

**PHASE FIELD EQUATIONS IN THE SINGULAR LIMIT
OF SHARP INTERFACE PROBLEMS**

By

Gunduz Caginalp

and

Xinfu Chen

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PHASE FIELD EQUATIONS IN THE SINGULAR LIMIT OF SHARP INTERFACE PROBLEMS

GUNDUZ CAGINALP[†] AND XINFU CHEN[‡]

Abstract. In one of the singular limits as interface thickness approaches zero, solutions to the phase field equations formally approach those of a sharp interface model which incorporates surface tension. Here, we use a modification of the original phase field equations and prove this convergence rigorously in the one-dimensional and radially symmetric cases.

Key words. Phase field equations, Stefan problems, sharp interface, undercooling, surface tension.

1. Introduction.

In this paper we consider solutions to a phase field model and prove that they are governed by solutions to a sharp interface model [see 7 and the references therein] (encompassing surface tension and kinetic undercooling) in a singular limit of vanishing interface thickness. The proof is restricted to the one-dimensional and radially symmetric cases, although a formal analysis indicates a more general result [6, 7].

The convergence of solutions to the phase field equations to those of sharp interface problems such as the Stefan model or the Hele–Shaw model was suggested by the asymptotic analysis [6–8]. It has already been proven rigorously in the special cases of steady state problem [1, 9, 21] and the traveling wave problem [11]. Related theorems also include in [4, 10, 16, 22].

The relevant sharp interface problems may be described as a material in a region $\Omega \subset \mathbb{R}^N$ which may be in either of two phases, e.g. solid and liquid (denoted by $-$ and $+$ respectively). The heat diffusion equation applies to each phase. Across the interface, Γ , the latent heat of fusion must be dissipated or absorbed in accordance with the conservation of energy. Since there is considerable practical, as well as theoretical, interest in these equations, we write these equations in the dimensional form as

$$\rho c_{\text{spm}} \frac{\partial T}{\partial t} = K_{\text{tc}} \Delta T \quad \text{in } \Omega \setminus \Gamma, \quad (1.1)$$

$$\rho l_m v = K_{\text{tc}} [\nabla T \cdot n]_{\pm}^- \quad \text{on } \Gamma \quad (1.2)$$

where T is the (absolute) temperature, ρ the density, c_{spm} the specific heat per mass, units of Energy(Mass · Degree)⁻¹, K_{tc} the thermal conductivity, units of Energy(Area · Time ·

[†]Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, PA 15260. The first author of this work is supported by the NSF Grant DMS-9002242.

[‡]School of Mathematics, University of Minnesota, Minneapolis, MN 55455. The second author would like to thank Professor Avner Friedman for his valuable discussions and the Sloan Doctoral Dissertation Fellowship (1990–1991) for its support.

Temp Grad) $^{-1} = \text{Energy}(\text{Length}^{N-2} \cdot \text{Degree} \cdot \text{Time})^{-1}$, l_m the latent heat per mass, units of Energy(Mass) $^{-1}$, v the normal velocity of the interface (positive if directed toward the liquid), n the unit vector normal to the interface (pointing to the liquid), and $[\]_+^-$ denotes the jump between solid and liquid. An additional condition which must be satisfied at the interface is given by

$$[s]_E (T(x, t) - T_m) = -\sigma\kappa(x, t) - \alpha\sigma v \quad \text{on } \Gamma \quad (1.3)$$

where s is the entropy per unit volume, units of Energy(Length $^N \cdot \text{Degree}$) $^{-1}$, $[s]_E$ the difference in entropy (in equilibrium) per unit volume between the “+” phase and the “-” phase, κ the sum of principle curvatures at the point on Γ , σ the surface tension, units of Energy(Area) $^{-1} = \text{Energy}(\text{Length}^{N-1})^{-1}$, and α the relaxation scaling, units of Time \cdot Length $^{-2}$.

If σ is set to be zero in (1.3), then (1.1)–(1.3) is known as the classical Stefan model [27], a key feature of which is the distinguishability of phases based on the temperature alone. That is, $T(x, t) > T_m$ implies that the point belongs to liquid, and vice versa. This simple criterion for determining phases is no longer possible with the introduction of more realistic physics embodied by (1.3) (with $\sigma \neq 0$) which allows for the possibility of supercooling (i.e., the presence of liquid below the melting temperature) and analogously superheating. The condition (1.3) with $\sigma \neq 0$ clearly is a stabilizing influence on the shape of interface since surface tension multiplies the curvature term, thereby inhibiting interfaces with high curvature.

A convenient dimensionless version of (1.1)–(1.3) which is often implemented in the physics literature uses a rescaled dimensionless temperature, u , diffusivity, D , and capillary length, d_0 , defined by

$$u := \frac{T - T_m}{l_m/c_{\text{spm}}}, \quad D := \frac{K_{\text{tc}}}{\rho c_{\text{spm}}}, \quad d_0 := \frac{\sigma}{[s]_E l_m/c_{\text{spm}}} \quad (1.4)$$

(provided s is measured in the original degree rather than dimensionless temperature), so that the equations can be written in the form

$$u_t = D\Delta u \quad \text{in } \Omega \setminus \Gamma, \quad (1.1')$$

$$v = D[\nabla u \cdot n]_+^- \quad \text{on } \Gamma, \quad (1.2')$$

$$u = -d_0\kappa - \alpha d_0 v \quad \text{on } \Gamma. \quad (1.3')$$

Equations (1.1')–(1.3') can be studied subject to appropriate initial and boundary conditions, e.g.,

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.4)$$

$$u(x, t) = u_\partial(x), \quad x \in \partial\Omega, t > 0. \quad (1.5)$$

Local existence and uniqueness of solutions to (1.1')–(1.3'), (1.4), (1.5) were recently proven by Chen and Reitich in [14].

It has been useful, from both theoretical and practical perspective, to study (sharp interface) free boundary problems as a limit of problems with finite interfacial thickness which incorporate some of the physical structure of the interface. Toward this end, we consider a phase field system based on the free energy

$$\mathcal{F} := \int d^N x \left\{ \frac{1}{2} \xi^2 |\nabla \varphi|^2 - \frac{1}{a} G(\varphi) - 2u\gamma f(\varphi) \right\} \equiv \int F(u, \varphi) d^N x \quad (1.6)$$

which differs from the usual phase field equations with the insertion of a function $\gamma f(\varphi)$ replacing φ (see [8, p. 211]). Here the function $G(\varphi)$ is a symmetric double well potential with minima at ± 1 , e.g., $\frac{1}{8}(\varphi^2 - 1)^2$, and $f(\varphi)$ is a function having the property that $f'(\pm 1) = 0$, thereby ensuring that the roots of

$$\frac{1}{a} G'(\varphi) + 2u\gamma f'(\varphi) = 0 \quad (1.7)$$

remain at ± 1 . The variable φ is dimensionless and is called the phase or order parameter. Since the term $u\gamma f(\varphi)$ must have units of energy per volume with u and $f(\varphi)$ dimensionless, γ must have dimensions of energy per volume. The numerical value of γ will be related to the macroscopic parameters introduced in §2 below.

