Two essays on dynamic models of firm financing

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Chapter 1

Introduction

How firms raise enough capital to sustain profitable production is a central question in corporate finance. Broadly speaking, equity and debt are the most common sources for companies' external finance. Firms' interaction with capital markets should be viewed as a repeated process rather than a one-shot game. In my dissertation, I explicitly take such a dynamic feature into consideration and investigate the firms' behaviors in the equity and debt markets separately.

In the first essay "Buying high and selling low: Stock repurchases and persistent asymmetric information", coauthored with my adviser Philip Bond, we study the consequences of allowing for repeated equity market transactions in a model with asymmetric information between firms and investors. All firms in the model possess a profitable project that they need to raise cash to undertake. However, equilibria exist in which firms return cash to investors via share repurchases. Consistent with managerial accounts, some repurchasing firms profit from repurchasing their stock. The ultimate source of these profits is that other firms buy high in order to improve the terms of their subsequent stock issues. Only equilibria with repurchases satisfy a relatively mild refinement. Repurchases lower social welfare by reducing the fraction of firms that invest, even though repurchasing itself is a costless signal. Our model generates a number of empirically consistent predictions.

In the second essay "A dynamic model of optimal creditor dispersion", I build a dynamic rollover model to analyze how firms choose the number of creditors and how this decision changes over time. Raising debt from more creditors may lead to inefficient liquidation caused by coordination failure. Potential coordination failure can, however, improve a firm's incentive to repay its debt, thus increasing its debt capacity. Given this trade-off between higher liquidation risk and enhanced pledgeability, I show that firms optimally increase the number of creditors when they perform badly. Even though having more creditors increases the liquidation probability, allowing for potential coordination failure from multiple creditors is valuable. Policies that commit the creditors to ex post efficient coordination exacerbate rollover difficulty and the reduction in firm value ex ante. Finally, if the firm can renegotiate its debt very frequently, the extra pledgeability from multiple creditors diminishes. The model also generates empirical implications for the firm value, the interest rates, and the probabilities of liquidation, renegotiation, and default.

The remainder of the dissertation is organized as follows. Chapter 2 contains the first essay "Buying high and selling low: Stock repurchases and persistent asymmetric information". Chapter 3 contains the second essay "A dynamic model of optimal creditor dispersion". Proofs of the results can be found in the appendix.

Chapter 2

Buying high and selling low: Stock repurchases and persistent asymmetric information

2.1 Introduction

An important idea in corporate finance is that firms have more information about their future cash flows than investors. A large body of research has studied the consequences of this asymmetric information for a firm's capital market transactions. However, the vast majority of such papers have restricted firms to a single round of capital market transactions.[1] In this paper, we study the implications of relaxing this assumption for what is arguably the best-known corporate finance model based on asymmetric information, namely Myers and Majluf (1984) model of equity financing to fund an investment.[2]

Our main finding is that allowing for multiple capital market transactions in Myers and Majluf generates the following equilibrium dynamics. Some firms repurchase their

[1] In exceptions such as Lucas and McDonald (1990, 1998), Chowdhry and Nanda (1994), and Hennessy et al. (2010), a firm's informational advantage only lasts one period. In contrast, in our paper the information asymmetry is persistent. In Constantinides and Grundy (1989), which we discuss in detail below, firms engage in two rounds of transactions, but the second transaction is a deterministic function of the first.

[2] As we detail below, we focus on the version of this model where firms know more about the value of their existing assets, but have no informational advantage with respect to growth options.

stock for strictly less than its fair value, consistent with managerial claims that repurchases are driven by undervaluation.[3] Other firms repurchase stock in order to lower the cost of subsequent equity issuance, consistent with empirical evidence (see Billett and Xue, 2007).[4]

Moreover, these dynamics are present in *all* equilibria satisfying a standard and arguably mild refinement: specifically "Never Dissuaded Once Convinced," Osborne and Rubinstein (1990), and discussed in detail below.

At first sight, the ability of firms to strictly profit from trading on their superior information would appear to violate the no-trade theorem (see, e.g., Milgrom and Stokey, 1982). Many existing models of share repurchases avoid this problem by introducing an assumption that firms (exogenously) care directly about an interim share price.[5] Our model avoids this assumption. Instead, in our model some firms strictly profit from repurchases because other inferior firms also repurchase, and make losses. This second group of firms "buy high" when they repurchase, i.e., buy their stock for more than it is worth.

Why does this second group of inferior firms repurchase at a loss? They do so in order to improve the terms at which they can subsequently issue stock to finance a profitable investment. This is consistent with the empirical findings of Billett and Xue (2007). Nonetheless, and as is standard in models of this type, even the improved issuance terms are still associated with a negative price response at issue (this is the "selling low" of the title). These firms can be viewed as "manipulating" their stock price: after they repurchase, their stock price increases, and although the price then declines with the issue announcement, the issue price is still higher than it would otherwise be.

Repurchases do not carry any deadweight loss in our model; in this, our model is very different from much of the prior literature, which assumes that payouts generate a deadweight loss either via increased taxes or via an increased need for (exogenously) costly external financing.[6] Nonetheless, the repurchases strictly lower social welfare

[3] Brav et al. (2005) survey managers. A very large fraction of managers agree (Table 6) that the "Market price of our stock (if our stock is a good investment, relative to its true value)" is an important factor.

[4] Related, in Brav et al. (2005), a very large fraction of managers agree (Table 3) that "Repurchase decisions convey information about our company to investors."

[5] See discussion of related literature below.

[6] See discussion of related literature below. Note that Brav et al. (2005) survey of managers finds

(meaning the total amount of profitable investment), in the sense that social welfare is lower in an equilibrium with repurchases than in an equilibrium of a benchmark one-period model without repurchases. The reason is that firms that issue to finance the profitable investment are forced to first repurchase to signal their quality, and this repurchase generates a loss (which, as discussed above, makes it possible for other firms to strictly profit from repurchases). Consequently, equilibrium repurchases raise the net cost of financing for firms that eventually invest; this in turn reduces the amount of equilibrium investment. Note that because repurchases have no deadweight loss, this welfare result is fundamentally different from the commonly-made observation (see, e.g., Arrow, 1973) that social welfare would be higher if a costly signal were prohibited.

Related literature:

Grullon and Ikenberry (2000) offer a good survey of the literature on repurchasing.

The idea that firms repurchase their stock to signal they are good is related to the old idea that *retaining* equity is a useful signal (Leland and Pyle, 1977). Also related, Example 1 of Brennan and Kraus (1987) has a good firm simultaneously repurchasing debt and issuing equity. The debt repurchase allows the firm to signal that it is good.

Our paper is related to the literature on signaling in static payout models. In one branch of this literature (e.g., Bhattacharya, 1980; Vermaelen, 1984; Miller and Rock, 1985), good firms repurchase to show that they have (or expect to have) high cash flow. Bad firms do not mimic because they have low cash flow, and so paying out cash necessitates either costly external financing or distorts investment. An important assumption in this branch of the literature is that a firm's objective (exogenously) includes the interim share price. Regarding this assumption, Allen and Michaely (2003) write "why would a management care so much about the stock price next period? Why is its horizon so short that it is willing to 'burn money' (in the form of a payout) just to increase the value of the firm now, especially when the true value will be revealed next period?" In contrast to this literature, we do not exogenously assume that the firm cares about the interim share price.

In a second branch of the literature (e.g., John and Williams, 1985; Ambarish et al.,

little support for the idea that repurchases are made to signal that a firm can bear such costs. For example, only a small fraction of managers (Table 3) say that "We use repurchases to show we can bear costs such as borrowing costly external funds or passing up investment..."

1987; Williams, 1988), firms pay out cash in a costly way, typically by issuing dividends, which are tax-inefficient. Firms then issue equity to finance an investment. Good firms pay out, while bad firms do not. Because of this separation, good firms are able raise the funds they need for investment in a less dilutive way. Bad firms do not mimic good firms because they would pay the same cost (inefficient cash pay outs), but benefit less because dilution is less costly to them then it is to good firms. The economic function of pay outs in these models is that they destroy value. This raises the question of whether other value-destroying actions would make better signals, and led the literature to consider multi-dimensional signaling models (see, e.g., Ofer and Thakor (1987), Viswanathan (1995); we briefly consider the robustness of our analysis to multi-dimensional signaling in Section 2.9). Because repurchases are generally regarded as a tax efficient way of making pay outs, and hence do not destroy value, the main focus of this branch of the literature is on dividends rather than repurchases.

Constantinides and Grundy (1989) study a model in which firms issue securities to fund an investment, and can commit to return any excess cash in the form of a repurchase. They give conditions under which full separation of firms is possible, and show that the commitment to repurchase plays an important role in supporting this separating equilibrium. Because the equilibria they study are fully separating, no firm profits from the repurchase transaction. Moreover, it is important that the original security issued differs from equity. In contrast, we study a case in which firms cannot commit to future transactions, and transact in the same security (equity) at all dates. We show that all equilibria entail some firms making strictly positive profits from stock repurchases.

An important assumption in any model of repurchasing based on signaling, including ours, is that a firm's repurchase decision is actually observable. Although regulatory mandates force this to be true in many markets, there has been some debate in the literature about the observability of repurchases in the United States. For example, in an early study of repurchases, Barclay and Smith (1988) find evidence that the announcement of a repurchase program is followed by an increased bid-ask spread, which they interpret as an increase in adverse selection, which they in turn interpret as investors being unsure about whether or not they are trading against the firm. However, in general subsequent research has not supported this original finding (see the discussion in Grullon and Ikenberry, 2000).

A relatively small literature studies dynamic models of trade under asymmetric information. Noldeke and Van Damme (1990) and Swinkels (1999) study a labor market model where education acts as a signal. Fuchs and Skrzypacz (2013) study trade of a single indivisible asset that is more highly valued by buyers than the seller. They focus on whether more trading opportunities increase or reduce welfare. Kremer and Skrzypacz (2007) and Daley and Green (2012) study a similar model in which information arrives over time. In contrast to these papers, in our model both sales and repurchases are possible; trade is in divisible shares; and the gains from trade arise from the possibility of financing a profitable investment. Perhaps closest to the current paper are Morellec and Schürhoff (2011) and Strebulaev et al. (2014). Both papers study dynamic models in which a firm with long-lived private information chooses a date to raise outside financing and invest. In both papers, issue and investment are tied together (by assumption), and the combination of repurchases with subsequent equity issue—which is our main focus—is not examined. Instead, the main results of both papers concern the timing of investment. Finally, a contemporaneous paper by Ordoñez et al. (2013) studies a dynamic model of debt issuance.

In a model with moral hazard in place of adverse selection, DeMarzo and Urošević (2006) study the dynamics of a large shareholder selling off his stake in a firm.

Bond and Eraslan (2010) study trade between differentially-informed parties in common-values setting. The no-trade theorem does not apply because the eventual owner of the asset takes a decision that affects the asset's final cash flow. Trade affects the information available to the party making the decision. In the current paper, trade of the asset (i.e., shares) at date 1 instead affects a firm's ability to raise finance at date 2. Related, Huang and Thakor (2013) give a model in which the gains from repurchase stem from reducing disagreement among a firm's shareholders.

2.2 Example

Firms have cash 1, and the opportunity to invest 9 at date 2 in a project that subsequently yields 11. Hence firms need to raise additional funds of 8 in order to invest. Firms can either repurchase (buy) or issue (sell) shares at each if dates 1 and 2. All uncertainty is resolved at date 3, and firms act to maximize their date 3 share price. The initial number of shares is normalized to 1.

Firm assets-in-place a are distributed over $[0,40]$, with a density that satisfies the following properties. First, there is a probability $\frac{1}{4}$ that the assets-in-place lie in each of the intervals $[0,2]$, $[2,4]$,$[4,21]$, and $[21,40]$. Second, the conditional expectation of a satisfies $E[a|\in[0,2]]=1$, $E[a|\in[2,4]]=2.2$, $E[a|\in[21,40]]=37.8$.[7]

The following is a Perfect Bayesian equilibrium (PBE):

- At date 1, firms with assets-in-place either in $[2,4]$ or $[21,40]$ spend all their cash 1 to repurchase $\frac{1}{22}$ shares for a price $P_1=22$. The remaining firms do nothing.

- At date 2, firms with assets-in-place below 2 raise funds 8 by issuing 2 shares at a price $P_2^D=4$. Firms with assets-in place in $[2,4]$ raise funds 9 by issuing $\frac{9}{4.4}$ shares at a price $P_2^{RI}=4.4$. The remaining firms do nothing.

We verify this is an equilibrium. First, conditional on firms behaving this way, the repurchase and issues prices are fair, as follows. The date 2 issue-after-repurchase price is fair, since it solves

$$P_2^{RI}=\frac{E[a|a\in[2,4]]+11}{1-\frac{1}{22}+\frac{9}{P_2^{RI}}},$$

i.e.,

$$P_2^{RI}=\frac{2.2+11-9}{1-\frac{1}{22}}=4.4.$$

The date 2 direct issue price is fair, since it solves

$$P_2^D=\frac{E[a|a\in[0,2]]+11}{1+\frac{8}{P_2^D}},$$

i.e.,

$$P_2^{RI}=1+11-8=4.$$

The date 1 repurchase price is fair, since with probability 1/2 the date 2 price will be $P_2^{RI}=4.4$ and with probability 1/2 it will be $\frac{E[a|a\in[21,40]]}{1-\frac{1}{22}}=39.6$, and so, conditional on date 1 repurchase, the expected date 2 price is 22.

[7] Note that there are an infinite number of distributions satisfying these properties. We stress that these properties are chosen only to produce a reasonably simple numerical example.

Second, firms respond optimally to the stated repurchase and issue prices. The date 3 share price of a firm with assets-in-place a from repurchasing then issuing is

$$\frac{11+a}{1-\frac{1}{22}+\frac{9}{4.4}}=\frac{11+a}{3},$$

the date 3 share price issuing directly at date 2 is

$$\frac{11+a}{1+2}=\frac{11+a}{3},$$

the date 3 share price from repurchasing at date 1 and then doing nothing is

$$\frac{a}{1-\frac{1}{22}}=\frac{22}{21}a$$

and the date 3 share price from doing nothing at both dates is simply

$$1+a.$$

Out of these three alternatives, firm with assets-place below 4 obtain the highest payoff from either repurchasing and then investing, or directly issuing and investing; they are between the two options. Firms with assets-in-place between 4 and 21 obtain the highest payoff from doing nothing. Finally, firms with assets-in-place above 21 obtain the highest payoff from repurchasing at date 1 and then doing nothing.[8]

Discussion:

Firms with assets-in-place $a > 21$ repurchase shares for strictly less than their true value, $a+1$, and so make strictly positive profits. The reason investors accept the lower price is that these firms pool with worse firms (namely, firms with a between 2 and 4). But this raises the question of why these worse firms are prepared to repurchase. They do so in order to improve the terms at which they can subsequently issue: if instead they attempt to issue equity directly, they obtain a worse price. Specifically, they issue shares at a price 4 rather than 4.4.

The intermediate interval of firms with between 4 and 21 find issue too dilutive, as in Myers and Majluf, and also find repurchase too expensive.

[8] We have established that firms act optimally when their choice set is limited to the four equilibrium strategies. This still leaves open the possibility that a firm could profitably deviate to some strategy other than these four strategies. However, there exist off-equilibrium beliefs such that all other strategies leave firms worse off. The proof of Proposition 3 below includes the description of one such set of off-equilibrium beliefs.

Firms with $a > 21$ strictly profit from their repurchase transactions, even though these transactions fail to create any value. The ultimate source of these profits is that the investing firms with $a \leq 4$ end up paying a premium to raise capital. By this, we mean that if firms $a \leq 4$ could all credibly pool and issue directly, the issue price P would satisfy $P = \frac{11+\frac{1}{2}(1+2.2)}{1+\frac{8}{P}}$, i.e., $P = 3 + 1.6 = 4.6$, and so the payoff of each firm $a < 4$ would be $\frac{11+a}{1+\frac{9}{4.6}}$, which is higher than they get in the above equilibrium.

A related observation is that the equilibrium of the Myers and Majluf setting, where repurchase is impossible, entails investment by firms with assets-in-place between 0 and a cutoff level strictly in excess of 4. In other words, repurchases lower total surplus in the economy (see Section 2.7). Nonetheless, and as we show below, when repurchase is possible, any equilibrium that satisfies NDOC features some repurchase.

2.3 Model and preliminary results

Our model is essentially the same as Myers and Majluf (1984). The only substantive difference is that whereas Myers and Majluf consider a firm's interactions with the equity market at just one date, we consider two possible dates. As we will show, this additional feature generates equilibrium share repurchases.

There are four dates, $t = 0, 1, 2, 3$; an all-equity firm, overseen by a manager; and at each of dates 1 and 2, a large number of risk-neutral investors who trade the firm's stock. We normalize the date-0 number of shares to 1.

At date 0, the manager of the firm privately learns the value of the firm's existing assets ("assets-in-place"). Write a for the expected value of these existing assets, where $a \in [\underline{a}, \overline{a}]$. Let μ be a measure on $[\underline{a}, \overline{a}]$, which determines the distribution of assets-in-place a. We assume a has full support on $[\underline{a}, \overline{a}]$, and has no atoms. In addition to assets a, the firm has cash (or other marketable securities) with a value S.

At the end of date 2, the firm has an opportunity to undertake a new project. (In Section 2.8, we extend the model to allow for a choice of investment timing, with the firm able to invest at either date 1 or date 2.) The project requires an initial investment I and generates an expected cash flow $I + b$. For simplicity, we assume that b is common knowledge; in other words, we focus on a version of the Myers and Majluf environment in which asymmetric information is about assets-in-place, not investment opportunities.

Throughout, we assume $I > S$, so that the firm needs to raise external financing to finance the investment I.

At each of dates $t = 1, 2$, the firm can issue new equity and/or repurchase existing equity. Equity issues and repurchases take place as follows. The manager makes a public offer to buy or sell a fixed dollar amount s_t of shares, where $s_t > 0$ corresponds to share repurchases and $s_t < 0$ corresponds to share issues. Investors respond by offering a quantity of shares in exchange. In other words, if $s_t > 0$ each investor offers a number of shares he will surrender in exchange for s_t; and if $s_t < 0$, each investor offers a number of shares he will accept in return for paying the firm $-s_t$.

(Note that both a and $I+b$ are expected values, so our model allows for very volatile cash flows. In particular, we assume that there is enough cash flow volatility that it is impossible for firms to issue risk free debt. In general, the choice between risky debt and equity under asymmetric information is non-obvious; see Fulghieri et al. (2012) for a recent characterization. In Section 2.9 we discuss the robustness of our analysis to allowing for other securities.)

At date 3, the true value of the firm is realized, including the investment return, and the firm is liquidated.

Write P_3 for the date-3 liquidation share price, and write P_1 and P_2 for the transaction price of the shares at dates $t = 1, 2$. Because the number of investors trading at each of dates 1 and 2 is large, competition among investors implies that the date t share price is

$$P_t = E\left[P_3 | \text{date } t \text{ information, including firm offer } s_t\right]. \tag{2.1}$$

The manager's objective is to maximize the date 3 share price, namely

$$P_3 = \frac{S - s_1 - s_2 + a + b\mathbf{1}_{\text{investment}}}{1 - \frac{s_1}{P_1} - \frac{s_2}{P_2}}, \tag{2.2}$$

where $\mathbf{1}_{\text{investment}}$ is the indicator function associated with whether the firm undertakes the new project, and the denominator reflects the number of shares outstanding at date 3. Note that in the case that only share issues are possible, the manager's objective function coincides with the one specified in Myers and Majluf (1984), which is to maximize the utility of existing ("passive") shareholders. In our setting, where repurchases are possible, the manager's objective function can be interpreted as maximizing the value of passive shareholders, who neither sell nor purchase the firm's stock at dates 1 and 2. Alternatively, the manager's objective can be motivated by assuming that the manager himself has an equity stake in the firm, and is restricted from trading the firm's shares on his own account.[9]

For use throughout, observe that (2.1) and (2.2), together with the fact that the firm invests whenever it has sufficient funds, imply that the date 2 share price conditional on s_1 and s_2 is

$$P_2\left(s_1, s_2\right) = \frac{S - s_1 + E\left[a|s_1, s_2\right] + b\mathbf{1}_{S-s_1-s_2\geq I}}{1-\frac{s_1}{P_1}}. \tag{2.3}$$

Iterating, (2.1) and (2.3), together with the law of iterated expectations, imply that the date 1 share price conditional on s_1 is

$$P_1\left(s_1\right) = S + E\left[a + b\mathbf{1}_{S-s_1-s_2\geq I}|s_1\right]. \tag{2.4}$$

From (2.3) and (2.4), the payoff of firm a from (s_1, s_2) is

$$\begin{aligned} &\frac{S-s_1-s_2+a+b\mathbf{1}_{S-s_1-s_2\geq I}}{\left(1-\frac{s_1}{P_1}\right)\left(1-\frac{s_2}{S-s_1+E[a|s_1,s_2]+b\mathbf{1}_{S-s_1-s_2\geq I}}\right)} \\ &= \frac{S-s_1-s_2+a+b\mathbf{1}_{S-s_1-s_2\geq I}}{\left(1-\frac{s_1}{S+E\left[a+b\mathbf{1}_{S-s_1-s_2\geq I}|s_1\right]}\right)\left(1-\frac{s_2}{S-s_1+E[a|s_1,s_2]+b\mathbf{1}_{S-s_1-s_2\geq I}}\right)}. \end{aligned} \tag{2.5}$$

We characterize the perfect Bayesian equilibria (PBE) of this game. We restrict attention to pure strategy equilibria in which all investors hold the same beliefs off-equilibrium. We focus on equilibria in which all firms play a best response (as opposed to equilibria in which almost all firms play a best response).[10]

Finally, we state here a simple result that we use repeatedly:

[9] Note that if the manager also put weight on a high date 1 share price this would further increase the manager's incentives to repurchase equity. On the other hand, it is important for our analysis that the manager does not fully internalize the welfare of date 0 shareholders who sell at date 1: in particular, our analysis requires that if a manager is able to repurchase shares at less than their true value, then he does so. As discussed in the text, one justification is that the manager seeks to maximize the value of his own equity stake. A second justification is that when a firm repurchases its own stock, it may not be its existing shareholders who sell shares to the firm; instead, the firm's repurchase offer may be filled by short-sellers of the firm's stock. Attaching zero welfare weight to short-sellers is analogous to the Myers and Majluf assumption of attaching zero welfare weight to new purchasers of the firm's shares.

[10] Given a perfect Bayesian equilibrium in which almost all firms play a best response, one can easily construct an equilibrium in which all firms play a best response by switching the actions of the measure zero set of firms who originally did not play a best response. Because only a measure zero set of firms are switched, the original set of beliefs remain valid.

Lemma 1 *If in equilibrium firms a' and a'' conduct capital transactions (s'_1, s'_2) and (s''_1, s''_2), with $S - s'_1 - s'_2 > S - s''_1 - s''_2$, then $a' < a''$.*

An immediate corollary of Lemma 1 is:

Corollary 1 *In any equilibrium, there exists $a^* \in [\underline{a}, \overline{a}]$ such that all firms $a < a^*$ invest and all firms $a > a^*$ do not invest.*

2.4 One-period benchmark

Before proceeding to our main analysis, we characterize the equilibrium of the benchmark model in which firms can only issue or repurchase shares at date 1, with the date 2 issue/repurchase decision s_2 exogenously set to 0. The main conclusion of this section is that the the Myers and Majluf conclusion holds: only the lowest asset firms issue and invest, and repurchases play no meaningful role. In other words, the addition of the possibility of repurchases to the Myers and Majluf environment is, by itself, inconsequential. Instead, our results further below are driven by the possibility of firms engaging in capital transactions at multiple dates.

The key reason that the firms do not take advantage of repurchases in a one-period model is the no-trade theorem (Milgrom and Stokey, 1982). Even though firms enjoy an informational advantage relative to investors, they are unable to profit from this advantage.

Proposition 1 *In the single stage benchmark game, the set of firms who repurchase and strictly profit relative to doing nothing is of measure* 0*.*

Proposition 1 establishes that, in the one-period benchmark, a firm's ability to repurchase its own stock plays no meaningful role. Accordingly, the equilibria of the one-period benchmark coincide with those of the standard Myers and Majluf (1984) setting, as formally established by the next result:

Proposition 2 *In any equilibrium, there exists $a^* \in (\underline{a}, \bar{a}]$ such that almost all firms below a^* issue the same amount s^* and invest, while almost all firms above a^* receive the same payoff as doing nothing (i.e., $P_3 = a + S$).*

Proposition 2 characterizes properties an equilibrium must possess. However, it does not actually establish the existence of an equilibrium. However, this is easily done. In particular, fix any s^* such that $S - s^* \geq I$, and define a^* by

$$a^* = \max\left\{a \in [\underline{a}, \bar{a}] : \frac{S - s^* + a^* + b}{1 - \frac{s^*}{S+E[a|a\in[\underline{a},a^*]+b]}} \geq S + a^*\right\}.$$

Then there is an equilibrium in which all firms with assets below a^* issue and raise an amount $-s^*$, while firms with assets above a^* do nothing. Off-equilibrium-path beliefs are such that any offer to issue (i.e., $s < 0$ and $s \neq s^*$) is interpreted as coming from the worst type $\underline{a}$, and any offer to repurchase (i.e., $s > 0$) is interpreted as coming from the best type $\bar{a}$.

Observe that if $\frac{I+\bar{a}+b}{1+\frac{I-S}{S+E[a]+b}} \geq S + \bar{a}$, this benchmark model has an equilibrium in which the socially efficient outcome of all firms investing is obtained. In order to focus attention on the case in which asymmetric information causes a social loss, for the remainder of the paper we assume instead that

$$\frac{I + \bar{a} + b}{1 + \frac{I-S}{S+E[a]+b}} < S + \bar{a}, \tag{2.6}$$

so that there is no equilibrium of the benchmark model in which all firms invest. For use below, note that (2.6) implies

$$\bar{a} > E[a] + b > \underline{a} + b. \tag{2.7}$$

2.5 Analysis of the dynamic model

We now turn to the analysis of the full model, in which the firm is able to engage in capital transactions at multiple dates.

2.5.1 Existence of a repurchase equilibrium

We first show that there is nothing "special" about the example we presented above. For all parameter values satisfying (2.6), there exists an equilibrium in which the best firms strictly profit from repurchasing, while worse firms repurchase their stock for more than it is worth—i.e., "buy high"—in order to improve the terms at which they can subsequently issue stock to finance the investment.

Proposition 3 *An equilibrium exists in which a strictly positive mass of firms pool and repurchase at date* 1*. A strict subset of these these firms make strictly positive profits from the repurchase, and do nothing at date* 2*. The remaining repurchasing firms repurchase their stock for more than it is worth, and then issue enough shares to finance investment at date* 2*.*

The proof of Proposition 3 is constructive. The equilibrium constructed is either similar to the above example; or else features all firms repurchasing at date 1, with a strict subset then issuing equity to fund investment at date 2.

2.5.2 Necessity of repurchases

As is common with games of asymmetric information, our model has multiple equilibria. However, we next show that the properties stated in Proposition 3 are possessed by *any* equilibrium satisfying a refinement known as "Never Dissuaded Once Convinced" (NDOC) (Osborne and Rubinstein, 1990). Hence the NDOC refinement selects precisely the set of equilibria that feature repurchases.

In our context, NDOC states that date 2 investor beliefs after observing firm actions (s_1, s_2) must satisfy the following: (I) if s_1 is an equilibrium action, then date 2 beliefs assign probability 1 to the firm's type lying in the set of firms who play s_1 in equilibrium, and (II) if s_1 is not an equilibrium action, and date 1 beliefs assign probability 1 to some subset A of firm types, date 2 beliefs likewise assign probability 1 to the same subset A. This restriction is highly intuitive and is typically regarded as mild; see, for example, Rubinstein (1985) and Grossman and Perry (1986), or more recently, its use as Assumption 1 in Ely and Välimäki (2003) and as Condition R in Feinberg and Skrzypacz (2005).

Proposition 4 *Any equilibrium satisfying NDOC has the properties stated in Proposition 3.*

The economics behind Proposition 4 is as follows. Under assumption (2.6), the best firms do not invest in equilibrium.[11] Consequently, if they do not repurchase, these firms do not make any profits, and the final payoff of a high-value firm a is simply

[11] Formally, this is established in Corollary A-2 in the appendix.

$S + a$. Consequently, for repurchases to be unattractive in equilibrium for the top firm $\bar{a}$, investors must charge at least $S + \bar{a}$ to surrender their shares; in turn, this requires investors to believe that (off-equilibrium) repurchase offers come from very good firms. But given these beliefs, a low-value firm could profitably deviate from its equilibrium strategy by repurchasing at date 1, thereby triggering beliefs that it is very good, and then (by NDOC) issue at a high price at date 2.

A second important implication of Proposition 4 is that the equilibrium outcome of the one-period benchmark economy is not an equilibrium outcome of the full model under NDOC. At first sight, this might seem surprising: one might imagine that one could take the equilibrium of the one-period economy and then assign off-equilibrium beliefs to make other actions, and in particular repurchases, unattractive. However, the dynamic nature of the model makes this impossible. The reason is that, as just illustrated, to deter repurchases, off-equilibrium beliefs must assign a large weight to a repurchasing-firm being a high type; but given these beliefs, a deviating firm can issue at attractive terms at date 2. In brief, under NDOC it is impossible to assign off-equilibrium beliefs that deter *both* date 1 repurchase and date 2 issue.

2.5.3 Existence of a repurchase equilibrium satisfying NDOC

A significant drawback of the NDOC restriction is that in some cases it eliminates *all* equilibria: see Madrigal et al. (1987). To see the issue, consider again the example of Section 2.2. In the equilibrium described, if a firm does nothing at date 1, the NDOC restriction implies that investors must believe the firm has a type $a \leq 21$, regardless of the firm's action at date 2. This in turn means that any firm that does nothing at date 1 is able to repurchase shares for a price of $1 + 21 = 22$, or less. In particular, firms with $a > 21$ would make strictly positive profits from the strategy of doing nothing at date 1, and then repurchasing at date 2.

It is important to note that—despite this concern—the actions described in the example of Section 2.2 are consistent with an equilibrium satisfying NDOC. The reason is that the deviation just discussed—namely doing nothing and then repurchasing—gives a firm a payoff of $\frac{a}{1-\frac{1}{22}}$ if investors associate the strategy of do-nothing-then-repurchase with the belief that a firm is type $a = 21$. (Note that this belief satisfies NDOC.) But this payoff is no better than the equilibrium payoff of firms $a > 21$, and hence is not a profitable deviation.

Nonetheless, we are unable to establish the general existence of an equilibrium satisfying NDOC. However, there are two straightforward perturbations of our model under which we are able to obtain such a result:

Proposition 5 *There exists an equilibrium satisfying NDOC if either:*

(I) There is a probability $\alpha > 0$ that a firm is exogenously unable to conduct any capital market transaction at date 1. In addition, suppose the density function f for firm's asset in place takes the value $\frac{1}{b}$ at most countably many times.

(II) The maximum repurchase size is $\bar{S}$, and $\bar{S}$ is sufficiently small.

Moreover, under each of these model perturbations, Proposition 4 continues to hold, i.e., any equilibrium satisfying NDOC has the properties stated in Proposition 3.

Perturbation (I) of Proposition 5 is motivated by the observation that the act of doing nothing at date 1 has too much signaling power in the above example. After all, it is easy to imagine that a firm does nothing at date 1 for some exogenous reason; for example, perhaps its manager failed to get approval for either an issue or repurchase. In this case, NDOC does not impose any restriction on investor beliefs about firms that do nothing at date 1, and the equilibrium constructed in Proposition 3 is an equilibrium of this perturbed game. In contrast, NDOC continues to have bite for firms that repurchase at date 1 and then issue at date 2: this is why Proposition 4 continues to hold. Finally, note that the the exogenous probability α can be made arbitrarily small.

Perturbation (II) is motivated by the fact that there may exist limits on how much a firm can repurchase. For example, not all of the firm's "cash" S may be immediately available for repurchase transactions. Instead, only an amount $\bar{S}$ may be truly liquid, while the remaining portion $S - \bar{S}$ can be liquidated before the investment I must be made.[12] Existence is guaranteed in this case for the same reason that the example satisfies NDOC: when the maximal repurchase size is small, the deviation of doing-nothing at date 1 and then repurchasing at date 2 does not generate strictly higher profits than the strategy of the equilibrium established in Proposition 3, namely repurchasing immediately at date 1.

[12] See Duchin et al. (2014) for a detailed empirical analysis of the nature of firms' cash holdings.

2.6 Stock price reactions

A large empirical literature has examined stock price reactions to repurchase and issuance announcements; see, e.g., Allen and Michaely (2003) for a survey. As documented by this literature, repurchase announcements are associated with price increases, and issue announcements are associated with price declines.

