Comparative Analysis of Markup and Markdown Pricing Policies in Revenue Management Problems

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Dedication

This is dedicated to my parents.

ABSTRACT

It is a common practice for many industries to bolster sales and revenue through dynamic pricing strategies. For instance, markup pricing is especially prevalent in the commercial airline industry. Meanwhile, for products sold on a seasonal basis, many stores generally decrease their prices toward the end of the season. In this thesis, we compare the expected revenue that can be generated from two representative pricing policies: markup only and markdown only, in a continuous-time single product revenue management model over a finite horizon. In our model, the initial inventory is fixed and there is no replenishment opportunity during the selling period. The demand follows a homogeneous Poisson process whose rate is controlled by price only. The decisions are to select a price from a set of predetermined prices at each time during the selling period. We show that the markdown policy is superior to the markup policy when the inventory level is low and the remaining time is ample; when the remaining time is short, the markup policy is superior. Our findings complement the previous studies in each of these policies and provide valuable insights for practitioners to choose among different pricing strategies. We verify our findings through numerical tests.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Dynamic pricing has become an effective means for managing revenue since its original implementation in the airline industry, where analysis has revealed that flexible pricing systems could increase revenue by as much as 2% (a proportion which can prove crucial to the profitability of a major airline), see Feldman [3]. Inspired by the successful application in the airline industry, researchers have since pursued mathematical methods of increasing sophistication to analyze and develop dynamic pricing strategies. Consequently, studies on this subject have garnered considerable attention within the operations research community and grown rapidly over the last twenty years (Talluri and van Ryzin [10]).

Among various forms of dynamic pricing strategies, there are two primary types of pricing strategies according to the direction of the price path: the markup pricing strategy, and the markdown pricing strategy. As the names imply, in the markup pricing strategy, the price is only allowed to move upwards over time while in the markdown pricing strategy, the price is only allowed to move downwards. Both strategies are seen in practice. For example, fashion good retailers typically adopt a markdown policy that lowers the prices of their products when the sales season is coming to a close and there is still much inventory left. In contrast, airline companies typically choose a markup policy in which the ticket price is adjusted upward as more and more seats are reserved.

The need to understand the performance of different pricing policies has long been recognized. Gallego and van Ryzin [8] consider the problem of dynamically pricing a given stock of items over a finite horizon when the uncertain demand is only price-sensitive. They establish a continuous-time dynamic control model to study this problem. An essential result in their work is that for general demand functions, setting the price fixed at the level determined by the deterministic solution of the problem throughout the entire horizon is asymptotically optimal as the expected sales tend to infinity. Specifically, Gallego and van Ryzin [8] report in their numerical experiments that the worst relative performance of the optimal fixed-price policy is only 5.5% below the optimal revenue, and is nearly optimal when the initial inventory is large. The drawbacks of the fixed-price policy, nevertheless, emerge evidently either when items in stock are relatively few or when there is insufficient time to sell the items. To address these cases, Sen [9] proposes two simple dynamic heuristics that continuously update prices based on the inventory and time left to compensate the loss caused by the fixed-price policy. Through numerical study, Sen demonstrates that these dynamic heuristics are able to accomplish near-optimal performance. Particularly, one of the heuristics can lead to a maximum of 0.2% optimality gap in all single-product problems.

However, continuous price changes are impractical in real-world application. To make the model more realistic, Gallego and van Ryzin [8] consider the case where the allowable price set is a discrete set and give a heuristic solution in that case. The resulting heuristic is no longer a fixed-price heuristic, but consists of two prices: one price is allocated a certain number of stocks and a certain amount of time, and once either the allocated stock or the allocated time is running out, the retailer should switch to the other price. They show that the ratio of the expected revenue achieved by this heuristic to the expected revenue of the policy that allows an arbitrary number of price changes converges to one as the remaining inventory and/or the remaining time go to infinity. Nonetheless, it is still a heuristic and similar to the fixed-price heuristic, the performance deteriorates when the remaining inventory and remaining time are both small. Moreover, the order that the two prices are used is arbitrary, which is not realistic in practice.

One milestone in studying the dynamic pricing problem with a fixed number of prices is Feng and Gallego [4]. In this seminal work, they consider the optimal markup and markdown pricing strategies when there are two predetermined prices in the allowable price set. The goal is to determine the optimal time to switch between the two prices in a finite sales season. In particular, they formulate this problem as a stochastic control problem, and for both markup and markdown cases, they derive an exact optimal solution featuring a sequence of time thresholds. Feng and Xiao [5] extend the results of [4] by incorporating a risk factor and analyze the effect of the risk attitude on the optimal markup and markdown policies in the two-price case. In a later work, the same authors [6] generalize the recursive algorithm from the two-price case to the K-price case.

Despite notable interest in studying optimal control decision for both markup and markdown strategies, to our best knowledge, few studies have been done on comparing the expected revenue between the markup and markdown pricing policies. This thesis thereupon seeks to provide insight on this issue. Specifically, we consider a single-product revenue management model with a finite set of predetermined prices. Demand for the product is assumed to obey a Poisson process with an intensity that is associated with the price. We adopt the optimal solutions derived by Feng and Xiao [6] for the markup and markdown pricing policies. Our objective is to determine which policy (markup or markdown) yields higher revenue given a specific set of input parameters.

To illustrate our problem, consider a retailer with 9 products to sell in a certain period T. We assume unsold items have no salvage value. Based on his prior experience, the retailer sets two allowable prices, $p_1 = 200$ and $p_2 = 300$. The corresponding demand follows a Poisson process with rate $\lambda_1 = 1$ and $\lambda_2 = 0.5$, respectively. Now, he has two choices: (1) To use a markup strategy, i.e., he starts selling at p_1, and if the sales go well in the beginning, he may switch to p_2 at some point later in the selling season; (2) To use a markdown strategy, i.e., he initially offers at p_2, with the option of decreasing the price to p_1 if the sales are slow.

We use R_1 and R_2 to denote the expected revenue generated by the optimal markup policy and the optimal markdown policy, respectively. Using the methods in Feng and Xiao [6], we are able to compute and compare R_1 and R_2 for each different T. The result is shown in Figure 1.1. Taking two examples from Figure 1.1: if $T = 9$, then we find $R_1 = 1598$, $R_2 = 1572$, i.e., the markup policy outperforms the markdown policy by 1.7%; and if $T = 16$, then we find $R_1 = 2190$, $R_2 = 2258$, i.e., the markdown policy outperforms the markup policy by 3.1%. Therefore, the choice of policy has a significant impact on the revenue, and the impact does not seem to have a trivial explanation. Thus, we are motivated to shed light on this impact and find out the conditions under which each policy achieves a higher revenue.

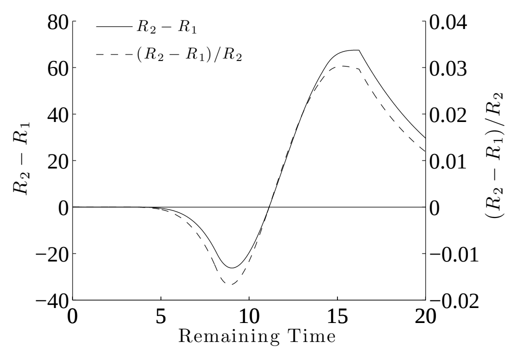


Figure 1.1: An example of the comparison between the markup and markdown policy

Our analysis is based on the results of Feng and Gallego [4] and Feng and Xiao [6]. We make use of the structures of the optimal solutions obtained in their papers. For the two-price case, we show that when the inventory is low and the remaining time is sufficiently long, the markdown policy results in a higher revenue than the markup policy. This is because in this situation, immediately increasing the price is optimal for the markup policy. However, pricing at the higher price in a two-price case reduces the markup policy to a fixed-price policy.

Meanwhile, the markdown policy begins at the higher price but is not limited to a fixed-price policy because it can decrease the price if the sales become slow. This added flexibility allows the markdown policy to generate at least as much revenue as the markup policy. In other situations, the markup policy is preferable. For the K-price case, we are unable to obtain the analytical results. However, numerical examples suggest that similar results hold. Namely, the markdown policy is superior for lower inventory and longer remaining time while inferior for shorter remaining time.

The remainder of the thesis is organized as follows: Chapter 2 reviews the prior literature on revenue management model that deals with multiple price changes and the optimal solutions for the markup and markdown pricing policies established by Feng and Xiao [6]. In Chapter 3, we focus on the comparison between the markup and markdown policies in a two-price model, and a pattern representing the gap in the expected revenue between the two policies is presented. Chapter 4 further verifies the derived pattern with numerical examples and extends our insights to the K-price model. Concluding remarks and future research topics are presented in Chapter 5.

Chapter 2

Model Description and Optimal Solutions

In this chapter, we describe the revenue management model under consideration and review the established results in the literature. We consider a finite horizon continuous-time single-product revenue management model. Suppose that (1) a set of $K(\geq 2)$ prices are predetermined; (2) demand follows a Poisson process whose rate only depends on the price; (3) the initial inventory is fixed and cannot be replenished during the selling season. Without loss of generality, we assume that there is no salvage value for the remaining inventory.

We denote the length of the sales horizon by T and the initial inventory by M. In particular, we index the time forward, i.e., we start from $t = 0$ and the sales season ends at $t = T$. Let $\mathcal{P} = \{p_1, \cdots, p_K\}$ be the set of predetermined prices and we suppose $p_1 < p_2 < \cdots < p_K$. The prices are presumably selected by taking into account business constraints such as competitor's prices. The Poisson demand process at $p_i \in \mathcal{P}$ has a constant intensity $\lambda_i = \lambda(p_i)$. Naturally, λ is a strictly decreasing function, i.e, $\lambda_i < \lambda_j$ whenever $p_i > p_j$. Notice that $\lambda_i p_i$ represents the expected revenue rate. We also assume that $\lambda_i p_i < \lambda_j p_j$ whenever $p_i > p_j$. This is because there is no need to offer a lower price with a smaller expected revenue rate under any circumstance.