The Euler–Lagrange equations coupled with the nonequilibrium ansatz [8, p. 211]

$$\alpha \xi^2 \varphi_t = -\delta \mathcal{F} / \delta \varphi$$

with $\alpha \xi^2$ as a relaxation time then implies the phase equation

$$\alpha \xi^2 \varphi_t = \xi^2 \Delta \varphi + \frac{1}{a} g(\varphi) + 2u\gamma f'(\varphi) \quad (1.8)$$

where $g(\varphi) := G'(\varphi)$. This equation is coupled with the conservation of energy equation

$$u_t + \frac{1}{2} \varphi_t = D \Delta u \quad (1.9)$$

using the units of (1.1')–(1.3'), so that (1.8), (1.9) can be studied subject to suitable initial and boundary conditions. Although a variety of such conditions may be imposed, a key feature must include the vanishing of φ_t far from the interface so that the usual heat equation is attained in these regions.

The interface is now specified implicitly as the set of points on which φ vanishes; i.e.,

$$\Gamma(t) := \{x \in \Omega \mid \varphi(t, x) = 0\} \quad (1.10)$$

comprises the interface.

2. The phase field model and the macroscopic parameters.

The difference between the previously studied phase field models [7 and the reference therein] and (1.8), (1.9) is the form of the entropy term arising from $-2u\gamma f(\varphi)$ in the free energy (1.6). The original equations in which $\gamma f(\varphi) = \varphi$ assume a linear approximation to the change in entropy density between phases. Although the linear approximation is convenient for many mathematical purposes, it is possible to consider nonlinear approximations within the transition region. The physical accuracy can be expected to be of the same order as the linear approximation if the value of γ is adjusted to reflect the (macroscopic) difference in entropy density of the pure phases. (A modified consistent phase field model was recently reformulated in [18].)

Noting that the entropy difference incorporates temperature units, which in this case have been scaled by l_m/c_{spm} , one has the thermodynamic identity for the entropy difference per unit volume,

$$[s]_E = -\left.\frac{\partial F}{\partial u}\right|_{\varphi=1} + \left.\frac{\partial F}{\partial u}\right|_{\varphi=-1} = 2\gamma f(1) - 2\gamma f(-1) = 2\gamma \int_{-1}^1 f'(\varphi) d\varphi. \quad (2.1)$$

Hence, the relation (2.1) defines γ .

Two scales emerge naturally from the phase equation (1.8): a length scale, ε , and a surface tension scale (Energy/Length ^{$N-1$}), $\bar{\sigma}$, given by

$$\varepsilon := \xi a^{\frac{1}{2}} \quad \text{and} \quad \bar{\sigma} := \xi a^{-\frac{1}{2}}. \quad (2.2)$$

Noting that ε^2 is the coefficient of the Laplacian in (1.8), we define a coordinate system in which r is signed distance (positive in liquid) from the interface, $\Gamma(t)$, and a stretched or “inner” coordinate

$$\rho := r/\varepsilon. \quad (2.3)$$

If we define Φ as the solution of the ordinary differential equation

$$\Phi'' + g(\Phi) = 0, \quad \Phi(\pm\infty) = \pm 1, \quad \Phi(0) = 0, \quad (2.4)$$

then $\Phi(\rho)$ is the $O(1)$ inner expansion for φ (see [7] for more details).

We now focus on the surface tension, σ , in terms of its relation with ξ and a . With a simply thermodynamic setting, the surface tension is obtained from a suitable local interpretation of the difference between the free energy of the system minus the average of the free energies in the pure phases (normalized with respect to surface area Λ); i.e.,

$$\sigma = \lim_{\text{measure of } \Lambda \rightarrow 0} \frac{\mathcal{F}_\Lambda(\Phi) - \frac{1}{2}(\mathcal{F}(+1) + \mathcal{F}(-1))}{\text{measure of } \Lambda}. \quad (2.5)$$

A calculation [8; p239] shows that, to leading order in ε , the surface tension is given by

$$\sigma = \|\Phi'\|_{L^2(\mathbf{R})}^2 \xi a^{-\frac{1}{2}} = m\bar{\sigma}, \quad m := \|\Phi'\|_{L^2(\mathbf{R})}^2. \quad (2.6)$$

Noting that $a = \varepsilon/\bar{\sigma} = \varepsilon m/\sigma$, one can write (1.8) as

$$\alpha\varepsilon^2\varphi_t = \varepsilon^2\Delta\varphi + g(\varphi) + 2\frac{\gamma m\varepsilon}{\sigma}u f'(\varphi). \quad (2.7)$$

Using (2.1) for γ and defining

$$n := \frac{\|\Phi'\|_{L^2(\mathbf{R})}^2}{\int_{-1}^1 f'(\varphi)d\varphi} = \frac{m}{f(1) - f(-1)}, \quad (2.8)$$

we obtain finally the system

$$\alpha\varepsilon^2\varphi_t = \varepsilon^2\Delta\varphi + g(\varphi) + \frac{n}{d_0}\varepsilon u f'(\varphi), \quad (2.9)$$

$$u_t + \frac{1}{2}\varphi_t = D\Delta u \quad (2.10)$$

where $d_0 = \sigma/[s]_{-}^{\pm}$ since the additional factor in (1.4) has been absorbed into the dimensionless temperature.

For the prototype double-well potential

$$G(\varphi) = \frac{1}{8}(\varphi^2 - 1)^2, \quad g(\varphi) = G'(\varphi) = \frac{1}{2}(\varphi - \varphi^3), \quad (2.11)$$

one has $m = 2/3$, while the choice of

$$f'(\varphi) = (1 - \varphi^2)^2 \quad (2.12)$$

implies $n = 5/4$. In this case the solution Φ to (2.4) is given by

$$\Phi(\rho) = \tanh \frac{\rho}{2}. \quad (2.13)$$

3. A formal asymptotic analysis.

We perform a preliminary asymptotic analysis for the equations (2.9), (2.10) which will establish the heuristic basis for the convergence of the phase field equations to the sharp interface model (1.1')–(1.3'). Sections 4–8 will then provide a rigorous proof of the convergence to this limit.

The basic strategy in attaining this limit is similar to that of Section IV of [6]. Using the scaling defined by (2.2), one considers the distinguished limits as $\varepsilon \rightarrow 0$ but $\bar{\sigma}$ is held fixed. (This corresponds to fixed d_0 in (2.9).)

Suppose that in (2.9), (2.10), φ varies much more rapidly across the interface (from -1 in solid to $+1$ in liquid) than does u , and that φ can be approximated by a function of the form $\phi((r - vt)/\varepsilon)$; that is, the independent derivative with respect to time is of high order. Then (2.9) can be written as

$$-\alpha\varepsilon v\phi_\rho = \phi_{\rho\rho} + \varepsilon\kappa\phi_\rho + \cdots + g(\phi) + \varepsilon\frac{n}{d_0}uf'(\phi) \quad (3.1)$$

where “ \cdots ” are terms of order ε^2 .

