CONTROLLING FOR OPTIMUM GROWTH WITH TIME DEPENDENT RETURNS

By

Raoul LePage

IMA Preprint Series # 875

October 1991

CONTROLLING FOR OPTIMUM GROWTH WITH TIME DEPENDENT RETURNS*

RAOUL LEPAGE†

Abstract. Kelly (1956) gave a gambling interpretation of Shannon's notion of channel capacity, in the course of which a strategy now known as the Kelly Principle emerged: invest so as to maximize the expected logarithm of the return. Optimality of the Kelly Principle was considered from several points of view by Breiman (1961) who established rapid growth and least expected time to attain a large return. In this note, a new principle for achieving rapid growth is obtained for investments in which the returns are obtained over random amounts of time. This new principle is: invest so as to maximize the ratio of the expected logarithm of the return to the expected time to obtain it. The Kelly Principle corresponds to the case of a degenerate distribution for the time component. Breiman's interesting asymptotic formula comparing the time τ required to attain a large fortune C, in repeated play, versus the time τ^* required by the Kelly strategy is:

$$\lim_{C \to \infty} E(\tau - \tau^*) = \Sigma \left[1 - \frac{E\ ln\ Y(\gamma_j, j)}{E\ ln\ Y(\gamma^*, 1)} \right],$$

where $Y(\gamma_j, j)$ (resp. $Y(\gamma^*, j)$) is the random multiplicative factor by which the fortune at state j is transformed into the fortune at state $j+1$, consequent to the predictably chosen strategic choice γ_j made for state j, asterisk denoting Kelly play. It is assumed that $ln\ Y(\gamma^*, 1)$ is non-lattice in the above. We obtain Breiman's formula as the corresponding special case of a new formula:

$$\lim_{C \to \infty} E(\tau - \tau^*) = (E\ t(\gamma^*, 1)) \Sigma \left[\frac{E\ t(\gamma_j, j)}{E\ t(\gamma^*, 1)} - \frac{E\ ln\ Y(\gamma_j, j)}{E\ ln\ Y(\gamma^*, 1)} \right],$$

in which $t(\gamma_j, j)$ (resp. $t(\gamma^*, j)$) is the random time required for the j-th state of play to be completed, asterisk denoting ratio play. The new principle has obvious applications to gambling and investment and potential for generating fast converging algorithms for real-time estimation and prediction, in which one must decide between high-quality/infrequent estimate revisions versus low-quality/frequent estimate revisions.

Introduction. We begin with the model considered by Kelly (1956). Consider a finite probability experiment $\{p_1, \ldots, p_k\}$ and non–negative numbers (multipliers) $\{m_1, \ldots, m_k\}$. The idea is to bet on the outcome of this experiment with the understanding that a bet of f on outcome i will return $m_i f$ if outcome i occurs, and will otherwise return zero.

Whatever the amount of one's fortune f, and regardless of the state of play, bets may be placed on any or all of the outcomes, and some amount may be held back. Having a positive fortune f, one may bet on the outcomes $\{1, \ldots, k\}$ by choosing amounts $\gamma_i \geq 0$, satisfying $\sum \gamma_i \leq 1$, and then proceeding to bet $f\gamma_i$ on outcome i. The return is

$$(\Sigma\ m_i\ \gamma_i(X = i) + (1 - \Sigma\ \gamma_i))f,$$

*Research partially supported by ONR Grants N00014–85–K–0150 and N00014–91–J–1087.

†Statistics and Probability, Michigan State University, East Lansing, MI 48824

where X denotes a random variable with $P(X = i) = p_i$, and the expression $(X = i)$ denotes the indicator random variable for the event $\{X = i\}$.

Following Kelly, assume an initial positive fortune F_0 and (after each play) reinvest the new fortune using the fixed betting fractions γ_i for the next (independent) play. Let F_n be the fortune after play n. Then

$$F_n = (\Sigma m_i \ \gamma_i(X_n = i) + (1 - \Sigma\gamma_i))F_{n-1},$$

where $\{X_n\}$ are i.i.d. with $P(X_n = i) = p_i$. Write $F_n = Y(\gamma, n)F_{n-1}$ for the line above. As observed by Kelly, if $P(Y(\gamma, 1) = 0) = 0$, then with probability one

$$n^{-1} ln \ (F_n/F_0) = \sum_{j=1}^{n} ln \ Y(\gamma, j)/n \to E \ ln \ Y(\gamma, 1).$$

and the limit is concave in γ. It follows that if $E \ ln \ Y(\gamma^*, 0) > E \ ln \ Y(\gamma, 1)$ then with probability one $F_n^* > F_n$ for all sufficiently large n. The Kelly Principle (broadly stated) is that, by choosing γ^* to maximize the expectation of the logarithm, the fortune F_n^* will eventually persist in higher returns than those consequent to any strategy which does not maximize the expected logarithm.