Our model provides a natural explanation of both these announcement effects. Issue announcements generate negative price responses because lower-value firms issue. This is the "selling low" of the paper's title, and is very much in line with the existing literature (again, see Allen and Michaely, 2003).

Repurchase announcements generate positive price reactions. The reason is that some of the firms repurchasing are high-value firms. This is an effect present in several existing models in the literature. With respect to this previous literature, the innovation of our paper is to obtain this effect without exogenously assuming that firms care about the interim stock price. Specifically, the reason high-value firms repurchase in our model is that they pool with low-value firms, and so are able to repurchase at an attractive price.

The reason low-value firms repurchase—and do so at a price that is high for them—is that by doing so they reduce the price of subsequent equity issues. This is one of the primary empirical implications of our model. Billett and Xue (2007) find evidence for this effect. They compare the issuance price reactions of firms that previously repurchased stock with the issuance price reactions of firms that did not previously repurchase. The price decline of the former group is smaller, consistent with our model.

The following result formalizes these predictions of our model:

Proposition 6 *Let* $s_1 \geq 0$ *be a date* 1 *repurchase decision used by a positive measure of firms. Then:*

(A, price drops at issue) A positive-measure subset of these firms issue an amount s_2 *such that* $S - s_1 - s_2 \geq I$ *at date* 2*, at a price* $P_2 \leq P_1$*. Moreover, the date-*2 *price of non-issuing firms exceeds* P_1*. Both relations are strict whenever* $\Pr(s_2|s_1) < 1$*.*

(B, repurchase increases subsequent issue price) Suppose that a positive measure of firms issue $s_1' < 0$ *at date* 1*. Then there exists* s_2' *such that* $s_2' \leq 0$*,* $S - s_1' - s_2' \geq I$*,* $\Pr(s_2'|s_1') = 1$*, and* $P_2(s_1', s_2') = P_1(s_1') \leq P_2(s_1, s_2)$*. Likewise, if* $(0, s_2')$ *with* $s_2' < 0$ *is*

played by a positive measure of firms, then $P_2(0, s_2') \le P_2(s_1, s_2)$*. Both price relations are strict if* $s_1 > 0$ *and* $\Pr(s_2|s_1) < 1$*.*

(C, price increases at repurchase) If, in addition, a positive measure of firms take no action at date 1*, then* $P_1(s_1) \ge P_1(0)$*, with the inequality strict under the same conditions as in Part (B).*

Our model also generates cross-sectional predictions between, on the one hand, the *size* of repurchases and issues, and on the other hand, the price response associated with these transactions. These predictions emerge in equilibria of the model in which multiple repurchase and issue levels coexist (in contrast to the example, which features just one repurchase level and one issue level).[13]

As one would expect, larger repurchases are associated with higher repurchase prices, since they are conducted by firms that are, on average, better. Similarly, larger issues are associated with lower issue prices. Both predictions are consistent with empirical evidence: see, for example, Ikenberry et al. (1995) for evidence on repurchases, and Asquith and Mullins (1986) for evidence on issues.

Proposition 7 *(A, repurchases) Consider an equilibrium in which* s' *and* $s'' > s'$ *are repurchase levels, with associated prices* P' *and* P''*, and such that there exist firms* a' *and* a'' *where firm* a' *(respectively,* a''*) repurchases* s' *(respectively,* s''*) and does not conduct any other capital transaction at any other date. Then (i)* $P'' \ge P'$*, (ii)* $s''/P'' > s'/P'$*, and (iii)* $a'' > a'$*. In particular, repurchase size is positively correlated with repurchase price.*

(B, issues) Let (s_1', s_2') *and* (s_1'', s_2'') *be equilibrium strategies such that* $S - s_1'' - s_2'' > S - s_1' - s_2'$*. Then* $P_2(s_1', s_2') > P_2(s_1'', s_2'')$*. In particular, if* $s_2' < 0$ *and* $s_2'' < 0$*, then greater cumulative issue is associated with lower date* 2 *issue prices.*[14]

2.7 Welfare

As we have established, our economy features equilibria in which some firm repurchase. Here, we ask how social welfare in such equilibria compares with social welfare in the

[13] One can show, via numerical simulation, that such equilibria exist.

[14] *It is also possible to establish that* $s_1' > s_1''$*, i.e., greater cumulative issue is associated with smaller initial repurchases. A proof is available upon request.*

equilibrium of the one-period benchmark. Because capital market transactions do not have any deadweight cost, social welfare is simply proportional to the fraction of firms that invest.[15] We obtain the following strong result:

Proposition 8 *Consider any equilibrium featuring repurchases, and a finite number of actions.* [16] *Then there exists an equilibrium of the benchmark one-period model that has strictly high welfare, and no repurchases.*[17]

The example illustrates the basic economics of this result. In the equilibrium of the example, some high-value firms strictly profit from repurchasing their stock for less than its true value. Because investors break even in expectation, the ultimate source of these profits is low-value firms who initially pool with high-value firms and repurchase, in order to reduce the cost of subsequent issues. Low-value firms lose money on the repurchase leg of this transaction. If repurchases are prohibited, low-value firms no longer have to endure this loss-making leg. This allows them to issue at better terms, which in turn means that a greater fraction of firms find issuance (and investment) preferable to non-issuance.

At least since Arrow (1973), it has been understood that the possibility of economic agents signaling their type by undertaking a socially costly action may result in lower welfare relative to a situation in which signaling is prohibited or otherwise impossible.[18]

In our setting, however, repurchases carry no deadweight cost, yet welfare is still reduced.

2.8 Extension: Investment timing

In our main model, the investment project can only be undertaken at date 2. Here, we consider an extension in which the investment can be undertaken at either date 1 or date 2 (though not both). We focus on the benchmark case in which the project available is exactly the same at each of the two dates.

[15] If each investor holds a diversified portfolio of shares, this welfare measure coincides with the Pareto welfare ranking.

[16] *This restriction is made for simplicity, to avoid mathematical complication. The result covers equilibria with an arbitrarily large (but finite) number of equilibrium actions.*

[17] *In particular, if the one-period benchmark has a unique equilibrium in the class of equilibria with* $S - s_1 + I$*, then welfare in this equilibrium exceeds welfare in any equilibrium of the full model.*

[18] For a recent result along these lines, see Hoppe, Moldovanu and Sela (2009).

Investment at date 1 moves both the cash outflow associated with investment (I) and the subsequent benefits ($I + b$) forward by one period. If the discount rate is positive, this means that date 1 investment is more expensive, but generates greater benefits, relative to investment at date 2. In our main model we normalize the discount rate to 0; or more precisely, the objects S, s_1, s_2, I, b, a are all expressed as date 3 future values. To incorporate the effect of the investment timing choice on investment costs and benefits, we write the investment cost at dates 1 and 2 as I_1 and I_2 respectively, and likewise write the present value generated as b_1 and b_2 respectively. Hence $\frac{I_1}{I_2} = \frac{b_1}{b_2} \geq 1$, where both ratios equal the one-period interest rate.

The flexibility of investment timing introduces an additional dimension in which firms can signal their type. In particular, if $b_1 > b_2$, then delaying investment is costly, and so there may exist equilibria in which bad firms issue and invest at date 1, while good firms signal their type by waiting until date 2 to issue and invest. (See Morellec and Schürhoff, 2011, for an analysis dedicated to this issue.) However, when b_1 and b_2 are sufficiently close, i.e., when the effect of discounting is small, one can show that no equilibrium of this type exists, and the best firms never invest in equilibrium. In this case, the economic forces behind our result that any equilibrium satisfying NDOC features repurchases (Proposition 4) remain unchanged. Formal proofs of the analogues of both Proposition 4 and Proposition 3 (on equilibrium existence) are available upon request.

Consequently, the extension of our model to endogenous investment timing leaves our main results unchanged, at least when discount rates are not too high. At the same time, endogenous investment timing introduces a new effect into our model: namely that the repurchase are associated with an inefficient delay of investment. Specifically, if repurchases are exogenously ruled-out, the one-period benchmark equilibrium remains an equilibrium of the two-period model, with all investment conducted at date 1.[19]

But when repurchases are feasible, any equilibrium satisfying NDOC features at least some investment at date 2. Hence, there are three distinct costs associated with investment: (i) inefficient delayed investment (the new effect of this section); (ii) the cross-subsidy from investing firms to repurchase-only firms (the effect stressed in the main model); and (iii) the cross-subsidy from better investing firms to worse investing

[19] Again, this is for the case in which b_1 and b_2 are sufficiently close.

firms (the standard Myers and Majluf effect).

2.9 Robustness

We have restricted attention to the case in which firms can only signal via equity repurchases. However, we do not believe this restriction is critical, as follows.

Our main equilibrium characterization result is that that any equilibrium satisfying NDOC must feature repurchases (Proposition 4). A key step ingredient in this result is that in any candidate equilibrium without repurchases, the best firms would obtain their reservation payoff of $S + a$. As discussed, this property implies that repurchases can only be deterred in equilibrium if off-equilibrium beliefs associate a repurchase offer with a high firm type. The dynamic setting, combined with NDOC, then implies that a firm that deviates and repurchases could issue at very good terms the following period, thereby undercutting the proposed equilibrium without repurchases.

This argument still works even if additional signaling possibilities are introduced, provided that any candidate equilibrium without repurchases has the best firms receiving their reservation payoffs. Indeed, the extension of Section 2.8 in which investment timing can potentially serve as a signal illustrates exactly this. Moreover, it may be possible to extend this argument to cover cases in which the best firms receive more than their reservation payoff, since in such a case, it is still necessary to assign very favorable beliefs to any firm that attempts to repurchase. Finally, note that in this generalization firms may repurchase a *different* security from equity; however, under the conditions described, some firms will repurchase some form of risky security.

2.10 Conclusion

We investigate the consequences of allowing for repeated capital market transactions in a model with asymmetric information between a firm and its investors. All firms in the model possess a profitable project that they need to raise cash to undertake. However, we show that there always exist equilibria in which firms return cash to investors via share repurchases. Consistent with managerial accounts, some repurchasing firms profit from repurchasing their stock. The ultimate source of these profits is that other firms buy “high” in order to improve the terms of subsequent stock issues. Moreover, only equilibria that feature repurchases satisfy the relatively mild NDOC restriction on off-equilibrium beliefs. Repurchases lowers social welfare by reducing the fraction of firms that invest, even though repurchasing itself is a costless signal. Our model generates a number of empirical predictions.

Chapter 3

A dynamic model of optimal creditor dispersion

3.1 Introduction

Many firms borrow simultaneously from multiple creditors. Having multiple creditors brings the disadvantage of coordination problems, which in bad times make it harder for a firm to restructure its debt to avoid liquidation. In good times, however, these same coordination problems enhance pledgeability by making it harder for a firm to opportunistically hold up its creditors.

In this paper, I study the trade-off between these two forces—liquidation risk and enhanced pledgeability—for a firm that seeks to roll over its existing debt. In contrast to the existing literature,[1] I focus on the case in which a firm has insufficient internal resources to repay its outstanding debt. Instead, the firm must issue new debt to repay the maturing debt—that is, roll over its debt. This case is empirically relevant. In reality, 47% of the Compustat firms during fiscal years 2012 and 2013 have insufficient operating cash flow to repay their maturing debt and thereby have to rely on debt rollover.[2]

[1] For example, Berglöf and Von Thadden (1994), Bolton and Scharfstein (1996), and Diamond (2004).

[2] I use the Compustat variables *total debt in current liability* (*DLC* — the total amount of short-term notes and the current portion of long-term debt that is due in one year) and *EBITDA* as the proxies for maturing debt and operating cash flow. For 47% of the firms, EBITDA is smaller than total

A firm's ability to roll over its debt is fundamentally a dynamic concept: the ability to roll over debt today depends on whether the firm's new creditors anticipate that they will be able to, in turn, roll over their debt in the future, which in turn depends on whether creditors anticipate that rollover will be possible even further in the future. I build a parsimonious dynamic model to analyze a firm's choice of the number of creditors in a rollover framework. Each period, a firm trades off the increase in liquidation risk with the enhanced pledgeability that a greater number of creditors engenders. Despite the model's parsimony, it is challenging to analyze and generates a rich set of predictions.

I use my model to make three main points. First, my model delivers predictions on how many creditors a firm has as well as when it decides to seek more creditors or consolidate the existing ones. I show that firms with higher growth prospects can support more creditors, which is consistent with cross-sectional empirical findings. In the time series, I show that firms increase the number of creditors when they perform badly and need to support a higher leverage, a point well illustrated by the following case. School Specialty Inc. is a distributor of classroom supplies that went bankrupt in 2013. Barrett (2013) writes, "The [subprime] recession and cuts in public spending severely affected school budgets and hurt School Specialty" (para. 17). The company increased the pool of creditors and borrowed \$64 million from a new lender, Bayside, in January 2012 after its current lenders refused to provide new loans to refinance its existing debt according to Dugan (2013). The firm indeed survived one more year until it breached a loan covenant set forth by Bayside and filed for bankruptcy protection.

Second, I challenge the received wisdom that having multiple creditors and the resulting coordination problems are responsible for firms' difficulties in rolling over their debt. In the School Specialty case, Bayside's demand for a full repayment after School Specialty's covenant violation triggered its bankruptcy filing. It is easy to conclude that introducing the additional lender Bayside and its high priority prevented private debt restructuring that could have led to a more efficient resolution. Implicit in such views is the idea that the firm would have had an easier time if it had had fewer creditors. But this counterfactual ignores the fact that borrowing from more creditors is an endogenous choice made by the firm in the past. Without the decision of borrowing from more creditors the firm could have failed even earlier. To make a more meaningful

debt in current liability.

comparison, I compare the expected liquidation probability and the firm value in my model to the ones in a counterfactual model in which the firm can borrow from only one creditor. I show that for a large range of fundamental values, firms with multiple creditors would have an even higher chance of liquidation and lower firm value, if they were forced to borrow from just one lender. An interesting policy implication is that ex post efficient reorganization processes, such as the automatic stay clause and Chapter 11 reorganization, that eliminate coordination failure among creditors may reduce a firm's ability to raise money ex-ante and result in lower welfare due to a more difficult debt rollover.

Finally, the model sheds light on how renegotiation frequency affects pledgeability. I show that in the limit when the firm can instantaneously renegotiate its debt, the enhanced pledgeability from more creditors becomes negligible. Although more creditors can indeed force more repayment, the source of this additional payout comes from the growth between two negotiation dates. With very frequent negotiation, per period growth vanishes, as does the additional debt capacity from having more creditors.

3.2 Related Literature

It has been well understood that having multiple creditors can cause coordination problems. Perhaps the most famous example is bank (creditor) run. Diamond and Dybvig (1983) show that in a static setting, socially inefficient bank run equilibria generally exist. Goldstein and Pauzner (2005) further characterize the probability of a bank run under a global game framework. He and Xiong (2012a) study the dynamic evolution of a panic-based run on staggered corporate debt.

If borrowing from multiple lenders is costly, then why do firms continue doing so? Berglöf and Von Thadden (1994) claim that having multiple lenders specialize in lending at different maturities is a superior structure. The short-term creditors can impose externalities on the long-term creditors at the renegotiation stage, thereby increasing the ex post repayment incentives and in turn the ex ante efficiency. Following this line of thinking, Diamond (2004) demonstrates that when enforcing a debt contract is difficult, a single lender with a large stake in the firm has limited or no incentive to take ex post disciplinary actions against the firm, since such actions also hurt the lender himself. The firm, knowing that disciplinary actions are not credible, will misbehave ex ante. In the case of multiple creditors, the creditor who takes the action can claim against the whole firm, thereby hurting the other creditors. The improved incentive for lenders to be active ex post forces the borrowers to behave and thus increases the amount of money that can be raised. These papers share the key insight that potential coordination failure with multiple creditors disciplines the firm and can potentially improve the ex ante outcome. However, they take the variation in the number of creditors exogenously and therefore are silent on when firms endogenously change the number of creditors.

Bolton and Scharfstein (1996) further develop this idea and study the optimal choice between one and two creditors. The firms in their model can strategically default and renegotiate the debt even when they have the money to repay. The creditor(s), upon (either strategic or fundamental-based) default by the firm, can sell the project to an inefficient outsider. Under a multilateral bargaining setup, the benefit of having multiple creditors is to increase the collective bargaining power against the firm following a strategic default. In this case, the creditors can extract higher repayments from the firm. However, the cost of introducing a second creditor is that it lowers the expected payoff following a bad state, where this stronger collective bargaining power makes it less likely for the creditors to get an outside investor. Although all of these papers study the benefit brought by coordination failure from multiple creditors, they are all static (i.e. one-shot negotiation). My model shares the classic idea that having multiple lenders is a costly mechanism to induce correct behavior from the borrowers, but instead I focus the optimal number of creditors with a dynamic model. This dynamic feature is particularly important since firms usually do not have sufficient operating cash flow to pay back the maturing debt and must rely on repeatedly rollover.

Several other papers have also explicitly investigated the cost and benefit of having multiple creditors from various perspectives. Brunnermeier and Oehmke (2013) extend this idea by allowing the borrower to choose the maturity of the debt contract and explain why an excessively short maturity structure prevails in equilibrium, despite the increased rollover risks. Detragiache et al. (2000) present a completely different trade-off. If banks can fail, then having multiple banking relationships is beneficial because financing is more robust in this case and will not fail unless all banks do. However, when all banks actually do fail, having more relationship banks is a stronger negative signal and therefore increases refinancing costs. Petersen and Rajan (1994) propose a model that illustrates how lenders' market power affects the quality of the financed firms and the cost of credit. They take the lenders' market power as an exogenous parameter. My paper endogenizes the variation of bargaining power by explicitly modeling the game between the firm and its creditors. Furthermore, their empirical studies in Petersen and Rajan (1994, 1995) suggest that having more creditors is associated with a higher cost of credit in equilibrium, which is consistent with my model's prediction.

The effects of debt rollover and renegotiation on credit risk and debt prices have been studied from an asset-pricing perspective. Mella-Barral and Perraudin (1997) and Mella-Barral (1999) study the asset-pricing implications when the firm can renegotiate and service the troubled debt, rather than just defaulting directly as in Leland (1998). He and Xiong (2012b) investigate how creditors with different maturities strategically interact with each other when they decide whether or not to roll over the maturing debt. Similar to the work of Diamond (2004), the creditors' decisions not to roll over pose externalities on other incumbent creditors with claims not yet matured. Hege and Mella-Barral (2005) look at an economy in which a firm can exchange liquidation rights for coupon concessions on debt and study how that feature affects the credit risk premia as the number of creditors changes. These papers focus on pricing the debt claims given the possibility of renegotiation or rollover frictions, assuming the creditors' structure is exogenously fixed. My paper, on the other hand, focuses on the optimal choice creditor dispersion.

3.3 The Model

3.3.1 The Project and Financing

Time t is discrete and the discount rate is normalized to 1. A risk-neutral penniless entrepreneur starts a firm at $t = 0$ with a project.[3] The project requires an upfront investment I_0 and generates no cash flow except for a final liquidating dividend at a random project maturity. At each date, the project matures with probability π. The actual realization of the final dividend depends on a stochastic firm-specific state

[3] Note that I do not distinguish between the entrepreneur and the firm in the model and use the two terms interchangeably.

$\theta_t \in \{G(ood), B(ad)\}$ and the fundamental $Y_t = Y_0 \Pi_{1 \le s \le t} z_s$, where z_s are i.i.d positive random variables with continuous density $g(z)$. Assume $g(z)$ has a compact support $[\underline{z}, \overline{z}]$. The random variables θ_t and z_s are independent. Denote the mean $E(z_s) = \mu > 1$ and assume $\underline{z} < 1$. If the project matures when the state is good ($\theta_t = G$), the realized final dividend is Y_t; otherwise, if the state is bad ($\theta_t = B$), the dividend is 0. The state θ_t follows a Markov process with transition probability $p^\theta = Prob(\theta_{t+1} = \theta | \theta_t = \theta)$ (for $\theta = G, B$), which can be interpreted, for example, as the demand shock for the firm's output or the firm-specific productivity shock. To ensure that the project has a finite value, I impose the following parameter assumption:

$$(1 - \pi)\mu < 1. \tag{3.1}$$

Denote τ_π to be the random project maturity date. Then given the initial state θ_1 and fundamental Y_0, the expected value of the project's final dividend can be naturally defined as

$$E(\mathbf{1}_{\theta_{\tau_\pi} = G} Y_{\tau_\pi} | \theta_1, Y_0). \tag{3.2}$$

Lemma 2 *If the project is carried through to its random maturity τ_π, then its expected value defined in (3.2) conditional on the current state θ and fundamental Y is given by*

$$\begin{aligned} E(\mathbf{1}_{\theta_{\tau_\pi} = G} Y_{\tau_\pi} | G, Y) &= \frac{\pi[1-(1-\pi)\mu p^B]\mu}{[1-(1-\pi)\mu][1-(1-\pi)\mu(p^G+p^B-1)]} Y, \\ E(\mathbf{1}_{\theta_{\tau_\pi} = G} Y_{\tau_\pi} | B, Y) &= \frac{\pi(1-p^B)(1-\pi)\mu^2}{[1-(1-\pi)\mu][1-(1-\pi)\mu(p^G+p^B-1)]} Y. \end{aligned} \tag{3.3}$$

At any time t, the project can be liquidated prematurely for λY_t. The liquidation value is assumed to be independent of θ because it is possible to sell the project to other firms that are not subject to this firm-specific shock. The liquidation coefficient $\lambda \le 1$ captures the inefficient separation of the project from its original developers. If liquidation is inefficient enough, i.e. $E(\mathbf{1}_{\theta_{\tau_\pi} = G} Y_{\tau_\pi} | B, Y) > \lambda Y$, then the project is always better off continuing even in the bad state. By (3.3), this is equivalent to

$$\lambda < \frac{\pi(1 - p^B)(1 - \pi)\mu^2}{[1 - (1 - \pi)\mu][1 - (1 - \pi)\mu(p^G + p^B - 1)]}, \tag{3.4}$$

which I assume throughout the paper. Under this assumption, the values specified by (3.2) are indeed first best. Denote them by $V_{FB}^{\theta\star}(Y)$.

If the entrepreneur has enough cash to finance the up-front investment I_0, then the project is optimally carried through to its maturity and the first best firm value is realized. However, as I have assumed, the firm does not have (sufficient) cash to begin with. In addition, to highlight the rollover and pledgeability frictions, I assume that the firm can only issue one-period debt to *short-lived* creditors. Since the project does not generate any interim cash flow, the firm must repeatedly issue new debt to finance the payment to the maturing creditors. The detailed game between the entrepreneur and the creditors will be defined later following a formal introduction of the timeline.

3.3.2 Timeline

Figure 1 outlines the timeline and the evolution of the state variables. The firm enters period t with N_t incumbent creditors and a total promised face value F_t. The current state θ_t and the previous fundamental Y_{t-1} are also publicly known. At period t, a new shock z_t (or equivalently Y_t) is realized, and then the project matures with probability π. If it matures, the game ends with a final dividend $Y_t \mathbf{1}_{\theta_t=G}$. Otherwise, the project continues to the *repayment stage*, and a new state θ_{t+1} is realized. The entrepreneur then has the following three options: (a) to voluntarily liquidate the project, (b) to make the promised repayment F_t, or (c) to initiate a repayment negotiation (described in the next subsection). If an agreement on the actual payment cannot be reached, the firm is forced into liquidation. Otherwise, if a repayment X_t is mutually accepted (in case (b) $X_t = F_t$ or in case (c) the negotiated amount), the firm enters the *refinancing stage* to raise *exactly* X_t from N_{t+1} identical creditors with an aggregate face value F_{t+1}. Both N_{t+1} and F_{t+1} are the firm's choice variables. Following a successful refinancing, the firm survives period t and the next period begins.

3.3.3 The Repayment Negotiation

At the repayment stage, the firm can choose to negotiate the payment (option (c) in the previous subsection). During a negotiation, the firm meets each creditor sequentially in a *random* order and makes a take-it-or-leave-it offer S_i to the ith creditor. Here, the index i reflects the realized random negotiation order. The offer history is public information. Each creditor, when it is his turn to negotiate, can either accept (A) the new promised payment or reject (R) the offer and exercise the liquidation right. If any creditor rejects the offer, the negotiation fails and the firm is liquidated. I assume that the rejecting creditor has priority over the liquidation proceeds and gets $\min(\frac{F_t}{N_t}, \lambda Y_t)$. The remaining creditors (who either previously accepted the offer or have not yet negotiated) get the remaining liquidation proceeds equally, $\min(\frac{F_t}{N_t}, \frac{1}{N_t-1}\max(0, \lambda Y_t - \frac{F_t}{N_t}))$. If all N_t creditors accept the new offers, the firm then enters the refinancing stage and tries to borrow $X_t = \sum_{i=1}^{N_t} S_i$.

3.3.4 The Firm's Refinancing Decision

Since the project does not generate any interim cash flow, the firm has to finance the repayment X_t and roll over this obligation to the next period. The firm chooses N_{t+1} new creditors and offers them the same one-period debt contract with total face value F_{t+1} in exchange for cash X_t to honor the repayment to the N_t incumbent creditors. The new creditors simultaneously accept or reject the new debt offerings. If anyone rejects, the new creditors get a reservation payoff of 0 and the firm is liquidated. The N_t incumbent creditors equally share the liquidation proceeds up to the face value and each gets $\frac{1}{N_t}\min(F_t, \lambda Y_t)$. On the other hand, if all N_{t+1} new creditors accept the offer, the firm survives period t and the game moves on to period $t+1$.

3.3.5 Terminal Payoffs, Markov Strategies, and Equilibrium Definition

The entrepreneur is long-lived and the creditors live for one period. The game ends at date t if one of the following events occurs: (a) the project matures, (b) the negotiating creditor forces liquidation, (c) the entrepreneur voluntarily liquidates the project, or (d) the refinancing offer is rejected. If (a) the project matures, each incumbent creditor (living from period t to $t+1$) gets $-\frac{X_{t-1}}{N_t} + \frac{1}{N_t}\min(\mathbf{1}_{\theta_t=G}Y_t, F_t)$ and the entrepreneur gets the remaining $\max(0, \mathbf{1}_{\theta_t=G}Y_t - F_t)$, where X_{t-1} is the amount of total repayment made by the firm in the previous period (or total funds borrowed from the current incumbent creditors). The first term $-\frac{X_{t-1}}{N_t}$ in the creditors' payoff captures the up-front cash lending in the previous period. If (b) one of the creditors forces liquidation, then the liquidating creditor gets $-\frac{X_{t-1}}{N_t} + \min(\frac{F_t}{N_t}, \lambda Y_t)$, every remaining creditor receives $-\frac{X_{t-1}}{N_t} + \min(\frac{F_t}{N_t}, \frac{1}{N_t-1}\max(0, \lambda Y_t - \frac{F_t}{N_t}))$, and the residual liquidation proceeds $\max(0, \lambda Y_t - F_t)$ go to the entrepreneur. If (c) the entrepreneur liquidates the project or (d) the refinancing offer is rejected, then each incumbent creditor receives $-\frac{X_{t-1}}{N_t} + \frac{1}{N_t}\min(\lambda Y_t, F_t)$ and the entrepreneur gets $\max(0, \lambda Y_t - F_t)$. Finally, if the firm survives each period, then the ith (i represents the realized negotiation order) short-lived incumbent creditor receives $-\frac{X_{t-1}}{N_t} + S_i$ in the case of having a negotiation and $-\frac{X_{t-1}}{N_t} + \frac{F_t}{N_t}$ otherwise.

A pure Markov strategy profile includes the following items. The firm has a negotiation strategy for the ith creditor $S_i^{\theta_{t+1}}(\sum_{j<i} S_j, F_t, Y_t, N_t) \in \mathbb{R}_+$ as a function of the total negotiated repayment in this period untill now $\sum_{j<i} S_j$, the originally promised face value F_t, the current fundamental Y_t, the realized next period state θ_{t+1}, and the number of incumbent creditors N_t. With a slight abuse of notation, $S_0^{\theta_{t+1}}(F_t, Y_t, N_t) \in \{L, F\}$ denotes a voluntary liquidation or a full repayment of F_t. The firm also has a set of financing strategies to choose the new number of creditors $N_+^{\theta_{t+1}}(X_t, F_t, Y_t) \in \mathbb{N}$ and the total face value $F_+^{\theta_{t+1}}(X_t, F_t, Y_t) \in \mathbb{R}_+$, as functions of the required financing amount X_t, the originally promised face value F_t, the fundamental Y_t, and the state θ_{t+1}. In addition, in each period, the ith incumbent creditor has an acceptance strategy after receiving an offer S_i: $s_i^{\theta_{t+1}}(\sum_{j<i} S_j, S_i, F_t, Y_t, N_t) \in \{A, R\}$. Finally, given any refinancing offer (F_+, N_+), the new creditors have acceptance strategies: $r_i^{\theta_{t+1}}(X_t, F_+, Y_t, N_+) \in \{A, R\}$ for all $i \leq N_+$.

In this paper, I focus on the Markov perfect equilibria, meaning the strategy profiles described above that are subgame perfect.

Remark 1: Rather than taking a contract design approach, as for example in Berglöf and Von Thadden (1994), Bolton and Scharfstein (1996), and Diamond (2004), I instead assume that only standard debt contracts are possible. I make this assumption because, unlike the other papers, I take the cross-externality among investors as given and investigate how firms choose exposure to this friction dynamically. In addition, I do not allow the firm to save. However, as will be discussed in section 3.6, I do not expect the possibility of savings to change the firm's choice of number of creditor.

Remark 2: One interpretation of the priority structure is that the rejecting creditor can partially liquidate the project to secure as much of the originally promised amount as possible. The project, however, is fundamentally impaired and will be forced into a full liquidation before the next creditor negotiates, in which case all other creditors share the remaining liquidation proceeds equally. Note that the results do not depend on the specific priority structure. As long as the liquidating creditor has some priority over the proceeds which can hurt other creditors, the story remains valid. The key economic force here is that the creditors can pose externalities on each other as in Berglöf and Von Thadden (1994), Bolton and Scharfstein (1996), and Brunnermeier and Oehmke (2013). As the number of creditors increases, each one of them can pose a larger externality on others by forcing a liquidation. Such an externality provides the creditors with stronger incentives to commit to an ex post liquidation and hence creates a stronger incentive for the firm to repay as well. The cost, on the other hand, is an early termination of the project as a result of coordination failure when the firm is in distress.

3.4 Equilibrium Construction

In this section, I explicitly construct an equilibrium. Before doing so, I introduce several key variables including debt value, debt capacity, and the total firm value.

3.4.1 Debt Value, Debt Capacity, and Firm Value

Given any strategy profile, I can define the total value of debt claims $D^{\theta_t}_{N_t}(F_t, Y_{t-1})$ at the beginning of each period. Here I keep the time indices to make the evolution of the state variables transparent.

$$\begin{aligned} D^{\theta_t}_{N_t}(F_t, Y_{t-1}) \quad = \quad & E\{\pi \min(F_t, Y_{t-1}z_t)\mathbf{1}_{\theta_t=G} + (1-\pi) \\ & [(\Pi_{i\leq N_t}\mathbf{1}_{s_i^{\theta_{t+1}}=A})\mathbf{1}_{S_0^{\theta_{t+1}}\neq L}(\Pi_{i\leq N_+^{\theta_{t+1}}}\mathbf{1}_{r_i^{\theta_{t+1}}=A})X_t+ \\ & [1-(\Pi_{i\leq N_t}\mathbf{1}_{s_i^{\theta_{t+1}}=A})\mathbf{1}_{S_0^{\theta_{t+1}}\neq L}(\Pi_{i\leq N_+^{\theta_{t+1}}}\mathbf{1}_{r_i^{\theta_{t+1}}=A})] \\ & \min(F_t, \lambda Y_{t-1}z_t)]\} \end{aligned} \tag{3.5}$$

The expectation is taken over the random variables z_t and θ_{t+1}. In the future, when the time indices are omitted, I use θ' to denote the next period state θ_{t+1}. The first term captures the payout to the debt holders upon project maturity, which happens with probability π. If the project does not mature, then the total (possibly negotiated) repayment X_t is honored if every player chooses not to liquidate. Otherwise, if anyone liquidates the project (rejects the offer), then the liquidation payoff is distributed. Let

τ_L be the stopping time when any player chooses liquidation depending, which can potentially depend on the entire history. Define $\tau_S = \min(\tau_\pi, \tau_L)$ to be the time when the game ends. The total value of the firm at the beginning of each period is

$$V_{N_t}^{\theta_t}(F_t, Y_{t-1}) = E\{\mathbf{1}_{\tau_S=\tau_\pi}\mathbf{1}_{\theta_{\tau_\pi}=G}Y_{\tau_\pi} + \mathbf{1}_{\tau_S=\tau_L}\lambda Y_{\tau_L}\},$$

which can be expressed recursively as:

$$\begin{aligned} V_{N_t}^{\theta_t}(F_t, Y_{t-1}) \quad = \quad & E\{\pi Y_{t-1} z_t \mathbf{1}_{\theta_t=G} + (1-\pi) \\ & [(\Pi_{i\le N_t}\mathbf{1}_{s_i^{\theta_{t+1}}=A})\mathbf{1}_{S_0^{\theta_{t+1}}\neq L}(\Pi_{i\le N_+^{\theta_{t+1}}}\mathbf{1}_{r_i^{\theta_{t+1}}=A})V_{N_+^{\theta_{t+1}}}^{\theta_{t+1}}(F_+^{\theta_{t+1}}, Y_{t-1}z_t) \\ & +[1-(\Pi_{i\le N_t}\mathbf{1}_{s_i^{\theta_{t+1}}=A})\mathbf{1}_{S_0^{\theta_{t+1}}\neq L}(\Pi_{i\le N_+^{\theta_{t+1}}}\mathbf{1}_{r_i^{\theta_{t+1}}=A})]\lambda Y_{t-1}z_t]\} \end{aligned} \tag{3.6}$$

It is convenient to define the debt capacity from N creditors as follows:

$$DC_{N_t}^{\theta_t}(Y_{t-1}) \equiv \max_{F_t} D_{N_t}^{\theta_t}(F_t, Y_{t-1}) \tag{3.7}$$

and the total debt capacity as

$$DC^{\theta_t}(Y_{t-1}) \equiv \max_{N_t} DC_{N_t}^{\theta_t}(Y_{t-1}). \tag{3.8}$$

Finally, define $\overline{F}_{N_t}^{\theta_t}$ to be the face value that maximizes the value of debt, given the number of creditors N_t, fundamental Y_{t-1}, and state θ_t. If several values of F deliver this maximum, $\overline{F}_{N_t}^{\theta_t}$ is the smallest one:

$$\overline{F}_{N_t}^{\theta_t}(Y_{t-1}) \equiv \min[\arg\max_{F_t} D_N^{\theta}(F_t, Y_{t-1})]. \tag{3.9}$$

As will be transparent in the next subsection, the entrepreneur has no incentive to pick a new face value $F_+ > \overline{F}_{N_+}$ because doing so would weakly reduce the firm value.