We assume an expansion of the form $\phi = \phi^0 + \varepsilon\phi^1 + \cdots$. Then equating the $O(1)$ terms in (3.1) gives

$$\phi_{\rho\rho}^0 + g(\phi^0) = 0. \quad (3.2)$$

For the prototype potentials given by (2.11), the solution is given by (2.13). The equation for the $O(\varepsilon)$ order terms in (3.1) is

$$\phi_{\rho\rho}^1 + g'(\phi^0)\phi^1 = H := -\alpha v\phi_\rho^0 - \kappa\phi_\rho^0 - \frac{n}{d_0}uf'(\phi^0). \quad (3.3)$$

Noting that the derivative of the $O(1)$ solution, ϕ_ρ^0 , satisfies the homogeneous equation corresponding to (3.3), one obtains the solvability condition

$$0 = (\phi_\rho^0, H) = \int_{-\infty}^{\infty} \phi_\rho^0 \left[-\alpha v\phi_\rho^0 - \kappa\phi_\rho^0 - \frac{n}{d_0}uf'(\phi^0) \right] d\rho. \quad (3.4)$$

Under the assumption that u varies slowly near the interface, i.e., when $f'(\phi^0)$ is of significant order, one has, upon using (2.8), the simplification (to leading order)

$$\int_{-\infty}^{\infty} \frac{n}{d_0}uf'(\phi^0)\phi_\rho^0 d\rho = \frac{n}{d_0}u \int_{-1}^1 f'(\phi^0)d\phi^0 = \frac{m}{d_0}u. \quad (3.5)$$

Finally, using the definition of m in (2.6), one obtains the interfacial relation (to leading order)

$$u = -d_0(\alpha v + \kappa) \quad \text{on } \Gamma. \quad (3.6)$$

Note that ϕ^0 has a transition layer behavior at the interface and attains constant values outside of a region of width ε from the interface (similar to the function $\tanh r/2\varepsilon$ in the original coordinates), so that φ_t vanishes. Then an asymptotic solution (u, φ) to (2.9), (2.10) must be governed, to leading order, by a solution to the heat equation (1.1') on $\Omega \setminus \Gamma$.

The latent heat condition (1.2') is obtained by integrating (2.10) across the interface,

$$\int_{-\delta}^{\delta} \left(u_t + \frac{1}{2} \varphi_t \right) dr = \int_{-\delta}^{\delta} \left(D u_{rr} + O(\varepsilon) \right) dr, \quad (3.7)$$

so that the boundedness of u_t and the approximation $\varphi_t = -v\varphi_r$ implies the relation

$$v = D[u_r]_+^- \quad (3.8)$$

which is equivalent to (1.2').

Hence, a solution to the generalized phase field equation (2.9), (2.10) is expected to have formal asymptotics which are governed, to leading order, by the sharp interface problem (1.1')–(1.3').

4. Statement of the rigorous result.

In this and subsequent sections we present the rigorous proof of the assertions made as a result of the formal asymptotics in Section 3. It is convenient to replace, without loss of generality, the coefficients α , n/d_0 , and D , in (2.9), (2.10) by unity since the constants do not influence the proof in a significant way. Also, we use the prototype $g(\varphi) = 2(\varphi - \varphi^3)$ since the general case is very similar. We then consider the system of equations

$$\varphi_t - \Delta \varphi = \frac{1}{\varepsilon^2} (1 - \varphi^2) \{ 2\varphi + \varepsilon u (1 - \varphi^2) \}, \quad (4.1)$$

$$u_t - \Delta u = -\frac{1}{2} \varphi_t \quad (4.2)$$

in $Q_T \equiv \Omega \times (0, T)$, $\Omega \equiv \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$, subject to the initial–boundary conditions

$$u(x, t) = u_0(x, t), \quad (x, t) \in \partial_T Q := \Omega \times \{0\} \cup \partial\Omega \times [0, T], \quad (4.3)$$

$$\varphi(x, 0) = \Psi\left(\frac{|x| - r_0}{\varepsilon}\right), \quad x \in \Omega, \quad (4.4)$$

$$\varphi_r = \frac{1}{\varepsilon} (1 - \varphi^2), \quad (x, t) \in \partial\Omega \times [0, T] \quad (4.5)$$

where r_1, r_0 and r_2 are three given constants satisfying $r_1 < r_0 < r_2$, and

$$\Psi(\rho) = \tanh \rho, \quad -\infty < \rho < \infty. \quad (4.6)$$

Although more general boundary conditions for u can be used without significant changes in the proofs, we shall use (4.3) for definiteness.

The initial condition for φ ensures that the initial shape is compatible with the basic length scale in the problem [7, 8]. The boundary condition for φ is compatible with the initial condition and with vanishing φ_t and φ_x near the external boundary which are necessary in order to attain the heat equation in the limit. While other conditions may also be used, this particular condition is technically convenient.

For any $1 \leq p \leq \infty$, $\alpha \in (0, 1)$, and $Q_t := \Omega \times (0, t)$, we introduce the norms:

$$\|f\|_{W_p^{2,1}(Q_t)} := \sum_{m+2k \leq 2} \|\partial_x^m \partial_t^k f\|_{L^p(Q_t)}, \quad (4.7)$$

$$\|f\|_{C^\alpha(Q_t)} := \|f\|_{C^0(Q_t)} + \sup_{\substack{(x_1, t_1), (x_2, t_2) \in Q_t \\ (x_1, t_1) \neq (x_2, t_2)}} \frac{|f(x_1, t_1) - f(x_2, t_2)|}{|x_1 - x_2|^\alpha + |t_1 - t_2|^\alpha}, \quad (4.8)$$

$$\|f\|_{C^{\alpha, \alpha/2}(Q_t)} := \|f\|_{C^0(Q_t)} + \sup_{\substack{(x_1, t_1), (x_2, t_2) \in Q_t \\ (x_1, t_1) \neq (x_2, t_2)}} \frac{|f(x_1, t_1) - f(x_2, t_2)|}{|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}}, \quad (4.9)$$

$$\|f\|_{C^{1+\alpha, (1+\alpha)/2}(Q_t)} := \|f\|_{C^{(1+\alpha)/2}(Q_t)} + \|\partial_x f\|_{C^{\alpha, \alpha/2}(Q_t)}. \quad (4.10)$$

We state a standard result concerning the system (4.1)–(4.5).

LEMMA 4.1. *Assume that $u_0 \in C^{2,1}(Q_T)$. Then for every $\varepsilon > 0$, the system (4.1)–(4.5) has a unique classical solution $(u^\varepsilon, \varphi^\varepsilon)$. Moreover,*

$$|\varphi^\varepsilon(x, t)| < 1 \quad \forall (x, t) \in \overline{Q_T} \equiv \overline{\Omega} \times [0, T]. \quad (4.11)$$

The proof of the existence of a unique solution is similar to that in [8], whereas (4.11) follows by applying the maximum principle to the parabolic equation (4.1) (treating u^ε as a known coefficient).

From now on we shall prove the following theorem:

THEOREM 4.1. *Assume that $u_0 \in C^{2,1}(Q_T)$ and is radially symmetric, and let $(u^\varepsilon, \varphi^\varepsilon)$ be the solution of (4.1)–(4.5). Then there exist functions $u(x, t) \in C^{\alpha, \alpha/2}(Q_{T^*})$ and $S(t) \in C^{1+\alpha/2}([0, T^*])$, such that*

$$\lim_{\varepsilon \rightarrow 0^+} u^\varepsilon(x, t) = u(x, t), \quad \forall (x, t) \in \overline{\Omega} \times [0, T^*), \quad (4.12)$$

$$\lim_{\varepsilon \rightarrow 0^+} \varphi^\varepsilon(x, t) = \begin{cases} 1 & \text{if } S(t) < |x| \leq r_2, 0 \leq t < T^*, \\ -1 & \text{if } r_1 \leq |x| < S(t), 0 \leq t < T^* \end{cases} \quad (4.13)$$

where $T^* > 0$ is the first time such that one of the following happens:

$$T^* = T; \quad S(T^*) = r_2; \quad S(T^*) = r_1. \quad (4.14)$$

Moreover, if we denote by Γ the set $\{(x, t) \in Q_{T^*} : |x| = S(t)\}$, then (u, Γ) is a solution to (1.1')-(1.3'); that is,

$$u_t = \Delta u \quad \text{in } Q_{T^*} \setminus \Gamma, \quad (4.15)$$

$$\dot{S}(t) = [u_r(x, t)]_+^- \quad \text{on } \Gamma, \quad (4.16)$$

$$\dot{S}(t) = -\frac{N-1}{S(t)} - \beta u(S(t), t), \quad \text{on } [0, T^*) \quad (4.17)$$

where β is a positive constant defined in Section 8 below.

