Magic Boxes and Related Topics

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There are many people that have earned my gratitude for their contribution to my time in graduate school.

Dedication

To those who held me up over the years.

Abstract

Magic boxes are a 3-dimensional generalization of magic rectangles, which in turn are a classical generalization of the magic square. In this paper, two new generalizations of the magic box are introduced: the magic box set and the magic hollow box. Several necessary conditions and several sufficient conditions for the existence of these structures are examined, as well as conditions which preclude the existence of these structures.

Contents

List of Figures

Chapter 1

Introduction

> There is no science that teaches the harmonies of nature more clearly than mathematics, and the magic squares are like a mirror which reflects the symmetry of the divine norm immanent in all things, in the immeasurable immensity of the cosmos and in the construction of the atom not less than in the mysterious depths of the human mind.
>
> —Paul Carus

Discovered by the ancients, studied for centuries, and still the topic of research today, the magic square has intrigued great minds throughout the world over the greater part of the last millennium [1]. The oldest known magic square is the *Lo Shu*, preserved in ancient Chinese literature. German artist *Albrecht Dürer* engraved the first known European magic square in his 1514 work *Melancholy. J.W. Göthe* presented a magic square of order 3 in his epic *Faust. Benjamin Franklin* was also known to have toyed with constructing magic squares. Rigorous mathematical study of magic square construction began in 1687 by the French aristocrat *Antoine de la Loubère* [2][3].

A *magic square* of order n is an $n \times n$ array containing the natural numbers $1, 2, 3, \ldots, n^2$ arranged such that the sum of the numbers along any row, column, or main diagonal is a fixed constant. The so-called *magic constant* for a magic square is the same for rows, columns, and main diagonals, and it is easy to see that this constant is $\frac{n(n^2+1)}{2}$ [4]. Magic squares are known to exist for every square where $n \neq 2$ [5].

One basic generalization of the magic square is the magic rectangle. An $m \times n$ *magic rectangle* is an $m \times n$ array containing the natural numbers $1, 2, \ldots, mn$ arranged such that the sum of the entries in each row is constant and the sum of each column is another constant (different constants if $m \neq n$) [6]. Note that there is no diagonal requirement as there are with magic squares. Magic rectangles can only exist when the two dimensions are the same parity and are both greater than 1, and it is known within these parameters that a magic rectangle exists when at least one dimension is greater than 2 [7].

Another natural generalization of the magic square is the magic cube. A *magic cube* of order n is an $n \times n \times n$ cubical array containing the natural numbers $1, 2, \ldots, n^3$ arranged such that the sums of the entries along each row, column, and pillar (rows in every dimension) and each of the four great diagonals are the same constant number. The *magic constant* for a magic cube is $\frac{n(n^3+1)}{2}$ [2]. It is known that a magic cube of order n exists for all $n \neq 2$ [4].

Mathematicians have relaxed and implemented restrictions for both magic squares and magic cubes to create new, similar structures. When we consider relaxing or augmenting the condition on the diagonal sums for squares and cubes, we derive the ideas of *semi-magic* (no diagonal sums are considered) and *super-magic* (all diagonals, not just the four great diagonals, of a cube must have the same constant sum as the rows) [3]. Semi-magic is an idea that applies to squares, rectangles, cubes, and rectangular prisms, whereas super-magic can only apply to squares, cubes, and hypercubes. Another analogue of the magic square and magic cube involves the relaxation of the requirement of using natural numbers. Kermes and Penner introduce the notion of using real numbers as entries of the semi-magic cube, and also discuss the use of entries from an Abelian group [3].

A further generalization is the magic box. As this is closely related to the topic of this paper, we will introduce more precise notation here. A magic box of size (a, b, c), which we will denote as $MB(a, b, c)$, is an $a \times b \times c$ array containing the natural numbers $1, 2, \ldots, abc$ arranged such that the sums of the entries along each of the a rows is some constant, the sums of the entries along each of the b columns is a (perhaps different) constant, and the sum along each of the ab pillars is also a (perhaps different) constant. Magic boxes can only exist when the three dimensions are the same parity. For even dimensions, a magic box exists if two of the dimensions are greater than 2 [6]. For odd dimensions, a magic box exists if two of the dimensions share a common factor [6]. Some magic boxes of pairwise relatively prime odd dimensions are known to exist, but it remains an open question whether there exists a magic box with arbitrary odd dimensions [8].

Another generalization is magic cubes and magic boxes of higher dimensions. Thomas Hagedorn refers to these shapes in general as magic n-rectangles [6]. A magic n-rectangle of size $(m_1, m_2, \ldots, m_n)$ is an $m_1 \times \cdots \times m_n$ array containing the natural numbers $1, 2, \ldots, m_1 m_2 \cdots m_n$ arranged such that the sums of the entries along each row in each of the n dimensions is some constant. As is the case with magic rectangles, the row sum for one dimension may be different from the row sum for a different dimension. Very little research has been done on these structures, but once again, all dimensions must be the same parity. For even dimensions, a magic n-rectangle exists if no two of the dimensions equal 2 [6]. There are no published results for odd dimensions for boxes of dimension 4 or higher.

We can make a further generalization of the magic square in two dimensions, called a magic rectangle set. A *magic rectangle set* $MRS(a, b; c)$ is a collection of c arrays of size $a \times b$ containing the natural numbers $1, 2, \ldots, abc$ arranged such that the sums of the entries along each row in each rectangle sums to the same constant and each column in each rectangle sums to another constant. It is known that a magic rectangle set $MRS(a, b; c)$ exists for every c if a and b are even and $ab > 4$ (i.e. both a and b

cannot equal 2) [9]. It is also known that $MRS(a, b; c)$ exists when a, b, and c are odd, $a > 1$, and $\gcd(a, b) > 1$ [10].

It is from the magic rectangle set that we make a new generalization in three dimensions, called the magic box set. A *magic box set* $MBS(a, b, c; d)$ is a collection of d arrays of size $a \times b \times c$ containing the natural numbers $1, 2, \ldots, abcd$ arranged such that the entries along each row in each rectangle sum to a constant number ρ, the entries along each column in each rectangle sum to a constant number σ, and the entries along each pillar in each rectangle sum to a constant number π. It is usually the case that $\rho \neq \sigma \neq \pi$; repeats can only happen when at least two of a, b, c are equal.

Yet another three-dimensional generalization of the magic rectangle arises when we consider a structure formed by linking n rectangles of size $a \times c$ along an edge, creating a regular n-gon with side length a and height c. The choice of notation reminds us that c is the height of each column. The middle of the structure is hollow; hence we will refer to this structure as a *magic hollow box*, and we will denote it as $MH(n, a, c)$. We look to fill this structure with the numbers $1, 2, \ldots, n(a-1)c$ (the number of total cells in such an object) such that each row sums to a constant ρ and each column sums to a constant σ.

Chapter 2

Problems

2.1 Problems Regarding Magic Box Sets

When we consider the magic box set, the chief question to answer is this: for which a, b, c and d does a magic box set $MBS(a, b, c; d)$ exist? We will now make an important observation which answers many cases of this question.

Observation 2.1. If a magic 4-rectangle $MR(a, b, c, d)$ exists, then a magic box set $MBS(a, b, c; d)$ exists.

This observation is clear: we can slice $MR(a, b, c, d)$ into d "slices" of size $a \times b \times c$. Since the magic 4-rectangle is magic along all four dimensions, it is certainly magic along any three dimensions. Hence, each $a \times b \times c$ slice is also magic along all three dimensions.

Using this observation, we will call on the following theorem proved by Thomas R. Hagedorn in [6] in order to partially solve our problem.

Theorem 2.2. *If m_i are positive even integers with $(m_i, m_j) \neq (2, 2)$ for $i \neq j$, then a magic n-rectangle $MR(m_1, m_2, \ldots, m_n)$ exists.*

Hagedorn proves in the same paper that the converse is also true, namely, that $MR(m_1, m_2, \ldots, m_{n-2}, 2, 2)$ does not exist. Hence we have our first result.

Theorem 2.3. *If a, b, c, and d are even and at most one of a, b, c, d equals 2, then $MBS(a, b, c; d)$ exists.*

Proof. This theorem follows directly from Theorem 2.2 and Observation 2.1. □

There are therefore only four cases left to examine for the existence of $MBS(a, b, c; d)$.

Problem 2.4. If c is even, does $MBS(2, 2, c; d)$ exist?

Problem 2.5. If a and b are even and greater than 2, does $MBS(a, b, 2; 2)$ exist?

Problem 2.6. If a, b, c are even and d is odd, does $MBS(a, b, c; d)$ exist?

Problem 2.7. If a, b, c are odd (in which case d must be odd), does $MBS(a, b, c; d)$ exist?

2.2 Problems Regarding Magic Hollow Boxes

This is an entirely new generalization of the magic square which has not been explored. Therefore, we focus our efforts on answering two questions.

Problem 2.8. Which values of n, a, or c guarantee that $MH(n, a, c)$ does not exist?

Problem 2.9. For which n, a, c does $MH(n, a, c)$ exist?

Chapter 3

New Results on Magic Box Sets

As we seek to determine the existence of certain magic box sets, it is useful to determine the constant row, column, and pillar sums that the magic box sets must have. The computation of these constants is trivial, and we list them here.

1. The row sum ρ is $\frac{b}{2}(abcd+1)$.

2. The column sum σ is $\frac{a}{2}(abcd+1)$.

3. The pillar sum π is $\frac{c}{2}(abcd+1)$.

We now examine Problems 2.4 and 2.5, where we consider the existence of $MBS(2,2,c;d)$ and $MBS(a,b,2;2)$. We arrive at the following theorems, which completely solve these problems.

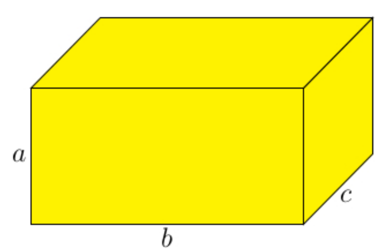


Figure 3.1: A box with a rows, b columns, and c layers

Theorem 3.1. *If c is even, $MBS(2,2,c;d)$ does not exist.*

Proof. Suppose that $a = b = 2$, and that $MBS(2,2,c;d)$ exists. Then $\rho = (2 \cdot 2 \cdot cd + 1) = (4cd + 1) = \sigma$. We need to construct four $2 \times 2 \times 4$ boxes using the numbers $1, 2, \dots, 4cd$ to create $MBS(2,2,c;d)$. Place 1 anywhere in any box. Then, to reach the necessary row sum, we must place $4cd$ in the adjacent cell. However, to achieve the necessary column sum, we must place $4cd$ in the cell directly below 1. This requires us to use the number $4cd$ twice, which violates the definition of a magic box. By contradiction, the proof is complete. □

Theorem 3.2. *Suppose $a \leq b$, a and b are even, and neither a nor b equals 2. Then $MBS(a,b,2;2)$ exists.*

Proof. We begin by taking a magic box $MB(a,b,2)$, which we know exists from [6]. Let $x_{i,j,k}$ denote the entry of $MB(a,b,2)$ in the $(i,j,k)^{th}$ cell, where $1 \leq i \leq a, 1 \leq j \leq b$, and $1 \leq k \leq 2$. We use this box to construct our base box $B_{a \times b \times 2}$, defining the $(i,j,k)^{th}$ entry of $B_{a \times b \times 2}$ as $y_{i,j,k} = 2(x_{i,j,k} - 1)$. It is clear that the sum of each row in $B_{a \times b \times 2}$ is $\frac{2ab(2ab-1)}{2} \cdot \frac{1}{a} = b(2ab-1)$, the sum of each column is $a(2ab-1)$, and the sum of each pillar is $2(2ab-1)$.

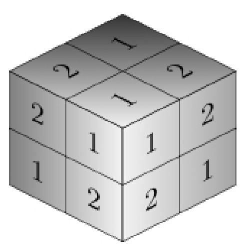


Figure 3.2: Latin cube of order 2

Now we construct a set of 2 residual boxes, RB^1 and RB^2, with entries $r^1_{i,j,k}$ and $r^2_{i,j,k}$, respectively, for $1 \leq i \leq a, 1 \leq j \leq b$, and $1 \leq k \leq 2$. We construct RB^1 by piecing together $\frac{a}{2} \times \frac{b}{2} = \frac{ab}{4}$ copies of the Latin cube in Figure 3.2, where $r^1_{1,1,1} = 1$. Then define RB^2 by $r^2_{i,j,k} = 1 + [(r^1_{i,j,k} + 1) \pmod 2]$. Now each row, column, and pillar of RB^1 and RB^2 contains exactly half of its entries as ones and the other half as twos. It is clear then that the rows sum to $\frac{3}{2}b$, the columns sum to $\frac{3}{2}a$, and the pillars sum to 3.

We finish our construction by defining a magic box set $MBS(a,b,2;2)$, where the $(i,j,k)^{th}$ entry in the d^{th} box is $r^d_{i,j,k}+y_{i,j,k}$, where $1 \leq d \leq 2, 1 \leq i \leq a, 1 \leq j \leq b$, and $1 \leq k \leq 2$. It should be clear by construction that the sum of every row in $MBS(a,b,2;2)$ is $b(2ab-1)+\frac{3}{2}b = \frac{b}{2}(4ab+1)$, the sum of every column is $a(2ab-1)+\frac{3}{2}a = \frac{a}{2}(4ab+1)$, and the sum of every pillar is $2(2ab-1)+3 = 4ab+1$. Moreover, by construction, every number between 1 and $4ab$ appears in the set exactly once. Therefore, the construction is complete. □

We can generalize this theorem and its proof to prove the existence of $MBS(a,b,c;d)$ when d divides a, b, and c, provided that the magic box $MB(a,b,c)$ exists. The real power in this theorem is that d can be even or odd.

Theorem 3.3. *If a magic box $MB(a,b,c)$ exists and d is a common divisor of a, b, and c, where $d \geq 2$, then $MBS(a,b,c;d)$ exists.*

Proof. We first take a magic box $MB(a,b,c)$. Let $x_{i,j,k}$ denote the entry of $MB(a,b,c)$ in the $(i,j,k)^{th}$ cell, where $1 \leq i \leq a, 1 \leq j \leq b$, and $1 \leq k \leq c$. We use this box to construct a base box $B_{a\times b\times c}$, defining the $(i,j,k)^{th}$ entry of $B_{a\times b\times c}$ as $y_{i,j,k} = d(x_{i,j,k}-1)$. It is clear that the sum of each row in $B_{a\times b\times c}$ is $\frac{abd}{2a}(abc-1) = \frac{bd}{2}(abc-1)$, the sum of each column is $\frac{ad}{2}(abc-1)$, and the sum of each pillar is $\frac{cd}{2}(abc-1)$.

Now we construct a set of n residual boxes, RB^n, where $n = 1, 2, \ldots, d$ with entries $r^n_{i,j,k}$, where $1 \leq i \leq a, 1 \leq j \leq b$, and $1 \leq k \leq c$. We will use the Latin cubes of order d to construct the residual boxes, where $LC^n(d)$ is a Latin cube of order d with entries $l^n_{e,f,g} = (n+e+f+g) \pmod d$, where $1 \leq n \leq d$, and $1 \leq e, f, g \leq d$, with the provision that all 0 entries are changed to d. We now construct each RB^n by "tiling" together $\frac{a}{d} \cdot \frac{b}{d} \cdot \frac{c}{d} = \frac{abc}{d^3}$ copies of $LC^n(d)$.

Since the sum along a row of $LC(d)$ is $\frac{d(d+1)}{2}$, we see that the sum of each row of RB^n is $\frac{b}{d} \cdot \frac{d}{2}(d+1) = \frac{b}{2}(d+1)$, the sum of each column is $\frac{a}{2}(d+1)$, and the sum of each pillar is $\frac{c}{2}(d+1)$. It should also be clear that for a fixed triple (i, j, k), the set of entries $\{r^n_{i,j,k}\}^d_{n=1}$ form the set $\{1, 2, \ldots, d\}$ (this follows directly from the definition of entries in $LC(d)$ and how they depend on n).

Now we build our magic box set $MBS(a, b, c; d)$, where the entries of the n^{th} box are defined as $z^n_{i,j,k} = r^n_{i,j,k} + y_{i,j,k}$. Then the sum of every row of each box is $\frac{bd}{2}(abc - 1) + \frac{b}{2}(d+1) = \frac{b}{2}(abcd+1)$, the sum of every column is $\frac{a}{2}(abcd+1)$, and the sum of every pillar is $\frac{c}{2}(abcd+1)$. We see from the construction that every number between 1 and $abcd$ appears in the set of boxes exactly once, since the base box has entries $\{0, d, 2d, \ldots, abcd - d\}$ and over the course of the set of boxes we add $1, 2, \ldots, d$ to every cell, filling in the "gaps." Moreover, since the row, column, and pillar sums of the n^{th} box in the set is independent of n, $MBS(a, b, c; d)$ is magic and the construction is complete. □

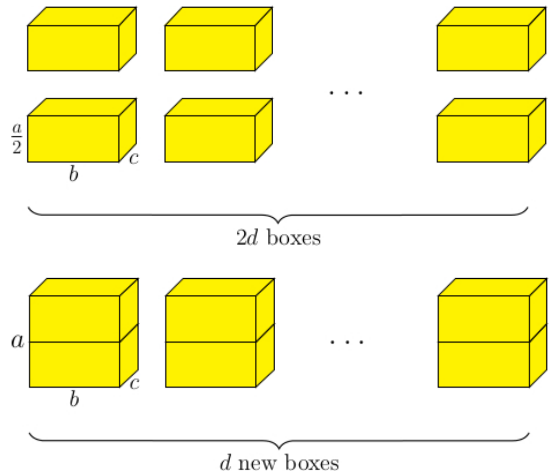
Observe that Theorem 3.2 also follows directly as a corollary to Theorem 3.3. Hence, Theorem 3.3 completely solves Problem 2.5, and partially solves Problems 2.6 and 2.7.

Now, we continue by solving further cases in Problem 2.6, where a, b, and c are even and d is odd.

Theorem 3.4. *If a, b, c are even and no two of a, b, c equal 2, then $MBS(a, b, c; d)$ is magic if, without loss of generality, $a \equiv 0 \pmod 4$ and $\{\frac{a}{2}, b, c\}$ contains no pair of 2's.*

Proof. Suppose a, b, and c are even, and no two of a, b, c equal 2. We know from [6] that $MBS(\frac{a}{2}, b, c; 2d)$ is magic. Then, we can rearrange these boxes by pairing the boxes together. From here, we can stack one box on top of the other in each pair, and the result is a set of d boxes of size $a \times b \times c$, where each box clearly has constant row, column, and pillar sums. Hence we have constructed $MBS(a, b, c; d)$. □

Theorem 3.4 covers quite a few cases: if $a, b, c > 2$ and one of a, b, c is divisible by 4, then $MBS(a, b, c; d)$ exists for any odd d. Recall that if $b = c = 2$, $MBS(a, b, c; d)$



does not exist by Theorem 3.1. Some cases excluded from the results of Theorem 3.4 are when $a, b, c \equiv 2 \pmod 4$, and when, without loss of generality, $a = 2, b = 4$, and $c \equiv 2 \pmod 4$. This does not preclude the existence of a magic box set in these cases: we simply have not been able to apply any of our previous results toward these cases.

We will now consider Problem 2.7, where a, b, c, and d are odd. We will prove a powerful result on the existence of a magic 4-rectangle, based on Hagedorn's analogous result in three dimensions in [6]. In order to reach this result, we must introduce the notions of a Kotzig Array and a Latin box, and prove some lemmas regarding their existence.