3.4.2 Equilibrium Characterization

Proposition 9 is the main result that characterizes the equilibrium strategies and the value functions.

Proposition 9 *Consider the following strategies:*

1. *The entrepreneur always makes an offer*
$$S_i^{\theta\star} = \min(\frac{F}{N}, \lambda Y) \tag{3.10}$$
to the i*th creditor.*

2. *The entrepreneur's financing strategy* $N_+^{\theta\star}(X,F,Y)$ *and* $F_+^{\theta\star}(X,F,Y)$ *solves*
$$\begin{aligned} &\max\nolimits_{N_+} V_{N_+}^{\theta}(F_+,Y) \\ s.t. \quad &F_+ \text{ is the smallest solution to } \quad D_{N_+}^{\theta}(F_+,Y) = X. \end{aligned} \tag{3.11}$$
If there is no combination of (N_+, F_+) *such that (3.11) holds, then the firm chooses* $N_+^{\theta\star} = 1$ *and* $F_+^{\theta\star} = 0$.[4]

3. *The* i*th creditor accepts the offer* S_i *(i.e.,* $s_i^{\theta\star}(\sum_{j<i} S_j, S_i, F, Y, N) = A$*) if and only if* $S_i \geq \min(\frac{F}{N}, \lambda Y)$ *and*
$$\sum_{j<i} S_j + S_i + \sum_{j>i} S_i^{\theta\star} \leq DC^{\theta}(Y). \tag{3.12}$$

4. *The potential new creditors accept the financing offers* $r_i^{\theta\star}(X, F_+, Y, N_+) = A$ *if and only if* $D_{N_+}^{\theta}(F_+,Y) \geq X$.

Under the proposed strategies, for any state $\theta = G, B$ *and any number of creditors* N*,*

1. *the value of debt* $D_N^{\theta}(F,Y)$ *is continuous and homogeneous of degree one (HD1) in* (F,Y)*;*

2. *the minimum face value that achieves the* N *creditor debt capacity from (3.9) is linear in* Y*, i.e.,* $\overline{F}_N^{\theta}(Y) = \overline{f}_N^{\theta} Y$ *for some constant* $\overline{f}_N^{\theta}$*;*

3. *the debt capacity from* N *creditors from (3.7) is linear in* Y*, i.e.,* $DC_N^{\theta}(Y) = \kappa_N^{\theta} Y$ *for some constant* κ_N^{θ}.

Define
$$\kappa^{\theta} \equiv \max_N \kappa_N^{\theta}. \tag{3.13}$$
If $\min(\kappa^G, \kappa^B) > \lambda$*, then the proposed strategies indeed constitute a subgame perfect equilibrium. In addition, the firm's value function* $V_N^{\theta}(F,Y)$ *satisfies*

[4] *In fact, in this case, the financing strategy can be arbitrary because it will be rejected by the potential new creditors.*

1. *$V_N^\theta(F,Y) \geq \kappa_N^\theta Y$, when $F \leq \overline{F}_N^\theta(Y)$;*

2. *$V_N^\theta(F,Y)$ is continuous, HD1 in (F,Y), weakly decreasing in F, and increasing in Y.*

The proof takes a guess and verify approach, with the full version in the appendix. However, outlining the procedures to establish the equilibrium is still helpful. The key to this construction lies in finding a consistent (κ^G, κ^B) that dictates the debt capacities in (3.8). Given a linear conjecture, the equilibrium strategies imply that rollover in state θ is possible only when the total offered repayment is feasible:

$$\min(F, N\lambda Y) \leq DC^\theta(Y) = \kappa^\theta Y. \tag{3.14}$$

With the continuation region explicitly expressed, the debt value (3.5) be rewritten as

$$\begin{aligned} D_N^\theta(F,Y) \quad = \quad & E\{\pi \min(F, Yz)\mathbf{1}_{\theta=G} + (1-\pi) \\ & [\mathbf{1}_{\min(F,N\lambda Yz)\leq \kappa^{\theta'}Yz} \min(F, N\lambda Yz) + \mathbf{1}_{\min(F,N\lambda Yz)>\kappa^{\theta'}Yz} \min(F, \lambda Yz)]\}, \end{aligned} \tag{3.15}$$

where the expectation is taken over z and θ'. It is easy to see that this is HD1 in (F,Y). Linearity of debt capacities is then just a simple corollary of HD1 with the coefficient:

$$\begin{aligned} \kappa_N^\theta \quad = \quad & \max_f E\{\pi \min(f, z)\mathbf{1}_{\theta=G} + (1-\pi) \\ & [\mathbf{1}_{\min(F,N\lambda z)\leq \kappa^{\theta'}z} \min(f, N\lambda z) + \mathbf{1}_{\min(F,N\lambda z)>\kappa^{\theta'}z} \min(f, \lambda z)]\} \end{aligned} \tag{3.16}$$

Clearly (3.16) depends on the initial conjecture of (κ^G, κ^B), and it has to arrive at the same debt capacity in equilibrium by equating (3.13). Any guess of (κ^G, κ^B) that survives this procedure is consistent and can be supported in an equilibrium.

If creditors expect a low debt capacity tomorrow, then the pledgeable amount today decreases today as in (3.16), resulting in a lower debt capacity today. Therefore, the self-fulfilling feature could result in multiple equilibria. It can be shown that the equilibrium is unique conditional on a fixed choice of κ^θ and the results of this paper do not depend on which κ^θ I choose.

Despite the potential multiplicity, the existence of any equilibrium is not obvious at all. This is because the right-hand side in (3.16) as a function of κ^θ is not continuous. For example, when $\kappa^\theta \geq \lambda N$, it is always possible to roll over. However, as soon as κ^θ decreases to just below λN, there is a nontrivial chance that the firm will be liquidated, which hurts the ex-ante borrowing capacity discontinuously. Fortunately, despite the discontinuity of (3.16), the right-hand side is still order preserving and Tarski's fixed point theorem guarantees a solution.

With a consistent conjecture of κ^θ held fixed, the firm's value function (3.6) reduces to the following dynamic programming problem:

$$\begin{aligned} V_N^\theta(F,Y) \quad &= \quad E\{\pi Y z \mathbf{1}_{\theta=G} + (1-\pi) \\ &\quad [\mathbf{1}_{\min(F,N\lambda Yz)\leq\kappa^{\theta'}Yz} V_{N_+}^{\theta'}(F_+,Yz) + \mathbf{1}_{\min(F,N\lambda Yz)>\kappa^{\theta'}Yz}\lambda Yz]\} \end{aligned} \tag{3.17}$$

where F_+ is the minimum solution to

$$D_{N_+}^{\theta'}(F_+,Y) = \min(F, N\lambda Yz). \tag{3.18}$$

Establishing continuity in V_N^θ is challenging, since a small change in (F,Y) can result in a discontinuous change in the minimum solution F_+. Therefore, the constraint correspondence $(F,Y) \mapsto \{(N_+,F_+)|$ s.t. (3.18) holds$\}$ is discontinuous and the standard theorem of maximum does not apply. Even so, one can show that the value function in equilibrium is indeed continuous. After proving the properties of the value functions V_N^θ, it is relatively straightforward to verify that the constructed strategy profile is indeed optimal.

Despite the complicated construction and verification, the equilibrium is quite intuitive. The entrepreneur has all the bargaining power, so he just needs to *credibly* offer each creditor his liquidation payoff $\min(\frac{F}{N},\lambda Y)$ as in (3.10). On the other hand, for an incumbent creditor to accept an offer S_n, it must be weakly higher than the liquidation payoff. In addition, condition (3.12) implies that the offer must be credible in the sense that following the proposed strategies, the total repayment can be financed.

The cost and benefit of having more creditors are immediately transparent in (3.14) and (3.15). With a higher N, the left-hand side of (3.14) weakly increases, causing a weakly higher chance of liquidation. On the other hand, having more creditors lowers the stake of an individual creditor relative to the whole firm and therefore effectively grants creditors higher bargaining power. The total actual repayment conditional on rollover in (3.15) weakly increases, as does the pledgeability.

3.4.3 Creditor Capacity and Safe Number of Creditors

Even though I do not pose any assumption on the transition probability p^θ, the debt capacities in the two states $\theta = G, B$ can be ordered in equilibrium.

Lemma 3 *The debt capacity is strictly higher in the good state, i.e.,* $\kappa^G > \kappa^B$.

To understand this relationship, one needs to remember the recursive nature of debt rollover. The maximum amount that the firm can borrow now depends on the maximum amount that the firm can borrow in the next period. Suppose that the firm is in a bad state now. With probability π, the firm dies without any payout. If the firm has a weakly higher debt capacity in the bad state, then the actual refinanceable repayment in the next period is bounded by the debt capacity in the bad state $\kappa^B Y_{t+1}$. The expected maximum repayment, however, is insufficient to support the debt capacity today $(1-\pi)E_t(\kappa^B Y_{t+1}) = (1-\pi)\mu\kappa^B Y_t < \kappa^B Y_t$. Therefore, the firm has no chance of repaying $\kappa^B Y_t$ in the bad state. In other words, in order to finance the debt capacity in a bad state, the firm must rely on the possibility a good state realization in the next period and utilize that higher borrowing capacity.

Given this lemma,[5] we can conveniently define

$$\overline{N} \equiv [\frac{\max(\kappa^G, \kappa^B)}{\lambda}] + 1 = [\frac{\kappa^G}{\lambda}] + 1. \tag{3.19}$$

When the number of creditors becomes large, namely $N > \overline{N}$, the equilibrium no longer depends on the number of creditors N. This is because when (3.19) holds, the liquidation threat becomes credible in both states $\theta = G, B$, and creditors reject any offer less than the full repayment of F. The firm then always repays the original face value whenever possible. The debt and the total firm values from (3.15) and (3.17) become

$$\begin{aligned} D_N^\theta(F,Y) &= E\{\pi \min(F, Yz)\mathbf{1}_{\theta=G} + (1-\pi) \\ &\quad [\mathbf{1}_{F \le \kappa^{\theta'} Yz} F + \mathbf{1}_{F > \kappa^{\theta'} Yz} \min(F, \lambda Y z)]\} \end{aligned}$$

and

$$\begin{aligned} V_N^\theta(F,Y) &= E\{\pi Y z \mathbf{1}_{\theta=G} + (1-\pi) \\ &\quad [\mathbf{1}_{F \le \kappa^{\theta'} Yz} V^{\theta'}_{N_+^{\theta\star}}(F_+^{\theta'\star}, Yz) + \mathbf{1}_{F > \kappa^{\theta'} Yz} \lambda Y z]\} \end{aligned}$$

[5] The bracket denotes the floor function: $[a]$= the largest natural number weakly smaller than a.

with the corresponding condition (3.11) replaced by $D^{\theta'}_{N_+}(F_+, Y) = F$. Since both the rollover (liquidation) region and the payoffs are independent of N, the firm's and the creditors' problems are no longer sensitive to N. I refer to $\overline{N}$ defined in (3.19) as the *creditor capacity* in the future. Without loss of generality, we can limit the firm's choice of the number of creditors to weakly below $\overline{N}$. This finite bound turns out to be a key piece in proving the general existence of the value functions V^{θ}_{N} in proposition 9.

Similarly, I define the *safe number of creditors*:

$$\underline{N} \equiv [\frac{\kappa^B}{\lambda}]. \tag{3.20}$$

When the number of creditors is lower than $\underline{N}$, condition (3.14) always holds, meaning that rollover is always possible. In this case, having more creditors enhances pledgeability without an immediate risk of liquidation. Despite this seemingly costless benefit, as we will see shortly, this does not imply that the firm always prefers to have at least $\underline{N}$ creditors.

3.5 Key Trade-offs and Empirical Predictions

Only in this section, I study the comparative statics of the exogenous changes in number of creditors. To do so, I change the incumbent number of creditors as if it is a parameter and keep the equilibrium continuation strategies. In other words, I study the outcome of a one shot deviation of the number of creditors in equilibrium. This exercise highlights the trade off between pledgeability and the liquidation risk that the firm faces when choosing creditor dispersion. Many empirical predictions can be carried through in equilibrium, whereas others may be reversed by the firm's selection effect. This topic will be discussed in section 3.6.3.

3.5.1 Pledgeability

Value of Debt, Debt Capacity, and Interest Rate

Having multiple creditors has two offsetting effects on the value of debt. On the one hand, the entrepreneur's payout incentive increases with more creditors, which in turn raises the value of debt for any given face value. On the other hand, having more creditors reduces the ex-post financial flexibility that leads to more liquidation, which in turn hurts the debt value. In some cases, however, an increase in the number of creditors has no effect on the liquidation probability, so the debt value increases.

Proposition 10 *Suppose that one of the following three conditions holds: (a)$\underline{N} \geq N_2 > N_1$, (b)$\overline{N} > N_2 > N_1 > \underline{N}$, or (c)$N_2 > N_1 = 1$. Then,*

1. *for any face value F and fundamental Y, the value of debt $D^{\theta}_{N_2}(F,Y) \geq D^{\theta}_{N_1}(F,Y)$,*
2. *as an immediate consequence of 1, the debt capacity is higher with more creditors, i.e., $\kappa^{\theta}_{N_2} \geq \kappa^{\theta}_{N_1}$,*
3. *also as an immediate consequence of 1, the required interest rate is lower with more creditors: for any θ and $X \leq DC^{\theta}_{N_1}(Y)$, let F^{θ}_k ($k = N_1, N_2$) be the minimum solution to $X = D^{\theta}_k(F^{\theta}_k, Y)$. Then the solutions exist and $F^{\theta}_{N_2} \leq F^{\theta}_{N_1}$.*

The three cases in proposition 10 are quite transparent. In case (a), as discussed following equation (3.20), rollover is always possible even in the bad state. Therefore the incumbent creditors pose no liquidation risk. In case (b), the firm is liquidated only in the bad state when the creditors cannot be paid in full. Case (c) is a little different. It states that the value of debt is the worst when there is just one creditor. It is the worst because with a single creditor, the actual repayment is just the liquidation payoff $\min(F, \lambda Y)$ regardless of whether or not rollover is possible.[6] The debt capacity is attained when $F \to \infty$:

$$\kappa^{\theta}_1 = [\pi \mathbf{1}_{\theta=G} + (1-\pi)\lambda]\mu. \tag{3.21}$$

It is easy to see from (3.15) that multiple creditors can at least secure a repayment of the liquidation value. Thus, having multiple creditors always weakly improves pledgeability.

In general, the benefit of having more creditors is the enhanced pledgeability, which lowers the required interest rate proxied by $\frac{F^{\theta}_N}{X}$ in proposition 10. The cost, as will become more clear in the next subsection, is a higher liquidation probability. Note here that one should not expect the negative correlation between the number of creditors and the interest rates to hold *in equilibrium*. I will postpone this discussion until section 3.6. As a preview, in equilibrium, the firms choose more creditors when they do badly.

[6] The actual repayment is conditional on the project not maturing.

In these cases, their debts are more likely to default so their creditors demand higher interest rates. Therefore, having more creditors is associated with poorer performance, which in turn causes higher interest rates.

Probability of Renegotiation and Default

I call it *renegotiation* whenever the firm *successfully* rolls over with an actual repayment that is strictly less than the promised face value. This occurs when $N\lambda Yz < \min(F, \kappa^{\theta} Yz)$ and the firm continues by repaying each creditor the liquidation value λYz. Similarly, I call it *default* whenever the creditors do not receive the full repayment F. Mathematically, default means $F > \min(\kappa^{\theta}, N\lambda)Yz$ when the project does not mature. In addition, the firm also defaults if the project matures and yet the final cash flow is insufficient to repay the creditors in full, namely, $Yz\mathbf{1}_{\theta=G} < F$. By definition, renegotiation is a special case of default. Notice that a firm can renegotiate or default multiple times over its life cycle. To avoid any confounding effect, I denote τ_R and τ_D to be the *first* time that the firm renegotiates or defaults and let

$$R_N^{\theta,T}(F,Y) = Prob(\tau_R \leq T \text{ and } \tau_R \leq \tau_\pi) \tag{3.22}$$

$$DFT_N^{\theta,T}(F,Y) = Prob(\tau_D \leq T \text{ and } \tau_D \leq \tau_\pi) \tag{3.23}$$

be the probability that firm does so at least once during the next $T \leq \infty$ periods before or when the project matures at τ_π. The probabilities of renegotiation and default must satisfy the following recursive formulation:

$$R_N^{\theta,T}(F,Y) = (1-\pi)E[R_{N_+^{\theta\star}}^{\theta',T-1}(F_+^{\theta'\star}, Yz)\mathbf{1}_{F\leq\min(\kappa^{\theta'},N\lambda)Yz} + \mathbf{1}_{N\lambda Yz<\min(F,\kappa^{\theta'}Yz)}] \tag{3.24}$$

and

$$\begin{aligned} DFT_N^{\theta,T}(F,Y) = \; & \pi[Prob(Yz<F)\mathbf{1}_{\theta=G} + \mathbf{1}_{\theta=B}] + (1-\pi) \\ & E[DFT_{N_+^{\theta\star}}^{\theta',T-1}(F_+^{\theta'\star}, Yz)\mathbf{1}_{F\leq\min(\kappa^{\theta'},N\lambda)Yz} + \mathbf{1}_{F>\min(\kappa^{\theta'},N\lambda)Yz}] \end{aligned} \tag{3.25}$$

The expression (3.24) is not difficult to understand. With probability $1-\pi$, the firm enters the repayment stage. Renegotiation occurs if $N\lambda Yz < \min(F, \kappa^{\theta} Yz)$; otherwise, if rollover is possible with a full repayment F, the continuation probability of renegotiation in the next $T-1$ periods is calculated by using the equilibrium refinancing strategies for the next period number of creditors $N_+^{\theta\star}$ and face value $F_+^{\theta\star}$. The expression (3.25) can be similarly interpreted.

If the project continues without a renegotiation or default, the creditors are paid F in full regardless of the number of creditors N. Therefore, the continuation probabilities $R^{\theta,T-1}_{N_+^{\theta\star}}(F_+^{\theta\star}, Yz)$ in (3.24) and $DFT^{\theta,T-1}_{N_+^{\theta\star}}(F_+^{\theta\star}, Yz)$ in (3.25) are independent of N as well. On the other hand, as N increases, the region in which the firm makes the full repayment widens, since $F \leq \min(\kappa^\theta, N\lambda)Yz$ is more likely to hold. Intuitively, more creditors collectively have more bargaining power and provide a higher immediate incentive for the firm to pay back its debt. This effect reduces both the probability of renegotiation and default. The result is summarized in the following proposition.

Proposition 11 *The probabilities of renegotiation and default are lower with more creditors, i.e.,* $R^{\theta,T}_{N_2}(F,Y) \leq R^{\theta,T}_{N_1}(F,Y)$ *and* $DFT^{\theta,T}_{N_2}(F,Y) \leq DFT^{\theta,T}_{N_1}(F,Y)$*, for all* $N_2 > N_1$*,* θ*,* F*, and* Y*.*

Proposition 11 is another way to demonstrate the pledgeability channel. Having more creditors provides a better repayment incentive and therefore reduces the probability that the firm willingly or unwillingly cuts debt repayment.

3.5.2 Liquidation Risk

Probability of Liquidation

Recall that τ_L and τ_π are the random times of liquidation and project maturity. Define

$$L^{\theta_t,T}_{N_t}(F_t, Y_{t-1}) = Prob(\tau_L \leq t+T \text{ and } \tau_L < \tau_\pi), \tag{3.26}$$

at the beginning of period period t, to be the expected probability of liquidation in the next $T \leq \infty$ periods before the project matures. Since liquidation occurs if and only if (3.14) is violated, the liquidation probability L must satisfy the recursive formulation:

$$L^{\theta,T}_N(F,Y) \quad = \quad (1-\pi)E[L^{\theta',T-1}_{N_+^{\theta\star}}(F_+^{\theta'\star}, Yz)\mathbf{1}_{\min(F,N\lambda Yz)\leq\kappa^{\theta'}Yz} + \mathbf{1}_{\min(F,N\lambda Yz)>\kappa^{\theta'}Yz}] \tag{3.27}$$

With probability $1-\pi$, the firm enters the repayment stage. A failed negotiation results in an immediate liquidation; otherwise, the continuation probability of liquidation in the next $T-1$ periods is calculated by using the equilibrium refinancing strategies for the next period number of creditors $N_+^{\theta\star}$ and face value $F_+^{\theta\star}$.

A direct consequence of having more creditors is that the immediate liquidation probability

$$L_N^{\theta,1}(F,Y) \quad = \quad (1-\pi)E[P(\min(F,N\lambda Yz) > \kappa^{\theta'}Yz)]$$

increases because the rollover condition (3.14) is less likely to hold with a bigger N. I state this simple result as a lemma.

Lemma 4 *The one-period-ahead liquidation probability increases with the number of creditors, i.e.,* $L_{N_2}^{\theta,1}(F,Y) \geq L_{N_1}^{\theta,1}(F,Y)$ *for all* $N_2 > N_1$, θ, F, *and* Y.

Lemma 4 highlights the cost of having more creditors arising from a higher chance of an immediate liquidation. It is also helpful to compare lemma 4 with a seemingly contradictory result proposition 11. Fundamentally unlike liquidation, renegotiation and default as I defined in subsection 3.5.1 pose no direct welfare loss, since they do not lead to an inefficient termination of the project. Instead, they (oppositely) reflect the entrepreneur's endogenous commitment level. With more creditors, the entrepreneur commits to make (more) repayment at the cost of a more likely ex post liquidation.

One can interpret renegotiation or default as financial distress and liquidation as a *costly* outcome (for example, failed private debt restructuring). Under this interpretation, the results in this subsection state that with more creditors, the firm ex ante is less likely to end up in distress. Once it is in distress, however, the creditors are less likely to strike a deal. This prediction is confirmed by Gilson et al. (1990), who find that financially distressed firms with more creditors are less likely to turn around and emerge from a private debt restructuring.

Firm Value

Having more creditors in general reduces the total firm value. An immediate consequence of having more creditors is a greater liquidation risk in the next period. The long-run effect is the higher actual repayment which permanently increases the future liquidation probability. Both effects lower the firm value.

Proposition 12 *The firm value is lower with more creditors:* $V^{\theta}_{N_1}(F, Y) \geq V^{\theta}_{N_2}(F, Y)$ *for any* θ*,* F*,* Y*, and* $N_1 < N_2$*.*

Note that, from proposition 10, the value of debt is in general higher with more creditors for any given face value. Therefore, the conclusion is a joint statement about both higher *market leverage* ($\frac{D^{\theta}_{N_i}}{Y}$) and more creditors. To focus on the net effect of creditor dispersion on the firm value, one can hold the value of debt constant. Recall that this is exactly the firm's refinancing problem (3.11). The next section analyzes this choice explicitly.

3.6 Creditor Dynamics

The dynamics associated with the number of creditors is determined by the firm's refinancing problem (3.11). As we have seen from the previous section, with more new creditors N_+, the benefit is a potentially lower refinancing cost F_+, as in proposition 10. On the other hand, the cost is a higher immediate liquidation threat in the next period, as in lemma 4. The firm optimally chooses N_+ by balancing the cost and benefit. Unfortunately, for a discrete choice problem like this one, an analytical solution is typically not available. However, all the numerical experiments that I have calculated unanimously show that the cost of having more creditors *always* outweighs the benefit. The firm chooses more creditors only when borrowing the required level of repayment from fewer creditors is infeasible.

3.6.1 A Numerical Example

In this subsection, I explicitly describe a numerical example based on the following parameter choices: the per period shock to the final dividend process z follows a uniform distribution on $(0.63, 1.83)$, the probabilities of the states staying unchanged $(p^G, p^B) = (0.8, 0.3)$, the per period probability of the project maturing $\pi = 0.2$, the liquidation coefficient $\lambda = 1$, and the required up-front investment $I_0 = 1$. Even with the choice of $\lambda = 1$, liquidation is still inefficient since the future growth opportunities are lost. The key equilibrium variables, debt capacities, are calculated to be $(\kappa^G_1, \kappa^G_2, \kappa^B_1, \kappa^B_2) = (1.23, 1.273, 0.984, 1.022)$ and $\kappa^{\theta} = \kappa^{\theta}_2$. Under this parameterization, the creditor capacity $\overline{N} = 2$, and therefore the relevant choice for the new creditors N_+ is between 1 and 2. The numerical example is not designed to match any data, and the qualitative features of this example are robust to parameter and distribution choices.

Figure 2 plots the total firm value normalized by fundamental ($\frac{V_N^\theta(F,Y)}{Y}$) against the normalized value of the debt ($\frac{D_N^\theta(F,Y)}{Y}$) or equivalently the amount that has to be borrowed $\frac{X}{Y}$ in problem (3.11). The solid (dashed) line is the firm value with a single creditor when the fundamental $\theta = G$ ($\theta = B$). The dotted (dash-dotted) line is the firm value with two creditors when the fundamental $\theta = G$ ($\theta = B$). The thick solid segments can be supported only by two creditors (the lower curves). A quick observation is that when the value of debt is low, the firm values for one and two creditors converge. This is because the firm has to honor the promised face value regardless of the fundamental realization and the number of lenders.[7] Thus, the choice of the number of creditors has no impact on the firm value. As the value of debt increases, the two lines diverge and, when both are feasible, the single creditor case always delivers a higher firm value. This pattern suggests that the cost of inefficient liquidation is greater than the benefit of interest reduction (lower continuation face value F_+). However, since the curves end on the x-axis at κ_N^θ,[8] the lower curves for two creditor cases indeed extend farther than their single creditor counterparts. This means that when the firm needs to borrow beyond its single creditor debt capacities, it has to seek two creditors.

Figure 3 is a typical sample path of the firm. Areas are shaded when the state is bad. The solid (dashed) line denotes the exogenous fundamental process Y_{t-1} (the face value process F_t determined in equilibrium). I use bold segments when the firm chooses two creditors. The values plotted at each period t are the state variables entering this period: number of creditors N_t, the promised face value F_t, state θ_t, and fundamental process Y_{t-1}. Finally, the dotted bars plot the interest rates $\frac{F_t}{D_{N_t}^{\theta_t}(F_t,Y_{t-1})}$ during each period.

The firm starts by borrowing the required investment $I_0 = 1$ from one creditor in a good state with an interest rate of 9% (a level of 1.09 in the plot). During period 1, the fundamental drops to 0.7. With a single creditor, the firm negotiates the actual

[7] This case is possible since per period shock z has a compact support. So when $\frac{F}{Y}$ is sufficiently small such that $\frac{F}{Y\underline{z}} \leq 1$, the firm will repay F in the next period in order to continue.

[8] This is because $\frac{D_N^\theta(F,Y)}{Y} \leq \kappa_N^\theta$ by definition.

payment down to the liquidation value 0.7 and issues new debt with a face value of 0.76 and an interest rate of 9% to finance the repayment. During period 2, the fundamental keeps deteriorating to 0.49 and the state θ_3 switches to bad. The firm again negotiates the actual payment down to 0.49. In a bad state, however, the firm must refinance this payment from two creditors, because the debt capacity with a single creditor is insufficient to cover the liquidation value. The interest rate soars to 63%. The firm enters period 3 with face value $F_3 = 0.8$. During period 3, even though the state θ is still bad, the fundamental dramatically improves and the firm is able to make the promised repayment 0.8 and roll over the debt with a single creditor. The required interest rate reduces to 49%. What happens during period 4 is very similar to period 1. The state θ returns to good and the firm pays out and refinances the liquidation value by borrowing from one creditor at an interest rate of 9%. On period 5, the fundamental continues to improve to 1.23, and the firm can even issue risk free debt to finance the 0.77 debt obligation. This is possible since even if the project matures with the lowest shock realization $z = \underline{z} = 0.63$, the full value of the debt can still be repaid.[9] Period 6 and 7 are similar to period 2 and 3: the state switches to bad, the financing costs for the firm increases and two creditors are eventually required. Finally, during period 8, the state θ returns to good and the realized fundamental improves to 1.14. Even so, the borrowing capacity is only $1.14 \times 1.27 = 1.45$, which is not high enough to cover the promised amount of 1.55 to the two creditors. The firm is then liquidated.

The first noticeable feature in figure 3 is that the firm switches to two creditors only in the bad state $\theta = B$ when the fundamental deteriorates and consolidates back to a single creditor structure when its performance improves. In the model, the firm is never liquidated with a single incumbent creditor. Therefore, the extra pledgeability from two creditors is costly, and the firm uses it only as a last line of defense. Second, the interest rates are higher in general with more creditors. Why does this not contradict with proposition 10, which states that having more creditors reduces interest rates? Even though an exogenous increase in the number of creditors may increase pledgeability and lower the interest rate, once the number of creditors is endogenized in equilibrium, the firm only chooses to have more creditors when higher debt capacity is needed, which occurs in worse states and causes higher interest rates. Empirically, Petersen and

[9] To be specific, $1.23 \times 0.63 > 0.77$.

Rajan (1994) find that companies with more banking relationships also have higher cost of credit.

3.6.2 When Do Firms Choose More Creditors?

Although dynamic programming discrete choice models generally do not deliver analytical tractability, I can provide a sufficient condition under which the firm increases the number of creditors. This result inherits the idea from the previous subsection that the firm has to borrow from more creditors when its debt capacity with fewer creditors is insufficient. The next result argues that one of these scenarios is the case in which the firm has performed poorly in the past. Here, I keep the time subscripts to avoid any confusion.

Proposition 13 *Suppose that the realized fundamental is low $F_t \geq N_t \lambda Y_t$, the state is bad $\theta_{t+1} = B$, and rollover is possible $N_t \lambda \leq \kappa^B$. Then the continuation number of creditors must strictly increase, $N_+^{B\star}(N_t \lambda Y_t, F_t, Y_t) > N_t$.*

Providing the proof here is worthwhile. Since $N_t \lambda Y_t \leq \kappa^B Y_t$ and $N_t \lambda Y_t \leq F_t$, the firm can roll over by paying the liquidation value to each creditor, totaling $N_t \lambda Y_t$. Because the realized repayment to each creditor at period $t+1$ is at most $\min\{\frac{F_{t+1}}{N_{t+1}}, \lambda Y_{t+1}\} \leq \lambda Y_{t+1}$, the debt capacity in the bad state with N_{t+1} creditors is bounded by

$$\kappa^B_{N_{t+1}} Y_t \leq (1-\pi) N_{t+1} E(\lambda Y_{t+1}) = (1-\pi)\mu(N_{t+1}\lambda Y_t).$$

Since $(1-\pi)\mu < 1$ by assumption (3.1), $\kappa^B_{N_{t+1}} Y_t < N_{t+1} \lambda Y_t$. The firm chooses a continuation number of creditors N_+ at least to finance the required repayment $N_t \lambda Y_t$. Thus,

$$N_t \lambda Y_t \leq \kappa^B_{N_+} Y_t < N_+ \lambda Y_t.$$

Therefore, $N_+ > N_t$.

