# DIFFUSION-CONVECTION-REACTION, FREE BOUNDARIES, AND AN INTEGRAL EQUATION

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**IMA Preprint Series # 816**

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B.H. GILDING* AND R. KERSNER**

**Abstract.** The property of finite speed of propagation for the general nonlinear diffusion-convection-reaction equation $u_t = (a(u))_{xx} + (b(u))_x + c(u)$ is characterized. This is achieved utilizing travelling-wave solutions of the equation. The study of the travelling waves is reduced to the analysis of an integral equation.

**1. Introduction.** In this paper we shall indicate how travelling waves can play an important role in the analysis of nonlinear diffusion-convection-reaction processes, and, how the study of travelling waves can be reduced to the study of an integral equation. To be specific, we show that in a process described by the equation

$$u_t = (a(u))_{xx} + (b(u))_x + c(u) \tag{1}$$

the occurrence of a free boundary is equivalent to the admission of a "finite" travelling wave, the study of which is the same as the study of a singular nonlinear Volterra integral equation.

In equation (1) subscripts denote partial differentiation, and the real coefficients $a$, $b$ and $c$ satisfy the following hypotheses:

$$a, b \in C([0,\infty)) \cap C^1(0,\infty), c \in C(0,\infty),$$
$$a'(s) > 0 \quad \text{for} \quad s > 0,$$
$$ca' \in L^1(0,\delta) \quad \text{for all} \quad \delta > 0,$$

and

$$a(0) = b(0) = c(0) = 0.$$

Suppose that $u(x,t)$ is a (generalized) solution of the Cauchy problem for (1) in the strip

$$S = (-\infty,\infty) \times (0,T]$$

where $T > 0$. To remove any ambiguity in those cases where the problem is not uniquely solvable, suppose furthermore that $u$ is the minimal solution. By definition $u \in C(\bar{S})$. Let

$$P[t] = \{x \in (-\infty,\infty) : u(x,t) > 0\}$$

*Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands.

**Computer and Automation Institute, Hungarian Academy of Sciences, H-1132, Budapest, Victor Hugo 18-22, Hungary.

and

$$\zeta(t) = \sup\{x \in (-\infty, \infty) : u(x,t) > 0\}$$

for any $t \in [0,T]$. We pose the following question for nonempty $P[0]$. If $\zeta(0) < \infty$, is it true that $\zeta(t) < \infty$ for all $t \in [0,T]$? When the latter is true, equation (7) is said to display *finite speed of propagation* (FSP). In this event, $\zeta(t)$ represents a free boundary denoting the right-hand side of the support of $u$. Note that we could also have defined FSP in terms of the infimum of the set $P[t]$. However, by a simple change of variables, this left-hand free boundary can always be transformed into a right-hand one.

In his treatise on the theory of heat published in 1835, Poisson [8] observed that the heat equation

$$u_t = u_{xx}$$

propagates disturbances with infinite speed. To quote: "Supposons que le barre n'a été échauffée primitivement que dans une portion limitée qui s'étendait depuis $x = -\varepsilon$ jusqu'à $x = \varepsilon$, de sorte qu'en dehors de ces limites sa température initiale $fx$ était zéro, comme la température extérieure ..... Cette expression de $u$ (the Poisson formula) nous montre que la chaleur communiquée à une portion de la barre se répand instantanément dans toute sa longueur; car, quelque grande que soit la distance $x$, et quelque petit que soit le temps $t$, il y aura toujours une valeur de $u$ qui ne sera pas rigoreusement nulle. Ce résultat tient à ce qu'en formant l'équation du mouvement de la chaleur, nous avons supposé instantanés les échanges de chaleur entre les tranches de la barre comprises dans l'étendue du rayonnement intérieur. Or, quelque rapides que soient ces échanges, ils ne peuvent avoir lieu dans la nature qu'en des intervalles de temps de grandeur finie; et si nous avions eu égard à cette circonstance, la conductibilité $k$ et par suit la quantité $a$ (the diffusion coefficient) ne seraient plus rigoreusement constantes....."