Time Dependent Returns. We now suppose that the factor $Y(\gamma, n)$ requires a random amount of time $t(\gamma, n)$ to be completed. The random vectors $(Y(\gamma, n),\ t(\gamma, n))$ are assumed to be i.i.d. for each γ, although Y and t need not be independent. The fortune $F_n = F_n(\gamma)$ is therefore available at time $T_n = \Sigma_1^n \ t(\gamma, j)$. For convenience, set $F_0 = 1$ and for each $C > 1$ define

$$\begin{aligned} N_c &= N_c(\gamma) = \min\{n : F_n \geq C\} \text{ (or } \infty), \\ \tau_C &= \tau_C(\gamma) = \Sigma_1^{N_c} \ t(\gamma_j, j). \end{aligned} \tag{1}$$

The following simple argument shows the correctness of the ratio principle.

PROPOSITION 1. *If* $(E \ ln \ Y(\gamma^*, 1))/E \ t(\gamma^*, 1) > (E \ ln \ Y(\gamma, 1))/E \ t(\gamma, 1) > 0$ *then, for every sufficiently large* C, $E \ \tau_C^* < E \ \tau_C$.

Proof. Wald's identity (Breiman, 1968) implies that the following are equivalent

$$\begin{aligned} &E(\tau_C^* - \tau_C) < 0 \\ &(E \ N_C^*)E \ t(\gamma^*, 1) < (E \ N_C)E \ t(\gamma, 1). \end{aligned} \tag{2}$$

For all sufficiently large C, the last line above is equivalent to $(E \ ln \ Y(\gamma^*, 1))/E \ t(\gamma^*, 1) > (E \ ln \ Y(\gamma, 1))/E \ t(\gamma, 1)$, since by standard renewal arguments, $\dfrac{E \ N_C^*}{E \ N_C} \to \dfrac{E \ ln \ Y(\gamma, 1)}{E \ ln \ Y(\gamma^*, 1)}$ as $C \to \infty$. □

The next result generalizes Theorem 1 of Breiman (1961) to the case of time–dependent returns. It closely parallels the arguments before Theorem 1 of Breiman (1961).

PROPOSITION 2. *If γ^* maximizes $(E\ ln\ Y(\gamma,1))/E\ t(\gamma,1)$ for γ, $E\ ln\ Y(\gamma^*,1) > 0$, and $ln\ Y(\gamma^*,1)$ is non-lattice, then for every sequence of predictable strategies $\{\gamma_n\}$,*

$$\lim_{C\to\infty} E(\tau_C^* - \tau_C)/(E\ t(\gamma^*,1)) = E\sum\left[\frac{E^{\mathcal{F}_{j-1}}\ t(\gamma_j,j)}{E\ t(\gamma^*,1)} - \frac{E^{\mathcal{F}_{j-1}}\ ln\ Y(\gamma_j,j)}{E\ ln\ Y(\gamma^*,1)}\right]. \tag{3}$$

Proof. Define N_C^* and τ_C^* as in (1) for $\gamma = \gamma^*$; and N_C, τ_C for the sequence $\{\gamma_n\}$. Let $t_j = t(\gamma_j,j), t_j^* = t(\gamma^*,j), N = N_C, N^* = N_C^*, Y_j = Y(\gamma_j,j), Y_j^* = Y(\gamma^*,j), E^j = E^{\mathcal{F}_j}$ then

$$E\left[\overset{N}{\Sigma} t_j - \overset{N^*}{\Sigma}\ t_j^*\right] = E\ \overset{N}{\Sigma}(t_j - t_j^*) + E\ \overset{N}{\Sigma}\ t_j^* - E\ \overset{N^*}{\Sigma}\ t_j^* \tag{4}$$

