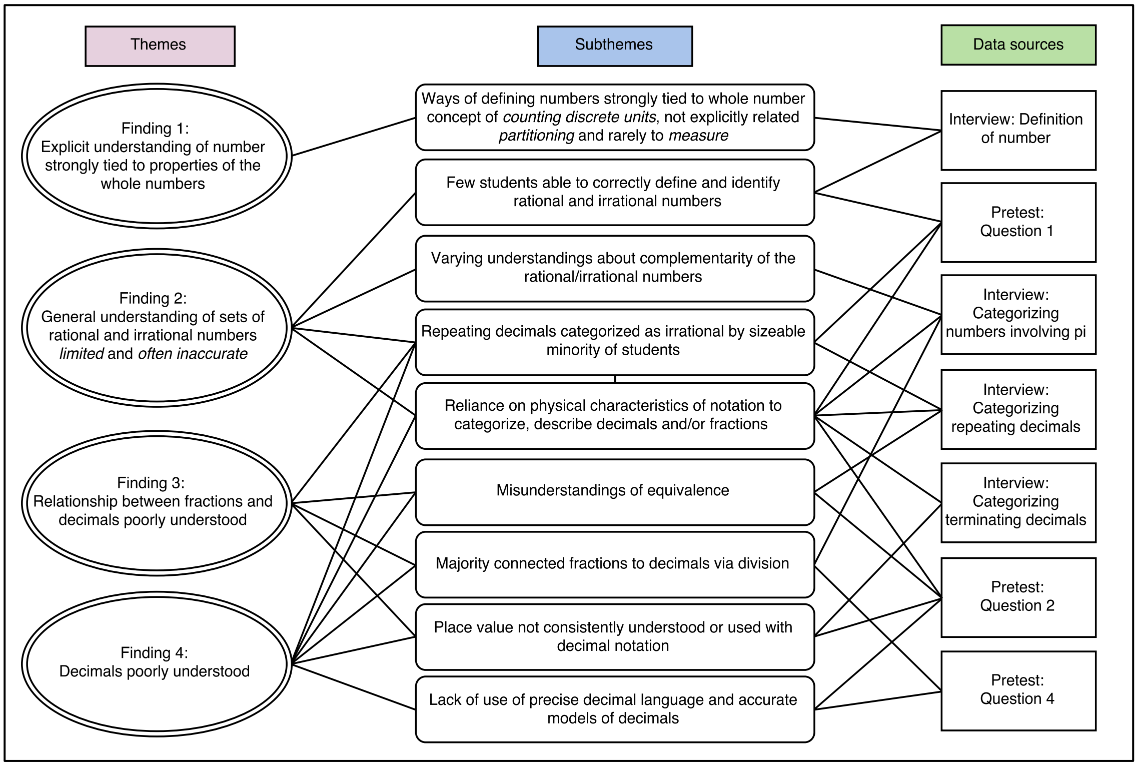


Figure 22. Themes, subthemes, and data sources related to PSTs' nature of understanding the *Number and Numeration System* and use of representations early in the unit.

The first theme that emerged from the data was that the preservice teachers' *explicit understandings of number were strongly tied to the properties of the set of whole numbers*. Data for this theme came from the ways the interviewees defined "number" during the first interview. In particular, all eight interviewees related numbers to *counts of discrete units*. This idea of number is appropriate and well-suited to understanding the set of whole numbers, but is not supportive of understanding more general sets of numbers, including the rational and real numbers. The idea of numbers as being related to *partitioning a given unit* was not addressed by any of the interviewees, a key conception related to understanding the set of rational numbers (Mack, 1993). Only two of the eight interviewees identified the idea that numbers can be thought of as *measures*, a key conception related to understanding the set of real numbers (Dantzig, 2005).

The second theme that emerged from the data was that the preservice teachers' *general understandings of the sets of rational and irrational numbers were limited and often inaccurate.* Data from this theme came from students' responses to pretest Question 1 (list examples and non-examples of rational and irrational numbers), pretest Question 2 (find a number between 0.66 and $\frac{2}{3}$), and the interview number sorting activity where they categorized repeating and terminating decimals and numbers involving pi. One subtheme related to this larger finding was that *few students were able to correctly define and identify rational (and irrational) numbers* at the start of the course. Students' examples and non-examples of rational numbers revealed that few of the preservice teachers began the course with a clear understanding of which numbers are rational. The initial stage of the interview during which interviewees defined and listed categories of numbers verified that rational and irrational numbers were not clearly understood by many students. Only two students in the course, including the interview participant Soren, gave examples of rational numbers that included integers and repeating and terminating decimals or their fraction equivalents. Soren was the only interviewee to clearly define rational and irrational numbers. A second subtheme was that a sizeable minority of students in the class, and four of the eight interviewees, *categorized a repeating decimal as irrational* at least once. A third subtheme was that four of the eight interviewees used the *physical characteristics of the notation to justify their categorization of a number as rational or irrational.* For instance, repeating decimals were categorized as irrational because they repeated, or did not terminate. The fraction

$\pi/1$ was categorized as irrational because π is not an integer so the fraction was not in the form a/b where a and b are integers. Finally, the interviewees' justifications of their categorization of numbers involving pi revealed that at least one student did not understand the sets of rational and irrational numbers to be non-overlapping subsets of the real numbers, and at least one student thought that a number could be real but neither rational nor irrational. Thus, the fourth related subtheme was that *not all students understood the complementarity of the sets of rational and irrational numbers and how they related to the set of real numbers.*

The third theme that emerged from the data was that the *relationship between fractions and decimals was poorly understood* by many students in the course. Data from this theme came from students' responses to pretest Question 2 (find a number between 0.66 and ⅔), pretest Question 4 (explain why ⅓ = 0.333…), and the interview number sorting activity where they categorized repeating and terminating decimals and numbers involving pi. One subtheme related to this was that *repeating decimals were categorized as irrational* by a sizeable minority of students. The fact that repeating decimals were categorized as irrational relates to the theme of the relationship between fractions and decimals being poorly understood because the standard definition of rational numbers uses the fraction notation as the defining characteristic. A second subtheme was that the *majority of students connected fractions to decimals by division*. The division algorithm is an opaque algorithm that hides the key ideas that relate fractions to decimals. In particular, the fact that both fractions and decimals can be related to the idea of partitioning is not apparent when one uses the division algorithm. Second, the fact that a decimal is a sum of fractions with denominators that are powers of ten is also not made obvious by the use of the standard division algorithm. The third subtheme was that *place value was not consistently used and/or understood by students*; in fact, four of the eight interviewees could not use place value to translate any terminating decimal into a fraction at the start of the course.

Finally, the fourth theme that emerged from the data was that *decimals were poorly understood.* Data from this theme came from students' responses to pretest Question 1 (list examples and non-examples of rational and irrational numbers), pretest Question 2 (find a number between 0.66 and ⅔), pretest Question 4 (explain why ⅓ = 0.333…), and the interview number sorting activity where they categorized repeating and terminating decimals and numbers involving pi. Several of the subthemes noted in the previous paragraphs were also related to this theme. First the fact that *repeating decimals were classified as irrational* was related to the fact that some students simply did not have a good understanding of repeating decimals. Second, even for those students who classified repeating decimals as rational, many *relied on the surface feature of the notation*, namely that it was infinite rather than terminating, *to justify the categorization*. Those students who *did not have a good understanding of decimal notation in terms of place value* certainly had little else to work with in terms of reasoning about the decimal. However, even some of those interviewees that did understand place value still reasoned about some decimals based on their notational characteristics (e.g. Jo referenced "predictability" repeatedly in her interview). Finally, two of the indicators of decimal understanding are the use of precise mathematical language and the use of accurate models for decimals (Cramer et al., 2015). There was *very little use of precise mathematical language* with regards to decimals on the pretest and in the interviews. The fact that four of the interviewees could not use place value with decimals certainly contributed to this. There was also *no use of accurate models for decimals on the pretest* (there was not a need for them during the interviews). While the lack of use of such language or models does not mean that the preservice teachers could not do so, the fact remains that they did not. However, the ability to do so is very important for the work of teaching elementary mathematics. Moreover, the questions on the pretest were worded so that they were set in the context of teaching (e.g. "Tell how you would explain to a student why ⅓ is equal to 0.333…").

Relationship of Findings to Research Questions: Early in the Unit

As summarized in the previous section, four themes and their related subthemes emerged from the data gathered early in the unit. Table 16 shows a summary of the major findings with their related research questions and data sources (see *Figure 22* for related sub-questions).

Table 16
Summary of Themes from Early in the Unit, with Related Research Questions and Data Sources

Themes	Related research question(s)	Data source(s)
1. Explicit understandings of number strongly tied to properties of the whole numbers	RQ1a	Interview 1: Def'n of rational/irrational numbers
2. General understanding of sets of rational and irrational numbers was *limited* and *often inaccurate*	RQ1a	Interview 1: Def'n of numbers Interview 1: Number sort Pretest Q1 Pretest Q2
3. The relationship between fractions and decimals was poorly understood	RQ1b RQ2a	Interview 1: Number sort Pretest Q2 Pretest Q4
4. Decimals in general were poorly understood	RQ1b RQ2a	Interview 1: Number sort Pretest Q1 Pretest Q2 Pretest Q4

Response to Research Questions 1a and 1b. The first research guiding this study asked: "What is the nature of preservice elementary teachers' (PSTs) understanding of the *Number and Numeration System* before and after participation in a unit designed to develop their understandings of this content?" This research question had two sub-questions that were related to data gathered early in the unit:

> 1a. What is the nature of PSTs' understandings of the sets of numbers generally, and *rational numbers* specifically, early in the unit?
>
> 1b. What is the nature of PSTs' understandings of fractions, decimals, and the connections between them early in the unit?

As shown in Table 16, Themes 1 and 2 relate to Research Question 1a. First, the ways the eight interviewees defined "number" early in their interviews supported the idea that the preservice teachers' *explicit understandings of number were strongly tied to the properties of the set of whole numbers*. Only the eight interviewees' were asked to define “number” but the consistency in their responses was notable. Second, the preservice teachers’ *general understandings of the sets of rational and irrational numbers were limited and often inaccurate.* Data supporting this theme came from the interviews as well as pretest Questions 1 and 2. Most notably, there was evidence that the majority of preservice teachers did not start the course with an accurate understanding of the sets of rational and irrational numbers. In particular, a large minority of the preservice teachers believed that repeating decimals and/or their corresponding fractions were irrational.

As shown in Table 16, Themes 3 and 4 relate to Research Question 1b. First, the *relationship between fractions and decimals was poorly understood* by many students in the course. Data supporting this came from the interviews as well as responses to pretest questions 2 and 4 wherein students were asked about 0.66 and ⅔, and $0.\overline{3}$ and ⅓. A variety of misunderstandings about the relationship between fractions and decimals were documented in these responses, including the idea that 0.66=⅔, 0.66=$0.\overline{6}$, and three groups of ⅓ equal one, but three groups of $0.\overline{3}$ are only “close” to one. No student gave a clear explanation for why ⅓ and $0.\overline{3}$ are equal. The interview tasks in which students were asked to categorize terminating and repeating decimals further revealed misunderstandings about the relationship between fractions and decimals, including the ideas that “all decimals must have a fraction form” and “no infinite decimal can have a fraction form.” Students’ responses to these all of these questions also supported the theme that, in general, *decimals were poorly understood*. In particular, half of the students interviewed *did not have a good understanding of decimal notation in terms of place value.*

Together, the four themes and their related subthemes all suggest that the nature of the preservice teachers' understanding of the Number and Numeration System as it relates to fractions, decimals, and the sets of rational and irrational numbers was *limited* and *often inaccurate* early in the course.

Response to Research Question 2a. The second research guiding this study asked, "What is the role of representations in the development of PSTs' understanding of the *Number and Numeration System*?" This research question had one subquestion that was related to data gathered early in the unit:

> 2a. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals before/early in the unit?

As shown in Table 16, Themes 3 and 4 relate to Research Question 2a. Overall, symbolic representations were the only representation used by the majority of participants in their responses to the pretest. One notable subtheme that emerged from the responses to the pretest questions and in explanations given during the first interview was that many students *relied on the surface features of a given notation* to reason about the notation.

However, the primary source of data for the ways the preservice teachers' utilized representations early in the unit came from their responses to pretest Question 4. This question asked them to describe how they would explain why ⅓ =0.333… to a student, so this question was the primary opportunity for students to utilize non-symbolic methods for the data presented in this study. The majority of preservice teachers responded that they would use the division algorithm and a minority suggested that they would use repeated addition as their method of explanation. In other words, *their chosen method of* *representation to explain this relationship was symbolic*. In fact, non-symbolic representations such as pictures or stories were only used by seven of the students and in all cases, the models chosen related to a circle model and showed or described the circle being split in thirds. In other words, there was *no use of a non-symbolic model that accurately represented the decimal*. The fact that some of the students could not accurately interpret decimal notation using place value is clearly related to this finding as it is not possible to draw pictures or otherwise represent numbers that have no substantive meaning.

Understandings After the Unit

The previous section presented data from early in the unit. The following sections present the data from selected questions on the posttest as well as relevant excerpts from the second set of individual interviews. Recall that as a descriptive case study (Yin, 2014) set in the second stage of Clements' (2007) Curriculum Research Framework, this study was designed to *describe* the nature of the preservice teachers' understandings related to the connected aspects of number and numeration at two different points in time during their teacher education coursework. It was not designed to *explain* changes in understanding during that time. Instead, the information about these understandings documented in this study are the necessary prerequisite for the design and testing of instructional sequences intended to impact learning (Clements, 2008b).

Notably, the data documented in this section are not completely parallel to the data from early in the unit. As the primary goal of this study was to document understandings that would support curriculum development, the second interview was used to gather information to support a richer description of the preservice teachers' understandings after the unit as they related to the instructional sequence. Thus, the tasks in the second interview were different from those in the first. Similarly, data from questions from the posttest related to these understandings that do not have parallel questions from the pretest will be presented as they relate to the nature of the preservice teachers' understandings after the unit. Where appropriate, comparisons between pre- and posttest data will be made, however, to give a more complete picture of the nature of the preservice teachers' understandings across the unit. This supports calls for research documenting how preservice teachers' mathematical knowledge for teaching may develop over time (Mewborn, 2001; Thanheiser et al., 2013).

The research sub-questions guiding this portion of the study are:

> 1c. What is the nature of PSTs' understandings of fractions, decimals, and the connections between them after the unit?
> 2b. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals after the unit?

To answer these questions, the data presented in this section focus on the aspects of the preservice teachers' understandings related to the connections between fractions and decimals, and between those notations and the set of rational numbers. Data for this section primarily come from the responses to posttest Questions 3, 5, and 6c and Bonus Questions 2 and 3 as the data from these questions relate directly to the research questions guiding this study. *Figure 23* shows these posttest questions. Data from the second interviews and posttest responses will be presented together.

Prior to the presentation of the data, a brief summary of the relevant activities from the instructional unit is provided. Following the presentation of the data, the findings related to the data collected after the instructional unit will be summarized. The section will then conclude with a summary of how the findings from this section relate to Research Questions 1c and 2b, the two sub-questions relevant to the data collected after the unit.

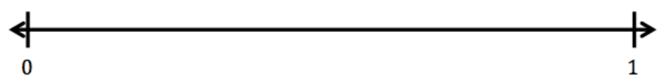
3.	Are there any rational numbers **between** $\frac{1}{3}$ and 0.333? If yes, give an example and justify how you know it is between $\frac{1}{3}$ and 0.333. If not, explain why not.
5.	Show/explain how you could help a student find the decimal representation of the fraction $\frac{1}{6}$ **without** using the standard division algorithm or a calculator.
6c.	Accurately locate each value on the given number line. When more than one number is given, position and label both numbers on the same number line. Briefly explain or show how you positioned each point. d. 0.1666... 
B2.	Write $\frac{1}{3}$ as a decimal in base-7. Show/explain how you got your answer.
B3.	Explain why $\frac{1}{3}$ written in base-7 decimals is similar to $\frac{2}{9}$ in base-10 decimals.

Figure 23. Posttest Questions 3, 5, 6c, and Bonus Questions 2 and 3.

Brief Summary of Relevant Activities from the Instructional Unit

Approximately five class days during the instructional unit were devoted to exploring the relationship between fractions and decimals and the sets of rational and irrational numbers in various ways. Two of these days were devoted to exploring different ways that *a* loaves of bread could be shared by *b* people and how those different ways of sharing the bread connected to various symbolic notations, including the standard fraction and decimal notations. These activities were called the *Breaking Bread* activities. An example of two different ways that three loaves of bread could be shared by four people so that the answer relates to either standard fraction or decimal notation is shown in *Figure 24*.

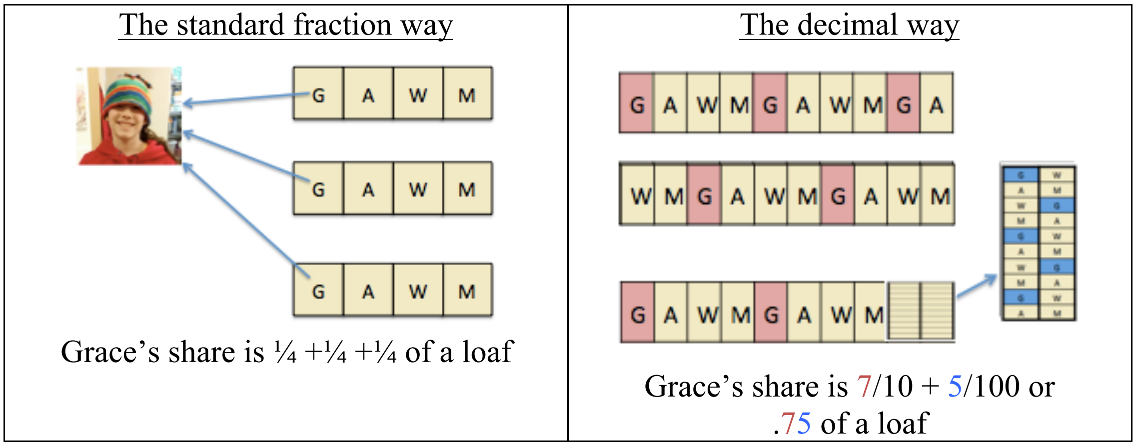


Figure 24. An example of the *Breaking Bread* activity showing how three loaves of bread could be shared by four people so that the answer could be denoted by a fraction (left) or decimal (right).

In addition to the *Breaking Bread* activities, two days were devoted to activities involving the number line and fractions and decimals. One day was devoted number line activities that involved locating fractions and rational decimals on a number line using partitioning. This day also included an exploration of the idea that $0.\overline{9}$ is located at the same spot on the number line as "1" and thus the two are considered equivalent. The second day was devoted to activities related to locating pi on a number line and how irrational numbers such as pi differ from rationals in how they may be located by a process of partitioning. Finally, one class day was spent looking at how the prime factorization of a fraction related to the decimal notation. The instructional sequence is described in more detail in Chapter 3.

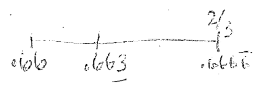
Posttest Question 3: Are There Numbers Between .333 and ⅓?

Posttest Question 3 asked: "Are there any rational numbers between ⅓ and 0.333? If yes, give an example and justify how you know it is between ⅓ and 0.333. If not, explain why not." This question was related to question two on the pretest, which asked: "Are there any rational numbers between ⅔ and 0.66? If yes, give an example and justify how you know it is between ⅔ and 0.66. If not, explain why not." The primary purpose of this question was to document the strategies used by the preservice teachers to compare and order a common fraction (⅓) and a terminating decimal that may functionally be used in place of the repeating decimal $0.\overline{3}$ in calculation (and so may be confused as being equivalent to ⅓). The terminating decimal 0.333 was chosen based on the fact that several of the preservice teachers had relied on their knowledge of percents to work accurately with decimals that terminated in one or two places. This problem forced them to work with and reason about decimal places beyond the hundredths place.

Interviewees' responses to posttest Question 3. Table 17 shows how the interviewees responded to both Question 3 on the posttest and the similar Question 2 on the pretest.

Table 17

Interviewees' Explanations of How to Find a Number Between a Terminating Decimal and Related Fraction on the Pre-and Posttest

Group	Student	Pretest (Question 2)	Posttest (Question 3)
MKT2	Soren	"0.662 is between 0.66 and ⅔ because ⅔ $=0.\overline{66}$."	"Yes, ⅓ = 0.333333…. 0.3333 is greater than 0.333 and < $0.\overline{33}$. It can be written as $\frac{3333}{10{,}000}$, so it is rational."
	Jo	"⅔ is really a number that continues after .66 when made into a decimal. Therefore .663 would be between .66 & ⅔ drawn on a number line."  	"Yes there are rational numbers between .333 and ⅓. Examples of this are 0.3331, 0.3332, 0.33305. I know these are in between ⅓ & 0.333 because they are greater than 0.333 and less than .3333 which is still less than ⅓ or .3333333…."
	Andie	"$\frac{1}{3} = .3\overline{3}$ $\frac{2}{3} = .6\overline{6}$ Yes because $\frac{2}{3} = .6\overline{6}$, therefore .66 ends with a 6 in the 100ths place, while $\frac{2}{3}$ or $.6\overline{6}$ continues infinitely."	"Yes →.3333 This number is between .333 and ⅓ because it terminates after the 1/10,000th place, so it is .0003 bigger than .333 but it is smaller than ⅓ because ⅓ continues infinitely as .3333…."
	Mei	"Yes. For example, 0.665 is between 066 and ⅔. Because ⅔ is irrational number, it's 0.66666…. There're numbers that are larger than 0.66 but smaller than 0.66666…. One example is 0.665."	"Yes. An example is 0.3331. I know ⅓ is 0.333… which a repeating 3 after the decimal. Any rational number that is smaller than 0.333… but larger than 0.333 – I only need the ten thousandth place or after is smaller than 3."
MKT1	Eva	"⅔ = .66 no numbers exist between them."	"Yes, there are numbers that exist between those 2 numbers. The number .3331 comes after .333 and before ⅓ or $.\overline{33}$."
	Nina	.66 3)200 18 20 $\frac{2}{3} \cdot \frac{100}{1}$ $\frac{200}{3}$ "No, ⅔ is equivalent to $0.66\overline{6}$. I know this because $\frac{2}{3} \cdot 100$ then by doing 200 ÷ 3 = $.66\overline{6}$."	"Yes, ex. .3331, .3332. I know this because ⅓ is a repeating decimal of .333… and .333 is terminating. If .333... goes on forever then [these are] between .333 & .333…."
	Willa	"Yes, 0.661 would be between them since ⅔ is $0.\overline{66}$."	"Yes, 0.3331 is between 0.333 and ⅓ because ⅓ is equal to $.\overline{333}$ and 0.3331 is less than that but greater than 0.333."
	Korey	"Yes, 0.665. I know this because ⅔ is $.66\overline{6}$ and decimal places can go on for awhile."	"Yes .3332 is between 0.333 and $0.33\overline{3}$."

Note. Minor changes to spelling, grammar, and punctuation were made to improve readability. Numbers are represented exactly as each student wrote them. If the student included a drawing or similar work, it is included.