Research in the 1950's showed that this property was common to all uniformly parabolic equations of the form (1) with smooth coefficients. However, in the same period an explicit solution of the equation

$$u_t = (u^m)_{xx} \text{ with } m > 1$$

was published [9] which indicated that this equation displays FSP. It is now known [4,6,7] that the general equation

$$u_t = (a(u))_{xx}$$

has FSP if and only if

(2) $$a'(s)/s \in L^1(0, \delta) \text{ for some } \delta > 0.$$

The situation becomes more involved when the coefficients in (1) are singular. For instance, for the equation

(3) $$u_t = (u^m)_{xx} + b_0(u^n)_x + c_0 u^p$$

with $b_0 = 0$, $c_0 < 0$ and $0 < p < 1$ it was indicated in [5] that there is FSP for all $m \geq 1$, whilst for this equation with $m \geq 1$, $0 < n < 1$ and $c_0 = 0$ it was shown in [1] that there is FSP if and only if $b_0 > 0$.

For a complete survey of previous results on FSP for equations of the type (1) one may consult [3] where a description of the methods used to obtain these results may also be found.

**2. The results.** Consider the nonlinear Volterra integral equation for $\theta(s)$ with $s \geq 0$:

$$\theta(s) = \lambda s + b(s) - \int_0^s c(r)a'(r)/\theta(r)dr. \tag{4}$$

Our principal results are the following.

THEOREM 1. *Suppose that there is a real parameter $\lambda > 0$ for which (4) has a nonnegative continuous solution such that*

$$a'(s)/\theta(s) \in L^1(0,\delta) \quad \text{for some} \quad \delta > 0. \tag{5}$$

*Then if $P[0]$ is bounded above there exists a $\tau \in (0, T]$ such that $P[t]$ is uniformly bounded above for all $t \in [0, \tau]$.*

THEOREM 2. *Suppose that there is no real parameter $\lambda > 0$ for which (4) has a nonnegative solution such that (5) holds. Then if $P[0]$ is not empty there exists a $\tau \in (0, T]$ such that $P[t]$ is nonempty, connected and unbounded above for all $t \in (0, \tau]$.*

It follows from these theorems that equation (1) displays FSP if and only if there is a parameter $\lambda > 0$ such that (4) has a solution satisfying (5).

The appearance of the integral equation (4) and the condition (5) may be motivated heuristically by the observation that a finite travelling wave solution of equation (1) can be defined by

$$u(x,t) = f(\lambda t - x)$$

where

$$\int_0^{f(\xi)} a'(s)/\theta(s)ds = \max\{0, \xi\} \tag{6}$$

for any function $\theta$ conforming to (4) and (5).

Because the integrand in (4) is singular in $\theta = 0$, in general this equation admits neither existence nor uniqueness. In fact, it is also possible that the equation has at least two solutions, one of which satisfies (5) and one which does not. To circumvent the difficulty of the singularity of the integrand in (4), the equation can be considered as the limit as $\varepsilon \downarrow 0$ of the regularized equation

$$\theta_\varepsilon(s) = \varepsilon + \lambda s + b(s) - \int_0^s c(r)a'(r)/\theta_\varepsilon(r)dr.$$

For every $\varepsilon > 0$ this equation has a maximal solution $\theta_\varepsilon$ which satisfies (5) on some interval of existence $[0, M_\varepsilon)$. Moreover, it can be shown that (4) has a solution satisfying (5) only if $M_\varepsilon \nrightarrow 0$ as $\varepsilon \to 0$. Whilst, if $M_\varepsilon \nrightarrow 0$, then the limit of the sequence $\theta_\varepsilon$ as $\varepsilon \to 0$ defines the maximal solution of (4). For further details, see [2,3].