2.1 Problem Formulation

To maximize the expected revenue, the prices have to be adjusted based on the remaining time and inventory. Let

$$S_i(t) = \begin{cases} 1, & \text{if } p_i \text{ is effective at } t, \\ 0, & \text{otherwise} \end{cases}$$

and a non-anticipating policy u to be

$$u = \{(S_1(t), \dots, S_K(t)) : 0 \le t \le T\}.$$

We denote the class of all non-anticipating pricing policies by $\mathcal{U}$. By the reason that only one price is active at a time and the total sold items cannot exceed the initial inventory, we impose

$$\sum_{i=1}^{K} S_i(t) \le 1, \quad 0 \le t \le T$$

and

$$\sum_{i=1}^{K} \int_0^T S_i(s) dN_i(s) \le M \quad (a.s.).$$

Here $N_i(s)$ is the total demand up to s at p_i, which follows a Poisson process with rate λ_i. We use $J_u(t, n)$ to denote the expected revenue over $[t, T]$ if there are n unsold items at time t, i.e.,

$$J_u(t, n) = \mathbb{E}\left[\sum_{i=1}^{K} \int_t^T S_i(s) p_i 1_{\{n(s^-)>0\}} dN_i(s)\right], \tag{2.1}$$

where $n(s^-)$ is the number of remaining items immediately before time s. The value function $V(t, n)$ is hereby defined as the optimal value of (2.1) over all allowable $u \in \mathfrak{U}$, i.e.,

$$V(t,n) = \sup_{u \in \mathfrak{U}} J_u(t,n).$$

Clearly, $V(t,0) = 0$ for $0 \le t \le T$ and $V(T,n) = 0$ for $n \ge 0$ since no revenue can be generated if all the items are sold or the time is exhausted. Some basic properties of the value function are shown below:

Proposition 1 *$V(t,n)$ is strictly decreasing in t, and strictly increasing in n.*

This proposition reveals that retailers will secure higher expected revenue by having more time to sell and/or by holding more inventory.

Proposition 2 *$V(t,n)$ is strictly concave in t and has decreasing difference in n, i.e.,*

$$\frac{\partial^2 V(t,n)}{\partial t^2} < 0 \quad \textit{and} \quad V(t,n+1) - V(t,n) < V(t,n) - V(t,n-1), \quad \forall t,n.$$

The proof of Proposition 1 and 2 can be found in Feng and Xiao [6].

2.2 Sufficient Optimality Conditions

From stochastic control theory (Bremaud [1]), a sufficient condition for $V(t,n)$ to be the optimal value of $J_u(t,n)$ is that it satisfies the Hamilton-Jacobi equation, namely,

$$\frac{\partial V(t,n)}{\partial t} + \max_{i=1,\ldots,K} \{\lambda_i[V(t,n-1) - V(t,n) + p_i]\} = 0 \tag{2.2}$$

with boundary conditions $V(t,0) = 0$, $0 \le t \le T$ and $V(T,n) = 0$, $n \ge 0$.

To interpret Condition (2.2), we rearrange the terms and suppose at time t, the maximum is attained at p_i. As a result, we obtain

$$\lambda_i p_i = -\left[\frac{\partial V(t,n)}{\partial t} + \lambda_i(V(t,n-1) - V(t,n))\right]. \tag{2.3}$$

The right side of (2.3) delineates the marginal loss in revenue. The loss consists of two parts: one because of the elapse of time, measured by $\partial V(t,n)/\partial t$, and the other one owing to the reduction in inventory, represented by $\lambda_i(V(t,n-1) - V(t,n))$. On the other hand, the left side of (2.3) represents the expected revenue at the rate of $\lambda_i p_i$. In other words, for $V(t,n)$ to be optimal, the marginal revenue should be exactly equal to the marginal loss.

In the next two sections, we introduce the optimal markup and markdown pricing policies derived by Feng and Xiao [6].

2.3 Optimal Markup Solutions

In the markup case, let $V_k^1(t,n)$ represent the maximum expected revenue generated over $[t,T]$ given p_k is effective at t with n unsold items. The key idea of the optimal markup policy derived in Feng and Xiao [6] is to compare $V_k^1(t,n)$ and $V_{k+1}^1(t,n)$.

First of all, $V_k^1(t,n) \geq V_{k+1}^1(t,n)$ holds since the decision maker can always choose to stay at the lower price p_k and not to markup p_{k+1}. When $V_k^1(t,n) > V_{k+1}^1(t,n)$, it means there is a premium gained by staying at p_k. Therefore, the decision maker should not markup at this moment; when $V_k^1(t,n) = V_{k+1}^1(t,n)$, the premium has reached zero thus the decision maker should switch to p_{k+1}. Indeed, it implies that staying at the lower price p_k will lose revenue in this case. Let

$$V_k^1(t,n) = V_{k+1}^1(t,n) + \bar{V}_k^1(t,n), \tag{2.4}$$

$\bar{V}_k^1(t,n)$ then represents the premium in excess of $V_{k+1}^1(t,n)$ due to the delay of switch. (2.4) reflects that the optimal expected revenue at a given price contains two parts: a revenue that can be secured by switching immediately and a premium resulting from the delay.

As shown in Figure 2.1, for a given inventory level n, there exists a time

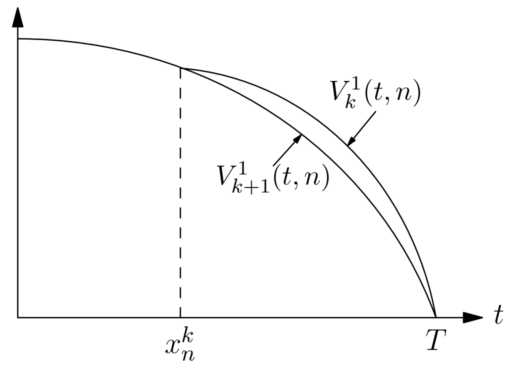


Figure 2.1: $V_k^1(t,n)$ and $V_{k+1}^1(t,n)$ for a given n in the markup case.

threshold x_n^k that partitions the sales horizon into two subintervals. If $t < x_n^k$, $V_k^1(t,n) = V_{k+1}^1(t,n)$, i.e., there is no premium in staying at the low price, therefore one should immediately switch. On the other hand, if $x_n^k \leq t \leq T$, there is a positive premium by keeping at the low price, which implies one should postpone the switch. Therefore, this x_n^k represents the latest switching time from p_k to p_{k+1} when there are n unsold items.

One property of x_n^k is its *monotonicity*, i.e. $x_1^k \geq x_2^k \geq \cdots \geq x_M^k$. Monotonicity cannot only be proved theoretically (see Feng and Xiao [6]), but is also consistent with business practice.

Feng and Xiao [6] establish a recursive algorithm to compute $\bar{V}_k^1(t,n)$ and x_n^k.

Theorem 3 (Theorem 2 in [6]) (Markup) *For* $1 \leq n \leq M$, $\bar{V}_k^1(t,n)$ *can be determined recursively by*

$$\bar{V}_k^1(t,n) = \begin{cases} \displaystyle\int_t^T L_k^1(s,n) e^{-\lambda_k(s-t)} ds, & \text{if } t \geq x_n^k, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$x_n^k = \inf\left\{0 \le t \le T : \int_t^T L_k^1(s,n)e^{-\lambda_k(s-t)}ds > 0\right\},$$

$$L_k^1(t,n) = (\lambda_{k+1} - \lambda_k)(V_{k+1}^1(t,n) - V_{k+1}^1(t,n-1)) + r_k - r_{k+1} + \lambda_k \bar{V}_k^1(t,n-1).$$

The proof of Theorem 3 (see [6]) shows that $L_k^1(t,n)$ changes its sign from negative to positive at most once, and so does the integral $\int_t^T L_k^1(s,n)e^{-\lambda_k(s-t)}ds$. Thus, x_n^k is uniquely determined.

Additionally, the fact $L_k^1(T,n) = r_k - r_{k+1} > 0$, along with the condition $\int_{x_n^k}^T L_k^1(s,n)e^{-\lambda_k(s-t)}ds = 0$ entails $L_k^1(x_n^k, n) < 0$. It is trivial to see $\bar{V}_k^1(t,n)$ satisfies

$$\frac{\partial \bar{V}_k^1(t,n)}{\partial t} = \lambda_k \bar{V}_k^1(t,n) - L_k^1(t,n)$$

on $t \ge x_n^k$. Thus, $\partial \bar{V}_k^1(x_n^k, n)/\partial t = 0 - L_k^1(x_n^k, n) > 0$, i.e., $\partial V_k^1(t,n)/\partial t|_{t \to x_n^k-} < \partial V_k^1(t,n)/\partial t|_{t \to x_n^k+}$, $V_k^1(t,n)$ is differentiable at all t except $t = x_n^k$.

Another important fact about $V_k^1(t,n)$ is presented below, which implies when $\bar{V}_k^1(t,n) > 0$, the marginal loss offsets the marginal revenue; when $\bar{V}_k^1(t,n) = 0$, the net marginal gain at state (t,n) is negative.

Theorem 4 (Theorem 1 in [6]) *When* $t \ge x_n^k$*,*

$$\frac{\partial V_k^1(t,n)}{\partial t} = \lambda_k(V_k^1(t,n) - V_k^1(t,n-1)) - \lambda_k p_k. \tag{2.5}$$

When $t < x_n^k$*,*

$$\frac{\partial V_k^1(t,n)}{\partial t} \le \lambda_k(V_k^1(t,n) - V_k^1(t,n-1)) - \lambda_k p_k.$$

2.4 Optimal Markdown Solutions

In the markdown case, let $V_k^2(t,n)$ denote the optimal expected revenue generated over $[t,T]$ given p_k is effective at t with n remaining items. The active price is offered sequentially from p_K to p_1, hence $V^2_{k+1}(t,n)$ is constructed based on $V^2_k(t,n)$. We herein denote the recursive formula to be $V^2_{k+1}(t,n) = V^2_k(t,n) + \bar{V}^2_k(t,n)$, and the computation of $\bar{V}^2_k(t,n)$ is accordingly adapted as shown in the following theorem.

Theorem 5 (Theorem 3 in [6]) (Markdown) *For* $1 \leq n \leq M$, $\bar{V}^2_k(t,n)$ *can be determined recursively by*

$$\bar{V}^2_k(t,n) = \begin{cases} \int_t^{y_n^k} L^2_k(s,n)e^{-\lambda_{k+1}(s-t)}ds, & \text{if } t \leq y_n^k, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$y_n^k = \inf\left\{0 \leq t \leq T : L^2_k(t,n) < 0\right\},$$
$$L^2_k(t,n) = (\lambda_k - \lambda_{k+1})(V^2_k(t,n) - V^2_k(t,n-1)) + r_{k+1} - r_k + \lambda_{k+1}\bar{V}^2_k(t,n-1).$$

Similarly, as shown in Figure 2.2, for a given inventory level n, there exists a time threshold y_n^k that divides the sales season into two parts. If $t < y_n^k$, $V^2_{k+1}(t,n) > V^2_k(t,n)$, due to a positive premium, the retailers delay the switch until y_n^k. If $y_n^k \leq t \leq T$, $V^2_{k+1}(t,n) = V^2_k(t,n)$. It means that the retailers should switch the price to p_k as the premium has dropped to zero. Thus, the role of y_n^k is the optimal switch time from p_{k+1} to p_k given n unsold items. In addition, y_n^k also possesses monotonicity, i.e., $y_1^k \geq y_2^k \geq \cdots \geq y_M^k$.