As can be seen in Table 17, the biggest change from the pre- to posttest for the four students in the MKT2 group was that three of the four specifically mentioned place value in their responses on the posttest (all but Jo). However, none of them effectively utilized their understanding of place value to give a complete description of why their answer was between 0.333 and ⅓. Soren only used his place value to show that the number he chose, 0.3333, could be written as a fraction and so was rational. Andie only noted that her number (also 0.3333) terminated "after the 1/10,000th place." Mei used place value to make the inaccurate statement that "I only need the ten-thousandth place or after is smaller than 3." This method would produce numbers that are greater than 0.333 and less than ⅓ but excludes numbers such as 0.33308 and 0.333019, both of which are greater than 0.333 and less than ⅓ but have digits in the "ten-thousandth place or after" that are greater than three. Jo did not reference place value at all in her answer. Recall that all four students in the MKT2 group were able to translate terminating decimals to fractions on the pretest and thus certainly were capable of meaningfully comparing these numbers by using either place value understandings or fractions. Decimals (and fractions) with understanding is only possible when their values can be meaningfully compared. Thus, their answers are disappointing in that they *did not apply their decimal understandings* to justify the ordering of the three numbers (including their answer).

Similarly, none of the students in the MKT1 group referenced place value in any way on this problem and so showed a similar lack of understanding of meaningful comparison of decimals and fractions. However, there were promising changes in understanding in this group from the pre- to the posttest. In particular, both Eva and Nina made a distinction between the terminating decimal 0.333 and the repeating decimal $0.\overline{3}$ on the posttest, whereas on the pretest they had not made a distinction between ⅔ and

0.66. Clearly, understanding that these numbers are distinct is an important understanding for elementary teachers who will be teaching their students about equivalence with regard to fractions and decimals. There was not substantial difference between Willa and Korey's explanations on the pre- and posttest for these problems. In both cases, their answers were technically correct, but they did not adequately justify why the number they chose was between the given terminating decimal and fraction.

Whole class response to posttest Question 3. In order to situate the interviewees' responses within the larger scope of the class, Table 18 summarizes the ways that all 32 students responded to question three on the posttest (Table 8, presented previously, summarizes the whole class response to the comparable question two on the pretest).

Table 18
*Summary of Ways the Whole Class Responded to Posttest Question 3 (*N*=32)*

Type	Reason	Number	Example
Correct	Stated the answer is between .333 and 1/3	15	"Yes—0.3333 is between 0.333 and ⅓ because the decimal equivalent of ⅓ is $0.\overline{3}$."
	Used place value to compare the terminating decimal to their answer	6	"$\frac{3333}{10,000} = .3333$, This number is 3 10,000ths larger than the given decimal but still smaller than ⅓ which repeats 3's infinitely."
	Referenced to decimal length	4	"Yes, .3331, .3332, .3333 are just a few examples. .333<.3331<$.\overline{333}$. I know this because adding another number to the end of the 3rd three makes the number larger than .333 but is still smaller than a continuous $.\overline{3}$."
	Used subtraction	1	'Yes. ⅓ = $.\overline{33}$. $\overline{33} - .333 = .00\overline{33}$ .3333 is in between because its bigger than .333 & smaller than $.33333\overline{33}$."
Incorrect	Found number between .3 and ⅓	2	"Yes. Example: 0.33. ⅓ is a repeating decimal ($0.\overline{3}$). On a number line, 0.33 would be between 0.3 and 0.333."
	Stated ⅓ is smaller than .333	4	"Yes, 0.31 is an example. This number is greater than ⅓ and smaller than 0.333. There are an infinite number of rational numbers between these values."

Note. Minor changes to spelling, grammar, and punctuation were made to improve readability. Numbers are represented exactly as each student wrote them. If the student included a drawing or similar work, it is included.

As shown in Table 18, 27 of 32 preservice teachers correctly found a rational number between 0.333 and ⅓. In every case, the answer was a decimal, with the most common answers being 0.3331, 0.3332, and 0.3333. Notably, *no* student argued that the two numbers were equal. In comparison, seven students had argued that 0.66 and ⅔ were equal on the pretest.

Of correct responses, fifteen students justified their answer by simply stating that their number was greater than 0.333 and smaller than ⅓ or $0.\overline{3}$. In other words, they gave no real justification for why their answer was between 0.333 and ⅓. Six students used place value ideas to compare their answer (always a terminating decimal) to the decimal 0.333. However, they did not extend this argument to explain why ⅓ was then larger than their answer, but instead drew on the idea that ⅓ written as a decimal has "infinitely many threes." For instance, Andie stated that .3333 was between .333 and ⅓ " because it terminates after the 1/1000ths place so it is .0003 bigger than .333 but it is smaller than ⅓ because ⅓ continues infinitely as .3333…." In other words, these six students partially justified their response, but only in terms of how their answer compared to 0.333. An additional four students gave a correct repsonse but justified their answer only by referencing the relative lenghts of the decimal notations. That is, they argued that 0.333 terminates after three places, while $0.\overline{3}$ has infinitely many threes, so appending digits to the end of 0.333 would result in numbers between 0.333 and ⅓ (see the example in Table 18 for "Referenced decimal length"). Finally, one student used subtraction to show that the difference bewteen ⅓ and 0.333 was $.000\overline{33}$, but then simply stated that 0.3333 was between .333 and ⅓ (see the example in Table 18 for "Used subtraction").

Six students gave an incorrect response to this question. Four explained that the number they gave was *greater than ⅓* and *less than 0.333*. Three of these students actually named a number that was less than both 0.333 and ⅓ (i.e. 0.331, 0.332, 0.31), and none of them named ⅓ as a decimal. The fourth student, "Ramona[8]," showed evidence of the "shorter is larger" misconception (Steinle, 2004). In particular, Ramona argued that .33333 is "smaller than .333 because it terminates in the $^{1}/_{100,000th}$ place instead of the $^{1}/_{1000}$, and it is bigger than ⅓ because ⅓ is .333…(repeating decimal)."

---

[8] Pseudonym.

Thus RE clearly understood decimal place value, but she was not using it to reason correctly about the relative size of the numbers. Notably, Ramona was the only student who correctly named a number between 0.333 and ⅓ but for an incorrect reason, and she was the only student who showed evidence of holding the "shorter is larger" misconception. The remaining two students with an incorrect answer also named a number less than both ⅓ and 0.333 (0.33 and 5/16=0.3125), but they justified their answer by explaining that they had found a *number between 0.3 and ⅓*. In other words, they treated 0.3 and 0.333 as if they could be used interchangably. Thus, six of the 32 students (19%) in the class showed evidence of misunderstandings of decimal notation, a marked improvement from the seventeen[9] showing misconceptions on the pretest, but still a large number given this is content they will be responsible for teaching in their future classrooms.

Finally, although 26 of 32 preservice teachers were able to correctly identify a rational number between ⅓ and 0.333 on the posttest, none of their answers could have been used to explain to the six students with incorrect answers *why* their answer was incorrect. Importantly, none of these preservice teachers used common denominators to compare and order the three numbers (including their answer) in this problem, despite this being the approach they will need to teach students to use. Troublingly, six of these students used the *length of the decimal* as their primary method to justify ordering decimals, a strategy that can support the common misconception among younger students that "longer is larger" (Steinle, 2004). While it is certainly possible that many or all of

---

[9] An additional student left the problem blank on the pretest, for a total of 18 students with an incorrect response on the pretest to the question "Are there numbers between 0.66 and ⅔?

these students could have used common denominators to make this comparison if asked to do so, the fact that they did not find that step necessary without prompting suggests that they do not yet have a full understanding of what it means to compare and order rational numbers. It should be noted that this work was done as part of the final test for this unit and was a large part of their final grade (25%), so it is reasonable to assume that they were trying to show their best work on this problem.

Summary of findings related to posttest Question 3. The primary theme was that emerged from the data related to the responses to posttest Question 3 was that *the preservice teachers did not consistently apply their decimal understandings* to justify the ordering of the three numbers in the problem (including their answer). A minority, six of the thirty-two, used their place value understandings to meaningfully compare their chosen decimal with the decimal 0.333 (see the example in Table 18 for "Used place value to compare the terminating decimal to their answer"). However, none of these students meaningfully compared their answer to the fraction ⅓. Instead, if a justification was given for the ordering of a decimal to the fraction ⅓, the students instead noted that $0.\overline{3}$ has an infinite number of threes. In other words, they *relied on the physical characteristics of the decimal notation to justify ordering of numbers*. Teaching students to meaningfully compare and order fractions and decimals means that teachers need to support the development of students' mental models of the quantities being compared (Cramer, Monson, Wyberg, Leavitt, & Whitney, 2009; Cramer, Post, & DelMas, 2002). The preservice teachers' responses to this question therefore did not demonstrate sufficient *mathematical knowledge for teaching* elementary students. However, one promising result from this section was that the two interviewees (Eva and Nina) who had failed to distinguish between 0.66 and ⅔ (or $0.\overline{6}$) on the pretest did note that 0.333 and ⅓ were different numbers. In fact, only two students in the course treated 0.3, 0.33, and/or $0.\overline{3}$ as equivalent in their justifications of ordering, so there were *limited examples of misunderstandings of equivalence* in the response to this problem.

Posttest Question 5: Find ⅙ as a Decimal

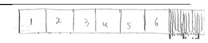
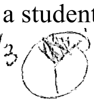
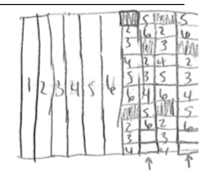
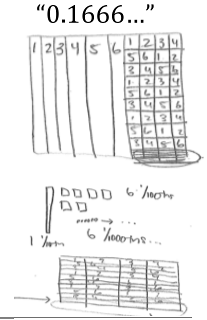
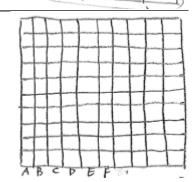
Posttest Question 5 asked, "Show/explain how you could help a student find the decimal representation of the fraction ⅙ without using the standard division algorithm or a calculator." This was related to pretest question four which asked, "Tell how you would help a student understand why ⅓=0.333… when written as a decimal." The wording, "without using the standard division algorithm," was used on the posttest in order to document if and in what ways students were able to describe this relationship other than by using division after the unit. The following sections will present a summary of the ways the eight interviewees responded to this question on the posttest, with their responses supplemented with relevant excerpts from their second interviews as appropriate. The data related to the four students in the MKT2 group will be presented first, followed by the data related to the four students in the MKT1 group. A summary of the whole class responses will be presented last.

Interviewees' responses to posttest Question 5: MKT2 group. As shown in Table 19, the four interviewees in the MKT2 group gave very similar answers to how to find ⅙ as a decimal on the posttest. In particular, they all utilized the context and picture introduced in the course where loaves of bread are shared fairly among a group of people (called the "Breaking Bread" stories). Each of these students clearly described a process of dividing one loaf of bread among six people, but doing so in a way that always creates partitions that are powers of ten (so the answer may be written as a decimal[10]) so that the answer may be denoted by a decimal. They each described one person's share using correct decimal-fraction language, such as one-*tenth* and six-*hundredths*, making clear connections between the decimal notation and size of the pieces each person would receive. Three of the four students (all but Mei) also clearly depicted this process pictorially, showing each person receiving one-tenth of the loaf, then six hundredths, and the remaining four hundredths that would still need to be shared. Mei did not show this clearly, but did show a square divided in hundredths, and also clearly described the process of sharing the pieces depicted. This was a notable change from the pretest where none of the four MKT2 students described or depicted the decimal notation non-symbolically in their explanation of why $\frac{1}{3}=0.\overline{3}$. The descriptions given by these four students to this problem show that all four would be capable of clearly describing and showing why the decimal form of the fraction one-sixth repeats and is denoted "0.166…," precisely the type of knowledge necessary to teach this content meaningfully to elementary students.

[10] Always partitioning in powers of ten (e.g. tenths, hundredths, thousandths, etc.) was deemed "decimal law" in the course).

Table 19

MKT2 Students' Work on Comparable Problems Relating a Fraction to a Repeating Decimal on the Pre- and Posttest

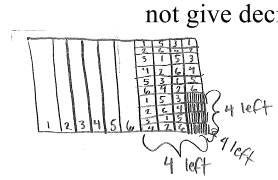
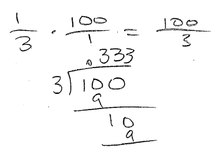
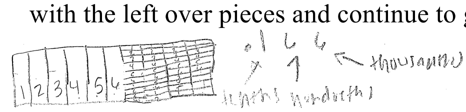
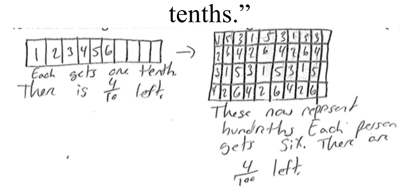
Student	Pretest (Question 4)	Posttest (Question 5)	
Soren	$\frac{1}{3}=\frac{3}{9}$ $\frac{1}{9}=0.\overline{111}$ $\frac{2}{9}=0.\overline{222}$ $\frac{3}{9}=0.\overline{333}$ $\frac{4}{9}=0.\overline{444}$	"If we want to find a decimal, we partition into powers of 10. 1 loaf of bread split between 6 people. We split into tenths, and each person gets 1. We split the remaining 4/10 into hundredths, and each person gets 6 of them. There are 4/100 left, so we split it into thousandths. There will always be 4 left, so the partitioning will never be equal. 1/10 + 6/100 + 6/1000 + … = 0.166…"	
Jo	"⅓ shows how much of a whole is present. One "whole" in terms of decimals in 1.0. I could show a pie chart. I could also use the percentages or division to show this to a student." .33 = 1/3 	"Let's think of sharing a loaf of bread with 6 people. However, we have to follow the 'decimal law.' That means every time we cut the bread, it has to be into 10 equal pieces. First everyone got 1 tenth of the bread. There are 4 tenths left over. Split those into 10 equal parts each (these are hundredths of the original loaf). Everyone got 6 hundredths of the loaf. There are 4 hundredths left over. Split these into 10 equal pieces. Continue doing this [and] the people will always get 6 pieces with 4 left over. As a decimal this is .1666… because each person gets 1 tenth, 6 hundredths, 6 thousandths, etc."	
Andie	"I would explain that when we divide 1 by 3 the answer is not whole, & the 3 repeats infinitely (or forever)."	"⅙ = 1 loaf of bread shared by 6 people." "Share 1 loaf of bread with 6 people, divide the bread into 1/10ths (decimals have to be thought of as 1/10ths). See that each person gets 1/10th and 4 1/10ths are left. Partition those 1/10ths into 1/100ths and share equally amongst 6 people. See that there are 4 left again. This will continue forever."	"0.1666…" 
Mei	"I would let them to use the math '1 ÷ 3' and let them try to solve the math. Once they see they will always get 3 after the decimal, we would make the predication that there will be always 3 after the decimal."	"Create a problem: 1 loaf of bread is shared by 6 people. What is the decimal of a loaf of bread each person can get? Then use base-10 model of cubes to help them understand how to solve the problem. Each person gets 10 of cubes first, the tenth place is 1. Divide the rest 40 cubes equally, each gets 6 cubes then, the hundredth place is 6. 4 cubes are left, divide them again to get thousandth place. Get $\frac{1}{6}$ = 0.16_ _…"	

Note. Minor changes to spelling, grammar, and punctuation were made to improve readability. Numbers are represented exactly as each student wrote them. If the student included a drawing or similar work, it is included.

Interviewees' responses to posttest Question 5: MKT1 group. As shown in Table 20, three of the four students in the MKT1 group used the *Breaking Bread* story and/or picture in order to explain how to find ⅙ as a decimal. Nina and Willa gave answers very similar to the four students in the MKT2 group. Both clearly explained in words and pictures both what one-sixth is when written as a decimal and why it repeats. Recall that all four students in the MKT1 group could not translate a terminating decimal into a fraction using place value during the first interview, so the fact that both Nina and Willa were able to clearly describe decimal numeration using precise mathematical language represented an important shift in their understanding and indicated that they were developing conceptual understanding of decimals (Cramer et al., 2015).

Table 20

MKT1 PSTs' Work on Comparable Problems Relating a Fraction to a Repeating Decimal on the Pre- and Posttest

Student	Pretest (Question 4)	Posttest (Question 5)
Eva	$\frac{3}{3} = 1$ pie "If you think about the fraction as a pie, there are 3 slices of pie total. We need to figure out how to think about how 1 slice of pie from 3 looks as a decimal. Divide 1 piece of pie by 3 and your number should be less than 1."	"We would have to keep partitioning and always have 4 left. This is why the number repeats." [Did not give decimal form of ⅙] 
Nina	"I would help the student by showing them to multiply by 100 and then if you divide 3 by 100 you get .333." 	"1 loaf of bread shared by six people, split into tenths. By decimal law each person get 1 tenth of the loaf then each person gets 6 hundredths of the loaf. If you continue to split the 4 hundredths left over into thousandths each person will get sixth thousandths and you would continue this process with the left over pieces and continue to get 6." 
Willa	"I would explain that 1 cannot be divided by 3 evenly so adding it 3 times written as 0.333… would get it closest to 1."	"Using decimal law, we must always partition into tenths."  "The process continues but will never end. So each person will get 1 tenth, 6 hundredths, 6 thousands [sic], and so on, which represented as a decimal would be $0.1\overline{666}$."
Korey	"To find the decimal of ⅓ you would divide the bottom number (denominator) into the top number (numerator) and you would get the decimal .333."	$\frac{1}{6} = 0.166\ldots$ "I don't know yet how to explain how to find the decimal form w/out using standard deviation [sic]."

Note. Minor changes to spelling, grammar, and/or punctuation were made to improve readability. All numbers are presented exactly as the student wrote them. If a student drew a picture, it is included.

During the first interview, Nina expressed confusion about whether two-thirds was equivalent to 0.6, 0.66 or $0.\overline{6}$. During the second interview, she explained how the *Breaking Bread* pictures helped her understand the distinction between these numbers.

> Well, I didn't understand/ because I think was confused about which one would be bigger or point six, point six six six. But now you can, I think one of the questions on the test was like point six six six one would be bigger. So decimals can just keep repeating. And I liked figuring out, with the pictures breaking it up, how there's always going to be six left and you can't/ I think that helped to show that they're different. (Nina, interview 2)

A few turns later, she then demonstrated how she could use the story and picture to determine that two-thirds was the same as $0.\overline{6}$.

> N: "Three people. Okay. [drawing] And you'd have ten because it's base ten. [drawing] Each person is going to have six so it's going to be point six. [counting then drawing]. Then this is what's left over. And you're going to split it up again, so it's going to be a repeating decimal if you would keep going. So two-thirds would be the point six repeating rather than the point six because these are what's left."
>
> I: "Okay, and how come it repeats? You know, it repeats as point six six six six? Why all sixes?"
>
> N: "I'm just going to make these bigger. So you would do this, and you would cut them again [drawing]. And then you each would get six again and there's going be two left over. And then each would get six again and there's going to be two left over. [drawing] And the number next would be the thousandths. And then do you want me to keep going?"
>
> I: "No. Do you see why it's always repeating?"
>
> N: "Yeah. It's so we have two left over every time. And you just keep splitting those into ten and you have the same six again. So it's going to just go on forever."

Nina also described showing 0.6 as "six loaves shared by ten people" and "each person would get one of the ten pieces." She furthermore was able to see those same six-tenths in the picture she had drawn for the two-thirds when prompted to do so and why this showed that two-thirds was greater than six-tenths. In other words, the nature of Nina's understanding of the relationship between repeating decimals and fractions after the unit was similar to the understandings shown by the four students in the MKT2 group.

Like Nina, during the first interview, Willa had also had difficulty distinguishing between repeating decimals and similar terminating decimals (e.g. 0.3 and $0.\overline{3}$). During the second interview, Willa also stated that the *Breaking Bread* story and picture had helped her understand the difference between the two numbers.

> I: "How about how does point three, this one right here, relate to one-third?"
> W: "That would be close, but not exactly the same. Because if it's point three repeating, you can still get things after the point three. But it would be pretty close on the number line."
> I: "Can you point at all what helped you figure that out a bit? This was something that you said several times at the beginning that point three and one-third, that that was the same. So, I'm curious what helped you figure/ because right now you're confident they're not."
> W: "Well, when we did the bread, we took like, the one third. We had one and we would split it into thirds. And we'd split it in thirds again if we were doing it by two people or whatever. So you get another third and another third, and it just keeps on going and you're not going to get rid of that last piece."

Note that Willa's inclination here was to describe splitting each rectangle into *thirds* rather than *tenths*, which, if she had drawn it, would not have resulted in a picture that depicted the repeating decimal $0.\overline{3}$. She also had some difficulty describing how she would use the *Breaking Bread* story to depict three-tenths (0.3).

> I think that would be if you give one and you have (…) That would be like it you have one slice and you have to give (…) It's split into, I want to say tenths? But I don't think that's right because you'd still have the one left if you gave it to three people. Unless you split it (…) I know it's three-tenths. So it's done. It just stops. (Willa, interview 2)

Despite her successful use of the *Breaking Bread* story and picture on the posttest, Willa's understanding of this story and context was still quite fragile. However, it did support for her the mental image of a sharing process that continued in the case of the repeating decimal versus a sharing process that "just stops" for the terminating decimal, which, in turn, she stated helped her understand why 0.3 and $0.\overline{3}$ were different numbers. Moreover, the above statements show that Willa clearly also understood that a repeating decimal could be related to a fraction, something that she did not believe during the first interview. Recall that during the first interview, Willa argued that 0.66 was equal to ⅔ whereas the repeating decimal $0.\overline{6}$ could not be exactly equal to two-thirds because "you can't represent it as a fraction because it doesn't have a set quantity." Thus the nature of Willa's understanding of the relationship between fractions and repeating decimals after the unit was fragile, but far more mathematically sound than it had been early in the unit.

Like Nina and Willa, Eva also drew a picture correctly showed how a rectangle could be divided evenly among six individuals using tenths, hundredths, and so on. However, unlike the other two, Eva did not show what one-sixth would be when written as a decimal based on this picture. During her second interview, it became clear that she while she could draw the picture, she was not connecting the picture to the decimal notation. Just prior to the exchange below, Eva was discussing repeating decimals and the picture she drew for question five on her posttest.

> I: "Can you tell me one thing? You did this all totally perfectly. But the one thing you didn't write down, is you didn't write the decimal. So can you tell me what that decimal would be that's in your picture?"
> E: "I think. I don't know. Um (…) I don't know if I remember how to. I remember/ isn't it point?"
> I: "The decimal tells how much of each piece they get."
> E: "Right."
> I: "So you had one loaf of bread shared by six people. And you split it into ten pieces because you're in decimal law. And you gave out one two three four five six of the tenths. So this person got this one, this person got this one. Each person got one tenth. Right here, that's what that is right there. So we would put a one in that first spot to say that each person got one of these big pieces."
> E: "Um hm."
> I: "And then four of them were left and you split those ones up."
> E: "And each person got [counting]. I know how to do it but I don't."
> [7 turns omitted. Eva counts one person's share of the "hundredths" in her picture]

E: "So then people are getting point, or I'm trying to think in terms of the decimal I suppose. Point oh six but they're still getting this point one here."
I: "So they're getting one tenth. And what's the name of these little pieces here?"
E: "Um, one fortieth? Or is that out of the whole?"

Note that Eva did not use precise mathematical language when referring to the decimal in this exchange, but instead naming six-hundredths as "point oh six" and one-tenth as "point one." She made it clear that she did not see the connection to the decimal partitions and the picture when she named the hundredths in her picture as fortieths (there were forty-hundredths in her picture). However, as our conversation continued and Eva began to see her picture as depicting hundredths and thousandths, her language shifted.