The intuition here is straightforward. For each individual creditor, the pledgeable amount is at most the expected liquidation value. In the bad state, with probability π, the firm dies without payout in the next period. The assumption $(1-\pi)\mu < 1$ implies that the expected liquidation value tomorrow is less than the liquidation value today. So for each liquidation value that the firm has to pledge today, it must seek more than one creditor on average. Therefore, the number of creditors must strictly increase. This result has also been empirically documented by Farinha and Santos (2002), who show that firms are more likely to abandon a single creditor structure when the performance measures are worse.[10]

3.6.3 Empirical Predictions Revisited

Recall that section 3.5 focused on the comparative statics of the number of creditors on interest rates, the probabilities of liquidation, renegotiation, and default, and the firm value. Now I discuss the corresponding implications in equilibrium, taking into account that the firm chooses more creditors when it is in bad shape. As we have seen in subsection 3.6.1, the implication of proposition 10 on the interest rate is reversed. The equilibrium selection effect dominates the pledgeability effect, resulting in higher interest rates associated with more creditors. However, as the direction predicted by proposition 12, the firm value is still lower with more creditors in equilibrium. The selection effect that links more creditors with bad performance reinforces the comparative static result in proposition 12. By the same reasoning, the liquidation probability jumps up with more creditors in equilibrium.

Growth and the Number of Creditors

When the per period shock to fundamental z_t on average improves, the future of the firm becomes more promising. This situation has several effects. A direct effect is that the firm has a higher liquidation value on average in the next period, which increases the bargaining position of the creditors. Second, the firm in the next period is more likely to have the resources to make the promised repayment or survive a negotiation. Both effects improve the debt value as well as the debt capacity, and more creditors can be supported.

Proposition 14 *Suppose g_i $(i = a, b)$ are two density functions for z, and g_a first-order stochastically dominates g_b. Then for any equilibrium under g_b, there exists an equilibrium under g_a such that $\kappa^{\theta,a} \geq \kappa^{\theta,b}$, where $\kappa^{\theta,i}$ are the corresponding debt capacity*

[10] The performance measures include liquidity, cash flow, leverage and so on.

coefficients.[11] *In addition, the creditor capacity and the safe number of creditors are both higher under g_a, i.e.,* $\overline{N^a} \geq \overline{N^b}$ *and* $\underline{N^a} \geq \underline{N^b}$.

Since first-order stochastic dominance implies that the average growth rate is higher, a direct implication is that firms with higher growth rates can be associated with more creditors. This is consistent with the empirical evidence documented by Farinha and Santos (2002), who find that firms with a better growth perspective, as measured by sales growth, tend to have more creditors.

3.7 The Value of Coordination Failure

3.7.1 Ex Post Efficient Policies

Coordination failure among creditors reduces the financial flexibility that the firm needs during a crisis. Quite often, firms in distress or even default are more valuable as going concerns than they are being liquidated piecemeal. In fact, because of the coordination problem among creditors, many policies are designed to reduce or eliminate liquidation. For example, the *automatic stay* clause, which halts creditors' actions to claim a debtor's assets, and *Chapter 11 reorganization*, which promotes a constructive renegotiation with all creditors collectively, both fall into this category. If the policies indeed eliminate all ex post coordination failure and force multiple creditors to negotiate the debt as one, then I show that such policies cause ex ante higher chances of liquidation and lower firm values.

Committing to an ex post efficient negotiation is equivalent to a counterfactual model in which the firm can borrow *only* from one creditor. With one creditor, the firm at most repays the liquidation value if the project does not mature, independent of the firm's ability to switch to multiple creditors. Therefore, it is easy to see that the debt capacities are still (κ_1^N, κ_1^B) given by (3.21) in this counterfactual case.

In the bad state, the debt capacity is $\kappa_1^B = (1-\pi)\mu\lambda < \lambda$, so when the realized fundamental Y is sufficiently weak ($F > \lambda Y$), repayment negotiation fails because the firm cannot credibly pledge the liquidation payoff $\min(F, \lambda Y) = \lambda Y > \kappa_1^B Y$. Therefore,

[11] *The opposite direction holds too. That is, for any equilibrium under g_a, there exists an equilibrium under g_b such that $\kappa^{\theta,a} \geq \kappa^{\theta,b}$.*

the single-creditor counterfactual case has effectively no room for negotiation, when the state is bad. On the contrary, in the true model if the firm is allowed to have multiple creditors, it can pledge at least λY (in fact, $\kappa^B Y$), so a single creditor never liquidates. Therefore, the expected probability of liquidation $L_1^{\theta,T}(F,Y)$ is lower for the true model compared with the counterfactual one.

Using the same example as in section 3.6, figure 4 plots the expected probability of liquidation $L_1^{\theta,\infty}(F,Y)$ against the expected value of the debt conditional on the current state $\theta = G$ (top panel) and $\theta = B$ (bottom panel). The solid (dashed) line is the liquidation probability with a single creditor (two creditors) in the full model. The dotted line is for the counterfactual model in which the number of creditors is exogenously fixed at one.

As figure 4 illustrates, having two creditors generally means a higher liquidation probability than having a single creditor in the true model because of the following two adverse effects. The short-term effect is a higher probability of an immediate liquidation in the next period, captured by lemma 4. The long-term effect is that more creditors can secure a bigger repayment, which requires a larger continuation face value, which in turn causes a higher liquidation probability in the future. Even so, the option of having two creditors is still beneficial in the sense that it uniformly reduces the firm's liquidation probability with a single creditor compared with the counterfactual. The possibility of having multiple creditors in the future and supporting a higher debt level prevent an even sooner liquidation when the firm initially gets into trouble.

Firm values tell a similar story. Although establishing strict inequalities in a dynamic programming framework requires some work, the economics behind it is intuitive. Without the costly mechanism to support a higher leverage by more creditors, the firm fails even sooner, lowering its value.

Proposition 15 *Let $V_{CF}^{\theta}(F,Y)$ be the firm value in the counterfactual world. Then for any $F > 0$, $V_{CF}^{\theta}(F,Y) < V_1^{\theta}(F,Y)$, and for any $N > 1$, there exists a nonempty set $\mathbb{F}$ (may depend on N) such that $V_{CF}^{\theta}(F,Y) < V_N^{\theta}(F,Y)$ for all $F \in \mathbb{F}$.*

In this economy, since the creditors always break even, the total value of the firm is a welfare criterion. As predicted by proposition 15, eliminating the possibility of a coordination failure is socially inefficient. More interestingly, the result suggests that mistakenly sticking with a single creditor may be even more inefficient than having the firm mistakenly end up with multiple creditors. This comparision between two types of mistakes is also confirmed by the liquidation probability. In figure 4, for a substantial range of fundamental values, the liquidation probability with *two creditors* in the true model is strictly lower compared with the *single* creditor counterfactual.

These findings raise caution regarding ex post efficient procedures such as automatic stay clause and Chapter 11 reorganization. These policies can be somewhat viewed as a commitment that the creditors will accept ex post efficient offers. Although eliminating ex post inefficiency, the policies also prevent the firm from utilizing enhanced pledgeability in the future. As we have seen, such policies lead to more likely liquidation, lower firm value and lower welfare ex ante.

3.7.2 Collateral

Collateral is typically viewed as a means of securing a creditor's position. It alleviates the ex post coordination problem because the liquidating creditor can no longer pose externalities on the secured creditors. In the extreme case, if all positions are secured, then no ex post coordination failure exists. In the model this case is equivalent to the previous counterfactual model in which the firm is exogenously restricted to borrowing from only one creditor. All results in the previous subsection still hold, with the striking prediction that firms that issue collateralized debts are more likely to be liquidated and have lower values. In addition, counterintuitively collateralized debt also leads to lower borrowing capacity compared with an uncollateralized instrument that is subject to ex post coordination failure.

3.8 Renegotiation Frequency

How does renegotiation frequency affect the equilibrium outcome? Since renegotiation happens each time the debt matures, it is equivalent to the debt maturity in the model. To highlight the economic intuition, I simplify the model such that the shock $z_t = \mu$ is a constant and the transition matrix is symmetric $p^G = p^B = p$. Instead of shrinking the debt maturity directly, I keep a stationary structure of one-period debt and extend the expected project duration. Letting

$$\hat{\pi} = \frac{\pi}{T}, \tag{3.28}$$

the expected project duration becomes $E(\tau_{\hat{\pi}}) = \frac{T}{\pi} = TE(\tau_{\pi})$, T times longer than in the original model. This structure effectively shrinks the debt maturity to $\frac{1}{T}$ period under the original calendar time. I then pick the new growth rate $\hat{\mu}$ and the switching probability $\hat{p}$ to match the first best firm values as defined in (3.3):

$$\hat{V}^{\theta\star}_{FB}(Y|\hat{\mu}, \hat{\pi}, \hat{p}) = V^{\theta\star}_{FB}(Y|\mu, \pi, p) \tag{3.29}$$

for both $\theta = G, B$, so the firm quality is unaffected by the change in timescale. The following lemma confirms that the proposed modifications are natural in the following sense. First, the parameters of the game after the timescale change are well defined. Second, when the period length is very small, the (probabilities of) changes in the state variables are also very small.

Lemma 5 *The new set of parameters after the timescale change $\hat{\pi} = \frac{\pi}{T} \in (0, 1)$, $\hat{\mu} = \frac{T\mu}{T\mu - \mu + 1} > 1$, and $\hat{p} = \frac{T - 1 + p(1-\pi)}{T - \pi} \in (0, 1)$ are well defined. In addition, as the effective debt maturity goes to 0, i.e., when $T \to \infty$, the new parameters satisfy $\hat{\pi} = \frac{\pi}{T} \to 0$, $\hat{\mu} \to 1$, $\hat{p} \to 0$, and $(1 - \hat{\pi})\hat{\mu} < 1$.*

From (3.21), an immediate implication of lemma 5 is $\hat{\kappa}^{\theta}_{1} \to \lambda$. The next result characterizes the debt capacity under the new timescale. The key feature is that with more frequent negotiation, in the limit, the pledgeable amount in the bad state $\theta = B$ approaches the liquidation value. Recall from the example in section 3.6 and proposition 13 that the debt capacity in the bad state is crucial for when the firm increases the number of creditors. Therefore, the benefit of having multiple creditors becomes negligible as the firm renegotiates more frequently. I denote the variables with hats as the ones after the timescale change.

Proposition 16 *When $T \to \infty$, the debt capacities $\hat{\kappa}^{B} \to \lambda$.*

To understand this result, recall that the firm can only pledge the liquidation value with a single creditor. Although more creditors indeed enforce more repayment by proposition 10, the ultimate source of this extra repayment is from the *growth between two negotiation dates*. If the expected per period growth of the fundamental diminishes ($\hat{\mu} \to 1$), then the incremental pledgeability vanishes as well. Since renegotiation is closely related to the debt maturity in the model and a troubled firm typically negotiates the repayment at maturity in practice, the renegotiation frequency can be interpreted as the debt maturity. With very short maturity,[12] having multiple creditors provides no extra pledgeability.

3.9 Possible Extensions

3.9.1 Staggered Debt

In this section, I explicitly consider the staggered debt structure. Everything stays the same except for the (re)financing stage. After the entrepreneur decides the number of new creditors, he specifies the order in which the new debt claims mature in the next period. The creditors know their position in the maturity sequence, which controls the binary renegotiation order. Denote n to be the creditor whose debt matures in the nth place. Clearly, same as before, the ones who renegotiate earlier are in better positions. Hence, the value of debt also depends on n:

$$\begin{aligned} D^{\theta}_{n,N}(F,Y) &= E\{\pi \frac{\min(F,Yz)}{N}\mathbf{1}_{\theta=G} \\ &+ (1-\pi)[\mathbf{1}_{\min(F,N\lambda Yz)\leq \kappa^{\theta'}Yz}\min(\tfrac{F}{N},\lambda Yz) \\ &\quad +\mathbf{1}_{\min(F,N\lambda Yz)>\kappa^{\theta'}Yz}\min(\tfrac{F}{N},\max(\lambda Yz-\tfrac{n-1}{N}F,0))]\}. \end{aligned}$$

The first two terms are the same as in (3.15). The last term captures the fact that if rollover fails, the previous $n-1$ creditors will reject the firm's offers and claim $\min(\frac{n-1}{N}F,\lambda Yz)$ against the firm's liquidation proceeds. The nth creditor can claim the remaining cash from liquidation up to the full value of his claim. Define the total value of debt as

$$D^{\theta}_{N}(F,Y) = \sum_{n=1}^{N} D^{\theta}_{n,N}(F,Y).$$

Then, it is easy to see that $D^{\theta}_{N}(F,Y)$ defined above coincides with (3.15). Therefore, the total value of debt and the firm's problem are essentially unchanged.

[12] For example, an overnight repo agreement.

3.9.2 Trading of the Debt Claims in the Secondary Market

Because debt claims may be more valuable if held by some different number of investors, so the creditors may have incentives to trade with others. Suppose after the firm issues new debt claims in the refinancing stage, the new creditors can trade these claims free of transaction costs. Then naturally trades will occur until the number of creditors eventually maximizes the debt value. In the numerical example from section 3.6.1, for instance, even if the firm issues debt to a single creditor, it is in the creditor's best interest to sell half of the claim to a second investor. As a matter of fact, it can be shown that the number of creditors is often suboptimally high, which hurts the firm value by Proposition 12.

This result is surprising because it suggests that better liquidity in the secondary debt market undermines the firm's ability to control its creditor dispersion and therefore could potentially be bad for the firm. However, I must highlight that I am not claiming a secondary corporate debt market is necessarily bad and should be banned all together. In fact, He and Xiong (2012b) have studied the consequence of a liquidity crunch in the secondary market. What is indeed worth noting here is that when the coordination problem among the creditors is a major concern, trades among them may render the pool of debt holders inefficiently large and thereby exacerbate rollover risks.

It is also interesting to contrast the result with Dewatripont's comment that the possibility of trading leads to ex post efficient consolidation of the claims. [13] To get Dewatripont's effect in this model, we need to allow for trades after the uncertainty is realized (*before* the renegotiation). When rollover is going to fail in the current model, the creditors have incentive to consolidate, similar to the discussion in section 3.7.1. Here the timing is different. Trading *after* the new debts are issued in the refinancing stage leads to sub-optimally large creditor pool, as dispersed ownership may make the debt more valuable.

3.9.3 Uneven Concentration

So far, I have assumed that the firm must evenly distribute the face value of the debt when refinancing from new creditors. However, this assumption is not crucial for the

[13] The comment is made at the Nobel foundation conference on corporate finance (Stockholm, August 1995). See page 410 in Tirole (2006) for detailed discussions.

intuition to work. The key economic force here is that having more creditors means that each one of them can pose greater externalities on the others, causing coordination problems, which, on the other hand, improves their collective bargaining position against the entrepreneur. Allowing creditors to have different shares of the loan does not eliminate these channels.

Of course, the exact amount of externalities they create certainly depends on the specific distribution of creditor size. For example, suppose there are two creditors. One is large and the other is small. Renegotiating with the smaller creditor will be more difficult, while forcing concession from the larger one will be easier. In the limit, if the large lender holds almost the entire outstanding debt, then the outcome approaches the single creditor case. The same economic forces can also potentially endogenize the optimal debt concentration, an aspect that can be investigated by future research.

3.9.4 Private Savings by the Entrepreneur

Suppose the entrepreneur can save; that is, instead of raising just enough money to roll over maturing debts, the firm can now borrow more and keep internal cash. The relevant question regarding the equilibrium creditor dispersion becomes whether the firm wishes to borrow from more creditors and save for the future. A rigorous analysis of this problem is beyond the scope of this paper, but intuitively I conjecture that the firm has no incentive to do so.

First, having more creditors increases the firm's probability of liquidation. Moreover, internal cash in the current model is unlikely to serve as a "cushion" that provides "the last source of repayment", as one might imagine. Recall that the project has no cash flow. Therefore, additional cash can only be raised by promising an even higher repayment (weakly positive interest rate). Because the internal cash can always be seized so, when the debt becomes due in the next period, this additional repayment may hardly be renegotiated down even when the liquidation value of the project is very low. Thus, the private cash savings will be insufficient to meet the associated additional repayment, let alone to be the source of funds for the original level of debt. To summarize, having internal cash may not benefit the firm. It gives each creditor a stronger bargaining position as the liquidation value of the firm (including both project and cash) increases, which in turn exacerbates the coordination problem among the lenders.

3.9.5 Entrepreneur's Liquidation Incentive

The endogenous parameter assumption $\kappa^{\theta} \geq \lambda$ in proposition 9 rules out the entrepreneur's incentive to voluntarily liquidate the project. Without this assumption, the entrepreneur may wish to liquidate in equilibrium. For example, in a bad state, if the entrepreneur definitely foresees a liquidation tomorrow, he is better off voluntarily liquidating today, since the liquidation payoff λY_t is higher than the continuation value $(1-\pi)\lambda E_t(Y_t z_{t+1}) = (1-\pi)\mu\lambda Y_t$. An interesting study would investigate how creditor dispersion interacts with the entrepreneur's liquidation decision. This topic is left for future research.

3.10 Conclusion

I build a dynamic model in which the firm must repeatedly roll over debt and can renegotiate repayment. Having more creditors brings the disadvantage of coordination problems, which in bad times make it harder for a firm to restructure its debt to avoid liquidation. In good times, however, these same coordination problems enhance pledgeability by making it harder for a firm to opportunistically hold up its creditors. In the model, the firm actively chooses the number of creditors over time by optimally trading off pledgeability with the liquidation probability.

Analysis of the model shows that firms increase the number of creditors when they perform badly. Doing so increases the liquidation probability and lowers the firm value. Allowing for coordination failure in equilibrium is valuable and policies that commit the creditors to ex post efficient coordination reduce the firm value and may raise the liquidation probability. If the firm can renegotiate the debt very frequently, the enhanced pledgeability from multiple creditors diminishes.

The model's implications highlight the potential for selection bias in empirical studies that investigate the effect of creditor dispersion. For example, an exogenous increase in the number of creditors lowers the required interest rate due to the firm's better repayment incentives. In equilibrium, however, this relationship is reversed because firms choose more creditors when they are in trouble, which in turn leads to higher interest rate.

Finally, having outstanding debt may provide the entrepreneur with the incentive to inefficiently continue the project, for example, risk shifting and gambling for survival. The received wisdom is that a higher level of debt exacerbates the problem and increases the inefficiency associated with such continuation bias. In this paper, I make parameter assumptions such that continuing the project is always efficient.[14] Therefore, there is no debt-equity conflict in continuing the project inefficiently. Instead, if abandoning the project is optimal in certain states, then having outstanding debt generates non-monotonic outcomes in my model, in contrast with the aforementioned intuition. When leverage is low, the entrepreneur implements the first best liquidation strategy. When leverage is high, the efficient liquidation can still be implemented. In this case, even though the entrepreneur is willing to gamble for survival, the creditors refuse to rollover and force an efficient termination. In addition, an intermediate case may exist, in which the debt level is high enough to distort the entrepreneur's liquidation incentive, but not too high to spur the creditors into action. Intuitively, having more creditors in this intermediate case may facilitate restoring the efficient liquidation strategy and correct the entrepreneur's continuation bias. A more rigorous analysis is required to further investigate this problem and I look forward to future research that can shed light on this issue.

[14] To be specific, condition (3.4).

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Appendix A

A.1 Proof of Results

Proof of Lemma 1: Suppose to the contrary that $a' \geq a''$. Since firms a' and a'' follow different strategies, $a' > a''$. Let P_1' and P_2' (respectively, P_1'' and P_2'') be the prices associated with s_1' and s_2' (respectively, s_1'' and s_2''). Also, let $\mathbf{1}'$ and $\mathbf{1}''$ be the investment decisions of firms a' and a''.

From the equilibrium conditions,

$$\frac{a'' + S - s_1'' - s_2'' + b\mathbf{1}''}{1 - \frac{s_1''}{P_1''} - \frac{s_2''}{P_2''}} \geq \frac{a'' + S - s_1' - s_2' + b\mathbf{1}'}{1 - \frac{s_1'}{P_1'} - \frac{s_2'}{P_2'}}. \tag{A.1}$$

By supposition, and given optimal investment decisions, the numerator of the LHS is strictly smaller than the numerator of RHS. Hence the denominator of the LHS must also be strictly smaller, i.e.,

$$1 - \frac{s_1''}{P_1''} - \frac{s_2''}{P_2''} < 1 - \frac{s_1'}{P_1'} - \frac{s_2'}{P_2'}. \tag{A.2}$$

Also from the equilibrium conditions,

$$\frac{a' + S - s_1' - s_2' + b\mathbf{1}'}{1 - \frac{s_1'}{P_1'} - \frac{s_2'}{P_2'}} \geq \frac{a' + S - s_1'' - s_2'' + b\mathbf{1}''}{1 - \frac{s_1''}{P_1''} - \frac{s_2''}{P_2''}}.$$

From (A.2),

$$\frac{a' - a''}{1 - \frac{s_1'}{P_1'} - \frac{s_2'}{P_2'}} < \frac{a' - a''}{1 - \frac{s_1''}{P_1''} - \frac{s_2''}{P_2''}},$$

which implies

$$\frac{a'' + S - s_1' - s_2' + b\mathbf{1}'}{1 - \frac{s_1'}{P_1'} - \frac{s_2'}{P_2'}} > \frac{a'' + S - s_1'' - s_2'' + b\mathbf{1}''}{1 - \frac{s_1''}{P_1''} - \frac{s_2''}{P_2''}},$$

contradicting the equilibrium condition (A.1) and completing the proof.

Proof of Corollary 1: Suppose to the contrary that the claim does not hold, i.e., there exists an equilibrium in which there are firms a' and $a'' > a'$ where a'' invests and a' does not invest. Since investment decisions are optimal, the capital transactions of firms a' and a'', say (s_1', s_2') and (s_1'', s_2''), must satisfy $S - s_1' - s_2' < I \leq S - s_1'' - s_2''$. This contradicts Lemma 1, completing the proof.

Proof of Proposition 1: Suppose otherwise. Let $s_1(a)$ be the strategy of firm a, and $A^{rep} = \{a : s_1(a) > 0\}$ be the set of firms who repurchase in equilibrium. By supposition, $\mu(A^{rep}) > 0$. On the one hand, a firm prefers repurchasing to doing nothing if and only if $\frac{a+S-s_1}{1-\frac{s_1}{P_1(s_1)}} \geq a + S$, or equivalently, $P_1(s_1) \leq a + S$. Since by supposition a strictly positive mass of repurchasing firms have a strict preference for repurchasing,

$$E\left[P_1(s_1(a)) - (a+S)\,|a \in A^{rep}\right] < 0.$$

One the other hand, investors only sell if $P_1(s_1) \geq E\left[\frac{a+S-s_1}{1-\frac{s_1}{P_1(s_1)}}|s_1\right]$, or equivalently, $P_1(s_1) \geq E[a|s_1] + S$. By the law of iterated expectations, this implies

$$E\left[P_1(s_1(a)) - (a+S)\,|a \in A^{rep}\right] \geq 0.$$

The contradiction completes the proof.

Proof of Proposition 2: Fix an equilibrium. From Proposition 1, there cannot be a positive mass of firms who repurchase and obtain $P_3 > a+S$. By a parallel proof, there cannot be a positive mass of firms who issue, do not invest, and obtain $P_3 > a + S$. By (2.4), any issue s that is enough for investment is associated with the price

$$P_1(s) = S + E[a|s] + b. \tag{A.3}$$

Given these observations, standard arguments then imply that there exists some $\varepsilon > 0$ such that almost all firms in $[\underline{a}, \underline{a} + \varepsilon]$ issue and invest: if an equilibrium does not have this property, then these firms certainly have the incentive to deviate and issue and invest, since this is profitable under any investor beliefs. So by Corollary 1, there exists $a^* > \underline{a}$ such that all firms in $[\underline{a}, a^*)$ issue and invest.

Finally, suppose that contrary to the claimed result that different firms in $[\underline{a}, a^*)$ issue different amounts. Given Lemma 1, it follows that there exists $\check{a} \in (\underline{a}, a^*)$ such that any firm in $[\underline{a}, \check{a})$ issues strictly more than any firm in $(\check{a}, a^*)$. Hence there must exist firms $a' \in [\underline{a}, \check{a})$ and $a'' \in (\check{a}, a^*)$ such that

$$P_1\left(s\left(a'\right)\right) \leq S + a' + b < S + a'' + b \leq P_1\left(s\left(a''\right)\right).$$

Since $-s\left(a'\right) > -s\left(a''\right)$, this combines with the equilibrium condition for firm a' to deliver the following contradiction, which completes the proof:

$$\frac{S - s\left(a''\right) + a' + b}{1 - \frac{s(a'')}{P_1(s(a''))}} \leq \frac{S - s\left(a'\right) + a' + b}{1 - \frac{s(a')}{P_1(s(a'))}} \leq \frac{S - s\left(a''\right) + a' + b}{1 - \frac{s(a'')}{P_1(s(a'))}} < \frac{S - s\left(a''\right) + a' + b}{1 - \frac{s(a'')}{P_1(s(a''))}}.$$

Proof of Proposition 3:

Preliminaries:

Given any date 1 repurchase level $s_1 > 0$, define $a^*(s_1)$ to be the smallest solution of

$$\frac{1}{1 + \frac{I - S + s_1}{S - s_1 + E[a|a \leq a^*] + b}}(I + a^* + b) - (S - s_1 + a^*) = 0. \tag{A.4}$$

We first show that $a^*(s_1)$ is well-defined, decreasing in s_1, and lies in $(\underline{a}, \bar{a})$. The proof is as follows. The LHS of (A.4) is strictly positive at $a^* = \underline{a}$. The LHS of (A.4) is strictly decreasing in s_1 for any $a^* > \underline{a}$. Consequently, (2.6) implies that the LHS of (A.4) is strictly negative at $a^* = \bar{a}$. Existence of $a^*(s_1)$ follows by continuity. The other two properties are immediate.

Observe that at $s_1 = 0$ and $a_1 = \underline{a}$,

$$\frac{1}{1 - \frac{s_1}{S + \bar{a}}} \frac{1}{1 + \frac{I - S + s_1}{S - s_1 + E[a|a \leq a_1] + b}}(I + a_1 + b) > S + a_1. \tag{A.5}$$

By continuity, choose $\bar{a}_1 > \underline{a}$ and $\bar{s}_1 > 0$ such that inequality (A.5) holds for all $(a_1, s_1) \in [\underline{a}, \bar{a}_1] \times [0, \bar{s}_1]$. Note that $a^*\left(s_1\right) > \bar{a}_1$.

Fix $s_1 \in (0, \min\left\{\bar{s}_1, \frac{S}{2}\right\}]$ sufficiently small such that

$$\max\left\{\frac{I - S + s_1}{S - s_1 + E\left[a|a < a^*(\frac{S}{2})\right] + b}, \frac{I - S + s_1}{S - s_1 + E\left[a|a < \bar{a}_1\right] + b}\right\} \leq \frac{I - S}{S + \underline{a} + b}. \tag{A.6}$$

Given s_1, we explicitly construct an equilibrium. There are two cases, corresponding to whether $S + a^*\left(s_1\right)$ is larger or smaller than $S + E\left[a\right] + b\Pr\left(a \leq a^*\left(s_1\right)\right)$. In the first case, all firms repurchase s_1 at date 1, and then a strict subset of firms issue $I + s_1 - S$

at date 2. In the second case, some firms repurchase s_1 at date 1, with a strict subset then issuing $I + s_1 - S$ at date 2; while other firms do nothing at date 1, with a strict subset then issuing $I - S$ at date 2. In both cases, any off-equilibrium repurchase offer triggers investor beliefs that the firm is type $\bar{a}$, while any off-equilibrium issue offer triggers beliefs that the firm is type $\underline{a}$.

Case 1: $S + a^*(s_1) \geq S + E[a] + b\Pr(a \leq a^*(s_1))$.

In this (easier) case, we show there is an equilibrium in which at date 1 all firms repurchase s_1; and at date 2 firms $a \leq a^*(s_1)$ issue $I - S + s_1$ and invest, while other firms do nothing at date 2. The date 1 repurchase price P_1 and date 2 issue price P_2 in such an equilibrium are

$$\begin{aligned} P_1 &= S + E[a] + b\Pr(a \leq a^*(s_1)) \\ P_2 &= \frac{S - s_1 + E[a|a \leq a^*(s_1)] + b}{1 - \frac{s_1}{P_1}}. \end{aligned}$$

Hence the payoff for a firm a from repurchase-issue is

$$\frac{1}{1 - \frac{s_1}{P_1} + \frac{I-S+s_1}{P_2}}(I + a + b) = \frac{1}{1 - \frac{s_1}{P_1}} \frac{I + a + b}{1 + \frac{I-S+s_1}{S-s_1+E[a|a\leq a^*(s_1)]+b}}. \tag{A.7}$$

By (A.6) and $a^*(s_1) > \bar{a}_1$, the payoff (A.7) is at least

$$\frac{1}{1 - \frac{s_1}{P_1}} \frac{I + a + b}{1 + \frac{I-S}{S+\underline{a}+b}} > \frac{I + a + b}{1 + \frac{I-S}{S+\underline{a}+b}}.$$

The RHS of this inequality is the payoff to issuing directly given out-of-equilibrium beliefs in which direct issue is associated with the worst firm $\underline{a}$. Hence all firms prefer the equilibrium repurchase-issue strategy to the off-equilibrium direct issue strategy.

Firms $a \geq a^*(s_1)$ prefer repurchase-do-nothing to do-nothing. To see this, simply note that the payoff for a firm a from repurchase-do-nothing is $\frac{S-s_1+a}{1-\frac{s_1}{P_1}}$, which exceeds the payoff from do-nothing, i.e., $S + a$, if and only if $P_1 \leq S + a$. Since we are in Case 1, this condition is satisfied for all firms $a \geq a^*(s_1)$.

Firms $a \geq a^*(s_1)$ prefer repurchase-do-nothing to repurchase-issue by the definition of $a^*(s_1)$.

Likewise, firms $a \leq a^*(s_1)$ prefer repurchase-issue to repurchase-do-nothing by the definition of $a^*(s_1)$.

Finally, firms $a \leq a^*(s_1)$ prefer repurchase-issue to do-nothing because this is true for firm $a^*(s_1)$; and is also true for firm $\underline{a}$, since this firm prefers direct issue to do-nothing. Since all payoffs are linear in firm type, it then follow that all firms between $\underline{a}$ and $a^\star(s_1)$ likewise prefer repurchase-issue to do-nothing.

Case 2: $S + a^*(s_1) < S + E(a) + b \Pr(a \leq a^*(s_1))$.

In this case, we show there exists a_1 and a_2, along with a partition A_0, A_1 of $[\underline{a}, a_1]$, such that the following is an equilibrium: At date 1 firms $A_1 \cup [a_2, \bar{a}]$ repurchase s_1, while other firms do nothing; and at date 2 firms A_1 issue $I - S + s_1$ and invest, firms A_0 directly issue $I - S$ (without previously repurchasing), and the remaining firms do nothing.

In such an equilibrium, the date 1 repurchase price P_1 and date 2 issue price P_2 following repurchase are

$$\begin{aligned} P_1 &= S + \frac{E[a|A_1]\mu(A_1) + E[a|a \geq a_2]\mu([a_2, \bar{a}]) + b\mu(A_1)}{\mu(A_1) + \mu([a_2, \bar{a}])} \\ P_2 &= \frac{S - s_1 + E[a|A_1] + b}{1 - \frac{s_1}{P_1}}. \end{aligned}$$

We show that there exist $a_1, a_2 \in [\underline{a}, \bar{a}]$ and $a_1 < a_2$, together with a partition A_0, A_1 of $[\underline{a}, a_1]$, that solve the following system of equations (where P_1 is as defined immediately above):

$$\frac{1}{1 - \frac{s_1}{P_1}} \frac{1}{1 + \frac{I - S + s_1}{S - s_1 + E[a|A_1] + b}} (I + a_1 + b) = S + a_1 \tag{A.8}$$

$$\frac{1}{1 - \frac{s_1}{P_1}} \frac{1}{1 + \frac{I - S + s_1}{S - s_1 + E[a|A_1] + b}} = \frac{1}{1 + \frac{I - S}{S + E[a|A_0] + b}} \tag{A.9}$$

$$P_1 = S + a_2 \tag{A.10}$$

Condition (A.8) states that firm a_1 is indifferent between repurchase-issue and do-nothing. Condition (A.9) states that firms are indifferent between repurchasing and then issuing, and issuing directly. Condition (A.10) states that firm a_2 is indifferent between repurchase-do-nothing and do-nothing.