Likewise, $\bar{V}^2_k(t,n)$ satisfies

$$\frac{\partial \bar{V}^2_k(t,n)}{\partial t} = \lambda_{k+1}\bar{V}^2_k(t,n) - L^2_k(t,n)$$

on $t \leq y_n^k$. Apparently, $\partial \bar{V}^2_k(y_n^k,n)/\partial t = 0$ because both $L^2_k(y_n^k,n) = 0$ and $\bar{V}^2_k(y_n^k,n) = 0$, which manifests that, unlike $V^1_k(t,n)$, $V^2_k(t,n)$ is continuously differentiable on $[0,T]$.

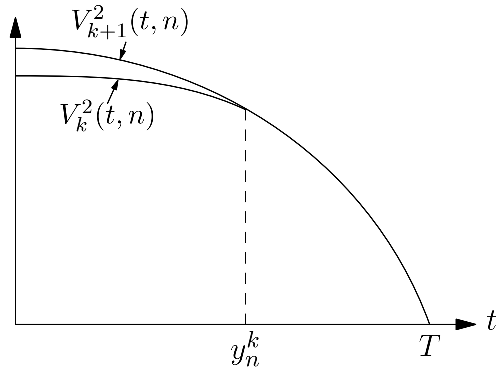


Figure 2.2: $V_k^2(t,n)$ and $V_{k+1}^2(t,n)$ for a given n in the markdown case.

In addition, because $L_k^2(t,n)$ is a decreasing function of t, having been proved by [6], it follows that

$$\begin{aligned}\frac{\partial \bar{V}_k^2(t,n)}{\partial t} &= \lambda_{k+1}\int_t^{y_n^k} L_k^2(s,n)e^{-\lambda_{k+1}(s-t)}ds - L_k^2(t,n)\\ &< \lambda_{k+1}L_k^2(t,n)\int_t^{y_n^k} e^{-\lambda_{k+1}(s-t)}ds - L_k^2(t,n)\\ &= -L_k^2(t,n)e^{-\lambda_{k+1}(y_n^k-t)}\\ &< 0\end{aligned}$$

when $t \le y_n^k$. Therefore, $\bar{V}_k^2(t,n)$ is monotonically decreasing on $t \le y_n^k$.

The counterpart of Theorem 4 for the markdown case is stated below.

Theorem 6 (Theorem 1 in [6]) *When* $t \le y_n^k$,

$$\frac{\partial V_{k+1}^2(t,n)}{\partial t} = \lambda_{k+1}(V_{k+1}^2(t,n) - V_{k+1}^2(t,n-1)) - \lambda_{k+1}p_{k+1}.$$

When $t > y_n^k$,

$$\frac{\partial V_{k+1}^2(t,n)}{\partial t} \le \lambda_{k+1}(V_{k+1}^2(t,n) - V_{k+1}^2(t,n-1)) - \lambda_{k+1}p_{k+1}.$$

The literature has given the optimal solutions for the markup and markdown pricing policies. In the next chapter, we compare the expected revenue of the optimal solution achieved by each pricing policy and a stable pattern of the revenue difference is identified for a two-price model.

Chapter 3

Comparative Analysis of Pricing Policies

Although the optimal strategies for the markup and markdown pricing policies have been previously studied, there is a higher level question that has not been answered: given a scenario at hand (e.g. fixed inventory, a certain selling period, and certain price choices) wherein the price path is constrained to be monotonic, which policy, markup or markdown, should the seller choose to employ?

From the example shown in Chapter 1, we find that for a given initial inventory, there can be a significant difference in revenue between the two optimal pricing policies at a certain time; neither policy consistently generates more revenue than the other. It is clear that the expected revenue is determined by both the length of remaining time and the amount of unsold items. In this chapter, we address the question of how the revenue difference between the optimal markup and markdown policies is affected by these two factors for a set of predetermined prices. It should be noted that the price set in our problem are assumed to be fixed and identical for both the markup and markdown strategies. We do not address the problem in which the prices themselves are allowed to be different in different strategies.

Using the notation we specify in Chapter 2, for $p_1 < p_2 < \cdots < p_K$, starting at time t and n items to sell, the expected revenue for the optimal markup and markdown strategies are $V_1^1(t,n)$ and $V_K^2(t,n)$, respectively. We further define the difference between the expected revenue of the optimal markdown policy and that of the optimal markup policy by

$$\delta V(t,n) := V_K^2(t,n) - V_1^1(t,n).$$

Our goal is to determine the conditions on t and n such that $\delta V(t,n)$ is positive or negative. We concentrate on the qualitative analysis of the variation of $\delta V(t,n)$ as a function of t and n, instead of developing an analytical form of the conditions.

3.1 Preliminary Analysis

Before we present our conclusion, we first introduce some preliminary results which are useful in our study.

Define

$$\Delta V_k^i(t,n) = V_k^i(t,n) - V_k^s(t,n), \quad i = 1,2$$

where $V_k^s(t,n)$ denotes the expected revenue generated by a fixed-price policy pricing at p_k, i.e.,

$$V_k^s(t,n) = p_k \mathbb{E} \min(n, N_k(T) - N_k(t)).$$

Obviously, with greater flexibility, either the markup or markdown policy is superior to the fixed-price policy, hence $\Delta V_k^i(t,n) \geq 0$ for $i = 1,2$. Furthermore, we have the following lemmas.

Lemma 7 *In the markup case, $\Delta V_k^1(t,n)$ is decreasing in t and n, i.e.,*

$$\Delta V_k^1(t,n) - \Delta V_k^1(t,n-1) \leq 0 \quad \textit{and} \quad \frac{\partial \Delta V_k^1(t,n)}{\partial t} \leq 0$$

when $t \geq x_n^k$ for $n = 1,2,\cdots,M$, and "=" holds if and only if $t \geq x_1^k$.

Proof See Appendix.

Lemma 8 *In the markdown case, $\Delta V_{k+1}^2(t,n)$ is increasing in t and n, i.e.,*

$$\Delta V_{k+1}^2(t,n) - \Delta V_{k+1}^2(t,n-1) > 0 \quad and \quad \frac{\partial \Delta V_{k+1}^2(t,n)}{\partial t} > 0$$

when $t \le y_n^k$ for $n = 1, 2, \cdots, M$.

Proof The proof of this Lemma is similar to that of Lemma 7.

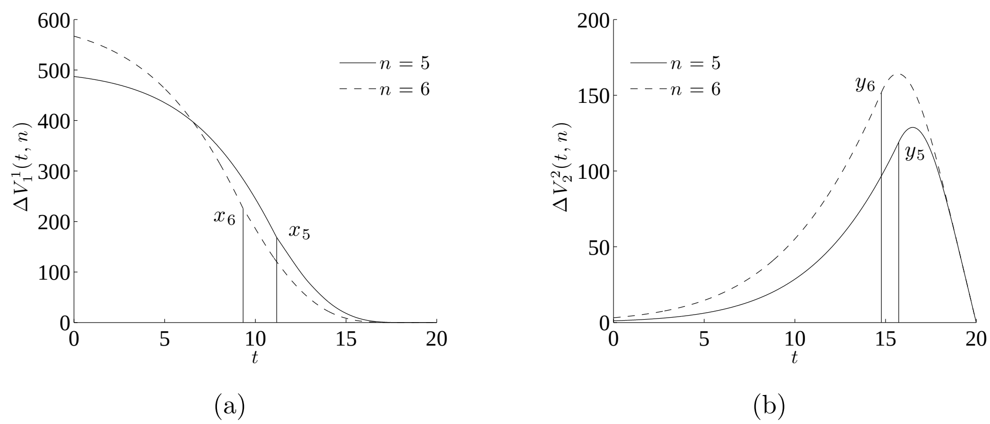


Figure 3.1: Validation of (a) Lemma 7 and (b) Lemma 8.

We can confirm our preliminary results using the numerical example from Chapter 1. For Lemma 7, Figure 3.1(a) shows that beyond the time threshold x_n, both curves are decreasing and convex in t. In addition, $\Delta V_1^1(t,n)$ is inversely related to the inventory size in this domain. For Lemma 8, Figure 3.1(b) shows that prior to the time threshold y_n, both curves are increasing and convex in t. Moreover, $\Delta V_2^2(t,n)$ is an increasing function of inventory size in this domain.

To intuitively understand the lemmas above, we note that the more items the retailer has to sell or the less time left in the sales horizon, the less likely the retailer will markup in the remaining time. Therefore, the difference of using a markup strategy and a fixed-price strategy is smaller. Likewise, with more items and less time remaining, the retailer is more likely to markdown before any item is actually sold. Hence, the difference of using a markdown strategy and a fixed-price strategy is larger.

Next, we have the following theorem about the relationship of the time thresholds between the two cases.

Theorem 9 *For* $n = 1, 2, \cdots$, $x_n^k < y_n^k$, $k = 1, 2, \cdots, K-1$.

Proof See Appendix.

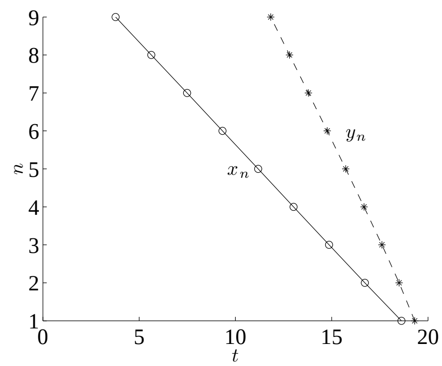


Figure 3.2: Validation of Theorem 9.

We validate Theorem 9 using the same example. Recall that given n items left, x_n^k is the latest time one should switch from p_k to p_{k+1} and y_n^k is the optimal switch time from p_{k+1} to p_k. Theorem 9 hence displays that with the same on-hand inventory, the markup decision from p_k to p_{k+1} should always be made earlier than the markdown decision from p_{k+1} to p_k. It implies that one should be conservative when making a switch decision since the switch is irreversible.

During the proof of Theorem 9, we obtain an additional result about $\Delta V_k^i(t,n)$, $i = 1, 2$: the higher the price is, the smaller the difference between an optimal markup policy and a fixed-price policy is; for the markdown case, the situation is reversed.

Corollary 10 *In the markup case,* $\Delta V_k^1(t,n) > \Delta V_{k+1}^1(t,n)$*; in the markdown case,* $\Delta V_k^2(t,n) < \Delta V_{k+1}^2(t,n)$, $k = 1, 2, \cdots, K-1$.

In the markup case, the higher the current price is, the less likely it is that the retailer will perform subsequent price markup(s) in the remaining time. Therefore, the difference between using a markup strategy and a fixed-price strategy becomes smaller as the current price increases. An extreme case is when the current price reaches the highest price and the markup policy becomes equivalent to a fixed-price policy. A similar line of reasoning holds for the markdown case.