By Wald's identity, (4) is equal to

$$\begin{aligned} &= E\ \overset{N}{\Sigma}(t_j - t_j^*) + (E\ t^*)E(N - N^*) \\ &= E\ \overset{N}{\Sigma}(E^{\mathcal{F}_{j-1}}t_j - E\ t^*) + \frac{E\ t^*}{E\ ln\ Y^*}\left[E\ \overset{N}{\Sigma}(E\ ln\ Y^* - E^{j-1} ln\ Y_j) + D\right], \end{aligned} \tag{5}$$

where $D = E\ \overset{N}{\Sigma}\ ln\ Y_j - E\ \overset{N^*}{\Sigma}\ ln\ Y_j^*$. As in Proposition 8 of Breiman (1961), on the event $\overset{N}{\Sigma}\ ln\ Y_j - \overset{N^*}{\Sigma}\ ln\ Y_j^* \to$ finite limit where $D \to 0$ a.s. . Elsewhere, the result (3) is trivial. Taking $C \to \infty$ in (4) gives

$$\begin{aligned} &\lim_{C\to\infty} E\left[\overset{N}{\Sigma}\ t_j - \overset{N}{\Sigma}\ t_j^*\right] \\ &= \lim_{C\to\infty}(E\ t^*)\ E\sum^{N}\left[\frac{E^{j-1}\ t_j}{E\ t^*} - \frac{E^{j-1}\ ln\ Y_j}{E\ ln\ Y^*}\right], \end{aligned} \tag{6}$$

which is equal to the right side of (3) since the bracketed term on the right of (6) is non-negative. □

Theorem 1 of Breiman (1961) is Proposition 2 above for the case $t_n \equiv t^* \equiv 1$.

Note. Breiman generalizes Kelly's model to allow for bets on events which are not disjoint. This generalization does not affect the above results or their proofs, which are valid for Breiman's model.

REFERENCES

[1] Bell, R. and Cover, T. (1980), *Competitive optimality of logarithmic investment*, Math. of O.R. 5 161–166.

[2] Breiman, L. (1961), *Optimal gambling systems for favorable games*, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, 1 65–78.

[3] Breiman, L. (1968), *Probability*, Addison Wesley, New York.

[4] Finkelstein, M. and Whitley, R. (1981), *Optimal Strategies for repeated games*, Adv. Appl. Prob. 13 415–428.

[5] Heath, D., Orey, S., Pestien, V. and Sudderth, W. (1987), *Minimizing or maximizing the expected time to reach zero*, SIAM J. of Control and Optimization, 25, 1, 195–205.

[6] Kelly, J. (1956), *A new interpretation of information rate*, Bell System Tech. J. 35 917–926.

[7] LePage, R. (1988), *Controlling a diffusion toward a large goal and the Kelly Principle, IMA Preprint Series #874, October 1991, U. of Minn.*

[8] LePage, R. and Schreiber, B., *A principle of Kelly–Breiman type for adaptive estimation*, Technical Rept. Center for Stochastic Processes, Univ. N. Carolina.