Notationally, define $\gamma_0 \equiv \frac{\mu(A_0)}{\mu([\underline{a}, a_1])}$ and $E_0 \equiv E[a|A_0]$, and note that $E[a|A_1] = \frac{E[a|a \leq a_1] - \gamma_0 E_0}{1 - \gamma_0}$. The system of equations (A.8)-(A.10) has a solution if and only if the following system has a solution in γ_0, E_0, a_1 and a_2:

$$\begin{aligned}\frac{1}{1-\frac{s_1}{S+a_2}}\frac{1}{1+\frac{I-S+s_1}{S-s_1+\frac{E[a|a\le a_1]-\gamma_0 E_0}{1-\gamma_0}+b}}(I+a_1+b)-(S+a_1) &= 0 \qquad \text{(A.11)}\\ \frac{1}{1+\frac{I-S}{S+E_0+b}}(I+a_1+b)-(S+a_1) &= 0 \qquad \text{(A.12)}\\ \frac{(E[a|a\le a_1]-\gamma_0 E_0)\,\mu([\underline{a},a_1])+E[a|a\ge a_2]\,\mu([a_2,\bar{a}])}{(1-\gamma_0)\,\mu([\underline{a},a_1])+\mu([a_2,\bar{a}])} & \\ +\frac{b(1-\gamma_0)\,\mu([\underline{a},a_1])}{(1-\gamma_0)\,\mu([\underline{a},a_1])+\mu([a_2,\bar{a}])}-a_2 &= 0 \qquad \text{(A.13)}\end{aligned}$$

along with the additional restriction that E_0 is consistent with γ_0 and a_1. (At $\gamma_0 = 0$ this consistency condition is simply that E_0 lies in the interval $[\underline{a}, a_1]$. As γ_0 increases, the lower bound of this interval increases and the upper bound decreases, with both continuous in γ_0.)

Claim (i): There exists $\hat{a} \in [\bar{a}_1, \bar{a}]$ *such that for* $\gamma_0 = 0$ *and* $a_1 \in [\hat{a}, a^*(s_1)]$, *equation (A.11) has a unique solution in* a_2, *which we denote* $a_2(a_1)$. *Moreover,* $a_2(a_1)$ *is continuous in* a_1, *with* $a_2(\hat{a}) = \bar{a}$ *and* $a_2(a^*(s_1)) = a^*(s_1)$, *and* $a_2(a_1) \in (a_1, \bar{a})$ *for* $a_1 \in (\hat{a}, a^*(s_1))$.

Proof of Claim (i): The LHS of (A.11) is strictly decreasing in a_2, so if a solution exists it is continuous. By the definition of $a^*(s_1)$, the LHS of (A.4) is positive for all $a_1 \in [\underline{a}, a^*(s_1)]$, and strictly so except for at $a_1 = a^*(s_1)$. Consequently, the LHS of (A.11) evaluated at $a_2 = a_1$ is greater than $\frac{S-s_1+a_1}{1-\frac{s_1}{S+a_1}} - (S+a_1) = 0$, and strictly so except for at $a_1 = a^*(s_1)$. So at $a_1 = a^*(s_1)$ we have $a_2(a_1) = a_1$, while for $a_1 < a^*(s_1)$ any solution to (A.11) must strictly exceed a_1.

Evaluated at $a_1 = \bar{a}_1$ and $a_2 = \bar{a}$, the LHS of (A.11) is strictly positive, by (A.5). Evaluated at $a_1 = a^*(s_1)$ and $a_2 = \bar{a}$, the LHS of (A.11) is

$$\frac{S-s_1+a^*(s_1)}{1-\frac{s_1}{S+\bar{a}}} - (S+a^*(s_1)) = (S+\bar{a})\,\frac{S-s_1+a^*(s_1)}{S-s_1+\bar{a}} - (S+a^*(s_1)) < 0.$$

So by continuity, there exists $\hat{a} \in (\bar{a}_1, a^*(s_1))$ such that, for all $a_1 \in (\hat{a}, a^*(s_1))$, the LHS of (A.11) evaluated at $a_2 = \bar{a}$ is strictly negative, while at $a_1 = \hat{a}$ it is exactly zero.

Consequently, for $a_1 \in [\hat{a}, a^*(s_1)]$ equation (A.11) has a unique solution in a_2. The solution lies in the interval $[a_1, \bar{a}]$; equals a_1 when $a_1 = a^*(s_1)$; equals $\bar{a}$ when $a_1 = \hat{a}$; and lies in $(a_1, \bar{a})$ otherwise. This completes the proof of the Claim (i).

Claim (ii): There exists $\bar{\gamma}_0 > 0$ such that (A.12) has a unique solution, $E_0(a_1)$ say, when $\gamma_0 \in [0, \bar{\gamma}_0]$ and $a_1 \in [\hat{a}, a^*(s_1)]$. Moreover, the solution $E_0(a_1)$ is independent of γ_0, and is consistent with a_1 and γ_0.

Proof of Claim (ii): From Claim (i), (A.11) has a unique solution in a_2 when $\gamma_0 = 0$ and $a_1 \in [\hat{a}, a^*(s_1)]$. A necessary condition for (A.11) to have a solution is that the LHS of (A.11) is weakly negative at $a_2 = \bar{a}$. From (A.6), and the fact that $a_1 \geq \hat{a} \geq \bar{a}_1$, we know $\frac{1}{1+\frac{I-S}{S+\underline{a}+b}} < \frac{1}{1-\frac{s_1}{S+\bar{a}}} \frac{1}{1+\frac{I-S+s_1}{S-s_1+E[a|a\leq a_1]+b}}$. Hence the LHS of (A.12) is strictly negative when $E_0 = \underline{a}$. Conversely, the LHS of (A.12) is strictly positive when $E_0 = a_1$. Finally, noting that the LHS of (A.12) is strictly increasing in E_0 completes the proof of Claim (ii).

Since (A.11) is strictly decreasing in a_2, it follows from Claims (i) and (ii) that there exist continuous functions $a_2(a_1; \gamma_0), \hat{a}(\gamma_0), a^*(s_1; \gamma_0)$ of $\gamma_0 \in [0, \bar{\gamma}_0]$ such that for all $a_1 \in [\hat{a}(\gamma_0), a^*(s_1; \gamma_0)]$, the unique solution of (A.11) and (A.12) is $(a_2(a_1; \gamma_0), E_0(a_1))$; and moreover, $(a_2(a_1; 0), \hat{a}(0), a^*(s_1; 0)) = (a_2(a_1), \hat{a}, a^*(s_1))$. Moreover, it is straightforward to see that for any $\gamma_0 \in [0, \bar{\gamma}_0]$, $a_2(a_1; \gamma_0)$ is continuous in a_1.

At $\gamma_0 = 0$, the LHS of (A.13) evaluated at $(a_1, a_2, E_0) = (\hat{a}(\gamma_0), a_2(\hat{a}(\gamma_0); \gamma_0), E_0(\hat{a}(\gamma_0)))$ equals $E[a|a \leq a_1] + b - \bar{a}$, which is strictly negative by (2.7); while evaluated at $(a_1, a_2, E_0) = (a^*(s_1; \gamma_0), a_2(a^*(s_1; \gamma_0); \gamma_0), E_0(a^*(s_1; \gamma_0)))$ it equals $E[a] + b\Pr(a \leq a^*(s_1)) - a^*(s_1)$, which is strictly positive since we are in Case 2. By continuity, the same two statements also hold for γ_0 small but strictly positive. Fix any such γ_0. By continuity, there then exists exists $(a_1, a_2(a_1; \gamma_0), E_0(a_1))$ that satisfies equations (A.11)-(A.13). This completes the treatment of this case, and hence the proof.

Lemma A-1 *There is no equilibrium in which almost all firms invest.*

Proof of Lemma A-1: Suppose to the contrary that there is an equilibrium in which almost all firms invest. By assumption (2.6), it follows that there is a firm a' that invests and such that $a' > E[a]$ and

$$S + a' > \frac{I + a' + b}{1 + \frac{I-S}{S+E[a]+b}}.$$

Let (s_1, s_2) be the strategy of firm a', and let (P_1, P_2) be the associated prices. So the equilibrium condition for firm a' implies

$$\frac{S - s_1 - s_2 + a' + b}{1 - \frac{s_1}{P_1} - \frac{s_2}{P_2}} \geq S + a' > \frac{I + a' + b}{1 + \frac{I-S}{S+E[a]+b}} \geq \frac{S - s_1 - s_2 + a' + b}{1 - \frac{s_1+s_2}{S+E[a]+b}},$$

where the final inequality makes use of $-s_1 - s_2 \geq I - S$ (since firm a' invests) and $a' > E[a]$. Since any firm has the option of following strategy (s_1, s_2), it follows that the equilibrium payoff of an arbitrary firm a is at least

$$\frac{S - s_1 - s_2 + a + b}{1 - \frac{s_1}{P_1} - \frac{s_2}{P_2}} > \frac{S - s_1 - s_2 + a + b}{1 - \frac{s_1+s_2}{S+E[a]+b}}.$$

Consequently, the unconditional expected firm payoff is *strictly* greater than

$$E\left[\frac{S - s_1 - s_2 + a + b}{1 - \frac{s_1+s_2}{S+E[a]+b}}\right] = S + E[a] + b.$$

But this violates investor rationality (formally, it violates (2.4)), giving a contradiction and completing the proof.

Corollary A-2 *In any equilibrium, there is a non-empty interval* $[\bar{a} - \delta, \bar{a}]$ *of firms that do not invest.*

Proof of Corollary A-2: Immediate from Corollary 1 and Lemma A-1.

Proof of Proposition 4:

Claim: There is a non-empty interval $[\bar{a} - \delta, \bar{a}]$ of firms that make strictly positive profits, i.e., obtain a payoff strictly in excess of $S + a$.

Proof of Claim: Suppose to the contrary that this is not the case. i.e., that one can find a firm a arbitrarily close to $\bar{a}$ that has a payoff of $S + a$.

Consider any repurchase offer $s_1 > 0$. If $P_1(s_1) < S + \bar{a}$, then by supposition one can find a firm that could strictly increase its payoff by repurchasing s_1, a contradiction. Hence $P_1(s_1) \geq S + \bar{a}$. So from (2.4), the beliefs associated with s_1 must be such that

$$E\left[a + b\mathbf{1}_{S-s_1-s_2 \geq I} | s_1\right] \geq \bar{a}. \tag{A.14}$$

There are two separate cases, which we deal with in turn. In the first case, $E[a|s_1] = \bar{a}$. By the NDOC restriction on beliefs, it follows that if the firm offers $s_2 = S - s_1 - I < 0$ so that investment is possible, the firm's equilibrium payoff (2.5) is

$$\frac{I + a + b}{\left(1 - \frac{s_1}{S+\bar{a}+b}\right)\left(1 - \frac{S-s_1-I}{S-s_1+\bar{a}+b}\right)} = \frac{I + a + b}{I + \bar{a} + b}(S + \bar{a} + b) > S + a,$$

where the inequality follows from $I > S$ and $b > 0$. Consequently, any firm a is able to achieve a payoff strictly in excess of $S + a$ by first repurchasing s_1 and then at date 2 issuing enough shares to fund investment I. The contradiction completes the proof of the claim.

The remainder of the proof deals with the second case, in which $E\left[a|s_1\right] < \bar{a}$. In this case, inequality (A.14) implies that $\Pr\left(s_2 \text{ s.t. } S - s_1 - s_2 \geq I|s_1\right) > 0$, and hence that there exists s_2 with $S - s_1 - s_2 \geq I$ such that $E\left[a + b|s_1, s_2\right] \geq \bar{a}$. So by (2.3), firm a's payoff from playing (s_1, s_2) is weakly greater than

$$\frac{S - s_1 - s_2 + a + b}{\left(1 - \frac{s_1}{P_1(s_1)}\right)\left(1 - \frac{s_2}{S - s_1 + \bar{a}}\right)}.$$

By the equilibrium condition, the unconditional expected equilibrium payoff of a firm is at least

$$\frac{S - s_1 - s_2 + E\left[a\right] + b}{\left(1 - \frac{s_1}{P_1(s_1)}\right)\left(1 - \frac{s_2}{S - s_1 + \bar{a}}\right)} \geq \frac{I + E\left[a\right] + b}{\left(1 - \frac{s_1}{P_1(s_1)}\right)\left(\frac{I + \bar{a}}{S - s_1 + \bar{a}}\right)}, \tag{A.15}$$

where the inequality follows from (2.7) and $S - s_1 - s_2 \geq I$.

Since $P_1\left(s_1\right)$ is bounded below by $S + \underline{a}$, the term $\frac{s_1}{P_1(s_1)}$ approaches 0 as s_1 approaches 0. Consequently, the limiting value of the RHS of (A.15) is

$$\left(I + E\left[a\right] + b\right)\frac{S + \bar{a}}{I + \bar{a}}. \tag{A.16}$$

Because the above argument holds for any initial choice of $s_1 > 0$, the unconditional expected equilibrium payoff of a firm is at least (A.16). Moreover, by (2.7), expression (A.16) is itself strictly greater than $S + E\left[a\right] + b$. But this violates investor rationality (formally, it violates (2.4)), giving a contradiction.

Completing the proof: By Corollary A-2 and the Claim, there exists $\delta' > 0$ such that all firms in $\left[\bar{a} - \delta', \bar{a}\right]$ make strictly positive profits and do not invest. Let $\varepsilon > 0$ be the minimum profits made by a firm in this interval. (Note that the minimum is well-defined because a firm's equilibrium payoff is continuous in a: if this is not the case, there is a profitable deviation for some a.) Then choose $\delta \in \left(0, \delta'\right)$ sufficiently small such that, for all $a \in [\bar{a} - \delta, \bar{a}]$, $a + \varepsilon > \bar{a}$,$a + b > \bar{a}$, and $\left(S + a\right)\frac{\bar{a}}{a} < S + a + \varepsilon$. To complete the proof, we show all firms in $[\bar{a} - \delta, \bar{a}]$ repurchase, and make strictly positive profits from the repurchase transaction.

Suppose to the contrary that there exists some firm $a \in [\bar{a}-\delta, \bar{a}]$ that either does not repurchase, or else makes weakly negative profits from the repurchase: formally, either $s_1(a) \leq 0$, or $s_1(a) > 0$ with $P_1(s_1(a)) \geq S + a$; and either $s_2(a) \leq 0$, or $s_2(a) > 0$ with $P_2(s_1(a), s_2(a)) \geq \frac{S-s_1(a)+a}{1-\frac{s_1(a)}{P_1(s_1(a))}}$.

We first show that firm a's payoff is bounded above by

$$\frac{S-s_1(a)+\bar{a}}{1-\frac{s_1(a)}{P_1(s_1(a))}}. \tag{A.17}$$

If $s_2(a) > 0$ this is immediate. Otherwise, (2.5) and the fact that by (2.3) (and using $a \leq \bar{a}$ and the firm does not invest) $P_2(s_1(a), s_2(a)) \leq \frac{S-s_1(a)+\bar{a}}{1-\frac{s_1(a)}{P_1(s_1(a))}}$ together imply that the firm's payoff is bounded above by

$$\frac{S-s_1(a)-s_2(a)+a}{\left(1-\frac{s_1(a)}{P_1(s_1(a))}\right)\left(1-\frac{s_2(a)}{S-s_1(a)+\bar{a}}\right)} = \frac{S-s_1(a)+\bar{a}}{1-\frac{s_1(a)}{P_1(s_1(a))}}\frac{S-s_1(a)-s_2(a)+a}{S-s_1(a)-s_2(a)+\bar{a}},$$

which is below expression (A.17).

If $s_1(a) > 0$, expression (A.17) is in turn bounded above by

$$\frac{S-s_1(a)+\bar{a}}{1-\frac{s_1(a)}{S+a}} = (S+a)\frac{S-s_1(a)+\bar{a}}{S-s_1(a)+a} \leq (S+a)\frac{\bar{a}}{a}.$$

But this is less than $S+a+\varepsilon$, a contradiction.

Consequently, it must be the case that $s_1(a) \leq 0$. Observe that if $P_1(s_1(a)) \leq S+a+\varepsilon$, from (A.17) and the fact that $a+\varepsilon > \bar{a}$, firm a's payoff is bounded above by

$$(S+a+\varepsilon)\frac{S-s_1(a)+\bar{a}}{S-s_1(a)+a+\varepsilon} < S+a+\varepsilon,$$

which again is a contradiction. Hence $P_1(s_1(a)) > S+a+\varepsilon > S+\bar{a}$. It then follows from (2.4) that there must exist s_2 such that $S - s_1(a) - s_2 \geq I$ and

$$E\left[S+a+b\mathbf{1}_{S-s_1-s_2\geq I} | s_1 = s_1(a), s_2\right] \geq P_1(s_1(a)). \tag{A.18}$$

On the one hand, the equilibrium payoff of firm $a \in A$ is—using (A.17), together with $P_1(s_1(a)) > S+\bar{a}$—bounded above by

$$P_1(s_1(a))\frac{S-s_1(a)+\bar{a}}{P_1(s_1(a))-s_1(a)} \leq P_1(s_1(a))\frac{I+\bar{a}}{P_1(s_1(a))+I-S}.$$

On the other hand, the payoff to firm a to instead deviating and using strategy $(s_1(a), s_2)$, where s_2 is as above, is bounded below by

$$\frac{S - s_1(a) - s_2 + a + b}{1 - \frac{s_1(a)+s_2}{P_1(s_1(a))}} \geq \min\left\{P_1(s_1(a)), P_1(s_1(a)) \frac{I + a + b}{P_1(s_1(a)) + I - S}\right\}.$$

Since this is strictly greater than the upper bound on firm a's equilibrium payoff, firm a has the incentive to deviate. This contradicts the equilibrium condition, and completes the proof.

Proof of Proposition 5:

Perturbation (I), exogenous probability of no capital market transaction at date 1*:*

Under this perturbation, if a firm undertakes transactions s_1 and $s_2 = -(I - S + s_1)$ at date 2, so that it is just able to invest, the date 2 price is

$$P_2 = \frac{S - s_1 + E[a|s_1, s_2] + b}{1 - \frac{s_1}{P_1}}$$

and the firm's payoff is

$$\frac{I + a + b}{1 - \frac{s_1}{P_1} + \frac{I-S+s_1}{P_2}} = \frac{1}{1 - \frac{s_1}{P_1}} \frac{I + a + b}{1 + \frac{I-S+s_1}{S-s_1+E[a|s_1,s_2]+b}}. \tag{A.19}$$

Preliminaries: Consider an open set K of $s_1 \in (0, \min\{\bar{s}_1, \frac{S}{2}\}]$ sufficiently small such that

$$\max\left\{\frac{I - S + s_1}{S - s_1 + E\left[a|a < a^*(\frac{S}{2})\right] + b}, \frac{I - S + s_1}{S - s_1 + E[a|a < \bar{a}_1] + b}\right\} \leq \frac{I - S}{S + \underline{a} + b}. \tag{A.20}$$

We next show that it is possible to pick some $s_1 \in K$ such that $S + a^*(s_1) \neq S + E[a] + b\Pr(a \leq a^*(s_1))$. Suppose otherwise,

$$S + a^*(s_1) = S + E[a] + b\Pr(a \leq a^*(s_1)) \tag{A.21}$$

holds identity on K. For any s_1 such that $a^*(s_1)$ is continuous, one can choose a sequence of $s_{1,n} \in K$ such that $\lim_{n\to\infty} s_{1,n} \to s_1$. By continuity, $a^*(s_{1,n}) \to a^*(s_1)$. Notice that (A.21) holds for all $s_{1,n}$ as well as s_1. Therefore, it must be $a^*(s_{1,n}) - a^*(s_1) = b[\Pr(a \leq a^*(s_{1,n})) - \Pr(a \leq a^*(s_1))]$, which implies $\frac{\Pr(a \leq a^*(s_{1,n})) - \Pr(a \leq a^*(s_1))}{a^*(s_{1,n}) - a^*(s_1)} = \frac{1}{b}$. Taking the limit, we have $f(a^*(s_1)) = \frac{1}{b}$ for any continuity point s_1. Because $a^*(s_1)$ is a strictly decreasing function, so it can have at most countably many discontinuities. Together, this argument suggests that $f(a) = \frac{1}{b}$ for uncountably many a. A contradiction. So we can choose an $s_1 \in K$ such that (A.21) does not hold.

Given s_1, we explicitly construct an equilibrium. There are two cases, corresponding to whether $S+a^*(s_1)$ is strictly larger or strictly smaller than $S+E[a]+b\Pr(a \leq a^*(s_1))$. In the first case, all firms repurchase s_1 at date 1, and then a strict subset of firms issue $I+s_1-S$ at date 2. In the second case, some firms repurchase s_1 at date 1, with a strict subset then issuing $I+s_1-S$ at date 2; while other firms do nothing at date 1, with a strict subset then issuing $I-S$ at date 2. In both cases, any off-equilibrium repurchase offer triggers investor beliefs that the firm is type $\bar{a}$, while any off-equilibrium issue offer triggers beliefs that the firm is type $\underline{a}$.

Formally, proof shows that: For any set of parameter values, for all $\alpha > 0$ sufficiently small, an equilibrium exists.

Case 1: $S + a^*(s_1) > S + E[a] + b\Pr(a \leq a^*(s_1))$.

We construct an equilibrium in which: At date 1, all active firms other than some subset A_0 repurchase an amount s_1. At date 2, firms that repurchased at date 1 and have $a \leq a_1$issue $I - S + s_1$ and invest, while firms that did not repurchase at date 1 and have $a \leq a_3$ issue $I - S$ and invest. The set A_0 and cutoffs a_1 and a_3 satisfy $A_0 \subset [\underline{a}, a_1] \subset [\underline{a}, a_3]$.

The date 1 repurchase price is

$$P_1 = S + E[a|a \notin A_0] + b\Pr(a \leq a_1 | a \notin A_0)$$

and the equilibrium indifference conditions are

$$\frac{1}{1-\frac{s_1}{P_1}} \frac{I + a_1 + b}{1 + \frac{I-S+s_1}{S-s_1+E[a|a\in[\underline{a},a_1]\setminus A_0]+b}} = \frac{S - s_1 + a_1}{1-\frac{s_1}{P_1}}$$

$$\left(1-\frac{s_1}{P_1}\right)\left(1+\frac{I-S+s_1}{S-s_1+E[a|a\in[\underline{a},a_1]\setminus A_0]+b}\right) =$$

$$1 + \frac{I-S}{S+\frac{\alpha\mu([\underline{a},a_3])E[a|a\leq a_3]+(1-\alpha)\mu(A_0)E[a|A_0]}{\alpha\mu([\underline{a},a_3])+(1-\alpha)\mu(A_0)}+b}$$

$$\frac{I + a_3 + b}{1 + \frac{I-S}{S+\frac{\alpha\mu([\underline{a},a_3])E[a|a\leq a_3]+(1-\alpha)\mu(A_0)E[a|A_0]}{\alpha\mu([\underline{a},a_3])+(1-\alpha)\mu(A_0)}+b}} = S + a_3.$$

Respectively, these three conditions say that: firm a_1 is indifferent between repurchase-issue and repurchase-do-nothing; firms are indifferent between repurchase-issue and direct-issue; firm a_3 is indifferent between direct issue and do-nothing.

Notationally, define $\gamma_0 \equiv \frac{\mu(A_0)}{\mu([\underline{a},a_1])}$ and $E_0 \equiv E[a|A_0]$. Note that $E[a|a \notin A_0] = \frac{E[a]-\gamma_0\mu([\underline{a},a_1])E_0}{1-\gamma_0\mu([\underline{a},a_1])}$ and $\Pr(a \le a_1|a \notin A_0) = \frac{(1-\gamma_0)\mu([\underline{a},a_1])}{1-\gamma_0\mu([\underline{a},a_1])}$ and $\mu([\underline{a},a_1]\setminus A_0) = (1-\gamma_0)\mu$ $([\underline{a},a_1])$ and $E[a|a \in [\underline{a},a_1]\setminus A_0] = \frac{E[a|a\le a_1]-\gamma_0 E_0}{1-\gamma_0}$.

Hence

$$P_1(a_1,E_0) = S + \frac{E[a] - \gamma_0\mu([\underline{a},a_1])E_0 + b(1-\gamma_0)\mu([\underline{a},a_1])}{1-\gamma_0\mu([\underline{a},a_1])} \tag{A.22}$$

and the equilibrium indifference conditions are

$$\frac{I+a_1+b}{1+\frac{I-S+s_1}{S-s_1+\frac{E[a|a\le a_1]-\gamma_0 E_0}{1-\gamma_0}+b}} = S - s_1 + a_1 \tag{A.23}$$

$$\begin{aligned}\left(1-\frac{s_1}{P_1(a_1,E_0)}\right)\left(1+\frac{I-S+s_1}{S-s_1+\frac{E[a|a\le a_1]-\gamma_0 E_0}{1-\gamma_0}+b}\right) &= \\ 1+\frac{I-S}{S+\frac{\alpha\mu([\underline{a},a_3])E[a|a\le a_3]+(1-\alpha)\gamma_0\mu([\underline{a},a_1])E_0}{\alpha\mu([\underline{a},a_3])+(1-\alpha)\gamma_0\mu([\underline{a},a_1])}+b}&\end{aligned} \tag{A.24}$$

$$\frac{I+a_3+b}{1+\frac{I-S}{S+\frac{\alpha\mu([\underline{a},a_3])E[a|a\le a_3]+(1-\alpha)\gamma_0\mu([\underline{a},a_1])E_0}{\alpha\mu([\underline{a},a_3])+(1-\alpha)\gamma_0\mu([\underline{a},a_1])}+b}} = S + a_3. \tag{A.25}$$

For any $\gamma_0 > 0$, let $E_0(a_1;\gamma_0)$ be the value of E_0 that solves (A.23) given a_1. (The LHS of (A.23) is strictly decreasing in E_0 for $\gamma_0 > 0$, so if a solution exists, it is unique.) Note that $E_0(a^*(s_1);\gamma_0) = E[a|a \le a^*(s_1)]$. Recall that $a^*(s_1)$ lies strictly between $\underline{a}$ and $\bar{a}$. Hence, for $\gamma_0 > 0$, the LHS of (A.23) strictly exceeds the RHS at $a_1 = a^*(s_1)$ and $E_0 = \frac{\underline{a}+E[a|a\le a^*(s_1)]}{2}$. Define $\bar{a}^*(\gamma_0) \ge a^*(s_1)$ by

$$\bar{a}^*(\gamma_0) = \max_{a_1}\left\{\frac{I+\tilde{a}_1+b}{1+\frac{I-S+s_1}{S-s_1+\frac{E[a|a\le\tilde{a}_1]-\gamma_0\frac{\underline{a}+E[a|a\le\tilde{a}_1]}{2}}{1-\gamma_0}+b}} - (S-s_1+\tilde{a}_1) \ge 0 \text{ for all } \tilde{a}_1 \in [a^*(s_1),a_1]\right\}.$$

Note that, by the definition of $a^*(s_1)$, $\bar{a}^*(0) = a^*(s_1)$. In addition, because the expression in the above definition is strictly increasing in γ_0, so $\bar{a}^*(\gamma_0) > a^*(s_1)$ for $\gamma_0 > 0$. Moreover, $\bar{a}^*(\gamma_0) \to a^*(s_1)$ as $\gamma_0 \to 0$.

For the remainder of the proof, fix $\gamma_0 > 0$ sufficiently small such that $\bar{a}^*(\gamma_0) \leq \bar{a}$, and moreover (and using the fact we are in Case 1), such that for all $a_1 \in [a^*(s_1), \bar{a}^*(\gamma_0)]$,

$$S + a_1 > S + \frac{E[a] - \gamma_0 \mu([\underline{a}, a_1])\underline{a} + b(1-\gamma_0)\mu([\underline{a}, a_1])}{1 - \gamma_0 \mu([\underline{a}, a_1])}. \tag{A.26}$$

Having fixed $\gamma_0 > 0$, we omit the γ_0 arguments in $\bar{a}^*$ and $E_0(a_1)$ for the remainder of the proof.

By the definition of $a^*(s_1)$, the LHS of (A.23) is strictly less than the RHS for $a_1 > a^*(s_1)$ and $E_0 = E[a|a \leq a_1]$. Hence $\frac{\underline{a}+E[a|a\leq a_1]}{2} \leq E_0(a_1) < E[a|a \leq a_1]$ for $a_1 \in (a^*(s_1), \bar{a}^*]$, with $E_0(\bar{a}^*) = \frac{\underline{a}+E[a|a\leq a^*(s_1)]}{2}$. Therefore the function $E_0(a_1)$ is well-defined and continuous over $[a^*(s_1), \bar{a}^*]$.

Define $a_3(a_1; \alpha)$ as the value of a_3 that solves (A.25), given a_1 and $E_0 = E_0(a_1)$. Observe that the LHS of (A.25) strictly exceeds the RHS at $\alpha = 0$, $E_0 \geq \underline{a}$ and $a_3 = \underline{a}$. Moreover, by assumption (2.6), the LHS is strictly less than the RHS at $\alpha = 0$, $E_0 \leq E[a]$ and $a_3 = \bar{a}$. Hence for $a_1 \in [a^*(s_1), \bar{a}^*]$, $E_0 = E_0(a_1)$, and $\alpha = 0$, there is a unique value of a_3 solving (A.25). By continuity, the same is true $a_1 \in [a^*(s_1), \bar{a}^*]$, $E_0 = E_0(a_1)$, and α sufficiently small. Note that $a_3(a_1; \alpha)$ is continuous in both a_1 and α.

For use below, we next establish that $a_3(a^*(s_1); 0) > a^*(s_1)$. By definition,

$$\frac{I + a^*(s_1) + b}{1 + \frac{I-S+s_1}{S-s_1+E[a|a\leq a^*(s_1)]+b}} = S - s_1 + a^*(s_1).$$

Since $a^*(s_1) > E[a|a \leq a^*(s_1)]$, it is straightforward to show that

$$\frac{I + a^*(s_1) + b}{1 + \frac{I-S}{S+E[a|a\leq a^*(s_1)]+b}} > S + a^*(s_1).$$

By definition, this last inequality is at equality if $a^*(s_1)$ is replaced by $a_3(a^*(s_1); 0)$. Consequently, $a_3(a^*(s_1); 0) > a^*(s_1)$.

We now turn to (A.24). Since $E_0(\bar{a}^*) = \underline{a} < E[a|a \leq \bar{a}^*]$, and $E[a|a \leq \bar{a}_1] < E[a|a \leq a^*(s_1)] < E[a|a \leq \bar{a}^*]$,

$$\frac{I - S + s_1}{S - s_1 + \frac{E[a|a\leq\bar{a}^*]-\gamma_0 E_0(\bar{a}^*)}{1-\gamma_0} + b} < \frac{I - S + s_1}{S - s_1 + E[a|a \leq \bar{a}^*] + b} < \frac{I - S + s_1}{S - s_1 + E[a|a \leq \bar{a}_1] + b}.$$

Since certainly $1-\frac{s_1}{P_1(a_1,E_0)}<1$, it follows from (A.20) that the LHS of (A.24) is strictly less than the RHS at $(a_1,E_0)=(\bar{a}^*,E_0(\bar{a}^*))$ and $\alpha=0$.

Next, we show that the LHS of (A.24) strictly exceeds the RHS at $(a_1,E_0)=(a^*(s_1),E_0(a^*(s_1)))$ and $\alpha=0$. The proof is by contradiction. Suppose to the contrary that, at $(a_1,E_0)=(a^*(s_1),E_0(a^*(s_1)))$,

$$\left(1-\frac{s_1}{P_1(a_1,E_0)}\right)\left(1+\frac{I-S+s_1}{S-s_1+\frac{E[a|a\leq a_1]-\gamma_0 E_0}{1-\gamma_0}+b}\right)\leq 1+\frac{I-S}{S+E_0+b}. \tag{A.27}$$

First, we show that at $(a_1,E_0)=(a^*(s_1),E_0(a^*(s_1)))$,

$$P_1(a_1,E_0)<S+a_1. \tag{A.28}$$

To establish (A.28), note that because $(a_1,E_0)=(a^*(s_1),E_0(a^*(s_1)))$ satisfies (A.23), the combination of (A.23) and (A.27) implies

$$\frac{S-s_1+a_1}{1-\frac{s_1}{P_1(a_1,E_0)}}\geq\frac{I+a_1+b}{1+\frac{I-S}{S+E_0+b}}.$$

Substituting in for $a_3(a_1)$, and using the earlier observation that $a_3(a^*(s_1);0)>a^*(s_1)=a_1$, we have

$$\frac{S-s_1+a_1}{1-\frac{s_1}{P_1(a_1,E_0)}}\geq\frac{(I+a_1+b)(S+a_3(a^*(s_1);0))}{I+a_3(a^*(s_1);0)+b}>S+a_1,$$

which is equivalent to (A.28).

