INTERIOR GRADIENT ESTIMATES FOR SOLUTIONS OF PRESCRIBED CURVATURE EQUATIONS OF PARABOLIC TYPE

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1 Introduction

In this paper we establish *a priori* interior gradient estimates for fully non-linear equations of the form

$$(1.1) \qquad F[D^2u] \equiv f(\kappa) = \frac{1}{w}\psi(x,t,u,\nu,u_t), \quad w = (1+|Du|^2)^{1/2}$$

where $\nu = (Du/w, -1/w)$ denotes the downward directed unit normal on the spatial graph S of the function u, $\kappa = (\kappa_1, \cdots, \kappa_n)$ is the n-tuple of principal curvatures of S with respect to the upward unit normal vector, $\psi \in C^1(R^n \times R^+ \times R \times S^n \times R)$ is a prescribed function, and f is a smooth symmetric function defined in an open convex symmetric cone $\Gamma \subset R^n$, with vertex at the origin, and containing the positive cone $\Gamma^+ \equiv \{\kappa \in R^n :$ each component $\kappa_i > 0\}$. The equation is assumed to be parabolic, i.e. (see [1])

$$(1.2) \qquad f_i \equiv \frac{\partial f}{\partial \kappa_i} > 0 \text{ in } \Gamma, \text{ for } 1 \le i \le n,$$

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and

$$\psi_\tau(x,u,t,\nu,\tau) > 0. \tag{1.3}$$

Equations of form (1.1) are naturelly related to the problem of the normal deformation of a hypersurface by a function of its curvatures. The corresponding elliptic equations have first been studied by Caffarelli, Nirenberg and Spruck [1]-[3]. Following them, we call a function u *admissible* if, at every point on the graph $S = S_t$ of $u(\cdot,t)$, $\kappa \in \Gamma$, for any t.

In order to obtain interior gradient bounds, we assume in addition there exist constants $c_0, c_1 > 0$ such that

$$f_j \geq c_0 + c_1 \sum f_i, \quad \text{for all } \kappa \in \Gamma \text{ with } \kappa_j < 0, \tag{1.4}$$

and

$$f \text{ is homogeneous of degree one.} \tag{1.5}$$

The prescribed function ψ is assumed to satisfy (1.3) and for some positive constants M

$$-\psi_u,\ \psi_\tau \leq M; \quad |D_x\psi|,\ |D_\nu\psi| \leq Mw, \tag{1.6}$$

and furthermore,

$$\psi \text{ is concave with respect to } \tau, \text{ and } \psi \geq -M \text{ on } \{\tau = 0\}. \tag{1.7}$$

We can now state our main result of this paper. Let B_1 be the open unit ball in R^n and $T > 0$.

Theorem 1.1. *Let $u \in C^3(B_1 \times (0,2T)) \cap C^1(\overline{B}_1 \times [0,2T))$ be an admissible solution of (1.1), and $u(0,T) = u_0$, $u < 0$ on $\overline{B}_1 \times [0,2T)$. Then, under the conditions (1.2)-(1.7),*

$$|Du(0,T)| \leq C \tag{1.8}$$

for some constant C depending only on n, c_0, c_1, M, u_0, T.

An interior gradient estimate for any bounded solution of (1.1) follows inmediately from Theorem 1.1.

Typical examples of functions f satisfying all the conditions of the theorem are $f(\kappa) = (\sigma^{(k)}(\kappa))^{1/k}$, where $\sigma^{(k)}$ is the k-th elementary symmetric function

$$\sigma^{(k)}(\kappa) = \sum_{i_1 < i_2 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k}.$$

For $(\sigma^{(k)})^{1/k}$ the cone $\Gamma = \Gamma_k$ consists of that component of its positivity set which contains Γ^+; see [1], and the condition (1.4) is verified in [9].

Estimate (1.8) has been obtained by Evans and Spruck [4] for the mean curvature evolution equation. In the stationary case, the corresponding estimates were proved by Korevaar [8] for Weingarten equations and by Li [9] for more general nonlinear elliptic equations of prescribed curvatures. All of the proofs of these results, including those of the present paper are based on the original method of Korevaar [8].

In order to prove the theorem we will need to differentiate equation (1.1). For technical convenience, we follow Trudinger [11] and do this on the graph S of u. Thus in §2 we develop the necessary calculus on a hypersurface that is relevant to prescribed curvature equations. Using this calculus, we construct in §3 the necessary comparison functions and complete the proof of Theorem 1.1. At the very end of the paper we make some final remarks indicating how our assumptions may be somewhat relaxed or modified. In particular, we recover the results of [9] in the stationary case.