Recent IMA Preprints

#	Author/s	Title

801 **Hi Jun Choe**, Regularity for solutions of nonlinear variational inequalities with gradient constraints
802 **Peter Shi and Yongzhi Xu**, Quasistatic linear thermoelasticity on the unit disk
803 **Satyanad Kichenassamy and Peter J. Olver**, Existence and non-existence of solitary wave solutions to higher order model evolution equations
804 **Dening Li**, Regularity of solutions for a two-phase degenerate Stefan Problem
805 **Marek Fila, Bernhard Kawohl and Howard A. Levine**, Quenching for quasilinear equations
806 **Yoshikazu Giga, Shun'ichi Goto and Hitoshi Ishii**, Global existence of weak solutions for interface equations coupled with diffusion equations
807 **Mark J. Friedman and Eusebius J. Doedel**, Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study
808 **Mark J. Friedman**, Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds
809 **Peter W. Bates and Songmu Zheng**, Inertial manifolds and inertial sets for the phase-field equations
810 **J. López Gómez, V. Márquez and N. Wolanski**, Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition
811 **Xinfu Chen and Fahuai Yi**, Regularity of the free boundary of a continuous casting problem
812 **Eden, A., Foias, C., Nicolaenko, B. and Temam, R.**, Inertial sets for dissipative evolution equations Part I: Construction and applications
813 **Jose–Francisco Rodrigues and Boris Zaltzman**, On classical solutions of the two-phase steady-state Stefan problem in strips
814 **Viorel Barbu and Srdjan Stojanovic**, Controlling the free boundary of elliptic variational inequalities on a variable domain
815 **Viorel Barbu and Srdjan Stojanovic**, A variational approach to a free boundary problem arising in electrophotography
816 **B.H. Gilding and R. Kersner**, Diffusion-convection-reaction, free boundaries, and an integral equation
817 **Shoshana Kamin, Lambertus A. Peletier and Juan Luis Vazquez**, On the Barenblatt equation of elasto-plastic filtration
818 **Avner Friedman and Bei Hu**, The Stefan problem with kinetic condition at the free boundary
819 **M.A. Grinfeld**, The stress driven instabilities in crystals: mathematical models and physical manifestations
820 **Bei Hu and Lihe Wang**, A free boundary problem arising in electrophotography: solutions with connected toner region
821 **Yongzhi Xu, T. Craig Poling, and Trent Brundage**, Direct and inverse scattering of time harmonic acoustic waves in an inhomogeneous shallow ocean
822 **Steven J. Altschuler**, Singularities of the curve shrinking flow for space curves
823 **Steven J. Altschuler and Matthew A. Grayson**, Shortening space curves and flow through singularities
824 **Tong Li**, On the Riemann problem of a combustion model
825 **L.A. Peletier & W.C. Troy**, Self-similar solutions for diffusion in semiconductors
826 **C.J. van Duijn, L.A. Peletier & R.J. Schotting**, On the analysis of brine transport in porous media
827 **Minkyu Kwak**, Finite dimensional description of convective reaction-diffusion equations
828 **Minkyu Kwak**, Finite dimensional inertial forms for the 2D Navier–Stokes equations
829 **Victor A. Galaktionov and Sergey A. Posashkov**, On some monotonicity in time properties for a quasilinear parabolic equation with source
830 **Victor A. Galaktionov**, Remark on the fast diffusion equation in a ball
831 **Hi Jun Choe and Lihe Wang**, A regularity theory for degenerate vector valued variational inequalities
832 **Vladimir I. Oliker and Nina N. Uraltseva**, Evolution of nonparametric surfaces with speed depending on curvature, II. The mean curvature case.
833 **S. Kamin and W. Liu**, Large time behavior of a nonlinear diffusion equation with a source
834 **Shoshana Kamin and Juan Luis Vazquez**, Singular solutions of some nonlinear parabolic equations
835 **Bernhard Kawohl and Robert Kersner**, On degenerate diffusion with very strong absorption
836 **Avner Friedman and Fernando Reitich**, Parameter identification in reaction-diffusion models
837 **E.G. Kalnins, H.L. Manocha and Willard Miller, Jr.**, Models of q-algebra representations I. Tensor products of special unitary and oscillator algebras
838 **Robert J. Sacker and George R. Sell**, Dichotomies for linear evolutionary equations in Banach spaces
839 **Oscar P. Bruno and Fernando Reitich**, Numerical solution of diffraction problems: a method of variation of boundaries
840 **Oscar P. Bruno and Fernando Reitich**, Solution of a boundary value problem for Helmholtz equation via variation of the boundary into the complex domain
841 **Victor A. Galaktionov and Juan L. Vazquez**, Asymptotic behaviour for an equation of superslow diffusion. The Cauchy problem
842 **Josephus Hulshof and Juan Luis Vazquez**, The Dipole solution for the porous medium equation in several

space dimensions

843 **Shoshana Kamin and Juan Luis Vazquez**, The propagation of turbulent bursts

844 **Miguel Escobedo, Juan Luis Vazquez and Enrike Zuazua**, Source-type solutions and asymptotic behaviour for a diffusion-convection equation

845 **Marco Biroli and Umberto Mosco**, Discontinuous media and Dirichlet forms of diffusion type

846 **Stathis Filippas and Jong-Shenq Guo**, Quenching profiles for one-dimensional semilinear heat equations

847 **H. Scott Dumas**, A Nekhoroshev-like theory of classical particle channeling in perfect crystals

848 **R. Natalini and A. Tesei**, On a class of perturbed conservation laws

849 **Paul K. Newton and Shinya Watanabe**, The geometry of nonlinear Schrödinger standing waves

850 **S.S. Sritharan**, On the nonsmooth verification technique for the dynamic programming of viscous flow

851 **Mario Taboada and Yuncheng You**, Global attractor, inertial manifolds and stabilization of nonlinear damped beam equations

852 **Shigeru Sakaguchi**, Critical points of solutions to the obstacle problem in the plane