Recall that the sum of principle curvatures of a ball of radius r is $\frac{N-1}{r}$, so that equation (4.17) is equivalent to (1.3').

To explain the idea of the proof of the theorem, we introduce a function $Z^\varepsilon(x, t) : Q_T \rightarrow \mathbb{R}^1$ defined by

$$Z^\varepsilon(x, t) = \varepsilon \Psi^{-1}(\varphi^\varepsilon(x, t)). \quad (4.18)$$

Since (4.11) implies that the value of φ^ε is in the range of Ψ , the function Z^ε is well-defined. Clearly the definition of Z^ε implies that

$$\varphi^\varepsilon(x, t) = \Psi\left(\frac{Z^\varepsilon(x, t)}{\varepsilon}\right), \quad (x, t) \in Q_T. \quad (4.19)$$

The overall strategy for the proof of the theorem is to show that Z^ε is approximately equal to $|x| - S(t)$ for some function $S(t) \in C^\alpha$, $\alpha \in (1/2, 1)$. Once we prove this, we can substitute (4.19) into (4.2) to obtain Hölder estimates, for u^ε , independent of ε , by applying a potential analysis to the Green's representation for the solution u^ε of the heat equation (4.2). Having known the Hölder continuity of the function u^ε , we then go back to the equation (4.1), which has been well-studied in the case u^ε being a constant [2, 5, 12, 23, 24, 25], a known function [17], or an unknown function satisfying a parabolic equation coupled with φ^ε [13, 15, 26]. The conclusion is expressed by equations (4.13) and (4.17). Finally, by using (4.13) and the distribution sense of equation (4.2), one obtains (4.15) and (4.16).

To prove that Z^ε is approximately equal to $|x| - S(t)$, we need, however, some regularity on u^ε . For this reason, we introduce the functions:

$$M_\varepsilon^0(t) := \|u^\varepsilon\|_{C^0(Q_t)}, \quad (4.20)$$

$$M_\varepsilon^1(t) := \|u_r^\varepsilon\|_{C^0(Q_t)}, \quad (4.21)$$

$$M_\varepsilon(t) := M_\varepsilon^0(t) + \varepsilon M_\varepsilon^1(t), \quad (4.22)$$

$$T^\varepsilon := \sup\{t \in [0, T] : M_\varepsilon(\tau) \leq \frac{1}{\sqrt{\varepsilon}} \text{ for all } \tau \in [0, t]\}. \quad (4.23)$$

By restricting oneself to on the interval $[0, T^\varepsilon]$, one can carry out all the steps described in the proceeding paragraph. Therefore, to complete the proof, one need only show that the

a priori estimate thus obtained in the interval $[0, T^\varepsilon]$ is independent of ε since this means that $T^\varepsilon = T$.

In the following we shall denote by C the various kinds of constants which are independent of ε . Also, we shall identify functions of variable (x, t) with functions of variable (r, t) with $r = |x|$ since all function in the sequel are radially symmetric. Finally, we shall assume, without loss of generality, that

$$M_\varepsilon(t) \geq 1, \quad t \in [0, T]. \quad (4.24)$$

5. $C^{(1+\alpha)/2}$ estimate for the interface.

In this section, we shall show that the interface which coincides with the zero level set of Z^ε is $C^{(1+\alpha)/2}$ for any $\alpha \in (0, 1)$.

LEMMA 5.1. *Let Z^ε and T^ε be defined as in (4.18) and (4.23). Then there exist positive constants ε_0 and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $(x, t) \in \overline{Q_{T^\varepsilon}}$, one has*

$$|Z^\varepsilon(x, t)| \leq 2r_2 + \frac{N}{r_1} T^\varepsilon, \quad (5.1)$$

$$\frac{1}{2} \leq Z_r^\varepsilon(x, t) \leq 2, \quad (5.2)$$

$$-C\varepsilon M_\varepsilon(t) \leq Z_r^\varepsilon(x, t) - 1 \leq C\varepsilon M_\varepsilon(t). \quad (5.3)$$

Note that (5.3) is stronger than (5.2).

Proof. Substituting (4.19) into (4.1) yields the equation

$$\left(Z_t^\varepsilon - \Delta Z^\varepsilon \right) \Psi' - \frac{|\nabla Z^\varepsilon|^2}{\varepsilon} \Psi'' - \frac{1}{\varepsilon} (1 - \Psi^2) \left(2\Psi + \varepsilon u^\varepsilon (1 - \Psi^2) \right) = 0. \quad (5.4)$$

Using the radial coordinates and the identities

$$\Psi' = 1 - \Psi^2, \quad \Psi'' = -2\Psi\Psi',$$

one can write (5.4) as

$$Z_t^\varepsilon - Z_{rr}^\varepsilon - \frac{N-1}{r} Z_r^\varepsilon + \frac{2}{\varepsilon} (Z_r^{\varepsilon 2} - 1) \Psi \left(\frac{Z^\varepsilon}{\varepsilon} \right) - u^\varepsilon \Psi' \left(\frac{Z^\varepsilon}{\varepsilon} \right) = 0, \quad (5.5)$$

and write the initial-boundary conditions (4.4), (4.5) as

$$Z^\varepsilon(r, 0) = r - r_0, \quad r \in (r_1, r_2), \quad (5.6)$$

$$Z_r^\varepsilon(r_1, t) = Z_r^\varepsilon(r_2, t) = 1, \quad t \in [0, T] \quad (5.7)$$

where we have identified the function $Z^\varepsilon(x, t)$ with the function $Z^\varepsilon(r, t)$ with $r = |x|$.

Noting that

$$\Psi'\left(\frac{Z^\varepsilon}{\varepsilon}\right) = \cosh^{-2}\left(\frac{Z^\varepsilon}{\varepsilon}\right) \leq 4e^{-\frac{|Z^\varepsilon|}{\varepsilon}} \leq 4\varepsilon^k \quad \text{if } |Z^\varepsilon| \geq k\varepsilon|\ln \varepsilon|,$$

one can directly verify that for ε sufficient small, the functions

$$Z^+ := r + \frac{N}{r_1}t$$

and

$$Z^- := r - 2r_2 - \varepsilon t$$

are, respectively, a supersolution and a subsolution to (5.5)–(5.7), so that

$$Z^-(r, t) \leq Z^\varepsilon(r, t) \leq Z^+(r, t), \quad r_1 \leq r \leq r_2, \quad 0 \leq t \leq T^\varepsilon,$$

and therefore the assertion (5.1) follows.