2 Calculus on the hypersurface S

Following [5], we let

$$\delta = \nabla - \nu\,(\nu \cdot \nabla)$$

denote the tangential gradient operator on S, where ∇ is the gradient operator in R^{n+1}. Let $e_1, \cdots, e_{n+1}$ denote the orthonormal coordinate frame of R^{n+1}, and set

$$\delta_i = e_i \cdot \delta, \quad \nu^i = e_i \cdot \nu, \quad 1 \le i \le n+1.$$

It follows (for example from [10]) that the curvature matrix $[\delta_i \nu^j]$ is symmetric with eigenvalues $\kappa_1, \cdots, \kappa_n, 0$ on S, where $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of S.

We need to extend the symmetric function f to some open, symmetric subset of R^{n+1}. There is a canonical way to do this. By a local version (see [7, p.108]) of a theorem of Glaeser [6], we can find a smooth function h which satisfies

$$f(\kappa) = h(\sigma^{(1)}(\kappa), \cdots, \sigma^{(n)}(\kappa)), \quad \text{for } \kappa \in \Gamma.$$

On the other hand, the elementary symmetric functions $\sigma^{(1)}, \cdots, \sigma^{(n)}$ are naturally defined on R^{n+1}. Set

$$\tilde{\Gamma} = \{\lambda \in R^{n+1} : (\sigma^{(1)}(\lambda), \cdots, \sigma^{(n)}(\lambda)) \in R^n \quad \text{is in the domain of } h\}.$$

Then $\tilde{\Gamma}$ is a symmetric open subset of R^{n+1}. Now define a smooth symmetric function $\tilde{f}$ on $\tilde{\Gamma}$ by

$$\tilde{f}(\lambda) = h(\sigma^{(1)}(\lambda), \cdots, \sigma^{(n)}(\lambda)), \quad \text{for } \lambda \in \tilde{\Gamma}.$$

We note that $\Gamma \times \{0\} \subset \tilde{\Gamma}$, and

$$\tilde{f}(\kappa, 0) = f(\kappa), \quad \text{for } \kappa \in \Gamma.$$

It is obvious that at a point $(\kappa, 0) \in \Gamma \times \{0\}$

$$\tilde{f}_i(\kappa, 0) = f_i(\kappa) \quad \text{for } 1 \le i \le n.$$

Let G denote the function on the linear space of real $(n+1) \times (n+1)$ symmetric matrices given by

$$G(A) = \tilde{f}(\lambda_1, \cdots, \lambda_{n+1}),$$

where $\lambda_1, \cdots, \lambda_{n+1}$ are the eigenvalues of the symmetric matrix A. Equation (1.1) may then be written in the form

$$G(\delta\nu) = \frac{1}{w}\psi. \tag{2.9}$$

To proceed further we note that if $\tilde{e}_1, \cdots, \tilde{e}_{n+1}$ form another orthonormal coordinate frame of R^{n+1} and if we set

$$\tilde{\nu}^i = \nu \cdot \tilde{e}_i, \quad \tilde{\delta}_i = \tilde{e}_i \cdot \delta, \quad 1 \le i \le n+1,$$

then

$$\nu^i = (e_i \cdot \tilde{e}_j)\tilde{\nu}^j, \; \delta_i = (e_i \cdot \tilde{e}_j)\tilde{\delta}_j,$$
$$\delta_i \nu^j = (e_i \cdot \tilde{e}_k)(e_j \cdot \tilde{e}_l)\tilde{\delta}_k \tilde{\nu}^l,$$

and

$$G^{ij}([\delta_k \nu^l]) = G^{ms}([\tilde{\delta}_k \tilde{\nu}^l])(e_i \cdot \tilde{e}_m)(e_j \cdot \tilde{e}_s), \tag{2.10}$$

where

$$G^{ij}(A) = \frac{\partial G}{\partial A_{ij}}(A), \quad A = [A_{ij}].$$

In particular, if we choose at a fixed point on the graph S, $\tilde{e}_{n+1} = -\nu$ (which implies $\tilde{\delta}_{n+1} = 0$) and $\tilde{e}_1, \cdots, \tilde{e}_n$ such that the matrix $[\tilde{\delta}_k \tilde{\nu}^l] = [\kappa_1, \cdots, \kappa_n, 0]$ is diagonal, then as pointed out in [3],

(2.11) $\qquad G^{ij}([\tilde{\delta}_k \tilde{\nu}^l])$ is diagonal and $= f_i$ when $1 \le i = j \le n$.