I: "Out of the whole loaf. If we continued across, there would be ten groups of ten."
E: "Or a hundred. So they're getting six out of a hundred, but it's six-hundredths, right?"
I: "Exactly. And that's how our decimal thing works. Literally all you have to write is the six, that's why you do it that way. So we're splitting this into tens so we can just do this. So we can just write it perfectly and you don't have to worry about anything. And so we got tenths and we write how many tenths you get. Then we split into hundredths and we write how many of those. And then you'll split those that are left up into ten pieces. So what will the name of those be?"
E: "Thousandths."
I: "Um hm. And how many of those would each person get?"
E: "They're getting six-thousandths. So that is why it is like that." [referencing the decimal]

Thus, although Eva had drawn a picture that could support reasoning about what one-sixth would be in decimal notation, she did not initially *interpret* that picture as relating to the decimal notation. Notably, once she did connect her picture to partitions showing powers of ten, she both immediately started using precise decimal language ("it's six-hundredths, right?") and stated about the decimal "So that's why it is like that." As with

Nina and Willa, this marked an important shift in Eva's understanding of decimal notation from the first to second interview, as she consistently stated throughout the first interview that she did not know how to relate a (terminating) decimal to a fraction.

Unlike the other seven interviewees, Korey did not make use of the *Breaking Bread* story and/or picture in answering posttest question five. Instead, she wrote, "$\frac{1}{6}$=0.166…," then stated that she was unable to explain this without the standard division algorithm ("standard deviation"). After completing her test, she approached the researcher about this problem and she and the researcher discussed how to use the *Breaking Bread* story and picture to solve the problem. Early in the second interview, she stated that she felt more confident about finding the decimal form of a fraction without division after our conversation.

> I like the fraction to decimal thing. Like with the "How do you get the one-sixth, how do you explain/ how does it go into a decimal?" I liked that. That kind of stuck with me, especially after the test, and you were like, and I was like, "Oh yeah, I forgot, that's how we're supposed to do it!" And it really stuck with me. (Korey, interview 2)

She was then asked to show how she would find one-sixth as decimal using the *Breaking Bread* context.

> K: "So, this is how I think of it. The one on top means how many of something you have. So we've been doing bread so I'll do bread [drawing]. So then you have one loaf of bread. And the number on bottom, the denominator is how many you need to share it with the one. Oh and then it's also in tenths because it's in its one place so it's in the tenths. Yeah, that's it [drawing]. So then I split it into six [drawing]. So first I split it into three and then I split into two."
> I: "So thirds and then halves make sixths?"
> K: "Yeah. Oh wait, is that how/?
> I: "How much would each person get right now?"
> K: "Each person would get, if I was splitting into six, each person would get one."

I: “Everyone would get one. And what’s the name of those parts?”
K: “Um, one tenth.”
I: “One?”
K: “Oh, one sixth!”

There are two notable understandings that Korey showed in this passage, despite the fact that she, like Soren, initially made sixths rather than tenths. First, she named the parts she created using correct fraction language, “six*ths*” and “ten*ths*.” This was a change from the first interview where she named all but the most common fractions (i.e. ½, ⅔) using whole number language only. For instance, during the first interview she named 2/429 as “two over four-twenty-nine” and 67/94 as “sixty-seven ninety-four.” Second, she knew that her picture needed to show *tenths* in order to relate to the *decimal*. Like the other three students in the MKT1 group, the fact that Korey knew that decimals involved partitions of powers of ten marked a major shift in her understanding of decimal notation between the first and second interviews.

After Korey partitioned her rectangle into sixths, a discussion ensued of how her picture related to the fraction notation ⅙, and that decimal notation required her to create tenths, hundredths, thousandths, and so on. The following exchange then occurred.

K: “Oh! Oh! That’s what I forgot. Okay. So if it’s in tenths. So then get one piece of bread because that’s still the same, you split in tenths, tenths.”
I: “Because you need tenths if you want decimals.”
K: “Let’s pretend it’s even.” [referencing her drawing]
I: “That’s fine.”
K: “So then each person would get one. And I know you guys do a, b, and c, but I/” [referring to how we labeled each person’s share of bread in pictures in class]
I: “You do whatever you want.”
K: “One, two, three, four, five, six [writing on drawing]. So then each person gets (…) point one because each person gets one of a tenth?”
I: “Exactly. And that’s what that says. It says ‘one tenth.’” [pointing to decimal]

K: "And then these you have to split. So there's only four left, so then you have to split this stuff into tenths again?"
I: "Right, because you want to be able to talk about these things in terms of decimals, and decimals always have tens, hundreds/ tenths, hundredths, thousandths."
[Six turns omitted, Korey describing how she is partitioning]
K: "Okay so then they get one, two, three, four, five, six."
I: "So everyone gets six, and what was the name of those little pieces?"
K: "Six, these ones are hundreds."
I: "How come?"
K: "Because if you split it all the way across, it turns into hundreds."
I: "There you go. And that let's you write it as a decimal //That's why you want to do that."
K: "Yes. And then there's four left over, so then again you'd have to split and split and split, then you can tell its/ there would always be four left over."

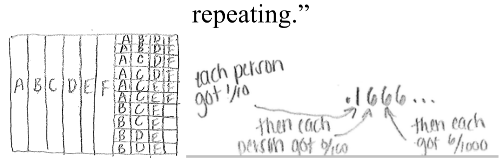
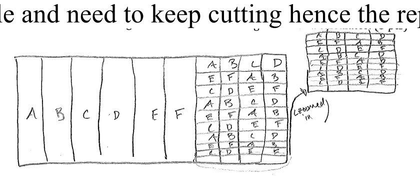
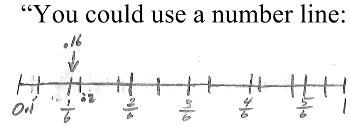
Note that in the above exchange, Korey referred to the pieces as "hundreds" rather than "hundred*th*s," indicating that her language use with regard to decimals was still fragile. However, also note that she was making clear connections between the picture, the decimal notation, and the reason why the decimal notation was repeating. Thus, the nature of Korey's understanding of the relationship between fractions and repeating decimals after the unit was fragile, but based on the idea that both fractions and decimals were notations that depicted *partitioning*. Furthermore, her understanding of decimal numeration after the unit was based on an understanding that decimal notation is related to partitions of powers of ten, a much more mathematically sound understanding of decimal numeration than she showed early in the unit.

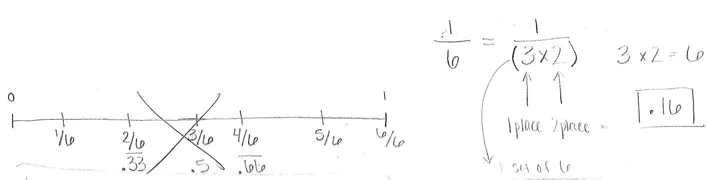
Whole class response to posttest Question 5. In order to situate the eight interviewees' responses within the larger frame of the class, in this section, a summary of the ways the whole class described how to find the decimal form of the fraction ⅙ without using the standard division algorithm will be presented. Recall that the majority of students (56%) used division as their primary or only method of describing this relationship on the pretest.

As shown in Table 21, the majority of the class (24 students, 75%) used the *Breaking Bread* story and/or picture to correctly solve this problem. Similarly, six out of eight interviewees (75%) also used this same method and solved the problem correctly. Two students in class, including Eva, created a correct picture for this situation, but did not write the corresponding decimal related to $\frac{1}{6}$. Three students in the class stated that they would use a number line to show how to find the decimal representation of $\frac{1}{6}$. All three students who used this method stated that $\frac{1}{6}=0.1\overline{6}$ and that they would locate $\frac{1}{6}$ and $0.1\overline{6}$ at the same point on the number line. In other words, these students showed how they could model the *equivalence* of these two notations, but not why the two were equal. None of the interviewees used this method. Finally, three students, including Korey, took a purely symbolic approach to this problem. As stated previously, Korey said that she could not do the problem without using the division algorithm. Another student showed the standard division algorithm on her page as her description of how to find the decimal form of $\frac{1}{6}$. The third student, "EL," showed a number line crossed out and the fraction $\frac{1}{6}$ written in its prime factorization form "$^{1}/_{(3x2)}$" (see Table 21). Overall, the whole class' responses to this question showed that, like the interviewees, the majority of students in the course showed evidence of a more flexible understanding of the relationship between fractions and decimals after the unit as 75% of them were able to use non-symbolic methods (picture, story, words) to show why one-sixth is the repeating decimal $0.1\overline{6}$.

Table 21

*Summary of Ways the Whole Class Showed How to Find ⅙ as a Decimal on the Posttest (*N*= 32)*

<table>
<tr><th>Type of response</th><th>Number</th><th>Example</th></tr>
<tr><td>Rectangle divided into tenths, hundredths, etc.</td><td>24</td><td>“You have 1 loaf of bread that is shared among 6 people. Decimal law states that you have to make cuts into tenths. Each person gets 1/10 and you have 4/10 remaining. You partition the 4/10 into tenths, which would actually represent hundredths. Then each person would receive 6/100. There are still 4/100 leftover so you would keep repeating this pattern over and over. There will always be 4/(some power of 10) leftover so you know your decimal is repeating.”

</td></tr>
<tr><td>Rectangle shown, no decimal answer given</td><td>2</td><td>“One way to explain is to represent 1 loaf of bread for 6 people. Since we’re using decimals, you divide the loaf into 10 pieces. Each person gets 6 piece, you are left with 4, which you further partition into 100ths. After assigning these pieces to 6 people, the student will realize that every will initially get 1 piece and then, you will constantly be partitioning because you be left with 4 pieces for 6 people and need to keep cutting hence the repeating decimal.”
</td></tr>
<tr><td>Number line</td><td>3</td><td>“You could use a number line:


“First partition the number line of 0→1 into sixths. Then partition it into tenths (decimals). Having located ⅙, now we see that $\frac{1}{6} = 0.\overline{16}$.”</td></tr>
<tr><td>Other</td><td>3</td><td>
</td></tr>
</table>

Note. The question prompted students to show how to find the decimal without using the standard division algorithm.

Summary of findings from responses to posttest Question 5. Three related themes emerged from the data related to the ways the preservice teachers explained how to find ⅙ as a decimal on the posttest. First, there was *widespread evidence of understanding of the relationship between the fraction ⅙ and the decimal* $0.1\overline{6}$. On the posttest, the majority of the students in the course and six of the eight interviewees *clearly described the division process as partitioning* a given unit into tenths, hundredths, and so on in order to find the decimal form of the fraction ⅙. The remaining two interviewees were able to successfully describe this process during the second interview (with some support).

Second, there was *widespread evidence of decimal understandings* in the preservice teachers' descriptions of how to find the decimal for ⅙. There was widespread use of *precise mathematical language* to describe the decimal $0.1\overline{6}$ as one-*tenth*, six-*hundredths*, six-*thousandths*, and so on. In fact, many students actually wrote these words out in their descriptions. Notably, two of the four interviewees (Nina and Willa) who were unable to use place value understandings to translate terminating decimals to a fraction during the first interview gave very clear description of the decimal places as "tenths," "hundredths," and so on in their responses, an encouraging shift in their understanding of decimal notation. Additionally, there was *widespread use of an accurate model for the decimal* in students' responses. The ability to make this division process transparent for students along with the ability to accurately model and describe decimals using precise mathematical language are important skills for elementary teachers to develop. The fact that so many of the preservice teachers showed these understandings on this problem makes the lack of use of such ideas on posttest Question 3 that much more striking.

Third, the use of *non-symbolic representations supported students in making connections between fractions and decimals and in developing decimal understandings.* In particular, during the second interview, Nina, Willa, and Korey all referenced the fact that the idea of loaves of bread being split in particular ways had helped them better understand concepts presented in the unit.

Posttest Question 6c: Locate 0.166… on a Number Line

Posttest Question 6 asked, "Accurately locate each value on the given number line…. Briefly explain or show how you positioned each point." Posttest question 6c asked students to locate the repeating decimal "0.1666…." This was related to posttest Question 4 which asked students to find ⅙ as a decimal. In the following sections, a summary of the ways the eight interviewees responded to this question on the posttest will be presented. This will be supplemented by their responses with relevant excerpts from their second interviews as appropriate. The data related to the four students in the MKT2 group will be presented first, followed by the data related to the four students in the MKT1 group. A summary of the whole class responses will be presented last.

Interviewees' responses to posttest Question 6c: MKT2 group. As shown in Table 22, three of the four students in the MKT2 group were able to apply their knowledge that ⅙ $=0.1\overline{6}$ to precisely locate the decimal as a number on a number line. Jo, Andie, and Mei all partitioned the unit interval into sixths and located $0.1\overline{6}$ at the tick mark for one-sixth. Andie and Jo even specifically noted that ⅙ is equivalent to $0.1\overline{6}$ in their explanations. Soren, on the other hand, tried to locate the repeating decimal using the decimal notation. His explanation shows that he clearly understood the decimal $0.1\overline{6}$

correctly as a sum of fractions with denominators that are powers of ten. His picture shows that he was correctly partitioning the interval into tenths, hundredths, and so on to locate the decimal. The problem with Soren's picture is that he did not depict the actual location of the number $0.1\overline{6}$ on the number line, but instead depicted the sequence of points that converge (on the lefthand side) to the point $0.1\overline{6}$. Interestingly, during the second interview, Soren stated that he would use the fraction form of a number to locate a repeating decimal on a number line. In particular, he was asked about the repeating decimal $0.\overline{583}$, which he had described locating by the same "successive partitions by ten" process that he used to locate $0.1\overline{6}$ on the posttest.

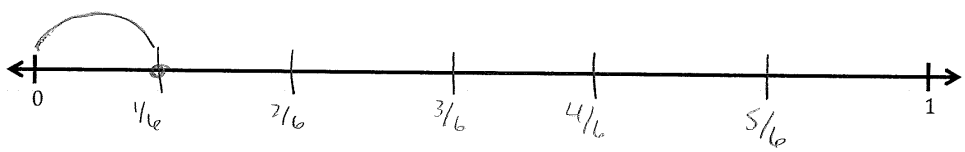
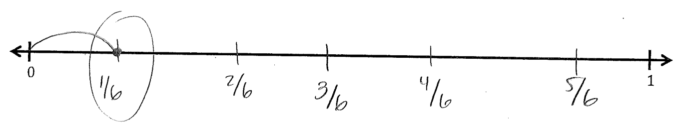
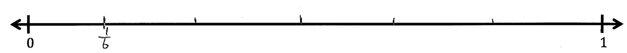
> I: "Okay. And then I asked you about a different type of repeating fraction, I think it was this one, and I said, where would that go? And you told me, well you could try to locate it. You get closer and closer and closer to it. You put five tenths and eight hundreds and three thousandths. That's the process you said. And that would approach, you would get close to the right number, it would kind of help you see where it was. So that was your strategy. Can you tell what would you do now?"
> S: "So that's, I think, five hundred eighty three over nine hundred ninety nine. Just like your general strategy. So, I mean if you wanted to partition that, that's how you would find it exactly."
> I: "Okay, what would you do?"
> S: "So I don't know if I can simplify that at all."
> I: "What if, could you do it without simplifying? I mean, I'm not going to make you do it."
> S: "Okay."
> I: "Let's be clear on that. I just want you to tell me how you'd do it if you had all the time in the world."
> S: "Then—and a really big line… Partition into nine hundred ninety-nine and count over five hundred eighty three."

First, the fact that Soren simply knew that $0.\overline{583}$ was equal to 583/999 speaks to his strength as a student of mathematics. Second, he demonstrated here that he did understand how to use the fraction form of a repeating decimal to locate it precisely on a number line. As he was not asked about his different answer to this similar question on the posttest, it is unclear why he did not use the fraction form in that case.

The notion that every decimal represents a precise location along the real number line is a key conception that supports understanding *decimals as numbers*. Interpreting the repeating decimal as an infinite sum instead supports the notion of the decimal as a *process* (Sfard, 1991). Between Jo, Andie, and Mei's answers on the posttest and Soren's explanation for how to locate $0.\overline{583}$ on a number line during his second interview, all four students demonstrated that they could use the number line as a tool for representing repeating decimals as a *number* rather than a *process* when they knew the equivalent fraction for a given repeating decimal. They also showed that they understood that a repeating decimal could name the *same point on the line* as the fraction, a key idea necessary for understanding why fractions and repeating (and terminating) decimals are part of the same set of numbers called the rationals.

Table 22

MKT2 Students' Work on Posttest Question 6c: Locate 0.166… on a Number Line

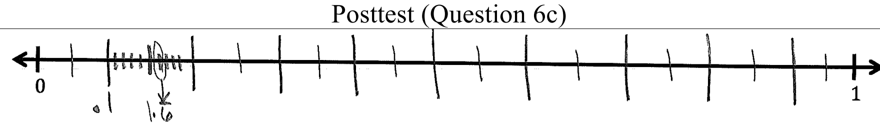
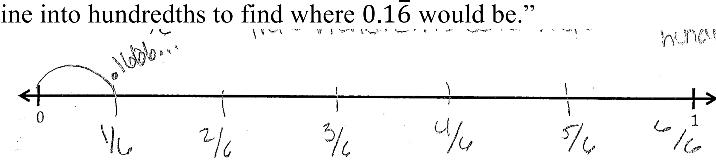
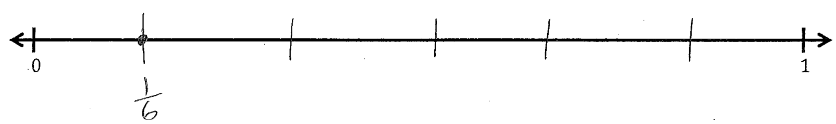
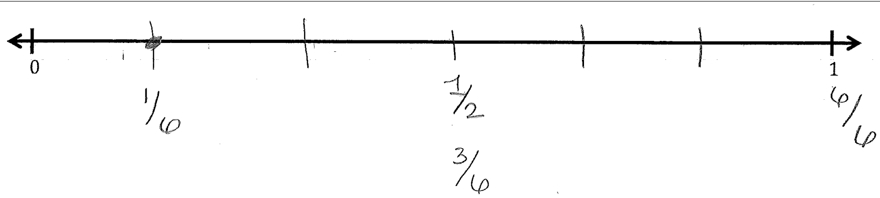
Student	Posttest (Question 6c)
Soren	 "Partition into tenths, and count over 1. Partition the second tenths again into tenths and count over 6 again. Repeat indefinitely."
Jo	 "⅙ is equivalent to .1666…. I partitioned the unit into 6 equal parts and counted over 1 interval."
Andie	 "I knew I couldn't locate .166… on the number line in this form. I knew that .166… is equivalent to ⅙th. I partitioned the unit into ⅙ths & was able to locate ⅙th."
Mei	 "Partition the unit interval [0,1] into 6 parts and locate ⅙.

Note. Spelling, grammar, and punctuation are preserved. Numbers are represented exactly as each student wrote them. If the student included a drawing or similar work, it is included.

Interviewees' responses to posttest Question 6c: MKT 1 group. As shown in Table 23, three of the four students in the MKT1 group were able to apply their knowledge that ⅙ =$0.1\overline{6}$ to precisely locate the decimal as a number on a number line. Nina, Willa, and Korey all partitioned the unit interval into sixths and located $0.1\overline{6}$ at the tick mark for one-sixth. Nina and Willa also specifically noted that ⅙ is equivalent to $0.1\overline{6}$ in their explanations. Korey did not write an explanation, but she did show that she understood the notion of equivalence on the number line with fractions both by locating $0.1\overline{6}$ at ⅙ and by writing "1/2 and 3/6" under the same point.

Table 23

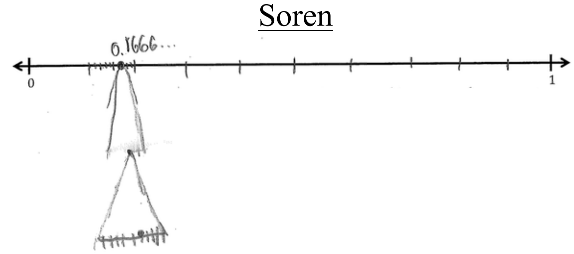
MKT1 Students' Work on Posttest Question 6c: Locate 0.166... on a Number Line

Student	Posttest (Question 6c)
Eva	 "I first partitioned the number line into tenths to find where 0.1 would be. I then partitioned the number line into hundredths to find where $0.1\overline{6}$ would be."
Nina	 "I knew the .1666…is equal to ⅙ so I partitioned the number line into sixths and hopped over one line."
Willa	 "I knew that 0.1666… is equivalent to ⅙, so I partitioned the line into sixths and found the first one."
Korey	 *No explanation written.*

Note. Minor changes to spelling, grammar, and/or punctuation were made to improve readability. All numbers are presented exactly as the student wrote them. If a student drew a picture, it is included.

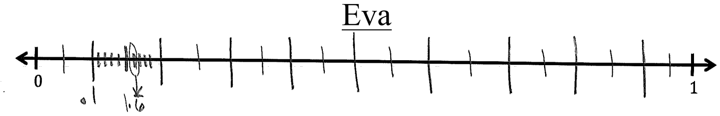
Unlike the other three, Eva tried to locate the repeating decimal using the decimal notation. Recall that Soren had also taken a similar approach (*Figure 25* shows Eva and Soren's work on this problem). However, there was a distinct difference between Eva and Soren's pictures and explanation for how they located the decimal. Soren clearly stated and showed that this process of partitioning subunits into tenths would continue "indefinitely." Eva, on the other hand, did not show her process as continuing, nor did she describe making any partitions beyond hundredths on the number line. Again, her correct usage of the terms "tenths" and "hundredths" marked an important shift in understanding for Eva from the first to second interview, but her number line and description here do not show that she understood this process to continue through successive partitions of each new subunit into tenths. Also note that she denoted the final decimal as "$1.\overline{6}$" rather than "$.1\overline{6}$," a subtle but important distinction given the limitations of her understanding of decimal notation at the start of the course.

Soren



"Partition into tenths, and count over 1. Partition the second tenths again into tenths and count over 6 again. Repeat indefinitely."

Eva



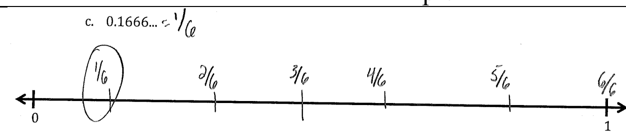
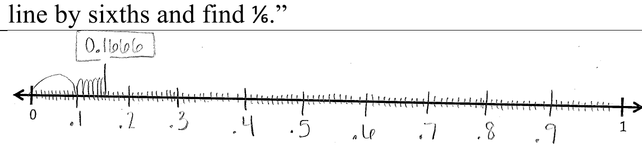
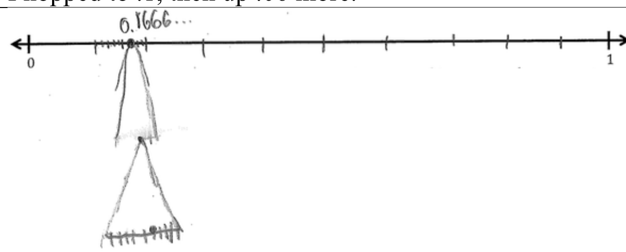
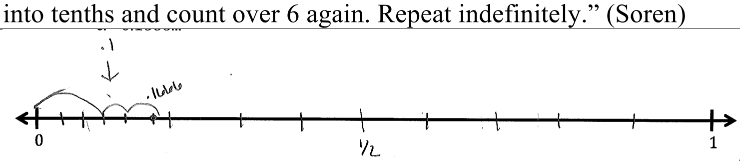
"I first partitioned the number line into tenths to find where 0.1 would be. I then partitioned the number line into hundredths to find where $0.1\overline{6}$ would be."

Figure 25. Soren and Eva use similar, but distinct, methods to locate the repeating decimal $0.1\overline{6}$ on a number line.