Definition 3.5. A *Kotzig Array* of size $a \times b$ is an $a \times b$ array with the property that each row contains the numbers $1, 2, \ldots, a$, each occurring exactly once, and the entries of each column sum to the same constant.

Definition 3.6. A *Latin box* of size $a \times a \times b$ has the properties that each $a \times a$ face of the box is a Latin square and the entries of each $1 \times b$ column sum to some constant c. This is a three-dimensional analogue to a Kotzig Array.

Lemma 3.7. *A Latin box of size $m \times m \times 2$ exists for m odd.*

Proof. The first layer is an $m \times m$ Latin square. The second layer is the complement of the first layer, such that the column sum is $m+1$ for each column. Since the mapping of numbers in $\{1,2,\ldots,m\}$ to $\{1,2,\ldots,m\}$ is a bijection, the second layer must also be a Latin square. $\square$

Lemma 3.8. *A Latin box of size $m \times m \times 3$ exists when m is odd.*

Proof. Begin by constructing a Latin square with entries l^1_{ij}, where $1 \le i, j \le m_1$. Then the second layer is computed by

$$l^2_{ij} = \begin{cases} l^1_{ij} + \lceil \frac{m}{2} \rceil & \text{if } l^1_{ij} < \lceil \frac{m}{2} \rceil \\ l^1_{ij} - \lfloor \frac{m}{2} \rfloor & \text{if } l^1_{ij} > \lfloor \frac{m}{2} \rfloor \end{cases}$$

To construct the third layer, we consider the column sums from the first two layers. We will notice that the set $\{l^1_{ij} + l^2_{ij} : 1 \le j \le m\}$ forms an arithmetic progression with a difference of one for each i. To see this, first consider the numbers in the first layer from 1 to $\lfloor \frac{m}{2} \rfloor$. The temporary column sums for these columns are formed by

$$\begin{aligned} l^1_{ij} + l^2_{ij} &= l^1_{ij} + (l^1_{ij} + \lceil \frac{m}{2} \rceil) \\ &= 2l^1_{ij} + \lceil \frac{m}{2} \rceil \end{aligned} \tag{3.1}$$

Similarly, the temporary column sums for the numbers $\lceil \frac{m}{2} \rceil$ to m are given by

$$2l^1_{ij} - \lfloor \frac{m}{2} \rfloor \tag{3.2}$$

Using (3.1) and (3.2), we fill out the following table of temporary sums.

l^1_{ij}	$2l^1_{ij} + \lceil \frac{m}{2} \rceil$	l^1_{ij}	$2l^1_{ij} - \lfloor \frac{m}{2} \rfloor$
1	$2 + \lceil \frac{m}{2} \rceil$	$\lceil \frac{m}{2} \rceil$	$1 + \lceil \frac{m}{2} \rceil$
2	$4 + \lceil \frac{m}{2} \rceil$	$\lceil \frac{m}{2} \rceil + 1$	$3 + \lceil \frac{m}{2} \rceil$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lfloor \frac{m}{2} \rfloor$	$(m-1) + \lceil \frac{m}{2} \rceil$	$m - 1$	$(m-2) + \lceil \frac{m}{2} \rceil$
		m	$m + \lceil \frac{m}{2} \rceil$

Hence we can arrange all of the integers from 1 to m, each occurring exactly once, in the i^{th} row of the third layer to obtain a constant sum for all columns. The column sum will be $\frac{3(m+1)}{2}$. Since the temporary column sums uniquely depend on l^1_{ij}, the third layer will also be a Latin square. □

3.1 Example: Construction of $5 \times 5 \times 3$ Latin box

We demonstrate the construction in the proof of Lemma 3.8 for a $5 \times 5 \times 3$ Latin box.

1. Construct any 5×5 Latin square as the top layer.

2. Construct the second layer based off of the entries in the top layer. Add $\lceil \frac{m_1}{2} \rceil = \lceil \frac{5}{2} \rceil = 3$ to the corresponding entry in the top layer if the top layer's number is 1 or 2. Subtract $\lfloor \frac{m_1}{2} \rfloor = 2$ from the corresponding entry in the top layer if the top layer's number is 3, 4, or 5.

   - i.e. $1 \mapsto 4, 2 \mapsto 5$ and $3 \mapsto 1, 4 \mapsto 2, 5 \mapsto 3$

3. Notice that the column sums (between layers) along each row (up-down and left-right along one layer) of the top two layers are 4, 5, 6, 7, and 8.

4. Assign 1, 2, 3, 4, and 5 in the columns whose sums are 8, 7, 6, 5, and 4, respectively.

   - Using the second layer's entries to build the third layer, $1 \mapsto 5, 2 \mapsto 3, 3 \mapsto 1, 4 \mapsto 4$, and $5 \mapsto 2$

						Sum
	1	2	3	4	5	15
	5	1	2	3	4	15
	4	5	1	2	3	15
	3	4	5	1	2	15
	2	3	4	5	1	15
Sum	15	15	15	15	15	

(a) Top layer

						Sum
	4	5	1	2	3	15
	3	4	5	1	2	15
	2	3	4	5	1	15
	1	2	3	4	5	15
	5	1	2	3	4	15
Sum	15	15	15	15	15	

(b) Middle layer

						Sum
	4	2	5	3	1	15
	1	4	2	5	3	15
	3	1	4	2	5	15
	5	3	1	4	2	15
	2	5	3	1	4	15
Sum	15	15	15	15	15	

(c) Bottom layer

Figure 3.3: A $5 \times 5 \times 3$ Latin box

5. Now each column has a sum of 9, and each row in every layer has a sum of 15.

Lemma 3.9. *A Latin box of size $m \times m \times n$ exists for m, n both odd.*

Proof. If $n = 1$, then we just have a Latin square. Suppose $n > 1$. Then we know that

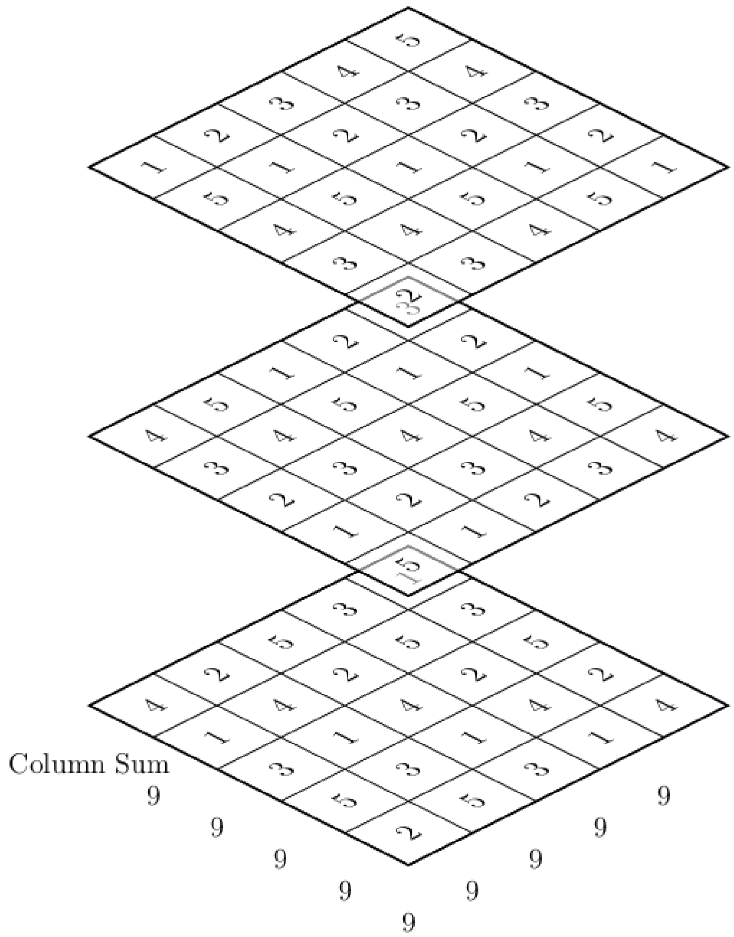


Figure 3.4: All rows sum to 15, all columns sum to 9

$n = 2k + 3$ for some k and we can construct the desired Latin box by stacking k copies of a Latin box of size $(m \times m \times 2)$ and one Latin box of size $(m \times m \times 3)$. The existence of these pieces follows directly from Lemmas 3.7 and 3.8. $\square$

These lemmas will now allow us to prove the following theorem about a specific class of 4-rectangles.

Theorem 3.10. *A 4-rectangle of size* (m, m, m, n) *is magic if* $m \geq 3, n \geq 2$, *and* $m \equiv n \pmod 2$.

Proof. If m, n are even and $m > 2$, then the 4-rectangle of size (m, m, m, n) is magic by [6]. Hence we will assume m, n are odd and greater than 1.

Define a base box $R_{m \times m \times n}$, which is magic [6], and let T be a Latin box of size $m \times m \times n$. Denote the entries of $R_{m \times m \times n}$ by $r_{i,j,k}$, and the entries of T be $t_{i,j,k}$, where $1 \leq i, j \leq m$ and $1 \leq k \leq n$. Then we will construct a 4-dimensional box S with dimensions $m \times m \times m \times n$ and with entries $s_{i,j,k,l} = r_{i+j,k,l} + m^2 n(t_{i-j,k,l} - 1)$. Observe that $1 \leq s_{i,j,k,l} \leq m^2 n + m^2 n(m_1 - 1) = m^3 n$. We will now show that each entry of S is distinct.

Suppose $s_{i,j,k,l} = s_{i',j',k',l'}$. Then, because of the entries where $t_{i-j,k,l} - 1 = 0$, we have $s_{i,j,k,l} = r_{i+j,k,l} = r_{i'+j',k',l'} = s_{i',j',k',l'}$. Since R is a magic box, its entries are distinct. This implies that $k = k', l = l'$, and $i + j \equiv i' + j' \pmod{m}$. It follows that $t_{i-j,k,l} = t_{i'-j',k',l'} = t_{i'-j',k,l}$, which implies that $i - j \equiv i' - j' \pmod{m}$, since m is odd and the entries in each column of T are distinct. These two congruences imply that $i = i'$ and $j = j'$. Hence S is a magic 4-rectangle of size (m, m, m, n). □

The following corollary applies Theorem 3.10 to the topic of magic box sets.

Corollary 3.11. *If $p \geq 3$ and q are both odd, or if $p \geq 3$ is even and q is anything, then $MBS(p, p, p; q)$ exists.*

Proof. Suppose $p \geq 4$. If p is even, then $MBS(p, p, p; q)$ exists according to Theorem 3.4.

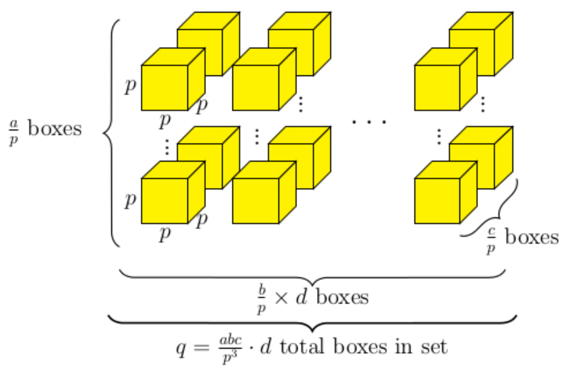
Now assume p, q are odd and $p \geq 3$. If $q = 1$, then $MBS(p, p, p; q)$ is equivalent to a magic cube of order 3, and exists according to [4]. Now consider $q \geq 3$. From Theorem 3.10, we know a magic 4-rectangle of size (p, p, p, q) exists. We can create a set of q boxes of size $p \times p \times p$ by separating the layers of the 4-rectangle along the fourth dimension. Each of these boxes will clearly have constant row sums, column sums, and pillar sums, since a $MBS(p, p, p; q)$ is less restrictive than a magic 4-rectangle of size (p, p, p, q). □

The next theorem is another partial solution to Problem 2.7, where a, b, c, and d are odd, and also can be applied to Problem 2.6, where a, b, and c are even and d is odd.

Theorem 3.12. *If $p \geq 3$ and p is a common divisor of a, b, and c, then $MBS(a, b, c; d)$ exists.*

Proof. Suppose a, b, c, and d are even. Then, if no two of a, b, c equal 2, $MBS(a, b, c; d)$ exists by [6] and Theorem 3.2.

Now suppose a, b, c and d are odd. Let $q = d \cdot \frac{abc}{p^3}$. Observe then that q is odd. By Corollary 3.11, we know that $MBS(p, p, p; q)$ exists. We will construct our magic box set by grouping together the boxes in $MBS(p, p, p; q)$.



We will "stack" $\frac{a}{p}$ boxes in one dimension, $\frac{b}{p}$ boxes in the second dimension, and $\frac{c}{p}$ boxes in the third dimension to create a set of boxes of size $a \times b \times c$. It is clear that the number of boxes in our constructed set is d:

$$\frac{q}{(abc)/p^3} = \frac{d \cdot (abc)/p^3}{(abc)/p^3} = d$$

Note that by construction, each of the new boxes must also be magic.

The remaining case is when a, b, and c are even and d is odd. The aforementioned construction works in this case as long as $\frac{abc}{p^3}$ is odd, that is, as long as $a \equiv b \equiv c \equiv 2 \pmod 4$. If, without loss of generality, $a \equiv 0 \pmod 4$, then $MBS(a, b, c; d)$ exists if $a > 4$, according to Theorem 3.4. We now consider the "worst" case where, say, $a = 4, b = 2$, and $c \equiv 2 \pmod 4$. This case is covered by neither Theorem 3.4 nor this proof so far; it is not necessary, however, since by assumption a, b, and c must share a common divisor greater than 2. Hence all cases have been covered and the proof is complete. □

We now turn our attention to a different type of construction in order to prove the existence of more magic box sets, this time where d has a "small" divisor that is greater than 1.

Theorem 3.13. *Suppose $a \leq b \leq c$, a, b, c, and d are odd, $MB(a, b, c)$ exists, and d has a divisor larger than 1 but smaller than a. Then $MBS(a, b, c; d)$ exists.*

Proof. This is a proof by construction.

Begin with a magic box $MB(a, b, c)$, using the entries $0, \ldots, abc - 1$ instead of the typical $1, 2, \ldots, abc$. Multiply every element of this box by d, and then add 1 to each entry. Call this our base box. Notice that the entries of the base box are $1, d + 1, 2d + 1, \ldots, abcd - d + 1$.

Now suppose $d = pq$, where $p \leq a$. Construct a $p \times q$ Kotzig Array (using the numbers $0, 1, \ldots, pq - 1$). Now lift the Kotzig Array by adding 0 to each element in the bottom row, q to each element in the second row, $2q$ to the third row, and so on, until finally adding $(p - 1)q$ to each element in the top row.

We will use the first $p \times 1$ column of the lifted Kotzig Array to determine a cube of order p which will have constant row, column, and pillar sums (somewhat analogous to a Latin Cube, although not using consecutive numbers). We will refer to this cube as an l-cube. We will then use different orderings of the rows of this l-cube to construct p distinct boxes of size $a \times b \times c$.

To construct one residual box out of one l-cube:

- Have an empty $a \times b \times c$ box.
- Place the l-cube in the front upper left corner of the box.

- The first p entries in the column immediately to the right of the l-cube are chosen so that, when added to the adjacent entry in the last column of the l-cube, the sum is $d-1$.

- The first p entries in the row immediately below the l-cube are chosen in exactly the same manner as the columns to the right.

- Fill the subsequent columns to the right (rows below) such that the sum of each entry with its immediate neighbor to the left (above) is $d-1$.

- Fill the remaining space on the front face of the box such that the sum of each entry with its immediate neighbor to the left is $d-1$. This completes the front face.

- Complete the next $p-1$ faces using the same process as above.

- Complete the top face using the process above.

- Complete subsequent faces working downward using the process above.

To create the second residual box, peel the top layer of the l-cube and place it at the back, and follow the process above to create a second residual box. Iterate this process to construct p total residual boxes.

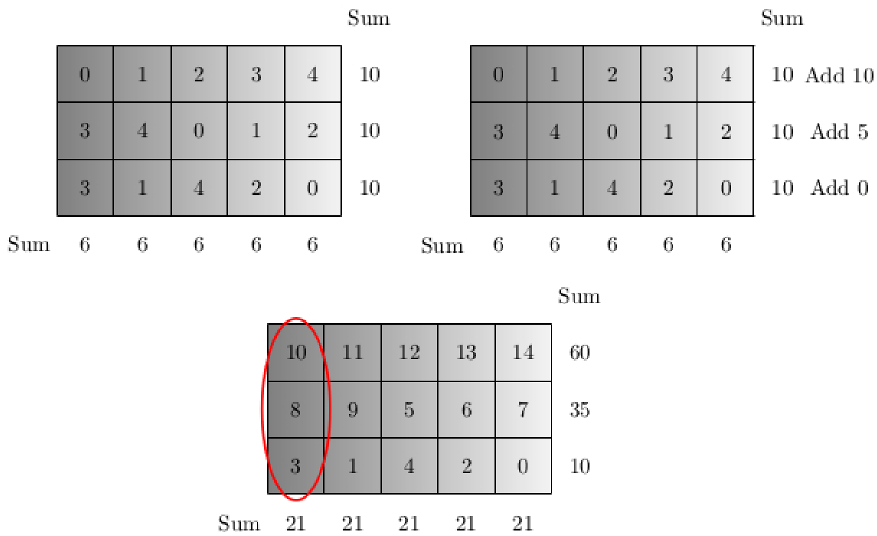
Iterate this process for the other $q-1$ columns of the lifted Kotzig Array in order to obtain $qp = d$ residual boxes. Since each residual box is determined by one distinct l-cube, the d residual boxes are also distinct (entrywise). Furthermore, each entry l_{ijk} takes on each of the values $0, 1, 2, \ldots, d-1$ exactly once in the set of d l-cubes, and therefore each entry r_{ijk} takes on each of the values $0, 1, \ldots, d-1$ in the set of residual boxes.

Observe that, by construction, the residual boxes all have the same constant row sum, column sum, and pillar sum.

