# SEPARATION OF VARIABLES FOR THE DIRAC EQUATION IN KERR NEWMAN SPACE TIME

By

**E.G. Kalnins**

and

**W. Miller, Jr.**

**IMA Preprint Series # 794**

May 1991

# SEPARATION OF VARIABLES FOR THE DIRAC EQUATION IN KERR NEWMAN SPACE TIME

E.G. KALNINS* AND W. MILLER, JR.†

**Abstract.** The Dirac equation is solved for an electron in a Kerr Newman geometry using an adaptation of the procedure of Chandrasekhar. The corresponding eigenfunctions obtained can then be represented as series of Jacobi polynomials. The spectrum of eigenvalues can be calculated using continued fraction techniques. Representations for the eigenvalues and eigenfunctions are obtained for various ranges of the parameters appearing in the Kerr Newman metric. Some comments concerning the bag model of nucleons are made.

**Introduction.** The Kerr Newman [1] space time represents the external gravitational field of a charged rotating black hole. The Kerr space time [2] has the remarkable property that many of the equations of mathematical physics are solvable by means of a separation of variables type ansatz [3]. This property allows one to study linear gravitational perturbations, for example, in the neighborhood of a Kerr space time solution. Solutions of Maxwell's equations and the Dirac equation can also be obtained by this method. The separable functions that arise have been studied by a number of authors [4],[5],[6],[7]. In this article we demonstrate that the separability of the Dirac equation can also be achieved in the Kerr Newman space time background [8]. Reducing the problem to two pairs of coupled ordinary differential equations, we develop methods for solving them. In particular we compute the spectrum of the separation parameter for small and large values of the parameter $a$. Using symmetry properties of the equations in the angular variable $\theta$ we reduce the problem to one involving a three term recurrence relation. This enables the spectrum to be computed in terms of continued fractions. In addition to developing properties of the $\theta$ dependent separation functions we derive a three term matrix recurrence relation for the separated $r$ dependent equations. Representations of these solutions for large $a$ and $r$ are then developed. For the particular case of flat space, expansion theorems are given for the $r$ dependent functions. Finally we comment on the applicability of these functions to bag models of nucleons.

**1. Separation of the Dirac equation in a Kerr Newman space time.** We use consistently the spinor notation of Penrose and Rindler [9] and, in particular, the null tetrad formalism. The Kerr Newman solution of Einstein's equations has the line element

$$(1.1)\qquad \begin{aligned} ds^2 =&(1-\frac{2Mr}{\rho^2})dt^2-\rho^2(\frac{dr^2}{\Delta}+d\theta^2)\\ &-((r^2+a^2)+\frac{2a^2Mr\sin^2\theta}{\rho^2})\sin^2\theta\ d\phi^2+\frac{4aMr\sin^2\theta}{\rho^2}\ dtd\phi \end{aligned}$$

*Department of Mathematics and Statistics, University of Waikato, Hamilton, New Zealand

†Institute for Mathematics and its Applications and School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, U.S.A. The work of this author was supported in part by the National Science Foundation under grant DMS 88-23054.

where $\Delta = r^2 + a^2 + e^2 - 2Mr$, $\rho^2 = r^2 + a^2\cos^2\theta$ and $\overline{\rho} = r + ia\cos\theta$. The electromagnetic field (vector potential) due to the charge of this solution is $(A_t, A_r, A_\theta, A_\phi) = (-er/\rho^2, 0, 0, era\sin^2\theta/\rho^2)$. Specifically we adopt the Kinnersley null tetrad [3] of vectors with components

$$
\begin{aligned}
l^i &= \frac{1}{\sqrt{2}\Delta}(r^2 + a^2, \Delta, 0, a), \\
n^i &= \frac{1}{\sqrt{2}\rho^2}(r^2 + a^2, -\Delta, 0, a), \\
m^i &= \frac{1}{\sqrt{2}\overline{\rho}}(ia\sin\theta, 0, 1, i\cos\sec\theta), \\
\overline{m}^i &= \frac{1}{\sqrt{2}\overline{\rho}^*}(-ia\sin\theta, 0, 1, -i\cos\sec\theta).
\end{aligned} \tag{1.2}
$$

In tetrad components the vector potential is

$$
\begin{aligned}
A_{00'} &= -\frac{er}{\sqrt{2}\Delta}, \\
A_{11'} &= -\frac{er}{\sqrt{2}\rho^2}, \\
A_{01'} &= A_{10'} = 0.
\end{aligned} \tag{1.3}
$$

The Dirac equation for spin 1/2 particles in an electromagnetic field is, in spinor form,

$$
\begin{aligned}
(\nabla^B{}_{B'} - ieA^B{}_{B'})\varphi_B &= \frac{im_e}{\sqrt{2}}\chi_{B'}, \\
(\nabla_B{}^{B'} - ieA_B{}^{B'})\chi_{B'} &= -\frac{im_e}{\sqrt{2}}\varphi_B
\end{aligned} \tag{1.4}
$$

In terms of the modified field components $\varphi_0 = \phi_0 e^{i(m\varphi+\sigma t)}$, $\overline{\rho}^*\varphi_1 = \phi_1 e^{i(m\varphi+\sigma t)}$, $\chi_{0'} = X_0\ e^{i(m\varphi+\sigma t))}$, $\overline{\rho}\chi_{1'} = X_1 e^{i(m\varphi+\sigma t)}$, these equations assume the form

$$
\begin{aligned}
-L_{1/2}\phi_0 + (D_0 - \frac{ier}{\sqrt{2}\Delta})\phi_1 &= -im_e(r - ia\cos\theta)X_0 \\
(\Delta D^\dagger_{1/2} + \frac{ier}{\sqrt{2}})\phi_0 + L^\dagger_{1/2}\phi_1 &= -im_e(r - ia\cos\theta)X_1 \\
-L^\dagger_{1/2}X_0 + (D_0 - \frac{ier}{\sqrt{2}\Delta})X_1 &= im_e(r + ia\cos\theta)\phi_0 \\
(\Delta D^\dagger_{1/2} + \frac{ier}{\sqrt{2}})X_0 + L_{1/2}X_1 &= im_e(r + ia\cos\theta)\phi_1.
\end{aligned} \tag{1.5}
$$