Whole class response to posttest Question 6c. As shown in Table 24, the overwhelming majority of students in the course (26 of 32, 81%) located the repeating decimal $0.1\overline{6}$ at the same point on the number line as the fraction ⅙. All of these students showed the number line partitioned into sixths only. Two students, including Eva, located $0.1\overline{6}$ at one-tenths and six-hundredths (0.16), an incorrect response. Soren was the only student in the class to try to show how to locate $0.1\overline{6}$ by progressively subdividing the unit into tenths, hundredths, and so on, while also indicating that this process would continue "indefinitely." Although this is a correct interpretation of the decimal as an infinite sum of fractions with denominators that are powers of ten, he did not show that the limit of this sum is ⅙ (a concept outside the scope of the course), and so did not locate the decimal at a precise point along the line. Two students in the course used a combination of tenths and other fractions to locate $0.1\overline{6}$. For instance, student "TI" located $0.1\overline{6}$ two-thirds of the way between one- and two-tenths. Overall, the students' work on this problem shows that the majority of preservice teachers in the course understood that the fraction ⅙ and repeating decimal $0.1\overline{6}$ were equivalent and so were located at the same point on the number line. The explanations of the students with incorrect answers generally showed that they understood decimal notation to be denoting fractional powers of ten.

Table 24

Summary of Ways the Whole Class Located the Repeating Decimal $0.1\overline{6}$ *on a Number Line on the Posttest (*N=*32)*

Type of response	Number	Example
Located at ⅙	26	 "I converted the decimal into a fraction as was able to partition the number line by sixths and find ⅙."
Located $0.1\overline{6}$ at 0.16	3	 "First, I partitioned the line into 1/10ths. Then I partitioned it into 1/100s. I hopped to .1, then up .06 more."
Showed 1/10, 6/100, and 6/1000, and indicated process would continue indefinitely	1	 "Partition into tenths and count over 1. Partition the second tenths again into tenths and count over 6 again. Repeat indefinitely." (Soren)
Other	2	 "First I partitioned the line in half. Then each half into 5ths to vreate 10ths. I found the point .1 and partitioned the following section into thirds. ⅔s of the way between .1 & .2 is $0.1\overline{66}$ so I counted up 2 notches past .1."

Note. Minor spelling, grammar, and/or punctuation corrections were made to improve readability. Numbers are presented exactly as written by students.

Summary of findings related to posttest Question 6c. Two related themes emerged from the data related to the ways the preservice teachers explained how to locate $0.1\overline{6}$ on a number line on the posttest. First, there was *widespread evidence of understanding of the relationship between the fraction ⅙ and the decimal* $0.1\overline{6}$. In particular, the overwhelming majority of students in the course *located the repeating* *decimal at the same position as ⅙ on the number line*. All who did so showed and/or *described partitioning the unit into sixths* in order to locate ⅙. The idea that equivalent numbers are located at the same point on the number line is an important understanding related to number lines. The use of partitioning to locate rational numbers on a number line is an important understanding related to understanding the set of rational numbers and to understanding fractions and decimals as numbers.

Second, as students described the process of partitioning the number line, there was widespread use of *precise mathematical language* to describe the fraction and/or decimal. Thus, there was *widespread evidence of decimal and fraction understanding* in students' responses to this problem.

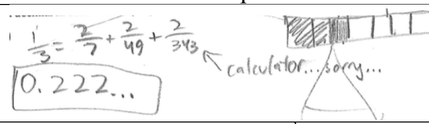
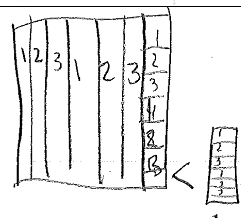
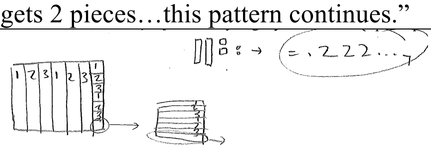
Posttest Bonus Questions 2 and 3: Find ⅓ as a Decimal in Base Seven and Explain How It Relates to 2/9 in Base Ten

Posttest Bonus Question 2 asked, "Write ⅓ as a decimal in base-7. Show/explain how you got your answer." Posttest Bonus Question 3 asked, "Explain why ⅓ written in base-7 decimals is similar to 2/9 in base-10 decimals." These questions were related to posttest question four which asked students to find ⅙ as a decimal as they asked students to generalize the process used to find ⅙ as a decimal in base ten and to make connections between finding decimals in base ten and finding decimals in other bases. The concept of generalizing the fraction-decimal relationship had been only briefly touched upon in a whole class discussion during the final ten minutes of one class period. Knowing if students can generalize concepts is a way to measure depth of understanding, but since this content had only been briefly included in the unit, the instructor for the course and the researcher agreed that making such questions required on the unit test would be unfair to students. Thus, these questions were included as optional “bonus” questions (worth up to two percentage points each). As a result, not all students responded to these questions. In the following sections, a summary of the ways the eight interviewees responded to these questions on the posttest will be presented. This will be supplemented by their responses with relevant excerpts from their second interviews as appropriate. The data related to the four students in the MKT2 group will be presented first, followed by the data related to the four students in the MKT1 group. A summary of the whole class responses will be presented last.

Interviewees’ responses to posttest Bonus Questions 2 and 3: MKT2 group. Table 25 shows how the four interviewees in the MKT2 group responded to Bonus Questions 2 and 3 on the posttest. Only three of the four attempted these problems, Mei left both blank.

Table 25

MKT2 Students' Posttest Work Showing how to Write the Fraction ⅓ as a "Decimal" in Base 7 (Bonus Question 2) and Why ⅓ in Base Seven and $^2/_9$ in Base Ten Have the Same Decimal Form (Bonus Question 3)

Student	Bonus question 2	Bonus question 3
Soren		*Blank*
Jo	 "1 loaf for 3 people, only split into 7 pieces. $.\overline{222}$ Split into sevenths & each person got 2 sevenths, split remaining seventh into sevenths (49ths of entire loaf) each person gets 2 pieces…this pattern continues."	"The denominator 9 is one less than the base 10. So every time you split the loaf there will be an extra piece left over. With $^2/_9$, there are 2 loaves of bread, so everyone gets 2 tenths, 2 hundredths, etc. ⅓ is equivalent to $^2/_6$. 6 is one less than the base 7 so the same situation occurs. Everyone gets 2 sevenths, 2 forty-ninths, etc…both $^2/_9$ and ⅓ is $.\overline{222}$ as a decimal."
Andie		"In base-7, ⅓ is equivalent to 2/6. 6 is one less than the base of 7, so we know that there will always be at least $1/7^{th}$ (or in this case $2/7^{th}$s) left over. In base-10, 9 in one less than the base, so there will always be a remainder. So ⅓ in base-7 is equivalent to 2/9 in base-10."
Mei	*Blank*	*Blank*

Note.

As shown in Table 25, in their responses to posttest bonus questions two and three, Andie and Jo showed that they were able to extend their understanding of decimal notation to non-base ten situations. Both utilized the "Breaking Bread" context to support their reasoning. During their interviews, both Jo and Andie stated that this context and pictures had been useful to them in making sense of the relationship between fractions and repeating decimals.

> I learned a lot. I really enjoyed going all the way back to learning about number systems and that whole concept because I think that is super foundational to every thing else that we learned. And we were going into fractions and understanding why one third is repeating. Things like that that I've always just assumed, a lot of

> those were explained and given concrete reasons, like, "Here's why this is the way it is," it's not just a mystery that you have to accept. You can actually look into it and understand that. So I really appreciated that. Kind of like the whole loaves of bread, splitting things / very eye-opening, I guess. (Jo, interview 2)
>
> The one thing that… I think I have a better understanding of it now is thinking about what fractions and then decimals would be in other base systems. And that took me a little while to figure out. The way that I started to think about it, because I think one of the extra credit problems was something, I don't remember what the fraction was and what the base system was [I give her the test question]. Yes, one-third in base seven would be. And so then I tried to think of it by doing the bread thing and that helped me understand why one-third in base seven is point two two repeating. (Andie, interview 2)

Andie went on to say that *Breaking Bread* story and picture were *necessary* tools for her to use in making sense of this relationship.

> A: "And so then I tried to think of it by doing the bread thing and that helped me understand why one-third in base seven is point two two repeating. But initially I had a really hard time. Without the bread thing, or without dividing, I don't think I would have been able to figure it out. I had to think of it this way. But that was something that I struggled with."
> I: "Does it seem like a problem to you that you had to think of it this way?"
> A: "No. No. I don't think so, but I didn't know how to conceptualize it otherwise, or how to think of it otherwise."

In other words, Andie was not just able to *describe* how to use the Breaking Bread story and picture in order to help students understand why, say ⅙ is denoted by the repeating decimal $0.1\overline{6}$ (in base ten), but she herself *actively used the story and picture as a tool for thinking* in order to find what ⅓ would be as a decimal in base seven. Solving this problem thus provided Andie with an idea that is relevant to elementary mathematics in a ways that was genuinely puzzling and that could act as a "proof-by-example" of how problems in context and pictures can be powerful tools for thinking about mathematics.

Soren also correctly found that ⅓ would be written as the base-7 decimal $0.\overline{2}_7$, but unlike Andie and Jo, he did not use the *Breaking Bread* story and picture to do so.

Instead he wrote "$\frac{1}{3}={}^{2}/_{7}+{}^{2}/_{49}+{}^{2}/_{343}$" with an arrow pointing to it that said, "Calculator, sorry." While this response shows a clear understanding of the places of base seven decimal notation, it was not clear how he found these numbers, so he was asked about this during the second interview.

> S: "Yeah, so different bases made sense. That was very helpful for understanding fractions and decimals and just being able to talk about it."
> I: "How was that helpful for fractions and decimals?"
> S: "Well, the realization that you can write decimals in base three or base five or whatever was/ I don't know, that was kind of cool. So after that, let's see, I'm trying to think of examples. That made the Egyptian breaking bread thing, that made a lot more sense after talking about bases a little bit more. I think we talked about bases after we did (incomprehensible)."
> I: "We did. We did it first with just doing two days with just writing fractions and then we came back and did it with/ now you can do it the decimal way. So that kind of came over (incomprehensible)."
> S: "Yeah, so that helped a lot. I'm still a little hesitant to rely completely on the bread thing in bases other than ten. I don't know why, I just am."
> I: "How come? On your test, you didn't use it actually. You used a calculator."
> S: "Yeah. I just knew I would want to check my work with my calculator anyway. And so I figured I would skip the breaking bread step because I didn't want to waste too much time. But it's also, I don't necessarily trust my ability to be exact when I'm drawing things out. So that's part of it too."
> I: "Does exactness matter there?"
> S: "It doesn't. (…) No, I don't think so. I don't remember with that problem specifically, but it usually doesn't for those problems."
> I: "I think …the last one I said was one-third in base seven."
> S: "Yeah. Okay. (…) So (…) now I don't even remember how I would do that with bread."

Notably, Soren was the only student in the course who correctly stated that ⅓ as a decimal in base seven is $0.\overline{2}_7$ *without* using the *Breaking Bread* story and picture. Furthermore, as the last line of the above exchange suggests, Soren actually struggled to use the *Breaking Bread* story and context to solve this problem during the interview. That is, despite his correct usage of the story and picture in his response to question five, his understanding of how to depict decimal notation pictorially was fragile. After he made the statement, “I don’t even remember how I would do that with bread,” the interviewer offered him paper to try and solve the problem, and the following exchange occurred.

> S: “Okay. (…)” [Sits silently, does not start drawing]
> I: “What would you do if it was base ten?”
> S: “You would, like what I did was split it into ten. Three-tenths plus three-hundredths plus three-thousandths.”
> I: “Right. And you know that that’s what that decimal representation is going to say.”
> S: “Yeah.”
> I: “What would that look like in a picture?”
> S: “Just partition into thirds, right?”

Note that despite Soren’s clear understanding that the decimal notation of one-third in base ten involved *tenths*, *hundredths*, and *thousandths*, his first inclination for depicting that decimal pictorially was to draw *thirds*. Following this statement, he and the interviewer then discussed how a picture depicting thirds would relate to the notation ⅓ whereas depicting tenths required partitioning in tenths. Soren then stated, “I think I remember how to do this,” and showed how to partition a rectangle in tenths, hundredths, and so on to depict one-third as a decimal pictorially. When he was then asked, “Alright so how about if I was going to do it the decimal base seven way?,” Soren responded by showing how to partition the rectangle in progressive powers of seven to get the base seven “decimal” $0.\overline{2}_7$. Moreover, doing so, prompted him to reflected upon the fact that it was not “exactness” that mattered in the picture.

> S: “And then so each person gets two pieces. One two three and there’s one left over. Two forty-ninths. And then you do that. Um, yeah. I guess … exactness doesn’t matter. I don’t know. I guess I was just thinking about that problem specifically wrong.”
> I: “What do you mean?”
> S: “I was forgetting the counting it out and doing it like that. I just skipped that

step and reverted back to before I took your class."

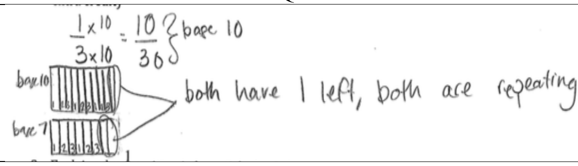
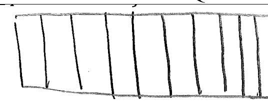
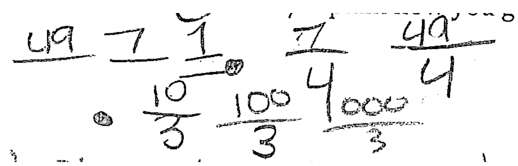
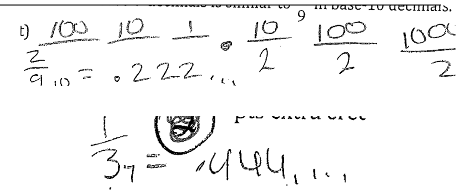
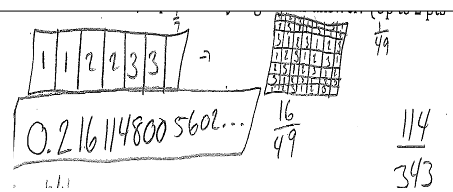
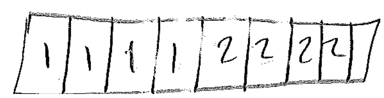
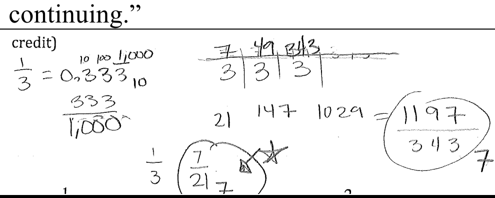
In his second interview, Soren showed that he was capable of using the *Breaking Bread* story and picture to meaningfully model what one-third would look like as a decimal in base ten and base seven. More importantly, however, he also showed that this understanding was still fragile. He clearly needed more time and support in understanding why solving a challenging problem on a calculator (when he could not explain what he did or why it worked) was not useful in terms of developing his mathematical knowledge *for teaching*.

Mei also was able to use the *Breaking Bread* story and context to correctly solve posttest problem five, but then she left blank the two bonus problems that generalized the idea of decimal notation to other bases. This was not discussed during her second interview, so it is not possible to say whether Mei was able to use the story and picture to generalize her understanding of decimal notation to bases other than ten.

Interviewees' responses to posttest Bonus Questions 2 and 3: MKT1 group. As shown in Table 26, none of the four students in the MKT1 group were able to generalize their understanding of decimal numeration and the fraction-decimal relationship to bases other than ten on the posttest. However, the answers they gave revealed that they were all developing flexibility in their understanding of decimal notation. Their answers also revealed some misunderstandings and/or limitations in their understanding as well.

Table 26

MKT1 Students' Posttest Work Showing how to Write the Fraction ⅓ as a "Decimal" in Base 7 (Bonus Question 2) and Why ⅓ in Base Seven and $^{2}/_{9}$ in Base Ten Have the Same Decimal Form (Bonus Question 3)

Student	Bonus Question 2	Bonus Question 3
Eva		
Nina	 "To solve this I first realized that ⅓ is .333… so that written in base ten would be three tenths" [rest cut off]	 "⅓ written in base seven is similar because it is a repeating decimal. No matter how many times you keep splitting the leftovers you will [rest cut off]."
Willa	 "Split the loaf into sevenths and after giving 2 to everyone there was one left which I split into 7ths making 49ths, I could give out 16 pieces to everyone where there would be one left over. This would keep continuing."	 "They are similar because no matter how many times you split them up there will always be one piece left."
Korey		"Because it is the reduced form."

As can be seen in Table 26, both Eva and Willa attempted to use the *Breaking Bread* story and/or picture to solve the problem. Recall that Willa was able to successfully use that context to explain why ⅙ would be written as $0.1\overline{6}$ in (base ten) decimal notation. Willa's picture and explanation for this problem show that she was able to correctly relate the story and picture to this generalized situation. She correctly divided her one “loaf” into sevenths and showed that each of the three people would get two of the sevenths. She then stated that she would split the remaining piece “into 7ths making 49ths,” a correct description of the next step of the process. However, her picture showed that she actually divided the next piece into forty-nine parts. This led to her then finding that each person would get “16 pieces.” Note that the first three digits after the decimal in her answer are “0.216,” a match to what she found in her picture. In other words, Willa was actually very close to being able to correctly generalize her understanding of decimal notation to bases other than ten. She also showed that she was making clear connections between the story, picture, and the written decimal in her work. She just had made the mistake of partitioning the remaining one-seventh into forty-nine pieces to create forty-ninths, rather than utilizing the idea that sevenths of sevenths make forty-ninths. Thus, although Willa was not able to correctly find that ⅓ as a decimal in base seven is $0.\overline{2}_7$, her work on this problem on the posttest showed that she was developing *flexibility* in her understanding of decimal notation, and that she was using the *Breaking Bread* story and picture as tools for thinking about the relationship between fractions and decimals. Given the fact that she could not even use place value to convert terminating decimals to fractions early in the course, this represents a substantial shift in understanding for Willa during the unit.

Eva also tried to use the *Breaking Bread* picture to find ⅓ as a decimal in base seven. Recall that Eva had not been able to connect her (correct) picture to a decimal on posttest question five (which asked her to show how to find ⅙ as a decimal), so it is not surprising that she was not able to find the “decimal” in base seven. However, as shown in *Figure 26*, Eva's work on the bonus question two demonstrated that she had some correct and some incorrect understandings of what was meant by "base 10" notation, and that she was starting to develop some understandings of the relationship between decimal notation and pictorial representations. In particular, her symbolic work showing that she multiplied ⅓ by $^{10}/_{10}$ to create the equivalent fraction $^{10}/_{30}$ which she then labeled "base 10" suggests that she still had misunderstandings about either the term "base ten" and/or the fact that decimal notation (in base ten) is specifically related to fractions with denominators that are *powers*, not *multiples*, of ten. On the other hand, her picture showing one rectangle split in tenths and then shared by three and another rectangle split in sevenths and shared by three, shows that she did understand at least the first position to the right of the decimal point in base ten and base seven as being related to tenths and sevenths, respectively. She was also able to correctly interpret that ⅓ in both bases would be a repeating decimal due to the one piece remaining in each case. Like Willa, this represented a substantial change in understanding of decimal notation and the relationship between fractions and decimals for Eva during the unit. It also showed that Eva had some understanding of how to use pictures as tools for thinking about decimals.

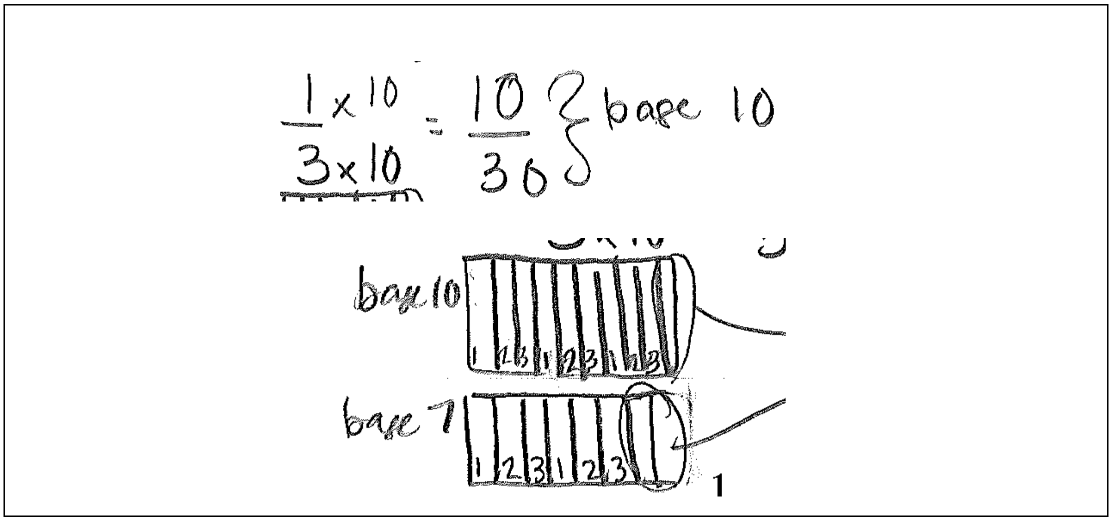


Figure 26. Eva's work showing her attempts to write ⅓ as a "decimal" in base 7.

Both Nina and Korey attempted to solve this problem symbolically, an extremely difficult task since the "decimal" is repeating. Thus, it is not surprising that neither woman was able to correctly determine what ⅓ would be when written as a decimal in base seven. However, as shown in *Figure 27*, in this process, both women did show that they understood the relationship between decimal notation and place value, and that they could generalize this understanding to bases other than base ten. Nina's work shows the symmetry of the place values around the ones place as well as the fact that the places in base seven are powers of seven. Her work also shows her relating the decimal places in base ten to the powers of ten. Korey's work shows only the decimal place values but again, shows that she was correctly relating the places with the powers of ten and seven. Recall that neither woman was not able to use place value to relate decimals to fractions during the first interview, so the fact they both showed an understanding of the place value and decimal notation and also generalized that understanding to decimals in other bases represents a substantial change in understanding for Nina and Korey over the course of the unit.

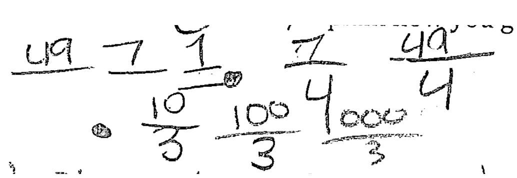
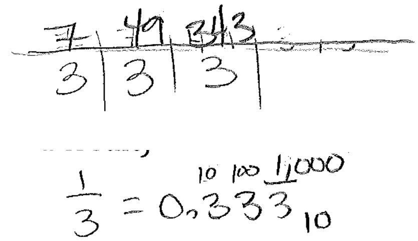
Nina	Korey
	

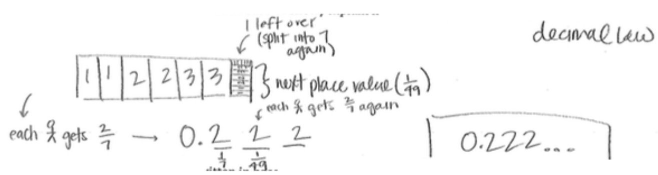
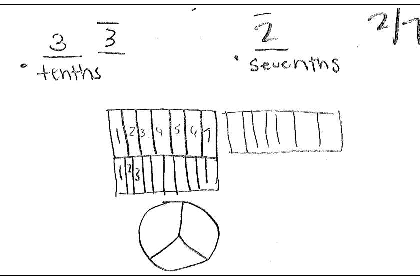
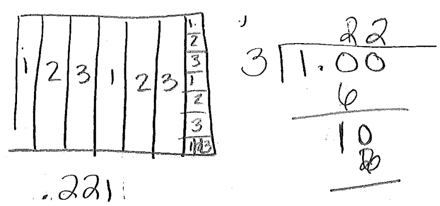
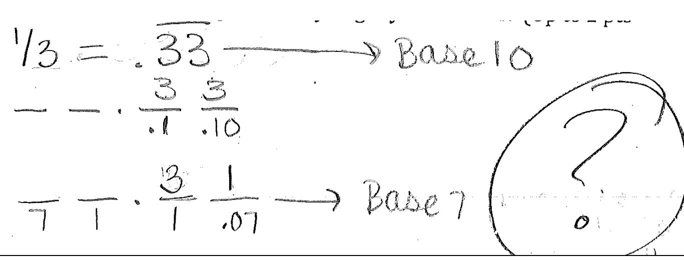
Figure 27. Nina and Korey's work showing their understanding of place value notation in base ten and base seven.