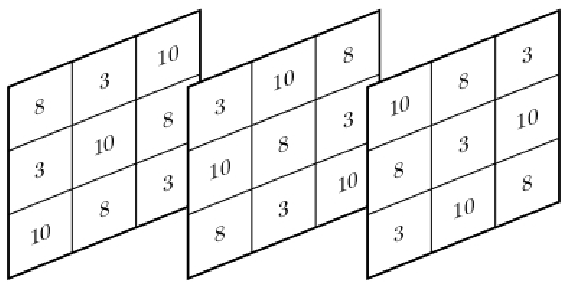
Use these d residual boxes to construct $MBS(a, b, c; d)$ by simply adding the base box to each residual box, using enrywise addition. It is easy to see from the construction that the entries in this set of boxes are $1, 2, \ldots, abcd$. Hence the set of boxes is magic. $\square$

3.2 Example of Construction in Proof of Theorem 3.13

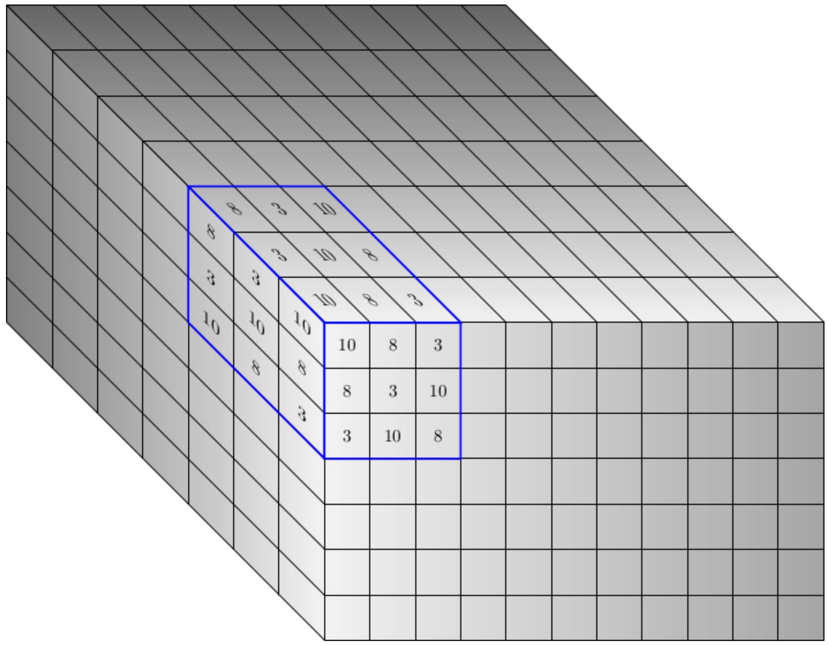
We will use the construction in the proof of Theorem 3.13 to complete most of the construction for $MBS(3, 7, 11; 15)$ in a series of images. We do not construct a base box here, but we do give examples of how to construct the residual boxes. The base box will be $MB(3, 5, 7)$ using numbers 0 through $3 \times 5 \times 7 - 1$, with every entry multiplied by 15, and then with 1 added to each cell. The construction is completed by adding the base box entry-wise to each of the 15 residual boxes.



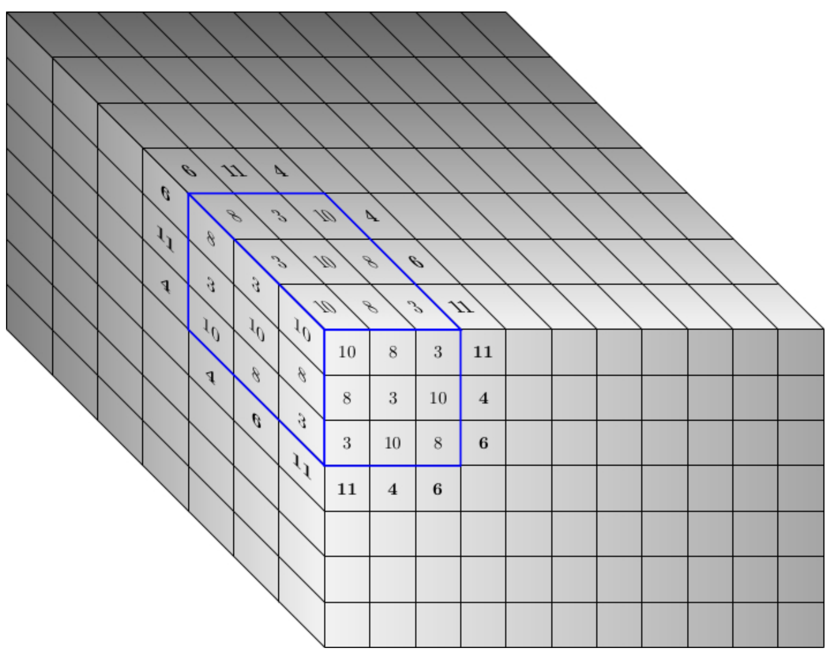
Lifting a 3×5 Kotzig Array and selecting the first column



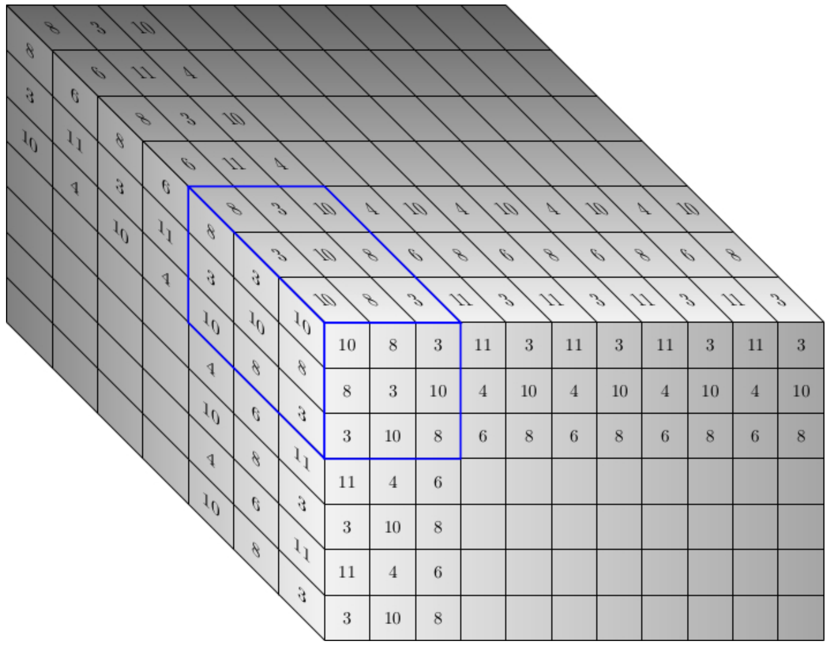
Latin cube of order 3 from the first column



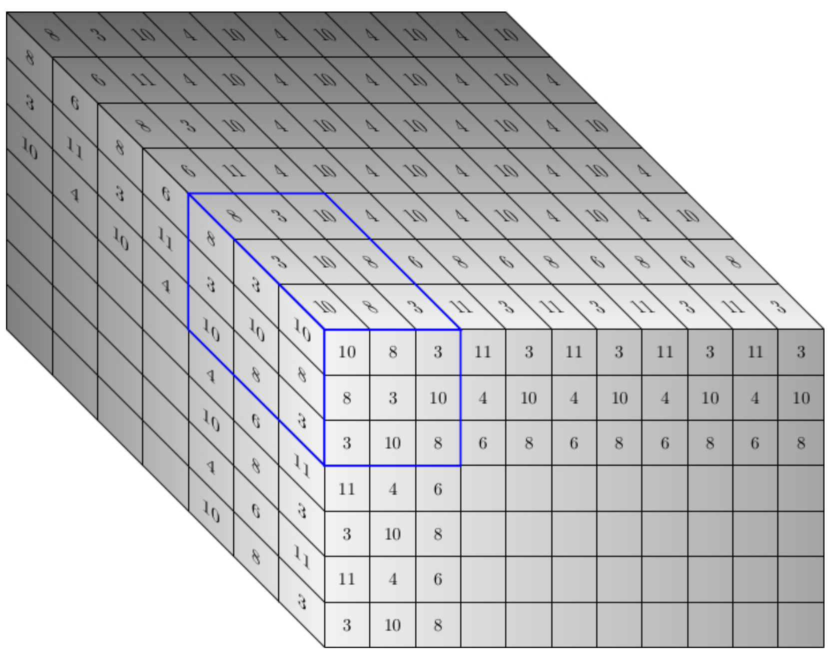
Latin cube in the corner of first residual box



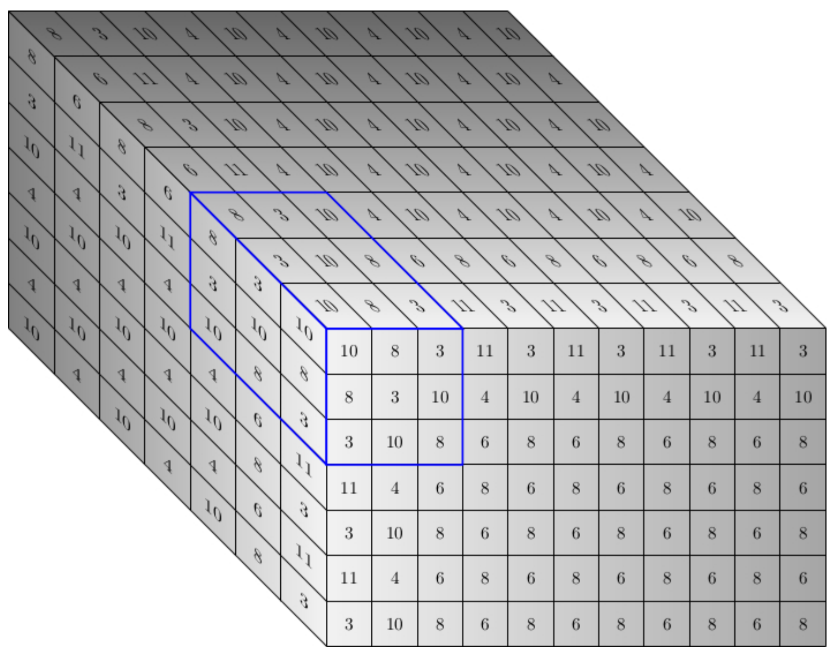
Filling in adjacent entries



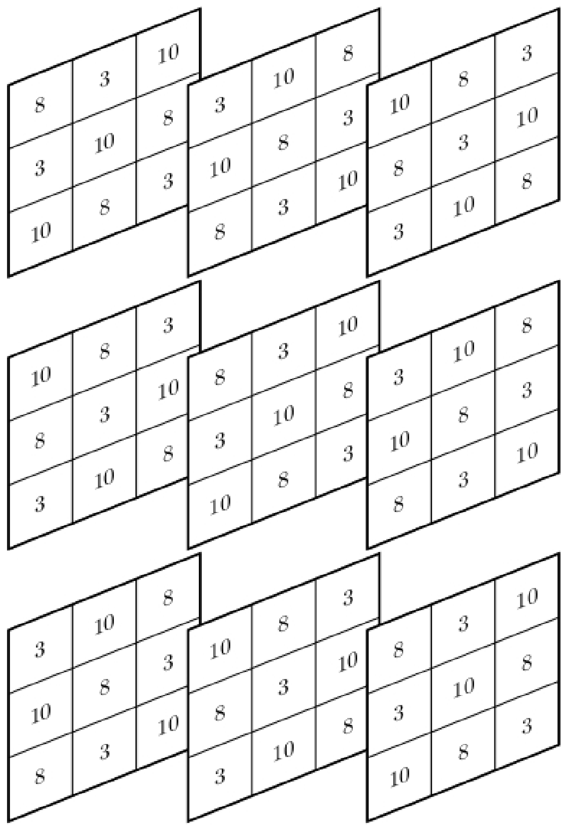
Filling out the edges of the box



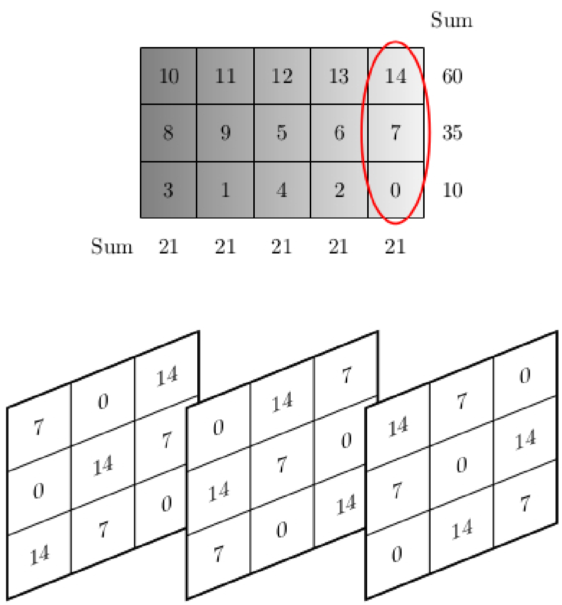
Filling in one face of the box



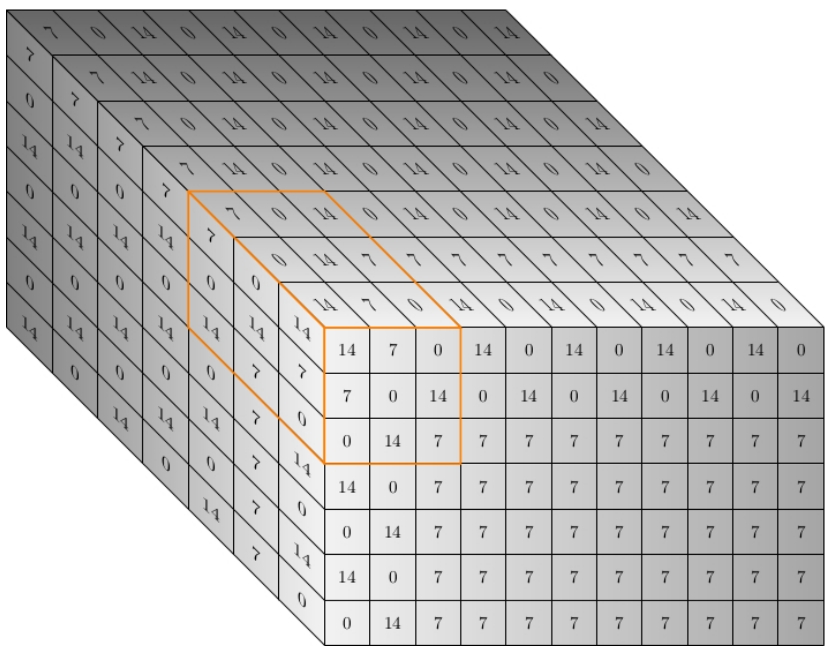
The first complete residual box



The three distinct Latin cubes from the first column of the Kotzig Array



The last column of the Kotzig Array and a Latin cube constructed from it



A residual box constructed from the last column of the Kotzig Array

We derive several useful corollaries from Theorem 3.13.

Corollary 3.14. *If d has a divisor greater than 1 and less than or equal to a, b, or c, and gcd$(a, b) > 2$, then $MBS(a, b, c; d)$ exists.*

Proof. This follows directly from Theorem 3.13 and Theorem 2.2 ($MB(a, b, c)$ exists if $\gcd(a, b) > 2$). □

For the next corollary, we need a lemma proven by Hagedorn in [6].

Lemma 3.15. *If α, a, b, and c are odd and $MB(a, b, c)$ exists, then $MB(\alpha a, b, c)$ exists.*

Corollary 3.16. *If α, a, b, and c are odd, $MB(a, b, c)$ exists, and $d'|d$ such that $1 < d' \leq \alpha a, b$, or c, then $MBS(\alpha a, b, c; d)$ exists.*

Proof. This follows directly from Lemma 3.15 and Theorem 3.13. □

Corollary 3.17. *For every odd a, b, c, d where gcd$(a, b) > 1$, there is an α such that $MBS(\alpha a, b, c; d)$ exists.*

We now use a construction similar to the construction in the proof of Theorem 3.13 in order to prove another powerful theorem. Theorem 3.18 will allow us to reduce Problem 2.7, the case where a, b, c, and d are odd.

Theorem 3.18. *If $MB(a, b, c)$ exists and a, b, c, d are odd, then $MBS(a, b, c; d)$ exists.*

Proof. Suppose that $a \leq b \leq c$. We will use $MB(a, b, c)$ to construct $MBS(a, b, c; d)$. Begin with a magic box $MB(a, b, c)$, using the entries $0, \dots, abc-1$ instead of the typical $1, 2, \dots, abc$. Multiply every element of this box by d, and then add 1 to each entry. Call this our base box B. Notice that the entries of the base box B are $1, d+1, 2d+1, \dots, abcd - d + 1$, and that B has constant row, column, and pillar sums, since it is based on $MB(a, b, c)$. It is easy to check this: for example, simply find the total sum of all the numbers in B and divide by the number of rows a and the number of layers c. This yields a row sum $\rho_B = \frac{b}{2}(abcd - d + 2)$. Similarly, B has column sum $\sigma_B = \frac{a}{2}(abcd - d + 2)$ and pillar sum $\pi_b = \frac{c}{2}(abcd - d + 2)$.

Now construct an $a \times d$ Kotzig Array, using the numbers $0, 1, 2, \ldots, d - 1$ in each of the a rows of the array. The constant column sum of this array is simply the total sum of all elements of the array divided by the number of columns, or

$$\frac{(d-1)d}{2} \cdot a \cdot \frac{1}{d} = \frac{a(d-1)}{2}.$$

We will use the first $a \times 1$ column of the Kotzig Array to determine a cube of order a which will have constant row, column, and pillar sums (somwhat analogous to a Latin cube, although we do not use consecutive numbers). We refer to this cube as an l-cube. Construct a single residual box R_1 using this l-cube following the method in the proof of Theorem 3.13. Use the second column of the Kotzig Array to construct a second l-cube and form R_2, and so on, until we have d residual boxes, $R_1, R_2, \ldots, R_d$. The row, column, and pillar sums of each R_i are simple to calculate. By construction, the sum of each row and pillar is the row/column sum of the l-cube plus several pairs of numbers, where each pair sums to $d - 1$. The column sum of R_i is simply the column sum of the l-cube. Using this method, we find the row, column, and pillar sums explicitly for each R_i:

$$\begin{aligned}
\rho_R &= \frac{a(d-1)}{2} + \frac{(b-a)(d-1)}{2} \\
&= \frac{(a+b-a)(d-1)}{2} \\
&= \frac{b(d-1)}{2} \\
\sigma_R &= \frac{a(d-1)}{2} \\
\pi_R &= \frac{a(d-1)}{2} + \frac{(c-a)(d-1)}{2} \\
&= \frac{(a+c-a)(d-1)}{2} \\
&= \frac{c(d-1)}{2}.
\end{aligned}$$

Now add each of the residual boxes R_i to B using entrywise addition to create a set of d boxes. Notice that by construction, when given any specific cell, each value $1, 2, \ldots, d$ appears exactly once in that cell among the set $\{R_1, R_2, \ldots, R_d\}$. For this reason, our set of d boxes (adding B to each R_i) contains every value $1, 2, \ldots, d, d+1, d+2, \ldots, 2d, \ldots, abcd-d+1, abcd-d+2, \ldots, abcd-d+d = abcd$, each occurring exactly once. Furthermore, each of the boxes has a constant row, column, and pillar sum:

$$
\begin{aligned}
\rho_{B+R} &= \frac{b(abcd-d+2)}{2} + \frac{b(d-1)}{2} \\
&= \frac{b}{2}(abcd-d+2+d-1) \\
&= \frac{b}{2}(abcd+1) \\
\sigma_{B+R} &= \frac{a(abcd-d+2)}{2} + \frac{a(d-1)}{2} \\
&= \frac{a}{2}(abcd+1) \\
\pi_{B+R} &= \frac{c(abcd-d+2)}{2} + \frac{c(d-1)}{2} \\
&= \frac{c}{2}(abcd+1).
\end{aligned}
$$

According to the general formulas for ρ, σ, and π for some $MBS(a, b, c; d)$, $\rho_{B+R} = \rho, \sigma_{B+R} = \sigma$, and $\pi_{B+R} = \pi$. Hence the construction of $MBS(a, b, c; d)$ is complete. □

Notice that Problem 2.7 is now reduced to the following: for which odd a, b, c does $MB(a, b, c)$ exist? Luckily, Hagedorn proved some results about this problem in [6], which we will incorporate into our next theorem. But first, we will demonstrate the construction in the proof of Theorem 3.18.