These equations can be solved by the usual ansatz

$$
\begin{aligned}
&\phi_0 = R_{1/2}S_{1/2},\ \phi_1 = R_{-1/2}S_{-1/2}, \\
&X_0 = -R_{1/2}S_{-1/2},\ X_1 = R_{-1/2}S_{-1/2}
\end{aligned} \tag{1.6}
$$

to give the coupled equations

$$
(1.7)\qquad
\begin{aligned}
&L_{1/2}S_{1/2} = (\lambda - am_e\cos\theta)S_{-1/2},\\
&L^\dagger_{1/2}S_{-1/2} = -(\lambda + am_e\cos\theta)S_{1/2},\\
&(D_0 - \frac{ier}{\sqrt{2}\Delta})R_{-1/2} = (\lambda + im_e r)R_{1/2},\\
&(\Delta D^\dagger_{1/2} + \frac{ier}{\sqrt{2}})R_{1/2} = (\lambda - im_e r)R_{-1/2}.
\end{aligned}
$$

where

$$
(1.8)\qquad
\begin{aligned}
L_{1/2} &= \frac{\partial}{\partial\theta} + Q + \frac{1}{2}\cot\theta,\\
L^\dagger_{1/2} &= \frac{\partial}{\partial\theta} - Q + \frac{1}{2}\cot\theta,\\
D_0 &= \frac{\partial}{\partial r} + i\frac{K}{\Delta},\\
D^\dagger_{1/2} &= \frac{\partial}{\partial r} - \frac{iK}{\Delta} + \frac{(r-M)}{\Delta},
\end{aligned}
$$

with

$$
Q = a\sigma\sin\theta + m\csc\theta,\ K = (r^2+a^2)\sigma + am.
$$

The first two of these equations are the same coupled equations that are obtained in the case of Kerr space time (and even flat space $M = 0$). The last two equations are generalizations of the $r$ dependent equations obtained for the electron. The main problem is to determine the eigenvalues $\lambda$. Looking for series solutions of the form

$$
(1.9)\qquad
\begin{aligned}
S_{1/2} &= \sum_{r=N}^{\infty} a_r u^r_{m,-1/2},\\
S_{-1/2} &= \sum_{r=N}^{\infty} b_r u^r_{m,1/2}
\end{aligned}
$$

where $u^j_{m,n}$, [10], are the matrix elements of the rotation group and $N = \min\{|m|, 1/2\}$

we obtain the recurrence formulas

$$\left[-\lambda + \frac{amm_e}{(2r-1)r}\right] b_r + am_e \left[\frac{\sqrt{(r^2-m^2)(r^2-\frac{1}{4})}}{(2r-1)r} b_{r-1} + \frac{\sqrt{((r+1)^2-m^2)(r^2-\frac{1}{4})}}{(2r+3)(r+1)} b_{r+1}\right] = -i\left[(r+\frac{1}{2}) - \frac{ia\sigma(r+\frac{1}{2})}{r(r+1)}\right] a_r + ia\sigma\left[\frac{\sqrt{(r^2-m^2)(r^2-\frac{1}{4})}}{r(2r-1)} a_{r-1} + \frac{\sqrt{((r+1)^2-\frac{1}{4})((r+1)^2-m^2)}}{(r+1)(2r+3)} a_{r+1}\right], \tag{1.10}$$

$$\left[\lambda - \frac{amm_e}{2r(r+1)}\right] a_r + am_e \left[\frac{\sqrt{(r^2-m^2)(r^2-\frac{1}{4})}}{r(2r-1)} a_{r-1} + \frac{\sqrt{((r+1)^2-m^2)((r+1)^2-\frac{1}{4})}}{(r+1)(2r+3)} a_{r+1}\right] = \left[-i(r+\frac{1}{2}) - \frac{iam(r+\frac{1}{2})}{r(r+1)}\right] b_r - ia\sigma\left[\frac{\sqrt{(r^2-\frac{1}{4})(r^2-m^2)}}{r(2r-1)} b_{r-1} - \frac{\sqrt{((r+1)^2-\frac{1}{4})((r+1)^2-m^2)}}{(r+1)(2r+3)} b_{r+1}\right]. \tag{1.11}$$

These two relations can be written in the form

$$\alpha_r C_{r-1} + \beta_r C_r + \gamma C_{r+1} = 0, \tag{1.12}$$

$$\text{where} \quad C_r = \begin{bmatrix} a_r \\ b_r \end{bmatrix}.$$

In order to compute the spectrum of $\lambda$ we could suitably redefine the vector $C_r = \Omega_r D_r$ such that the three term vector recurrence relation has the form $-\delta_r D_{r-1} + D_r + \varepsilon_r D_{r+1} = 0$, $D_0 + \varepsilon_0 D_1 = 0$. The corresponding spectrum for $\lambda$ can then be calculated from the determinant of the matrix infinite continued fraction:

$$\det(I + \varepsilon_0(I + \varepsilon_1(I + \varepsilon_2(I + \cdots \delta_2)^{-1}\delta_1)^{-1})\delta_0) = 0. \tag{1.13}$$

To obtain a three term recurrence relation we observe that the equations admit the discrete symmetry obtained by the transformation

$$(1.14)\qquad \begin{aligned} &P : \theta \to \pi - \theta. \\ &PS_{1/2}(\theta) = S_{1/2}(\pi - \theta) = \varepsilon S_{-1/2}(\theta) \\ &PS_{-1/2}(\theta) = S_{-1/2}(\pi - \theta) = \varepsilon S_{1/2}(\theta). \end{aligned}$$

Therefore we can impose the symmetry requirements that

$$(1.15)\qquad \begin{aligned} S_{1/2}(\pi - \theta) &= \varepsilon S_{-1/2}(\theta), \\ S_{-1/2}(\pi - \theta) &= \varepsilon S_{1/2}(\theta), \end{aligned}$$

where $\varepsilon = \pm 1$.

Using the relation $u^j_{m,-1/2}(-\cos\theta) = (-1)^{j-m} i u^j_{m,+1/2}(\cos\theta)$ and requiring that our solutions be eigenfunctions of $P$ with eigenvalue $\varepsilon$, we see that $a_r = i\varepsilon b_r(-1)^{r+1-m}$. Consequently the three term matrix recurrence relations satisfied by the vector $\begin{bmatrix} a_r \\ b_r \end{bmatrix}$ become a single three term recurrence relation

$$(1.16)\qquad a(\varepsilon_r\sigma - m_e)\frac{h(r,m)}{r(2r-1)}\, b_{r-1} - a(\varepsilon_r\sigma + m_e)\frac{h(r+1,m)}{(r+1)(2r+3)}\, b_{r+1} + \left[\lambda - \frac{amm_e}{2r(r+1)} + \varepsilon_r\left(r + \frac{1}{2} + a\sigma m\frac{r+\frac{1}{2}}{r(r+1)}\right)\right]\, b_r = 0,$$

where $h(j,m) = \sqrt{(r^2 - m^2)(r^2 - 1/4)}$ and $\varepsilon_r = (-1)^r\varepsilon$.