Whole class response to posttest Bonus Questions 2 and 3. As shown in Table 27, a total of thirteen students in the course (41%) were able to determine that one-third as a "decimal" in base seven would be denoted $0.\overline{2}_7$. Ten of these students (including Jo and Andie) used the *Breaking Bread* story and/or picture to show how they found their answer. An additional three students (including Eva and Willa) drew a picture showing a rectangle partitioned in sevenths with pieces being given out, but were not able to correctly determine the base seven "decimal" using their picture.

Three students in the class (including Soren) correctly found that the "decimal" would be $0.\overline{2}_7$ but gave an unclear explanation for how they determined their answer. Of these, only Soren showed purely symbolic work. The other two drew pictures but their pictures were not clearly related to the decimal notation they found (for example, see work sample for "⅓=$0.\overline{2}_7$, unclear how answer found" in Table 27). In addition to Soren, six other students also showed purely symbolic work but all six of these students failed to find one-third as a "decimal" in base seven. Thus, Soren's ability to find the "decimal" in base seven without using a story or picture as a tool for thinking was unique in the class.

Table 27

*Summary of Ways the Whole Class Found ⅓ as a Decimal in Base 7 on the Posttest (*N*=32)*

Type of response	Number	Typical explanation
⅓ = $0.\overline{2}$ in base 7 with picture clearly related to answer	10	
⅓ = $0.\overline{2}$ in base 7, unclear how answer found	3	
Drew fully or partially correct picture, wrote decimal incorrectly	3	
Incorrect, symbolic work only	6	
Blank[a]	10	

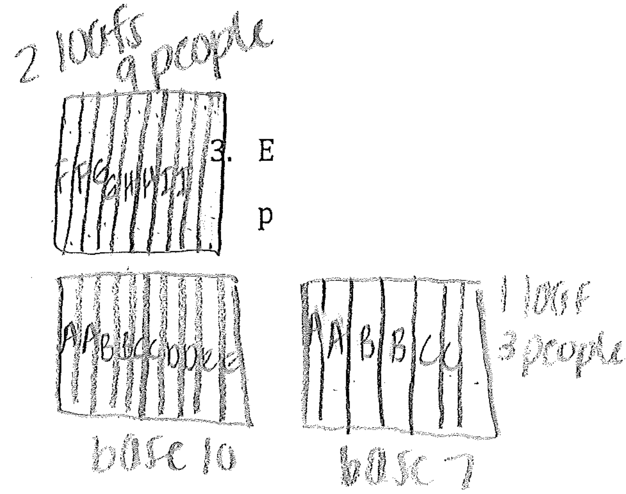
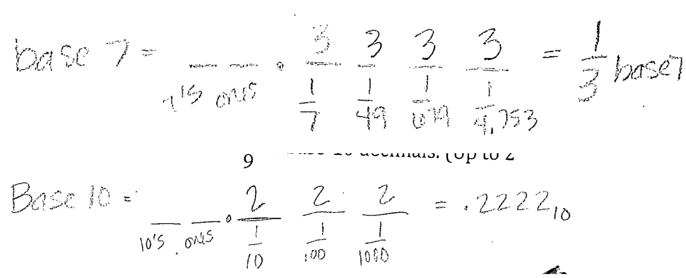
Note. Minor corrections to spelling and grammar were made to improve readability. All numbers are written exactly as the student wrote them.

As shown in Table 28, nine students in the class noted the structural similarity between finding $^{2}/_{9}$ as a decimal in base ten and ⅓ as a decimal in base seven. In particular, all nine noted that both would be denoted "$0.\overline{2}$" in their respective bases because the process of partitioning and "sharing the bread" would lead to analogous situations. Notably, all nine also used the *Breaking Bread* strategy to find ⅓ in base seven

(Bonus Question 2). One additional student noted that both would be denoted $0.\overline{2}$ but did not explain why this was the case. Six students noted that both would be repeating decimals but gave no further explanation of their relationship.

Table 28

*Summary of Ways the Whole Class Explained Why ⅓ in Base 7 is Similar to $^2/_9$ in Base 10 on Posttest (*N*=32)*

Type of response	Number	Sample explanation
Both $^1/_3$ and $^2/_9$ would be written as $0.\overline{2}$ in base 7 and 10 respectively with clear explanation	9	 "It is similar because each person either gets 2/10 or 2/7 of bread and there is always a remainder that repeats and everyone always ends up receiving 2 of that remainder, causing both decimals to be $.\overline{222}$."
Both are $0.\overline{2}$, no explanation	1	"It is. $\overline{2}$ for both of them."
Description of both decimals as repeating without direct reference to both being written as $0.\overline{2}$	6	 "These numbers are similar because they are both repeating rational numbers."
Other (incorrect)	2	"They are similar in value to one another because base 10 has larger numbers because it is power of tens, and in base 7 the numbers are smaller in value because the powers of 7 are smaller than 10."
Blank[a]	14	

Note. Minor corrections to spelling and grammar were made to improve readability. All numbers are written exactly as the student wrote them.

Summary of findings related to posttest Bonus Questions 2 and 3. Two related themes emerged from the data related to the preservice teachers' responses to Bonus Questions 2 and 3 on the posttest. These questions were "bonus" questions because this content was only briefly discussed for approximately ten minutes during one class period and was not otherwise supported in the course. However, a sizeable minority of students in the course (thirteen of thirty-two students, 41%) were able to find that ⅓ as a decimal in base seven would be $0.\overline{2_7}$. The fact that so many students were able to *generalize their understanding of the fraction-decimal relationship to non-base-ten situations* supports the idea that there was *widespread evidence of understanding of the relationship between fractions and decimals* on the posttest. The ability to generalize within the domain of number is an important part of developing "a profound understanding of fundamental mathematics" (Ma, 1999).

Of the thirteen students who found ⅓ as a decimal in base seven, ten used the same *Breaking Bread* story and picture that was used by the majority of students when finding ⅙ as a decimal (in base ten) on posttest Question 6. Moreover, nine of the ten students who used this model were able to explain why ⅓ as a decimal in base seven and $^2/_9$ as a decimal in base ten would both be denoted $0.\overline{2}$ in their respective bases. Thus, the second theme to emerge from this data was that *non-symbolic representations supported students in making connections between fractions and decimals and in generalizing decimal understandings.* That it was, in fact, the non-symbolic representations that were supporting their thinking was demonstrated during the interviews with Soren and Willa, as each used the model to find and make sense of why the decimal was denoted $0.\overline{2}_7$.

Finally, both Jo and Andie stated outright that it was the context and model that had helped them to solve this problem. In fact, Andie stated that she did not know how to think of it otherwise, suggesting that she was truly using the model as a tool for thinking about this problem.

Summary of Findings From After the Unit

In the previous sections, data from the posttest and second interview were presented that documented the nature of eight preservice elementary teachers' understanding of number and notation as it related to fractions and decimals and the relationships between them. Data documenting how all thirty-two members of the class responded to the posttest questions were also presented in order to give a more complete picture of these eight teachers' understandings within the context of their classroom learning community. In particular, the whole class data showed that the responses given by the eight interviewees on the pretest were typically similar to responses given by other members of the same class.

This study focused on preservice elementary teachers' understanding of fractions, decimals, the sets of rational and irrational numbers, and the relationships between them within the larger frame of number and numeration. The primary source of data from the above sections came from a posttest given immediately after the conclusion of the unit, eight weeks into the fifteen-week course. Data from interviews conducted in the month following the posttest were also used to give a more complete picture of the nature of the preservice teachers' understandings.

Four primary themes and eight supporting subthemes emerged from this data that relate to the nature of the preservice teachers' understanding of these areas of number and numeration after the unit. *Figure 28* summarizes these findings as well as the data sources to which they relate. This section will conclude with a statement of how these findings relate to the research questions guiding this study.

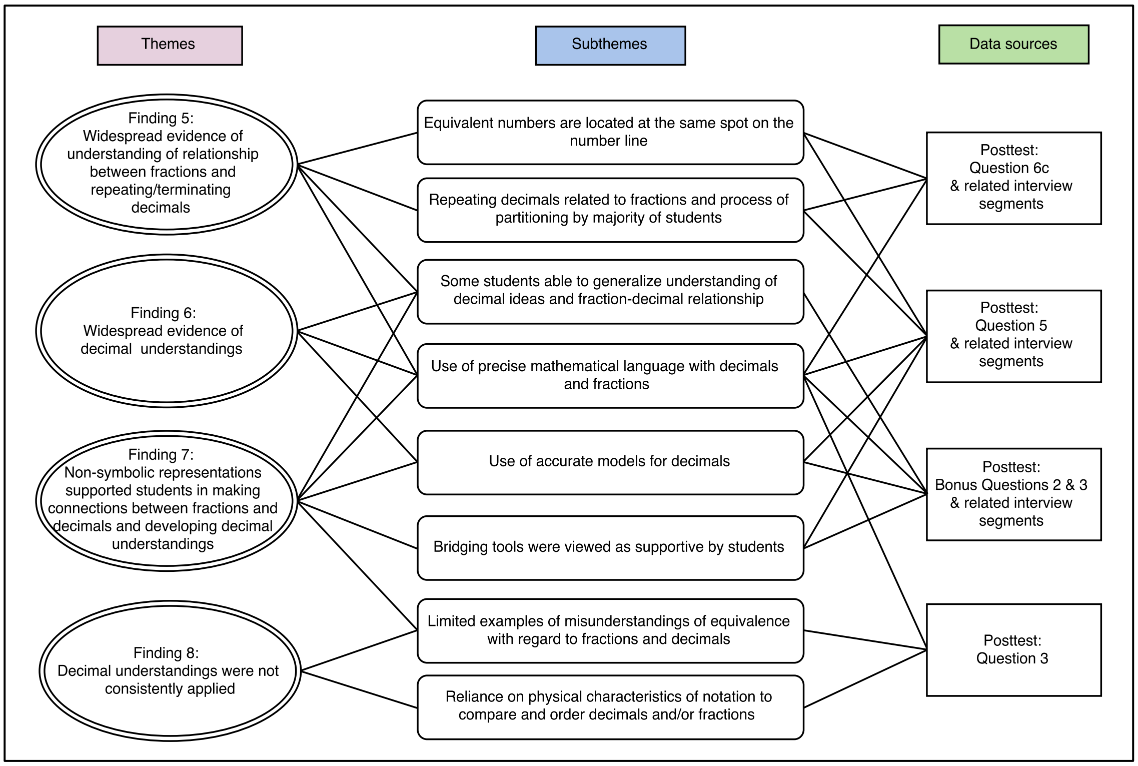


Figure 28. Themes, subthemes, and data sources related to PSTs' nature of understanding the *Number and Numeration System* and use of representations after the unit.

The first theme that emerged from the data was that there was *widespread evidence of understanding of the relationship between the fractions and repeating or terminating decimals.* Data for this theme came from all five of the posttest questions reported here as well as from the individual interviews conducted after the unit. The majority of students (24 of 32, 75%) were able to transparently connect the fraction ⅙ to the repeating decimal $0.1\overline{6}$. Moreover, the preservice teachers' descriptions of how to connect the fraction to the decimal clearly showed and described the division process and how it related to the repeating decimal $0.1\overline{6}$. Recall that division was the primary method that the preservice teachers said they would use to explain to a student why ⅓ equals 0.333… on the pretest, but none explained the process transparently. Notably, this transparent division process foregrounded the role of *partitioning* as it related to the repeating decimal. Thus, an important subtheme was that *repeating decimals were related to fractions by a process of partitioning* by a majority of students. Understanding the idea of creating a number by *partitioning a given unit* is a key idea necessary for understanding the set of rational numbers. Given the difficulties that many of the preservice teachers had in understanding repeating decimals as rational, as documented in earlier in this chapter, this is a promising result.

A second related subtheme was that the *equivalent numbers* $0.1\overline{6}$ *and ⅙ were located at the same point on the number line* by the majority of students. One potential issue with the partitioning process used by most of the students to connect ⅙ to the decimal $0.1\overline{6}$ is that it depicts the decimal as an infinite process. Understanding the equivalence of ⅙ and $0.1\overline{6}$ requires that the students view both ⅙ and $0.1\overline{6}$ as the same *number*. Viewing a notation as a *process* can interfere with this understanding (Dubinsky et al., 2013; Sfard, 1991). In these results, the majority of the preservice teachers located $0.1\overline{6}$ at the same point on the number line as the fraction ⅙, and again explicitly used a process of *partitioning a given unit* in order to do so. These results should be viewed with caution, however. The idea of using the fraction-equivalent of a repeating decimal in order to locate it on a number line was explicitly taught in the course, so it is unclear if the preservice teachers deeply understood the idea that the two notations were indeed *equivalent*, or if they were simply copying what had been modeled in the class. However, the fact that they were willing and able to use partitioning to locate a repeating decimal is an idea that can support them in understanding why repeating decimals are rational.

A third important subtheme related to finding that there was widespread evidence of understanding of the relationship between fractions and repeating decimals was the fact that a substantial minority of students (41%) were *able to generalize their understanding to relate a fraction to a decimals in another base.* The ability to generalize suggests that these students were attending to the important structural features of positional notation rather than simply replicating a process they had been taught.

The second theme that emerged from the data was that there was *widespread evidence of decimal understandings* in the preservice teachers' work on the posttest and during the interviews. Data for this theme again came from all five of the posttest questions reported here as well as from the individual interviews conducted after the unit. Two important subthemes were the widespread *use of precise mathematical language* and *use of accurate models* for decimals. Both are indicators of decimal understandings (Cramer et al., 2015). Moreover, as stated above, a substantial minority were able to generalize to their understandings to non-base ten decimals. Importantly, students did not just find ⅓ as a decimal in base seven, they also continued to use precise mathematical language to name the base-seven decimal places as "sevenths," "forty-ninths," and so on, and to draw accurate models of these decimals. This is particularly promising since the ability to generalize suggests strong understandings of the important structural features of our base ten positional numeration system.

The third theme that emerged from the data was that the *non-symbolic representations supported students in making connections between fractions and decimals and in developing decimal understandings.* Data from this theme came primarily from the second interviews, with supporting evidence coming from students' work on the posttest Questions 5 and 6c and Bonus Questions 2 and 3. First, during their interviews, the interviewees described the model as supportive, and several drew on it to explain their thinking at various points. Second, as noted above, the majority of students used the *Breaking Bread* and number line models to meaningfully connect the ⅙ and $0.1\overline{6}$ on the posttest. As students used the *Breaking Bread* model, they used precise mathematical language to describe the decimal as one-*tenth*, six-*hundredths*, and so on. They also referenced the model to explain why the decimal repeated (e.g. "There's always four pieces left over so that's why it repeats"). Furthermore, twelve out of the thirty-two students[11] (38%) were able to utilize the *Breaking Bread* model to meaningfully connect the fraction ⅓ to the decimal $0.\overline{2_7}$. Importantly, only the students who utilized the *Breaking Bread* model explained why ⅓ as a decimal in base seven and $^2/_9$ as a decimal in base ten would both be denoted $0.\overline{2}$ in their respective bases.

Finally, the fourth theme that emerged from the data was that *decimal understandings were inconsistently applied.* Data from this theme came primarily from

[11] Ten students did this on the pretest. Soren and Willa were not among these ten but made this connection during their second interviews.

from students' responses to posttest Question 3, as well as two students' responses to posttest Question 6c. One related subtheme was that there were *still some examples of students misunderstanding equivalence with regard to fractions and decimals*. On posttest Question 6c, two students located $0.1\overline{6}$ at 0.16 on the number line. Both described their process of locating $0.1\overline{6}$ as finding one-tenth and six-hundredths, so they were clearly treating the two numbers as equivalent. On posttest Question 3, two students used 0.3, 0.33, 0.333, and/or $0.\overline{3}$ interchangeably. Both of these misconceptions were more prevalent early in the unit, but clearly some students in the course were still struggling with these ideas. The other related subtheme was that there was still some *reliance on the physical characteristics of decimal notation to compare and order decimals*. In particular, on posttest Question 3, no student meaningfully compared their chosen decimal (which needed to be between 0.333 and ⅓) with ⅓ or $0.\overline{3}$. Instead, of those that did offer justification for their comparison, they all referenced the fact that notation for $0.\overline{3}$ had an infinite number of threes while their chosen decimal terminated. Thus, although the results from the work on the other problems on the posttest showed that the overwhelming majority of the preservice teachers could meaningfully interpret and model decimals, none of them applied those understandings to this problem.

Relationship of Findings to Research Questions: After the Unit

As summarized in the previous section, four themes and their related subthemes emerged from the data gathered after the unit. Table 29 shows a summary of the major findings with their related research questions and data sources (see *Figure 28* for related sub-questions).

Table 29

Summary of Themes from After the Unit, with Related Research Questions and Data Sources

Themes	Related research question(s)	Data source(s)
5. Widespread evidence of understanding of the relationship between fractions and repeating/terminating decimals	RQ1c RQ2b	Posttest Q3 Posttest Q5 Posttest Q6c Posttest Bonus Q2 Posttest Bonus Q3 Interview 2
6. Widespread evidence of decimal understandings	RQ1c RQ2b	Posttest Q3 Posttest Q5 Posttest Q6c Posttest Bonus Q2 Posttest Bonus Q3 Interview 2
7. Non-symbolic representations supported students in making connections between fractions and decimals, and in developing decimal understandings	RQ2b	Interview 2 Posttest Q5 Posttest Q6c Posttest Bonus Q2 Posttest Bonus Q3
8. Decimal understandings were not consistently applied	RQ1c	Posttest Q3 Posttest Q6c

Response to Research Question 1c. The first research guiding this study asked: "What is the nature of preservice elementary teachers' (PSTs) understanding of the *Number and Numeration System* before and after participation in a unit designed to develop their understandings of this content?" This research question had one sub-question that was related to data gathered after the unit:

> 1c. What is the nature of PSTs' understandings of fractions, decimals, and the connections between them after the unit?

As shown in Table 29, Themes 5, 6, and 8 relate to Research Question 1c. Together, these three themes suggest that the nature of the preservice teachers' understanding of the *Number and Numeration System* as it relates to fractions, decimals, and the sets of rational and irrational numbers was *more connected* after the unit. In particular, the majority of preservice teachers and all of the interviewees showed that they were able to use multiple representations in order to find and explain connections between fractions and decimals. Every interviewee demonstrated the ability to use precise mathematical language when it came to fractions and decimals, as well as the ability to accurately model decimals. The overwhelming majority of students in the class also used precise mathematical language and an accurate model of decimals at least once on their posttest as well. However, the inconsistent application of these understandings to a question that essentially asked students to compare and order three rational numbers suggests that there were still *limitations in their understanding* of these ideas in areas that matter for the work of teaching.

Response to Research Question 2b. The second research question asked, "What is the role of representations in the development of PSTs' understanding of the *Number and Numeration System*?" This research question had one sub-question that was related to data gathered after the unit:

> 2b. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals after the unit?

As shown in Table 29, Themes 5, 6, and 7 relate to Research Question 2b. Together these three themes and their related subthemes (*Figure 28*) suggest that non-symbolic representations played an important role in the development of preservice teachers' understanding of the *Number and Numeration System*. Recall that the primary form of representation that students used during the first interview and on the pretest was symbolic notation. After the unit, a variety of representations were used to reason about and describe fractions, decimals, and the relationship between them. These included using precise mathematical language for decimals and fractions, drawing accurate rectangle models to represent a decimal, locating fractions and decimals on a number line using partitioning, and utilizing a real-world context (sharing bread). Many responses to posttest Question 4 and Bonus Questions 2 and 3 showed evidence of students making connections between these representations as well. The correct responses to Bonus Questions 2 and 3, along with the supporting interview segments, suggested that the non-symbolic representations supported the development of several preservice teachers' understandings of the relationship between fractions and decimals, as well as their understanding of decimal notation.

There was also evidence that symbolic representations continued to play an important role in the ways preservice teachers were reasoning about aspects of number and numeration after the unit. Some instances of reasoning based solely on symbolic notation were present, most notably in the responses to posttest Question 3 This question asked if there were rational numbers between 0.333 and ⅓, and if so, asked students to give an example and justify how they knew it was between the two. Thus, this question essentially asked the students to compare and order three rational numbers. In the responses to this question, there was a widespread lack of use of both verbal and pictorial representations of the numbers, even though both would have helped to meaningfully compare and order the numbers. Furthermore, several students *relied on the surface features of the decimal notation* to justify their ordering of decimals in the responses to this question.

Conclusion

This chapter presented the results of preservice teachers' responses to selected tasks on a pre- and posttest as well as during two hour-long individual interviews with eight of the preservice teachers. The first interview was conducted in the first two weeks of the unit, and the second was conducted in the month following the completion of the unit. The nature of the preservice teachers' understanding early in and after the unit was documented and summarized. The role of representations early in and after the unit was documented and summarized as well. In the next chapter, these results will be discussed, along with the limitations and implications of the study. Recommendations for future research will be made as well.

Chapter 5

Summary, Conclusions, and Implications

"*If one intends to convey to the students the feeling of the structurality of mathematics, one has to emphasize, first of all, the coherent picture of the number system with its strict hierarchy"* (Fischbein et al., 1995, p. 30).

The mathematical knowledge need for teaching is a specialized form of mathematical knowledge, different from the mathematical knowledge necessary in other professions (Ball et al., 2008). Elementary teachers need a "profound understanding of fundamental mathematics" (Ma, 1999) in order to support their students in developing the deep, conceptual understanding of the mathematical ideas introduced in the elementary years. A large body of research has documented the fact that, in the United States, the overwhelming majority of preservice elementary teachers do not have the mathematical knowledge necessary for the work of teaching (Tatto & Senk, 2011; Thanheiser et al., 2013). This includes virtually all areas of understanding related to the sets of rational and real numbers (Dubinsky et al., 2013; Fischbein et al., 1995; LeSage, 2011; Post et al., 1991; Sinclair et al., 2006; Tobias, 2009; Zazkis & Sirotic, 2010). It is clear that teacher education mathematics coursework plays a vital role in developing preservice teachers mathematical knowledge for teaching, and that addressing preservice teachers' understanding of number and numeration as it relates to whole numbers, fractions, and decimals is of critical importance (Kilpatrick, Swafford, & Findell, 2001a)

One challenge for teacher education is supporting preservice teachers in developing this knowledge during the limited time available in preservice teacher programs. One possible method of addressing this challenge is to take advantage of the fact that the concepts of whole numbers and place value, fractions, decimals, rational, and real numbers are deeply interconnected mathematically and design curricula for preservice teachers that builds on and emphasizes these connections. For instance, rather than treating whole numbers, fractions, and decimals as isolated units, curricula could be designed to highlight the ways that fractions and decimals build on and extend whole number ideas. One potential advantage to this "connected" approach to curriculum design is that it would foreground the fact that mathematics in general, and number and numeration specifically, are a coherent and cohesive set of ideas rather than a disconnected set of facts and procedures. Necessary understandings in the individual areas of number and numeration could be supported and developed while at the same time ensuring that teachers are knowledgeable about the ways that the concepts they are teaching are connected to one another and to concepts that come before and after them (Ball & Bass, 2003).