3.3 Example of Construction in proof of Theorem 3.18

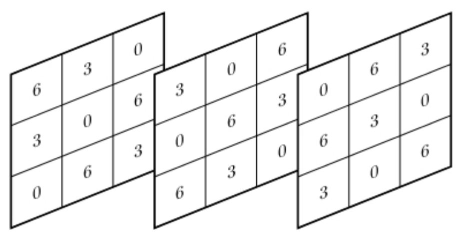
In this subsection, we will partially demonstrate the construction in the proof of Theorem 3.18 (to build $MBS(3, 9, 11; 7)$) using a series of images. Since it is almost exactly the same as the construction in the proof of Theorem 3.13, we will construct only one residual box. The construction is completed by adding the base box entry-wise to each of the 7 residual boxes. The base box is $MB(3, 9, 11)$ using numbers 0 through $3 \times 9 \times 11 - 1$, with every entry being multiplied by 7, then adding 1 to each cell. We are left with a set of 7 distinct boxes which satisfy the definition for $MBS(3, 9, 11; 7)$.

								Sum
	0	1	2	3	4	5	6	21
	6	4	2	0	5	3	1	21
	3	4	5	6	0	1	2	21
Sum	9	9	9	9	9	9	9	

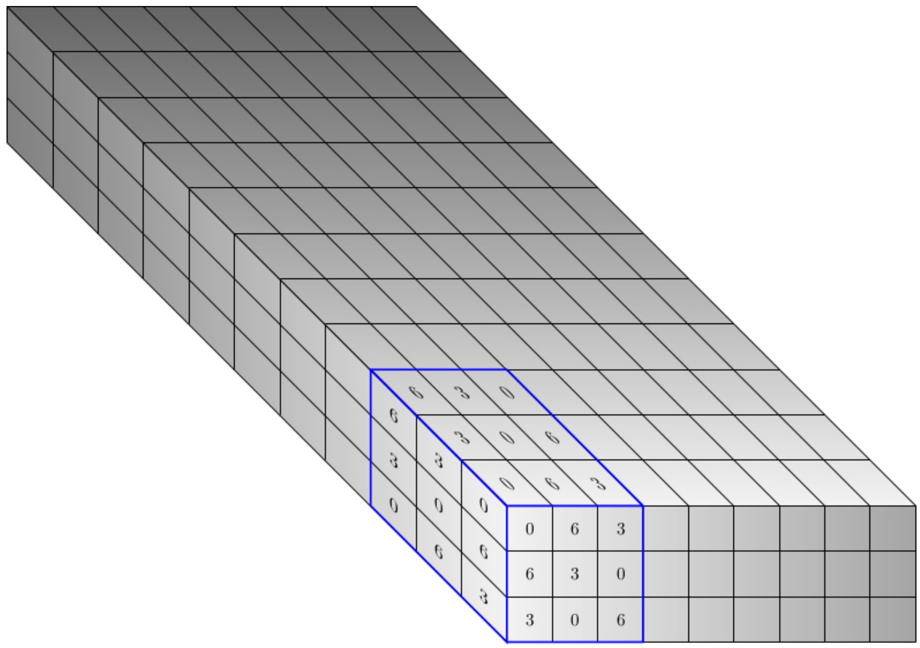
A 3×7 Kotzig Array

								Sum
	0	1	2	3	4	5	6	21
	6	4	2	0	5	3	1	21
	3	4	5	6	0	1	2	21
Sum	9	9	9	9	9	9	9	

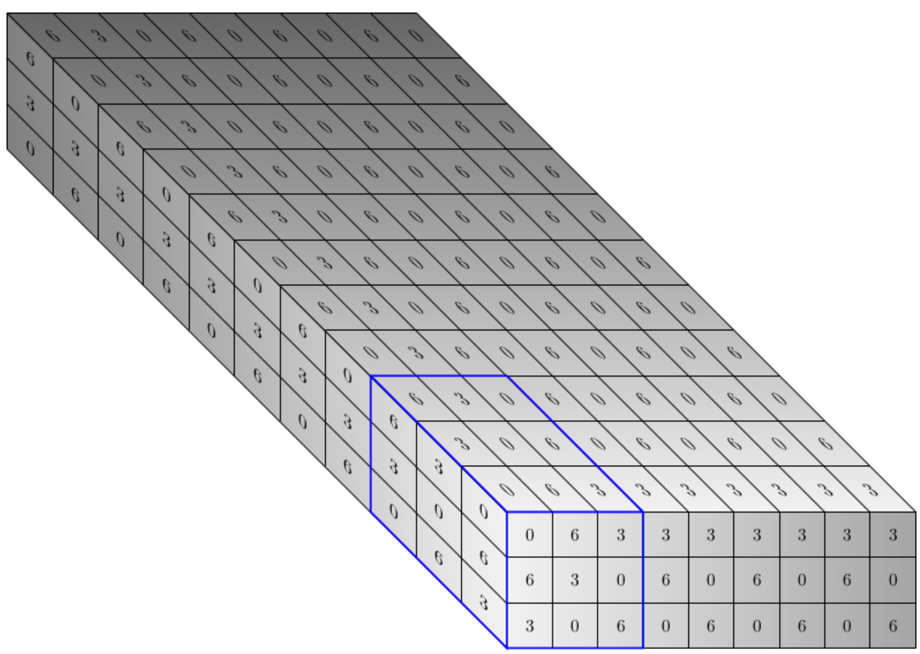
Selecting the first column



Latin cube constructed from the first column



Latin cube in the corner of first residual box



The first complete residual box

Theorem 3.19. *If $a \leq b \leq c$, a, b, c, d are odd, and gcd$(a, b) > 1$ or gcd$(b, c) > 1$, then $MBS(a, b, c; d)$ exists.*

Proof. Since $\gcd(a, b) > 1$ or $\gcd(b, c) > 1$, $MB(a, b, c)$ exists according to Theorem 2.2. Then, if $d \leq a$, $MBS(a, b, c; d)$ exists according to Theorem 3.13. If $d > a$, then $MBS(a, b, c; d)$ exists according to Theorem 3.18. □

We now make an interesting observation: the construction in the proof of Theorem 3.18 does not use the fact that a, b, and c are odd: it was included in the assumptions simply because that was the case we were focused on. Hence, we make a powerful generalization.

Theorem 3.20. *If a, b, and c are the same parity, d is odd, and $MB(a, b, c)$ exists, then $MBS(a, b, c; d)$ exists.*

Proof. Follow the construction in the proof of Theorem 3.18. □

This concludes our new results regarding magic box sets. For a summary of how these results compare to each other and to known results, and to see which problems remain open in this area, refer to Chapter 5.

Chapter 4

Magic Hollow Boxes

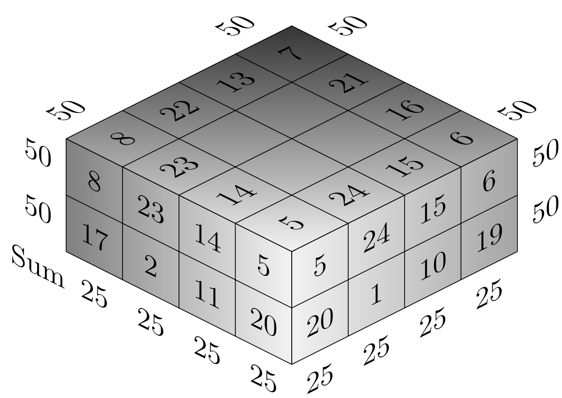


Figure 4.1: You can think of this as 4 joined rectangles, or as 2 stacked squares

In this section we examine a new generalization of magic squares, the *magic hollow box*. Recall from the introduction that this structure can be imagined as n rectangles of dimension $a \times c$, joined along the length c edges to form a stack of c regular n-gons. We look to fill this structure with the numbers $1, 2, \ldots, n(a-1)c$ (the number of total cells in such an object) such that each row sums to ρ and each column sums to σ. See Figure 4.1 for an example of a square ($n = 4$), with $a = 4$ and $c = 2$.

It turns out that Figure 4.1 will prove very useful in constructing squares of more layers. For convenience, we will refer to these objects as $MH(n, a, c)$. It is also expedient to view this object as a stack and look at it from the top. This way we can distinguish layers that all have an equal number of cells filled with numbers.

This is a completely new generalization of the magic square, and the first natural question to arise is that of existence.

Problem 4.1. For which n, a, and c does some $MH(n, a, c)$ exist?

We observe from Figure 4.1 that $MH(4, 4, 2)$ exists. We will use this figure in a construction to prove the existence of $MH(4, 4, c)$.

Theorem 4.2. *$MH(4, 4, c)$ exists for all even c.*

Proof. Given some even c, we will form a general construction using Figure 4.1.

It is useful to know the column and row sums.

$$\begin{aligned}
\sigma &= \frac{\text{sum of all cells}}{\#\text{ of columns}} \\
&= \frac{\frac{1}{2}(n(a-1)c)(n(a-1)c+1)}{n(a-1)} \\
&= \frac{c(nc(a-1)+1)}{2} \\
&= \frac{c(4c(3)+1)}{2} \\
&= \frac{c}{2}(12c+1) \\
\rho &= \frac{a \times \text{column sum}}{\#\text{ of layers}} \\
&= \frac{4\sigma}{c} \\
&= \frac{4 \cdot \frac{c}{2}(12c+1)}{c} \\
&= 2(12c+1).
\end{aligned}$$

We start the construction with a set of c unlabeled square layers stacked on top of each other. We will fill in the boundaries of each of these squares in a manner which will produce $MH(4,4,c)$. For convenience, we will refer to the cells of each square layer (from top to bottom) as either corners or edges (non-corners). Refer again to Figure 4.1. Place this in the middle two layers of the unlabeled stack. Now, we need the row sums of both of these layers to equal ρ, but it is clear that as is, the row sums are 50. We will address this by adding the same constant to each entry in Figure 4.1. This constant must be

$$\frac{\text{col sum} - (\text{col sum of 4.1}) \cdot (\#\text{ of copies of 4.1 that fit in the empty structure})}{\#\text{ of layers}}$$

$$\begin{aligned}
&= \frac{\sigma - 25 \cdot \frac{c}{2}}{c} \\
&= \frac{\frac{c}{2}(12c+1) - \frac{25c}{2}}{c} \\
&= \frac{\frac{c}{2}(12c+1-25)}{c} \\
&= \frac{\frac{c}{2}(12c-24)}{c} \\
&= 6c - 12.
\end{aligned}$$

Observe that the row sum of each of these two middle layers will be $50+4(6c-12) = 2+24c = 2(12c+1) = \rho$. We must now fill in the remaining unlabeled layers. Fill in the corners of the layer directly below the middle by adding 12 to the numbers in the corners immediately above it, and fill in the edges by subtracting 12 from the numbers in the edges immediately above them. We iterate this process, working down to the bottom layer. We then start from the middle again and work up, but this time we will subtract 12 from the corners in the layer immediately below and add 12 to the number in the edges in the layer immediately below. Iterate this process, working up.

Observe that in each layer, the row sum differs from ρ by $12 - 12 - 12 + 12 = 0$, so each row sums to ρ. Now we check the column sum. We note that the pairwise column sum of the middle two layers is $25 + 2(6c - 12) = 12c + 1$. Now move up one layer from the temporary top (the top of the filled structure) and down one layer from the temporary bottom (the bottom of the filled structure). The pairwise column sum of these two layers must also be $12c + 1$, since we add 12 to the corners below and subtract them up top, and we add 12 to the edges up top and subtract 12 from the edges below. If we iterate this process of checking the pairwise column sums, we see that each pair has column sums of $12c + 1$. Therefore, the total column sum is $(12c + 1) \cdot (\#\text{ of pairs}) = (12c + 1) \cdot \frac{c}{2} = \sigma$.

Since all rows sum to ρ and all columns sum to σ, the final step is to show that each of the numbers from $1, \ldots, 12c$ appears exactly once. It is convenient to classify the numbers $\{1, 2, \ldots, 12\}$ as b numbers (since they are in the bottom half of $\{1, 2, \ldots, 24\}$), and the numbers $\{13, 14, \ldots, 24\}$ as t numbers (occurring in the top half of $\{1, 2, \ldots, 24\}$). Then, if we look at the completed structure in terms of b's and t's, we see that the corners in the top $\frac{c}{2}$ layers and the edges in the bottom $\frac{c}{2}$ layers are all b numbers plus a constant (where the constant is a multiple of 12). The spaces not occupied by b numbers are occupied by t numbers plus a constant (where the constant is a multiple of 12). We now look at what values we have in the structure. In the top and bottom layers, the b values take on every number in the set $\{1, 2, \ldots, 12\}$. In the second from the top and second from the bottom layers, the b values take on the original b values plus a constant of 12, therefore taking on every number in the set $\{13, 14, \ldots, 24\}$. The b values in the third from the top and third from the bottom layers are the original b values plus 24, taking on every number in the set $\{25, 26, \ldots, 36\}$. Continuing with this reasoning, we look at the middle two layers. The b values of these two layers are the original b values plus $12(\frac{c}{2} - 1) = 6c - 12$, which comes out to every number in the set $\{6c - 11, 6c - 10, \ldots, 6c\}$. Hence the b values cover every number in the set $\{1, 2, \ldots, 6c\}$ exactly once.

Now we look at the t values. The original t values are $13, 14, \ldots, 24$, but by construction, the t values in the center two layers take on the original t values plus

$12(\frac{c}{2}-1) = 6c-12$, or every number in the set $\{6c+1, 6c+2, \ldots, 6c+12\}$. If we move up one layer and down one layer, the t values of these next two layers take on the original t values plus $12(\frac{c}{2}) = 6c$, or every number in the set $\{6c+13, 6c+14, \ldots, 6c+24\}$. Following this reasoning, we see that the t values in the top and bottom layers take on the original t values plus a constant of $12(c-2) = 12c-24$, or every number in the set $\{12c-11, 12c-10, \ldots, 12c\}$. Hence, between the b's and the t's throughout the layers, we find every number from $1, 2, \ldots, 12c$ exactly once in the structure. Hence our construction does indeed produce $MH(4,4,c)$. □

See Figures 4.2 and 4.3 for an example of this construction.

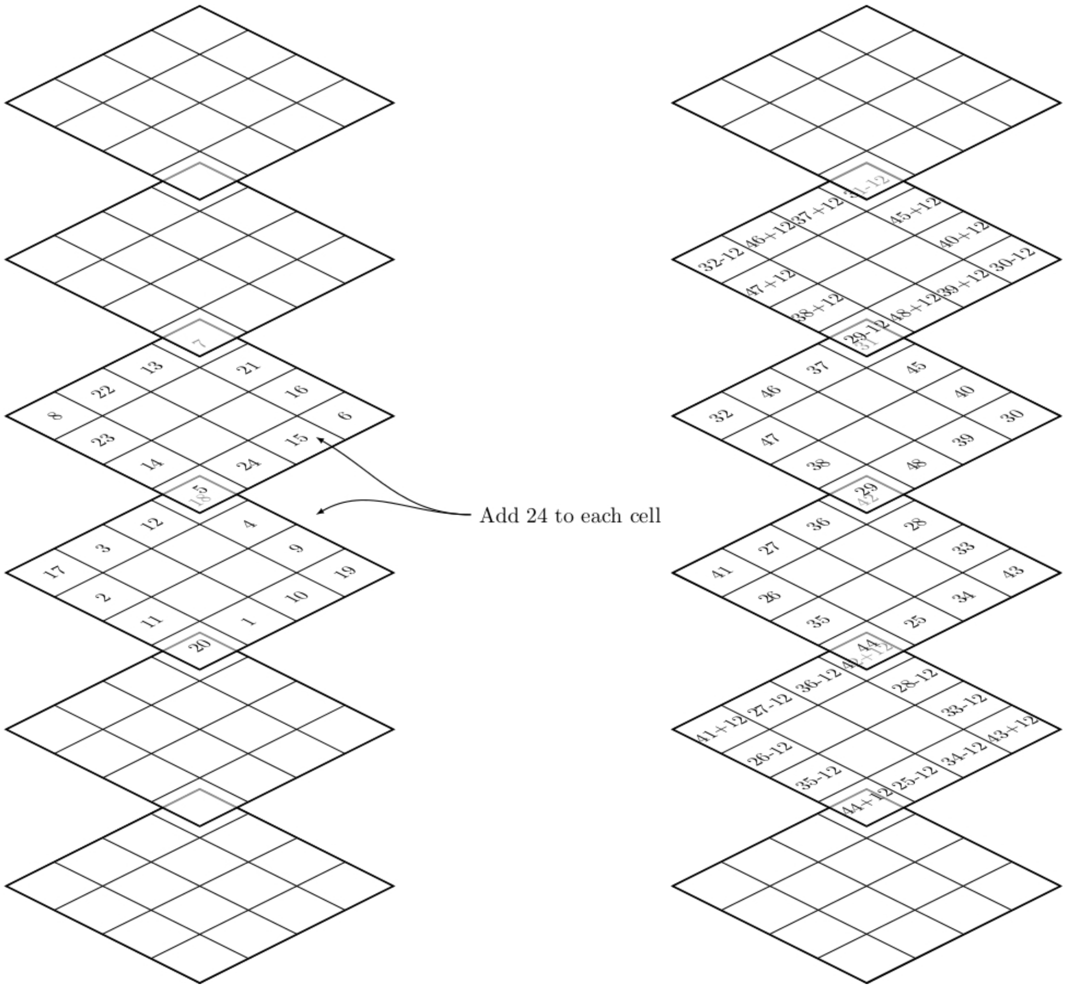
A natural next step is to explore what happens if we keep the side lengths at 4 ($a = 4$) and change the shape from a square to polygon with more sides. We now prove several results regarding the existence of $MH(n,4,2)$.

Theorem 4.3. *$MH(n,4,2)$ exists for $n \equiv 2 \pmod 4$, where $n \geq 6$.*

Proof. This is a proof by construction. Note that there are $(4-1)n = 3n$ numbers per layer, and $6n$ numbers in the overall structure. The second layer will be the complements of the first layer, such that the column sums all add up to $6n+1$. We observe that the magic row sum ρ is $2(3 \cdot n \cdot 2 + 1) = 2(6n+1)$.

We will begin by selecting the n corner pieces for the top layer. Write down a list of the numbers $1, 2, \ldots, 3n$; the numbers we will put in the corners will be the middle n numbers from this list ($n+1, \ldots, 2n$). Now, place the lowest corner number, $n+1$ (which is odd), in any arbitrary corner. Next, travel around the n-gon counterclockwise, visiting every other corner. Place the odd corner numbers in these corners, so that when we travel counterclockwise around the n-gon, we will have $n+1$, gap, $n+3$, gap, $\ldots$, gap, $3n-1$.