To prove (5.2), we differentiate (5.5) with respect to r and set $Z^\varepsilon_r = w$, obtaining a nonlinear parabolic equation, for w ,

$$\begin{aligned} \mathcal{N}w := w_t - w_{rr} - \frac{N-1}{r}w_r + \frac{N-1}{r^2}w + \frac{4}{\varepsilon}\Psi w w_r \\ + \frac{2}{\varepsilon^2}\Psi'(w^2 - 1)w - u_r^\varepsilon\Psi' + \frac{2}{\varepsilon}u^\varepsilon\Psi\Psi'w = 0, \quad (x, t) \in Q_T, \quad (5.8) \end{aligned}$$

$$w = 1, \quad (x, t) \in \partial_T Q. \quad (5.9)$$

Set

$$w^+(r, t) := 1 + 2\varepsilon M_\varepsilon(\bar{t}), \quad r \in [r_1, r_2], \quad 0 \leq t \leq \bar{t} \leq T^\varepsilon$$

and

$$w^-(r, t) := 1 - 2\varepsilon M_\varepsilon(\bar{t}) - \frac{N-1}{r_1^2}t, \quad r \in [r_1, r_2], \quad 0 \leq t \leq \bar{t} \leq T_1^\varepsilon$$

where $T_1^\varepsilon \in (0, T^\varepsilon]$ is any constant which can ensure $w^- \geq \frac{1}{2}$ for all $t \in [0, T_1^\varepsilon]$. One can verify, by using the definition of M_ε and the fact that $\frac{1}{2} < w^- < w^+ < 2$, that w^+ and w^- are, respectively, a supersolution and a subsolution of (5.8), (5.9). Therefore, a comparison principle for parabolic equation implies that $w^- \leq Z^\varepsilon_r \leq w^+$, which, in turn, implies that (5.2) is valid in $[0, T_1^\varepsilon]$.

We shall now use (5.2) to prove (5.3). Clearly, we need only prove the first inequality in (5.3) since the inequality $w \leq w^+$ implies the second inequality in (5.3).

We claim that, for a suitable pair of constants k_1 and k_2 which are independent of ε , the function

$$\underline{w} := 1 - k_1\varepsilon\sqrt{Z^\varepsilon + k_1^2\varepsilon^2} - k_2\varepsilon M_\varepsilon \quad (5.10)$$

is a subsolution to (5.8), (5.9).

One can compute

$$\begin{aligned}
I &:= \underline{w}_t - \underline{w}_{rr} - \frac{N-1}{r} \underline{w}_r + \frac{N-1}{r^2} \underline{w} \\
&= -\frac{k_1 \varepsilon Z^\varepsilon}{(Z^{\varepsilon^2} + k_1^2 \varepsilon^2)^{1/2}} \left(Z^\varepsilon_t - Z^\varepsilon_{rr} - \frac{N-1}{r} Z^\varepsilon_r \right) + \frac{k_1^3 \varepsilon^3}{(Z^{\varepsilon^2} + k_1^2 \varepsilon^2)^{3/2}} Z^{\varepsilon^2}_r + \frac{N-1}{r^2} \underline{w} \\
&= -\frac{k_1 \varepsilon Z^\varepsilon}{(Z^{\varepsilon^2} + k_1^2 \varepsilon^2)^{1/2}} \left(-\frac{2}{\varepsilon} (Z^{\varepsilon^2}_r - 1) \Psi + u^\varepsilon \Psi' \right) + \frac{k_1^3 \varepsilon^3}{(Z^{\varepsilon^2} + k_1^2 \varepsilon^2)^{3/2}} Z^{\varepsilon^2}_r + \frac{N-1}{r^2} \underline{w} \\
&\leq 2k_1 \max\{Z^{\varepsilon^2}_r - 1, 0\} |\Psi| + k_1 \varepsilon |u^\varepsilon| |\Psi'| + 4 + \frac{N-1}{r^2} \\
&\leq k_1 \varepsilon M_\varepsilon (8|\Psi| + \Psi') + 4 + \frac{N-1}{r^2},
\end{aligned}$$

where in the third equation, we have used equation (5.5), in the first inequality, we have used the fact that $Z^\varepsilon \Psi \geq 0$, $\frac{Z^\varepsilon}{(Z^{\varepsilon^2} + k_1^2 \varepsilon^2)^{1/2}} \leq 1$, and $Z^{\varepsilon^2}_r \leq 4$, and in the last inequality, we have used the fact that $Z^{\varepsilon^2}_r - 1 \leq w^{+2} - 1 \leq 4\varepsilon M_\varepsilon$ and $|u^\varepsilon| \leq M_\varepsilon$. One can also compute

$$\begin{aligned}
II &:= \frac{4}{\varepsilon} \Psi \underline{w} \underline{w}_r + \frac{2}{\varepsilon^2} \Psi' (w^2 - 1) \underline{w} \\
&= -4k_1 \Psi \frac{\underline{w} Z^\varepsilon Z^\varepsilon_r}{(Z^{\varepsilon^2} + k_1^2 \varepsilon^2)^{1/2}} + \frac{2}{\varepsilon^2} \Psi' (\underline{w} + 1) (\underline{w} - 1) \underline{w} \\
&\leq -k_1 |\Psi| \chi_{\{|Z^\varepsilon| \geq k_1 \varepsilon\}} - \frac{2k_2 M_\varepsilon}{\varepsilon} \Psi'
\end{aligned}$$

since $Z^\varepsilon_r > 1/2$. Finally,

$$III := -u_r^\varepsilon \Psi' + \frac{2}{\varepsilon} u^\varepsilon \Psi \Psi' \underline{w} \leq 3 \frac{M_\varepsilon}{\varepsilon} \Psi'.$$

Therefore,

$$\begin{aligned}
\mathcal{N} \underline{w} &= I + II + III \\
&\leq k_1 \varepsilon M_\varepsilon (8|\Psi| + \Psi') + 4 + \frac{N-1}{r^2} - k_1 |\Psi| \chi_{\{|Z^\varepsilon| \geq k_1 \varepsilon\}} - \frac{2k_2 M_\varepsilon}{\varepsilon} \Psi' + 3 \frac{M_\varepsilon}{\varepsilon} \Psi' \\
&\leq C k_1 \sqrt{\varepsilon} + C - k_1 \min \left\{ \Psi(k_1), \frac{k_2 M_\varepsilon}{k_1 \varepsilon} \Psi'(k_1) \right\} \leq 0
\end{aligned}$$

if we first take k_1 large enough, and then k_2 large enough and ε small enough. Therefore by comparison, $\underline{w} \leq Z^\varepsilon_r$, which implies that the first inequality in (5.3) holds for $t \in [0, T_1^\varepsilon]$.

Repeating the above proof in the interval $[T_1^\varepsilon, T^\varepsilon]$, one can easily extend, step by step, the valid interval for (5.2), (5.3) up to $[0, T^\varepsilon]$, and therefore complete the proof of the lemma. \square

We can now obtain L^p and Hölder estimates for Z^ε (based on M_ε).