The expressions for the coefficients $b_r$ can be calculated iteratively using the recurrence formula, expressing the eigenvalue $\lambda$ in the form $\lambda = \sum_{r=0}^{\infty} \lambda_r a^r$ and using the expansion $\frac{b_{j\pm r}}{b_j} = \sum_{k=r}^{\infty} {}_jA^{\pm r} a^k$ , $j = 0, 1, 2, 3, \ldots$ . The first few terms in these series are

$$(1.17)\qquad \lambda_0 = -\frac{\varepsilon(2j+1)}{2}$$

$$\lambda_1 = -\frac{m(\varepsilon\sigma(2j+1) - m_e)}{2j(j+1)}$$

$$\lambda_2 = -\frac{2\varepsilon}{(2j+1)^2}\left[\frac{(\varepsilon\sigma - m_e)^2 h^2(j,m)}{j^2(2j-1)} + \frac{(\varepsilon\sigma + m_e)^2 h(j+1,m)^2}{(j+1)^2(2j+3)}\right]$$

$$\lambda_3 = -\frac{4(\varepsilon\sigma - m_e)^2 h^2(j,m)}{j^2(2j-1)(2j+1)^3}\left[\frac{m(-m_e + \sigma\varepsilon)}{(j-1)j(j+1)} - \frac{2\varepsilon\sigma m j}{(j-1)(j+1)}\right] - \frac{4(\varepsilon\sigma + m_e)^2 h^2(j+1,m)}{(j+1)^2(2j+3)(2j+1)^3}\left[-\frac{m(m_e + \sigma\varepsilon)}{j(j+1)(j+2)} + \frac{2\varepsilon\sigma m(j+1)}{j(j+2)}\right]$$

for the expansion of the eigenvalue $\lambda$.

$$(1.18)\qquad {}_jA_1^1 = -\frac{2\varepsilon(m_e - \sigma\varepsilon)h(j+1,m)}{(2j+1)^2(j+1)}$$

$${}_jA_1^{-1} = \frac{h(j,m)(\varepsilon\sigma - m_e)}{2\varepsilon j(2j+1)}$$

$${}_jA_2^2 = \frac{h(j+1,m)h(j+2,m)(\sigma^2 - m_e^2)}{(j+1)(j+2)(2j+3)(2j+1)^2}$$

$${}_jA_2^{-2} = \frac{h(j-1,m)h(j,m)(\sigma^2 - m_e^2)}{j(j-1)(2j-1)(2j+1)^2}$$

$${}_jA_2^1 = \frac{4(\varepsilon\sigma + m_e)h(j+1,m)}{(j+1)(2j+1)^3}\left[\frac{2\varepsilon\sigma m(j+1)}{j(j+2)} - \frac{m(m_e + \sigma\varepsilon)}{j(j+1)(j+2)}\right]$$

$${}_jA_2^{-1} = \frac{4(\varepsilon\sigma - m_e)h(j,m)}{(j+1)(2j+1)^3}\left[\frac{m(-m_e + \varepsilon\sigma)}{(j-1)j(j+1)} - \frac{2\varepsilon\sigma mj}{(j-1)(j+1)}\right].$$

where $\varepsilon = (-1)^{j-m+1}$.

To justify the preceding perturbation computation we note that the eigenvalue equation can be written in the form $LS = \lambda S,\ L = L_0 + aV$ where

$$(1.19)\qquad L_0 = \begin{bmatrix} 0 & -\frac{\partial}{\partial\theta} - \frac{m}{\sin\theta} - \frac{\cot\theta}{2} \\ \frac{\partial}{\partial\theta} - \frac{m}{\sin\theta} + \frac{\cot\theta}{2} & 0 \end{bmatrix}$$

$$(1.20)\qquad V = \begin{bmatrix} -m_e\cos\theta & -\sigma\sin\theta \\ -\sigma\sin\theta & +m_e\cos\theta \end{bmatrix}$$

$$S = \begin{bmatrix} S_{-1/2} \\ S_{+1/2} \end{bmatrix} \qquad \sigma^2 - m_e^2 > 0.$$

The boundary conditions are [1.15]. The inner product is

$$(1.21)\qquad (T,S) = \int_0^\pi (T_{1/2}^* S_{1/2} + T_{-1/2}^* S_{-1/2})\sin\theta\ d\theta.$$

We restrict the argument that follows to the case when $\varepsilon = (-1)^{j-m}$; the case when $\varepsilon = (-1)^{j-m+1}$ can be treated similarly. It is easy to verify that $L$ is formally self-adjoint with respect to the inner product. Moreover, the operator $L_0$ on this space is self adjoint with discrete spectrum

$$(1.22)\qquad L_0 S_j^0 = \mu_j S_j^0, \quad \mu_j = -j - \frac{1}{2}; \quad j = 0,1,2,\dots\ .$$

Thus the eigenvalue decomposition for the resolvent operator $(L_0 - \lambda I)^{-1}$ takes the form

$$(L_0 - \lambda I)^{-1} S_j^0 = \frac{-1}{j + \lambda + \frac{1}{2}} S_j^0, \quad j = 0, 1, 2, \dots . \tag{1.23}$$

It follows that $(L_0 - \lambda I)^{-1}$ is a compact self adjoint operator for real $\lambda$ not in the spectrum of $L_0$. The perturbing operator $V$ is bounded and self adjoint with operator norm $\|V\| = \sigma$. From the identity

$$(L - \lambda I)^{-1} = (L_0 - \lambda I)^{-1} - a(L_0 - \lambda I)^{-1} V (L - \lambda I)^{-1} . \tag{1.24}$$

and the facts that 1) the product of a compact operator and a bounded operator is compact, and 2) the sum of two compact operators is compact, it follows that the resolvent operator $(L - \lambda I)^{-1}$ is self-adjoint and compact for real $\lambda$ not in the spectrum. Thus $L$ can be defined uniquely as a self-adjoint operator with discrete spectrum [11]. It is easy to check that the spectrum is of multiplicity one. It follows from Chapter VII of [11] that $L = L_0 + aV$ is a so called "self-adjoint holomorphic family of type (A)" defined for all real $a$. The radius of convergence for each power series expansion of the perturbed eigenvalues and eigenfunctions about $a = 0$ is at least $1/(2\sigma)$ in the complex $a$ plane. Moreover, the eigenfunctions found from the perturbation process form a Hilbert space basis for all real $a$.