Find $n+1$ again. Travel on a line through the center of the n-gon to the opposite corner. In this corner, place $n+2$. Work in a counterclockwise cycle, placing the even



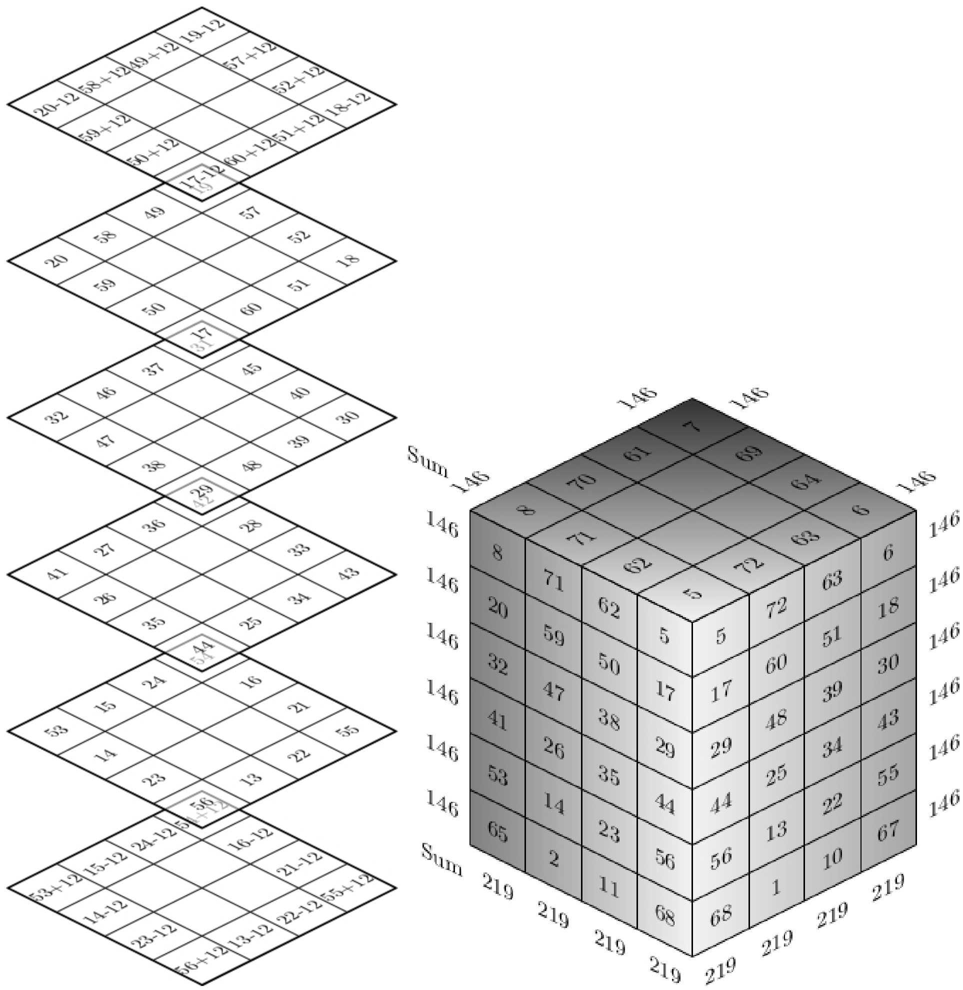
(a) Start with $MH(4,4,2)$

(b) Use the middle to build up and down

Figure 4.2: First steps: construction of $MH(4,4,6)$.

corner numbers in ascending order in the empty corners. Write in the corners of the second layer by taking the complements of the first layer.

We will now deal only with the second layer. We need to fill in the edges of this second layer such that the entries of each row sum to ρ. Let's call $\rho - \sum_{i=1}^{2} \text{corner}_i$ the needed *edge sum* for a given side of the n-gon. Notice that the placement of the



(a) Use the top and bottom to build the new layers

(b) A compressed look at $MH(4,4,6)$

Figure 4.3: Conclusion: construction of $MH(4,4,6)$.

corner pieces in this second layer gives us the needed distinct edge sums of $\frac{5n}{2}+2, \frac{5n}{2}+4, \ldots, 3n+1, 3n+3, \ldots, 3n+\frac{n}{2}=\frac{7n}{2}$, and we have two sides that require each given needed edge sum. We have already used the numbers $n+1, n+2, \ldots, 2n$ and their complements: we will use the numbers $1, 2, \ldots, n, 2n+1, 2n+2, \ldots, 3n$ to fill in the edges of the second layer, and their complements in the first layer. We will meet the required edge sums and place them in their appropriate locations:

- $(\frac{n}{2}-1)+(2n+3)$ and $(\frac{n}{2}-2)+(2n+4)$ equal $\frac{5n}{2}+2$.
- $(\frac{n}{2}-3)+(2n+7)$ and $(\frac{n}{2}-4)+(2n+8)$ equal $\frac{5n}{2}+4$.

  ⋮

- $2+(3n-3)$ and $1+(3n-2)$ equal $3n-1$.
- $n+(2n+1)$ and $(n-1)+(2n+2)$ equal $3n+1$.
- $(n-2)+(2n+5)$ and $(n-3)+(2n+6)$ equal $3n+3$.
- $(n-4)+(2n+9)$ and $(n-5)+(2n+10)$ equal $3n+5$.

  ⋮

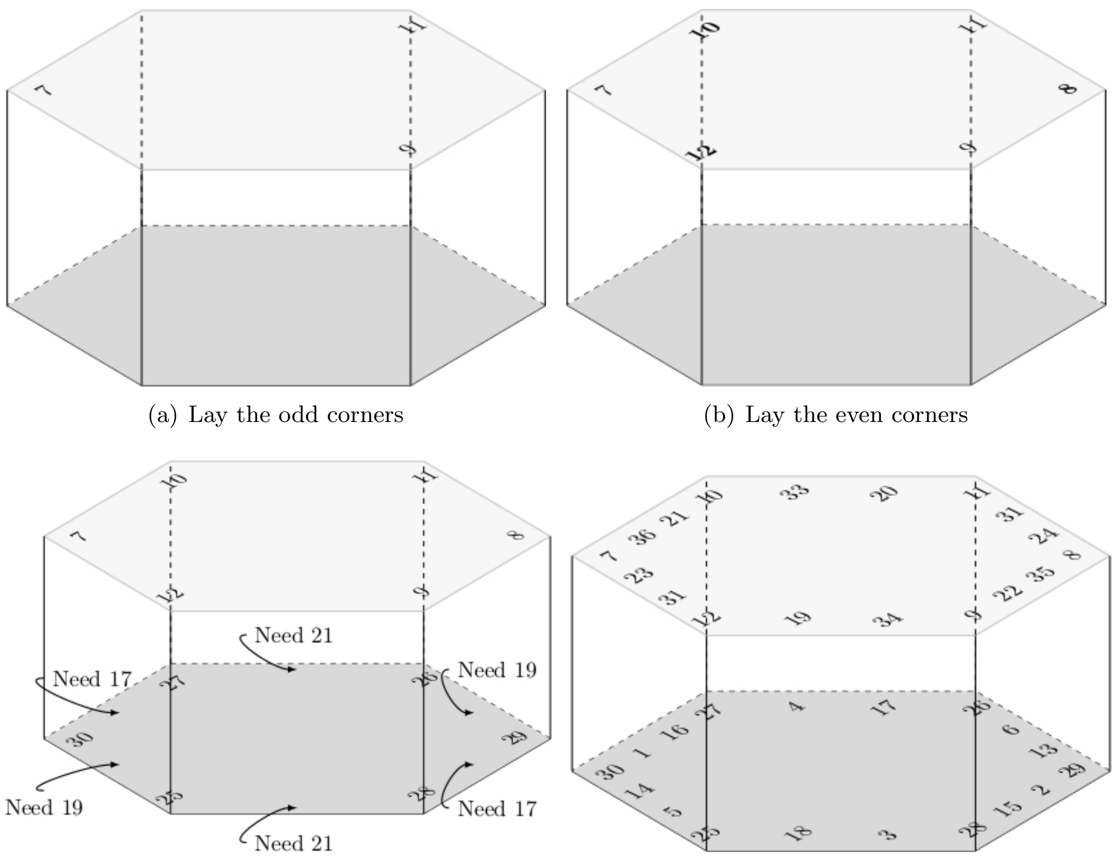
- $(\frac{n}{2}+1)+(3n-1)$ and $\frac{n}{2}+3n$ equal $\frac{7n}{2}$.

We then place these pairs of numbers along the appropriate row of the second layer of the n-gon, and their complements directly above them in the first layer. It is clear that the column sum for every column is $6n+1$, and by construction, each row has a row sum of $2(6n+1)$. Thus the construction is complete. □

4.1 Example of Construction in Proof of Theorem 4.3

We will use the construction in the proof of Theorem 4.3 to build $MH(6,4,2)$. Refer to Figure 4.4 for a visual construction.

1. We use the numbers $n+1,\ldots,2n$ (that is, $7,8,\ldots,12$) for the corners.
2. Starting with 7 in an arbitrary corner, go counterclockwise to every other corner, placing 9 and 11.
3. We then go straight across from 7 and place 8.
4. Travel counterclockwise, placing 10 and 12 in the open corners.



(a) Lay the odd corners

(b) Lay the even corners

(c) Place the corners of the bottom

(d) Find the appropriate pairs of edge pieces

Figure 4.4: Construction of $MH(6,4,2)$, following the proof of 4.3.

5. Fill in the corners of the second layer, which are 30, 27, 26, 29, 28, 25 when read clockwise.

6. Notice that the remaining sums that we need are $\frac{5n}{2}+2$, $\frac{5n}{2}+2$, $\frac{5n}{2}+4$, $\frac{5n}{2}+4$, $\frac{5n}{2}+6$, $\frac{5n}{2}+6$, which are $17, 17, 19, 19, 21, 21$.

7. Notice that $2+15$ and $1+16$ equal 17.

8. Notice that $6+13$ and $5+14$ equal 19.

9. Notice that $4+17$ and $3+18$ equal 21.

10. Place the above edge pieces in the second layer where appropriate.

11. Place the edge piece complements in the first layer.

Now the construction is complete.

Theorem 4.4. *$MH(n, 4, 2)$ exists for $n \equiv 0 \pmod 4$.*

Proof. We once again do a proof by construction. Draw a regular n-gon. Separate the numbers $1, 2, \ldots, 6n$ into two categories: $1, 2, \ldots, 3n$ and their complements, which are $6n, 6n-1, \ldots, 3n+1$, respectively. Reserve the numbers $n+1, n+2, \ldots, 2n$ for the corners of the top layer, their respective complements for the second layer, and the numbers $1, 2, \ldots, n$ and $2n+1, \ldots, 3n$ for the edges of the second layer.

Construct the first layer by laying the corners first. Pick an arbitrary corner, and lay down the odds in ascending order (beginning with $n+1$) at every other corner, traveling counterclockwise. Now find the location of $n+1$. Traveling clockwise from here, fill in all of the gaps with the even corner numbers in ascending order (beginning with $n+2$). Now fill in the second layer's corners using the complements of the first layer.

Observe that for every row, the edge sum that we need in the second layer is equivalent to the sum of the corners in the first layer of that row (this is true because the second layer's corners are the complements of the first). These needed edge sums in the second layer are $(n+1)+(n+2) = 2n+3$ (looking to the right of $n+1$), and $(n+1)+2n = 3n+1$ (looking to the left of $n+1$), and $(\frac{3n}{2}+1)+(\frac{3n}{2}+2) = 3n+3$. From the construction, it is possible to see that we need one sum of $2n+3$, $\frac{n}{2}-1$ sums of $3n+3$, and $\frac{n}{2}$ sums of $3n+1$ at every other edge. This is actually very easy to see if we consider the n-gon as a 3-colorable n-cycle. Color one edge with the "color" $2n+3$. Then color the remaining edges with "colors" $3n+1$ and $3n+3$, where no two adjacent edges have the same color. In this manner, it is easy to see that we end up with $\frac{n}{2}$ sums of $3n+1$ and $\frac{n}{2}-1$ sums of $3n+3$. By the placement of the corners, we see that the needed edge sums are indeed placed as was just described. We will now examine how to meet these needed edge sums in the second layer.

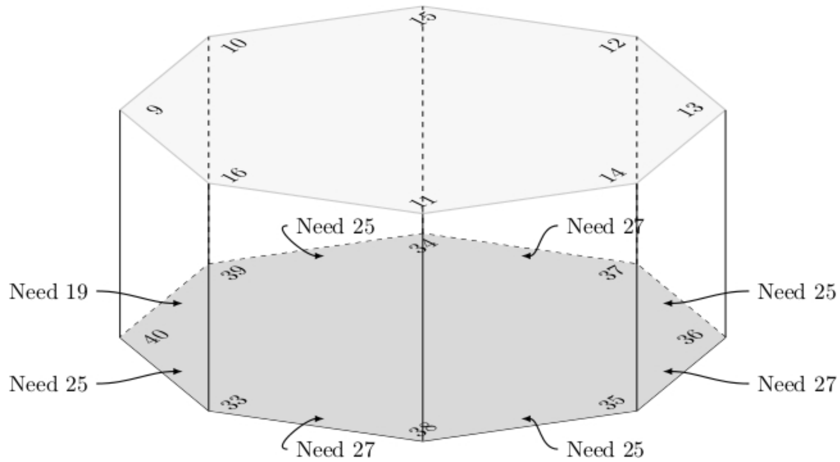
- $1 + (2n + 2) = 2n + 3$.
- $2 + (3n - 1), 4 + (3n - 3), \dots, n + (2n + 1)$ all equal $3n + 1$. These are pairs of evens from $2, \dots, n$ and odds from $2n + 1, \dots, 3n - 1$, so we have found $\frac{n}{2}$ sums of $3n + 1$.
- $3 + (3n), 5 + (3n - 2), \dots, (n - 1) + (2n + 4)$ all equal $3n + 3$. These are pairs of odds from $3, \dots, n - 1$ and evens from $2n + 4, \dots, 3n$, so we have found $\frac{n}{2} - 1$ sums of $3n + 3$.

Place the pairs of numbers in the appropriate edges of the second layer in order to satisfy the row sum $\rho = 2(6n + 1)$. Place their complements directly above them in the first layer to complete the construction. □

4.2 Example of Construction in Proof of Theorem 4.4

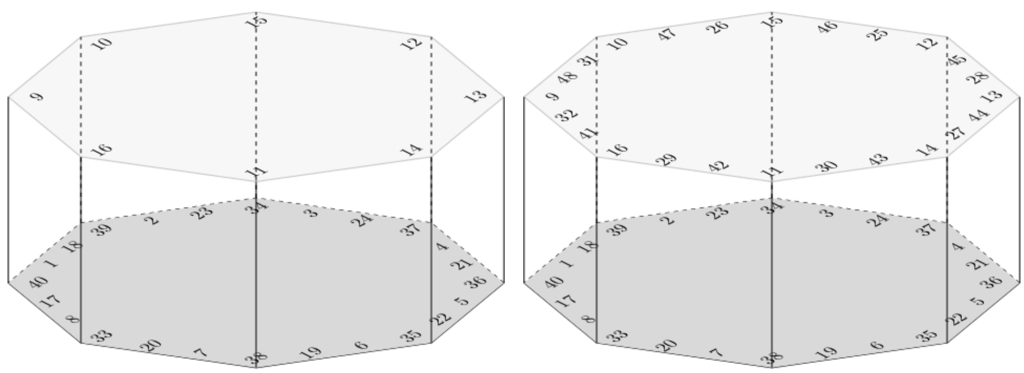
(a) Lay the odd corners (b) Lay the even corners

Figure 4.5: First steps: construction of $MH(8, 4, 2)$.



(a) Place the corners of the bottom

Figure 4.6: Further construction of $MH(8,4,2)$.



(a) Find the appropriate pairs of edge pieces (b) Place the complements of the edge pieces on the top

Figure 4.7: $MH(8,4,2)$, following the proof of 4.4.

We will use the construction in the proof of Theorem 4.4 to build $MH(8,4,2)$. Refer to Figures 4.5, 4.6, and 4.7 for a visual construction.

1. We use the numbers $n+1,\ldots,2n$ (which are $9,10,\ldots,16$) for the corners of the first layer.

2. Place 9 in an arbitrary corner. Working counterclockwise and visiting every other corner, place 11, 13, 15.

3. Now place 10 immediately to the right of 9, and working clockwise, place 12, 14, 16 in the open corners.

4. Fill in the corners of the second layer with the complements of the corners of the first (column sum $\sigma = 49$), which read clockwise as 40, 39, 34, 37, 36, 35, 38, 33.

5. Notice that the edge sums that we need for the second layer are $2n+3, 3n+1, 3n+3$, or $19, 25, 27$.

6. We need one sum of 19, $\frac{n}{2} = 4$ sums of 25, and $\frac{n}{2} - 1 = 3$ sums of 27.

7. Notice that $1 + 18 = 19$.

8. Notice that $2 + 23, 4 + 21, 6 + 19, 8 + 17$ all equal 25.

9. Notice that $3 + 24, 5 + 22, 7 + 20$ all equal 27.

10. Place the above edge pieces in the second layer where appropriate.

11. Place the complements of the edge pieces in the first layer.

Now the construction is complete and the resulting object is $MH(8, 4, 2)$.

Theorem 4.3 and Theorem 4.4 prove that $MH(n, 4, 2)$ exists for all even $n \geq 4$. We will see from Theorem 4.5 that $MH(n, 4, 2)$ also exists for all odd $n \geq 3$.

Theorem 4.5. *$MH(n, 4, 2)$ exists for all odd n, where $n \geq 3$.*

Proof. This is another proof by construction, and somewhat simpler than the last two. Draw a regular n-gon. Separate the numbers $1, 2, \ldots, 6n$ into two categories: $1, 2, \ldots, 3n$ and their complements, which are $6n, 6n - 1, \ldots, 3n + 1$, respectively. Reserve the numbers $n+1, n+2, \ldots, 2n$ for the corners of the top layer, their respective complements for the second layer, and the numbers $1, 2, \ldots, n$ and $2n+1, \ldots, 3n$ for the edges of the second layer.

Construct the first layer by laying the corners first. Place $n+1$ in an arbitrary corner, and travel around the n-gon clockwise, placing $n+2, n+3, \ldots, 2n$ in each corner, in ascending order. Now fill in the second layer's corners using the complements of the first layer.