853 **F. Abergel, D. Hilhorst and F. Issard-Roch**, On a dissolution-growth problem with surface tension in the neighborhood of a stationary solution

854 **Erasmus Langer**, Numerical simulation of MOS transistors

855 **Haïm Brezis and Shoshana Kamin**, Sublinear elliptic equations in $\mathbb{R}^n$

856 **Johannes C.C. Nitsche**, Boundary value problems for variational integrals involving surface curvatures

857 **Chao–Nien Chen**, Multiple solutions for a semilinear elliptic equation on $\mathbb{R}^N$ with nonlinear dependence on the gradient

858 **D. Brochet, X. Chen and D. Hilhorst**, Finite dimensional exponential atttractor for the phase field model

859 **Joseph D. Fehribach**, Mullins-Sekerka stability analysis for melting-freezing waves in helium-4

860 **Walter Schempp**, Quantum holography and neurocomputer architectures

861 **D.V. Anosov**, An introduction to Hilbert's 21st problem

862 **Herbert E Huppert and M Grae Worster**, Vigorous motions in magma chambers and lava lakes

863 **Robert L. Pego and Michael I. Weinstein**, A class of eigenvalue problems, with applications to instability of solitary waves

864 **Mahmoud Affouf**, Numerical study of a singular system of conservation laws arising in enhanced oil reservoirs

865 **Darin Beigie, Anthony Leonard and Stephen Wiggins**, The dynamics associated with the chaotic of tangles two dimensional quasiperiodic vector fields: theory and applications

866 **Gui–Qiang Chen and Tai–Ping Liu**, Zero relaxation and dissipation limits for hyperbolic conservation laws

867 **Gui–Qiang Chen and Jian–Guo Liu**, Convergence of second–order schemes for isentropic gas dynamics

868 **Aleksander M. Simon and Zbigniew J. Grzywna**, On the Larché–Cahn theory for stress-induced diffusion

869 **Jerzy Łuczka, Adam Gadomski and Zbigniew J. Grzywna**, Growth driven by diffusion

870 **Mitchell Luskin and Tsorng-Whay Pan**, Nonplanar shear flows for nonaligning nematic liquid crystals

871 **Mahmoud Affouf**, Unique global solutions of initial-boundary value problems for thermodynamic phase transitions

872 **Richard A. Brualdi, Keith L. Chavey and Bryan L. Shader**, Rectangular L-matrices

873 **Xinfu Chen, Avner Friedman and Bei Hu**, The thermistor problem with zero–one conductivity II

874 **Raoul LePage**, Controlling a diffusion toward a large goal and the Kelly principle

875 **Raoul LePage**, Controlling for optimum growth with time dependent returns

876 **Marc Hallin and Madan L. Puri**, Rank tests for time series analysis a survey

877 **V.A. Solonnikov**, Solvability of an evolution problem of thermocapillary convection in an infinite time interval

878 **Horia I. Ene and Bogdan Vernescu**, Viscosity dependent behaviour of viscoelastic porous media

879 **Kaushik Bhattacharya**, Self-accommodation in martensite

880 **D. Lewis, T. Ratiu, J.C. Simo and J.E. Marsden**, The heavy top: a geometric treatment

881 **Leonid V. Kalachev**, Some applications of asymptotic methods in semiconductor device modeling

882 **David C. Dobson**, Phase reconstruction via nonlinear least-squares

883 **Patricio Aviles and Yoshikazu Giga**, Minimal currents, geodesics and relaxation of variational integrals on mappings of bounded variation

884 **Patricio Aviles and Yoshikazu Giga**, Partial regularity of least gradient mappings

885 **Charles R. Johnson and Michael Lundquist**, Operator matrices with chordal inverse patterns

886 **B.J. Bayly**, Infinitely conducting dynamos and other horrible eigenproblems

887 **Charles M. Elliott and Stefan Luckhaus**, 'A generalised diffusion equation for phase separation of a multi-component mixture with interfacial free energy'

888 **Christian Schmeiser and Andreas Unterreiter**, The derivation of analytic device models by asymptotic methods

889 **LeRoy B. Beasley and Norman J. Pullman**, Linear operators that strongly preserve the index of imprimitivity

890 **Jerry Donato**, The Boltzmann equation with lie and cartan

891 **Thomas R. Hoffend Jr., Peter Smereka and Roger J. Anderson**, Method for resolving the laser induced local heating of moving magneto-optical recording media