For large $a$ the asymptotic form of the eigenvalues and eigenfunctions can be computed. In order to do this we look for solutions of the form

$$\begin{aligned} S_{\pm 1/2} &= e^{a\phi(\cos\theta)} \psi_{\pm 1/2}(\theta)\left(1 + \frac{f_1^{\pm}(\theta)}{a} + \frac{f_2^{\pm}(\theta)}{a^2} + \cdots\right), \\ \lambda &= a\mu + \lambda_0 + \frac{\lambda_1}{a} + \frac{\lambda_2}{a^2} + \cdots . \end{aligned} \tag{1.25}$$

Equating powers of $a$ we find the the leading order condition

$$\phi'^2 + \mu^2 - \sigma^2 + (\sigma^2 - m_e^2)\cos^2\theta = 0. \tag{1.26}$$

In order to obtain a single-valued expression for $\phi$ we choose $\mu = m_e$ for which $\phi = \pm\sqrt{\sigma^2 - m_e^2}\cos\theta = \beta\cos\theta$, and the condition

$$m_e(1 + \cos\theta)\psi_{1/2} = (\beta - \sigma)\sin\theta\,\psi_{-1/2}. \tag{1.27}$$

Further conditions can be solved to give

$$\begin{aligned} \psi_{-1/2} &= m_e\left(\sin\frac{\theta}{2}\right)^{-n-1}\left(\cos\frac{\theta}{2}\right)^n, \\ \psi_{1/2} &= (\beta - \sigma)\left(\sin\frac{\theta}{2}\right)^{-n}\left(\cos\frac{\theta}{2}\right)^{n-1} \end{aligned} \tag{1.28}$$

and

$$2\lambda_0 m_e - (2m+1)\sigma = \beta n, \tag{1.29}$$

where the integer $n = j - \frac{1}{2}\left[|m - \frac{1}{2}| + |m + \frac{1}{2}|\right]$.

The functions $f_1^\pm$ are given by

$$\begin{aligned} f_1^- &= A\frac{\sin\frac{\theta}{2}}{\cos^3\frac{\theta}{2}} + B\tan\frac{\theta}{2} + C\sec^2\frac{\theta}{2} + D\csc^2\frac{\theta}{2} + E\cot\theta \\ &\quad + Fcsc\theta + G\sec\frac{\theta}{2} + H\sec\theta, \\ f_2^+ &= -f_1^-\frac{(n+m+\frac{1}{2})}{(\beta-\sigma)\sin^2\theta} - \frac{\lambda_0}{2m_e\cos^2\frac{\theta}{2}}, \end{aligned} \tag{1.30}$$

where

$$\begin{aligned} A &= -\frac{m_e n(2m-1)}{6\beta}, \quad B = -\frac{2m_e n(m+1)}{3\beta}, \\ C &= -\frac{\lambda_0(\sigma+\beta)(n+\frac{1}{2})}{8\beta m_e} - \frac{\lambda_0(\sigma-\beta)(m-1/2)}{4\beta m_e} + \frac{\lambda_0^2}{4\beta}, \quad D = \frac{n(n+2)}{16\beta} \\ E &= -\frac{m_e m(n+1)}{4\beta}, \quad F = -\frac{m_3(n+1)}{8\beta} \\ G &= \frac{(n-3)}{2\beta}\left[\frac{n}{4} + \frac{\lambda_0(\beta-\sigma)}{2m_e}\right], \quad H = -\frac{(m+\frac{1}{2})(n-3)}{8\beta}. \end{aligned}$$

The next term in the asymptotic series for the eigenvalue $\lambda$ is given by

$$\lambda_1 m_e = -\frac{\lambda_0}{4m_e}[(\frac{7}{2} + 2n + 2m)\sigma + (\frac{3}{2} - 2m)\beta] - \frac{(n^2 + \frac{n}{2} - 1 - 3mn)}{16}. \tag{1.31}$$

The higher order terms in the asymptotic series become increasingly complex. The next term is given by

$\lambda_2 m_e =$

(1.32)

$$\left[-\frac{\lambda_1(2m+2n+1)\beta}{4m_e}+\frac{\lambda_0^2(\sigma+\beta)(2m+2n+1)}{16m_e^2}+\frac{\lambda_0^2 n(\beta-\sigma)}{8m_e^2}+\frac{(3n-1)(2m+2n+1)}{32m_e}\right.$$

$$\left.-\frac{\lambda_1 n(\beta-\sigma)}{4m_e}\right]+\frac{1}{32}[2n(n+2)+(m+\frac{1}{2})(n+1)]\left[-\frac{(\beta-\sigma)(8\beta\lambda_1+(1-2n)\lambda_0)}{16m_e}-\frac{\lambda_0^2}{8}+\right.$$

$$\left.\frac{\lambda_0(\sigma+\beta)(2m+3)}{m_e}-\frac{1}{128}(6m+7-2n)(2m+1-2n)\right]+\frac{(n-3)}{2}\left[\frac{n}{4}+\frac{\lambda_0}{2m_e}(\beta-\sigma)\right]\times$$

$$\left[\frac{\lambda_0(\sigma+\beta)(2m+n+1)}{\beta m_e}+\frac{\lambda_1 m_3}{\beta}+\frac{\lambda_0 n(\beta-\sigma)}{\beta m_e}\frac{1}{64}(10m+2n+1)(2m-2n+1)\right]+$$

$$\frac{\lambda_0(\beta-\sigma)}{16}(n+1)(2m+1)-\left[\frac{\lambda_0(\sigma+\beta)(n+\frac{1}{2})}{m_e}+\frac{\lambda_0(\sigma-\beta)(m-\frac{1}{2})}{m_e}-\frac{\lambda_0^2}{4}\right.$$

$$\left.+\frac{(m+\frac{1}{2})(n+1)}{32}\right]\left[\frac{\lambda_0(\sigma+\beta)(2m+2n+1)}{m_e}+\frac{\lambda_1 m_e}{\beta}+\frac{\lambda_0 n(\beta-\sigma)}{4\beta m_e}\right.$$

$$\left.+\frac{1}{32\beta}(2n+6m-1)(2n-2m-1)\right]-\frac{(m+\frac{1}{2})(n-3)}{8}\left[\frac{\lambda_0(\sigma+\beta)(2m+2n+1)}{8\beta m_e}+\frac{\lambda_1 m_e}{\beta}\right.$$

$$\left.+\frac{\lambda_0 n(\beta-\sigma)}{4\beta m_e}-\frac{1}{8\beta}(2m-2n+1)(n+2m-1)\right].$$

To obtain the connection with the functions which are eigenfunctions of $P$ we need only take suitable combinations of the two independent functions whose asymptotic properties we have computed here. The second solution can be obtained by taking the transformation $\theta \to \pi - \theta$.