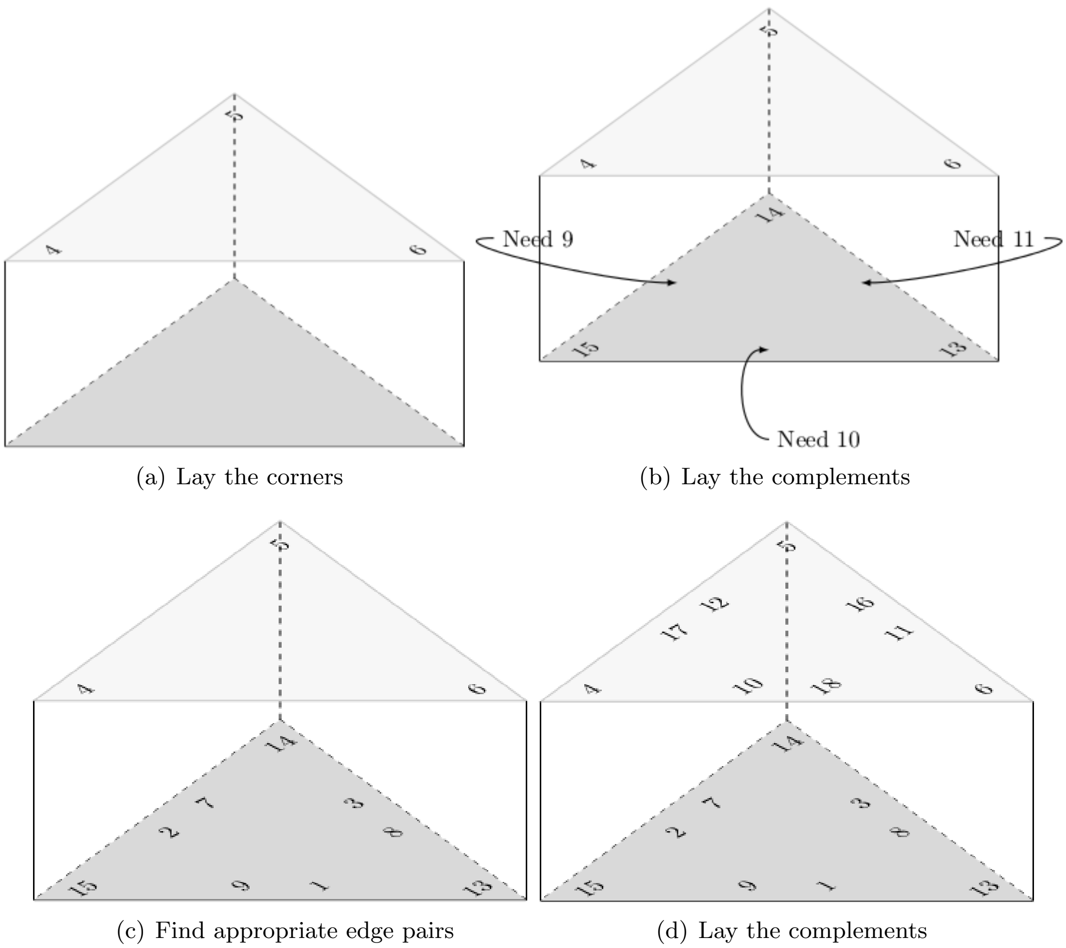
Observe that for every row, the edge sum that we need in the second layer is equivalent to the sum of the corners in the first layer of that row (this is true because the second layer's corners are the complements of the first). These needed edge sums in the second layer are $(n+1)+(n+2) = 2n+3, (n+2)+(n+3) = 2n+5, \ldots, (2n-1)+2n = 4n-1$ and $2n+(n+1) = 3n+1$. We satisfy the needed edge sums as follows:

- $2 + (2n+1) = 2n+3$
- $3 + (2n+2) = 2n+5$
- $4 + (2n+3) = 2n+7$
  $\vdots$
- $n + (3n-1) = 4n-1$
- $1 + 3n = 3n+1$.

Place these pairs of edges where needed in the second layer, and then complete the first layer with the complements of the second layer. By construction, all columns have equal sums and all n rows on both layers have the same sum. Therefore the construction is complete. $\square$

4.3 Example of Construction in proof of Theorem 4.5

We use the construction in the proof of Theorem 4.5 to build $MH(3, 4, 2)$. Refer to Figure 4.8 for a visual construction.



(a) Lay the corners (b) Lay the complements

(c) Find appropriate edge pairs (d) Lay the complements

Figure 4.8: Construction of $MH(3,4,2)$.

1. We use the numbers $n+1,\ldots,2n$ (which are $4,5,6$) for the corners of the first layer.

2. Place 4 in an arbitrary corner. Working clockwise, place 5 and 6.

3. Fill in the corners of the second layer with the complements of the first layer (column sum $\sigma = 19$), which read clockwise as 15, 14, 13.

4. Notice that the edge sums that we need for the second layer are $2n+3,\ldots,4n-$

$1, 3n+1$ (which are $9, 11, 10$.

5. Notice that $2+7=9$.

6. Notice that $3+8=11$.

7. Notice that $1+9=10$.

8. Place the above pairs of edges in the appropriate row of the second layer.

9. Place the complements of the edge pieces in the first layer.

Now the construction is complete and the resulting object is $MH(3,4,2)$.

It is particularly interesting to note that this construction works for all n, odd as well as even. The first two constructions are much more challenging, although certainly worth mentioning as it is also interesting to consider how many different constructions there are for such objects. For an example of the construction in the proof of Theorem 4.5, but with an even n, we will construct $MH(8,4,2)$ using this method.

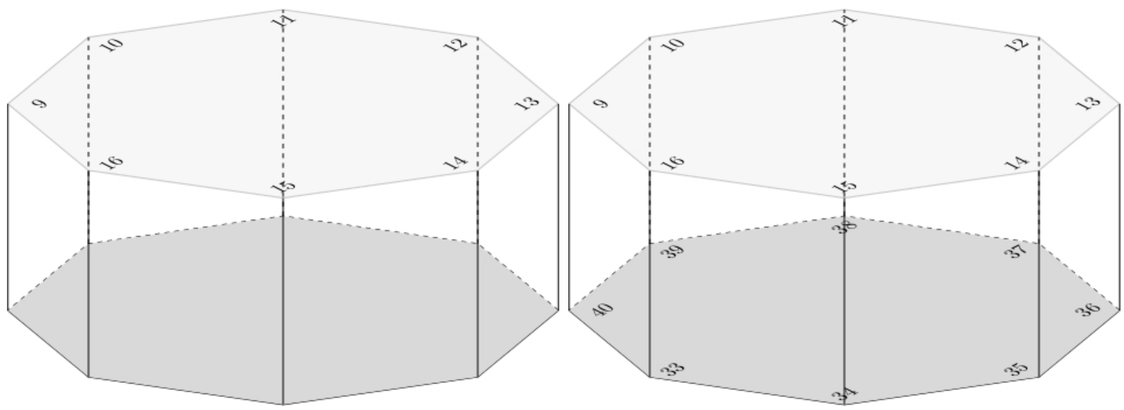
4.4 Example of Construction in Proof of Theorem 4.5, where n is even

Here we will use the construction in the proof of Theorem 4.5 to construct $MH(8,4,2)$. Refer to Figures 4.9, 4.10, and 4.11 for a visual construction.

1. We use the numbers $n+1, \ldots, 2n = 9, 10, \ldots, 16$ for the corners of the first layer.

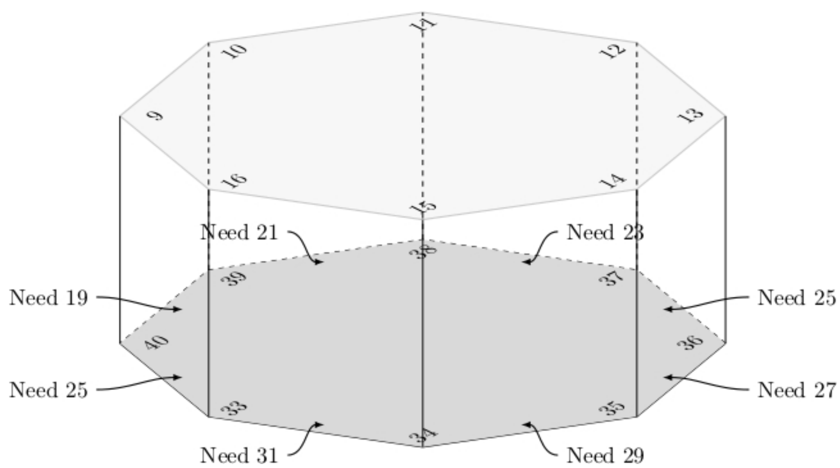
2. Place 9 in an arbitrary corner. Working clockwise, place $10, 11, \ldots, 16$.

3. Fill in the corners of the second layer with the complements of the first layer (column sum $\sigma = 49$).



(a) Lay the corners (b) Place complements on the bottom

Figure 4.9: First steps: construction of $MH(8,4,2)$.



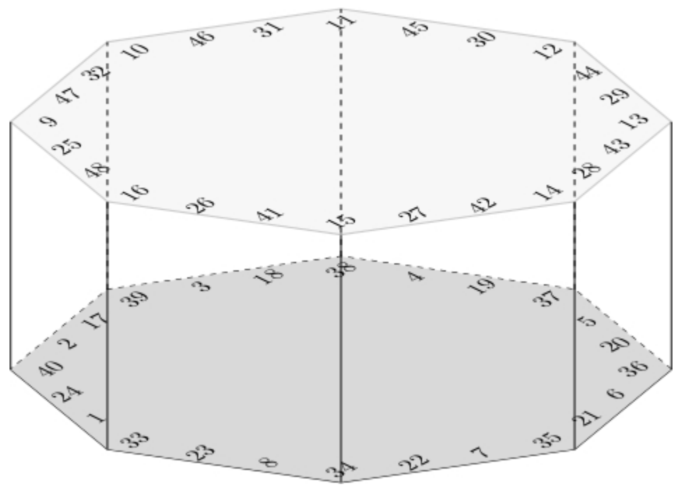
(a) Identify needed edge sums

Figure 4.10: Next step: construction of $MH(8,4,2)$.

4. Notice that the edge sums that we need for the second layer are $2n+3, 2n+5\ldots, 4n-1, 3n+1$ (which are 19, 21, 23, 25, 27, 29, 31, 25).

5. Notice that $2+17=19$.

6. Notice that $3+18=21$.



(a) Find the appropriate pairs of edge pieces

Figure 4.11: $MH(8,4,2)$, following the proof of Theorem 4.5.

7. Notice that $4 + 19 = 23$.
8. Notice that $5 + 20 = 25$.
9. Notice that $6 + 21 = 27$.
10. Notice that $7 + 22 = 29$.
11. Notice that $8 + 23 = 31$.
12. Notice that $1 + 24 = 25$.
13. Place the above pairs of edges in the appropriate row of the second layer.
14. Place the complements of the edge pieces in the first layer.

Now the construction is complete and the resulting object is $MH(8,4,2)$.

Corollary 4.6. *$MH(n,4,2)$ exists for all $n \geq 3$.*

We will use this result and an analogue to the proof of Theorem 4.2 to achieve a much stronger result.

Theorem 4.7. *$MH(n,4,c)$ exists for all n and all even c.*

Proof. Given some even c, we will construct $MH(n,4,c)$ using Corollary 4.6.

We will first derive the column sum and row sum of $MH(n,4,c)$, which can be done as follows:

$$\begin{aligned}
\sigma &= \frac{\text{sum of all cells}}{\#\text{ of columns}} \\
&= \frac{\frac{1}{2}(n(a-1)c)(n(a-1)c+1)}{n(a-1)} \\
&= \frac{c(nc(a-1)+1)}{2} \\
&= \frac{c(nc(3)+1)}{2} \\
&= \frac{c}{2}(3nc+1) \\
\rho &= \frac{a \times \text{column sum}}{\#\text{ of layers}} \\
&= \frac{4\sigma}{c} \\
&= \frac{4 \cdot \frac{c}{2}(3nc+1)}{c} \\
&= 2(3nc+1).
\end{aligned}$$

We start the construction with a set of c unlabeled layers of regular n-gons stacked on top of each other. We will fill in the boundaries of each of these n-gons in a manner which will produce $MH(n,4,c)$. For convenience, we refer to the cells of each layer (from top to bottom) as either corners or edges (non-corners). Refer again to the construction in Theorem 4.5, and use it to construct $MH(n,4,2)$. Place this in the middle two layers of the unlabeled stack. Now, we need the row sums of both of these layers to equal $\rho = 2(3nc+1)$, but it is clear that as is, the row sums are equal to $2(6n+1)$. We achieve a row sum of ρ by adding the same constant to each entry in the middle of the stack. This constant must be

$$\frac{\text{col sum} - (\text{col sum of } MH(n,4,2)) \cdot (\#\text{ of copies of } MH(n,4,2) \text{ that fit})}{\#\text{ of layers}}$$

$$\begin{aligned}
&= \frac{\sigma - (6n+1)\cdot \frac{c}{2}}{c} \\
&= \frac{\frac{c}{2}(3nc+1) - \frac{(6n+1)c}{2}}{c} \\
&= \frac{\frac{c}{2}(3nc+1-(6n+1))}{c} \\
&= \frac{\frac{c}{2}(3nc-6n)}{c} \\
&= \frac{3n(c-2)}{2}.
\end{aligned}$$

Observe that the row sum of each of these two middle layers will be

$$\begin{aligned}
&2(6n+1) + 4\left(\frac{3n(c-2)}{2}\right) \\
&= 12n + 2 + 2(3nc - 6n) \\
&= 12n + 2 + 6nc - 12n \\
&= 6nc + 2 \\
&= 2(3nc+1) \\
&= \rho.
\end{aligned}$$

We must now fill in the remaining unlabeled layers. Fill in the corners of the layer directly below the middle by adding $3n$ to the numbers in the corners immediately above it, and subtract $3n$ from the numbers in the edges immediately above it. We iterate this process, working down to the bottom layer. We then start from the middle again and work up, but this time we will subtract $3n$ from the corners in the layer immediately below and add $3n$ to the number in the edges in the layer immediately below. Iterate this process, working up.

Observe that in each layer, the row sum differs from ρ by $3n - 3n - 3n + 3n = 0$, so each row sums to ρ. Now we check the column sum. We note that the pairwise column sum of the middle two layers is

$$(6n+1) + 2\left(\frac{3n(c-2)}{2}\right)$$
$$= 6n + 1 + 3n(c-2)$$
$$= 6n + 1 + 3nc - 6n$$
$$= 3nc + 1.$$

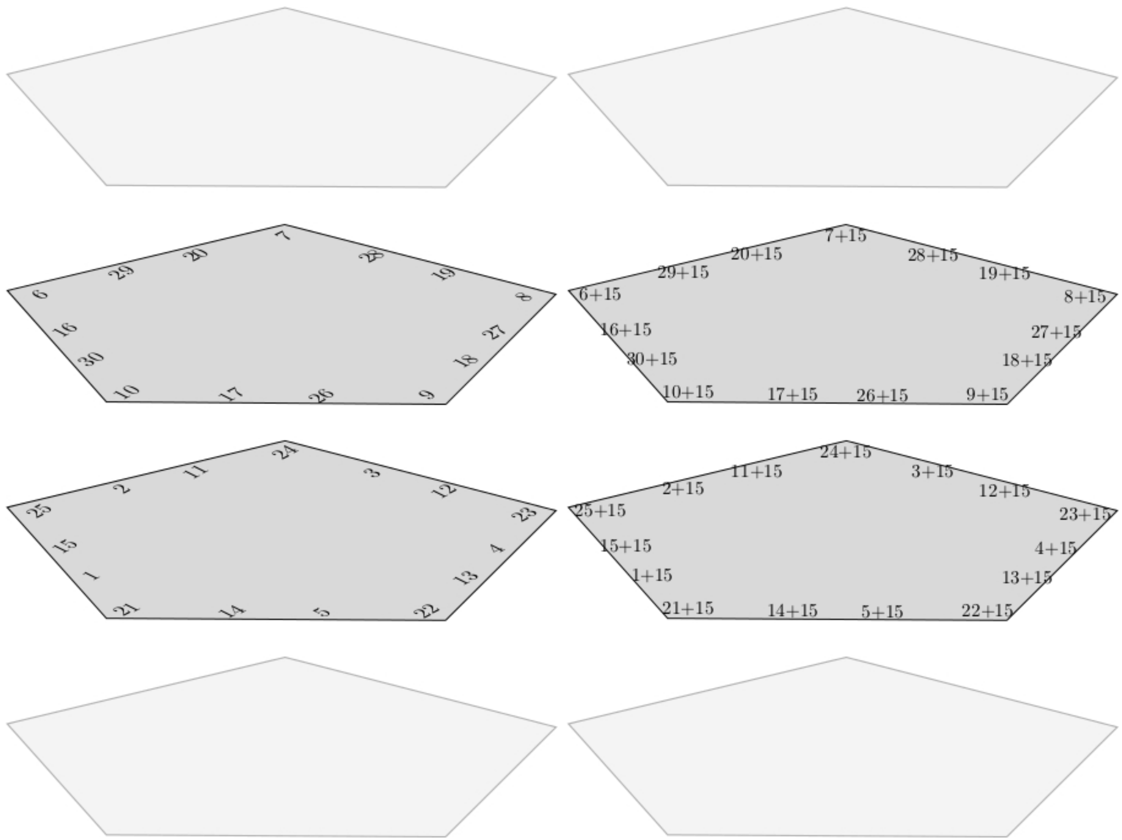
Now consider the next layer up and the next layer down. The pairwise column sum of these two layers must also be $3nc + 1$, since we add $3n$ to the corners below and subtract them up top, and we add $3n$ to the edges up top and subtract $3n$ from the edges below. If we iterate this process of checking the pairwise column sums, we see that each pair has column sums of $3nc + 1$. Therefore, the total column sum is $(3nc+1) \cdot (\# \text{ of pairs}) = (3nc+1) \cdot \frac{c}{2} = \sigma$.

Since all rows sum to ρ and all columns sum to σ, the final step is to show is that each of the numbers from $1, \ldots, 3nc$ appears exactly once. It is convenient to classify the numbers $\{1, 2, \ldots, 3n\}$ as b numbers (since they are in the bottom half of $\{1, 2, \ldots, 6n\}$), and the numbers $\{3n+1, 3n+1, \ldots, 6n\}$ as t numbers (occurring in the top half of $\{1, 2, \ldots, 6n\}$). Then, if we look at the completed structure in terms of b's and t's, we see that the corners in the top $\frac{c}{2}$ layers and the edges in the bottom $\frac{c}{2}$ layers are all b numbers plus a constant (where the constant is a multiple of $3n$). The spaces not occupied by b numbers are occupied by t numbers plus a constant (where the constant is a multiple of $3n$). We now look at what values we have in the structure. In the top and bottom layers, the b values take on every number in the set $\{1, 2, \ldots, 3n\}$. In the second from the top and second from the bottom layers, the b values take on the original b values plus a constant of $3n$, therefore taking on every number in the set $\{3n+1, 3n+2, \ldots, 6n\}$. The b values in the third from the top and third from the bottom layers are the original b values plus $2 \cdot 3n$, taking on every number in the set $\{6n+1, 6n+2, \ldots, 9n\}$. Continuing with this reasoning, we look at the middle two layers. The b values of these two layers are the original b values plus $3n(\frac{c}{2} - 1) = 3n\frac{c}{2} - 3n$, which comes out to every number in the set $\{3n\frac{c}{2} - 3n + 1, 3n\frac{c}{2} - 3n + 2, \ldots, 3n\frac{c}{2} - 3n + 3n = 3n\frac{c}{2}\}$. Hence the b values cover every number in the set $\{1, 2, \ldots, 3n\frac{c}{2}\}$ exactly once.