3.2 Markdown versus Markup

In this section, we study the property of $\delta V(t,n)$. For the ease of analysis, we illustrate our work with a two-price model. Let $p_1 < p_2$, $\lambda_1 > \lambda_2$, and we omit the superscript 1 on x_n and y_n for simplicity.

In view of $x_n < y_n$ from Theorem 9, $\delta V(t,n)$ can be expressed piecewisely by

$$\delta V(t,n) = \begin{cases} V_2^2(t,n) - V_2^s(t,n) = \Delta V_2^2(t,n), & 0 \le t \le x_n, \\ V_2^2(t,n) - V_1^1(t,n), & x_n < t < y_n, \\ V_1^s(t,n) - V_1^1(t,n) = -\Delta V_1^1(t,n), & y_n \le t \le T. \end{cases}$$

From Lemma 7 and Lemma 8, we obtain that $\Delta V_1^1(t,n)$ is decreasing in t and n in $(y_n, T]$, while $\Delta V_2^2(t,n)$ is increasing in t and n in $[0, x_n)$. We will prove in Theorem 11 that $\delta V(t,n)$ only transitions from positive to negative once on $t \in [x_n, y_n]$, denoted by w_n. Moreover, since $\partial\delta V(w_n,n)/\partial t < 0$ and $\partial\delta V(y_n,n)/\partial t = -\partial\Delta V_1^1(y_n,n)/\partial t > 0$, the continuity of $\partial\delta V(t,n)/\partial t$ in $[w_n, y_n]$ secures that there exists a time, denoted by u_n, at which $\partial\delta V(u_n,n)/\partial t = 0$. It can be proved that $\delta V(u_n,n)$ is the only local minimum in $[w_n, y_n]$.

Overall, the full pattern of $\delta V(t,n)$ in a two-price case is described as below. We can conclude that: (1) When the inventory level is low and the remaining time is sufficient, the markdown policy yields more revenue than the markup policy; otherwise, the markup policy is superior. (2) The maximum benefit of the markdown policy occurs no earlier than x_n while the maximum benefit of the markup

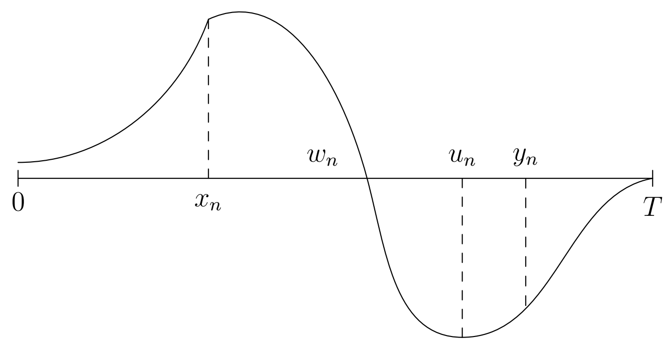


Figure 3.3: Pattern of $\delta V(t,n)$ for a given n in the two-price case before x_n shifts to zero.

policy occurs before y_n. An intuitive explanation is that when the inventory level is low and the remaining time is sufficiently large, a markup policy will immediately choose to increase the price, thus losing flexibility since no subsequent price changes can be made thereafter. A similar explanation applies for the other part of the conclusion.

Theorem 11 *In the two-price case, when $x_n > 0$,*
(1) $\delta V(t,n)$ transitions from positive to negative on $t \in [x_n, y_n]$ only once, denoted by w_n, i.e., $\delta V(t,n) > 0$ on $[x_n, w_n)$ and $\delta V(t,n) < 0$ on $(w_n, y_n]$.
(2) The sequence of $\{w_n : n = 1, 2, \cdots\}$ is strictly decreasing in n.

Proof Taking the derivative of $\delta V(t,n)$ in $[x_n, y_n]$, we have

$$\begin{aligned}
\frac{\partial \delta V(t,n)}{\partial t} &= \frac{\partial V_2^2(t,n)}{\partial t} - \frac{\partial V_1^1(t,n)}{\partial t} \\
&= \lambda_2(V_2^2(t,n) - V_2^2(t,n-1) - p_2) - \lambda_1(V_1^1(t,n) - V_1^1(t,n-1) - p_1) \\
&= \lambda_1 \delta V(t,n) - \lambda_1 \delta V(t,n-1) - (\lambda_1 - \lambda_2)(V_2^2(t,n) - V_2^2(t,n-1)) - r_2 + r_1.
\end{aligned}$$

Let $H(t,n) = G(t,n) + \lambda_1 \delta V(t,n-1)$, where $G(t,n) = (\lambda_1 - \lambda_2)(V_2^2(t,n) - V_2^2(t,n-1)) + r_2 - r_1$. Thus,

$$\frac{\partial \delta V(t,n)}{\partial t} = \lambda_1 \delta V(t,n) - H(t,n)$$

with boundary conditions $\delta V(x_n, n) = \Delta V_2^2(x_n, n)$ and $\delta V(y_n, n) = -\Delta V_1^1(y_n, n)$. Thus, the existence of w_n follows from $\delta V(x_n, n) > 0$ and $\delta V(y_n, n) \leq 0$.

When $n = 1$, we find $\delta V(t, 1) = 0$ has only one root on $[x_1, y_1]$. By induction, we assume $\delta V(t, n) = 0$ has only one root w_n on $[x_n, y_n]$. Based on this assumption, we are able to prove that $H(t, n+1) = 0$ also has unique root and it falls prior to w_n (see Appendix).

Suppose there are more than one root for $\delta V(t, n+1) = 0$. Without loss of generality, we assume w_{n+1}, $\hat{w}_{n+1}$ and $\tilde{w}_{n+1}$ are three roots satisfying $\delta V(t, n+1) = 0$ on $[0, T)$ where $w_{n+1} < \hat{w}_{n+1} < \tilde{w}_{n+1}$. Accordingly,

$$\frac{\partial \delta V(w_{n+1}, n+1)}{\partial t} < 0, \quad \frac{\partial \delta V(\hat{w}_{n+1}, n+1)}{\partial t} > 0, \quad \frac{\partial \delta V(\tilde{w}_{n+1}, n+1)}{\partial t} < 0.$$

As a consequence,

$$H(w_{n+1}, n+1) > 0, \;\; H(\hat{w}_{n+1}, n+1) < 0, \;\; H(\tilde{w}_{n+1}, n+1) > 0.$$

On the other hand,

$$\frac{\partial \delta V(y_{n+1}, n+1)}{\partial t} = -\frac{\partial \Delta V_1^1(y_{n+1}, n+1)}{\partial t} > 0$$

results in $H(y_{n+1}, n+1) = \lambda_1 \delta V(y_{n+1}, n+1) - \partial \delta V(y_{n+1}, n+1)/\partial t < 0$.

Based on the continuity of $H(t, n+1)$, there are at least one root of $H(t, n+1) = 0$ in $(w_{n+1}, \hat{w}_{n+1})$, one root in $(\hat{w}_{n+1}, \tilde{w}_{n+1})$ and one root in $(\tilde{w}_{n+1}, y_{n+1})$, which contradicts the fact that $H(t, n+1) = 0$ has only one root.

Therefore, w_{n+1} is the unique root of $\delta V(t, n+1) = 0$ on $[0, T)$. The root of $H(t, n+1) = 0$, denoted by h_{n+1}, drops in (w_{n+1}, y_{n+1}). Thus, $w_{n+1} < h_{n+1} < w_n$. Therefore, $\{w_n : n = 1, 2, \cdots\}$ is a strictly descent sequence in t. ∎

In the next chapter, we will show that although the pattern for a K-price might be complicated in some cases, similar results hold.

Chapter 4

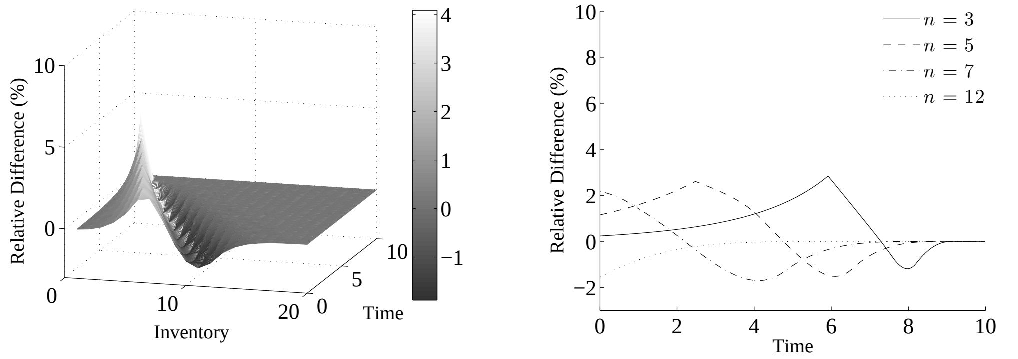
Numerical Results

In this chapter, we conduct several numerical experiments to verify the qualitative results proved in the previous chapter. Additionally, further studies are made to examine the revenue gap between the markup and markdown policies in a multi-price case.

4.1 Two-Price Model

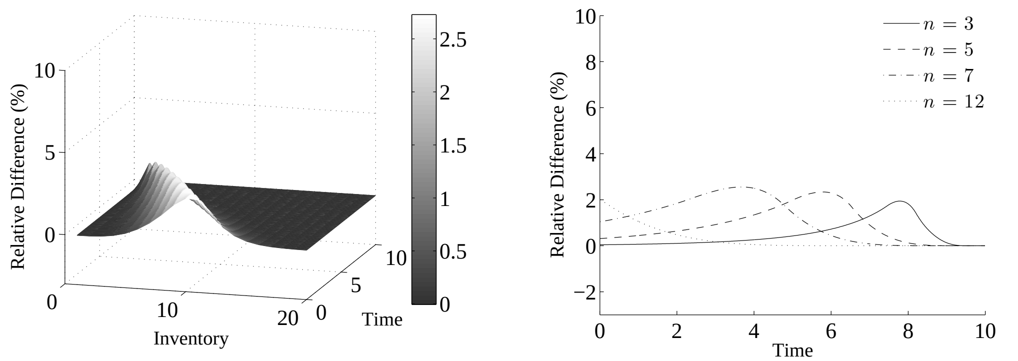
We first consider a two-price model. The prices are selected as $p_1 = 200$, $p_2 = 350$. The intensity is correspondingly chosen to be $\lambda(p) = e^{1-0.005p}$, i.e., $\lambda_1 = 1$, $\lambda_2 = 0.472$. The sales horizon and the initial inventory are set to be $T = 10$ and $M = 20$. In this case, the optimal fixed-price is $p^* = 200$. Figure 4.1 demonstrates the relative expected revenue between the markup and markdown policies. The figure on the left presents the normalized revenue difference as a function of the elapsed time and remaining inventory, while the figure on the right is the difference when the inventory is fixed at different levels.