Second, straightforward algebra implies

$$\begin{aligned}P_1(a_1,E_0) &= \frac{(1-\gamma_0)\mu([\underline{a},a_1])}{1-\gamma_0\mu([\underline{a},a_1])}\frac{I+\frac{E[a|a\leq a_1]-\gamma_0 E_0}{1-\gamma_0}+b}{\left(1-\frac{s_1}{P_1}\right)\left(1+\frac{I-S+s_1}{S-s_1+\frac{E[a|a\leq a_1]-\gamma_0 E_0}{1-\gamma_0}+b}\right)}\\ &+ \frac{1-\mu([\underline{a},a_1])}{1-\gamma_0\mu([\underline{a},a_1])}\frac{S-s_1+E[a|a\geq a_1]}{1-\frac{s_1}{P_1}}.\end{aligned}$$

Then (A.27) and (A.28) imply that at $(a_1, E_0) = (a^*(s_1), E_0(a^*(s_1)))$,

$$\begin{aligned} P_1(a_1, E_0) &> \frac{(1-\gamma_0)\mu([\underline{a}, a_1])}{1-\gamma_0\mu([\underline{a}, a_1])} \frac{I + \frac{E[a|a\leq a_1]-\gamma_0 E_0}{1-\gamma_0} + b}{1+\frac{I-S}{S+E_0+b}} + \\ &\quad \frac{1-\mu([\underline{a}, a_1])}{1-\gamma_0\mu([\underline{a}, a_1])}(S + E[a|a \geq a_1]) \\ &= \frac{(1-\gamma_0)\mu([\underline{a}, a_1])}{1-\gamma_0\mu([\underline{a}, a_1])}(S + E[a|a\leq a_1] + b) + \\ &\quad \frac{1-\mu([\underline{a}, a_1])}{1-\gamma_0\mu([\underline{a}, a_1])}(S + E[a|a \geq a_1]) \\ &= S + \frac{E[a] - \gamma_0\mu([\underline{a}, a_1]) E[a|a\leq a_1] + (1-\gamma_0)\mu([\underline{a}, a_1]) b}{1-\gamma_0\mu([\underline{a}, a_1])} \\ &= P_1(a_1, E_0), \end{aligned}$$

where the equality is simply (A.22). The contradiction completes the proof that the LHS of (A.24) strictly exceeds the RHS at $(a_1, E_0) = (a^*(s_1), E_0(a^*(s_1)))$ and $\alpha = 0$.

By continuity, it follows that, for $\alpha = 0$, there exists $a_1^{**} \in (a^*(s_1), \bar{a}^*)$ such that $(a_1, E_0, a_3) = (a_1^{**}, E_0(a_1^{**}), a_3(a_1^{**}; \alpha))$ satisfies the required conditions (A.23), (A.24) and (A.25). By continuity, the same statement holds true for all $\alpha > 0$ sufficiently small.

Finally, to complete the proof of Case 1, we must show that all firms prefer the equilibrium action described to doing nothing. It suffices to show this for firm a_1. We must show that firm a_1 indeed profits from repurchasing its own stock, i.e., $S + a_1 > P_1(a_1, E_0)$. This follows from (A.26), together with the fact that P_1 satisfies (A.22), $E_0 > \gamma_0$, and $a_1 \in [a^*(s_1), \bar{a}^*]$.

Case 2: $S + a^*(s_1) < S + E(a) + b\Pr(a \leq a^*(s_1))$.

In this case, we show there exists a_1, a_2 along with a partition A_0, A_1 of $[\underline{a}, a_1]$, such that the following is an equilibrium: At date 1 firms $A_1 \cup [a_2, \bar{a}]$ repurchase s_1, while other firms do nothing; and at date 2 firms A_1 issue $I - S + s_1$ and invest, firms A_0 directly issue $I - S$ (without previously repurchasing), along with inactive firms $[\underline{a}, a_1]$, and the remaining firms do nothing.

In such an equilibrium, the date 1 repurchase price P_1 and date 2 issue price P_2

following repurchase are

$$\begin{aligned} P_1 &= S + \frac{E[a|A_1]\,\mu(A_1) + E[a|a \geq a_2]\,\mu([a_2,\bar{a}]) + b\mu(A_1)}{\mu(A_1) + \mu([a_2,\bar{a}])} \\ P_2 &= \frac{S - s_1 + E[a|A_1] + b}{1 - \frac{s_1}{P_1}}. \end{aligned}$$

We show that there exist $a_1, a_2 \in [\underline{a}, \bar{a}]$, together with a partition A_0, A_1 of $[\underline{a}, a_1]$, that solve the following system of equations (where P_1 is as defined immediately above):

$$\begin{aligned} S + a_1 &= \frac{1}{1 - \frac{s_1}{P_1}} \frac{1}{1 + \frac{I - S + s_1}{S - s_1 + E[a|A_1] + b}} (I + a_1 + b) && \text{(A.29)} \\ S + a_2 &= P_1 && \text{(A.30)} \\ \frac{1}{1 - \frac{s_1}{P_1}} \frac{1}{1 + \frac{I - S + s_1}{S - s_1 + E[a|A_1] + b}} &= \frac{1}{1 + \frac{I - S}{S + \frac{\alpha\mu([\underline{a},a_1])E[a|a \leq a_1] + (1-\alpha)\mu(A_0)E[a|A_0]}{\alpha\mu([\underline{a},a_1]) + (1-\alpha)\mu(A_0)} + b}} && \text{(A.31)} \end{aligned}$$

Condition (A.29) states that firm a_1 is indifferent between repurchase-issue and do-nothing. Condition (A.30) states that firm a_2 is indifferent between repurchase-do-nothing and do-nothing. Condition (A.31) states that firms are indifferent between repurchasing and then issuing, and issuing directly.

Notationally, define $\gamma_0 \equiv \frac{\mu(A_0)}{\mu([\underline{a},a_1])}$ and $E_0 \equiv E[a|A_0]$, and note that $E[a|A_1] = \frac{E[a|a \leq a_1] - \gamma_0 E_0}{1 - \gamma_0}$. The system of equations (A.29)-(A.31) has a solution if and only if the following system has a solution in γ_0, E_0, a_1 and a_2:

$$\begin{aligned} \frac{1}{1 - \frac{s_1}{S + a_2}} \frac{I + a_1 + b}{1 + \frac{I - S + s_1}{S - s_1 + \frac{E[a|a \leq a_1] - \gamma_0 E_0}{1 - \gamma_0} + b}} - (S + a_1) &= 0 && \text{(A.32)} \\ \frac{I + a_1 + b}{1 + \frac{I - S}{S + \frac{\alpha E[a|a \leq a_1] + (1-\alpha)\gamma_0 E_0}{\alpha + (1-\alpha)\gamma_0} + b}} - (S + a_1) &= 0 && \text{(A.33)} \\ \frac{(E[a|a \leq a_1] - \gamma_0 E_0)\,\mu([\underline{a}, a_1]) + E[a|a \geq a_2]\,\mu([a_2,\bar{a}])}{(1 - \gamma_0)\,\mu([\underline{a}, a_1]) + \mu([a_2,\bar{a}])} & \\ + \frac{b(1 - \gamma_0)\,\mu([\underline{a}, a_1])}{(1 - \gamma_0)\,\mu([\underline{a}, a_1]) + \mu([a_2,\bar{a}])} - a_2 &= 0 && \text{(A.34)} \end{aligned}$$

along with the additional restriction that E_0 is consistent with γ_0 and a_1. (At $\gamma_0 = 0$ this consistency condition is simply that E_0 lies in the interval $[\underline{a}, a_1]$. As γ_0 increases, the lower bound of this interval increases and the upper bound decreases, with both continuous in γ_0.) Note that equations (A.32) and (A.34) are simple rewritings of (A.29) and (A.30), while (A.33) is obtained from combining (A.29) and (A.31).

Claim (i): There exists $\hat{a} \in [\bar{a}_1, \bar{a}]$ *such that for* $\gamma_0 = 0$ *and* $a_1 \in [\hat{a}, a^*(s_1)]$*, equation (A.32) has a unique solution in* a_2*, which we denote* $a_2(a_1)$*. Moreover,* $a_2(a_1)$ *is continuous in* a_1*, with* $a_2(\hat{a}) = \bar{a}$ *and* $a_2(a^*(s_1)) = a^*(s_1)$*, and* $a_2(a_1) \in (a_1, \bar{a})$ *for* $a_1 \in (\hat{a}, a^*(s_1))$*.*

Proof of Claim (i): The LHS of (A.32) is strictly decreasing in a_2, so if a solution exists it is continuous. By the definition of $a^*(s_1)$, the LHS of (A.4) is positive for all $a_1 \in [\underline{a}, a^*(s_1)]$, and strictly so except for at $a_1 = a^*(s_1)$. Consequently, the LHS of (A.32) evaluated at $a_2 = a_1$ is greater than $\frac{S-s_1+a_1}{1-\frac{s_1}{S+a_1}} - (S+a_1) = 0$, and strictly so except for at $a_1 = a^*(s_1)$. So at $a_1 = a^*(s_1)$ we have $a_2(a_1) = a_1$, while for $a_1 < a^*(s_1)$ any solution to (A.32) must strictly exceed a_1.

Evaluated at $a_1 = \bar{a}_1$ and $a_2 = \bar{a}$, the LHS of (A.32) is strictly positive, by (A.5). Evaluated at $a_1 = a^*(s_1)$ and $a_2 = \bar{a}$, the LHS of (A.32) is

$$\frac{S - s_1 + a^*(s_1)}{1 - \frac{s_1}{S+\bar{a}}} - (S + a^*(s_1)) = (S + \bar{a}) \frac{S - s_1 + a^*(s_1)}{S - s_1 + \bar{a}} - (S + a^*(s_1)) < 0.$$

So by continuity, there exists $\hat{a} \in (\bar{a}_1, a^*(s_1))$ such that, for all $a_1 \in (\hat{a}, a^*(s_1))$, the LHS of (A.32) evaluated at $a_2 = \bar{a}$ is strictly negative, while at $a_1 = \hat{a}$ it is exactly zero.

Consequently, for $a_1 \in [\hat{a}, a^*(s_1)]$ and $\gamma_0 = 0$, equation (A.32) has a unique solution in a_2. The solution lies in the interval $[a_1, \bar{a}]$; equals a_1 when $a_1 = a^*(s_1)$; equals $\bar{a}$ when $a_1 = \hat{a}$; and lies in $(a_1, \bar{a})$ otherwise. This completes the proof of the Claim (i).

Claim (ii): There exist constants $\bar{\gamma}_0 > 0$ and κ such that: If $\gamma_0 \le \bar{\gamma}_0$, $\alpha \le \frac{\gamma_0}{\kappa+\gamma_0}$, $a_1 \in [\hat{a}, a^*(s_1)]$, then there exists a unique $E_0(a_1; \gamma_0, \alpha)$ that solves (A.33), and moreover, $E_0(a_1; \gamma_0, \alpha)$is consistent with a_1 and γ_0.

Proof of Claim (ii): Fix $a_1 \in [\hat{a}, a^*(s_1)]$. As a preliminary, note that, from Claim (i), (A.32) has a unique solution in a_2 when $\gamma_0 = 0$ and $a_1 \in [\hat{a}, a^*(s_1)]$. A necessary condition for (A.32) to have a solution is that the LHS of (A.32) is weakly negative at $a_2 = \bar{a}$. From (A.20), and the fact that $a_1 \ge \hat{a} \ge \bar{a}_1$, we know $\frac{1}{1+\frac{I-S}{S+\underline{a}+b}} < \frac{1}{1-\frac{s_1}{S+\bar{a}}} \frac{1}{1+\frac{I-S+s_1}{S-s_1+E[a|a\le a_1]+b}}$. Consequently,

$$\frac{I + a_1 + b}{1 + \frac{I-S}{S+\underline{a}+b}} - (S + a_1) < 0.$$

From this inequality, the LHS of (A.33) is strictly negative when $\alpha = 0$, $\gamma_0 > 0$ and $E_0 = \underline{a}$. Conversely, the LHS of (A.33) is strictly positive when $\alpha = 0$, $\gamma_0 > 0$ and $E_0 = a_1$. The LHS of (A.33) is strictly increasing in E_0. Consequently, for $\alpha = 0$ and any γ_0, there is a unique solution E_0 to (A.33).

Moreover, there exists $\bar{\gamma}_0$ (independent of a_1) such that, for $\gamma_0 \leq \bar{\gamma}_0$, the solution E_0 is consistent with a_1 and γ_0.

By continuity, there exists κ such that the same statement is true provided $\frac{1-\alpha}{\alpha}\gamma_0 \geq \kappa$, i.e., $\alpha \leq \frac{\gamma_0}{\kappa+\gamma_0}$, cmpleting the proof of Claim (ii).

Since (A.32) is strictly decreasing in a_2, it follows from Claims (i) and (ii) that there exist functions $a_2\left(a_1;\gamma_0,\alpha\right), \hat{a}\left(\gamma_0\right), a^*\left(s_1;\gamma_0\right)$, continuous in γ_0 and α, such that for all $a_1 \in \left[\hat{a}\left(\gamma_0\right), a^*\left(s_1;\gamma_0\right)\right]$, the unique solution of (A.32) and (A.33) is $(a_2\left(a_1;\gamma_0,\alpha\right),$ $E_0\left(a_1;\gamma_0,\alpha\right))$; and moreover, $\lim_{\gamma_0\to 0}\left(a_2\left(a_1;\gamma_0,0\right), \hat{a}\left(\gamma_0\right), a^*\left(s_1;\gamma_0\right)\right) = (a_2\left(a_1\right), \hat{a},$ $a^*\left(s_1\right))$. Define $a_2\left(a_1;0,0\right) \equiv \lim_{\gamma_0\to 0} a_2\left(a_1;\gamma_0,0\right)$. Moreover, it is straightforward to see that for any $\gamma_0 \in [0,\bar{\gamma}_0]$, $a_2\left(a_1;\gamma_0,0\right)$ is continuous in a_1.

At $\gamma_0 = 0$, the LHS of (A.34) evaluated at $(a_1,a_2,E_0) = (\hat{a}\left(\gamma_0\right), a_2\left(\hat{a}\left(\gamma_0\right);\gamma_0,0\right),$ $E_0\left(\hat{a}\left(\gamma_0\right)\right))$ equals $E\left[a|a \leq a_1\right] + b - \bar{a}$, which is strictly negative by (2.7); while evaluated at $(a_1,a_2,E_0) = (a^*\left(s_1;\gamma_0\right), a_2\left(a^*\left(s_1;\gamma_0\right);\gamma_0,0\right), E_0\left(a^*\left(s_1;\gamma_0\right)\right))$ it equals $E\left[a\right] + b\Pr\left(a \leq a^*\left(s_1\right)\right) - a^*\left(s_1\right)$, which is strictly positive since we are in Case 2. By continuity, the same two statements also hold for γ_0 small but strictly positive. Fix any such γ_0. By continuity, there then exists $\left(a_1, a_2\left(a_1;\gamma_0,0\right), E_0\left(a_1\right)\right)$ that satisfies equations (A.32)-(A.34).

By a further application of continuity, for all α sufficiently small, there exists $\left(a_1, a_2\left(a_1;\gamma_0,\alpha\right), E_0\left(a_1;\alpha\right)\right)$ that satisfies equations (A.32)-(A.34).

This completes the treatment of this case, and hence the construction proof. Finally, we finish by proposing the following off equilibrium belief. Write $(\tilde{s}_1, \tilde{s}_2)$ for an arbitrary off-equilibrium action.

Off-equilibrium beliefs are as follows. Date 2 repurchases $\tilde{s}_2 > 0$ are associated with the best firm $\bar{a}$ and issues $\tilde{s}_2 < 0$ are associated with the worst firm $\underline{a}$. At date 1, repurchases $\tilde{s}_1 > 0$ are associated with the best firm $\bar{a}$ with probability $1-\varepsilon$ and the worst firm with probability ε; while issues $\tilde{s}_1 < 0$ are associated with the best firm $\bar{a}$ with probability ε and the worst firm $\underline{a}$ with probability $1-\varepsilon$. Note that these date 1 beliefs, together with the fact that with probability $\alpha > 0$ all firm types do nothing at date 1, mean that the specification of date 2 beliefs satisfies NDOC.

Write $\tilde{P}_1$ and $\tilde{P}_2$ for the associated off-equilibrium prices. Given the stated off-equilibrium beliefs, there exists some $\kappa > 0$ such that

$$\tilde{P}_1 \begin{array}{ccc} \geq & S + \bar{a} - \varepsilon\kappa & \text{if} \quad \tilde{s}_1 > 0 \\ \leq & S + \underline{a} + b + \varepsilon\kappa & \text{if} \quad \tilde{s}_1 < 0 \end{array}. \tag{A.35}$$

Moreover,

$$\tilde{P}_2 = \begin{cases} \frac{S - \tilde{s}_1 + \bar{a} + b\mathbf{1}_{S - \tilde{s}_1 - \tilde{s}_2 \geq I}}{1 - \frac{\tilde{s}_1}{\tilde{P}_1}} & \text{if} \quad \tilde{s}_2 > 0 \\ \frac{S - \tilde{s}_1 + \underline{a} + b\mathbf{1}_{S - \tilde{s}_1 - \tilde{s}_2 \geq I}}{1 - \frac{\tilde{s}_1}{\tilde{P}_1}} & \text{if} \quad \tilde{s}_2 < 0 \end{cases}. \tag{A.36}$$

From the proof of Proposition 3, the equilibrium payoff of any firm $a \in [\underline{a}, \bar{a}]$ strictly exceeds the payoff from direct issue under investor beliefs $\underline{a}$, namely $\frac{I + a + b}{1 + \frac{I - S}{S + \underline{a} + b}}$. Moreover, for firms a sufficiently close to $\bar{a}$, the equilibrium payoff also strictly exceeds the payoff from doing nothing, namely $S + a$. (Of course, this relation holds weakly for *all* firms.) Hence it is possible to choose $\varepsilon > 0$ such that, for all firms $a \in [\underline{a}, \bar{a}]$,

$$\max\left\{\frac{I + a + b}{1 + \frac{I - S - \varepsilon\kappa}{S + \underline{a} + b + \varepsilon\kappa}}, a\frac{S + \bar{a} - \varepsilon\kappa}{\bar{a} - \varepsilon\kappa}\right\} < \text{equilibrium payoff of firm } a. \tag{A.37}$$

Moreover, and using $b > 0$ and inequality (2.7), choose $\varepsilon > 0$ sufficiently small such that, in addition to inequality (A.37), the following pair of inequalities holds:

$$\frac{a}{\underline{a} + b} \leq \frac{I + a + b}{I + \underline{a} + b} \quad \text{if} \quad a \in [\underline{a} + b, \underline{a} + b + \varepsilon\kappa], \tag{A.38}$$

$$\underline{a} + b + \varepsilon\kappa \leq \bar{a} - \varepsilon\kappa. \tag{A.39}$$

Firm a's payoff from an arbitrary off-equilibrium strategy $(\tilde{s}_1, \tilde{s}_2)$ is

$$\frac{S - \tilde{s}_1 - \tilde{s}_2 + a + b\mathbf{1}_{S - \tilde{s}_1 - \tilde{s}_2 \geq I}}{1 - \frac{\tilde{s}_1}{\tilde{P}_1} - \frac{\tilde{s}_2}{\tilde{P}_2}}.$$

First, observe that

$$-\frac{\tilde{s}_2}{\tilde{P}_2} \geq -\frac{\tilde{s}_2}{S - \tilde{s}_1 + \underline{a} + b}\left(1 - \frac{\tilde{s}_1}{\tilde{P}_1}\right).$$

This follows directly from (A.36) if $\tilde{s}_2 < 0$, and from (A.36) together with (2.7) if $\tilde{s}_2 > 0$. Second, observe that

$$-\frac{\tilde{s}_1}{\tilde{P}_1} \geq -\frac{\tilde{s}_1}{S + \underline{a} + b + \varepsilon\kappa}.$$

This follows directly from (A.35) if $\tilde{s}_1 < 0$, and from (A.35) together with (A.39) if $\tilde{s}_1 > 0$.

Consequently, firm a's payoff is bounded above by

$$\frac{S - \tilde{s}_1 - \tilde{s}_2 + a + b\mathbf{1}_{S-\tilde{s}_1-\tilde{s}_2 \geq I}}{\left(1 - \frac{\tilde{s}_1}{S+\underline{a}+b+\varepsilon\kappa}\right)\left(1 - \frac{\tilde{s}_2}{S-\tilde{s}_1+\underline{a}+b}\right)}$$
$$= \frac{S - \tilde{s}_1 - \tilde{s}_2 + a + b\mathbf{1}_{S-\tilde{s}_1-\tilde{s}_2 \geq I}}{S - \tilde{s}_1 - \tilde{s}_2 + \underline{a} + b} \frac{S - \tilde{s}_1 + \underline{a} + b}{S - \tilde{s}_1 + \underline{a} + b + \varepsilon\kappa} (S + \underline{a} + b + \varepsilon\kappa).$$

To complete the proof, by (A.37) it is sufficient to show that expression (A.40) is bounded above by either the LHS of (A.37), or by $S + a$. There are four cases:

If $S - \tilde{s}_1 - \tilde{s}_2 \geq I$ it is immediate that (A.40) is bounded above by $\frac{I+a+b}{I+\underline{a}+b}(S + \underline{a} + b + \varepsilon\kappa)$, which is the first term in the LHS of (A.37).

If $S - \tilde{s}_1 - \tilde{s}_2 < I$ and $a \leq \underline{a} + b$ then (A.40) is bounded above by $(S + \underline{a} + b + \varepsilon\kappa)$.

If $S - \tilde{s}_1 - \tilde{s}_2 < I$ and $a \in [\underline{a} + b, \underline{a} + b + \varepsilon\kappa]$ then (A.40) is bounded above by $\frac{a}{\underline{a}+b}(S + \underline{a} + b + \varepsilon\kappa)$, and the result then follows from (A.38).

Finally, consider the case $S - \tilde{s}_1 - \tilde{s}_2 < I$ and $a > \underline{a} + b + \varepsilon\kappa$. Note first that since $S - \tilde{s}_1 - \tilde{s}_2 < I$, the off-equilibrium beliefs imply that the firm weakly loses money on its date 2 transactions, so that its payoff is bounded above by

$$\frac{S - \tilde{s}_1 + a}{1 - \frac{\tilde{s}_1}{\tilde{P}_1}} = \tilde{P}_1 \frac{S - \tilde{s}_1 + a}{\tilde{P}_1 - \tilde{s}_1}.$$

If $\tilde{s}_1 > 0$, this expression is bounded above by $\max\left\{S + a, \frac{a\tilde{P}_1}{\tilde{P}_1 - S}\right\}$, which by (A.35) is bounded above by $\max\left\{S + a, a\frac{S+\bar{a}-\varepsilon\kappa}{\bar{a}-\varepsilon\kappa}\right\}$. If instead $\tilde{s}_1 < 0$ this expression is bounded above by $\max\left\{S + a, \tilde{P}_1\right\}$, which by (A.35) is bounded above by $\max\{S + a, S + \underline{a} + b + \varepsilon\kappa\} = S + a$. This completes the proof .

Perturbation (II), exogenous upper bound $\bar{S}$ on repurchase size:

When the equilibrium of the proof of Proposition 3 falls in Case 1, off-equilibrium beliefs are defined in an identical way to Part (I) above, and the proof is identical.

For the remainder of the proof suppose that the equilibrium of the proof of Proposition 3 falls in Case 2. As a preliminary step, recall that the proof of Proposition 3 entails choosing s_1 to lie below some bound (defined in the proof). Here, choose $\bar{S}$ to lie below this same bound. Then set $s_1 = \bar{S}$.

Choose the sets A_0 and A_1 so that A_1 contains $\underline{a}$ and A_0 contains a point $\underline{a}^+$ that is close to $\underline{a}$. Off-equilibrium beliefs are identical to Part (I), with the exception of off-equilibrium beliefs following $s_1 = 0$: now, these beliefs put probability 1 on type a_2 if $\tilde{s}_2 > 0$, and put probability 1 on type $\underline{a}^+$ if $\tilde{s}_2 < 0$. Note that these beliefs satisfy NDOC.

Given these beliefs, a firm's payoff from deviating to $(s_1 = 0, \tilde{s}_2)$, where $\tilde{s}_2 > 0$, is

$$\frac{S - \tilde{s}_2 + a}{1 - \frac{\tilde{s}_2}{S+a_2}}.$$

For $a \le a_2$ this expression is below the do-nothing payoff of $S+a$. If instead $a > a_2$, this expression is below $\frac{S-\bar{S}+a}{1-\frac{\bar{S}}{S+a_2}} = \frac{S-s_1+a}{1-\frac{s_1}{S+a_2}}$, which is the payoff from following the equilibrium strategy $(s_1, 0)$ (recall the repurchase price is $S + a_2$). Hence no deviation of this type strictly improves a firm's payoff relative to the equilibrium payoff.

Finally, a parallel proof to Part (I) establishes that provided $\underline{a}^+$ is chosen sufficiently close to $\underline{a}$, no deviation of the type $(s_1 = 0, \tilde{s}_2)$ with $\tilde{s}_2 < 0$ strictly improves a firm's payoff relative to the equilibrium payoff.

All other deviations are handled exactly as in Part (I), completing the proof.

Lemma A-2 *If an equilibrium features capital transactions* (s_1', s_2') *and* (s_1'', s_2'') *with* $S - s_1' - s_2' = S - s_1'' - s_2''$, *then the associated transaction prices* P_1', P_2', P_1'', P_2'' *are such that*

$$1 - \frac{s_1''}{P_1''} - \frac{s_2''}{P_2''} = 1 - \frac{s_1'}{P_1'} - \frac{s_2'}{P_2'}. \tag{A.40}$$

Proof of Lemma A-2: Let a' and a'' be firms that play (s_1', s_2') and (s_1'', s_2'') respectively. The equilibrium conditions for firm a' include

$$\frac{S - s_1' - s_2' + a' + b\mathbf{1}_{S-s_1'-s_2' \ge I}}{1 - \frac{s_1'}{P_1'} - \frac{s_2'}{P_2'}} \ge \frac{S - s_1'' - s_2'' + a' + b\mathbf{1}_{S-s_1''-s_2'' \ge I}}{1 - \frac{s_1''}{P_1''} - \frac{s_2''}{P_2''}},$$

which simplifies to $1 - \frac{s_1''}{P_1''} - \frac{s_2''}{P_2''} \ge 1 - \frac{s_1'}{P_1'} - \frac{s_2'}{P_2'}$. The symmetric equilibrium condition for a firm a'' playing (s_1'', s_2'') then implies (A.40). QED

Proof of Proposition 6:

Part (A): Firms that repurchase s_1 at date 1 are, at date 2, in exactly the situation characterized by Proposition 2. Consequently, at date 2 a positive-measure subset of these firms must issue an amount s_2 such that investment is possible, i.e., $S - s_1 - s_2 \geq I$ at date 2. If almost all firms that repurchase s_1 also issue s_2, then $P_1 = P_2$, and the proof is complete. Otherwise, let $A_1^{s_1}$ denote the set of firms that repurchase s_1 at date 1. From Proposition 2, there exists a^* such that almost all firms in $A_1^{s_1} \cap [a^*, \bar{a}]$ choose not to issue s_2 at date 2. The equilibrium condition for any firm $a \in A_1^{s_1} \cap [a^*, \bar{a}]$ in this non-issuing set is

$$\frac{S - s_1 + a}{1 - \frac{s_1}{P_1}} \geq \frac{S - s_1 - s_2 + a + b}{1 - \frac{s_1}{P_1} - \frac{s_2}{P_2}}.$$

Hence

$$E\left[\frac{S - s_1 + a}{1 - \frac{s_1}{P_1}} | a \in A_1^{s_1} \cap [a^*, \bar{a}]\right] > E\left[\frac{S - s_1 - s_2 + a + b}{1 - \frac{s_1}{P_1} - \frac{s_2}{P_2}} | a \in A_1^{s_1} \cap [\underline{a}, a^*]\right],$$

so that the date 2 share price of non-issuing firms strictly exceeds the date 2 share price of issuing firms, i.e., P_2. Since the date 1 share price equals the conditional expectation of the date 2 share price, it follows that $P_2 < P_1$.

Part (B): First, suppose a positive measure of firms issue $s_1' < 0$. If $S - s_1' \geq I$, then by the argument of Proposition 2, almost all firms play $s_2' = 0$. If instead $S - s_1' < I$, then by the argument of Proposition 2, there exists s_2' such that $S - s_1' - s_2' \geq I$ and such that a positive measure of firms play (s_1', s_2'), and almost all the remainder play $(s_1', 0)$. Moreover, $\Pr(s_2'|s_1') = 1$, as follows. Suppose to the contrary that $\Pr(s_2'|s_1') < 1$. The equilibrium condition for a firm a that plays $(s_1', 0)$ is

$$\frac{S - s_1' + a}{1 - \frac{s_1'}{P_1(s_1')}} \geq \frac{S - s_1' - s_2' + a + b}{\left(1 - \frac{s_1'}{P_1(s_1')}\right)\left(1 - \frac{s_2'}{E[S - s_1 + a + b | s_1', s_2']}\right)},$$

which simplifies (using $s_2' < 0$) to

$$\frac{S - s_1' + a}{E\left[S - s_1' + a + b | s_1', s_2'\right]} \geq 1 - \frac{b}{s_2'}.$$

Hence any firm a that plays $(s_1', 0)$ must satisfy $a > E\left[a + b | s_1', s_2'\right]$. By Lemma 1, firms that play $(s_1', 0)$ are better than firms that play (s_1', s_2'). Hence $P_1(s_1') < S + \sup\{a : a \text{ plays } s_1'\}$; and almost all firms sufficiently close to $\sup\{a : a \text{ plays } s_1'\}$ play $(s_1', 0)$, and would obtain a higher payoff by doing nothing, a contradiction. This establishes that $P_2(s_1', s_2') = P_1(s_1')$.

We next establish the price comparison with firms that issue after previously repurchasing, i.e., $P_2(s_1, s_2)$. Given the first step, we handle the two cases in the proposition together: let (s_1', s_2') be a strategy with $S - s_1' - s_2' \geq I$ and $s_1', s_2' \leq 0$. At any date with strictly positive issue, the price is $P_2(s_1', s_2') = E[S + a + b | s_1', s_2']$. We first show that

$$S - s_1' - s_2' \geq S - s_1 - s_2. \tag{A.41}$$

The proof is by contradiction: suppose instead that $S - s_1' - s_2' < S - s_1 - s_2$. So by Lemma 1, $E[a|s_1', s_2'] > E[a|s_1, s_2]$. By Part (A), $P_1(s_1) \geq P_2(s_1, s_2) = \frac{S - s_1 + E[a|s_1,s_2] + b}{1 - \frac{s_1}{P_1(s_1)}}$, and so $P_1(s_1) \geq S + E[a|s_1, s_2] + b$. Hence

$$\left(1 - \frac{s_1}{P_1(s_1)}\right)\left(\frac{1}{S - s_1 + E[a|s_1, s_2] + b}\right) \geq \frac{1}{S + E[a|s_1, s_2] + b}.$$

So the payoff to firm a from (s_1, s_2) is

$$\frac{S - s_1 - s_2 + a + b}{\left(1 - \frac{s_1}{P_1(s_1)}\right)\left(1 - \frac{s_2}{S - s_1 + E[a|s_1,s_2] + b}\right)} \leq \frac{S - s_1 - s_2 + a + b}{\frac{S - s_1 - s_2 + E[a|s_1,s_2] + b}{S + E[a|s_1,s_2] + b}}.$$

Fix a firm playing (s_1, s_2) with $a > E[a|s_1, s_2]$. By the supposition $S - s_1' - s_2' < S - s_1 - s_2$, the payoff from (s_1, s_2) for firm a is strictly less than

$$\frac{S - s_1' - s_2' + a + b}{\frac{S - s_1' - s_2' + E[a|s_1,s_2] + b}{S + E[a|s_1,s_2] + b}},$$

which since $E[a|s_1', s_2'] > E[a|s_1, s_2]$ is in turn strictly less than

$$\frac{S - s_1' - s_2' + a + b}{\frac{S - s_1' - s_2' + E[a|s_1',s_2'] + b}{S + E[a|s_1',s_2'] + b}} = \frac{S - s_1' - s_2' + a + b}{1 - \frac{s_1' + s_2'}{S + E[a|s_1',s_2'] + b}}.$$

But this contradicts the equilibrium condition, since the RHS is firm a's payoff from deviating and playing (s_1', s_2'), and establishes inequality (A.41).

To complete the proof of Part (B), we consider in turn the cases in which (A.41) holds with equality, and in which it holds strictly. First, if (A.41) holds with equality, Lemma A-2 implies

$$-\frac{s_1}{P_1(s_1)} - \frac{s_2}{P_2(s_1, s_2)} = -\frac{s_1' + s_2'}{P_2(s_1', s_2')}.$$

From Part (A), $P_2(s_1, s_2) \leq P_1(s_1)$, and since $s_1 \geq 0$, this implies

$$-\frac{s_1 + s_2}{P_2(s_1, s_2)} \leq -\frac{s_1' + s_2'}{P_2(s_1', s_2')},$$

which since (A.41) holds with equality, implies $P_2(s_1, s_2) \geq P_2(s_1', s_2')$.