We also note that

$$\tilde{\nu}^i = 0, \quad 1 \le i \le n, \quad \tilde{\nu}^{n+1} = -1;$$
$$\nu^i = -e_i \cdot \tilde{e}_{n+1}, \quad 1 \le i \le n+1.$$

It is now not hard to see that from the above follow the formulae

$$G^{ij}\nu^j \delta_i = 0 \tag{2.12}$$

$$G^{ij}(\delta_{ij} - \nu^i\nu^j) = \sum f_i \tag{2.13}$$

$$G^{ij}\delta_i\nu^j = \sum f_i\kappa_i \tag{2.14}$$

$$G^{ij}\delta_i\nu^k\delta_j\nu^k = \sum f_i\kappa_i^2. \tag{2.15}$$

We also record for later reference the formulae

$$\frac{1}{w}G^{i\,n+1}\nu^i + G^{n+1\,n+1} = \sum f_i(e_{n+1}\cdot\tilde{e}_i)^2 \geq 0; \tag{2.16}$$

and

$$G^{i\,n+1}\delta_i|x|^2 = \sum f_i(e_{n+1}\cdot\tilde{e}_i)(e_\alpha\cdot e_i)x_\alpha \leq \frac{1}{w}\sum f_i, \tag{2.17}$$

which also follow easily.

Now differentiate (2.9) on S with respect to δ, using the commutator formula [10]

$$\delta_i\delta_j - \delta_j\delta_i = (\nu^i\delta_j\nu - \nu^j\delta_i\nu)\cdot\delta,$$

to obtain

$$G^{ij}\delta_i\delta_j\nu + \nu G^{ij}\delta_i\nu^k\delta_j\nu^k = \frac{1}{w}\delta\psi - \frac{1}{w^2}\psi\delta w. \tag{2.18}$$

Here we have used the formula (2.12).

Noting that $\nu^{n+1} = -1/w$, the (n+1)-component of (2.18) yields

$$G^{ij}\delta_i\delta_j w - \frac{2}{w}G^{ij}\delta_i w\delta_j w - w\sum f_i\kappa_i^2 = w\delta_{n+1}\psi - \psi\delta_{n+1}w. \tag{2.19}$$

We compute

$$\delta_{n+1}\psi = \frac{1}{w}\nu^\alpha\psi_{x_\alpha} + (1 - \frac{1}{w^2})\psi_u + \frac{1}{w^2}\psi_{\nu^i}\delta_i w + \frac{1}{w}\psi_\tau w_t. \tag{2.20}$$

Set

$$L = G^{ij}\delta_i\delta_j - \frac{1}{w}\psi_{\nu^i}\delta_i + \psi\delta_{n+1}$$

and

$$\widetilde{L} = L - \frac{2}{w}G^{ij}\delta_j w\delta_i.$$

From (2.19), (2.20) and (1.6) we derive the important formula

$$\widetilde{L}w - \psi_\tau w_t \geq -2Mw. \tag{2.21}$$

3 The proof of Theorem 1.1

Following Korevaar [8] we define

$$\eta(x,t) \equiv g(\phi(x,t)); \quad g(\phi) = e^{K\phi} - 1, \tag{3.22}$$

with the constant $K > 0$ to be determined and, as in [4]

$$\phi(x,t) = \left(\frac{u(x,t)}{2u_0} + \frac{t}{T}(1 - |x|^2)\right)^+ .$$

Let $v \equiv w\eta$. Then, where $\phi > 0$:

$$\widetilde{L}v - \psi_\tau v_t = w(L\eta - \psi_\tau \eta_t) + \eta(\widetilde{L}w - \psi_\tau w_t).$$

Thus from (2.21),

$$\widetilde{L}v - \psi_\tau v_t \geq w(L\eta - \psi_\tau \eta_t - 2M\eta). \tag{3.23}$$

We have

$$L\eta = g' L\phi + g'' G^{ij} \delta_i \phi \delta_j \phi, \tag{3.24}$$

whereas

$$L\phi = \frac{1}{2u_0} Lu - \frac{t}{T} L|x|^2.$$

We compute

$$\delta_i |x|^2 = \begin{cases} 2x_i - 2\nu^i \nu^\alpha x_\alpha, & 1 \leq i \leq n \\ -2\nu^{n+1} \nu^\alpha x_\alpha, & i = n+1, \end{cases}$$

and, for $1 \leq i \leq n+1$

$$\delta_i \delta_j |x|^2 = \begin{cases} 2(\delta_{ij} - \nu^i \nu^j) - 2\nu^\alpha x_\alpha \delta_i \nu^j - 2\nu^j \delta_i(\nu^\alpha x_\alpha), & 1 \leq j \leq n, \\ -2\nu^\alpha x_\alpha \delta_i \nu^{n+1} - 2\nu^{n+1} \delta_i(\nu^\alpha x_\alpha), & j = n+1. \end{cases}$$