In Figure 4.1(b), it can be seen that owing to the monotonicity of x_n, w_n and y_n, the entire characterized pattern of the revenue gap shift towards zero as the inventory rises and the leftmost part is progressively removed. This is consistent with the results we obtain in Chapter 3.



(a) Relative difference (b) Relative difference for fixed inventory

Figure 4.1: Relative difference (normalized by the revenue from the mixed policy) between the markup and markdown policies for $(p_1, p_2) = (200, 350)$.

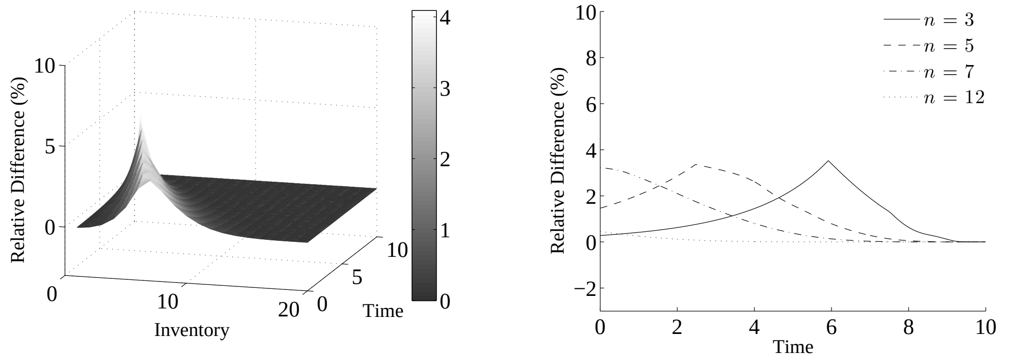


(a) Relative difference (b) Relative difference for fixed inventory

Figure 4.2: Relative difference (normalized by the revenue from the mixed policy) between the mixed and markdown policies for $(p_1, p_2) = (200, 350)$.

Feng and Xiao [7] give an algorithm for calculating the mixed pricing policy, in which the prices are allowed to change in both directions. With greater flexibility, the mixed pricing policy is always superior to the markup or markdown pricing policy. Figure 4.2 and 4.3 demonstrate the additional revenue generated by the mixed policy given the same discrete price set beyond the revenue generated by the markdown and markup policy, respectively.

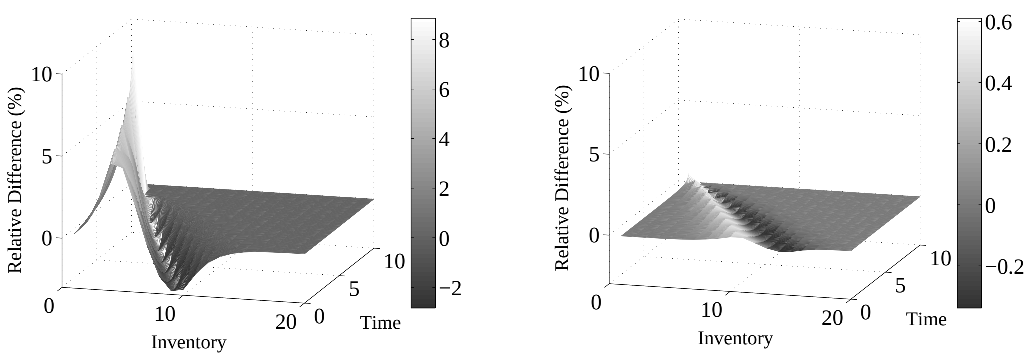
In order to better understand the relative revenue between different pricing



(a) Relative difference (b) Relative difference for fixed inventory

Figure 4.3: Relative difference (normalized by the revenue from the mixed policy) between the mixed and markup policies for $(p_1, p_2) = (200, 350)$.

policies, Table 4.1 displays the maximum relative difference (in absolute value) between the optimal markup, markdown and mixed policies over the entire horizon at each inventory level. In this example, it shows that the maximum percentage gap between the markdown and markup cases could be up to 4.09% in the positive side and down to 1.84% in the negative side. Moreover, the revenue generated by both the markup and markdown policies approaches that of the mixed policy as n increases and eventually, the relative difference reaches zero.



(a) Relative difference for $(p_1, p_2) = (200, 450)$ (b) Relative difference for $(p_1, p_2) = (200, 250)$

Figure 4.4: Relative difference between the markup and markdown policies.

Table 4.1: Maximum relative revenue difference (in absolute value) between different pricing strategies (%)

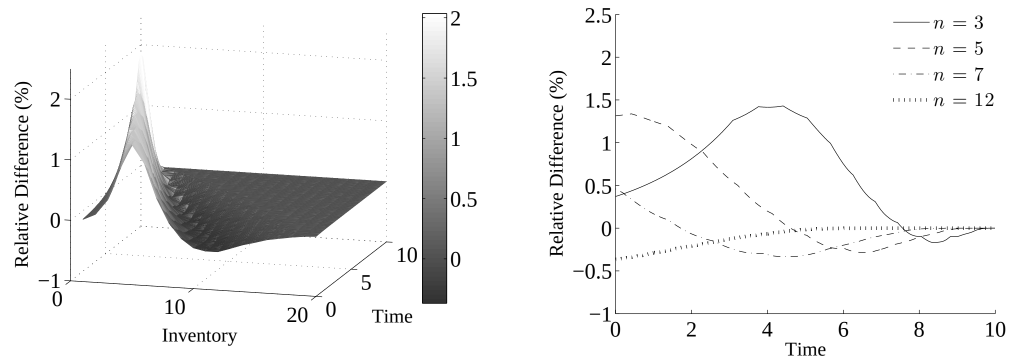
Inventory	C1	C2	C3	Inventory	C1	C2	C3
1	4.09	0.00	4.09	11	-1.84	2.58	0.74
2	3.09	1.50	3.66	12	-1.52	1.97	0.45
3	2.81	1.90	3.50	13	-0.86	1.12	0.26
4	2.68	2.13	3.41	14	-0.46	0.61	0.15
5	2.58	2.28	3.34	15	-0.24	0.32	0.08
6	2.51	2.40	3.27	16	-0.12	0.16	0.04
7	2.16	2.49	3.19	17	-0.06	0.08	0.02
8	-1.72	2.56	2.55	18	-0.03	0.03	0.01
9	-1.77	2.61	1.76	19	-0.01	0.02	0.00
10	-1.81	2.66	1.16	20	0.00	0.00	0.00

[1] C1: Markdown vs. Markup, C2: Mixed vs. Markdown, C3: Mixed vs. Markup

To see the effect of the choice of the price set on the revenue difference, we carry out another group of simulations by varying the allowable price set. We first consider a case with $p_1 = 200$, $p_2 = 450$ which corresponds to a case with wider price difference, and then consider a case with $p_1 = 200$, $p_2 = 250$ which corresponds to a case with more narrow price differences. The results are shown in Figure 4.4. In Figure 4.4, we observe a significant revenue difference change along with the increasing or decreasing price difference, respectively. Specifically, in this example, the maximum percentage gap could escalate up to 8.6% for $(p_1, p_2) = (200, 450)$ and drop drastically down to 0.6% for $(p_1, p_2) = (200, 250)$. Therefore, we can conclude that in general, the relative difference between the markup and markdown policies increases as the price difference in a two-price set increases.

4.2 K-Price Model

In the K-price model, the predetermined price set is selected as $p_k = 200 + 15(k-1)$, $k = 1, 2, \cdots, 11$. Our demand function is chosen to be $\lambda_1(p) = e^{1-0.005p}$ and $\lambda_2(p) = 2 - 0.005p$, respectively. The other setting remains the same. The optimal fixed-prices for $\lambda_1(p)$ and $\lambda_2(p)$ are both $p^* = 200$.



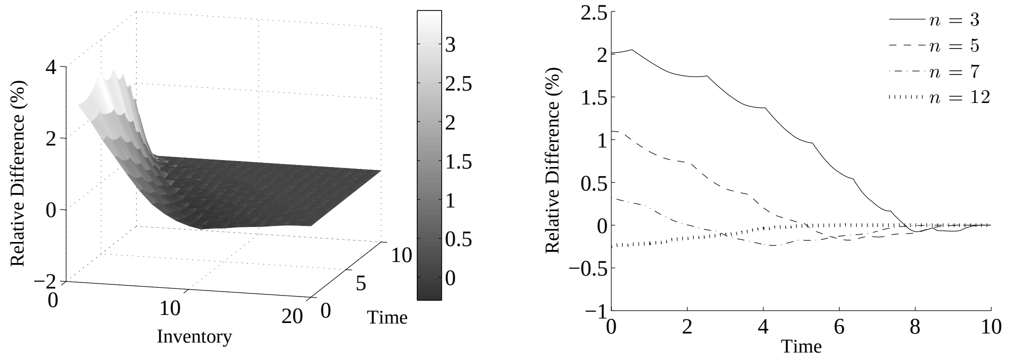
(a) Relative difference

(b) Relative difference for fixed inventory

Figure 4.5: Relative difference (normalized by the revenue from the markup policy) between the markup and markdown policies in a multi-price model under the exponential demand.

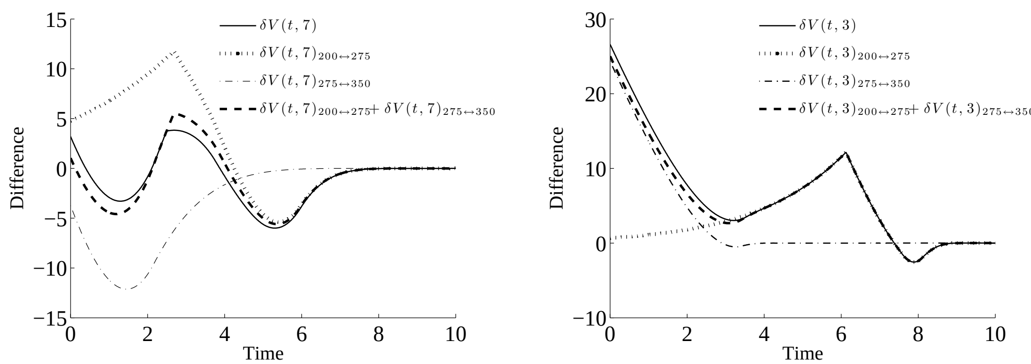
Figure 4.5 and 4.6 both show that at a low inventory level, when the remaining time is sufficient, the markdown policy outperforms the markup policy; as time elapses, the performance of markup policy will eventually exceed the markdown policy. Although strict proof for any K-price model is not presented at this stage, numerical tests suggest that a K-price case can be closely approximated as the addition of $K-1$ two-price cases, thus the major insights of our analysis about the revenue gap for a two price case still hold for a K-price case.