**2. The functions $R_{\pm 1/2}$ and their properties.** The coupled equations for the functions $R_{\pm 1/2}$ can be solved by methods similar to those adopted for the $\theta$ dependent functions $S_{\pm 1/2}$. Choosing new functions $V_{\pm}$ defined according to

$$V_+ = \Delta^{1/2} R_{+1/2}, \quad V_- = R_{-1/2} \tag{2.1}$$

and a new variable $A$ defined by

$$r - M = \sqrt{a^2+e^2-M^2}\sinh A \tag{2.2}$$

we can write these equations as

$$\left[\frac{\partial}{\partial A} + R\cosh A + \left(S+\frac{1}{2}\right)\tanh A + \frac{T}{\cosh A}\right] V_- = \tag{2.3}$$

$$(\lambda + im_e(M+\sqrt{a^2+e^2-M^2}\sinh A))V_+,$$

$$\left[\frac{\partial}{\partial A} - R\cosh A - \left(S-\frac{1}{2}\right)\tanh A - \frac{T}{\cosh A}\right] V_+ =$$

$$(\lambda - im_e(M+\sqrt{a^2+e^2-M^2}\sinh A))V_-,$$

where $R = i\sigma\sqrt{a^2+e^2-M^2}$, $S = 2i\sigma M - ie/\sqrt{2}$ and

$$T = \frac{i[\sigma(2M^2-e^2)+am] - \frac{ieM}{\sqrt{2}}}{\sqrt{a^2+e^2-M^2}}. \tag{2.4}$$

Solutions of these equations can be sought in the form

$$V_+ = \sum_r B_r U^{\nu+r}_{\rho\eta+\frac{1}{2}} \tag{2.5}$$

$$V_- = \sum_r A_r U^{\nu+r}_{\rho\mu+\frac{1}{2}}$$

where $\rho = iT$, $\eta = -S$.

The functions $U^\alpha_{\mu,\varepsilon}$ are analytic continuations of the matrix elements of the rotation group, viz. $U^\alpha_{\mu,\varepsilon} = u^\alpha_{\mu,\varepsilon}(i\sinh A)$. These functions satisfy the recurrence relations induced from the rotation group matrix elements. This can be seen by invoking the relations given by Gelfand et. al [10] for the rotation group in their complexified form. Typically such relations are

$$\left[\frac{\partial}{\partial A} - \frac{(i\rho+(\eta+\frac{1}{2})\sinh A)}{\cosh A}\right] U^{\nu+r}_{\rho\eta+\frac{1}{2}} = -\sqrt{(\nu+r+\frac{1}{2})^2 - \eta^2} U^{\nu+r}_{\rho\eta-\frac{1}{2}} \tag{2.6}$$

$$\left[\frac{\partial}{\partial A} + \frac{(i\rho+(\eta-\frac{1}{2})\sinh A)}{\cosh A}\right] U^{\nu+r}_{\rho\eta-\frac{1}{2}} = -\sqrt{(\nu+r+\frac{1}{2})^2 - \eta^2} U^{\nu+r}_{\rho\eta+\frac{1}{2}}$$

Consequently we obtain the recurrence relations

$$\begin{aligned}
(2.7)\qquad & \left[-\lambda - im_eM + \frac{m_e\sqrt{a^2+e^2-M^2}\,\rho(\eta-\frac{1}{2})}{(\nu+r)(\nu+r+1)}\right] A_r \\
& - m_e\sqrt{a^2+e^2-M^2}\left[\frac{\sqrt{[(\nu+r)^2-\rho^2][(\nu+r)^2-(\eta-\frac{1}{2})^2]}}{(2(\nu+r)-1)(\nu+r)} A_{r-1}\right. \\
& \left. + \frac{\sqrt{[\nu+r+1)^2-\rho^2][(\nu+r+1)^2-(\eta-\frac{1}{2})^2]}}{(\nu+r+1)(2(\nu+r)+3)} A_{r+1}\right] = \\
& - \left[\sqrt{(\nu+r+\frac{1}{2})^2-\eta^2} + iR\frac{\sqrt{(\nu+r+\frac{1}{2})^2-\eta^2\rho}}{(\nu+r+1)(\nu+r)}\right] B_r \\
& + iR\left[\frac{\sqrt{[(\nu+r-\eta)^2-\frac{1}{4}][(\nu+r)^2-\rho^2]}}{(\nu+r)(2(\nu+r)-1)} B_{r-1}\right. \\
& \left. - \frac{\sqrt{[(\nu+r+\eta+1)^2-\frac{1}{4}][(\nu+r)^2-\rho^2]}}{(\nu+r+1)(2(\nu+r)+3)} B_{r+1}\right].
\end{aligned}$$

$$\begin{aligned}
(2.8)\qquad & \left[\lambda - im_eM + \frac{m_e\sqrt{a^2+e^2-M^2}\,\rho(\eta+\frac{1}{2})}{(\nu+r)(\nu+r+1)}\right] B_r \\
& + m_e\sqrt{a^2+e^2-M^2}\left[\frac{\sqrt{[(\nu+r)^2-\rho^2][(\nu+r)^2-(\eta+\frac{1}{2})^2]}}{(\nu+r)(2(\nu+r)-1)} B_{r-1}\right. \\
& \left. + \frac{\sqrt{[(\nu+r+1)^2-\rho^2][(\nu+r+1)^2-(\eta+\frac{1}{2})^2]}}{(\nu+r+1)(2(\nu+r)+3)} B_{r+1}\right] = \\
& - \left[\sqrt{(\nu+r+\frac{1}{2})^2-\eta^2} + iR\frac{\sqrt{(\nu+r+\frac{1}{2})^2-\eta^2}}{(\nu+r)(\nu+r+1)}\right] A_r \\
& - \left[\frac{\sqrt{[\nu+r+\eta)^2-\frac{1}{4}][(\nu+r)^2-\rho^2]}}{(\nu+r)(2(\nu+r)-1)} A_{r-1}\right. \\
& \left. - \frac{\sqrt{[(\nu+r-\eta+1)^2-\frac{1}{4}][(\nu+r)^2-\rho^2]}}{(\nu+r)(2(\nu+r)+1)} A_{r+1}\right].
\end{aligned}$$

These relations are of the same type as derived for the functions $S_{\pm 1/2}$ with recurrence relations of the form

$$\zeta_r Z_{r-1} + \eta_r Z_r + \omega_r Z_{r+1} = 0, \tag{2.9}$$

where $Z_r = \begin{bmatrix} A_r \\ B_r \end{bmatrix}$.