Now we look at the t values. The original t values are $3n + 1, 3n + 2, \ldots, 6n$, but by construction, the t values in the center two layers take on the original t values plus $3n(\frac{c}{2} - 1) = 3n\frac{c}{2} - 3n$, or every number in the set $\{3n\frac{c}{2} + 1, 3n\frac{c}{2} + 2, \ldots, 3n\frac{c}{2} + 3n\}$. If we move up one layer and down one layer, the t values of these next two layers take on the original t values plus $3n\frac{c}{2}$, or every number in the set $\{3n\frac{c}{2} + 3n + 1, 3n\frac{c}{2} + 3n + 2, \ldots, 3n\frac{c}{2} + 6n\}$. Following this reasoning, we see that the t values in the top and bottom layers take on the original t values plus a constant of $3n(c - 2) = 3nc - 6n$, or every number in the set $\{3nc - 6n + 3n + 1 = 3nc - 3n + 1, 3nc - 3n + 2, \ldots, 3nc\}$. Hence, between the b's and the t's throughout the layers, we find every number from $1, 2, \ldots, 3nc$ exactly once in the structure. Hence our construction does indeed produce $MH(n, 4, c)$. □

4.5 Example of Construction in Proof of Theorem 4.7

As an example of this construction, we will build $MH(5, 4, 4)$ (see Figures 4.12 and 4.13 for a visual of the construction). We can think of this as either four stacked pentagons or as five 4×4 squares attached along edges. The first step in the construction requires us to have a copy of $MH(5, 4, 2)$. We will use one that can be obtained using the construction in the proof of Theorem 4.5. Following the construction in the proof of Theorem 4.7, we proceed with the following steps:



(a) Start with $MH(5, 4, 2)$ (b) Add 15 everywhere

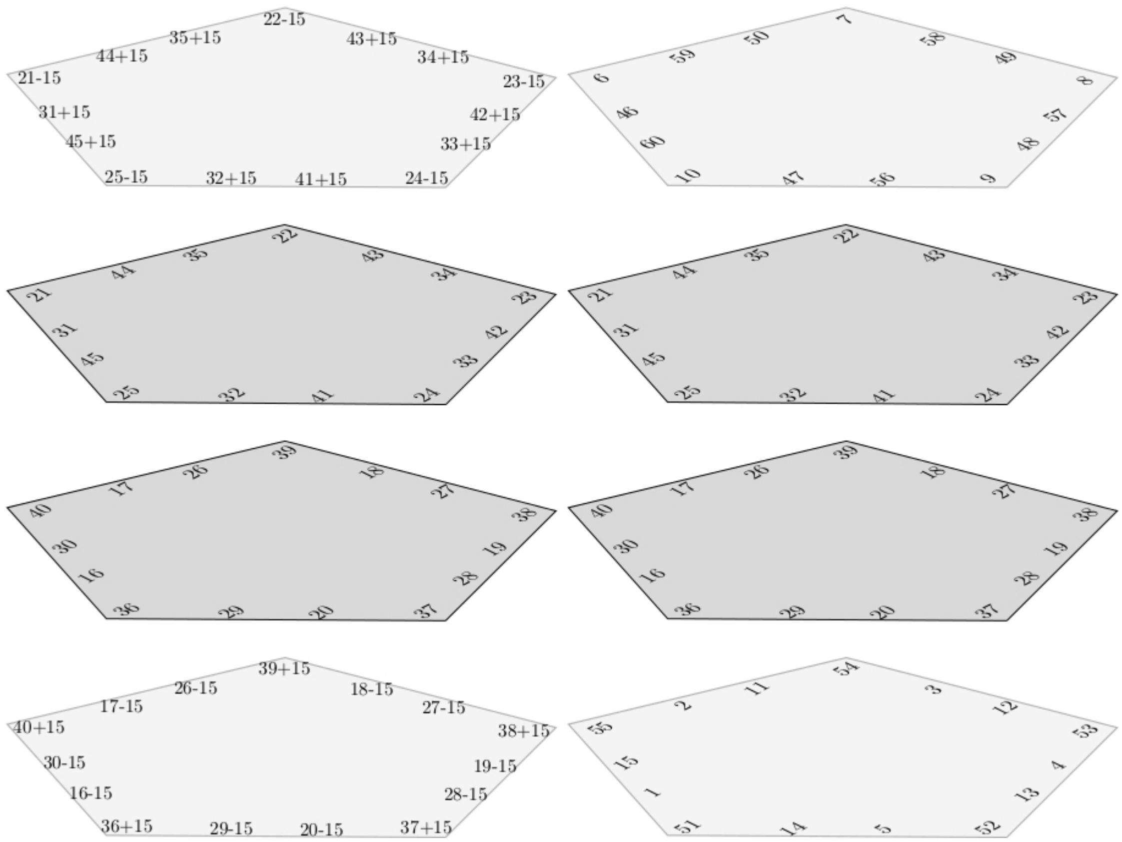
Figure 4.12: First steps: construction of $MH(5, 4, 4)$.

1. Begin with an stack of 4 empty pentagons. Fill in the middle with a copy of $MH(5, 4, 2)$.

2. Add $\frac{3n(c-2)}{2} = \frac{3 \cdot 5(4-2)}{2} = 15$ to every entry in the middle two layers.

3. To get the bottom layer, add $3n = 15$ to each of the corners of the layer above it, and subtract 15 from each of the edges above it.

4. To get the top layer, subtract $3n = 15$ from each of the corners in the layer below it, and add 15 to each of the edges below it.



(a) Use the middle to build the top and bottom (b) $MH(5,4,4)$

Figure 4.13: Conclusion: construction of $MH(5,4,4)$.

The resulting object is $MH(5,4,4)$.

At this point, we have considered only even values for c. We now consider odd c values.

Lemma 4.8. *If n is even and c is odd, then $MH(n,4,c)$ does not exist.*

Proof. We must only consider the column sum σ for this proof.

$$\begin{aligned}\sigma &= \frac{\text{sum of all cells}}{\#\text{ of columns}}\\ &= \frac{nc(4-1)\cdot(nc(4-1)+1)}{2\cdot n(4-1)}\\ &= \frac{c}{2}(nc(3)+1).\end{aligned}$$

Since c is odd and n is even, $(3nc+1)$ is odd. Hence, σ is not an integer. Thus, $MH(n,4,c)$ does not exist. □

Another natural generalization arises here: what do we know about the existence of $MH(n,a,c)$ for $a > 4$? We will first explore which values of n, a, and c preclude the existence of $MH(n,a,c)$, and then study existence of $MH(4,a,c)$.

Theorem 4.9. *If a is odd, then $MH(n,a,c)$ does not exist.*

Proof. Suppose a is odd and $MH(n,a,c)$ does exist. We will consider the column and row sums.

$$\begin{aligned}\sigma &= \frac{\text{sum of all cells}}{\#\text{ of columns}}\\ &= \frac{nc(a-1)\cdot(nc(a-1)+1)}{2\cdot n(a-1)}\\ &= \frac{c}{2}(nc(a-1)+1)\end{aligned}$$

and

$$\begin{aligned}\rho &= \frac{a\sigma}{c}\\ &= \frac{a(nc(a-1)+1)}{2}.\end{aligned}$$

Since a is odd, $nc(a-1)$ is even, and $nc(a-1)+1$ is odd. Hence,

$$a(nc(a-1)+1) = \text{odd} \times \text{odd} = \text{odd}$$

and therefore ρ is not an integer. This contradicts the assumption that $MH(n, a, c)$ exists, and the proof is complete. □

Theorem 4.10. *If n and a are even and c is odd, then $MH(n, a, c)$ does not exist.*

Proof. Suppose that a and n are even and c is odd, and $MH(n, a, c)$ exists. Consider σ given in the proof of Theorem 4.9:

$$\sigma = \frac{c}{2}(nc(a-1)+1).$$

Since n is even, $nc(a-1)$ is even and $nc(a-1)+1$ is odd. Then, since c is odd,

$$c(nc(a-1)+1) = \text{odd} \times \text{odd} = \text{odd}$$

and therefore σ is not an integer. This contradicts the assumption that $MH(n, a, c)$ exists, and the proof is complete. □

Because of Theorem 4.9, we will henceforth consider a as an even number greater than 2. We will now make two observations based on all of our previous theorems.

Observation 4.11. If n is even and c is even, then $MH(n, a, c)$ may exist.

Observation 4.12. If n is odd, then $MH(n, a, c)$ may exist as long as $c \geq 3$ (i.e. c can perhaps be odd).

Our next step is to explore the existence for $MH(4, a, c)$, where a and c are even (Theorem 4.9, Theorem 4.10). We will break this down further, considering the cases where $a \equiv 2 \pmod 4$ and $a \equiv 0 \pmod 4$.

Lemma 4.13. *If $a \equiv 2 \pmod 4$, then $MH(4, a, 2)$ exists.*

Proof. We will consider the case where $a = 6$ at the end of the proof. For now, suppose $a > 6$. Begin by building exactly half of each layer in the first step, using the numbers

$1, 2, \ldots, 4(a-1)$ (the entire completed structure will contain the first $2 \cdot 4(a-1)$ natural numbers, so in this first step we are placing the lower half of the numbers). Place 12 of these numbers in the layers as shown in Figure 4.14.

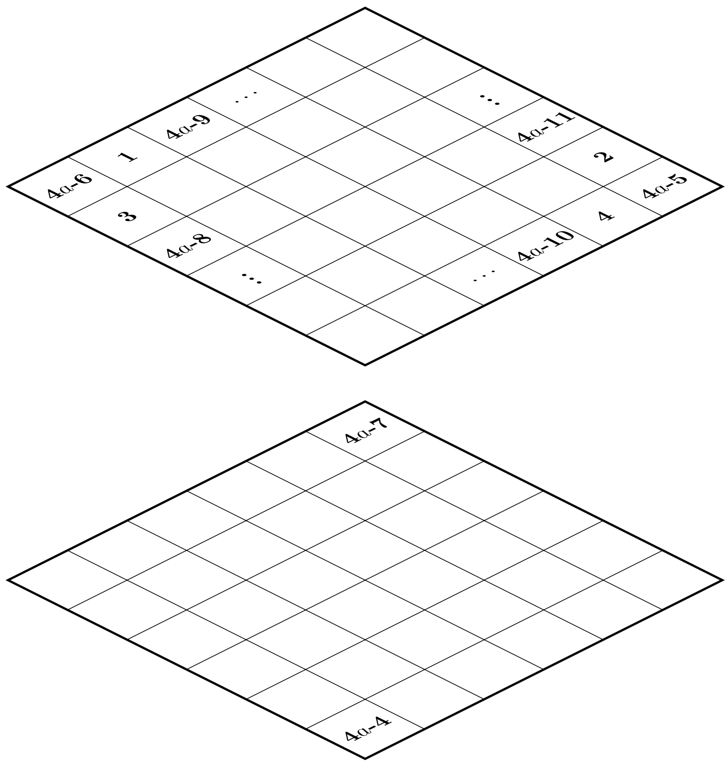


Figure 4.14: The first 12 numbers placed.

We now turn our attention to placing the following numbers: $a+3, a+4, \ldots, 3a-10$. We see that this list has $3a - 10 - (a + 3) + 1 = 2a - 12$ numbers. Create the pairs $(a+3, 3a-10), (a+4, 3a-11)$, etc., where each pair has a sum of $4a-7$. The number of pairs that we have is $a - 6 \equiv 2 - 6 \equiv 0 \pmod 4$. Place one quarter $((a/2-3)/2)$ of these pairs in each of the four edges of the top layer.

Now we have to place the numbers $5, 6, \ldots, a + 2, 3a - 9, \ldots, 4a - 12$, which all go in the bottom layer. Notice that we can also group these numbers into the pairs $(5, 4a-12), \ldots, (a+2, 3a-9)$, with each pair summing to $4a-7$. There are $4 \cdot (a/2-1)/2$

pairs in total. We will place one quarter $((a/2-1)/2)$ of these pairs into each of the four edges of the bottom layer.

To complete our construction, we place the complements of all of the numbers we have already placed into the corresponding cell in the opposite layer. We define the complements as we have before: each pair of complements will sum to $1+2\cdot 4(a-1) = 8a-7$ (this is σ from the proof of Theorem 4.9). Our structure clearly has a constant column sum, which is σ. We check the row sum (where $\rho = \frac{a}{2}(8a-7)$, from the proof of Theorem 4.9), starting with the top row of the top layer.

$$\begin{aligned}
\text{First part} &= (4a-6)+1+(4a-9)+\frac{a/2-3}{2}\cdot(4a-7) \\
&= \left(2+\frac{a/2-3}{2}\right)(4a-7) \\
&= \left(\frac{a/2+1}{2}\right)(4a-7).
\end{aligned}$$

$$\begin{aligned}
\text{Second part} &= \frac{a/2-1}{2}\cdot(\text{complement of } 4a-7)+4a \\
&= \frac{a/2-1}{2}(12a-7)+4a.
\end{aligned}$$

$$\begin{aligned}
\text{Top row sum} &= \frac{a/2+1}{2}(4a-7)+\frac{a/2-1}{2}(12a-7)+4a \\
&= \frac{a/2-1}{2}(4a-7+12a-7)+4a-7+4a \\
&= \frac{a/2-1}{2}(16a-14)+8a-7 \\
&= (\frac{a}{2}-1)(8a-7)+8a-7 \\
&= \frac{a}{2}(8a-7) \\
&= \rho.
\end{aligned}$$

It is easy to check that the other three sides also sum to ρ, and by construction, each layer will have the same row sum. Hence, the object we have created is $MH(4, a, 2)$, and our construction is complete. If $a = 6$, then $a + 3 = 9$ and $3a - 10 = 8$, so we cannot follow the second step of the construction. Hence, we ignore that step, realizing that it does not alter the guaranteed magicness of the construction (we can still place $(a/2 - 3)/2 = 0/2 = 0$ pairs along each edge of the top layer, as the construction calls for). Thus, this construction is also valid when $a = 6$. □

4.6 Example of construction in the proof of Lemma 4.13

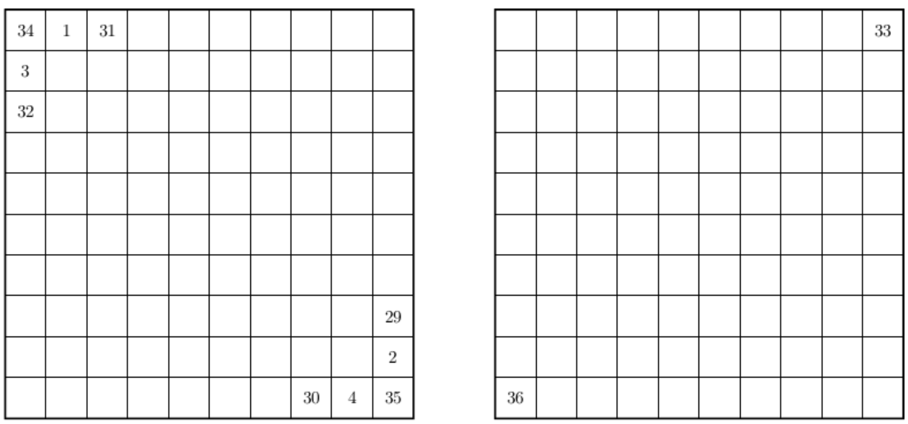


Figure 4.15: Follow the template in Figure 4.14.

Refer to Figures 4.15, 4.16, and 4.17 to view the construction of $MH(4, 10, 2)$ following the construction in the proof of Lemma 4.13.

Lemma 4.14. *If* $a \equiv 0 \pmod 4$*, then* $MH(4, a, 2)$ *exists.*

Proof. This proof is similar to that of Lemma 4.13, but a bit trickier. The case $MH(4, 4, 2)$ is given in Figure 4.1. Now consider $a \geq 8$. Begin by building exactly half of each layer in the first step, using the numbers $1, 2, \ldots, 4(a-1)$ (the entire completed structure will contain the first $2 \cdot 4(a-1)$ natural numbers, so in this first step

34	1	31	13	20					
3									
32									
14									
19									
									18
									15
									29
									2
					17	16	30	4	35

					27	6	28	5	33
									7
									26
									8
									25
21									
12									
22									
11									
36	9	24	10	23					

Figure 4.16: Fill out the first half of each side with pairs that sum to 33.

34	1	31	13	20	46	67	45	68	40
3									66
32									47
14									65
19									48
52									18
61									15
51									29
62									2
37	64	49	63	50	17	16	30	4	35

39	72	42	60	53	27	6	28	5	33
70									7
41									26
59									8
54									25
21									55
12									58
22									44
11									71
36	9	24	10	23	56	57	43	69	38

Figure 4.17: Filling in complements. Row sum is 365, column sum is 73.

we are placing the lower half of the numbers). Place 28 of these numbers in the layers as shown in Figure 4.18.

We now turn our attention to placing the following numbers, which is very similar to the second step of the construction in Lemma 4.13: $a+1, a+2, \ldots, 3a-8$, excluding the eight numbers $2a-7, \ldots, 2a$. We see that this list contains $2a-16$ elements. Create the pairs $(a+1, 3a-8), (a+2, 3a-9), \ldots, (2a-8, 2a+1)$, where each pair sums to $4a-7$. Notice that we have created $a-8 \equiv 0 \pmod 4$ pairs. Place one quarter $(a/4-2)$ of these pairs in each of the four edges of the top layer.

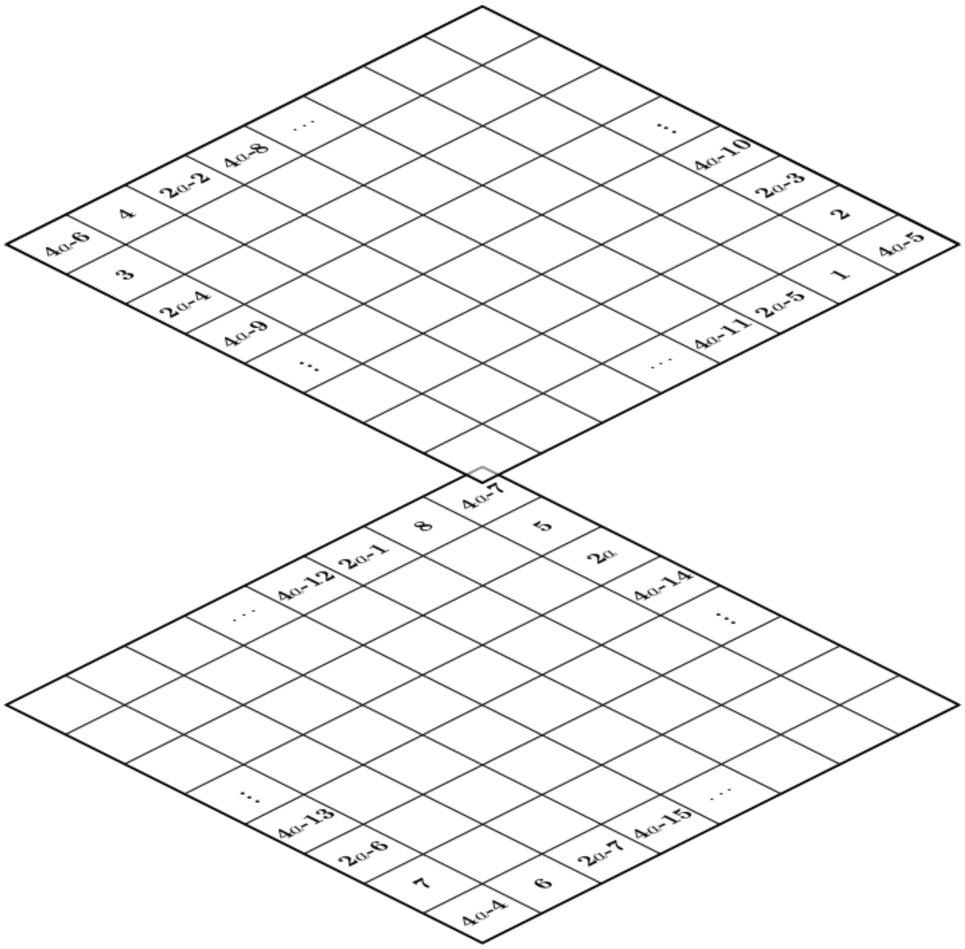


Figure 4.18: The first 28 numbers placed.

Now we must place the numbers $9, 10, \ldots, a, 3a-7, 3a-6, \ldots, 4a-16$, which all go in the bottom layer. Notice that we can group these numbers into the pairs $(9, 4a-16), (10, 4a-17), \ldots, (a, 3a-7)$, where each pair sums to $4a-7$. The number of pairs that we have is $a-9+1 = a-8 \equiv 0 \pmod 4$. Place one quarter $(a/4-2)$ of these pairs in each of the four edges of the bottom layer.