The following two numerical experiments serve as examples to illustrate the revenue gap of a K-price model being approximated by the addition of the revenue gaps of $K-1$ two-price models. In order to exhibit our results clearly, we choose a price set consisting of only three prices, $(p_1, p_2, p_3) = (200, 275, 350)$. The demand function is still chosen to be $\lambda_1(p)$ and $\lambda_2(p)$, respectively. In the case of the exponential demand function, the greatest discrepancy between the actual revenue difference and the revenue difference computed by the addition of two two-price models occurs for $n = 7$. Plotting both revenue differences over time in Figure 4.7(a), we find that the revenue difference $\delta V(t, 7)$ generated by the markup and markdown policies from set $(p_1, p_2, p_3) = (200, 275, 350)$ is still well approximated by the addition of the revenue difference $\delta V(t, 7)_{200 \leftrightarrow 275}$ from $(p_1, p_2) = (200, 275)$



(a) Relative difference (b) Relative difference for fixed inventory

Figure 4.6: Relative difference (normalized by the revenue from the markup policy) between the markup and markdown policies in a multi-price model under the linear demand.



(a) At $n = 7$ under the exponential demand (b) At $n = 3$ under the linear demand

Figure 4.7: The comparison of the revenue difference from $(p_1, p_2, p_3) = (200, 275, 350)$ and the addition of revenue difference from price set $(p_1, p_2) = (200, 275)$ and $(p_2, p_3) = (275, 350)$.

and $\delta V(t, 7)_{200\leftrightarrow 275}$ from $(p_2, p_3) = (275, 350)$. Similarly, the greatest discrepancy in the case of the linear demand function $\lambda_2(p)$ occurs for $n = 3$, as Figure 4.7(b) shows. As in the first case, we find that the actual revenue gap of the three-price model is closely approximated by the additive revenue gap of two two-price models.

Chapter 5

Conclusions

This work investigates a fundamental problem in revenue management that has nonetheless been neglected by previous studies: given a fixed price set and the constraint that the pricing path be monotonic, which pricing strategy, markup or markdown will yield the highest expected revenue? We formulate this problem in a straightforward manner by comparing the expected revenue generated between the optimal markup and markdown pricing policies. The optimal markup and markdown pricing policies we adopt are derived analytically by Feng and Xiao in [6], rather than heuristically as in the most of the literature.

For the two-price case, a clear pattern of the revenue gap is identified. When the inventory is low, the full pattern of the revenue gap has exactly one local maximum and one local minimum over time. As the inventory increases, the entire pattern shifts towards zero gradually and the left part of the pattern at $t = 0$ is hence truncated. The relation among the remaining time, inventory, and the revenue gap between the markup and markdown pricing policies is thus revealed: when the inventory is low and the remaining time is ample, the markdown policy achieves higher revenue than the markup policy. The reason is that, on one hand, increasing the price is optimal in this circumstance from the standpoint of a markup strategy to maximize the revenue. On the other hand, charging a higher price in a two price case will degrade the markup strategy to a fixed-price policy, and thus lose the flexibility to generate at least as much revenue as the markdown policy. In the other situations, the markup policy is superior, and an analogous explanation applies.

Our conclusion for a two-price model is verified by simulations. Although the K-price model is not amenable to the same analytical inspection as in the two price case, we present numerical tests of the revenue difference and the similar results hold, i.e., the markdown policy is superior with lower inventory and sufficient remaining time while inferior with insufficient remaining time. It is worth noting that there is one caveat of our analysis, that is, the allowable prices are assumed to be predetermined. In reality, practitioners may want to choose different price sets under different conditions and the result might be different.

As markup and markdown cases reflect a fair part of industry practice, the managerial value of insights provided by this work should be of interest to decision markers in a variety of domains. Future work may include the comparison between the two cases by taking time-dependent demand into account. Since time-dependent demand is a more realistic assumption, research results along these lines will be more applicable to the real world.

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Appendix A

Proof of Lemma 7 According to (2.5), when $t \geq x_n^k$,

$$\frac{\partial V_k^1(t,n)}{\partial t} = \lambda_k(V_k^1(t,n) - V_k^1(t,n-1)) - \lambda_k p_k. \tag{A.1}$$

We notice that

$$\frac{\partial V_k^s(t,n)}{\partial t} = \lambda_k(V_k^s(t,n) - V_k^s(t,n-1)) - \lambda_k p_k. \tag{A.2}$$

Let (A.1) $-$ (A.2), we have

$$\frac{\partial \Delta V_k^1(t,n)}{\partial t} = \lambda_k(\Delta V_k^1(t,n) - \Delta V_k^1(t,n-1)).$$

Taking an integral from t to T yields

$$\Delta V_k^1(t,n) = \lambda_k \int_t^T \Delta V_k^1(s,n-1) e^{-\lambda_k(s-t)} ds.$$

Accordingly, when $n = 1$,

$$\Delta V_k^1(t,1) = \begin{cases} 0, & t \geq x_1^k, \\ V_{k+1}^1(t,1) - V_k^s(t,1), & t < x_1^k. \end{cases}$$

Hence, $\Delta V_k(t,1) - \Delta V_k(t,0) = 0$ when $t \geq x_1^k$.

Assume that $\Delta V_k^1(t,n) - \Delta V_k^1(t,n-1) \le 0$, or equivalently, $\partial \Delta V_k^1(t,n)/\partial t \le 0$ for n on $t \ge x_n^k$. We will show that the same is true for $n+1$.

Through integration by parts, we have

$$\begin{aligned}\Delta V_k^1(t,n+1) &= \lambda_k \int_t^T \Delta V_k^1(s,n) e^{-\lambda_k(s-t)} ds \\ &= -\int_t^T \Delta V_k^1(s,n) d e^{-\lambda_k(s-t)} \\ &= \Delta V_k^1(t,n) + \int_t^T \frac{\partial \Delta V_k^1(s,n)}{\partial s} e^{-\lambda_k(s-t)} ds.\end{aligned} \tag{A.3}$$

Undoubtedly, $\Delta V_k^1(t,n+1) - \Delta V_k^1(t,n) \le 0$ on $t \ge x_{n+1}^k$ based on our assumption, and "=" holds if and only if $t \ge x_1^k$.

To prove Theorem 9, we first introduce the following lemma.

Lemma 12 *Let $\Delta_k(t,n) = V_k^s(t,n) - V_{k+1}^s(t,n)$ and t_n^k be the root of $\Delta_k(t,n) = 0$. Then $\Delta_k(t,n)$ has an unique sign change on $[0,T)$ at t_n^k with the sequence $\{t_n^k : n = 1, 2, \cdots\}$ strictly decreasing in n. Moreover, $\Delta_k(t,n)$ is monotone increasing on $t \le t_n^k$.*

Proof It can be seen that $\Delta_k(t,n)$ is the solution to the differential equation

$$\frac{\partial \Delta_k(t,n)}{\partial t} = \lambda_k \Delta_k(t,n) - M_k(t,n)$$

with boundary conditions $\Delta_k(t,0) = 0$, $\Delta_k(T,n) = 0$, where

$$\begin{aligned}M_k(t,n) &= G_k^s(t,n) + \lambda_k \Delta_k(t,n-1), \\ G_k^s(t,n) &= (\lambda_{k+1} - \lambda_k)(V_{k+1}^s(t,n) - V_{k+1}^s(t,n-1)) + r_k - r_{k+1}.\end{aligned}$$

The continuity of $\Delta_k(t,n)$ follows from the continuity of $V_k^s(t,n)$ and $V_{k+1}^s(t,n)$. The existence of t_n^k follows owing to $\Delta_k(t,n) > 0$ in the neighborhood of T except for T and $\lim_{T-t\to\infty} \Delta_k(t,n) = n(p_k - p_{k+1}) < 0$. Accordingly,

$$\Delta_k(t,n) = \int_t^T M_k(s,n)e^{-\lambda_k(s-t)}ds \tag{A.4}$$

or

$$\Delta_k(t,n) = \int_t^{t_n^k} M_k(s,n)e^{-\lambda_k(s-t)}ds.$$

Let s_n^k be the root of $M_k(t,n) = 0$. s_n^k exists because both $G_k^s(t,n)$ and $\Delta(t,n-1)$ are continuous, and positive in the neighborhood of T and negative for $T - t \to \infty$.

Note that $\Delta_k(t,1) > 0 = \Delta_k(t,0)$ on $t_1^k < t < T$, $M_k(t,1)$ is increasing in t for $t \leq s_1^k$, where $s_1^k > t_1^k$ and $\Delta_k(t,1)$ is increasing in $t \leq s_1^k$.

Thus, we assume that $\Delta_k(t,n) > \Delta_k(t,n-1)$ on $t_n^k \leq t < T$, $M_k(t,n)$ is increasing in t for $t \leq s_n^k$, where $s_n^k > t_n^k$, and $\Delta_k(t,n)$ is increasing in $t \leq s_n^k$. In other words, $\Delta_k(t,n)$ is increasing in $t \leq t_n^k$ and $\Delta_k(t,n) = 0$ has an unique root.

It is easy to see

$$G_k^s(t,n) = (\lambda_{k+1} - \lambda_k)p_{k+1}P(N_{k+1}(T-t) \geq n) + r_k - r_{k+1}.$$

Due to $p_k < p_{k+1}$, $\lambda_k > \lambda_{k+1}$, hence $G_k^s(t,n+1) > G_k^s(t,n)$. Together with $\Delta_k(t,n) > \Delta_k(t,n-1)$ on $t_n^k \leq t < T$, we obtain $M_k(t,n+1) > M_k(t,n)$ on $t_n^k \leq t < T$, and thus $s_{n+1}^k < s_n^k$. Meanwhile, through Eq. (A), $\Delta_k(t,n+1) > \Delta_k(t,n) \geq 0$ on $t_n^k \leq t < T$ holds, so $t_{n+1}^k < t_n^k$. Furthermore, we have $\Delta_k(t,n+1) \geq 0 > \Delta_k(t,n)$ on $t_{n+1}^k \leq t < t_n^k$, so $\Delta_k(t,n+1) > \Delta_k(t,n)$ on $t_{n+1}^k \leq t < T$.