LEMMA 5.2. For all $p \in (1, \infty)$ and $\alpha \in (0, 1)$, there exist constants C_p and C_α , which are independent of ε , such that for all $t \in [0, T^\varepsilon]$ one has the bounds

$$\|Z^\varepsilon\|_{W_p^{2,1}(Q_t)} \leq C_p M_\varepsilon(t), \quad (5.11)$$

$$\|Z^\varepsilon\|_{C^{1+\alpha, (1+\alpha)/2}(Q_t)} \leq C_\alpha M_\varepsilon(t). \quad (5.12)$$

Proof. The equation (5.5), along with the inequality (5.3), implies the inequalities

$$|Z^\varepsilon_t - Z^\varepsilon_{rr}| \leq \frac{N-1}{r} |Z^\varepsilon_r| + 2|Z^\varepsilon_r + 1| \frac{|Z^\varepsilon_r - 1|}{\varepsilon} |\Psi| + |u^\varepsilon| |\Psi'| \leq C M_\varepsilon(t).$$

Then (5.11) follows from the classical L^p estimates while (5.12) follows from Sobolev embedding Theorem. \square

Since Z^ε is strictly increasing (by (5.2)), one can define the inverse, $r = \widetilde{R}^\varepsilon(z, t)$, of the function $z = Z^\varepsilon(r, t)$. It is convenient to extend $\widetilde{R}^\varepsilon$ to R^ε on $\mathbb{R}^+ \times [0, T^\varepsilon]$ defined by

$$R^\varepsilon(z, t) := \begin{cases} \widetilde{R}^\varepsilon(z, t) & \text{if } z \in [Z^\varepsilon(r_1, t), Z^\varepsilon(r_2, t)], \\ r_1 & \text{if } z < Z^\varepsilon(r_1, t), \\ r_2 & \text{if } z > Z^\varepsilon(r_2, t). \end{cases} \quad (5.13)$$

The estimate (5.3) then implies that

$$Z^\varepsilon(r, t) = [r - R^\varepsilon(0, t)] [1 + O(\varepsilon M_\varepsilon)]. \quad (5.14)$$

LEMMA 5.3. For all $\alpha \in (0, 1)$ there exists a constant $C_\alpha > 0$ such that

$$\|R^\varepsilon\|_{C^\alpha(Q_t)} \leq C_\alpha M_\varepsilon(t) \quad \forall t \in (0, T^\varepsilon]. \quad (5.15)$$

This lemma follows from Lemma 5.2 and the estimate (5.2).

6. A L^∞ bound on u^ε using Green's Function.

We shall now use the heat equation (4.2) to estimate the L^∞ bound for u^ε .

Let $G(x, t; \xi, \tau)$ be the Green's function corresponding to the boundary conditions imposed on u ; that is, G satisfies

$$G_\tau + \Delta_\xi G = 0, \quad (x, \xi) \in \Omega \times \Omega, \quad 0 \leq \tau < t, \quad (6.1)$$

$$G(x, t; \xi, \tau) = 0, \quad (x, t) \in Q_T, \quad (\xi, \tau) \in \partial\Omega \times [0, T], \quad (6.2)$$

$$\lim_{\tau \rightarrow t^-} G(x, t; \xi, \tau) = \delta(x - \xi), \quad (x, \xi, t) \in \Omega \times \Omega \times (0, T] \quad (6.3)$$

where the δ function in the last equation has the standard interpretation in the sense of distributions. By Green's formula, a solution u^ε to (4.2), (4.3) is given by

$$\begin{aligned} u^\varepsilon(x, t) &= \int_{\Omega} G(x, t; \xi, 0) u_0(x, 0) d\xi - \int_0^t \int_{\partial\Omega} \frac{\partial G}{\partial \xi n}(x, t; \xi, \tau) u_0(\xi, \tau) dS_\xi d\tau \\ &\quad - \frac{1}{2} \int_0^t \int_{\Omega} G(x, t, \xi, \tau) \varphi_\tau^\varepsilon(\xi, \tau) d\xi d\tau \\ &\equiv u_1(x, t) + u_2(x, t) - \frac{1}{2} u_3(x, t) \end{aligned} \quad (6.4)$$

where n is the normal to the surface element dS of $\partial\Omega$.

Note that $u_1 + u_2$ is a solution to

$$v_t - \Delta v = 0 \quad \text{in } Q_T, \quad (6.5)$$

$$v = u_0 \quad \text{on } \partial_T Q, \quad (6.6)$$

so that one has, for a constant M_0 (depending only on g , Ω , and T), the bound

$$\|u_1 + u_2\|_{C^{1,1/2}(Q_T)} \leq M_0. \quad (6.7)$$

Hence, we need only analyze the regularity of u_3 in order to obtain the L^∞ or Hölder estimate for u^ε .

The following lemma establishes a recursive relation for $M_\varepsilon^1(t)$.

LEMMA 6.1. *For any $\alpha \in (0, 1)$ and $q \in (1, \frac{3}{2+\alpha})$, there exists a constant $C_{\alpha, q}$, which is independent of ε , such that for all $t \in [0, T^\varepsilon]$,*

$$M_\varepsilon^1 := \left\| \frac{\partial u^\varepsilon}{\partial r} \right\|_{C^0(Q_t)} \leq C_{\alpha, q} \varepsilon^{\alpha-1} M_\varepsilon(t) t^{\frac{3-(\alpha+2)q}{2q}} + M_0. \quad (6.8)$$

Proof. We need only estimate $|u_{3,r}|$. Denoting

$$\tilde{G}(r, t; r', \tau) := \int_{\{|\xi|=r'\}} G(x, t; \xi, \tau) \Big|_{|x|=r} dS_\xi d\tau, \quad (6.9)$$

one gets

$$u_3(r, t) = \int_0^t \int_{r_1}^{r_2} \tilde{G}(r, t, r', \tau) \varphi_\tau^\varepsilon(r', \tau) dr' d\tau = \frac{1}{\varepsilon} \int_0^t \int_{r_1}^{r_2} \tilde{G} \Psi' \left(\frac{Z^\varepsilon}{\varepsilon} \right) Z^\varepsilon_\tau dr' d\tau.$$

Differentiating both sides with respect to r yields

$$\begin{aligned}
\left| \frac{\partial u_3}{\partial r} \right| &\leq \frac{1}{\varepsilon} \int_0^t \int_{r_1}^{r_2} \left| \frac{\partial \tilde{G}}{\partial r} \right| \left| \Psi' \left(\frac{Z^\varepsilon}{\varepsilon} \right) \right| Z^\varepsilon_\tau |dr' d\tau \\
&\leq \frac{1}{\varepsilon} \|Z^\varepsilon_\tau\|_{L^{q'}} \left\{ \int_0^t \int_{r_1}^{r_2} |\tilde{G}_r|^q |\Psi'|^q dr' d\tau \right\}^{1/q} \\
&\leq \frac{1}{\varepsilon} \|Z^\varepsilon_\tau\|_{L^{q'}} \left\{ \left(\sup_{\tau \in [0, t]} \int_{r_1}^{r_2} |\Psi'|^{pq} \right)^{1/p} \int_0^t \left(\int_{r_1}^{r_2} \tilde{G}_r^{qp'} \right)^{1/p'} \right\}^{1/q}
\end{aligned} \tag{6.10}$$

where $1/q + 1/q' = 1$ ($q > 1$) and $1/p + 1/p' = 1$ ($p > 1$).