For large $a$ representations of the solutions can be achieved as follows. The expansion of the eigenvalue $\lambda$ takes the form

$$\lambda = am_e + \lambda_0 + \frac{\lambda_1}{a} + \frac{\lambda_2}{a^2} + \frac{\lambda_3}{a^3} + \cdots . \tag{2.10}$$

For large $a$ we seek solutions

$$V_\pm = \psi_\pm e^{\beta r}\{1 + \frac{f_1^\pm(r)}{a} + \frac{f_2^\pm(r)}{a^2} + \frac{f_3^\pm(r)}{a^3} + \cdots\}. \tag{2.11}$$

In order to obtain solutions of this type the constants $\psi_\pm$ must satisfy

$$i(\beta + \sigma)\psi_- = m_e \psi_+ \tag{2.12}$$

where $\beta = \sqrt{\sigma^2 - m_e^2}$.

Without loss of generality we can assume $f_n^\pm(r) = \sum_{i=0}^{n} a_{n,i}^\pm r^i$ with $a_{n,0}^+ = 0$, $n = 0, 1, 2, \dots$ . The first few terms in this expansion are

$$a_{1,0}^- = \frac{2\lambda_0\sigma - (2m+1)m_e}{2m_e\beta} \tag{2.13}$$

$$a_{1,1}^- = i(m+1) - \frac{i\lambda_0 m_e}{\beta} + \frac{i(2m+1)m_e^2}{2(\sigma+\beta)\beta}$$

$$a_{1,1}^+ = im - \frac{4\lambda_0 m_e \sigma(\sigma - \beta) - (2m+1)m_e^2(\sigma+\beta) - 2\lambda_0 m_e^3}{\beta(2\sigma(\sigma+\beta) - m_e^2)}.$$

For the functions $V_\pm$ an asymptotic expansion with respect to $r$ can be obtained. We search for solutions of the form

$$V_\pm = \phi_\pm e^{\varphi(r)} r^\rho (1 + \frac{v_1^\pm}{r} + \frac{v_2^\pm}{r^2} + \frac{v_3^\pm}{r^3} + \cdots) \tag{2.14}$$

for suitable functions $\varphi(r)$, and constants $\phi_\pm, \rho$. An expansion of this type is possible if

$$(\beta + \sigma)\phi_- = m_e\phi_+, \tag{2.15}$$

$$\rho = -\frac{\sigma i e}{\sqrt{2}\beta} + iM\left(2\beta + \frac{m_e^2}{\beta}\right).$$

The first nonzero terms of this expansion are given by the coefficients

$$v_1^- = -M - \frac{i\lambda}{m_e}, \tag{2.16}$$

$$v_2^+ = \frac{-e}{\sqrt{2}\beta} + \frac{m_e^2 M}{\beta(\beta+\sigma)}.$$

In the case of flat space, i.e., $e = M = 0$ it is possible to expand one set of complete eigenfunctions of the Dirac equation in terms of another. For the Dirac equation this is achieved using the Majorana representation of the Dirac gamma matrices viz.

$$\gamma = \begin{bmatrix} \mathbf{0} & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{bmatrix}, \ \gamma^5 = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \ \gamma^0 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{2.17}$$

¿From our knowledge of the separation of variables in oblate coordinates (i.e., inserting the conditions $e = M = 0$) the separable solutions are characterized as eigenfunctions of the operator

$$Q = \gamma^5 \boldsymbol{\gamma} \cdot \mathbf{L} + \gamma^1 + a(\gamma^5\gamma^1 \frac{\partial}{\partial z^2} + \gamma^5\gamma^0 \frac{\partial}{\partial z^3}) \tag{2.18}$$

where $L_i = \varepsilon_{ijk} z^j \partial / \partial z^k$, $i = 1, 2, 3$.

We now look for solutions of the eigenvalue equation $Q\Psi = \lambda\Psi$ which are also solutions of the Dirac equation

$$i\gamma^\mu \frac{\partial}{\partial z^\mu} \psi = m_e \psi. \tag{2.19}$$

Here $z^\mu$, $\mu = 0, 1, 2, 3$ are cartesian coordinates in Minkowski spacetime. If a standard choice of spherical coordinates is made, i.e., $z^0 = t$, $z^1 = \omega \sin\alpha \cos\gamma$, $z^2 = \omega \sin\alpha \sin\gamma$, $z^3 = \omega\cos\alpha$, and a formal Fourier transform ${}_f\psi = \int \exp(ik^0 z^0 + i\mathbf{k}\cdot\mathbf{z})\psi dz^0 d\mathbf{z}$ taken with $k^0 = \sigma$, $\mathbf{k} = (k^1, k^2, k^3) = \sqrt{\sigma^2 - m_e^2}(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$, and $d\mathbf{x} = \sin\alpha\ d\alpha\ d\gamma$ these two conditions are equivalent to the four equations

$$\left[\frac{\partial}{\partial\theta} - \frac{im}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\cot\theta}{2} + \sigma a \sin\theta\right] {}_f\overline{\psi}_4 = -(\lambda - a m_e \cos\theta)_f \overline{\psi}_1 \tag{2.20}$$

$$\left[\frac{\partial}{\partial\theta} + \frac{im}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\cot\theta}{2} - \sigma a \sin\theta\right] {}_f\overline{\psi}_3 = (\lambda - a m_e \cos\theta)_f \overline{\psi}_2$$

$$\left[\frac{\partial}{\partial\theta} + \frac{im}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\cot\theta}{2} - \sigma a \sin\theta\right] {}_f\overline{\psi}_1 = (\lambda - a m_e \cos\theta)_f \overline{\psi}_4$$

$$\left[\frac{\partial}{\partial\theta} - \frac{im}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\cot\theta}{2} + \sigma a \sin\theta\right] {}_f\overline{\psi}_2 = -(\lambda - a m_e \cos\theta)_f \overline{\psi}_3$$

where ${}_f\psi = R_1(-\theta)R_3(-\varphi){}_f\overline{\psi}$ and $R_1$, $R_3$ are rotations about the indicated coordinate axes in three-space.