We also note that

$$\delta_i u = \frac{1}{w} \nu^i, \quad 1 \leq i \leq n, \quad \delta_{n+1} u = 1 + \frac{1}{w} \nu^{n+1};$$

$$\delta_i\delta_j u = \frac{1}{w}\delta_i\nu^j + \nu^j\delta_i(\frac{1}{w}).$$

It follows, by means of (2.12)-(2.14), that

$$Lu = \frac{1}{w}\sum f_i\kappa_i + (1-\frac{1}{w^2})\psi - \frac{1}{w^2}\psi_{\nu^i}\nu^i - \frac{1}{w}\psi_{\nu^{n+1}};$$

$$\begin{aligned} L|x|^2 &= 2\sum f_i - 2\nu^\alpha x_\alpha(\sum f_i\kappa_i - \frac{1}{w}\psi) \\ &\quad -2\,(\frac{1}{w}G^{i\,n+1}\nu^i + G^{n+1\,n+1}) - \frac{2}{w}x_\alpha(\psi_{\nu^\alpha} - \nu^\alpha\psi_{\nu^i}\nu^i). \end{aligned}$$

Thus, utilizing (1.5), (1.6) and (2.16) one obtains

$$L\phi \geq \frac{1}{2u_0}\psi - 4\,(\sum f_i + M + \frac{M}{4u_0}).$$

Now plug this into (3.24) and use (1.6) and (1.7) to obtain

$$L\eta - \psi_\tau\eta_t \geq g''G^{ij}\delta_i\phi\delta_j\phi - 4g'(\sum f_i + C_0), \tag{3.25}$$

with $C_0 = (1 + \frac{1}{2u_0} + \frac{1}{T})M$.

Concerning the first term of the righthand side of (3.25), one calculates

$$\begin{aligned} G^{ij}\delta_i\phi\delta_j\phi &= \frac{t^2}{T^2}G^{ij}\delta_i|x|^2\delta_j|x|^2 - \frac{t}{u_0T}G^{i\,n+1}\delta_i|x|^2 \\ &\quad + \frac{1}{4u_0^2}(\frac{1}{w}G^{i\,n+1}\nu^i + G^{n+1\,n+1}). \end{aligned}$$

It follows therefore from (2.16) and (2.17) that

$$G^{ij}\delta_i\phi\delta_j\phi \geq \frac{1}{4u_0^2}\sum f_i(e_{n+1}\cdot\tilde{e}_i)^2 - \frac{2}{u_0w}\sum f_i. \tag{3.26}$$

On the other hand, the non-negative function v vanishes on the parabolic boundary

$$\partial B_1 \times [0, 2T) \bigcup \overline{B}_1 \times \{t = 0\}.$$

Thus it achieves its maximum value at some point $(x_0, t_0) \in B_1 \times (0, 2T]$. Now at the point $Q = (x_0, u(x_0, t_0)) \in S = S_{t_0}$:

$$\delta_i v \equiv w\delta_i \eta + \eta \delta_i w = 0,$$

and hence, if $\tilde{e}_1, \cdots, \tilde{e}_{n+1}$ are chosen at point Q as before,

$$g'\delta_i \phi = -v\delta_i \nu^{n+1} = -v\sum_{j=1}^{n}(e_i \cdot \tilde{e}_j)(e_{n+1} \cdot \tilde{e}_j)\kappa_j. \tag{3.27}$$

In view of (2.10) and (2.11), one thus obtains

$$G^{ij}\delta_i\phi\delta_j\phi = \frac{v^2}{(g')^2}\sum f_i \kappa_i^2 (e_{n+1} \cdot \tilde{e}_i)^2. \tag{3.28}$$

We show next, using condition (1.4), that (3.26) and (3.28) imply at point Q

$$G^{ij}\delta_i\phi\delta_j\phi \geq \frac{1}{32nu_0^2}(c_0 + c_1 \sum f_i) \tag{3.29}$$

provided that

$$w \geq 2 + 16u_0 \max\left\{1, \frac{4n}{c_1}\right\} \equiv C_1. \tag{3.30}$$

First of all, we observe that since

$$\begin{aligned} \delta_{n+1}\phi &= \frac{1}{2u_0}(1 - \frac{1}{w^2}) - \frac{2t}{Tw}\nu^\alpha x_\alpha \\ &\geq \frac{1}{2u_0}(1 - \frac{1}{w^2}) - \frac{4}{w}(1 - \frac{1}{w^2})^{\frac{1}{2}} \\ &\geq \frac{1}{4u_0}(1 - \frac{1}{w^2}) \end{aligned}$$

when $w \geq 1 + 16u_0$, the (n+1)-component of (3.27) gives

$$-v\sum_{j=1}^{n}(e_{n+1} \cdot \tilde{e}_j)^2\kappa_j = g'\delta_{n+1}\phi \geq \frac{g'}{4u_0}(1 - \frac{1}{w^2}). \tag{3.31}$$