**3. A sketch of the proof of Theorem 1.** Let

$$\alpha = \frac{1}{2}\int_0^\delta a'(s)/\theta(s)ds$$

and $\Omega = \{(x,t) \in S : x > \lambda t + \zeta(0) - \alpha\}$. Define $v(x,t)$ on $\overline{\Omega}$ by

$$\nu(x,t) = f(\lambda t - x + \zeta(0) + \alpha).$$

with $f$ given by (6). One can show that $v$ is a generalized solution of equation (1) in any domain $D = (\zeta(0), \infty) \times (0, \tau] \subseteq \Omega$. Moreover, noting that $v(\zeta(0), 0) > 0 = u(\zeta(0), 0)$, using the continuity of $u$ and $v$, one can choose $\tau$ so small that $v(\zeta(0), t) \geq u(\zeta(0), t)$ for all $t \in [0, \tau]$. Plainly though $v(x, 0) \geq 0 = u(x, 0)$ for all $x \in [\zeta(0), \infty)$. Whence, by a comparison principle argument, $v(x,t) \geq u(x,t)$ for all $(x,t) \in D$. This implies $u(x,t) = 0$ for all $x \geq \zeta(0) + \alpha + \lambda t$ and $t \in [0, \tau]$.

**4. A sketch of the proof of Theorem 2.** By the continuity of $u$ there is a $\tau \in (0, T]$ such that $P[t]$ is nonempty for all $t \in [0, \tau]$. Furthermore, given any $t_1 \in (0, \tau]$ and $x_1 \in P[t_1]$ there is a $\mu > 0$ and a $t_0 \in [0, t_1)$ such that

$$u(x_1, t) \geq \mu > 0 \text{ for all } t \in [t_0, t_1].$$

Set $D = (x_0, \infty) \times (t_0, t_1]$.

Now, for any $\lambda > 0$ and $\varepsilon > 0$ we can define a function $f$ on $(-\infty, \infty)$ by

$$f(\xi) = M_\varepsilon \quad \text{for} \quad \int_0^{M_\varepsilon} a'(s)/\theta_\varepsilon(s)ds \leq \xi$$

$$\int_0^{f(\xi)} a'(s)/\theta_\varepsilon(s)ds = \xi \quad \text{for} \quad 0 < \xi < \int_0^{M_\varepsilon} a'(s)/\theta_\varepsilon(s)ds$$

$$f(\xi) = 0 \quad \text{for} \quad \xi \leq 0.$$

Furthermore, it can be checked that $v(x,t)$ defined by

$$v(x,t) = f(\lambda(t-t_0) - x + x_1)$$

is a generalized strict subsolution of equation (1) in a suitable subdomain of $S$ [3]. The idea is now that since (4) has no solution satisfying (5), we can choose $\varepsilon$ so small that $v$ becomes defined on $\overline{D}$ and

$$v(x_1,t) \le \mu \quad \text{for all} \quad t \in [t_0, t_1].$$

Since though $v(x,t_0) = 0 \le u(x,t_0)$ for all $x \in [x_0, \infty)$, by comparison we then have $v(x,t) \le u(x,t)$ for all $(x,t) \in D$. This yields $u(x,t_1) \ge v(x,t_1) > 0$ for all $x \in [x_1, x_1 + \lambda(t_1 - t_0))$. Whence in the limit $\lambda \to \infty$, $u(x,t_1) > 0$ for all $x \in [x_1, \infty)$.

We distinguish two cases.

(i) The solution of (4) exists but does not satisfy (5). In this case, without loss of generality we may assume that $\mu < M_\varepsilon$ for all $\varepsilon > 0$. For small enough $\varepsilon$ there must then hold

$$\int_0^\mu a'(s)/\theta_\varepsilon(s)ds > \lambda(t_1 - t_0)$$

for otherwise in the limit $\varepsilon \to 0$ the maximal solution of (4) would satisfy (5) with $\delta = \mu$. This yields the required properties for $v$.

(ii) The solution of (4) does not exist. In this case, necessarily $M_\varepsilon \to 0$ as $\varepsilon \to 0$. Subsequently we merely have to choose $\varepsilon$ so small that $M_\varepsilon < \mu$, and $v$ has the required properties.