The solutions of the eigenvalue equations in the transformed space of Dirac spinors ${}_f\Psi$ are of the form

$$
\begin{aligned}
{}_f\overline{\psi}_1 &= (\beta - i\sigma)S_{-1/2}(\theta)e^{i(\sigma t + m\varphi)}, \\
{}_f\overline{\psi}_2 &= im_e S_{1/2}(\theta)e^{i(\sigma t + m\varphi)}, \\
{}_f\overline{\psi}_3 &= im_e S_{-1/2}(\theta)^{i(\sigma t + m\varphi)}, \\
{}_f\overline{\psi}_4 &= (\beta - i\sigma)S_{1/2}(\theta e^{i(\sigma t + m\varphi)}.
\end{aligned}
\tag{2.21}
$$

If the solutions are also eigenfunctions of the discrete transformation $P$ then the functions $S_{\pm 1/2}$ appearing in these expressions are just those we have already studied. The basic idea is the following: From the expressions for ${}_f\overline{\psi}$ recover the expressions for ${}_f\psi$. Using the expansion of the function $e^{i\mathbf{k}\cdot\mathbf{x}}$ in terms of spherical Bessel functions, as for instance found in [12], the form of $\psi$ is recovered. The Dirac spinors that result are eigenfunctions of $Q$ and $P$, are solutions of Dirac's equation and are represented relative to the cartesian coordinate and spin frames given in the definition above. Then transforming the spinor basis used in oblate spheroidal coordinates, we obtain expressions for solutions of the Dirac equation, in terms of series of spherical Bessel functions, which are eigenfunctions of $Q$ and $P$. The expressions for the components are given by

$$
\begin{aligned}
\psi_1 = \sum_{L=0}^{\infty}\sum_{r=0}^{\infty}\Big[&(\beta - i\sigma)b_r C(\tfrac{1}{2}, -\tfrac{1}{2}; r, m \mid L, K)C(\tfrac{1}{2}, -\tfrac{1}{2}; r, \tfrac{1}{2} \mid L, 0) - \\
&m_e a_r C(\tfrac{1}{2}, -\tfrac{1}{2}; r, m \mid L, K)C(\tfrac{1}{2}, -\tfrac{1}{2}; r, -\tfrac{1}{2} \mid L, 0)\Big] \\
&\cdot i^L j_L(\beta r)\; u^L_{m-\frac{1}{2}\;0}(\cos\alpha)\; e^{i(m-\frac{1}{2})\gamma}, \\
\psi_2 = \sum_{L=0}^{\infty}\sum_{r=0}^{\infty}\Big[&i(\beta - i\sigma)\; b_r\; C(\tfrac{1}{2}, \tfrac{1}{2}; r, m \mid L, K)C(\tfrac{1}{2}, -\tfrac{1}{2}; r, \tfrac{1}{2} \mid L, 0) + \\
&im_e\; a_r C(\tfrac{1}{2}, \tfrac{1}{2}; r, m \mid L, K)C(\tfrac{1}{2}, \tfrac{1}{2}; r, -\tfrac{1}{2} \mid L, 0)\Big] \\
&\cdot i^L J_L(\beta r)\; u_{m+\frac{1}{2}\;0}(\cos\alpha)\; e^{i(m+\frac{1}{2})\gamma},
\end{aligned}
\tag{2.22}
$$

where $C(j, m; l, n \mid L, M)$ are the Clebsch Gordan coefficients of the rotation group. The expressions for $\psi_3$ and $\psi_4$ can be obtained by interchanging the expressions $(\beta - i\sigma)$ and $im_e$ in the expressions for $\psi_1$ and $\psi_2$ to obtain $\psi_3$ and $\psi_4$, respectively. The change in spin frame from cartesian coordinates to the frame specified by the null tetrad given above can be readily computed. We standardize the choice of oblate coordinates as $z^0 = t$, $z^1 = \sqrt{r^2 + a^2}\sin\alpha\cos\gamma$, $z^2 = \sqrt{r^2 + a^2}\sin\alpha\sin\gamma$, $z^3 = r\cos\alpha$. The Lorentz transformation that maps the vector fields $D^\mu = \partial/\partial z^\mu$ into the vector fields

$$\text{(2.23)} \qquad \begin{aligned} \mathrm{D}^0 &= \frac{1}{\sqrt{2}}(\ \ell^\mu + n^\mu\ )\frac{\partial}{\partial z^\mu}, \\ \mathrm{D}^1 &= \frac{1}{\sqrt{2}}(\ \ell^\mu - n^\mu\ )\frac{\partial}{\partial z^\mu}, \\ \mathrm{D}^2 &= \frac{1}{\sqrt{2}}(\ m^\mu + \overline{m}^\mu\ )\frac{\partial}{\partial z^\mu}, \\ \mathrm{D}^3 &= \frac{1}{\sqrt{2}}(\ m^\mu - \overline{m}^\mu\ )\frac{\partial}{\partial z^\mu} \end{aligned}$$

according to $\mathrm{D}^\mu = L^\mu{}_\nu D^\nu$ can readily be calculated. The matrix elements $L^\mu{}_\nu$ are given by

$$\text{(2.24)} \qquad L^0{}_0 = \frac{1}{\sqrt{2}}\left[1 + \frac{(r^2+a^2)}{\rho^2}\right]$$

$$L^0{}_1 = \frac{1}{\sqrt{2}}\left[r\sin\alpha\cos\gamma\left[\frac{1}{\sqrt{r^2+a^2}} - \frac{\sqrt{r^2+a^2}}{\rho^2}\right] + \right.$$

$$\left. a\sin\alpha\sin\gamma\left[\frac{-1}{\sqrt{r^2+a^2}} - \frac{\sqrt{r^2+a^2}}{\rho^2}\right]\right]$$

$$L^0{}_2 = \frac{1}{\sqrt{2}}\left[r\sin\alpha\sin\gamma\left[\frac{1}{\sqrt{r^2+a^2}} - \frac{\sqrt{r^2+a^2}}{\rho^2}\right] + \right.$$

$$\left. a\sin\alpha\cos\gamma\left[\frac{1}{\sqrt{r^2+a^2}} + \frac{\sqrt{r^2+a^2}}{\rho^2}\right]\right]$$

$$L^0{}_3 = \frac{1}{\sqrt{2}}\ \cos\alpha\left[1 - \frac{r^2+a^2}{\rho^2}\right]$$

$$L^1{}_0 = \frac{-1}{\sqrt{2}} ia\sin\alpha\left[\frac{1}{\rho} + \frac{1}{\rho^*}\right]$$

$$L^1{}_1 = \frac{1}{\sqrt{2}}\left[i\sqrt{r^2+a^2}\cos\alpha\cos\gamma\left[\frac{1}{\rho} - \frac{1}{\rho^*}\right] - \sqrt{r^2+a^2}\sin\gamma\left[\frac{1}{\rho} + \frac{1}{\rho^*}\right]\right]$$

$$L^1{}_2 = \frac{1}{\sqrt{2}}\left[i\sqrt{r^2+a^2}\cos\alpha\sin\gamma\left[\frac{1}{\rho} - \frac{1}{\rho^*}\right] + \sqrt{r^2+a^2}\cos\gamma\left[\frac{1}{\rho} - \frac{1}{\rho^*}\right]\right]$$

$$L^1{}_3 = \frac{1}{\sqrt{2}} r \sin\alpha \left[\frac{1}{\rho} - \frac{1}{\rho^*}\right]$$