We also note that

$$
\begin{aligned}
\frac{\partial G_k^s(t,n+1)}{\partial t} &= (\lambda_{k+1}-\lambda_k)\left(\frac{\partial V_{k+1}^s(t,n+1)}{\partial t} - \frac{\partial V_{k+1}^s(t,n)}{\partial t}\right) \\
&= (\lambda_{k+1}-\lambda_k)\lambda_{k+1}[(V_{k+1}^s(t,n+1)-V_{k+1}^s(t,n)) - (V_{k+1}^s(t,n)-V_{k+1}^s(t,n-1))] \\
&= (\lambda_{k+1}-\lambda_k)\lambda_{k+1}p_{k+1}(P(N_{k+1}(T-t)\geq n+1) - P(N_{k+1}(T-t)\geq n)) \\
&= -(\lambda_{k+1}-\lambda_k)r_{k+1}P(N_{k+1}(T-t)=n) \\
&> 0,
\end{aligned}
$$

i.e., $G_k^s(t,n+1)$ is increasing on $[0,T]$. Therefore, $M_k(t,n+1) = G_k^s(t,n+1) + \lambda_k\Delta_k(t,n)$ is increasing on $t \leq s_n^k$. Since $s_{n+1}^k < s_n^k$, $M_k(t,n+1)$ is increasing on $t \leq s_{n+1}^k$. On the other hand, $\Delta_k(t,n+1) > 0$ on $s_{n+1}^k < t < T$ implies $s_{n+1}^k > t_{n+1}^k$.

It remains to be shown that $\Delta_k(t,n+1)$ is increasing in $t \leq s_{n+1}^k$. When $t_{n+1}^k < t \leq s_{n+1}^k$, $\Delta_k(t,n+1) > 0$ and $M_k(t,n+1) \leq 0$, thus $\partial\Delta_k(t,n+1)/\partial t > 0$; when $t \leq t_{n+1}^k$, since $M_k(t,n+1) < 0$ and is increasing,

$$
\begin{aligned}
\frac{\partial \Delta_k(t,n+1)}{\partial t} &= \lambda_k \int_t^{t_{n+1}^k} M_k(s,n+1)e^{-\lambda_k(s-t)}ds - M_k(t,n+1) \\
&> \lambda_k M_k(t,n+1)\int_t^{t_{n+1}^k} e^{-\lambda_k(s-t)}ds - M_k(t,n+1) \\
&= -M_k(t,n+1)e^{-\lambda_k(T-t)} \\
&> 0.
\end{aligned}
$$

Therefore, $\Delta_k(t,n+1)$ is increasing in $t \leq t_{n+1}^k$ and $\Delta_k(t,n+1) = 0$ has an unique root at t_{n+1}^k on $[0,T)$, where $t_{n+1}^k < t_n^k$. ∎

Now we present the proof of Theorem 9 with the aid of Lemma 12.

Proof of Theorem 9 We first show that $x_n^k \leq t_n^k$. To facilitate comparison, we rewrite $L_k^1(t,n)$ the same manner as $M_k(t,n)$, i.e.,

$$
L_k^1(t,n) = G_k^1(t,n) + \lambda_k \bar{V}_k^1(t,n-1),
$$

where

$$G_k^1(t,n) = (\lambda_{k+1} - \lambda_k)(V_{k+1}^1(t,n) - V_{k+1}^1(t,n-1)) + r_k - r_{k+1}.$$

By Lemma 7, we obtain that

$$V_{k+1}^1(t,n) - V_{k+1}^1(t,n-1) \leq V_{k+1}^s(t,n) - V_{k+1}^s(t,n-1)$$

when $t \geq x_n^{k+1}$ and "=" holds if and on if $t \geq x_1^{k+1}$, which indicates $G_k^1(t,1) = G_k^{1s}(t,1)$ on $t \geq x_1^{k+1}$, and $G_k^1(t,n) \geq G_k^{1s}(t,n)$ on $t \geq x_n^{k+1}$ where "=" holds if and only if $t \geq x_1^{k+1}$ when $n \geq 2$.

Thus for $n = 1$, we have $L_k^1(t,1) = M_k^1(t,1)$, so $\bar{V}_k^1(t,1) = \Delta_k^1(t,1)$ on $t \geq x_1^k$ and $t_1^k = x_1^k$. When $t < x_1^k$, $\bar{V}_k^1(t,1) = 0 > \Delta_k^1(t,1)$.

For $n = 2$, we see that $L_k^1(t,2) = G_k^1(t,2) + \lambda_k \bar{V}_k^1(t,1) = G_k^{1s}(t,2) + \lambda_k \Delta_k^1(t,1) = M_k^1(t,2)$ on $t \geq x_1^k$, and hence $\bar{V}_k(t,2) = \Delta_k^1(t,2)$ on $t \geq x_1^k$.

When $x_2^k \leq t < x_1^k$, $L_k^1(t,2) > M_k^1(t,2)$, and

$$\begin{aligned}
\bar{V}_k^1(t,2) &= \int_t^{x_1^k} L_k^1(s,2) e^{-\lambda_k(s-t)} ds + \int_{x_1^k}^T L_k^1(s,2) e^{-\lambda_k(s-t)} ds \\
&> \int_t^{x_1^k} M_k^1(s,2) e^{-\lambda_k(s-t)} ds + \int_{x_1^k}^T M_k^1(s,2) e^{-\lambda_k(s-t)} ds \\
&= \Delta_k^1(t,2).
\end{aligned}$$

Therefore $\Delta_k^1(x_2^k,2) < \bar{V}_k^1(x_2^k,2) = 0$ implying $x_2^k < t_2^k$.

When $t < x_2^k$, $\bar{V}_k^1(t,2) = 0 > \Delta_k^1(t,2)$, thus $\bar{V}_k^1(t,2) \geq \Delta_k^1(t,2)$ on $[0,T]$.

Assume that $x_n^k < t_n^k$ and $\bar{V}_k^1(t,n) \geq \Delta_k^1(t,n)$ with "=" if and only if $t \geq x_1^k$. Then $L_k^1(t,n+1) = M_k^1(t,n+1)$ on $t \geq x_1^k$ and $L_k^1(t,n+1) > M_k^1(t,n+1)$ on $x_{n+1}^k \leq t < x_1^k$, leads to $\bar{V}_k^1(t,n+1) = \Delta_k^1(t,n+1)$ on $t \geq x_1^k$ and $\bar{V}_k^1(t,n+1) > \Delta_k^1(t,n+1)$ on $x_{n+1}^k \leq t < x_1^k$, so $\Delta_k^1(x_{n+1}^k, n+1) < \bar{V}_k^1(x_{n+1}^k, n+1) = 0$ implies $x_{n+1}^k < t_{n+1}^k$. When $t < x_{n+1}^k$, $\bar{V}_k^1(t,n+1) = 0 > \Delta_k^1(t,n+1)$. In general, $\bar{V}_k^1(t,n+1) \geq \Delta_k^1(t,n+1)$ on $[0,T]$.

In the markdown case, note that $\Delta_k(t,n) = \Delta_k^2(t,n) = V_{k+1}^s(t,n) - V_k^s(t,n)$. Clearly, t_n^k is still the unique positive root of $\Delta_k^2(t,n) = 0$. Likewise, $\Delta_k^2(t,n)$ satisfies

$$\frac{\partial \Delta_k^2(t,n)}{\partial t} = \lambda_{k+1}\Delta_k^2(t,n) - M_k^2(t,n),$$

where

$$M_k^2(t,n) = G_k^{2s}(t,n) + \lambda_{k+1}\Delta_k^2(t,n-1),$$
$$G_k^{2s}(t,n) = (\lambda_k - \lambda_{k+1})(V_k^s(t,n) - V_k^s(t,n-1)). + r_{k+1} - r_k.$$

Therefore,

$$\Delta_k^2(t,n) = \int_t^T M_k^2(s,n)e^{-\lambda_{k+1}(s-t)}ds.$$

$L_k^2(t,n)$ is rewritten in the similar manner by

$$L_k^2(t,n) = G_k^2(t,n) + \lambda_{k+1}\bar{V}_k^2(t,n-1),$$

where

$$G_k^2(t,n) = (\lambda_k - \lambda_{k+1})(V_k^2(t,n) - V_k^2(t,n-1)) + r_{k+1} - r_k.$$

Lemma 8 implies that

$$V_k^2(t,n) - V_k^2(t,n-1) > V_k^s(t,n) - V_k^s(t,n-1)$$

when $t \leq y_n^k$. As a result, $G_k^2(t,n) > G_k^{2s}(t,n)$ on $t \leq y_n^k$.

Consequently for $n = 1$, $L_k^2(y_1^k, 1) = 0 = G_k^2(y_1^k, 1) > G_k^{2s}(y_1^k, 1) = M_k^2(y_1^k, 1)$, resulting in $\Delta_k^2(y_1^k, 1) < 0$. Lemma 12 demonstrates that in markdown case,

$\Delta_k^2(t,n)$ has a unique sign change from $+$ to $-$ at t_n^k, so $t_1^k < y_1^k$, also

$$\begin{aligned}
\bar{V}_k^2(t,1) &= \int_t^{y_1^k} L_k^2(s,1)e^{-\lambda_{k+1}(s-t)}ds = \int_t^{y_1^k} G_k^2(s,1)e^{-\lambda_{k+1}(s-t)}ds \\
&> \int_t^{y_1^k} G_k^{2s}(s,1)e^{-\lambda_{k+1}(s-t)} = \int_t^{y_1^k} M_k^2(s,1)e^{-\lambda_{k+1}(s-t)}ds \\
&= \int_t^{T} M_k^2(s,1)e^{-\lambda_{k+1}(s-t)}ds - \int_{y_1^k}^{T} M_k^2(s,1)e^{-\lambda_{k+1}(s-t)}ds \\
&= \Delta_k^2(t,1) - \Delta_k^2(y_1^k,1)e^{-\lambda_{k+1}(y_1^k-t)} \\
&> \Delta_k^2(t,1).
\end{aligned}$$

The inductive hypothesis is that $t_n^k < y_n^k$ and $\bar{V}_k^2(t,n) > \Delta_k^2(t,n)$ on $t \le y_n^k$. It follows that $L_k^2(t,n+1) > M_k^2(t,n+1)$ on $t \le y_n^k$. Thus, $L_k^2(y_{n+1}^k, n+1) = 0 > M_k^2(y_{n+1}^k, n+1)$, and as a consequence, $\Delta_k^2(y_{n+1}^k, n+1) < 0$ on $t \le y_{n+1}^k$ implying that $t_{n+1}^k < y_{n+1}^k$. Moreover,

$$\begin{aligned}
\bar{V}_k^2(t,n+1) &> \int_t^{y_{n+1}^k} M_k^2(s,n+1)e^{-\lambda_{k+1}(s-t)}ds \\
&= \Delta_k^2(t,n+1) - \Delta_k^2(y_{n+1}^k,n+1)e^{-\lambda_{k+1}(y_{n+1}^k-t)} \\
&> \Delta_k^2(t,n+1)
\end{aligned}$$

on $t \le y_{n+1}^k$. Obviously, $\bar{V}_k^2(t,n+1) = 0 > \Delta_k^2(t,n+1)$ on $t > y_{n+1}^k$.