One may write a basic inequality [20, chapt 1] involving the Green's function as

$$\begin{aligned}
\int_0^t \left(\int_{r_1}^{r_2} |\tilde{G}_r|^{qp'} dr' \right)^{1/p'} d\tau &\leq C \int_0^t \left\{ \int_{r_1}^{r_2} \left(1 + \frac{|r-r'|}{|t-\tau|^{3/2}} e^{-\frac{|r-r'|^2}{4|t-\tau|}} \right)^{qp'} dr' \right\}^{1/p'} \\
&\leq Ct^{\frac{1}{2p} + 1 - q}
\end{aligned} \tag{6.11}$$

where C depends only on Ω . Since $0 < \Psi' < 1$ and $Z^\varepsilon_\tau > 1/2$, one has

$$\int_{r_1}^{r_2} |\Psi'|^{pq} \leq \int_{r_1}^{r_2} \Psi' = \varepsilon \int_{Z^\varepsilon(r_1, \tau)}^{Z^\varepsilon(r_2, \tau)} \Psi' \left(\frac{Z^\varepsilon}{\varepsilon} \right) \frac{1}{Z^\varepsilon_r} d\left(\frac{Z^\varepsilon}{\varepsilon} \right) \leq 4\varepsilon. \tag{6.12}$$

Setting $p = \frac{1}{\alpha q}$, substituting (6.11), (6.12) into (6.10), and using the L^p estimate for Z^ε [(5.11)], one obtains the bound

$$\left| \frac{\partial u_3}{\partial r} \right| \leq C \frac{1}{\varepsilon} M_\varepsilon(t) t^{[\frac{1}{2p} + 1 - q]/q} \varepsilon^\alpha = C \varepsilon^{\alpha-1} M_\varepsilon(t) t^{\frac{3-(\alpha+2)q}{2q}},$$

and therefore the lemma follows. \square

To get the L^∞ bound for u_3 , we utilize the identities

$$\begin{aligned}
u_3(x, t) &= \int_0^t \int_\Omega G(x, t; \xi, \tau) \frac{\partial}{\partial \tau} [\varphi^\varepsilon(\xi, \tau) - \varphi^\varepsilon(\xi, t)] d\xi d\tau \\
&= \int_\Omega G[\varphi^\varepsilon(\xi, \tau) - \varphi^\varepsilon(\xi, t)] \Big|_{\tau=0}^{\tau=t} d\xi - \int_0^t \int_\Omega G_\tau [\varphi^\varepsilon(\xi, \tau) - \varphi^\varepsilon(\xi, t)] d\xi d\tau \\
&= \int_\Omega G(x, t; \xi, 0) [\varphi^\varepsilon(\xi, t) - \varphi^\varepsilon(\xi, 0)] d\xi + \int_0^t \int_\Omega \Delta_\xi G [\varphi^\varepsilon(\xi, \tau) - \varphi^\varepsilon(\xi, t)] d\xi d\tau \\
&= \int_\Omega G(x, t; \xi, 0) [\varphi^\varepsilon(\xi, t) - \varphi^\varepsilon(\xi, 0)] + \int_0^t \int_\Omega \frac{\partial G}{\partial r'}(x, t; \xi, \tau) [\varphi^\varepsilon(\xi, \tau) - \varphi^\varepsilon(\xi, t)] d\xi d\tau \\
&\quad - \int_0^t \int_\Omega \frac{\partial G}{\partial r'}(x, t; \xi, \tau) \left[\frac{\partial \varphi^\varepsilon}{\partial r'}(\xi, \tau) - \frac{\partial \varphi^\varepsilon}{\partial r'}(\xi, t) \right] d\xi d\tau \\
&\equiv A(x, t) + B(x, t) + C(x, t)
\end{aligned} \tag{6.13}$$

where integration by parts in t , the heat equation for G , and Green's theorem have been used.

LEMMA 6.2. For any $\alpha \in (1/2, 1)$, there exists a positive constant C_α , such that for any positive constant δ and for all $(x, t) \in Q_{T^\varepsilon}$ one has the bounds

$$|A(x, t) + B(x, t)| \leq 2, \quad (6.14)$$

$$|C(x, t)| \leq 8 + C_\alpha \left(\delta^{-\frac{3}{2}} + M_\varepsilon(t) \delta^{\alpha - \frac{1}{2}} \right), \quad (6.15)$$

$$M_\varepsilon^0 \leq C + C_\alpha \left(\delta^{-\frac{3}{2}} + M_\varepsilon(t) \delta^{\alpha - \frac{1}{2}} \right). \quad (6.16)$$

Proof. We need only prove (6.14) and (6.15) since (6.16) follows from (6.4), (6.7), (6.14) and (6.15).

Using the bound $|\varphi| < 1$ (Lemma 4.1), one obtains

$$|A + B| \leq 2 \sup |\varphi| \left\{ \int_\Omega G(x, t; \xi, 0) d\xi + \int_0^t \int_{\partial\Omega} \frac{\partial G}{\partial n}(x, t; \xi, \tau) dS_\xi d\tau \right\} \leq 2.$$

Write $C(x, t)$ as

$$\begin{aligned} C(x, t) &= \left(\int_0^{\max\{0, t-\delta\}} + \int_{\max\{0, t-\delta\}}^t \right) \int_\Omega \dots d\xi d\tau = C^{(1)} + C^{(2)}, \\ C^{(1)} &:= \int_0^{\max\{0, t-\delta\}} \int_{r_1}^{r_2} \tilde{G}_{r'}(r, t; r', \tau) \left[\varphi_{r'}^\varepsilon(r', t) - \varphi_{r'}^\varepsilon(r', \tau) \right] dr' d\tau, \\ C^{(2)} &:= \int_{\max\{0, t-\delta\}}^t \int_{r_1}^{r_2} \tilde{G}_{r'}(r, t; r', \tau) \left[\varphi_{r'}^\varepsilon(r', t) - \varphi_{r'}^\varepsilon(r', \tau) \right] dr' d\tau. \end{aligned} \quad (6.17)$$

Integrating by parts for the integral $C^{(1)}$ and using the bound

$$|\tilde{G}_{r'r'}(r, t, r', \tau)| \leq C(t - \tau)^{-\frac{3}{2}},$$

one finds that $C^{(1)}$ is bounded by $C\delta^{-\frac{3}{2}}$.

To estimate $C^{(2)}$, we substitute φ^ε by $\Psi(Z^\varepsilon/\varepsilon)$ in (6.17) and use the change of variables $\eta = Z^\varepsilon/\varepsilon$, obtaining

$$\begin{aligned} C^{(2)}(x, t) &= \int_{\max\{0, t-\delta\}}^t \int_{Z^\varepsilon(r_1, t)/\varepsilon}^{Z^\varepsilon(r_2, t)/\varepsilon} \tilde{G}_{r'}(r, t; R^\varepsilon(\varepsilon\eta, t), \tau) \Psi'(\eta) d\eta d\tau \\ &\quad - \int_{\max\{0, t-\delta\}}^t \int_{Z^\varepsilon(r_1, \tau)/\varepsilon}^{Z^\varepsilon(r_2, \tau)/\varepsilon} \tilde{G}_{r'}(r, t; R^\varepsilon(\varepsilon\eta, \tau), \tau) \Psi'(\eta) d\eta d\tau. \end{aligned}$$