Second, if instead (A.41) holds strictly, taking the expectation over the equilibrium condition for all firms a playing (s_1, s_2), together with the implication of Lemma 1 that $E[a|s_1, s_2] > E[a|s_1', s_2']$, yields

$$\begin{aligned}
\frac{S - s_1 - s_2 + E[a|s_1, s_2] + b}{1 - \frac{s_1}{P_1(s_1)} - \frac{s_2}{P_2(s_1,s_2)}} &\geq \frac{S - s_1' - s_2' + E[a|s_1, s_2] + b}{1 - \frac{s_1'}{P_1(s_1')} - \frac{s_2'}{P_2(s_1',s_2')}} \\
&> \frac{S - s_1' - s_2' + E[a|s_1', s_2'] + b}{1 - \frac{s_1'}{P_1(s_1')} - \frac{s_2'}{P_2(s_1',s_2')}}.
\end{aligned}$$

Since the first and last terms in this inequality are simply $P_2(s_1, s_2)$ and $P_2(s_1', s_2')$ respectively, this establishes $P_2(s_1, s_2) > P_2(s_1', s_2')$.

Part (C): By the argument of Proposition 2, either almost all firms that play 0 at date 1 also play 0 at date 2; or there exists s_2' such that $S - s_2' \geq I$ and almost all firms play either 0 or s_2' at date 2. The date 1 price for do-nothing firms satisfies

$$P_1(0) = E[S + a|(0,0)] \Pr(0|0) + P_2(0, s_2') \Pr(s_2'|0).$$

From Parts (A) and (B), we know $P_2(0, s_2') \leq P_2(s_1, s_2) \leq P_1(s_1)$. From the equilibrium condition, and firm a that plays $(0,0)$ satisfies $S + a \leq P_1(s_1)$, since otherwise firm a would be strictly better off playing $(s_1, 0)$. The result then follows, completing the proof.

Proof of Proposition 7: *Part (A):* By the equilibrium condition for firm a'',

$$\frac{S - s'' + a''}{1 - \frac{s''}{P''}} \geq \frac{S - s' + a''}{1 - \frac{s'}{P'}}. \tag{A.42}$$

Since $s'' > s'$, it is immediate that $s''/P'' > s'/P'$, establishing (ii). By the equilibrium condition for firm a',

$$\frac{S - s' + a'}{1 - \frac{s'}{P'}} \geq \frac{S - s'' + a'}{1 - \frac{s''}{P''}}. \tag{A.43}$$

Multiplying (A.43) by -1 and combining with (A.42) yields

$$\frac{a'' - a'}{1 - \frac{s''}{P''}} \geq \frac{a'' - a'}{1 - \frac{s'}{P'}}.$$

If $a' > a''$ then this inequality contradicts (ii); hence $a'' \geq a'$, which (since $a'' \neq a'$) establishes (iii).

Firm a' also has the choice of doing nothing, and so the equilibrium condition implies $S + a' \geq P'$, i.e., firm a' pays weakly less than its stock is worth. Consequently,

$$\frac{S - s'' + a'}{1 - \frac{s''}{P'}} \geq \frac{S - s' + a'}{1 - \frac{s'}{P'}},$$

i.e., if firm a' were able to repurchase more stock at the constant price P', it would weakly prefer to do so. Combined with (A.43), it then follows that $P'' \geq P'$, establishing (i), and completing the proof of Part (A).

Part (B): The proof is exactly the same as the final paragraph of the proof of Part (B) of Proposition 6.

Proof of Proposition 8: By hypothesis, there are only a finite number of strategies played in equilibrium. Throughout the proof, we ignore any firm that plays a strategy that is played by only a measure zero set of firms. Partition the remaining firms so that if two firms share the same $s_1 + s_2$ and make the same investment decision, then they lie in the same partition element. Let $A^1, \ldots, A^M$ be the partition elements in which firms invest. Let A^0 be the set of non-investing firms. Without loss, order the sets $A^1, \ldots, A^M$ so that $i > j$ is equivalent to $S - s_1 - s_2$ being smaller for firms in A^i than A^j. By Lemma 1, it follows that A^i are intervals, with $A^i > A^j$ if $i > j$. By Corollary 1, $\inf A^1 = \underline{a}$. Define $s^i = s_1 + s_2$ for all firms in A^i, and by Lemma A-2, and define $N^i = 1 - \frac{s_1(a)}{P_1(s_1(a))} - \frac{s_2(a)}{P_2(s_1(a), s_2(a))}$ for all firms $a \in A^i$.

If $s_1 \leq 0$ for some firm in A^i, an easy adaption of the arguments of Propositions 1 and 2 implies that all firms that use this action at date 1 take the same date 2 action, s_2. Moreover, by the definition of A^i, all such firms invest. So for these firms, the date 1 and 2 transaction prices coincide, and by (2.4), both equal $E\left[S + a + b | s_1(a), s_2(a)\right]$. Hence in this case $N^i = 1 - \frac{s^i}{E[S+a+b|s_1(a),s_2(a)]}$.

If instead $s_1 > 0$ for some firm in $a \in A^i$, then by Proposition 6, the date 1 and 2 transaction prices satisfy $P_1 \geq P_2$, and so using $s_2(a) < 0$, $N^i \geq 1 - \frac{s^i}{P_2(s_1(a), s_2(a))}$. Since $P_2(s_1(a), s_2(a)) = E\left[\frac{S - s^i + a + b}{N^i} | s_1(a), s_2(a)\right]$, we know

$$P_2(s_1(a), s_2(a)) \leq E\left[\frac{S - s^i + a + b}{1 - \frac{s^i}{P_2(s_1(a), s_2(a))}} | s_1(a), s_2(a)\right].$$

and hence

$$P_2\left(s_1\left(a\right), s_2\left(a\right)\right) \leq E\left[S + a + b | s_1\left(a\right), s_2\left(a\right)\right],$$

and so

$$N^i \geq 1 - \frac{s^i}{E\left[S + a + b | s_1\left(a\right), s_2\left(a\right)\right]}.$$

Moreover, by Proposition 6, the inequality is strict whenever $\Pr\left(\text{invest} | s_1\left(a\right)\right) < 1$.

The above observations imply

$$N^i \geq 1 - \frac{s^i}{E\left[S + a + b | a \in A^i\right]}, \tag{A.44}$$

with the inequality strict whenever $\Pr\left(\text{invest} | s_1 \in s_1\left(A^i\right)\right) < 1$.

We next show that $\Pr\left(\text{invest} | s_1 \in s_1\left(A^i\right)\right) < 1$ for at least some i. Suppose to the contrary that this is not the case. Then $\Pr\left(\text{not invest} | s_1 \in s_1\left(A^0\right)\right) = 1$. So $E\left[P_3 | s_1 \in s_1\left(A^0\right)\right] = E\left[P_3 | a \in A^0\right] = S + E\left[a | a \in A^0\right]$. But a straightforward adaptation of the proof of Proposition 4 implies that there exists an upper interval of firms who obtain a payoff strictly in excess of $S + a$, and by Corollary A-2, this upper interval has a non-null intersection with A^0. But then $E\left[P_3 | a \in A^0\right] > S + E\left[a | a \in A^0\right]$, a contradiction.

Boundary firms $a^{i*} \equiv \sup(A^i)$ must be indifferent across two adjacent issue paths s^{i-1} and s^i, i.e., for all $i < M$,

$$\frac{1}{N^i}(a^{i*} + S + b - s^i) = \frac{1}{N^{i+1}}(a^{i*} + S + b - s^{i+1}). \tag{A.45}$$

The heart of the proof is to establish that inequality (A.44), with the inequality strict for at least some i, implies

$$N^M > 1 - \frac{s^M}{E[a + S + b | a \in [\underline{a}, a^{M*}]]}. \tag{A.46}$$

We establish (A.46) by showing inductively that for any $i = 1, \ldots, M$,

$$N^i \geq 1 - \frac{s^i}{E[a + S + b | a \in [\underline{a}, a^{i*}]]}. \tag{A.47}$$

The initial case $i = 1$ is immediate from (A.44) and the earlier observation that $\inf A_1 = \underline{a}$. For the inductive step, suppose (A.47) holds at $i = K - 1 < M$. We show that (A.47) also holds at $i = K$.

Observe first that inequality (A.47) at $i = K - 1$ is equivalent to

$$\frac{S + a^{(K-1)*} + b - s^{K-1}}{N^{K-1}} \leq \frac{S + a^{(K-1)*} + b - s^{K-1}}{1 - \frac{s^{K-1}}{E[a+S+b|a\in[\underline{a},a^{(K-1)*}]]}}.$$

Since $a^{(K-1)*} \geq E[a|a \in [\underline{a}, a^{(K-1)*}]]$, the RHS of this inequality is increasing in s^{K-1}, i.e., if the share price is $E[a + S + b|a \in [\underline{a}, a^{(K-1)*}]]$, the best firm $a^{(K-1)*}$ in pool $[\underline{a}, a^{(K-1)*}]$ would be better off raising fewer funds than $S - s^{K-1}$. We know $S - s^K < S - s^{K-1}$, and so

$$\frac{S + a^{(K-1)*} + b - s^{K-1}}{N^{K-1}} < \frac{S + a^{(K-1)*} + b - s^{K}}{1 - \frac{s^{K}}{E[a+S+b|a\in[\underline{a},a^{(K-1)*}]]}}.$$

Combined with the indifference condition (A.45) at $i = K - 1$, it follows that

$$N^K \geq 1 - \frac{s^K}{E[a + S + b|a \in [\underline{a}, a^{(K-1)*}]]}.$$

Combined with (A.44), it then follows that

$$N^K \geq 1 - \frac{s^K}{E[a + S + b|a \in A^K \cup [\underline{a}, a^{(K-1)*}]]} = 1 - \frac{s^K}{E[a + S + b|a \in [\underline{a}, a^{K*}]]},$$

which establishes the inductive step. Moreover, this inequality must hold strictly for at least one step.

To complete the proof, note that in equilibrium, for all $a \in A^M$,

$$\frac{S - s^M + a + b}{N^M} \geq S + a.$$

So by (A.46)

$$\frac{S - s^M + a^{M*} + b}{1 - \frac{s^M}{E[S+a+b|a\in[\underline{a},a^{M*}]]}} > S + a^{M*}.$$

Consequently, by continuity together with (2.6), there exists $\tilde{a}^* > a^{M*}$ such that

$$\frac{S - s^M + \tilde{a}^* + b}{1 - \frac{s^M}{E[S+a+b|a\in[\underline{a},\tilde{a}^*]]}} = S + \tilde{a}^*.$$

It is straightforward to show that there is an equilibrium of the one-period benchmark in which firms in $[\underline{a}, \tilde{a}^*]$ issue shares at a price $E\left[S + a + b|a \in [\underline{a}, \tilde{a}^*]\right]$ to raise funds $-s^M$ and invest, while firms $a \in (\tilde{a}^*, \bar{a}]$ do nothing. This completes the proof.

Lemma A-3 *(Multi-dimensional Blackwell's Sufficient Condition) Let $X \subseteq \mathbb{R}^K$ and $B^L(X)$ be the space of bounded vector-valued functions: $v = (v_1, v_2, ..., v_L) : X \to \mathbb{R}^L$, where $L < \infty$. Equipe $B^L(X)$ with the sup norm over coordinates, i.e. $||v|| = \max_{i \leq L}\{\sup_x v_i(x)\}$. Suppose $v, w \in B^L(X)$, and define $v \geq w$ if and only if $v_i \geq w_i$ for all $i \leq L$. If the operator $T : B^L(X) \to B^L(X)$ satisfies that*

1. *(monotonicity) if $v \geq w$, then $T(v) \geq T(w)$, and*
2. *(discounting) there exists a constant β such that for any constant a, $T(v + a) \leq T(v) + \beta a$,*

then T is a contraction mapping with coefficient β, namely $||Tv - Tw|| \leq \beta||v - w||$ for any $v, w \in B^L(X)$.

Proof. Since $w \leq v + ||w - v||$, so monotonicity of T implies $T(w) \leq T(v + ||w - v||)$. The latter expession is in turn bounded by $T(v) + \beta||w - v||$ by discounting. Therefore,

$$T(w) - T(v) \leq \beta||w - v||.$$

Similarly, one can derive the opposite side $T(v) - T(w) \leq \beta||w - v||$. By the definition of the norm on $B^L(X)$, $||T(w) - T(v)|| \leq \beta||w - v||$. T is therefore a contraction mapping with coefficient β. ■

Proof of Lemma 2: In order to be consistent with the notations in the main text following the lemma, denote the values given in (3.2) by $V^{\theta\star}_{FB}(Y)$. They can be recursively formulated as following:

$$\begin{aligned} V^{G\star}_{FB}(Y) &= E\{\pi Yz + (1-\pi)[p^G V^{G\star}_{FB}(Yz) + (1-p^G)V^{B\star}_{FB}(Yz)]\} \\ V^{B\star}_{FB}(Y) &= (1-\pi)E[p^B V^{B\star}_{FB}(Yz) + (1-p^B)V^{G\star}_{FB}(Yz)]. \end{aligned} \tag{A.48}$$

The first part πYz captures the final dividend, which is materialized only in the good state $\theta = G$. This case occurs with probability π. The second part captures the continuation payoff taking into account a potential switch in the state θ. Normalizing by Y and letting $v^{\theta}_{FB}(Y) = \frac{V^{\theta\star}_{FB}(Y)}{Y}$, (A.48) becomes

$$\begin{aligned} v^{G}_{FB}(Y) &= E\{\pi z + (1-\pi)[p^G v^{G}_{FB}(Yz)z + (1-p^G)v^{B}_{FB}(Yz)z]\} \\ v^{B}_{FB}(Y) &= (1-\pi)E[p^B v^{B}_{FB}(Yz)z + (1-p^B)v^{G}_{FB}(Yz)z]. \end{aligned} \tag{A.49}$$

For any bounded continuous functions on $\mathbb{R}_+$: $v^{\theta}_{FB} \in B^1(\mathbb{R}_+)$, $(\theta = G, B)$, it is easy to check that the right hand side of (A.49) induces a natural operator $T : C^2_B(\mathbb{R}_+) \to C^2_B(\mathbb{R}_+)$ as following:

$$T(v^G_{FB}, v^B_{FB}) = \begin{cases} E\{\pi z + (1-\pi)[p^G v^G_{FB}(Yz)z + (1-p^G)v^B_{FB}(Yz)z]\} \\ (1-\pi)E[p^B v^B_{FB}(Yz)z + (1-p^B)v^G_{FB}(Yz)z]. \end{cases}$$

Clearly T satisfies the monotonicity condition in lemma A-3. To verify the discounting condition, notice

$$T(v_{FB} + a) = T(v_{FB}) + (1-\pi)E(az) = T(v_{FB}) + (1-\pi)\mu a.$$

By assumption (3.1) and lamma A-3, T is a contraction. Therefore, Banach fixed point theorem states that T has a unique fixed point, which implies (A.49) and thereby (A.48) have a unique solution. Finally, to find this solution, observe that (A.49) has a constant solution $(v^{G\star}_{FB}, v^{B\star}_{FB})$ that satisfies:

$$\begin{aligned} v^G_{FB} &= \pi\mu + (1-\pi)\mu[p^G v^G_{FB} + (1-p^G)v^B_{FB}]\} \\ v^B_{FB} &= (1-\pi)\mu[p^B v^B_{FB} + (1-p^B)v^G_{FB}]. \end{aligned}$$

Solving the above system for (v^G_{FB}, v^B_{FB}) gives (3.3).

Proof of Proposition 9: The proof contains three parts to verify the proposed equilibrium. First, given the conjectured properties stated in the proposition, I show that the conjectured strategy profile indeed constitutes a subgame perfect equilibrium. Part II (III) proves that the conjectured properties for the value of debt (firm) indeed hold in this equilibrium. In the following proof, the time indices and the arguements in the strategies are sometimes omitted when there is no confusion.

Part I: Given the stated properties of D^{θ}_N and V^{θ}_N, I check that the proposed strategy profile is subgame perfect. If the firm survives the period t stage game, then following the equilibrium strategies, the expected payoff to the entrepreneur is

$$V^{\theta_{t+1}}_{N^\star_+}(F^\star_+, Y_t) - \min(F_t, N\lambda Y_t) = \max_{N_+} V^{\theta_{t+1}}_{N_+}(F_+(X^*), Y_t) - X^*,$$

where $X^* = \min(F_t, N\lambda Y_t)$ and $F_+(X)$ is the smallest solution to $D^{\theta}_{N_+}(F_+, Y) = X$. By the definition of κ^{θ}, the conjectured property that $V^{\theta}_{N_+}(F_+, Y) \geq \kappa^{\theta}_{N_+} Y$, and the endogenous assumption $\kappa^{\theta} \geq \lambda$, the above equality implies:

$$V^{\theta_{t+1}}_{N^\star_+}(F^\star_+, Y_t) - \min(F_t, N\lambda Y_t) \geq \kappa^{\theta_{t+1}} Y - F_t \geq \lambda Y_t - F_t.$$

Therefore, the continuation payoff is weakly higher than the liquidation payoff. Thus the firm has no strict incentive to voluntarily liquidate nor to offer $S_i < \min(\frac{F}{N}, \lambda Y)$ and induce an immediate liquidation. Suppose the firm offers $S_i > \min(\frac{F}{N}, \lambda Y)$. Two possible cases can happen. If the offer is infeasible, i.e., $\sum_{j\leq i} S_j + (N-i)\min(\frac{F}{N}, \lambda Y) > DC^\theta(Y)$, then the creditor rejects the offer and the project is liquidated. This case is clearly dominated by the equilibrium outcome as discussed before. Alternatively if the offer is feasible. Let X be the total negotiated repayment following S_i. Clearly, it must be $X > X^*$, which implies $F_+(X) > F_+(X^*)$ for any given N_+. Because we have conjectured that $V_N^\theta(F, Y)$ is weakly decreasing in F, so

$$V_{N_+}^{\theta_{t+1}}(F_+(X^*), Y_t) - X^* \geq V_{N_+}^{\theta_{t+1}}(F_+(X), Y_t) - X^* > V_{N_+}^{\theta_{t+1}}(F_+(X), Y_t) - X,$$

for any N_+. Therefore, the entrepreneur is strictly worse off by offering any $S_i > \min(\frac{F}{N}, \lambda Y)$. In all, the offering strategy $S_i^\star = \min(\frac{F}{N}, \lambda Y)$ is optimal.

The entrepreneur's financing strategy $(N_+^{\theta\star}, F_+^{\theta\star})$ is just a repetition of the equilibrium definition. The ith incumbent creditor clearly has no incentive to accept any offer lower than the liquidation payoff. On the other hand, if the payoff is not feasible such that (3.12) fails, the project will be liquidated following the equilibrium strategies by other creditors. In this case, creditor i either gets $\min(\frac{F_t}{N_t}, \frac{1}{N_t-1}\max(0, \lambda Y_t - \frac{F_t}{N_t}))$ or $\frac{1}{N_t}\min(F_t, \lambda Y_t)$, both are weakly dominated by $\min(\frac{F_t}{N_t}, \lambda Y_t)$. Finally, the optimality of the new creditors' strategies $r_i^{\theta\star}$ is trivial to verify.

Part II: Given the above strategies, I now show that there exists a consistent linear conjecture of the debt capacities, i.e. $DC^\theta(Y) = \kappa^\theta Y$ for some constants κ^θ. In addition, the value of debt $D_N^\theta(F, Y)$ is continuous and HD1 in (F, Y).

Under the conjecture $DC^\theta(Y) = \kappa^\theta Y$, the equilibrium strategies (condition (3.12) in particular) imply that rollover is possible if and only if (3.14) holds. Under this condition, the value of debt can be rewritten as (3.15). The value of debt $D_N^\theta(F, Y)$ is clearly HD1, because one can verify that

$$D_N^\theta(F, Y) = Y D_N^\theta(\frac{F}{Y}, 1) \equiv Y D_N^\theta(f, 1).$$

where $f \equiv \frac{F}{Y}$. The ratio $\frac{\overline{F}_N^\theta(Y)}{Y}$ being a constant independ of Y is a simple corollary of HD1. In fact, one can readily see $\overline{f}_N^\theta(Y) = \arg\max_f D_N^\theta(f, 1)$. In addition, the debt capacity with N creditors is linear as given by (3.16). Finally, $D_N^\theta(f, 1)$ is continuous in

f, since it can be expressed as the sum of integrals in the form of $\int_{B(f)}^{A(f)} C(f,z)dz$, where A, B, C are continuous functions in their arguments. For example, when $N > \frac{\max_\theta \kappa^\theta}{\lambda}$,

$$\begin{aligned} D_N^\theta(f,1) \equiv{} & \pi \int_{\underline{z}}^{\overline{z}} \min(f,z)\mathbf{1}_{\theta=G} g(z)dz + (1-\pi)\textstyle\sum_{\theta'=G,B} \\ & P(\theta'|\theta)[\int_{\max(\underline{z},\frac{f}{\kappa^{\theta'}})}^{\max(\overline{z},\frac{f}{\kappa^{\theta'}})} g(z)fdz + \int_{\min(\underline{z},\frac{f}{\kappa^{\theta'}})}^{\min(\overline{z},\frac{f}{\kappa^{\theta'}})} \lambda z g(z)dz] \end{aligned} \tag{A.50}$$

which is clearly continuous in f. The remaining cases are similar. Finally, I show that there exists a consistent conjecture of $\{\kappa_N^\theta, \kappa^\theta\}_{N\in\mathbb{N}}^{\theta=G,B}$. Notice that (3.16) is a function of (κ^G, κ^B). Denote $\hat{\kappa}_N^\theta(\kappa^G,\kappa^B)$ to be this function and let $L^\theta(\kappa^G,\kappa^B) \equiv \max\{\hat{\kappa}_1^\theta, ..., \hat{\kappa}^\theta_{[\frac{\max_\theta k^\theta}{\lambda}]+1}\}$. So a consistent conjecture of $\{\kappa_N^\theta, \kappa^\theta\}_{N\in\mathbb{N}}^{\theta=G,B}$ is a solution to (3.13) which is in turn a fixed point of $L \equiv (L^G, L^B) : \mathbb{R}_+^2 \to \mathbb{R}_+^2$. Equipe $\mathbb{R}_+^2$ with the usual partial order $\leq$ such that $x \leq y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. Apparently L is order-preserving, since D_N^θ is weakly increasing in κ^θ. I shall then construct a complete lattice $\Omega \subseteq \mathbb{R}_+^2$ such that $L(\Omega) \subseteq \Omega$. By Tarski's fixed point theorem, L has a fixed point and therefore a solution to (3.13) exists. The remainder of the proof is to construct such an Ω.

By (3.1), it is possible to choose M big enough such that

$$(\pi + (1-\pi)M)\mu < M. \tag{A.51}$$

Let $\Omega \equiv [0,M]\times[0,M]$ be a complete lattice. Suppose $(\kappa^G,\kappa^B) \in \Omega$, (3.16) and (A.51) imply that for all $N \leq [\frac{M}{\lambda}] + 1$,

$$\begin{aligned} \hat{\kappa}_N^\theta ={} & \max_f E\{\pi \min(f,z)\mathbf{1}_{\theta=G} + (1-\pi) \\ & [\mathbf{1}_{\min(f,N\lambda z)\leq\kappa^{\theta'}z} \min(f,N\lambda z) + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta'}z} \min(f,\lambda z)]\} \\ \leq{} & \pi E(z) + (1-\pi)E[\mathbf{1}_{\min(f,N\lambda z)\leq\kappa^{\theta'}z}\kappa^{\theta'}z + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta'}z}\lambda z] \\ \leq{} & \pi\mu + (1-\pi)M\mu \\ <{} & M. \end{aligned}$$

Therefore, $L^\theta(\kappa_N^\theta) < M$, which implies that Ω is invariant under L . This completes the proof.

Part III: Finally, for any pair of $\kappa^\theta \geq \lambda$, I will show there exists a unique continuous HD1 function $V_N^\theta(F,Y)$ which is increasing in Y, weakly decreasing in F and $V_N^\theta(F,Y) \geq \kappa_N^\theta Y$ for any $F \leq \overline{F}_N^\theta$. By the discussion following proposition 9, in the conjectured equilibrium, the firm's problem can be rewritten as a dynamic programming problem (3.17) and (3.18). By the definition of $\overline{N}$ in (3.19) and the discussion following it, we can confine the choice of N_+^θ to $\{1,2,...,\overline{N}\}$ without loss of generality.

Define an auxiliary problem:

$$\begin{aligned} v_N^\theta(f) \;\; = & \\ & E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\leq\kappa^{\theta'}z} \max_{N_+} v_{N_+}^{\theta'}(f_+)z + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \end{aligned} \tag{A.52}$$

where $f_+(\frac{f}{z},N)$ is the minimum solution to

$$D_{N_+}^{\theta'}(f_+,1) = \min(\frac{f}{z}, N\lambda). \tag{A.53}$$

By the definition of $\overline{f}_N^\theta$ in proposition 9, it must be $f_+ \leq \overline{f}_{N_+}^{\theta'}$. Denote $T_N^\theta : B^{2\overline{N}} \to B$ to be the operator on $(v_i^\theta)_{i\leq\overline{N}}^{\theta=G,B}$ induced by the right-hand side of (A.52) and let $T \equiv (T_N^\theta) : B^{2\overline{N}} \to B^{2\overline{N}}$.

First, notice that if $v \in B^{2\overline{N}}$ is bounded by some $M > 1$, then $||Tf(v)|| \leq \pi(1+\mu) + (1-\pi)M(1+\mu)$ is also bounded. So T is indeed well-defined. Then I prove that T is a contraction mapping by verifying monotonicity and discounting conditions in lemma A-3. Monotonicity is trivial. For any constant a, $T(v+a) \leq T(v) + (1-\pi)(1+\mu)a$. So the discounting condition holds by (3.1).

Denote $C_{a,l} = \{v \;:\; v \text{ is bounded, continuous, decreasing, and } v|_{[0,a]} \geq l\} \subseteq B^1$ to be the subset of all bounded continuous decreasing functions taking values in $[l,\infty)$ when restricted to $[0,a]$. Consider $C \equiv \times_{N\leq\overline{N},\theta=G,B} C_{\overline{f}_N^\theta,\kappa_N^\theta}$. Clearly C is a closed subset of $B^{2\overline{N}}$. Next I show $T(C) \subseteq C$. Suppose $v \in C$ and $f_1 \leq f_2$. By the definition of f_+, we have $f_+(\frac{f_1}{z},N) \leq f_+(\frac{f_2}{z},N)$. To simplify notation, let $f_{1+} \equiv f_+(\frac{f_1}{z},N)$. The following inequalities must hold:

$$\begin{aligned} T_N^\theta(v)(f_1) &= E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f_1,N\lambda z)\leq\kappa^{\theta'}z} \max_{N_+} v_{N_+}^{\theta'}(f_{1+})z + \\ &\quad \mathbf{1}_{\min(f_1,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\ &\geq E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f_1,N\lambda z)\leq\kappa^{\theta'}z} \max_{N_+} v_{N_+}^{\theta'}(f_{2+})z + \\ &\quad \mathbf{1}_{\min(f_1,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\ &\geq E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f_2,N\lambda z)\leq\kappa^{\theta'}z} \max_{N_+} v_{N_+}^{\theta'}(f_{2+})z + \\ &\quad \mathbf{1}_{\min(f_2,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\ &= T_N^\theta(v)(f_2). \end{aligned}$$

The last inequality is because that $v^{\theta}_{N_+}(f_{2+}) \geq \kappa^{\theta} \geq \lambda$ and $\{z| \min(f_1, N\lambda z) \leq \kappa^{\theta} z\} \supseteq \{z| \min(f_2, N\lambda z) \leq \kappa^{\theta} z\}$ for $\theta = G, B$. So each coordinate in $T(v)$ is also a decreasing function. In addition,

$$\begin{aligned}
T^{\theta}_N(v)(\overline{f}^{\theta}_N) &= E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(\overline{f}^{\theta}_N, N\lambda z) \leq \kappa^{\theta'} z} \max\nolimits_{N_+} v^{\theta'}_{N_+}(f_+) z + \\
&\quad \mathbf{1}_{\min(\overline{f}^{\theta}_N, N\lambda z) > \kappa^{\theta'} z} \lambda z]\} \\
&\geq E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(\overline{f}^{\theta}_N, N\lambda z) \leq \kappa^{\theta'} z} \kappa^{\theta'} z + \mathbf{1}_{\min(\overline{f}^{\theta}_N, N\lambda z) > \kappa^{\theta'} z} \lambda z]\} \\
&\geq E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(\overline{f}^{\theta}_N, N\lambda z) \leq \kappa^{\theta'} z} \min(\overline{f}^{\theta}_N, N\lambda z) + \\
&\quad \mathbf{1}_{\min(\overline{f}^{\theta}_N, N\lambda z) > \kappa^{\theta'} z} \min(\overline{f}^{\theta}_N, N\lambda z)]\} \\
&= \kappa^{\theta}_N
\end{aligned}$$

The first inequality uses the fact $\max_{N_+} v^{\theta}_{N_+}(f_+) \geq \max_{N_+} \kappa^{\theta}_{N_+} = \kappa^{\theta}$ for both $\theta = G, B$, since $v \in C$. The second inequality holds because $\min(\overline{f}^{\theta}_N, N\lambda z) \leq \kappa^{\theta} z$ over the relevant region. The last equality is by the definition of $\overline{f}^{\theta}_N$ and (3.16). Because $T^{\theta}_N(v)$ is a decreasing function, so $T^{\theta}_N(v)|_{[0, \overline{f}^{\theta}_N]} \geq \kappa^{\theta}_N$. Finally, I show that $T^{\theta}_N(v)$ must be a continuous function. Consider $\frac{f_2}{f_1} = 1 + \delta$. By definition (A.52)

$$\begin{aligned}
T^{\theta}_N(v)(f_2) &= \pi\mu \mathbf{1}_{\theta=G} + (1-\pi) \textstyle\sum_{\theta'=G,B} P(\theta'|\theta) \\
&\quad [\textstyle\int_{\min(f_2, N\lambda z) \leq \kappa^{\theta'} z} \max_{N_+} v^{\theta'}_{N_+}(f_{2+}) z g(z) dz + \int_{\min(f_2, N\lambda z) > \kappa^{\theta'} z} \lambda z g(z) dz] \\
&= \pi\mu \mathbf{1}_{\theta=G} + (1-\pi) \textstyle\sum_{\theta'=G,B} P(\theta'|\theta) \\
&\quad [\textstyle\int_{\min(f_1, N\lambda z') \leq \kappa^{\theta'} z'} \max_{N_+} v^{\theta'}_{N_+}(f_{1+})(1+\delta)^2 z' g[z'(1+\delta)] dz' \\
&\quad + \textstyle\int_{\min(f_1, N\lambda z') > \kappa^{\theta'} z'} \lambda(1+\delta) z' g[z'(1+\delta)] dz']
\end{aligned}$$

where the change of variable $z = (1+\delta)z'$. Notice that, by assumption, v^{θ}_N are bounded by some constant M and g is a density function, so, as $\delta \to 0$, the functions under the integrals in the above expression are dominated by $2Mz'g(2z')$. Because the random variable z has a finite mean, so $\int 2Mz'g(2z')dz' < \infty$. The dominated convergence theorem then implies that as $\delta \to 0$, the last expression converges to $T^{\theta}_N(v)(f_1)$. Therefore, the function $T^{\theta}_N(v)$ is continuous. In all, I have established that the contraction mapping T maps C into itself.

By contraction mapping theorem, the operator T has a unique fixed point $v^{\star} \in B^{2\overline{N}}$. Furthermore, this fixed point must belong to C. Define

$$V^{\theta}_N(F, Y) = v^{\theta\star}_N(\frac{F}{Y})Y. \tag{A.54}$$

which is decreasing in F. It is very easy to verify that the constructed solution satisfies the original recursive problem (3.17) with (3.18). Because $v_N^{\theta\star}(\frac{F}{Y})$ is increasing in Y, so V_N^θ as defined above is also increasing in Y. This completes the full proof of this proposition.