$$L^2{}_0 = \frac{1}{\sqrt{2}} ia \sin\alpha \left[\frac{1}{\rho} - \frac{1}{\rho^*}\right]$$

$$L^2{}_1 = \frac{1}{\sqrt{2}} \left[\sqrt{r^2+a^2}\cos\alpha\cos\gamma\left[\frac{1}{\rho} + \frac{1}{\rho^*}\right] + \sqrt{r^2+a^2} i \sin\gamma\left[\frac{1}{\rho} + \frac{1}{\rho}\right]\right]$$

$$L^2{}_2 = \frac{1}{\sqrt{2}} \left[\sqrt{r^2+a^2}\cos\alpha\sin\gamma\left[\frac{1}{\rho} + \frac{1}{\rho^*}\right] - i\sqrt{r^2+a^2}\cos\gamma\left[\frac{1}{\rho} - \frac{1}{\rho^*}\right]\right]$$

$$L^2{}_3 = \frac{1}{\sqrt{2}} r \sin\alpha \left[\frac{1}{\rho} - \frac{1}{\rho^*}\right]$$

$$L^3{}_0 = \frac{1}{\sqrt{2}} \left[1 - \frac{(r^2+a^2)}{\rho^2}\right]$$

$$L^3{}_2 = \frac{1}{\sqrt{2}} \left[r\sin\alpha\cos\gamma\left[\frac{1}{\sqrt{r^2+a^2}} - \frac{\sqrt{r^2+a^2}}{\rho^2}\right] + \right.$$

$$\left. a\sin\alpha\sin\gamma\left[\frac{-1}{\sqrt{r^2+a^2}} + \frac{\sqrt{r^2+a^2}}{\rho^2}\right]\right]$$

$$L^3{}_2 = \frac{1}{\sqrt{2}} \left[r\sin\alpha\sin\gamma\left[\frac{1}{\sqrt{r^2+a^2}} + \frac{\sqrt{r^2+a^2}}{\rho^2}\right] + \right.$$

$$\left. a\sin\alpha\cos\gamma\left[\frac{1}{\sqrt{r^2+a^2}} - \frac{\sqrt{r^2+a^2}}{\rho^2}\right]\right]$$

$$L^3{}_3 = \frac{1}{\sqrt{2}} \cos\alpha \left[1 + \frac{(r^2+a^2)}{\rho^2}\right]$$

The solutions of Dirac's equation, relative to the frame [1.2], denoted by ${}_0\psi = \begin{bmatrix} \phi_A \\ \chi_{B'} \end{bmatrix}$ is obtained by applying $\Lambda(r,\alpha,\gamma)$, the representative of $L(r,\alpha,\gamma)$ acting on the Dirac spinors $\Psi$. But these are just the solutions of Dirac's equation found in terms of Teukolsky functions above. What has been achieved here is expressions for these solutions in terms of sums of spherical Bessel functions and spherical harmonics. Clearly these expressions will in general be quite complex. We content ourselves with the derivation of an expansion formula for the functions $R_{\pm 1/2}$. Indeed if we take $\alpha = \theta = \pi/2$, $\omega = \sqrt{r^2+a^2}$ then $L$ can be factored in the form $R_3(\varphi)N_1(A)R_3(\delta)$ where $e^A = 1 + \frac{a^2}{2r^2} + \frac{a}{r}\left[1+\frac{a^2}{4r^2}\right]^{1/2}$, $e^{i\delta} = i\left[\frac{(r+ia)(r-ia/2)}{(r-ia)(r-ia/2)}\right]^{1/2}$. The corresponding transformation matrix acting on the Dirac spinor then has the form $\Lambda(r,\pi/2,\gamma) = e^{\gamma^1\gamma^2\varphi/2}\, e^{\gamma^0\gamma^1 A/2}\, e^{\gamma^1\gamma^2\delta/2}$ and, consequently, ${}_0\psi = \Lambda(r,\pi/2,\gamma)\psi$.

The small $a$ expansion of $P_\pm = V_+ \pm V_-$ can be expressed in terms of Bessel functions. The first few terms in the expressions are

$$P_+ = (\sigma+m_3)^{1/2}J_j(\beta r)r^{1/2} - \frac{am}{2(j+1)}(\sigma-m_e)^{1/2}J_{j+1}(\beta r)r^{-1/2} + \frac{a\beta(\sigma+m_e)^{1/2}}{16j^2}\left[j(j+1)+2m^2\right]J_{j-1}(\beta r)r^{-1/2}+\cdots \tag{2.25}$$

$$P_- = (m_e-\sigma)^{1/2}J_{j+1}(\beta r)r^{1/2} + iam\frac{(\sigma+m_e)^{1/2}}{2j}J_j(\beta r)r^{-1/2} + \frac{a^2\beta(m_e-\sigma)^{1/2}}{16(j+1)^2}\left[(j+2)(j+1)-2m^2\right]J_{j+2}(\beta r)r^{-1/2}+\ldots$$

An interesting application of these solutions would be the solution of the MIT bag model of confinement for which the boundary is the surface of an oblate spheroid. For this problem the appropriate boundary condition is

$$i\gamma^\mu n_\mu\psi = \psi \quad \text{for} \quad r = r_0, \tag{2.26}$$

where $n^\mu n_\mu = -1$ and $r = r_0$ is the surface of the spheroidal bag. In the tetrad formalism the non zero components of the unit normal spacelike vector $n_\mu$ are $n_{00'} = \sqrt{\rho^2/2\Delta}$, $n_{11'} = -\sqrt{\Delta/2\rho^2}$. If we naively apply the boundary conditions to a single solution of the Dirac equation as found in section 1 this would require that $R_{1/2} = \sqrt{-\overline{\rho}^*/\overline{\rho}}\Delta^{-1/2}R_{-1/2}$ for $r = r_0$, which is clearly impossible. If however, the boundary conditions are modified so as to be of the form

$$i\gamma^\mu n_\mu\psi = (\cos\alpha + i\gamma^0\sin\alpha)\psi \tag{2.27}$$

where $e^{i\alpha} = \sqrt{-\overline{\rho}^*/\overline{\rho}}$ then this boundary value problem can be more readily solved, as it reduces to the requirement that $R_{1/2} = i\Delta^{-1/2}R_{-1/2}$ for $r = r_0$. The boundary conditions [2.27] do not adequately describe a quantum bag although they do imply that the probability density vanishes on the bag surface. In order to solve the bag model conditions a solution must be represented as a sum of eigenfunctions of the type developed above. There are then no problems in principle with the bag boundary conditions. We shall return to this problem subsequently.