But

$$\sum_{j=1}^{n}(e_{n+1} \cdot \tilde{e}_j)^2 = 1 - (e_{n+1} \cdot \tilde{e}_{n+1})^2 = 1 - \frac{1}{w^2},$$

so it follows from (3.31) that, at the point $Q \in S$

$$\min_{1\le i\le n} \kappa_i \le -\frac{g'}{4u_0 v} \equiv -\mu < 0. \tag{3.32}$$

Now assume first there exists some j such that

$$\kappa_j \le -\mu \ \text{ and } \ \kappa_j(e_{n+1}\cdot \tilde{e}_j)^2 \le -\frac{\mu}{2n}. \tag{3.33}$$

Then (3.29) follows directly from (3.28) and (1.4). Secondly, if (3.33) does not hold, we claim there exists some j such that

$$\kappa_j < 0 \ \text{ and } \ (e_{n+1}\cdot \tilde{e}_j)^2 \ge \frac{1}{4n}. \tag{3.34}$$

For otherwise we would have

$$\begin{aligned}\sum_{\kappa_i<0} \kappa_i(e_{n+1}\cdot\tilde{e}_i)^2 &= \sum_{\kappa_i\le-\mu} \kappa_i(e_{n+1}\cdot\tilde{e}_i)^2 + \sum_{-\mu<\kappa_i<0} \kappa_i(e_{n+1}\cdot\tilde{e}_i)^2 \\ &> -\frac{\mu}{2} - \frac{\mu}{4} = -\frac{3\mu}{4}.\end{aligned}$$

But this contradicts (3.31) when $w \ge 2$ and so (3.34) must hold. Now we can use (1.4), (3.26) and (3.30) to obtain (3.29).

We can now employ the maximum principle for second order parabolic equations to conclude that

$$w \le C_1 \ \text{ at } \ (x_0, t_0). \tag{3.35}$$

For when the constant K in (3.22) is chosen sufficiently large, say when

$$K > 128 n u_0^2 \cdot \max\left\{\frac{1}{c_0}, \frac{C_0+M}{c_1}\right\} \tag{3.36}$$

(C_0 as in (3.25)) it follows from (3.23), (3.25) and (3.29) that

$$\widetilde{L}v - \psi_\tau v_t > 0 \tag{3.37}$$

holds in some neighborhood of (x_0, t_0), contradicting the maximum principle. Finally, from $v(0,T) \le v(x_0,t_0)$ follows the *a priori* bound:

$$|Du(0,T)| \le C_1 \frac{\eta(x_0,t_0)}{\eta(0,T)} \le C_1 e^{2K} \equiv C. \tag{3.38}$$

The proof of Theorem 1.1 is complete.

We conclude with some remarks.

Remarks.1. From (3.36) we see the *a priori* bound in Theorem 1.1 is of the form $C = O(e^{Au_0^2 T^{-1}})$ for some constant A independent of u_0 and T. Compare with that of [8].

2. Without essential change of the proof, one can obtain local estimates for the gradiet independent of T near $\{t = 0\}$ provided one knows an estimate for the gradient on $\{t = 0\}$. Moreover, the condition $\psi_\tau \le M$ in (1.6) can be released in this case.

3. It is clear from the proof that the condition (1.5) can be replaced by

$$-M \le f(\kappa) - \sum f_i \kappa_i \le M \tag{3.39}$$

with respect to the admissible solution u, i.e. it holds as κ varies on the subset of Γ consisting of the n-tuple of principal curvatures of the (spatial) graph of u.

The first inequality in (3.39) is trivially implied by the assumptions that

$$f \text{ is concave} \tag{3.40}$$

and that

$$\liminf_{\substack{\kappa \to 0 \\ \kappa \in \Gamma}} f(\kappa) \ge -M. \tag{3.41}$$

The second inequality in (3.39) obviously holds if the prescribed function ψ is bounded above and $\sum f_i \kappa_i$ is bounded below, which is in turn implied by the concavity of f and a condition in [1]. Compare with the relevant conditions of [8] and those of [9].

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