Proof of Lemma 3: Suppose otherwise if $\kappa^G \leq \kappa^B$, then

$$\begin{aligned} & \mathbf{1}_{\min(F,N\lambda z)\leq\kappa^G z}\min(f,N\lambda z)+\mathbf{1}_{\min(F,N\lambda z)>\kappa^G z}\min(f,\lambda z) \\ \leq\ & \mathbf{1}_{\min(F,N\lambda z)\leq\kappa^B z}\min(f,N\lambda z)+\mathbf{1}_{\min(F,N\lambda z)>\kappa^B z}\min(f,\lambda z). \end{aligned}$$

So (3.16) implies:

$$\begin{aligned} \kappa^B &\leq \max_{N,f}(1-\pi)E[\mathbf{1}_{\min(F,N\lambda z)\leq\kappa^B z}\min(f,N\lambda z)+\mathbf{1}_{\min(F,N\lambda z)>\kappa^B z}\min(f,\lambda z)] \\ &\leq (1-\pi)E\kappa^B z=(1-\pi)\mu\kappa^B \\ &< \kappa^B. \end{aligned}$$

Contradiction! So it must be $\kappa^G > \kappa^B$.

Proof of Proposition 10: First, if $N_1 = 1$, then by (3.15),

$$\begin{aligned} D^\theta_{N_2}(F,Y) &= E\{\pi\min(F,Yz)\mathbf{1}_{\theta=G}+(1-\pi)[\mathbf{1}_{\min(F,N_2\lambda Yz)\leq\kappa^{\theta'}Yz}\min(F,N_2\lambda Yz)+ \\ &\qquad \mathbf{1}_{\min(F,N_2\lambda Yz)>\kappa^{\theta'}Yz}\min(F,\lambda Yz)]\} \\ &\geq E[\pi\min(F,Yz)\mathbf{1}_{\theta=N}+(1-\pi)\min(F,\lambda Yz)] \\ &= D^\theta_1(F,Y). \end{aligned}$$

If $\underline{N} \geq N_2 > N_1$, by the definition of $\underline{N}$ in (3.20), then the liquidation region $\{z|\min(F, N_i\lambda Yz) > \kappa^\theta Yz\} = \emptyset$ for $\theta = G, B$. Therefore,

$$\begin{aligned} D^\theta_{N_2}(F,Y) &= E[\pi\min(F,Yz)\mathbf{1}_{\theta=G}+(1-\pi)\min(F,N_2\lambda Yz)] \\ &\geq E[\pi\min(F,Yz)\mathbf{1}_{\theta=G}+(1-\pi)\min(F,N_1\lambda Yz)] \\ &= D^\theta_{N_1}(F,Y). \end{aligned}$$

Finally, if $\overline{N} > N_2 > N_1 \geq \underline{N}$, then $\{z|\min(F, N_i\lambda Yz) > \kappa^G Yz\} = \emptyset$ and $\{z|\min(F, N_i\lambda Yz) > \kappa^B Yz\} = \{z|F > \kappa^B Yz\}$. Therefore,

$$\begin{aligned} D^\theta_{N_2}(F,Y) &= E\{\pi\min(F,Yz)\mathbf{1}_{\theta=G}+(1-\pi)\{P(G|\theta)\min(F,N_2\lambda Yz) \\ &+ P(B|\theta)[\mathbf{1}_{F\leq\kappa^B Yz}\min(F,N_2\lambda Yz)+\mathbf{1}_{F>\kappa^B Yz}\min(F,\lambda Yz)]\}\} \\ &\geq E\{\pi\min(F,Yz)\mathbf{1}_{\theta=G}+(1-\pi)\{P(G|\theta)\min(F,N_1\lambda Yz) \\ &+ P(B|\theta)[\mathbf{1}_{F\leq\kappa^B Yz}\min(F,N_1\lambda Yz)+\mathbf{1}_{F>\kappa^B Yz}\min(F,\lambda Yz)]\}\} \\ &= D^\theta_{N_1}(F,Y). \end{aligned}$$

So statement 1 holds. Higher debt capacity with N_2 in each category ($\kappa^\theta_{N_2} \geq \kappa^\theta_{N_1}$) is a direct implication of the previous statement.

Finally, to show the last statement, by definition (3.15), $D^\theta_N(F, Y)$ is continuous in F with $D^\theta_N(0, Y) = 0$. Intermediate value theorem guarantees the existence of the solutions $F^\theta_{N_i}$. Utilizing statement 1,

$$D^\theta_{N_2}(F^\theta_{N_2}, Y) = S = D^\theta_{N_1}(F^\theta_{N_1}, Y) \leq D^\theta_{N_2}(F^\theta_{N_1}, Y).$$

Again by intermediate value theorem, the minimum solution to $D^\theta_{N_2}(F^\theta_{N_2}, Y) = S$ must be within $(0, F^\theta_{N_1}]$, completing the proof of the proposition.

Proof of Proposition 12: For any continuation number of creditors N_+, define F_{+,N_i} $(i = 1, 2)$ to be the minimum solution such that $D^\theta_{N_+}(F_{+,N_i}, Yz) = \min(F, N_i\lambda Yz)$. For a given N_+

$$\begin{aligned} D^\theta_{N_+}(F_{+,N_2}, Yz) &= \min(F, N_2\lambda Yz) \\ &\geq \min(F, N_1\lambda Yz) \\ &= D^\theta_{N_+}(F_{+,N_1}, Yz). \end{aligned}$$

Thus $F_{+,N_2} \geq F_{+,N_1}$, so for any given continuation number of creditors N_+, having more incumbent creditors $N_2 > N_1$ implies higher continuation face value $F_{+,N_2} \geq F_{+,N_1}$. By the recursive formulation (3.17) and proposition 9 we have:

$$\begin{aligned} V^\theta_{N_2}(F, Y) &= E\{\pi Yz\mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(F,N_2\lambda Yz)\leq\kappa^{\theta'}Yz} \max\nolimits_{N_+} V^{\theta'}_{N_+}(F_{+,N_2}, Yz) \\ &+ \mathbf{1}_{\min(F,N_2\lambda Yz)>\kappa^{\theta'}Yz}\lambda Yz]\} \\ &\leq E\{\pi Yz\mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(F,N_1\lambda Yz)\leq\kappa^{\theta'}Yz} \max\nolimits_{N_+} V^{\theta'}_{N_+}(F_{+,N_2}, Yz) \\ &\quad +\mathbf{1}_{\min(F,N_1\lambda Yz)>\kappa^{\theta'}Yz}\lambda Yz]\} \\ &\leq E\{\pi Yz\mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(F,N_1\lambda Yz)\leq\kappa^{\theta'}Yz} \max\nolimits_{N_+} V^{\theta'}_{N_+}(F_{+,N_1}, Yz) \\ &\quad +\mathbf{1}_{\min(F,N_1\lambda Yz)>\kappa^{\theta'}Yz}\lambda Yz]\} \\ &= V^\theta_{N_1}(F, Y). \end{aligned}$$

The first equality is by definition. The second inequality is because $\{z| \min(F, N_2\lambda Yz) \leq \kappa^\theta Yz\} \subseteq \{z| \min(F, N_1\lambda Yz) \leq \kappa^\theta Yz\}$ and $V^\theta_{N_+}(F_+, Yz) \geq \lambda Yz$ by proposition 9. The third inequality is because $F_{+,N_2} \geq F_{+,N_1}$ and the fact that $V^\theta_{N_+}(F_+, Yz)$ is decreasing in F_+ by proposition 9. Thus $V^\theta_{N_2}(F, Y) \leq V^\theta_{N_2}(F, Y)$.

Proof of Proposition 14: The proof shares the same spirit as the existence proof of κ^θ in proposition 9 part II. Define the same order-preserving function $L : \mathbb{R}^2_+ \to \mathbb{R}^2_+$

as in the proof of proposition 9 with the expectations taken under the distribution g_a. Pick any pair of $\kappa^{\theta,b}$. I shall prove that there exists a fixed point $\kappa^{\theta,a} \in \Omega$ of L, where $\Omega = [\kappa^{G,b}, M] \times [\kappa^{B,b}, M]$ is a complete lattice and M is given by (A.51). For any N and $\kappa^{\theta,a} \in \Omega$,

$$
\begin{aligned}
\hat{\kappa}_N^{\theta,a} &= \max_f E_{g_a}\{\pi \min(f,z)\mathbf{1}_{\theta=G} \\
&+ (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\le\kappa^{\theta',a}z}\min(f,N\lambda z) + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta',a}z}\min(f,\lambda z)]\} \\
&\ge \max_f E_{g_a}\{\pi \min(f,z)\mathbf{1}_{\theta=G} \\
&+ (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\le\kappa^{\theta',b}z}\min(f,N\lambda z) + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta',b}z}\min(f,\lambda z)]\} \\
&\ge \max_f E_{g_b}\{\pi \min(f,z)\mathbf{1}_{\theta=G} \\
&+ (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\le\kappa^{\theta',b}z}\min(f,N\lambda z) + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta',b}z}\min(f,\lambda z)]\} \\
&= \kappa_N^{\theta,b}.
\end{aligned}
$$

The first inequality is because $\min(f,N\lambda z) \ge \min(f,\lambda z)$ and $\{z|\min(f,N\lambda z) \le \kappa^{\theta,b}z\} \subseteq \{z|\min(f,N\lambda z) \le \kappa^{\theta,a}z\}$. The second inequality uses first order stochastic dominance and the fact that the function under the expectation is weakly increasing in z. Therefore, for any $\kappa^{\theta,a} \in \Omega$, $L^\theta(\kappa^{\theta,a}) = \max_N \hat{\kappa}_N^{\theta,a} \ge \max_N \kappa_N^{\theta,b} \ge \kappa^{\theta,b}$. So $L(\Omega) \subseteq \Omega$ and Tarski's fixed point theorem completes the argument. The omitted proof for the other direction is very similar, with the auxiliary set $\Omega = [0, \kappa_N^{\theta,a}]$.

Proof of Proposition 15: First I show $V_{CF}^\theta(F,Y) < V_1^\theta(F,Y)$. Recall the function space C and the mapping T defined in proposition 9 part III. Define a new closed subset of functions in $B^{2(\overline{N}+1)}$: $C_A = \{(v_{CF}^\theta, v_N^\theta)_{N\le\overline{N}}^{\theta=G,B} | (v_N^\theta)_{N\le\overline{N}}^{\theta=G,B} \in C \text{ and } v_{CF}^\theta \le v_1^\theta\} \subset B^{2(\overline{N}+1)}$. Let $C_B = \{(v_{CF}^\theta, v_N^\theta)_{N\le\overline{N}}^{\theta=G,B} \in C_A | v_{CF}^\theta < v_1^\theta \text{ for all } f > 0\} \subset C_A$. Finally let $C_\beta = \{(v_{CF}^\theta, v_N^\theta)_{N\le\overline{N}}^{\theta=G,B} \in C_A | v_{CF}^\theta(f) < v_1^\theta(f) \text{ for all } f > \beta\}$. Clearly $C_B = C_0 \subset C_{\beta_2} \subset C_{\beta_1} \subset C_\infty = C_A$ for any $\beta_2 < \beta_1$. Define a new mapping $TC(v_{CF}^\theta, v_N^\theta)_{N\le\overline{N}}^{\theta=G,B} = \Big(T_{CF}[(v_{CF}^\theta)^{\theta=G,B}], T[(v_N^\theta)_{N\le\overline{N}}^{\theta=G,B}]\Big)$ on $B^{2(\overline{N}+1)}$, where $T_{CF} = (T_{CF}^G, T_{CF}^B)$ is given by

$$
T_{CF}^\theta(v_{CF}) = \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\le\kappa_1^{\theta'}z} v_{CF}^{\theta'}(f_{+,1})z + \mathbf{1}_{\min(f,\lambda z)>\kappa_1^{\theta'}z}\lambda z]
$$

where $f_{+,N}$ is an abbreviation for $f_+(\frac{f}{z}, N)$, the minimum solution to $D_N^\theta(f_+,1) = \min(\frac{f}{z}, \lambda N)$ as before. Similar to the proof in proposition 9 part III, it is straight forward to check that TC defined above satisfies the monotonicity and discounting conditions stated in lemma A-3. So TC must have a unique fixed point $v^\star$ in $B^{2(\overline{N}+1)}$. Our goal is to show this $v^\star \in C_B$.

Claim: there exists a decreasing sequence of $\beta_n \to 0$ such that $\beta_0 = \infty$ and $TC(C_{\beta_n}) \subseteq C_{\beta_{n+1}}$.

Given this claim, we have

$$TC(C_A) = TC(C_\infty) \subseteq C_{\beta_1} \subseteq C_A. \tag{A.55}$$

The contraction mapping theorem states that the unique fixed point can be derived from repeated iterations starting from any point v, i.e., $v^\star = \lim_{n\to\infty} TC^{(n)}(v)$. Because the set C_A is closed, one can start the iteration from any point $v \in C_A$ and the limiting point $v^\star$ will stay in C_A by (A.55). Furthermore, for any n, one can argue $v^\star \in TC^{(n)}(C_A) \subseteq C_{\beta_n}$. As $n \to \infty$, $v^\star \in \lim_{n\to\infty} TC^{(n)}(C_A) \subseteq \lim_{n\to\infty} C_{\beta_n} = C_0 = C_B$. Therefore, $v^\star \in C_B$. Let $V^\theta_{CF}(F,Y) = v^{\theta\star}_{CF}(\frac{F}{Y})Y$. Following the same procedures in proposition 9 part III, one can check that it is indeed the firm's value function in the counterfactual case. The fact $v^\star \in C_B$ implies $V^\theta_{CF}(F,Y) < V^\theta_1(F,Y)$ for all $F > 0$, completing the first half of the statement in the proposition.

Finally, when $F < \lambda\underline{z}Y < \kappa^\theta Y$, the actual repayment in the true model must be $F = \min(F, \lambda NzY)$ regardless of the number of incumbent creditors N. The firm always survives the next period. Therefore, it is easy to see from (3.17) and (3.18) that the firm values do not depend on N when $F < \lambda\underline{z}Y$. Combining with the result we just proved, it is immediate that $V^\theta_N(F,Y) = V^\theta_1(F,Y) > V^\theta_{CF}(F,Y)$ for all N, establishing the proposition.

Proof of the claim: Suppose $(v^\theta_{CF}, v^\theta_N)^{\theta=G,B}_{N\le\overline{N}} \in C_A$. By the construction of the operator T^θ_{CF}, we have

$$\begin{aligned} T^\theta_{CF}(v_{CF}) &= \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\le\kappa_1^{\theta'}z} v^{\theta'}_{CF}(f_{+,1})z + \mathbf{1}_{\min(f,\lambda z)>\kappa_1^{\theta'}z}\lambda z] \\ &\le \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\le\kappa_1^{\theta'}z} v^{\theta'}_{1}(f_{+,1})z + \mathbf{1}_{\min(f,\lambda z)>\kappa_1^{\theta'}z}\lambda z] \end{aligned} \tag{A.56}$$

Because $\kappa^\theta_1 \le \kappa^\theta$ and $\kappa^B_1 = (1-\pi)\mu\lambda < \lambda < \kappa^B$, so whenever $f > (1-\pi)\mu\lambda\underline{z}$ there is a positive probability that $f > \kappa^B_1 z$. In addition, because $\max_{N_+} v^\theta_{N_+}(f_{+,N_+}) \ge \max_{N_+}\kappa^\theta_{N_+} = \kappa^\theta > \lambda$, the last expression in (A.56) is strictly dominated by

$$\begin{aligned} \text{(A.56)} \quad &< \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\le\kappa^{\theta'}z} \max_{N_+} v^{\theta'}_{N_+}(f_{+,N_+})z \\ &\quad +\mathbf{1}_{\min(f,\lambda z)>\kappa^{\theta'}z}\lambda z] \\ &= \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\max_{N_+} v^{\theta'}_{N_+}(f_{+,N_+})z] \\ &= T^\theta_1((v^\theta_N)^{\theta=G,B}_{N\le\overline{N}}). \end{aligned} \tag{A.57}$$

Therefore, $TC((v^{\theta}_{CF}, v^{\theta}_{N})^{\theta=G,B}_{N\leq\overline{N}}) \in C_{(1-\pi)\mu\lambda\underline{z}}$ and we can pick $\beta_1 = (1-\pi)\mu\lambda\underline{z}$. Let $\beta_{n+1} = (1-\pi)\beta_n$. I shall prove that $TC(v) \in C_{(1-\pi)\beta_n}$ for all $v \in C_{\beta_n}$. Suppose $(v^{\theta}_{CF}, v^{\theta}_{N})^{\theta=G,B}_{N\leq\overline{N}} \in C_{\beta_n}$ and consider any $f \in (\beta_n(1-\pi), \beta_n]$. On one hand, from the rollover condition (A.53) and the fact that $\frac{f}{z} \leq \frac{f}{\underline{z}} \leq \frac{\beta_n}{\underline{z}} < \frac{\beta_1}{\underline{z}} = (1-\pi)\mu\lambda < \lambda$, we have

$$D^B_1(f^B_+, 1) = \min(\frac{f}{z}, \lambda) = \frac{f}{z}.$$

On the other hand, from expression (3.15) and the fact $f \leq \beta_n \leq \lambda\underline{z}$, we have

$$D^B_1(f^B_+, 1) = (1-\pi)f^B_+.$$

The above two equalities together imply that $f^B_{+,1} = \frac{f}{z(1-\pi)} > \frac{\beta_n}{z}$, which in turn implies that there is positive possibility that $f^B_{+,1} > \beta_n$. By the construction of the set C_{β_n}, $v^B_{CF}(f_{+,1}) < v^B_1(f_{+,1})$ holds strictly when $f_{+,1} > \beta_n$. Therefore, the inequality (A.56) holds strictly in this case. On the other hand, the weak inequality between (A.56) and (A.57) is trivial, so we again have $T^{\theta}_{CF}(v_{CF})(f) < T^{\theta}((v^{\theta}_{N})^{\theta=G,B}_{N\leq\overline{N}})(f)$ for all $f > (1-\pi)\beta_n$. Therefore, we have established $TC(C_{\beta_n}) \subseteq C_{\beta_{n+1}}$ for the constructed sequence of β_n that converges to zero, completing the proof of the claim and the whole proposition.

Proof of Lemma 5: By definition (3.28), $\lim_{T\to\infty}\hat{\pi} = \lim_{T\to\infty}\frac{\pi}{T} = 0$ is obvious. Rewrite (3.29) using (3.3):

$$\frac{\hat{\pi}[1-(1-\hat{\pi})\hat{\mu}\hat{p}]\hat{\mu}}{[1-(1-\hat{\pi})\hat{\mu}][1-(1-\hat{\pi})\hat{\mu}(2\hat{p}-1)]} = \frac{\pi[1-(1-\pi)\mu p]\mu}{[1-(1-\pi)\mu][1-(1-\pi)\mu(2p-1)]}, \tag{A.58}$$

$$\frac{\hat{\pi}(1-\hat{p})(1-\hat{\pi})\hat{\mu}^2}{[1-(1-\hat{\pi})\hat{\mu}][1-(1-\hat{\pi})\hat{\mu}(2\hat{p}-1)]} = \frac{\pi(1-p)(1-\pi)\mu^2}{[1-(1-\pi)\mu][1-(1-\pi)\mu(2p-1)]}. \tag{A.59}$$

Adding the above two equations, we have

$$\frac{\hat{\pi}\hat{\mu}}{1-(1-\hat{\pi})\hat{\mu}} = \frac{\pi\mu}{1-(1-\pi)\mu}. \tag{A.60}$$

Plugging in $\hat{\pi} = \frac{\pi}{T}$ from (3.28), one can solve for $\hat{\mu} = \frac{T\mu}{T\mu-\mu+1} \to 1$ as $T \to \infty$. Finally, in order to calculate $\hat{p}$, divide (A.58) by (A.59) and then we have

$$\frac{1-(1-\hat{\pi})\hat{\mu}\hat{p}}{(1-\hat{p})(1-\hat{\pi})\hat{\mu}} = \frac{1-(1-\pi)\mu p}{(1-p)(1-\pi)\mu}.$$

Subtract 1 from both sides and multiply it by (A.60),

$$\frac{\hat{\pi}}{(1-\hat{p})(1-\hat{\pi})} = \frac{\pi}{(1-p)(1-\pi)}.$$

Plug in $\hat{\pi} = \frac{\pi}{T}$ and we can solve for $\hat{p} = \frac{T-1+p(1-\pi)}{T-\pi}$. Clearly, when $T \geq 1$, $\hat{\pi}, \hat{p} \in (0,1)$. In addition, $\lim_{T\to\infty} \hat{\pi} = 0$ and $\lim_{T\to\infty} \hat{p} = 1$. Finally,

$$(1-\hat{\pi})\hat{\mu} = \frac{\mu(T-\pi)}{T\mu-\mu+1} = 1 - \frac{1-\mu(1-\pi)}{T\mu-\mu+1} < 1,$$

by assumption (3.1). Therefore, the new parameters are well defined.

Proof of Proposition 16: First, notice $\hat{\kappa}^G$ must be bounded as $T \to \infty$. This is because

$$\hat{\kappa}_N^G Y = \max_F D_N^G(F,Y) \leq V_{FB}^{G\star}(Y).$$

So $\hat{\kappa}^G = \max_N \hat{\kappa}_N^G$ must be bounded by some upper bound M ($\frac{V_{FB}^{G\star}(Y)}{Y}$ for example) that is independent of T. Let $\hat{N}$ be the number of creditors such that $\hat{\kappa}_{\hat{N}}^B$ attains the total debt capacity $\hat{\kappa}^B$, then

$$\begin{aligned}
\hat{\kappa}^B &= \max_f (1-\hat{\pi})\{\hat{p}[\mathbf{1}_{\min(f,\hat{N}\lambda\mu)\leq\hat{\kappa}^B\hat{\mu}} \min(f,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f,\hat{N}\lambda\hat{\mu})>\hat{\kappa}_2^B\hat{\mu}} \min(f,\lambda\hat{\mu})] \\
&+ (1-\hat{p})[\mathbf{1}_{\min(f,\hat{N}\lambda\hat{\mu})\leq\hat{\kappa}_2^G\hat{\mu}} \min(f,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f,\hat{N}\lambda\mu)>\hat{\kappa}_2^G\hat{\mu}} \min(f,\lambda\hat{\mu})]\}. \quad \text{(A.61)} \\
&\leq \max_f (1-\hat{\pi})\{\hat{p}[\mathbf{1}_{\min(f,\hat{N}\lambda\mu)\leq\hat{\kappa}^B\hat{\mu}} \min(f,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f,\hat{N}\lambda\hat{\mu})>\hat{\kappa}_2^B\hat{\mu}} \min(f,\lambda\hat{\mu})] \\
&+ (1-\hat{p})\min(f,\hat{N}\lambda\hat{\mu})\}. \quad \text{(A.62)}
\end{aligned}$$

Let $f^\star$ be the optimal f such that (A.61) attains $\hat{\kappa}^B$. Suppose $f^\star \leq \hat{\kappa}^B\hat{\mu}$. Notice that the expression in (A.62) is increasing in $f \in [0, \hat{\kappa}^B\hat{\mu}]$ and $(1-\hat{\pi})\hat{\mu} < 1$ by lemma 5, so

$$\hat{\kappa}^B \leq (1-\hat{\pi})\min(\hat{\kappa}^B\hat{\mu}, \hat{N}\lambda\hat{\mu}) < \hat{\kappa}^B.$$

Contradiction! On the other hand, if $\hat{N}\lambda \leq \hat{\kappa}^B$, then it is optimal to set $f^\star$ arbitrarily large in (A.62) and $\hat{\kappa}^B = (1-\hat{\pi})\hat{N}\lambda\hat{\mu} < \hat{N}\lambda$. Again a contradiction! Therefore, at $f = f^\star$, it must be $\min(f^\star, \hat{N}\lambda\hat{\mu}) > \hat{\kappa}^B\hat{\mu}$ and (A.61) becomes:

$$\begin{aligned}
\hat{\kappa}^B &= (1-\hat{\pi})\{\hat{p}\min(f^\star,\lambda\hat{\mu}) \\
&+ (1-\hat{p})[\mathbf{1}_{\min(f^\star,\hat{N}\lambda\hat{\mu})\leq\hat{\kappa}^G\hat{\mu}} \min(f^\star,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f^\star,\hat{N}\lambda\hat{\mu})>\hat{\kappa}^G\hat{\mu}} \min(f^\star,\lambda\hat{\mu})]\} \\
&\leq (1-\hat{\pi})\{\hat{p}\min(f^\star,\lambda\hat{\mu}) + (1-p)\hat{\kappa}^G\hat{\mu}\} \\
&\leq (1-\hat{\pi})\{\hat{p}\lambda\hat{\mu} + (1-\hat{p})\hat{\mu}M\}, \quad \text{(A.63)}
\end{aligned}$$

where the last inequality uses the fact that $\hat{\kappa}^G \leq M$, which is independent of T. As $T \to \infty$, lemma 5 states $\hat{\pi} \to 0$, $\hat{\mu} \to 1$, and $\hat{p} \to 1$, so the upper bound given by (A.63) approaches λ. Finally, because $\hat{\kappa}^B \geq \hat{\kappa}_1^B \to \lambda$ as $T \to \infty$, so we conclude $\lim_{T\to\infty} \hat{\kappa}^B = \lambda$.

A.2 Figures

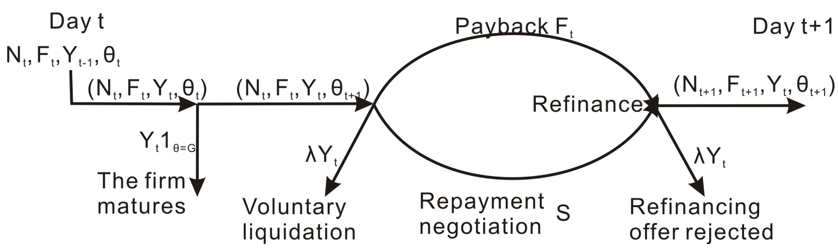

Figure 1 *The timeline and the evolution of the state variables.*

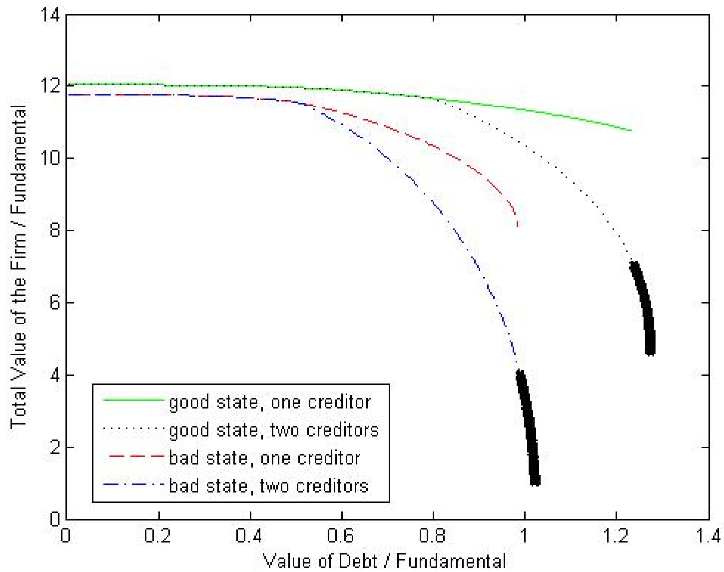

Figure 2 *The figure plots the expected total firm value against the expected value of the debt. The solid (dashed) line is the firm value with a single creditor when the fundamental* $\theta = G$ $(\theta = B)$*. The dotted (dash-dotted) line is the firm value with two creditors when the fundamental* $\theta = G$ $(\theta = B)$*. The thick solid black segments can be supported only by two creditors. Although the firm values are comparatively much lower along the thick lines, the firm cannot even reach that portion with just one creditor. When the value of debt is very low the choice between one and two creditors is irrelevant. As the fundamental worsens, the two groups of lines diverge and, when both are feasible, the single creditor structure always delivers a higher firm value.*

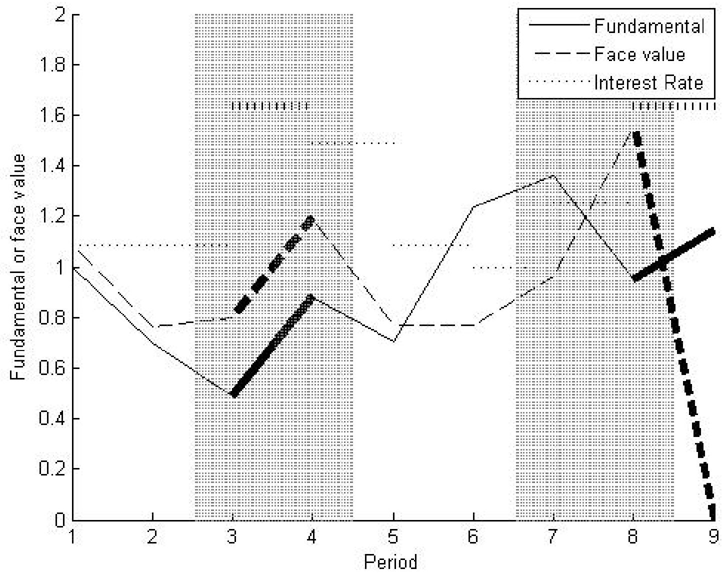

Figure 3 *The figure plots a typical sample path of the firm. Areas are shaded when the state is bad. The solid (dashed) line denotes the exogenous fundamental process* Y_{t-1} *(the face value process* F_t *determined in equilibrium). I use bold segments when the firm chooses two creditors. The values plotted at each period* t *are the state variables entering period* t*: number of creditors* N_t*, the promised face value* F_t*, state* θ_t*, and fundamental process* Y_{t-1}*. Finally, the dotted bars plot the interest rates* $\frac{F_t}{D^{\theta_t}_{N_t}(F_t, Y_{t-1})}$ *during each period.*

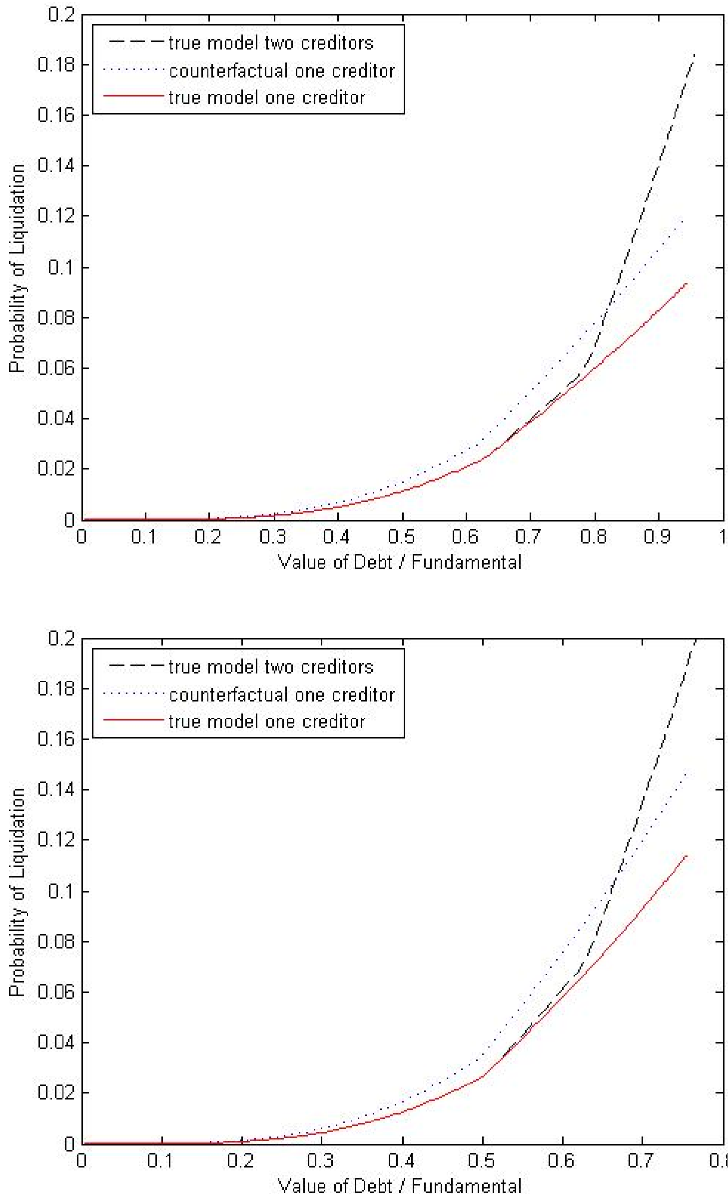

Figure 4 *The figure plots the expected probability of liquidation* $L_1^{\theta,\infty}(F,Y)$ *against the expected value of the debt conditional on the current state* $\theta = G$ *(top panel) and* $\theta = B$ *(bottom panel). The solid (dashed) line is the liquidation probability with a single creditor (two creditors) in the full model. The dotted line is for the counterfactual model in which the number of creditors is exogenously fixed at one. It is easy to see that having a single*

creditor in the true model means a lower liquidation probability compared to having two creditors as well as the counterfactual one creditor model. For a substantial range of fundamental values, the liquidation probability with two creditors *in the true model is strictly lower compared with the* single *creditor counterfactual.*