In conclusion, $x_n^k < y_n^k$. ∎

Supplement to Proof of Theorem 11 Since

$$\begin{aligned}
L^1(x_n,n) &= (\lambda_2-\lambda_1)(V_2^s(x_n,n) - V_2^s(x_n,n-1)) - r_2 + r_1 + \lambda_1\bar{V}_1^1(x_n,n-1) < 0, \\
L^2(y_n,n) &= (\lambda_1-\lambda_2)(V_1^s(y_n,n) - V_1^s(y_n,n-1)) + r_2 - r_1 + \lambda_1\bar{V}_1^2(y_n,n-1) = 0,
\end{aligned}$$

and when $n \geq 2$,

$$\begin{gathered}
V_2^2(x_n, n) - V_2^2(x_n, n-1) > V_2^s(x_n, n) - V_2^s(x_n, n-1), \\
V_2^2(y_n, n) - V_2^2(y_n, n-1) < V_1^s(y_n, n) - V_1^s(y_n, n-1), \\
\bar{V}_1^1(x_n, n-1) = 0, \ \ \bar{V}_1^2(y_n, n-1) > 0,
\end{gathered}$$

we have

$$\begin{gathered}
(\lambda_1 - \lambda_2)(V_2^2(x_n, n) - V_2^2(x_n, n-1)) + r_2 - r_1 > 0, \\
(\lambda_1 - \lambda_2)(V_2^2(y_n, n) - V_2^2(y_n, n-1)) + r_2 - r_1 < 0.
\end{gathered}$$

Therefore, the monotonicity of $V_2^2(t, n) - V_2^2(t, n-1)$ in $[x_n, y_n]$ ensures that there must exist a time, denoted by q_n, $x_n < q_n < y_n$, at which $G(q_n, n) = 0$.

When $n = 1$, we calculate that

$$V_2^2(t, 1) = \begin{cases} V_2^s(t, 1) + \Delta V_2^2(t, 1), & 0 \leq t \leq y_1, \\ V_1^s(t, 1), & y_1 < t \leq T, \end{cases}$$

$$V_1^1(t, 1) = \begin{cases} V_2^s(t, 1), & 0 \leq t < x_1, \\ V_1^s(t, 1), & x_1 \leq t \leq T. \end{cases}$$

Therefore,

$$\delta V(t, 1) = \begin{cases} \Delta V_2^2(t, 1), & 0 \leq t < x_1, \\ \bar{V}_1^2(t, 1) - (\bar{V}_1^1(t, 1) - \Delta_1^1(t, 1)), & x_1 \leq t \leq y_1, \\ 0, & y_1 < t \leq T. \end{cases}$$

Obviously, $\delta V(t, 1) > 0$ and strictly increases in $t \in [0, x_1)$. The proof of Theorem 9 shows that $\bar{V}_1^1(t, 1) = \Delta_1^1(t, 1)$ when $t \geq x_1$. Hence, $\delta V(t, 1) = \bar{V}_1^2(t, 1) > 0$ and strictly decreases in $t \in [x_1, y_1]$, which implies $w_1 = y_1$. Moreover,

$$L^2(y_1, 1) = (\lambda_1 - \lambda_2)V_1^s(y_1, 1) + r_2 - r_1 = 0,$$

thus $V_2^2(y_1, 1) = (r_1 - r_2)/(\lambda_1 - \lambda_2)$. This means $q_1 = y_1 = w_1$.

We assume w_n is the only root of $\delta V(t, n) = 0$ except T, i.e., $\delta V(t, n) > 0$ on $[0, w_n)$ and $\delta V(t, n) < 0$ on (w_n, T) and there is only one local maximum on $[0, w_n)$, denoted by v_n. Clearly, $x_n \leq v_n$. We also assume $q_n \leq w_n$ ("=" holds if and only if $n = 1$).

Since $(\lambda_1 - \lambda_2)(V_2^2(q_n, n+1) - V_2^2(q_n, n)) + r_2 - r_1 < 0$, it entails $q_{n+1} < q_n$, hence $q_{n+1} < w_n$. By this reason,

(1) When $t < q_{n+1}$, $G(t, n+1) > 0$, $\delta V(t, n) > 0$, $H(t, n+1) > 0$;

(2) When $t > w_n$, $G(t, n+1) < 0$, $\delta V(t, n) < 0$, $H(t, n+1) < 0$;

(3) When $q_{n+1} \leq t \leq w_n$, we classify the possible position of q_{n+1} into three categories:

1° When $q_{n+1} < x_n$, $[q_{n+1}, w_n]$ is partitioned into three intervals: $[q_{n+1}, x_n]$, $(x_n, v_n]$ and $(v_n, w_n]$. On $t \in [q_{n+1}, x_n]$, $H(t, n+1)$ is expressed by

$$H(t, n+1) = (\lambda_1 - \lambda_2)(V_2^2(t, n+1) - V_2^2(t, n)) + r_2 - r_1 + \lambda_1(V_2^2(t, n) - V_2^s(t, n)).$$

Differentiating $H(t, n+1)$ with respect to t,

$$\frac{\partial H(t, n+1)}{\partial t} = (\lambda_1 - \lambda_2)\frac{\partial V_2^2(t, n+1)}{\partial t} + \lambda_2 \frac{\partial V_2^2(t, n)}{\partial t} - \lambda_1 \frac{\partial V_2^s(t, n)}{\partial t}.$$

It can be found that the sign of $\partial H(t, n+1)/\partial t$ is not consistently positive or negative. Thus, let $t = m_{n+1}$ satisfy $\partial H(t, n+1)/\partial t = 0$, we have

$$\frac{\partial V_2^2(m_{n+1}, n+1)}{\partial t} = -\frac{\lambda_2}{\lambda_1 - \lambda_2}\frac{\partial V_2^2(m_{n+1}, n)}{\partial t} + \frac{\lambda_1}{\lambda_1 - \lambda_2}\frac{\partial V_2^s(m_{n+1}, n)}{\partial t}. \quad \text{(A.5)}$$

To determine whether $H(m_{n+1}, n+1)$ is maximum or minimum, we examine

$\partial^2 H(m_{n+1}, n+1)/\partial t^2$. Combing with (A.5),

$$\begin{aligned}
\frac{\partial^2 H(m_{n+1}, n+1)}{\partial t^2} =&(\lambda_1 - \lambda_2)\frac{\partial^2 V_2^2(m_{n+1}, n+1)}{\partial t^2} + \lambda_2 \frac{\partial^2 V_2^2(m_{n+1}, n)}{\partial t^2} - \lambda_1 \frac{\partial^2 V_2^s(m_{n+1}, n)}{\partial t^2} \\
=&(\lambda_1 - \lambda_2)\lambda_2 \left(\frac{\partial V_2^2(m_{n+1}, n+1)}{\partial t} - \frac{\partial V_2^2(m_{n+1}, n)}{\partial t} \right) + \lambda_2^2 \left(\frac{\partial V_2^2(m_{n+1}, n)}{\partial t} \right. \\
&\left. - \frac{\partial V_2^2(m_{n+1}, n-1)}{\partial t} \right) + \lambda_1 \lambda_2 \left(\frac{\partial V_2^s(m_{n+1}, n)}{\partial t} - \frac{\partial V_2^s(m_{n+1}, n-1)}{\partial t} \right) \\
=& -\lambda_2 \left[(\lambda_1 - \lambda_2)\frac{\partial V_2^2(m_{n+1}, n)}{\partial t} + \lambda_2 \frac{\partial V_2^2(m_{n+1}, n-1)}{\partial t} - \lambda_1 \frac{\partial V_2^s(m_{n+1}, n-1)}{\partial t} \right] \\
=& -\lambda_2 \frac{\partial H(m_{n+1}, n)}{\partial t}.
\end{aligned}$$

When $n = 1$,

$$\frac{\partial^2 H(m_2, 2)}{\partial t^2} = -\lambda_2 \frac{\partial H(m_2, 1)}{\partial t} = -\lambda_2(\lambda_1 - \lambda_2)\frac{\partial V_2^2(m_2, 1)}{\partial t} > 0.$$

Assume $\partial^2 H(m_n, n)/\partial t^2 > 0$. Since $m_{n+1} < x_n < m_n$, $\partial H(m_{n+1}, n)/\partial t < 0$, and further $\partial^2 H(m_{n+1}, n+1)/\partial t^2 > 0$. Therefore, $H(m_{n+1}, n+1)$ is the only local minimum on $[q_{n+1}, x_n]$. Consequently,

$$\begin{aligned}
H(t, n+1) \geq& H(m_{n+1}, n+1) \\
=&(\lambda_1 - \lambda_2)\left(\frac{1}{\lambda_2} \frac{\partial V_2^2(m_{n+1}, n+1)}{\partial t} + p_2 \right) + r_2 - r_1 \\
&+ \lambda_1 (V_2^2(m_{n+1}, n-1) - V_2^s(m_{n+1}, n-1)) \\
=&(\lambda_1 - \lambda_2)(V_2^2(m_{n+1}, n) - V_2^2(m_{n+1}, n-1)) + r_2 - r_1 \\
&+ \lambda_1 (V_2^2(m_{n+1}, n-1) - V_2^s(m_{n+1}, n-1)) \\
>&(\lambda_1 - \lambda_2)(V_2^2(x_n, n) - V_2^2(x_n, n-1)) + r_2 - r_1 \\
>&0.
\end{aligned}$$

On $t \in (x_n, v_n]$, due to $\partial^2 G(t, n+1)/\partial t^2 < 0$, $\partial^2 \delta V(t, n)/\partial t^2 < 0$,

$$\frac{\partial^2 H(t, n+1)}{\partial t^2} < 0,$$

indicating that $\partial H(t, n+1)/\partial t$ is decreasing in t with boundary condition $H(x_n, n+1) > 0$.

Subsequently, on $t \in (v_n, w_n]$, because $G(t, n+1)$ and $\delta V(t, n)$ are both strictly decreasing, $H(t, n+1)$ is strictly decreasing.

2° When $x_n \leq q_{n+1} < v_n$, according to the analysis above, $\partial H(t, n+1)/\partial t$ is decreasing in t on $[q_{n+1}, v_n]$ with boundary condition $H(q_n, n+1) > 0$ and $H(t, n+1)$ is strictly decreasing in t on $(v_n, w_n]$.

3° When $q_{n+1} \geq v_n$, $H(t, n+1)$ is strictly decreasing in t on $[q_{n+1}, w_n]$.

For case 3°, it is trivial to see $H(t, n+1)$ has only one root for $H(t, n+1) = 0$ owing to the decreasing property of $H(t, n+1)$ in $t \in [q_{n+1}, w_n]$ with $H(t, n+1) > 0$ when $t < q_{n+1}$ and $H(t, n+1) < 0$ when $t > w_n$; for case 1° and 2°, $H(t, n+1) > 0$ before $H(t, n+1)$ is strictly decreasing to w_n in $t \in [q_{n+1}, w_n]$. In conclusion, there is only one root for $H(t, n+1) = 0$, and it falls in (q_{n+1}, w_n).