**5. Applications.** The criterion for FSP that equation (4) has a solution satisfying (5) can be formulated in more explicit terms for special forms of equation (1) of interest. For example the following results are corollaries.

THEOREM 3. *If $c \equiv 0$ then equation (1) displays FSP if and only if*

$$\max\{0, -b(s)\} = O(s) \text{ as } s \downarrow 0$$

*and*

$$a'(s)/\max\{s, b(s)\} \in L^1(0,\delta) \quad \textit{for some} \quad \delta > 0.$$

THEOREM 4. *If $b \equiv 0$ and $c(s) \le 0$ for all $s > 0$ then equation (1) has FSP if and only if*

$$a'(s)/\max\left\{s, \left|\int_0^s c(r)a'(r)dr\right|^{\frac{1}{2}}\right\} \in L^1(0,\delta) \quad \textit{for some} \quad \delta > 0.$$

THEOREM 5. *If $b \equiv 0$ and $c(s) \geq 0$ for all $s > 0$ then equation (1) has FSP if and only if*

$$\int_0^s c(r)a'(r)dr = O(s^2) \text{ as } s \downarrow 0$$

*and (2) holds.*

THEOREM 6. *The model equation (3) with $m > 0$, $n > 0$, $m + p > 0$, $b_0 \neq 0$ and $c_0 \neq 0$ has FSP if and only if one of the following hold:*

(i) $c_0 < 0, n \geq 1, m > \min\{1, p\}$;

(ii) $c_0 < 0, n < 1, b_0 < 0, \min\{m, n\} > p$;

(iii) $c_0 < 0, n < 1, b_0 > 0, m > \min\{n, p\}$;

(iv) $c_0 > 0, n \geq 1, m > 1$ *and* $m + p > 2$;

(v) $c_0 > 0, n < 1, 0 < b_0 < 2\sqrt{mc_0/n}, m > n$ *and* $m + p > 2n$;

(vi) $c_0 > 0, n < 1, b_0 \geq 2\sqrt{mc_0/n}, m > n$ *and* $m + p \geq 2n$.

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## REFERENCES

[1] J.I. DIAZ & R. KERSNER, *Non existence d'une des frontières libres dans une équation dégénéree en théorie de la filtration*, C.R. Acad. Sci. Paris Sér. I Math. 296 (1983), pp. 505–508.

[2] B.H. GILDING, *A singular nonlinear Volterra integral equation*, submitted for publication. Appearing in provisional form as: University of Twente Faculty of Applied Mathematics Memorandum 955 (1991).

[3] B.H. GILDING & R. KERSNER, *The characterization of reaction-convection-diffusion processes by travelling waves*, in preparation.

[4] A.S. KALASHNIKOV, *On equations of the nonstationary-filtration type in which the perturbation is propagated at infinite velocity*, Moscow Univ. Math. Bull. 27 (1972), pp. 104–108.

[5] L.K. MARTINSON & K.B. PAVLOV, *The problem of the three-dimensional localization of thermal perturbations in the theory of non-linear heat conduction*, U.S.S.R. Comput. Math. and Math. Phys. 12 (1972), pp. 261–268.

[6] O.A. OLEINIK, A.S. KALASHNIKOV & CHZHOU Y.-L., *The Cauchy problem and boundary problems for equations of the type of nonstationary filtration*, Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), pp. 667–704.

[7] L.A. Peletier, *On the existence of an interface in nonlinear diffusion processes*, Ordinary and Partial Differential Equations, (edited by B.D. Sleeman & I.M. Michael), Lecture Notes in Mathematics 415, Springer-Verlag, Berlin (1974), pp. 412–416.

[8] S.D. Poisson, *Théorie Mathématique de la Chaleur*, Bachelier, Paris (1835).

[9] Ya. b. Zel'dovich & A.S. Kompaneets, *On the theory of propagation of heat with the heat conductivity depending upon the temperature*, Collection in Honour of the Seventieth Birthday of Academician A.F. Ioffe, Izdat Akad. Nauk SSSR, Moscow (1950), pp. 61–71.

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