This descriptive, qualitative case study (Yin, 2014) seeks to contribute to this knowledge base by documenting preservice elementary teachers' ways of understanding the relationships between fractions and decimals, and between those notations and the sets of rational and irrational numbers. This study was a case study where the unit of analysis was a single class of preservice elementary teachers participating in an instructional unit designed to support them in making explicit connections between concepts related to number and numeration. The study also documents how the preservice teachers' use of representations during the course supported and/or constrained the development of their understanding of these relationships.

The previous chapter presented the results of preservice teachers' responses to selected tasks presented during interviews and on a pre- and posttest. The remainder of this chapter is organized into five parts. First, the purpose and research questions guiding the study are restated and summarized. Second, the results of the study are briefly summarized and discussed. Third, limitations of the study are described. Fourth, implications from the study are presented and discussed, and recommendations for future research are made. Finally, a conclusion is presented.

Restatement of Purpose and Research Questions

This study focused on preservice elementary teachers' understanding of fractions, decimals, the sets of rational and irrational numbers, and the relationships between them within the larger frame of number and numeration. Data for this study came from a pre- and posttest completed on the first and last day of the seventeen-class unit on number and numeration within a mathematics content course for future elementary teachers. The primary goal of this study was to document the understandings and use of representations by a group of preservice teachers at two different points in their mathematical content coursework in order to support the design of curricula that promotes the development of preservice elementary teachers' mathematical understandings from a *connected* perspective. The purpose of documenting their ways of thinking before and after the instructional unit was two-fold. First, effective curricula must build on and extend learners' current ways of knowing and "be generative in students' development of future understanding" (Clements, 2007, p. 40). Knowing what preservice elementary teachers know about the connections between fractions and decimals, and these notations and the sets of rational and real numbers, is a necessary prerequisite for such work. Second, the content areas of interest in this study—fractions, decimals, rational numbers—are all familiar topics for preservice teachers. Other research has documented that pre-service teachers' familiarity with a topic can be an obstacle to their learning, particularly when they do not realize they lack the understanding necessary to teach the topic meaningfully to children (Markovits & Sowder, 1990; Sinclair et al., 2006; Sowder et al., 1998). Thus, documenting preservice teachers ways of understanding and interacting with these concepts during their teacher education coursework is also needed to support the design of curricula. In particular, noting ways that preservice teachers do, and do not, take up understandings developed in the course can help curriculum designers identify characteristics of potentially productive (or non-productive) tasks and activities.

Two research questions and their related sub-questions guided this study:

1. What is the nature of preservice elementary teachers' (PSTs) understanding of the *Number and Numeration System* before and after participation in a unit designed to develop their understandings of this content?
    a. What is the nature of PSTs' understandings of the sets of numbers generally, and *rational numbers* specifically, early in the unit?
    b. What is the nature of PSTs' understandings of fractions, decimals, and the connections between them early in the unit?
    c. What is the nature of PSTs' understandings of fractions, decimals, and the connections between them after the unit?
2. What is the role of representations in the development of PSTs' understanding of the *Number and Numeration System*?
    a. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals before/early in the unit?
    b. What representations were used by the PSTs in reasoning about the relationship between fractions and decimals after the unit?

The *Number and Numeration System* (*Figure 29*) is a conceptual model developed for this study depicting the ways that knowledge of number, notation, and representation are interconnected. The *NNS* diagram highlights the fact that one's understanding of numbers as *quantities* is a different from, and yet connected to, one's understanding of *notations* and other non-symbolic *representations* that are used to denote those quantities. It also highlights the fact that the interconnected nature of these concepts means that learners' understandings (or lack thereof) of one area can impact understanding in the others. This study focuses specifically on the interconnected aspects of the *NNS* related to the set of rational and real numbers, including the connections between fractions and decimals and their connections to the sets of rational and irrational numbers. Since the ways that the preservice teachers used representations as tools for thinking about number and notation on the test were naturally tied to the ways they understood those numbers and notations, this summary will address the nature of understandings and role of representations in the development of those understandings (research questions 1 and 2) in an integrated fashion.

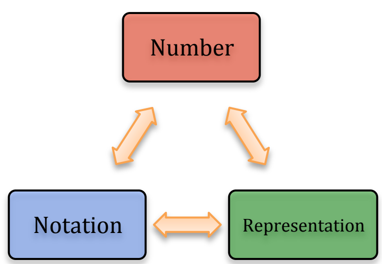


Figure 29. The *Number and Numeration System* depicts the ways knowledge of number, notation, and (non-symbolic) representations are interconnected.

Summary and Discussion of Results

Table 30 summarizes the results of this study for each of the tasks reported on in the previous chapter. Table 31 summarizes the major themes that emerged from the data shown in Table 30, along with the research question(s) to which each finding relates.

Table 30

Summary of Results by Data Source

Data source	Summary
Interview 1 Definition of number	• Understandings of number strongly tied to idea of *counts of discrete units* (whole numbers) • No inclusion of concept of *partitioning* • Only two included concept of *measure*
Interview 1 Number sort	• Half of interviewees (all MKT1 group) unable to meaningfully interpret decimals using place value • 6 of 8 interviewees believed repeating decimals to be related to fractions; none knew why, some reasons showed misconceptions (e.g. all decimals come from fractions) • 1 interviewee believed repeating decimals could *not* be equivalent to a fraction • Limited understanding of set of irrationals • Limited understanding of π; some misconceptions • Rational/irrational numbers not viewed as complementary subsets of the real numbers by all interviewees
Interview 1 Definition of rational/irrational numbers	• Both sets most often defined by decimal notation, with majority of interviewees using whether or not decimal terminated to characterize the difference between the two sets • Little evidence of understanding the rational numbers as a set
Pretest Q1 (examples/non of rational/irrational) & related excerpts from Interview 1	• Repeating decimals were considered irrational/non-rational by sizeable minority • Only two students in class listed examples and non-examples that showed understanding of different numbers that are part of rational numbers
Pretest Q2 (find a number between 0.66 and ⅔) & related excerpts from Interview 1	• Some students believed there were no numbers between 0.66 and ⅔ (non-equivalent numbers viewed as equivalent) • If given, justifications for ordering numbers based on $0.\overline{6}$ having infinite number of digits • No use of place value understandings to meaningfully compare, order numbers
Pretest Q4 (⅓ = 0.333…) & related excerpts from Interview 1	• Large majority stated they would use division (algorithm) to explain this equivalence; no one role of partitioning into powers of ten explicit • 7 students would use argument that relied on believing $0.\overline{9} = 1$; 3 of these stated they did not believe this equivalence (others not asked) • If non-symbolic models were used, they were all circle models showing only the fraction ⅓
Posttest Q3 (0.333 and ⅓) & related excerpts from Interview 2	• All students believed there were numbers between 0.333 and ⅓ • Minority used decimal understandings to meaningfully compare, order two terminating decimals (e.g. 0.333 and 0.3333) • No use of decimal understandings to meaningfully compare terminating decimal to repeating decimal (e.g. 0.3333 and $0.\overline{3}$) • 2 students persisted in using non-equivalent numbers as if they were equivalent (e.g. replacing 0.333 with 0.3)

Data source	Summary
Posttest Q5 (⅙ as a decimal) & related excerpts from Interview 2	• Large majority showed evidence of decimal understandings including use of precise decimal language and picturing decimal accurately • Majority explained connection between fraction and repeating decimal using transparent description of partitioning process; reason decimal repeated transparently explained • *Breaking bread* strategy described as supportive by interview participants
Posttest Q6c ($0.1\overline{6}$ on number line) & related excerpts from Interview 2	• Large majority located $0.1\overline{6}$ at same point as ⅙ • All who located ⅙ and $0.1\overline{6}$ at same point showed/described partitioning unit into sixths • 2 students located $0.1\overline{6}$ at 0.16 (treated non-equivalent numbers as equivalent)
Posttest Bonus Q2 & Q3 (⅓ as a decimal in base 7, how related to $\frac{2}{9} = 0.\overline{2}_{10}$) & related excerpts from Interview 2	• 22 students attempted the problem, 13 found ⅓ as decimal in base 7 • 10 showed evidence of generalized decimal understandings including use of precise mathematical language and picturing non-base-ten decimal accurately • All who stated that $\frac{1}{3} = 0.\overline{2}_7$ and $\frac{2}{9} = 0.\overline{2}_{10}$ also used non-symbolic representations (pictures and words) to describe ⅓ as a decimal in base 7

Table 31
Summary of Findings, and Related Research Questions and Data Sources

Findings	Related research question(s)	Data source(s)
1. Explicit understandings of number strongly tied to properties of the whole numbers	RQ1a	Interview 1: Definition of rational/irrational numbers
2. General understanding of sets of rational and irrational numbers was *limited* and *often inaccurate*	RQ1a	Interview 1: Definition of numbers Interview 1: Number sort Pretest Q1 Pretest Q2
3. The relationship between fractions and decimals was poorly understood	RQ1b RQ2a	Interview 1: Number sort Pretest Q2 Pretest Q4
4. Decimals in general were poorly understood	RQ1b RQ2a	Interview 1: Number sort Pretest Q1 Pretest Q2 Pretest Q4
5. Widespread evidence of understanding of the relationship between fractions and repeating/terminating decimals	RQ1c RQ2b	Posttest Q3 Posttest Q5 Posttest Q6c Posttest Bonus Q2 Posttest Bonus Q3 Interview 2
6. Widespread evidence of decimal understandings	RQ1c RQ2b	Posttest Q3 Posttest Q5 Posttest Q6c Posttest Bonus Q2 Posttest Bonus Q3 Interview 2
7. Non-symbolic representations supported students in making connections between fractions and decimals, and in developing decimal understandings	RQ2b	Interview 2 Posttest Q5 Posttest Q6c Posttest Bonus Q2 Posttest Bonus Q3
8. Decimal understandings were not consistently applied	RQ1c	Posttest Q3 Posttest Q6c

Note. RQ1=Research Question 1: "What is the nature of preservice elementary teachers' (PSTs) understanding of the *Number and Numeration System* before and after participation in a unit designed to develop their understandings of this content?"
RQ2=Research Question 2: "What is the role of representations in the development of PSTs' understanding of the *Number and Numeration System*?"

Based on the pretest given on the first day of the unit and hour-long interviews with eight preservice teachers in the first two weeks of the course, the preservice teachers' understanding of the *NNS* early in the unit as it related to rational and real numbers could be characterized as *limited* and *often inaccurate.* First, the preservice teachers in this study mostly knew little about what rational numbers are or why they are a considered an important set of numbers. Many listed fractions and decimals as examples of numbers, but generally did not define the concept of number in ways that included the types of quantities that could be denoted by fractions and/or decimals. What numbers are, how they look, and how they work *changes* with the introduction of the (positive) rationals, and this makes the rational numbers difficult for children to understand. Elementary teachers are the educators tasked with helping children learn and understand the whole and rational numbers, so having a clear understanding of what these two sets of numbers are, how they work, and how they are related is relevant for the work of teaching. Thus, while defining "number" is admittedly a difficult task, the fact that none of the preservice teachers in this study could cogently articulate a conception of number that includes both rational and real quantities suggests that there is a serious need for attention to their understanding of number in mathematics education coursework.

Second, limited understandings of decimal notation emerged as problematic in multiple ways early in the unit. This study documented that at least four of the preservice teachers in the class did not begin the course with an understanding of how place value related to decimal notation. Since only the interview participants were asked questions that documented their understanding of place value, it is unclear how many of the members of the class did not have this understanding at the start of the course. For those students who did understand place value with decimal notation, uncertainties about what infinite decimals represented and how to interpret them were prevalent. Several students knew that repeating decimals could be written as fractions but none could explain why.

Not understanding place value with regard to decimal notation makes meaningfully understanding why some decimals denote rational numbers, while others do not, impossible because the fraction-decimal relationship cannot be understood. This study documented the fact that preservice teachers often reasoned about and categorized decimals based on what they *looked like*; for instance, whether the decimal terminated or repeated, whether the decimal was “predictable.” In fact, asking students who do not understand decimal notation to categorize a decimal as rational (or irrational, etc.) likely promoted categorizing decimals based on what they look like because the transparent aspects of the notation itself were the only representation to which the student had clear access. Similarly, asking students to categorize infinite decimals as rational who were unable to explain why repeating decimals can be written as fractions (while non-repeating infinite decimals cannot) also may have promoted categorizing decimals based on the surface features of the notation, again because the properties of the notation itself were the most clearly accessible features that could be used as a basis for categorization. It is unclear how students without a deep understanding of decimal notation could make sense of the fact that we sometimes use “different symbols to represent the same idea and similar-looking symbols to represent different ideas" (Markovits & Sowder, 1990, p. 5). Yet having an understanding of what rational numbers are and why fractions and some decimals may be used to denote them is the type of understanding that supports elementary teachers in the work of teaching these concepts meaningfully to children.

In contrast, after the unit, the majority of the preservice teachers were able to demonstrate understandings of decimal notation and the relationship between fractions and (repeating) decimals in ways that could support reasoning about why fractions and some decimals are part of the set of rational numbers. Most notably, a strong majority of the preservice teachers were able to transparently connect the fraction ⅙ to the repeating decimal $0.1\overline{6}$ in two different ways. They could interpret the fraction as the quotient "1 ÷ 6," and draw and describe the partitioning process in a way that clearly showed why the resulting decimal was $0.1\overline{6}$, including why it was a repeating decimal. They also showed $0.1\overline{6}$ as being located at the same point on a number line as ⅙, which they located by a process of partitioning the unit into sixths. Both of these methods foregrounded the partitioning process as well as the equivalence of the two numbers, precisely the type of understanding of the relationship between fractions and decimals missing early in the unit. Furthermore, students who transparently showed the partitioning on the number line and with the *Breaking Bread* rectangle model also used precise mathematical language to describe the resultant partitions. That is, they wrote "I partitioned the number line into *sixths*," and "I cut the loaf of bread into *tenths*," and so on. This is the type of mathematical language that elementary teachers need to use to promote decimal and fraction understandings in their students (Cramer et al., 2015), so it is promising that so many of the preservice teachers used that precise language on these problems on the posttest. Similarly, the ability draw accurate models for fractions and decimals is also a skill needed by teachers, so again, it is promising that so many of the preservice teachers did so on these problems. Thus, based on the posttest given on the last day of the unit and hour-long interviews with the same eight preservice teachers in the month following the completion of the unit, the preservice teachers' understanding of the *NNS* after the unit as it related to rational and real numbers could be characterized as *more connected* than it had been earlier.

However, while there was widespread evidence that a strong majority of the preservice teachers could model decimals accurately and could use place value understandings to interpret decimals, there was also a widespread lack of use of these understandings on a question that asked the teachers to compare and order three rational numbers (posttest Question 3). Meaningfully teaching students to compare and order fractions and decimals is part of the work of teaching elementary mathematics and involves accurately modeling and representing these numbers (National Council of Teachers of Mathematics, 2000b; National Governors Association, 2010). Thus, in their responses to this question, the preservice teachers did not demonstrate evidence of mathematical knowledge *for teaching*. It should be noted that posttest Question 5 (a question on which most preservice teachers showed decimal understandings) explicitly asked the preservice teachers to describe how they would *help a student* to find the decimal representation of ⅙. In contrast, posttest Question 3 was not set in the work of teaching, but rather simply asked the preservice teachers to find a number between 0.333 and ⅓ and justify their answer. It is possible that more of the preservice teachers may have applied their decimal understandings if they had been instructed to explain how they would justify their ordering to a student. Regardless, it is notable that the preservice teachers mostly did not utilize decimal understandings in their responses to this question since those understandings are necessary to justify their responses. In terms of curriculum design for mathematics education, this raises the question of how to support students learning not just that mathematical ideas are connected, but also that they should *draw on those connections* when solving problems.

In summary, there were differences and similarities in the ways that the preservice teachers understood the connections between fractions and decimals and the set of rational numbers early in and after a unit designed to develop their understandings of these connections. There were also differences and similarities in the ways they used representations to explain and support their thinking on some of the problems. Early in the unit, there were many ways in which the preservice teachers' understandings were disconnected and not supportive of viewing the various aspects of number and numeration relevant to elementary teachers as connected. In contrast, after the unit, many of the preservice teachers showed understandings that were supportive of viewing the various aspects of number and numeration relevant to elementary teachers as *connected*.

Limitations

There are several limitations that may be identified in this study. First, the researcher designed, implemented, and analyzed the study. The researcher also acted as the co-instructor in the course during the unit. It is to be expected that student interactions with the instructors may be influenced by their roles as evaluators of student performance. Although the other course instructor was the one who assigned grades for the course, students may still have perceived the researcher as being in an evaluative position due to the co-instruction, which in turn, could have impacted the results. As this study was a case study of preservice teachers in a single mathematics content class, generalizability is limited. The results may not be the same with a different group of students, or with a different instructor teaching the same material. Also, the researcher did all the coding of the data so there was no inter-rater reliability. Finally, while one level of member checking was built into the study during the second interview, there was not an opportunity to follow up with member checking with the research participants following analysis of all of the data. Sharing and discussing the findings with the participants would have added further credibility to this research.

Implications and Recommendations for Future Research

This study adds to the limited body of research that directly addresses preservice teachers' understanding of the connections between fractions and decimals, and their connection to the sets of rational and real numbers (Amato, 2005, 2006; LeSage, 2011; Sinclair et al., 2006). This study also extends prior work on bridging tools (Abrahamson & Wilensky, 2007; Abrahamson, 2006; Fuson & Abrahamson, 2005) by documenting how two bridging tools were used to promote understanding of the connections between fraction and decimal notation and their relationship to the concepts of *partitioning* and *measure.* In doing so, this study addresses Kastberg and Morton's (2014) call for more research on how to develop preservice teachers' understandings *productively* and *efficiently* by using activity sequences that develop more than one concept at a time (p. 329). In this section, five general implications for the design of curricula in mathematics content courses for preservice elementary teachers that arose out of this study will be presented. Recommendations for related future research will be made as well.

The first implication to arise from this study was related to the bridging tools used in the instructional unit. Over the past two decades, there have been multiple and continued calls for content coursework that supports preservice teachers in better understanding fractions, decimals, place value, and the operations of multiplication and division (Chick et al., 2006; Kastberg & Morton, 2014; Kilpatrick, Swafford, & Findell, 2001b; Mewborn, 2001; Olanoff, 2011; Post et al., 1991; Tatto & Senk, 2011; Thanheiser, 2014). The fact that the overwhelming majority of preservice teachers were able to successfully use both the *Breaking Bread* story/picture and a number line as a tool for connecting fractions and decimals is therefore a promising finding since both tools have the potential to simultaneously support preservice teachers in developing understandings in multiple areas. "Fair sharing" situations, like the one used in the *Breaking Bread* tool, are useful for developing fraction understandings in elementary students (Empson & Levi, 2011). Extending such situations to include fair sharing "the decimal way"—that is, by partitioning so the pieces created may always be expressed using decimal notation—can build meaning for the decimal notation and for why a given fraction and decimal are equivalent. Moreover, the partitioning process necessary to produce shares that may be notated by a decimal foregrounds the role that the powers of ten play in decimal notation and therefore meaningfully models place value. This process also naturally and meaningfully models the steps of the standard division algorithm. The number line is a supportive tool for building meaning for fractions and decimals as *numbers that are measures*, understandings of equivalence and ordering, and understandings of the role of partitioning and units in fraction and rational decimal notation (Pearn & Stephens, 2007; Saxe et al., 2007).

Multiple avenues of research could be pursued with regard to bridging tools. First, the tools used in this study could be examined in other situations and with different populations of students to determine if and how they support preservice teachers in developing understandings of fractions, decimals, rational numbers, and the connections between all three. Second, the number line has potential in supporting the concept of equivalence between repeating and terminating decimals and fractions, as well as developing preservice teachers' understanding of why only *some* decimals denote rational numbers, but *all* decimals denote real numbers. Research that develops and examines activities that promote decimal understandings using a number line are needed. Finally, only two bridging tools have been proposed and used in this study, but the area of number and numeration is replete with connections for which bridging tools could be designed. Studies that document the creation and testing of bridging tools in various settings could support the development of more curricula for preservice teachers which promotes a *connected* understanding of mathematical concepts.

The second implication to arise from this study was also related to the use of the *Breaking Bread* bridging tool and the preservice teachers' decimal understandings. In this study, preservice teachers' decimal understandings emerged as a linchpin that held, or failed to hold, many interrelated ideas together. Early in the unit, not understanding decimal notation made connecting fractions to decimals and decimals to the sets of rational and irrational numbers difficult or impossible. After the unit, decimal understandings were completely intertwined with the ways preservice teachers made connections related to rational numbers. This study is not designed to support the making of claims about whether improved decimal understandings supported making these connections or if making the connections improved decimal understandings. However, it can be noted that in the instructional unit, the two concepts (decimals and how they related to fractions and rational numbers) were developed simultaneously. Every lesson involving rational decimals in the unit explicitly tied those decimals to fractions. The one lesson involving irrational decimals was a lesson about building meaning for the number π. This lesson worked to develop the idea that π was a point on a number line, and emphasized the fact that it could not be reached by the process of partitioning and so could not be represented as a fraction.

In their review of the research on preservice teachers' understanding of decimals, Kastberg and Morton (2014) argued that more research is needed that investigates how preservice teachers develop decimal understandings and how teacher educators can support that development. One issue with deepening preservice teachers' knowledge of decimals is that they are familiar territory and preservice teachers do not always know that they lack the understanding necessary to teach decimals meaningfully to students. The results of this study suggest that taking a connected approach to developing fraction and decimal understandings could be a productive approach as asking students to make connections between fractions and decimals revealed many areas of limited or inaccurate decimal understandings. Furthermore, activities that ask students to make connections between fractions and *non-base-ten* decimals have the advantage of making the familiar strategy of using the division algorithm to convert a fraction to a decimal untenable. Converting to non-base-ten decimals forces students to think about the role that partitioning by powers of the base plays in decimal notation. The fact that a large minority of students in this study were able to figure out how to write a fraction as a non-base-ten decimal (and a repeating one, no less!) despite having only limited exposure to the idea in the unit suggests that such activities could be viable for use teacher education coursework. One area of possible future research would be to use the *Breaking Bread* activity with the extension to non-base-ten decimals with other groups of preservice teachers and examine how it supports, or fails to support, their understandings of base-ten decimals. Another possibility for future research would be to examine more generally how taking a connected approach to developing fraction and decimal understanding impacts learners' understandings in the separate areas of fractions and decimals.

The third implication to arise from this study was related to the number sorting activity completed during the first interview. The number sort was used in order to elicit the ways that the preservice teachers reasoned about the various subsets of the real numbers and their connections to different notations. In the number sort, preservice teachers were given numbers written in various notations (e.g. fractions, decimals, whole numbers, etc.) and asked to place them into all the categories to which they believed the numbers belonged. As they placed the numbers in categories, they were asked to explain their thinking. Prior to completing the activity, the preservice teachers were given definitions for each of the sets of numbers. In particular, rational numbers were defined as "any number that can be written in the form $^{a}/_{b}$ where *a* and *b* are integers and *b* does not equal zero." This is a common definition of rational numbers.