To complete our construction, place the complements of all of the numbers we have already placed into the corresponding cell in the opposite layer. We define the complements as we have before: each pair of complements will sum to $1 + 2 \cdot 4(a-1) = 8a-7$

(this is σ from the proof of Theorem 4.9). Our structure clearly has a constant column sum, which is σ. We check the row sum (where $\rho = \frac{a}{2}(8a-7)$, from the proof of Theorem 4.9), starting with the top row of the top layer.

$$\begin{aligned}
\text{First part} &= (4a-6)+4+(2a-2)+(4a-8)+(a/4-2)\cdot(4a-7) \\
&= 10a-12+(a/4-2)(4a-7). \\
\text{Second part} &= (4a+5)+(6a-6)+(8a-15)+4a+(a/4-2)(12a-7) \\
&= 22a-16+(a/4-2)(12a-7). \\
\text{Top row sum} &= 10a-12+(a/4-2)(4a-7)+22a-16+(a/4-2)(12a-7) \\
&= 32a-28+(a/4-2)(16a-14) \\
&= 4(8a-7)+(a/2-4)(8a-7) \\
&= \frac{a}{2}(8a-7) \\
&= \rho.
\end{aligned}$$

It is easy to check that the other three sides also sum to ρ, and by construction, each layer will have the same row sum. Hence, the object we have created is $MH(4,a,2)$, and our construction is complete. □

Theorem 4.15. *$MH(4,a,c)$ exists if and only if a and c are even.*

Proof. To prove the only if direction, suppose $MH(4,a,c)$ exists. Then a and c must be even according to Theorems 4.9 and 4.10.

To prove the if direction, suppose a and c are even. We know that $MH(4,a,2)$ exists as a direct result from Lemmas 4.13 and 4.14. We will mimic the construction in the proof of Theorem 4.7 to achieve our result.

We will first derive the column sum and row sum of $MH(4,a,c)$, which can be done as follows:

$$\begin{aligned}
\sigma &= \frac{\text{sum of all cells}}{\text{\# of columns}} \\
&= \frac{\frac{1}{2}(n(a-1)c)(n(a-1)c+1)}{n(a-1)} \\
&= \frac{c(nc(a-1)+1)}{2} \\
&= \frac{c(4c(a-1)+1)}{2} \\
&= \frac{c}{2}(4c(a-1)+1).
\end{aligned}$$

$$\begin{aligned}
\rho &= \frac{a \times \text{column sum}}{\text{\# of layers}} \\
&= \frac{a\sigma}{c} \\
&= \frac{a \cdot \frac{c}{2}(4c(a-1)+1)}{c} \\
&= \frac{a}{2}(4c(a-1)+1).
\end{aligned}$$

Start the construction with a set of c unlabeled layers of regular n-gons stacked on top of each other. We will fill in the boundaries of each of these n-gons in a manner which will produce $MH(4, a, c)$. As before, it is convenient to classify the numbers $\{1, 2, \ldots, 4(a-1)\}$ as b numbers (since they are in the bottom half of $\{1, 2, \ldots, 8(a-1)\}$), and the numbers $\{4a-3, 4a-2, \ldots, 8(a-1)\}$ as t numbers (occurring in the top half of $\{1, 2, \ldots, 8(a-1)\}$). Refer again to the constructions in Lemmas 4.13 and 4.14, and use one of them to construct $MH(4, a, 2)$. Place this in the middle two layers of the unlabeled stack. We now identify the locations of the b numbers and the t numbers, and we will henceforth refer to these specific cells (and their corresponding cells in each layer) as either b cells or t cells. Now, we need the row sums of both of these layers to equal $\rho = \frac{a}{2}(4c(a-1)+1)$, but it is clear that as is, the row sums are equal to $\frac{a}{2}(8a-7)$. We achieve a row sum of ρ by adding the same constant to each entry in the middle of the stack. This constant must be

$$\frac{\text{col sum} - (\text{col sum of } MH(4,a,2)) \cdot (\#\text{ of copies of } MH(4,a,2) \text{ that fit})}{\#\text{ of layers}}$$

$$\begin{aligned}
&= \frac{\sigma - (8a-7)\cdot\frac{c}{2}}{c} \\
&= \frac{\frac{c}{2}(4c(a-1)+1) - (8a-7)\frac{c}{2}}{c} \\
&= \frac{\frac{c}{2}((4c(a-1)+1) - (8a-7))}{c} \\
&= \frac{\frac{c}{2}(4c(a-1) - 8a + 8)}{c} \\
&= 2c(a-1) - 4a + 4.
\end{aligned}$$

Observe that the row sum of each of these two middle layers are

$$\begin{aligned}
&\frac{a}{2}(8a-7) + a(2c(a-1) - 4a + 4) \\
&= \frac{a}{2}(8a-7) + \frac{a}{2}(4c(a-1) - 8a + 8) \\
&= \frac{a}{2}(4c(a-1) - 8a + 8 + 8a - 7) \\
&= \frac{a}{2}(4c(a-1) + 1) \\
&= \rho.
\end{aligned}$$

We must now fill in the remaining unlabeled layers. Fill in the t cells of the layer directly below the middle by adding $4(a-1)$ to the numbers in the t cells immediately above it, and fill in the b cells in this layer by subtracting $4(a-1)$ from the numbers in the b cells immediately above it. We iterate this process, working down to the bottom layer. We then start from the middle again and work up, adding $4(a-1)$ to the t cells in the layer immediately below and subtracting $4(a-1)$ from the number in the b cells in the layer immediately below. Iterate this process, working up.

Observe that in each layer, there is an equal number of b cells and t cells. Hence, the row sum differs from ρ by $4(a-1)+4(a-1)\cdots-4(a-1)\cdots-4(a-1)=0$, so each row sums to ρ. Now we check the column sum. We note that the pairwise column sum of the middle two layers is

$$\begin{aligned}&(8a-7)+2(2c(a-1)-4a+4)\\&=(8a-7)+4c(a-1)-8a+8\\&=4c(a-1)+1.\end{aligned}$$

Now consider the next layer up and the next layer down. The pairwise column sum of these two layers must also be $4c(a-1)+1$, since we add $4(a-1)$ to the t cells below and subtract them up top, and we add $4(a-1)$ to the b cells up top and subtract $4(a-1)$ from the b cells below. If we iterate this process of checking the pairwise column sums, we see that each pair has column sums of $4c(a-1)+1$. Therefore, the total column sum is $(4c(a-1)+1)\cdot(\#\text{ of pairs})=(4c(a-1)+1)\cdot\frac{c}{2}=\sigma$.

Since all rows sum to ρ and all columns sum to σ, the final step is to show is that each of the numbers from $1,\ldots,4c(a-1)$ appears exactly once. If we look at the completed structure, we see that the every entry is simply a value of $MH(4,a,2)$ plus a constant (where the constant is a multiple of $4(a-1)$). In the top and bottom layers, the b values take on every number in the set $\{1,2,\ldots,4(a-1)\}$. In the second from the top and second from the bottom layers, the b values take on the original b values (from $MH(4,a,2)$) plus a constant of $4(a-1)$, therefore taking on every number in the set $\{4a-3,4a-2,\ldots,8a-8\}$. The b values in the third from the top and third from the bottom layers are the original b values plus $2\cdot 4(a-1)$, taking on every number in the set $\{8a-7,8a-6,\ldots,12a-12\}$. Continuing with this reasoning, we look at the middle two layers. The b values of these two layers are the original b values plus $4(a-1)(\frac{c}{2}-1)=4(a-1)\frac{c}{2}-4(a-1)$, which comes out to every number in the set

$\{4(a-1)\frac{c}{2}-4(a-1)+1, 4(a-1)\frac{c}{2}-4(a-1)+2, \ldots, 4(a-1)\frac{c}{2}-4(a-1)+4(a-1) = 2c(a-1)\}$. Hence the b values cover every number in the set $\{1, 2, \ldots, 2c(a-1)\}$ exactly once.

Finally, look at the t values. The original t values are $4a-3, 4a-2, \ldots, 8a-8$, but by construction, the t values in the center two layers take on the original t values plus $4(a-1)(\frac{c}{2}-1) = 4(a-1)\frac{c}{2}-4(a-1)$, or every number in the set $\{2c(a-1)+1 = 4(a-1)\frac{c}{2}+1, 4(a-1)\frac{c}{2}+2, \ldots, 4(a-1)\frac{c}{2}+4(a-1)\}$. If we move up one layer and down one layer, the t values of these next two layers take on the original t values plus $4(a-1)\frac{c}{2}$, or every number in the set $\{4(a-1)\frac{c}{2}+4(a-1)+1, 4(a-1)\frac{c}{2}+4(a-1)+2, \ldots, 4(a-1)\frac{c}{2}+8a-8\}$. Following this reasoning, we see that the t values in the top and bottom layers take on the original t values plus a constant of $4(a-1)(c-2) = 4c(a-1)-8a-8$, or every number in the set $\{4c(a-1)-8a-8+4(a-1)+1 = 4c(a-1)-4a-3, 4c(a-1)-4a-2, \ldots, 4c(a-1)\}$. Hence, between the b's and the t's throughout the layers, we find every number from $1, 2, \ldots, 4c(a-1)$ exactly once in the structure. Hence our construction does indeed produce $MH(n, 4, c)$. $\square$

Chapter 5

Conclusion and Discussion

5.1 Magic Box Sets

In Chapter 2, we stated several open problems regarding magic box sets. We now recapitulate how the theorems in Chapter 3 have answered these problems.

Result 5.1. If a, b, c are even, and at least two of a, b, c equal 2, $MBS(a, b, c; d)$ does not exist.

This is a result of Theorem 3.1, and completely answers Problem 2.4.

Result 5.2. If a, b, c are even, exactly one of a, b, c equals 2, and $d = 2$, $MBS(a, b, c; d)$ exists.

This is a result of Theorem 3.2, and completely answers Problem 2.5.

Result 5.3. If a, b, c are even (where at most one equals 2) and d is odd, $MBS(a, b, c; d)$ exists.

This is a result of Theorem 3.20, and along with Result 5.1 completely answers Problem 2.6.

Result 5.4. If a, b, c are odd (in which case d must be odd), and $MB(a, b, c)$ exists, then $MBS(a, b, c; d)$ exists.

This requires a, b, and c to be greater than 1. We also know that $MB(a, b, c)$ exists if $\gcd(a, b) > 1$. This result is Theorem 3.18, and partially solves Problem 2.7. This leaves the following open problems.

Problem 5.5. Does $MBS(a, b, c; d)$ exist for any arbitrary quadruple of odd integers a, b, c, d (where a, b, c are greater than 1)?

It is an open question whether $MB(a, b, c)$ exists for any arbitrary triple of odd integers a, b, c (all greater than 1). When a, b, and c are odd and greater than 1, then the existence of $MB(a, b, c)$ implies the existence of $MBS(a, b, c; d)$. Hence, these two questions, while not equivalent, go hand in hand. It also leads us to our next problem.

Problem 5.6. Does there exist a quadruple of odd integers a, b, c, and d (all greater than 1) for which $MBS(a, b, c; d)$ exists, but $MB(a, b, c)$ does not exist?

Of course, this problem becomes trivial if $MB(a, b, c)$ exists for all triples of odd integers a, b, c, where a, b, and c are greater than 1.

5.2 Magic Hollow Boxes

We turn our attention to the subject of Chapter 4, the magic hollow box. We summarize our results here. Keep in mind that $n \geq 3$, otherwise our structure would not be 3-dimensional.

Result 5.7. If a is odd, $MH(n, a, c)$ does not exist.

This is the result of Theorem 4.9.

Result 5.8. If n is even and c is odd, $MH(n, a, c)$ does not exist.

This is the result of Theorem 4.10.

Result 5.9. $MH(n, 4, c)$ exists for all n and all even c.

This is the result of Theorem 4.7.

Result 5.10. $MH(4, a, c)$ exists for all even a and all even c.

This is the result of Theorem 4.15.

These results leave the following open problems.

Problem 5.11. If n is odd, a is even, and c is odd, does $MH(n, a, c)$ exist?

Problem 5.12. If a and c are even, for which n does $MH(n, a, c)$ exist?

I have not been successful at finding any examples to help formulate even a conjecture regarding Problem 5.11.

With regard to Problem 5.12, I have constructed $MH(3, 6, 2)$, $MH(6, 6, 2)$, $MH(8, 6, 2)$, and $MH(3, 8, 2)$ (refer to the figures in Appendix A). These objects all share the same very general structure (exactly half of the entries in each layer are b's and t's, as described in the proof of Theorem 4.15). They can be used to construct $MH(3, 6, c)$, $MH(6, 6, c)$, $MH(8, 6, c)$, and $MH(3, 8, c)$ for any even c by generalizing the construction in the proof of Theorem 4.15.

Because of these results, I make the following conjecture.

Conjecture 5.13. If a and c are even, then $MH(n, a, c)$ exists for all $n \geq 3$.

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Appendix A

More Magic Hollow Boxes

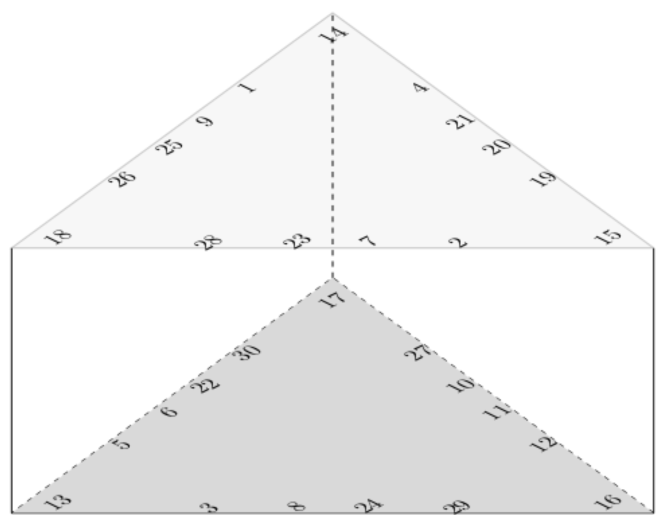

Figure A.1: $MH(3, 6, 2)$.

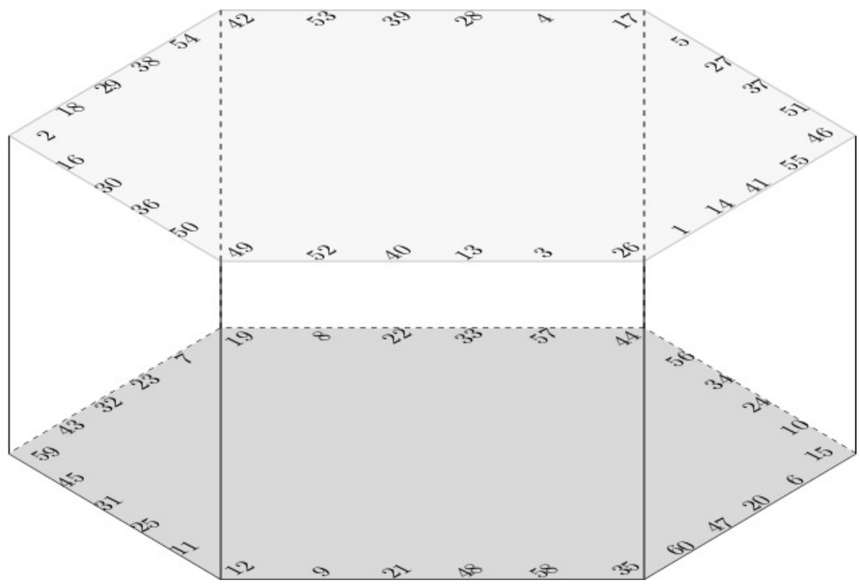

Figure A.2: $MH(6,6,2)$.

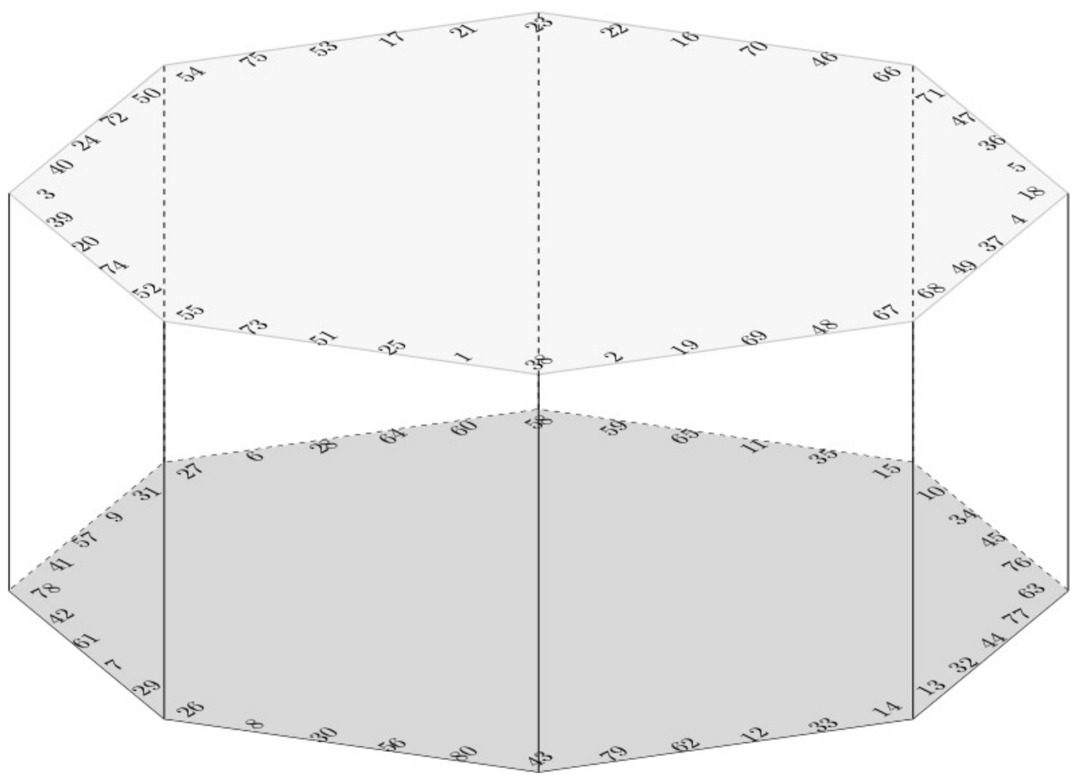

Figure A.3: $MH(8,6,2)$.

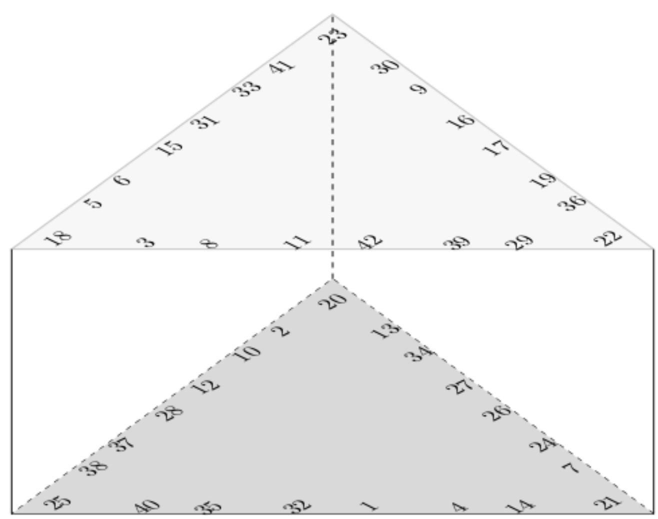

Figure A.4: $MH(3, 8, 2)$.