One potential problem with this definition is that it presumes that the learner understands fraction notation well enough to understand why the ability to write a given number in fraction form is sufficient for establishing it as rational. However, research on preservice elementary teachers' knowledge of fractions has consistently found that most do not understand fractions well at all (Thanheiser et al., 2013). A second potential problem with the common definition of rational numbers is that the definition promotes a very procedural approach to thinking about the set of rational numbers. In particular, it encourages learners to translate non-fractions into fraction-form, a process that can be completed without thinking deeply about the properties of a fraction that make it a rational number. Procedures such as "put every whole number or integer over one" may be easily used to "show" that the whole numbers and integers can be written in fraction form, but they do little to promote understanding of the idea that whole numbers and integers are subdomains of the rational numbers because they may be conceived of as partitions of a unit, where the partition is simply to take the whole unit. Defining the set of rational numbers in terms of fraction notation therefore seems ill-suited for use with preservice elementary teachers. As the educators tasked with teaching students about what rational numbers are and how they work, it is important for elementary teachers to have a clear understanding of why the rational numbers are considered a set of numbers, what properties define the set, and how those properties build on, extend, and differ from properties of the set of whole numbers. This study documented the fact that the preservice teachers indeed did have many difficulties classifying numbers as rational (or not) even after being given this definition. Another implication of this study is therefore that the ways that rational numbers are defined as a set in content courses for preservice teachers may need to be reconsidered. In particular, using a definition that highlights the common *properties* of rational numbers rather than the *notations* would be much more useful for the work of teaching. Future studies could work to establish such a definition (or suitable definitions) and test their utility in building mathematical knowledge for teaching.

The fourth implication to arise from this study was also related to the number sorting activity, namely the fact that the number sorting activity itself had some problematic elements that were revealed during the interview. As documented in Chapter 4, many misconceptions about the sets of rational and irrational (and by extension, real) numbers arose during the activity. In many cases, the preservice teacher categorized a number *correctly*, and their misconceptions were only made apparent in the *justifications* given for the categorizations. In other cases, the preservice teachers categorized numbers *incorrectly* by applying a heuristic that could sometimes be used to correctly categorize numbers. For instance, Nina categorized of the decimal form of pi as rational based on the idea that rational numbers were fractions and fractions could be divided to create decimals. Since the preservice teachers were not generally aware that they were making such mistakes, the number sorting activity may have supported some of the preservice teachers in solidifying misconceptions, particularly when they got the “right answer” for the wrong reason. Thus, the results of this study do not support using activities like the number sort as a *teaching tool* intended to support students in understanding how various numbers related to the subsets of the real numbers. The number sort was, however, an effective *assessment tool* for eliciting students’ understandings and misunderstandings about fraction and decimal notation and about the various sets of numbers as long as the categorizations were justified. Similar number sorting tasks should be closely examined and perhaps reconsidered as an activity used in mathematics courses teaching students about the structure of the real number system.

The fifth implication to arise from this study was related to more generally to the study's focus on examining and promoting preservice teachers' mathematical understandings from a *connected perspective*. The ideas of mathematics are deeply interconnected. This study draws on the idea that conceptual understanding of mathematics may be defined by the learners' understandings of the *connections* between mathematical ideas (Hiebert & Lefevre, 1986; Lesh, Post, & Behr, 1987). The National Council of Teachers of Mathematics states that, "Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a *coherent* and *connected* enterprise" (2000, p. 17, emphasis added). Ball and Bass (2003) argue,

> If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students. We believe that it is mathematical understandings of the latter type that serve as a necessary condition for teachers to teach for students' high-quality understanding. (2003, pp. 4–5)

In other words, the importance of supporting teachers in developing an understanding of the *connections* between mathematical ideas is widely accepted. Yet, studies that are designed to document, measure, and/or assess understanding from a *connected* perspective are surprisingly rare (see Wilensky, 1993 for a notable exception). While it is undoubtedly easier to document understanding in isolated areas of mathematics, more work needs to be done that directly addresses the fact that building connections between concepts is at least as important as building understanding of the concepts themselves. With regard to number and numeration and elementary teaching, there is a need to support teachers in developing an understanding of how the concept of number develops and changes mathematically as students are introduced to the sets of whole and rational numbers. As documented in the previous chapter, the preservice teachers in this study were not able to articulate these connections between the whole and rational numbers. In fact, most were only able to clearly articulate conceptions of number related to the set of whole numbers. Another implication of this study is therefore that more research needs to be devoted to examining the ways that preservice teachers are making *connections* between mathematical ideas. Common ways of measuring and discussing such connections are needed to facilitate and promote such studies as well.

The present study was designed with the intention of supporting such research. This study is situated in the second phase of Clements' (2008) curriculum research framework and used grounded theory methods and clinical interviews to examine preservice teachers' understanding of number and numeration from a *connected* perspective. Clements argues that such work can support the development of cognitive models, which herein relate to the ways preservice teachers develop a connected understanding of number and numeration. Once such models are developed, they can "be tested and extended with teaching experiments, to build models of … thinking and learning that also suggest guidelines for instructional activities" (Clements, 2008, p. 413). Thus, future studies that directly address the ways that preservice teachers are building the *coherent* and *connected* knowledge of fundamental mathematical concepts could help facilitate the development of models for how such knowledge may be supported and promoted. These models, in turn, could support the development of instructional strategies appropriate for use in mathematics content courses for preservice teachers.

Conclusion

"Number" is a complex idea that grows and changes over time as new number systems are introduced. Developing a *coherent* and *connected* understanding of number and notation as it relates to the whole numbers, fractions, and decimals is important for the work of elementary teaching and must be supported by mathematics content coursework. This study suggests that preservice elementary teachers may be well-served by activity sequences that intentionally develop their understandings of multiple aspects of number and notation simultaneously. Such learning sequences can potentially promote understandings of the individual aspects of number and notation that need to be addressed during teacher education coursework, while at the same time, helping make explicit the ways that these concepts are connected mathematically. As a case study that documented the nature of preservice elementary teachers' understandings of number and numeration from a *connected* perspective, this study can support the initial stages of such research.

References

Abrahamson, D. (2004). *Keeping meaning in proportion: The multiplication table as a case of pedagogical bridging tools* (Unpublished doctoral dissertation). Evanston, IL.

Abrahamson, D. (2006). Mathematical representations as conceptual composites: Implications for design. In *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psycholog of Mathematics Education*.

Abrahamson, D., & Wilensky, U. J. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning*, *12*(2007), 23–55. doi:10.1007/s10758-007-9110-6

Amato, S. A. (2005). Developing students' understanding of fractions as numbers. *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 49–56.

Amato, S. A. (2006). Improving student teachers' understanding of fractions. *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 41–48.

Andreasen, J. B. (2006). *Classroom mathematical practices in a preservice elementary mathematics education course using an instructional sequence related to place value and operations* (Unpublished doctoral dissertation). University of Central Florida, Orlando, FL.

Arcavi, A., Bruckheimer, M., & Ben-zvi, R. (1987). History of mathematics for teachers: The case of irrational numbers. *For the Learning of Mathematics*, *7*(2), 18–23.

Ball, D. L. (1990a). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, *21*(2), 132–144.

Ball, D. L. (1990b). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, *90*(4), 449–466.

Ball, D. L. (2002). Knowing mathematics for teaching: Relations between research and practice. *Mathematics and Education Reform Newsletter*, *14*(3), 1–5.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In E. Simmt & B. Davis (Eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3–14). Edmonton, AB.

Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433–456). New York:

Macmillan.
Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. doi:10.1177/0022487108324554
Battista, M. T. (2012). *Cognition-based assessment and teaching of place value: Building on students' reasoning*. Portsmouth, NH: Heinemann.
Boaler, J. (2015). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages, and innovative teaching*. San Francisco, CA: Jossey-Bass.
Borko, H., Eisenhart, M. A., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education*, *23*(3), 194–222.
Bruner, J. (1971). Bruner on the learning of mathematics: A "process" orientation. In D. Aichele & R. Reys (Eds.), (pp. 166–192). Boston, MA: Prindle, Weber & Schmidt. Retrieved from https://moodle.umn.edu/file.php/21887/Week_4/Bruner_1971_.pdf
Burroughs, E. A., & Yopp, D. (2010). Prospective teachers' understanding of decimals with single repeating digits. *Investigations in Mathematics Learning*, *3*(I), 23–42.
Chick, H., Baker, M., Pham, T., & Cheng, H. (2006). Aspects of teachers' pedagogical content knowledge for decimals. *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 297–304.
Clements, D. H. (2007). Curriculum research: Toward a framework for "research-based curricula." *Journal for Research in Mathematics Education*, *38*(1), 35–70.
Clements, D. H. (2008a). Design experiments and curriculum research. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 410–422). New York: Routledge.
Clements, D. H. (2008b). Design experiments and curriculum research. In A. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 410–422). New York: Routledge.
Cobb, P. (2003). Investigating students' reasoning about linear measurement as a paradigm case of design research. *Journal for Research in Mathematics Education, Monograph*, *12*, 1–16.
Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3/4), 175–190.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, *23*(1), 2–33.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit, 135–164.

Cramer, K. A. (2003). Using a translation model for curriculum development and classroom instruction. In J. L. Richard A & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 449–463). Mahwah, NJ: Lawrence Erlbaum Associates.

Cramer, K. A., & Lesh, R. A. (1988). Rational number knowledge of preservice elementary education teachers. In M. Behr (Ed.), *Proceedings of the 10th Annual Meeting of the North American Chapter of the International Group for Psychology of Mathematics Education* (pp. 425–431). DeKalb: Il: PME.

Cramer, K. A., Monson, D. S., Ahrendt, S., Colum, K., Wiley, B., & Wyberg, T. (2015). Five indicators of decimal understandings. *Teaching Children Mathematics*, *3*(22), 186–195.

Cramer, K. A., Monson, D. S., Wyberg, T., Leavitt, S., & Whitney, S. B. (2009). Models for initial decimal ideas. *Teaching Children Mathematics*, *16*(2), 106–117.

Cramer, K. A., Post, T. R., & DelMas, R. C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the Rational Number Project curriculum. *Journal for Research in Mathematics Education*, *33*(May 2015), 111–144. doi:10.2307/749646

Cramer, K. A., Wyberg, T., & Levitt, S. (2009). *Rational number project: Fraction operations and initial decimal ideas*. Retrieved from http://www.cehd.umn.edu/ci/rationalnumberproject/rnp2.html

Curtice, F. E. (2010). *Preservice teachers: Fraction operations and error patterns* (Unpublished masters thesis). Texas A&M, Corpus Christi.

Dantzig, T. (2005). *Number: The language of science*. New York: Pearson Education.

Dienes, Z. P. (1960). *Building up mathematics*. London: Hutchinson Educational. Retrieved from http://www.nrdc.org.uk/anr_details.asp?ID=1540

Diezmann, C. M., & English, L. D. (2001). Promoting the use of diagrams as tools for thinking. In A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 77–89). Reston, VA: NCTM.

Domoney, B. (2001). Student teachers' understanding of rational numbers. *Proceedings of the British Society for Research into Learning Mathematics*, *21*(November), 13–18.

Doritou, M., & Gray, E. (2009). Teachers' subject knowledge: The number line representation. *Proceedings of the 6th Conference of the European Society for Research in Mathematics Education CERME 6*, 139–141.

Dubinsky, E., Arnon, I., & Weller, K. (2013). Preservice teachers' understandings of the relation between a fraction or integer and its decimal expansion: The case of $0.\overline{9}$ and 1. *Canadian Journal of Science, Mathematics and Technology Education*, *13*(3), 232–258.

Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals*. Portsmouth, NH: Heinemann.

Fasteen, J., Melhuish, K., & Thanheiser, E. (2015). Multiplication by 10five : Making sense of place value structure through an alternate base. *Mathematics Teacher Educator*, *3*(2), 83–98.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, *16*(1), 3–17.

Fischbein, E., Jehiam, R., & Cohen, D. (1995). The concept of irrational numbers in high-school students and prospective teachers. *Educational Studies in Mathematics*, *29*, 29–44.

Fischer, R. M. (2014). *Rational numbers and the Common Core State Standards: A descriptive case study* (Unpublished doctoral dissertation). Montana State University, Bozeman.

Fuson, K. C., & Abrahamson, D. (2005). Understanding ratio and proportion as an example of the apprehending zone and conceptual-phase problem-solving models. *Handbook of Mathematical Cognition*, 213–234.

Goldin, G. A., & Steingold, N. (2001). Systems of Representations and the Development of Mathematical Concepts. *The Roles of Representation in School Mathematics*, 1–23.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). New York: Macmillan.

Hiebert, J. C. (1988). A theory of developing competence with written mathematical symbols. *Educational Studies in Mathematics*, *19*, 333–355.

Hiebert, J. C., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Erlbaum.

Hopkins, T. M., & Cady, J. A. (2007). What is the value of @#?: Deepening teachers' understanding of place value. *Teaching Children Mathematics*, 434–437.

Kajander, A. (2010). Elementary mathematics teacher preparation in an era of reform: The Development and assessment of mathematics for teaching. *Canadian Journal of Education*, *33*(1), 228–255.

Kastberg, S., & Morton, C. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on decimals. *The Mathematics Enthusiast*, *11*(2), 311–332.

Khoury, H. A., & Zazkis, R. (1994). On fractions and non-standard representations: Pre-service teachers' concepts. *Educational Studies in Mathematics*, *27*, 191–204.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001a). *Adding it up: Helping children learn mathematics*. Washington, D.C.: National Academy Press.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001b). Executive summary. In *Adding it up: Helping children learn mathematics*. Washington, D.C.: National Academy Press.

Kinach, B. M. (2002a). A cognitive strategy for developing pedagogical content knowledge in the secondary mathematics methods course: Toward a model of effective practice. *Teaching and Teacher Education*, *18*(1), 51–71. doi:10.1016/S0742-051X(01)00050-6

Kinach, B. M. (2002b). Understanding and learning-to-explain by representing mathematics: Epistemological dilemmas facing teacher educators in the secondary mathematics "'methods'" course. *Journal of Mathematics Teacher Education*, *5*, 153–186.

LeSage, A. (2011). So much to do, so little time: Supporting prospective elementary teachers' sense of rational numbers. *Proceedings of the IADIS International Conference E-Learning 2011*, *1*, 409–420.

Lesh, R. A., Behr, M., & Post, T. R. (1987). Rational number relations and proportions. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 41–58). Hillsdale, NJ: Lawrence Erlbaum.

Lesh, R. A., & Doerr, H. M. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3–33). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R. A., Post, T. R., & Behr, M. (1987). Representations and translations among representation in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33–40). Hillsdale, NJ: Lawrence Erlbaum.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Newbury Park, CA: SAGE Publications.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. New York: Routledge.

Mack, N. K. (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research*. Mahwah, NJ: Lawrence Erlbaum.

Markovits, Z., & Sowder, J. T. (1990). Students' understanding of the relationship between fractions and decimals. *Focus on Learning Problems in Mathematics*, *13*(1), 3–11.

McClain, K. (2003). Supporting preservice teachers' understanding of place value and multidigit arithmetic. *Mathematical Thinking and Learning*, *5*(4), 281–306.

Merriam, S. B. (2009). *Qualitative research*. (Jossey-Bass, Ed.). San Francisco.

Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Education Research Journal*, *3*, 28–36.

Minnesota academic standards in mathematics. (2007). Retrieved from https://goo.gl/pJmh0B

Mitchell, R. N., Charalambous, C. Y., & Hill, H. C. (2013). Examining the task and knowledge demands needed to teach with representations. *Journal of Mathematics Teacher Education*, 37–60. doi:10.1007/s10857-013-9253-4

Moore, T. J., Miller, R. L., Lesh, R. A., Stohlmann, M. S., & Kim, Y. R. (2013). Modeling in engineering: The role of representational fluency in students' conceptual understanding. *Journal of Engineering Education*, *102*(1), 141–178. doi:10.1002/jee.20004

Morris, A. K., Hiebert, J. C., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn. *Journal for Research in Mathematics Education*, *40*(5), 491–529.

Moseley, B. J. (2005). Students' early mathematical representation knowledge: The effects of emphasizing single or multiple perspectives of the rational number domain in problem solving. *Educational Studies in Mathematics*, *60*(1), 37–69.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122–147.

Murawska, J. M. (2013). *Preservice elementary school teachers' conceptual understanding of place value within a constructivist framework* (Unpublished doctoral dissertation). Northern Illinois University, DeKalb.

Nataraj, M. S., & Thomas, M. O. J. (2009). Developing understanding of number system structure from the history of mathematics. *Mathematics Education Research Journal*, *21*(2), 96–115.

National Council of Teachers of Mathematics. (2000a). Executive summary. In *Principles and standards for school mathematics* (pp. 1–6). Retrieved from http://www.nctm.org/uploadedFiles/Math_Standards/12752_exec_pssm.pdf

National Council of Teachers of Mathematics. (2000b). *Principles and standards for school mathematics*. Reston, VA: Author.

National Governors Association. (2010). *Common core state standards for mathematics*. Washington, D.C.: Author.

Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal*, *45*(4), 1080–1110.

Novick, L. R. (2004). Diagram literacy in preservice math teachers, computer science majors, adn typical undergraduates: The case of matrices, networks, and hierarchies. *Mathematical Thinking and Learning*, *6*(3), 307–342.

O'Connor, M. C. (2001). "Can any fraction be turned into a decimal?" A case study of a mathematical group discussion. *Educational Studies in Mathematics*, *46*, 143–185.

Olanoff, D. E. (2011). *Mathematical knowledge for teaching teachers: The case of multiplication and division of fractions* (Unpublished doctoral dissertation). Syracuse University, Syracuse.

Patton, M. Q. (2002). *Qualitative research and evaluation methods* (3rd ed.). Thousand Oaks, CA: SAGE Publications, Inc.

Pearn, C. S., & Stephens, M. (2007). Whole number knowledge and number lines help to develop fraction concepts. *Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia*, 601–610.

Post, T. R., Harel, G., Behr, M. J., & Lesh, R. A. (1991). Intermediate teachers' knowledge of rational number concepts. In *Integrating research on teaching and learning mathematics* (pp. 194–219).

Powell, A. B., Francisco, J. M., & Maher, C. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, *22*(4), 405–435.

Radin, D. J. (2007). *Coursework on base numeration systems and its influence on pre-service elementary teachers' understanding of place value concepts* (Unpublished doctoral dissertation). Oklahoma State University, Stillwater, OK.

Richardson, K., Berenson, S., & Staley, K. (2009). Prospective elementary teachers use of representation to reason algebraically. *The Journal of Mathematical Behavior*, *28*(2-3), 188–199.

Richman, F. (1999). Is 0.999...=1? *Mathematics Magazine*, *72*(5), 396–400.

Roche, A., & Clarke, D. M. (2013). Primary teachers' representations of division: Assessing mathematical knowledge that has pedagogical potential. *Mathematics Education Research Journal*, *25*(2), 257–278. doi:10.1007/s13394-012-0060-5

Rowan, B., Schilling, S. G., Ball, D. L., Miller, R., Atkins-Burnett, S., & Camburn, E. (2001). *Measuring teachers' pedagogical content knowledge in surveys: An exploratory study*. Ann Arbor.

Roy, G. J. (2008). *Prospective teachers' development of whole number concepts and operations during a classroom teaching experiment* (Unpublished doctoral dissertation). University of Central Florida, Orlando, FL.

Safi, F. (2009). *Exploring the understanding of whole number concepts and operations: A case study analysis of prospective elementary school teachers* (Unpublished doctoral dissertation). University of Central Florida, Orlando, FL.

Saldana, J. (2013). *The coding manual for qualitative researchers*. Los Angeles, CA: SAGE Publications.

Saxe, G. B., Diakow, R., & Gearhart, M. (2012). Towards curricular coherence in integers and fractions: a study of the efficacy of a lesson sequence that uses the number line as the principal representational context. *Zdm*, *45*(3), 343–364.

Saxe, G. B., Shaughnessy, M. M., Shannon, A., Langer-Osuna, J. M., Chinn, R., & Gearhart, M. (2007). Learning about fractions as points on a number line. In *The learning of mathematics: Sixty-ninth yearbook* (pp. 221–236).

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36.

Shaughnessy, M. M. (2009). *Student's flexible use of multiple representations for rational number: Decimals, fractions, parts of area, and number lines* (Unpublished doctoral dissertation). University of California, Berkeley, Berkeley, CA.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–22.

Siegler, R. S., & Lortie-Forgues, H. (2015). Conceptual knowledge of fraction arithmetic. *Journal of Educational Psychology*.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–96.

Sierpinska, A. (1994). *Understanding in mathematics*. London: Falmer Press.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, *11*, 499–511.

Simon, M. A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, *24*(3), 233–254.

Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, *8*(4), 359–371.

Sinclair, N., Liljedahl, P., & Zazkis, R. (2006). A coloured window on pre-service teachers' conceptions of rational numbers. *International Journal of Computers for Mathematical Learning*, *11*(2), 177–203.

Sirotic, N., & Zazkis, A. (2006). Irrational numbers: The gap between formal and intuitive knowledge. *Educational Studies in Mathematics*, *65*(1), 49–76.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 20–26.

Skemp, R. R. (2006). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School*, *12*(2), 88–95.

Sowder, J. T., Armstrong, B., Lamon, S., Simon, M. A., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education*, *1*(2), 127–155.

Stacey, K., Helme, S., Steinle, V., Baturo, A., Irwin, K., & Bana, J. (2001). Preservice teachers' knowledge of difficulties in decimal numeration. *Journal of Mathematics Teacher Education*, *4*, 205–225.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*(5), 503–518.

Steiner, C. J. (2009). *A study of pre-service elementary teachers' conceptual understanding of integers* (Unpublished doctoral dissertation). Kent State University, Kent, OH.

Steinle, V. (2004). *Changes with age in students' misconceptions of decimal numbers* (Unpublished doctoral dissertation). The University of Melbourne, Melbourne.

Strauss, A. L., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks, CA: SAGE Publications, Inc.

Suh, J. M., Johnston, C., Jamieson, S., & Mills, M. (2008). Promoting decimal number sense and representational fluency. *Mathematics Teaching in the Middle School*, *14*(1), 44–50.

Tatto, M. T., & Senk, S. (2011). The mathematics education of future primary and secondary teachers: Methods and findings from the Teacher Education and Development Study in Mathematics. *Journal of Teacher Education*, *62*(2), 121–137.

Thanheiser, E. (2009). Preservice elementary teachers' conceptions of multidigit whole numbers. *Journal for Research in Mathematics Education*, *40*(3), 251–281.

Thanheiser, E. (2014). Developing prospective teachers' conceptions with well-designed tasks: Explaining successes and analyzing conceptual difficulties. *Journal of Mathematics Teacher Education, 18*(2), 141-172.

Thanheiser, E., Browning, C., Edson, A., Kastberg, S., & Lo, J. J. (2013). Building a knowledge base: Understanding prospective elementary teachers ' mathematical content knowledge. *International Journal for Mathematics Teaching and Learning*, (July).

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education*, *10*(4-6), 415–432.

Timmerman, M. A. (2010). Making connections: Elementary teachers' construction of division word problems and representations, *114*.

Tobias, J. M. (2009). *Preservice elementary teachers' development of rational number understanding through the social perspective and the relationship among social and individual environments* (Unpublished doctoral dissertation). University of Central Florida, Orlando, FL.

Tobias, J. M. (2012). Prospective elementary teachers' development of fraction language for defining the whole. *Journal of Mathematics Teacher Education*, *16*(2), 85–103.

Tsao, Y.-L. (2005). The number sense of preservice elementary teachers. *College Student Journal*, 647–679.

Utley, J., & Reeder, S. (2012). Prospective elementary teachers' development of fraction number sense. *Investigations in Mathematics Learning*, *5*(2), 1–14.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, *14*(5), 453–467.

Wheeldon, D. A. (2008). *Developing mathematical practices in a social context: An instructional sequence to support prospective elementary teachers' learning of fractions* (Unpublished doctoral dissertation). University of Central Florida, Orlando, FL.

Widjaja, W., Stacey, K., & Steinle, V. (2008). Misconceptions about density of decimals: Insights from Indonesian pre-service teachers' work. *Journal for Science and Mathematics Education in Southeast Asia*, *31*(2), 117–131.

Widjaja, W., Stacey, K., & Steinle, V. (2011). Locating negative decimals on the number line: Insights into the thinking of pre-service primary teachers. *The Journal of Mathematical Behavior*, *30*(1), 80–91.

Wilensky, U. J. (1993). *Connected mathematics: Building concrete relationships with mathematical knowledge* (Unpublished doctoral dissertation). Massachusetts Institute of Technology, Cambridge, MA.

Yackel, E., Underwood, D., & Elias, N. (2007). Mathematical tasks designed to foster a reconceptualized view of early arithmetic. *Journal of Mathematics Teacher Education*, *10*(4-6), 351–367.

Yin, R. K. (2014). *Case study research: Design and methods*. Thousand Oaks, CA: SAGE Publications, Inc.

Yopp, D., Burroughs, E. A., & Lindaman, B. J. (2011). Why it is important for in-service elementary mathematics teachers to understand the equality .999…=1. *The Journal of Mathematical Behavior*, *30*(4), 304–318.

Young, E., & Zientek, L. R. (2011). Fraction operations: An examination of prospective teachers' errors, confidence, and bias. *Investigations in Mathematics Learning*, *4*(I), 1–23.

Zazkis, R., & Gadowsky, K. (2001). Attending to transparent features of opaque representations of natural numbers. In A. Cuoco (Ed.), *The roles of representation in school mathematics* (pp. 146–165). Reston, VA: NCTM.

Zazkis, R., & Sirotic, N. (2010). Representing and defining irrational numbers: Exposing the missing link. *Research in Collegiate Mathematics Education*, *16*(7), 1–27.

Zazkis, R., & Whitkanack, D. (1993). Non-decimals: fractions in bases other than ten. *International Journal of Mathematical Education in Science and Technology*, *24*, 77–83.

Appendix A: Overview of Activities in Instructional Unit

Number and Numeration Unit Overview			
Day 1: Intro to course; Pretest; Intro to mathematical knowledge for teaching (Ball video); Guess my number	Day 2: Base dancing; Writing number using **base-X positional** notation	Day 3: Features of number systems (symbols, grouping, base); More practice writing numbers in other bases; Historical (Egyptian, Traditional Chinese, Mayan) numeral systems jigsaw, day 1: Learning about assigned number system and designing learning tasks; Addition (subtraction) in other number systems	Day 4: Historical numeral systems jigsaw, day 2: Teaching others about assigned number system, learning about other systems
Day 5: Review/summary of all 3 historical number systems; General introduction to models for thinking about numbers (pictures, unifix cubes, paper, etc).; Looking at student work on number lines; Important features of number lines	Day 6: QUIZ; **Number line understanding;** Equivalence on a number line; Interval and unit on number line; Partitioning a number line; Definition of rational numbers and integers	Day 7: **Finding rational numbers on number lines** that already have partitions (common denominators); Breaking bread and Egyptian Fractions: Two ways of thinking about the same amount; Introduction to *fraction form*, division as equal sharing	Day 8: Review different ways of representing the same number by grouping in different ways—related to number systems studied; Multiplication: Egyptian multiplication; whole number multiplication stories
Day 9: Fraction multiplication with stories, pictures;	Day 10: Division as measurement, equal sharing Fraction division with stories, pictures	Day 11: Fraction multiplication and division—stories and pictures	12: **Breaking bread and decimals**: Introduction to decimals and relationship to fractions; "Decimals" in other bases
Day 13: **Rational numbers** as *all* repeating, terminating **decimals** and vice-versa (**fraction-decimal equivalence**); 0.999…=1	Day 14: Differences between **Rationals** and **Irrationals**; Finding π on a number line; Irrationals as points on **number line** NOT able to be reached by partitioning/iterating	Day 15: Integers Introduction to integers as "opposites"; Student difficulties with integer subtraction; Floats and anchors as prototype story and way to use chip model to make sense of integer addition and subtraction problems	Day 16: Putting it together—Review stations Day 17: Unit Test

Appendix B: Pretest

Name: ___________________

Pre-Reflection on Number and Operations Unit

1. Give some examples and non-examples for each kind of number. If you are unsure whether a number is a given type, put it in the "unsure" category.

Whole Numbers	NOT whole numbers	Unsure

Integers	NOT integers	Unsure

Rational Numbers	NOT rational numbers	Unsure

Irrational Numbers	NOT irrational numbers	Unsure

2. Are there any numbers between 0.66 and $\frac{2}{3}$? If so, give an example and briefly explain how you know it is between 0.66 and $\frac{2}{3}$. If not, explain why not.

3. The number 121_3 is a number written in base-3. Write 121_3 as a **base-10** number. Show and/or explain how you found your answer.

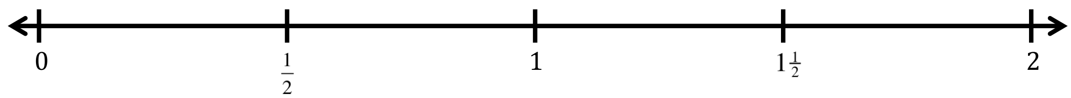
4. Tell how you would help a student understand why $\frac{1}{3} = 0.333\ldots$ when written as a decimal.

5. Two common student mistakes with addition are shown below. What do these mistakes **have in common**?

$$\frac{2}{3} + \frac{4}{5} = \frac{6}{8}$$

$$\begin{array}{r} 0.43 \\ \underline{+0.6} \\ 0.49 \end{array}$$

6. Accurately locate $\frac{3}{5}$ on the given number line **without using a ruler** or other measuring device. Briefly explain or show how you located the number.



7. For each open number sentence, (a) solve, (b) write a story problem that matches the sentence *exactly as written*, and (c) show how to solve by drawing pictures.

    a. $\frac{2}{5} \times 1\frac{1}{4}$

Solution:

Story:

Picture:

For each open number sentence, (a) solve, (b) write a story problem that matches the sentence *exactly as written*, and (c) show how to solve by drawing pictures.

b. $1\frac{1}{4} \div \frac{3}{4}$

Solution:

Story:

Picture:

c. $^{-}3 - {}^{-}5$

Solution:

Story:

Picture:

Appendix C: Posttest

Name__________________________ Score______/100

Number and Numeration Exam
Spring 2015

1. Use the definition of *rational numbers* and *integers* to answer the following (6 pts)

a. Which of the following numbers are *rational* numbers? Circle all that apply.

$\frac{16}{4}$ 5 .153274 $-\frac{2}{3}$ 13.29 $.\overline{216}$ 0 .121231234...

b. Which of the following numbers are *integers*? Circle all that apply.

$\frac{16}{4}$ 5 15.000 $-\frac{2}{3}$ -13.29 2,340,643 0 -8

2. The "Ooktal" numeral system is shown below, along with examples of how to write the numbers 83 and 600 using the Ooktal system. (5 pts each)

Quantity	1	8	64	512
Ooktal symbol	/	<	×	♮

83: ×<<///   600: ♮×<<<

a. What base does the Ooktal system use? How do you know?

b. Is the Ooktal numeral system a *simple grouping* system (like the Ancient Egyptian), a *multiplicative grouping* system (like the Ancient Chinese), or a *positional* system (like the Mayan and Hindu-Arabic systems)? Briefly justify your answer.

3. Are there any rational numbers **between** $\frac{1}{3}$ and 0.333? If yes, give an example and justify how you know it is between $\frac{1}{3}$ and 0.333. If not, explain why not. (4 pts)

4. For the problem 1.2 ÷ .3 = ____, one student gave the following answer:

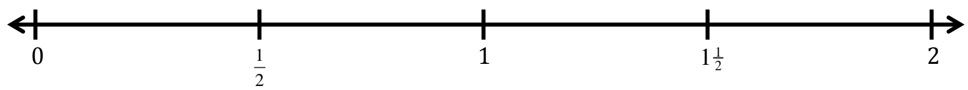
> "The answer is 4. I thought of 1.2 as $\frac{12}{10}$ and .3 as $\frac{3}{10}$. Then I used the common denominator algorithm for division of fractions. Since 12 ÷ 3 = 4, I know that $\frac{12}{10} \div \frac{3}{10} = 4$."

   Is this answer *completely correct*, *partially correct*, or *incorrect*. Justify your answer. (5 pts)

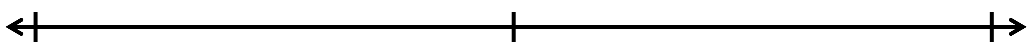
5. Show/explain how you could help a student find the decimal representation of the fraction ⅙ **without** using the standard division algorithm or a calculator. (5 pts)

6. Accurately locate each value on the given number line. When more than one number is given, position and label both numbers on the same number line. Briefly explain or show how you positioned each point. (5 pts each)

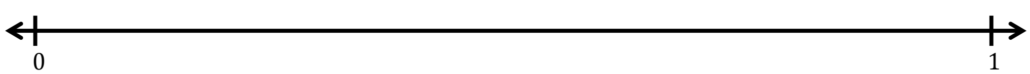
   a. $\frac{4}{3}$



   b. 1.25 and 1.7



   c. 0.1666...



7. The number 41_5 is a number written in base-5.

a. Write 41_5 as a **base-3** number. Show and/or explain how you found your answer. (5 pts)

b. The number 112_3 is a number written in base-3. Which number is larger, 112_3 or 41_5? Explain how you know. (5 pts)

8. For the open number sentence below, (a) write a story problem that matches the sentence *exactly as written*, (b) show how to solve by drawing pictures, and (c) clearly state the solution to your problem.

$$1\frac{1}{4} \div \frac{3}{4} = \square$$

Story (6 pts):

Picture (6 pts):

Solution (3 pts): ________________

9. For the open number sentence below, (a) write a story problem that matches the sentence *exactly as written*, (b) show how to solve by drawing pictures, and (c) clearly state the solution to your problem.

$$^{-}3 - {}^{-}5 = \_\_\_\_$$

Story (6 pts):

Picture (6 pts):

Solution (3 pts): ____________________

10. For the open number sentence below, (a) write a story problem that matches the sentence *exactly as written*, (b) show how to solve by drawing pictures, and (c) clearly state the solution to your problem.

$$\frac{2}{5} \times 1\frac{1}{4} = \square$$

Story (6 pts):

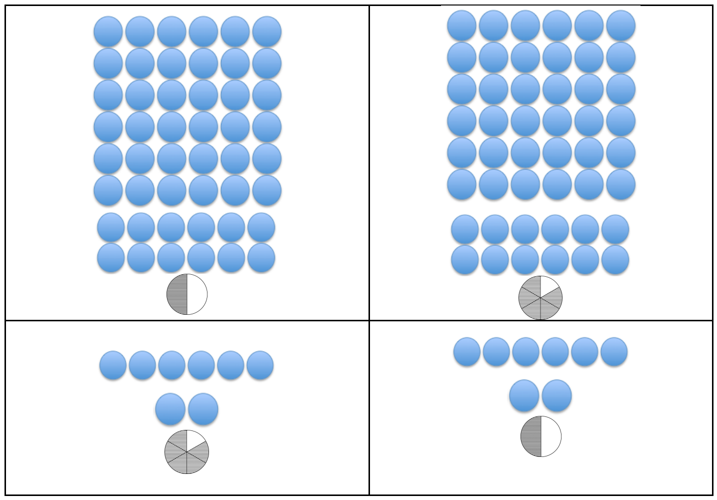
Picture (6 pts):

Solution (3 pts): ______________

BONUS PROBLEMS (Optional)

1. Circle the picture below that best shows the quantity represented by this **base-6** number: **12.5_6.** (1 pt extra credit)

a.



b. Briefly justify your answer. (1 pt extra credit)

2. Write $\frac{1}{3}$ as a decimal in base-7. Show/explain how you got your answer. (Up to 2 pts extra credit)

3. Explain why $\frac{1}{3}$ written in base-7 decimals is similar to $\frac{2}{9}$ in base-10 decimals. (Up to 2 pts extra credit)

Appendix D: Introductory Questionnaire

Tell Me About Yourself

Name: ___________________________

Please call me: ____________________

Can I email you without using "blind copy bcc:" when I send a class email? YES NO

Describe your favorite math class.

Describe your least favorite math class.

What is your favorite subject to learn about? Why?

What about teaching interests you?

What questions do you have about teaching math to elementary students?

Is there anything else you would like me to know about you?

Appendix E: Description of Interview Participants

Soren Soren was a male, sophomore student. He the only male interviewee. His highest level of mathematics was AP calculus in high school. He reported liking math. He stated that he enjoys "approaching things from different angles if I can. Try and figure it out that way, and just try and logic it through. I did, in calc, I didn't memorize all the things I was supposed to memorize but I found ways I could do it." Soren felt he was good at math "in the settings that I've had to do math," and that he was good at mental math. He remembered having a hard time memorizing facts and formulas in elementary math but that he was still "pretty quick" with his multiplication facts. He also stated, "I enjoy fractions." When asked to explain, he said, "It was just cool. I had a revelation with multiplying and dividing, I could just flip it. And finding common denominators, for some reason, I got a kick out of that."

Jo Jo was a female, sophomore student. She took both AP calculus and statistics in high school (junior and senior year, respectively). She reported getting a 4 on the AP calculus exam. These were the last two mathematics classes she had taken. She reported she was excited to take the class for elementary education majors. She had tutored a younger student during high school and that experience got her interested in pursuing teaching as a career. In elementary school, she reported enjoying taking timed tests and always wanted to be finished first. She reported that when she was younger, she liked that math was "the one subject where I could see what the right answer was and how my answer was wrong or right." As she progressed in math, she began to see math as less black-and-white but still enjoyed the "math high that you get when you work on a problem and finally get it right." She described herself as good at math "when I know what I need to do." She added, "When I have to be the one to kind of think creatively, if a problem is presented differently, I don't always possess the skills to get there my own way or to be creative enough. If I get stuck, it's sometimes hard for me to get unstuck."

Andie Andie was a female, sophomore student. Andie's highest level of mathematics taken was calculus II which she took in high school. She had not taken any mathematics while at the university. She reported doing well in the class and that she got a 5 on the AP exam. She said that math was a subject that she was always good at but that she preferred English and history. She said that a challenge for her was explaining mathematics to other people and "understanding why I do what I do when I'm solving a problem." She said that knowing more than one way to solve a problem was important when working with others, particularly if they did not understand the way she solved the problem.

Mei Mei was a female, junior student. She grew up in an Asian country and had been living in the United States for three years at the time of the interview. She planned to return to her home country after completing her degree. English was her second language and her English was excellent. There were a few times both during the interviews and in class when language was a barrier. Sometimes Mei did not know an English word to describe something, other times either she or the interviewer had difficulties understanding something the other had said. The highest mathematics class she took was calculus I, which she took after high school but before beginning her work at the university. What she liked about math the fact that different processes could be used to get the same result and "you can discover and explore as many processes as you can." She also stated that she did not believe that she learned this from her elementary teachers, but rather, she discovered it herself. She thought she was "kind of good at math," because she could solve some problems, but compared to her peers at home, she could not solve as many difficult problems.

Eva Eva was a female, sophomore student. Eva's highest level of math taken was college algebra and statistics. When asked to tell a little bit about her math background, Eva responded, "I've had a bad relationship with math since day one, I would say. Just, I had no interest in it." She reported that math was a subject she struggled with throughout her schooling, starting learning her basic facts in elementary school. She said that she performed poorly on timed tests of her facts and so had to go to summer school. However, once there she credited timed tests and flash cards (along her realization, "Ok, I obviously need to do this") with helping her to learn her facts. She reported working hard at math after elementary school, and that she liked math when she understood it. She stated that she liked when she could work with manipulatives but strongly disliked word problems. "I love stories but I don't like numbers, so putting the two together just frustrated me." Eva believed that math was something that came naturally to some people. She also reported that it was important for her to feel that she was not alone when she did not understand something, to know that others felt the same way.

Nina Nina was a female sophomore student. Nina had taken the highest level of math of the interviewees. She was originally a math major and so had taken at least two courses beyond calculus in college including Sequences and Series (which she dropped) and Linear Algebra. She was in advanced math in elementary school and took calculus in high school. She said that she thought she was good at math until she struggled with some courses at the university and that now her "confidence is shot down." Nina reporting liking math. When asked what she liked about math, she responded, "I like being able to if you struggle through a problem, knowing there's one concrete answer. And when you get to the answer, it just feels good too. Especially with higher math it just takes forever to do something, a problem to finally get to an answer an understand it. I like that." She reported liking timed tests of basic facts in elementary school and said she had always done well on them.

Willa Willa was a female, sophomore student. Willa's highest level of math class taken was pre-calculus in high school. When asked about her math background, Willa stated, "I was terrible at math. I'm still pretty bad." She reported having a particularly bad experience with her fifth grade teacher but that she did better in high school math, enjoying both algebra and geometry. When asked about what she liked about algebra, she explained, "It just makes sense. You followed a set process and you got to the answer. There was no you don't know what you do next, really. It always made sense." She reported feeling neutral towards mathematics, "It's that I like it, but I also don't dislike it." She did not consider herself to be good at math.

Korey Korey was a female, non-traditional student, having returned for her degree after several years out of school. She was a junior. Her highest level of mathematics taken was college algebra or "math for everyday life." When asked if she liked math, Korey responded, "I don't like to say I don't like math because it doesn't interest me because it's hard for me." She did not remember much about elementary math other than "a lot of addition and subtraction and using like pictures, like 3 apples and then you take away 2 apples." She felt that she was not good at math because "I usually get help from people who like tell me kind of how to get the answers, but it's just like I want to understand the whole process of getting it, so I don't think I'm good at math because I don't know like all the steps." She did state, "When I understand it, I really like it. But I don't like it because I don't understand it."

Appendix F: Interview Protocols

Interview 1 Sample Protocol

Part 1: Math background

- Tell me a little about your math background. (Highest class taken? Last class taken?)
- What do you remember about elementary math?
- Do you like math? Why or why not?
- Do you consider yourself good at math? Why or why not?

Part 2: *Number* domain

- How would you define "number"—what is a "number?"
  - Follow up: Is there anything else that could be a number?
  - Are there different kinds of numbers? Why/why not?
  - Can you give me a definition for what you mean by each category of number?
  - Can you give me some examples of each kind of number?
- Give interviewee sheets with categories of subsets of real numbers, and definition for each subset (shown below). Talk through and check for understanding.

  Provided definitions:

Whole numbers (sometimes called the ***counting*** or ***natural*** numbers): The positive integers, {1, 2, 3, ...}.
Integers: A number that can be written without a fractional part *or* The positive and negative whole numbers and 0.
Rational Numbers: All numbers that can be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.
Real Numbers: A number that can represent a quantity along a continuous line.
Irrational Numbers: Real numbers that are *not* rational, or cannot be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

- Give number cards one at a time and ask them to put in as many categories as they think appropriate. Ask to justify thinking.
  - Include at least one of each of the following categories:
    - Common terminating decimal (0.3, 72.3, 0.25)
    - Uncommon terminating decimal (0.000003400, 0.6215, 3.14)
    - Common fraction (⅔, ⅓, ¼)
    - Improper fraction ($^{5}/_{4}$)
    - Uncommon fraction ($^{0}/_{4}$, $^{40}/_{41}$, $^{2}/_{492}$, $^{3}/_{-5}$)
    - Non-standard fraction ($^{0}/_{0}$, $^{5}/_{0}$, $^{\pi}/_{3}$)
    - Integer (17, -3, $^{6}/_{2}$)
- If time permits: How are all these different kinds of numbers related? Can you draw me a picture that shows how you see the relationships?

Interview 1 Sample Protocol, cont.

Part 3: *Representation* domain

- What is the relationship between fractions and decimals?
- How do you feel about working with fractions? Why?
- How do you feel about working with decimals? Why?
- Can all fractions be written as decimals? Why/why not? (If not, give some examples)
- Why do you think we teach kids about fractions in elementary school? About decimals?

Part 4: *Operations*

- How would you define addition?
    - Is addition the same thing for all these different kinds of numbers? Why/why not?
- How would you define subtraction?
    - Is subtraction the same thing for all these different kinds of numbers? Why/why not?
- How would you define multiplication?
    - Is multiplication the same thing for all these different kinds of numbers? Why/why not?
    - As time permits: Ask student to relate stated definition of multiplication to following problems:
        - $3 \times \frac{2}{3}$
        - $\frac{2}{3} \times \frac{3}{4}$
        - $\pi \times 3$
        - $\pi \times \pi$
- How would you define division?
    - Is division the same thing for all these different kinds of numbers? Why/why not?
    - As time permits: Ask student to relate stated definition of multiplication to following problems:
        - $12 \div 3$
        - $12 \div \frac{1}{3}$
        - $\frac{1}{3} \div 12$
        - $\frac{3}{4} \div \frac{2}{3}$

Interview 2 Sample Protocol

Format: Semi-structured individual interviews.
Time expected: 45 min – 1 hour each
When: After the post-test for the number and numeration portion of the course.

Part 1: Standard

- Thinking back over the unit, what stands out to you as things you learned?
- What questions do you have?
- Show the unit overview. Go over and ask student to talk through what they remember learning and/or struggling with from the unit.
- Do you have advice about what you would say we should definitely keep or definitely change in the unit?
- What is the relationship between fractions and decimals?

Part 2: Individualized

- Individualized questions based on questions from initial interview that they found challenging, ask what they think now.
    - Sample questions:
        - What do you know about these three numbers? 0.6, 0.66..., and $\frac{2}{3}$?
        - What do you know about $0.\overline{583}$? How would you locate it on a number line?
        - Can you turn 1.352 into a fraction?
        - How would you locate $^5/_4$ on a number line? □?
        - What do you know about π? How would you locate π on a number line?
        - What do you know about $^0/_0$?
- Individualized questions based on questions from posttest that they found challenging, where work was unclear, or that I would like more information about how they thought about the problem.
    - Sample questions:
        - Explain how to find $^1/_6$ as a decimal (or how you would show a student).
        - Explain how you could find $\frac{1}{3}$ as a decimal in base seven.
        - Explain how you located $0.1\overline{6}$ on the number line. Why did you use that method?
        - Can you write a story to match $\frac{2}{3} \times \frac{3}{4}$? Show how you would draw a picture to match your story.
        - (For problems with drawings included) Did you solve the problem before drawing a picture? How did you solve it?
        - Does your solution match your picture? How do you know?

Appendix G: First Cycle Codes

Nested Hierarchy of First Cycle Codes

- Understanding
    - Making connections
        - Flexible/inflexible
    - Good understanding
    - Good memory
    - Difficulty describing concept taught in elementary grades
    - Procedural understanding
    - Conceptual understanding
- Representation
    - Words
    - Using partitioning
    - Concrete/manipulative
    - Symbolic
    - Number line
    - Picture
    - Story/context
    - Breaking Bread
- Notation
    - Place value/positional notation
        - Non base 10
        - Base 10
    - Fraction-decimal connection
    - Decimal
        - Terminating decimal
        - Repeating decimal
        - Non-terminating, non-repeating decimal
        - Preference for decimals
        - Misunderstandings
    - Fraction
        - Misunderstandings
        - Preference for fractions
- Number
    - Historical
    - Definition of number
        - Domains of number as nested subsets
        - Domains of number understood using definitions
    - Zero
    - Real numbers
    - Negatives/integers
    - Whole numbers
    - Rational number
    - Irrational numbers
        - Pi
- Intentional connection
- Critical event