By dividing the η integration of the first integral into the three parts:

$$\int_{Z^\varepsilon(r_1, t)/\varepsilon}^{Z^\varepsilon(r_1, \tau)/\varepsilon} + \int_{Z^\varepsilon(r_1, \tau)/\varepsilon}^{Z^\varepsilon(r_2, \tau)} + \int_{Z^\varepsilon(r_2, \tau)/\varepsilon}^{Z^\varepsilon(r_2, t)/\varepsilon},$$

one obtains the bound

$$\begin{aligned}
& |C^{(2)}(x, t)| \\
& \leq \int_{\max\{0, t-\delta\}}^t d\tau \int_{\frac{Z^\varepsilon(r_1, \tau)}{\varepsilon}}^{\frac{Z^\varepsilon(r_2, \tau)}{\varepsilon}} \left| \tilde{G}_{r'}(r, t; R^\varepsilon(\varepsilon\eta, t), \tau) - \tilde{G}_{r'}(r, t, R^\varepsilon(\varepsilon\eta, \tau), \tau) \right| \Psi'(\eta) d\eta \\
& \quad + 2 \int_{-\infty}^{\infty} d\eta \Psi'(\eta) \int_{\max\{0, t-\delta, 0\}}^t |\tilde{G}_{r'}(r, t, R^\varepsilon(\varepsilon\eta, t), \tau)| d\tau \\
& \equiv C^{(21)} + C^{(22)}.
\end{aligned}$$

The integral $C^{(22)}$ is bounded by 8 since for all $r, r' \in (r_1, r_2)$, one has

$$\int_{\max\{0, t-\delta\}}^t |\tilde{G}_{r'}(r, t, r', \tau)| d\tau \leq 2.$$

The integral $C^{(21)}$ can be estimated by

$$\begin{aligned}
C^{(21)} & \leq \int_{\max\{t-\delta, 0\}}^t d\tau \int_{\frac{Z^\varepsilon(r_1, t)/\varepsilon}{}^{\frac{Z^\varepsilon(r_2, t)/\varepsilon}{} d\eta \Psi'(\eta) |\tilde{G}_{r'r'}(r, t; \bar{r}', \tau) | R^\varepsilon(\varepsilon\eta, \tau) - R^\varepsilon(\varepsilon\eta, t) |} \\
& \leq \int_{\max\{t-\delta, 0\}}^t d\tau \int_{-\infty}^{\infty} d\eta \Psi'(\eta) C |t - \tau|^{-3/2} C_\alpha M_\varepsilon(t) |t - \tau|^\alpha \\
& \leq \tilde{C}_\alpha M_\varepsilon(t) \delta^{\alpha-1/2}
\end{aligned}$$

where the mean value theorem and Lemma 5.3 have been used. Combining all the estimates, one obtains (6.15) and the lemma. \square

THEOREM 6.1. *There exist positive constants ε_0 and C such that for all $\varepsilon \in (0, \varepsilon_0)$, one has the bound*

$$\|u^\varepsilon\|_{L^\infty(Q_T)} \leq C. \tag{6.18}$$

Proof. Using (6.8) (with $q = \frac{1}{2}(1 + \frac{3}{2+\alpha})$), (6.16), and the definition of M_ε in (4.22), one has

$$M_\varepsilon(t) \leq C_{\alpha, T} \left[1 + \delta^{-3/2} + M_\varepsilon(t) \delta^{\alpha-1/2} + \varepsilon^\alpha M_\varepsilon(t) \right]. \tag{6.19}$$

Choosing ε_0 and δ satisfying

$$\varepsilon_0^\alpha C_{\alpha, T} \leq \frac{1}{4}, \quad C_{\alpha, T} \delta^{\alpha-1/2} \leq \frac{1}{4},$$

one has, from (6.19), the bound

$$M_\varepsilon(t) \leq 2C_{\alpha, T} \left[1 + \delta^{-\frac{3}{2}} \right] \equiv \widetilde{M}_0, \quad \text{for all } t \in [0, T^\varepsilon], \quad \varepsilon \in (0, \varepsilon_0]. \tag{6.20}$$

Further choosing ε_0 sufficient small such that

$$\widetilde{M}_0 \leq \frac{1}{2\sqrt{\varepsilon_0}},$$

one concludes, from the definition of T^ε in (4.23) and the estimate (6.20), that

$$T^\varepsilon = T$$

if $\varepsilon \leq \varepsilon_0$. This completes the proof of Theorem 6.1. \square

Having the estimate (6.20), Lemmas 5.2 and 5.3 can be strengthened as follows:

THEOREM 6.2. *There exists a constant $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in (0, 1)$, one has*

$$\|R^\varepsilon\|_{C^\alpha(\mathbf{R}^1 \times [0, T])} \leq C_{\alpha, T}, \quad (6.21)$$

$$\|Z^\varepsilon\|_{C^{1+\alpha, (1+\alpha)/2}(Q_T)} \leq C_{\alpha, T}. \quad (6.22)$$

7. Hölder estimates for u^ε .

Theorem 6.2 implies that the interface (determined by $Z^\varepsilon = 0$ or $r = R^\varepsilon(0, t)$) does not intersect the external walls ($r = r_1$ or $r = r_2$) for a certain amount of time; that is, for any sufficient small positive constant a , the constant T_a defined by

$$T_a := \sup\{t \in (0, T] : Z^\varepsilon(r_1, \tau) \leq -a, Z^\varepsilon(r_2, \tau) \geq a, \forall \varepsilon \in (0, a^2], \tau \in [0, t]\} \quad (7.1)$$

is positive.

THEOREM 7.1. *For any $\alpha \in (0, 1)$ and $a > 0$ sufficiently small, there exists a positive constant $C_{\alpha, a}$ such that, for all $\varepsilon \in (0, a^2]$, one has*

$$\|u^\varepsilon\|_{C^{\alpha, \alpha/2}(Q_{T_a})} \leq C_{\alpha, a} \quad (7.2)$$

where T_a is as in (7.1).

Proof. The definition of T_a implies that φ^ε is exponentially close to ± 1 at the external walls; i.e.,

$$\varphi^\varepsilon(x, t)^2 = \Psi^2\left(\frac{Z^\varepsilon(x, t)}{\varepsilon}\right) = 1 + O(e^{-a/\varepsilon}), \quad \forall \varepsilon \in (0, a^2], t \in [0, T_a]. \quad (7.3)$$

Therefore, the right-hand side of equation (4.1) is uniformly (in ε) bounded in the set $\{(x, t) \in Q_{T_a} : |x| < r_1 + a/4 \text{ or } |x| > r_2 - a/4\}$, so that (7.2) holds in the set $\{(x, t) \in Q_{T_a} : |x| < r_1 + a/8 \text{ or } |x| > r_2 - a/8\}$, by the standard parabolic estimates [20].

It now remains to consider the case when

$$(x, t) \in \Omega_a \times [0, T_a], \quad \Omega_a := \{x \in \mathbf{R}^N : r_1 + a/8 \leq |x| \leq r_2 - a/8\}. \quad (7.4)$$

Write u^ε as the sum of u_1 , u_2 , and u_3 as in (6.4). In view of the estimate (6.7), we need only consider u_3 .

Decompose u_3 into the sum of A , B , and C as in (6.13). One can easily conclude that B is smooth since its kernel $\frac{\partial \tilde{G}}{\partial n}(x, t; \xi, \tau)$ is smooth when $\xi \in \partial\Omega$ and $x \in \Omega_a$.

Next we estimate $C(x, t)$. As in the previous section, we can write C as

$$C(x, t) = I + \dots$$

where

$$I := \int_0^t \int_{-\frac{a}{4\varepsilon}}^{\frac{a}{4\varepsilon}} \Psi'(\eta) \left[\tilde{G}_{r'}(r, t; R^\varepsilon(\varepsilon\eta, t), \tau) - \tilde{G}_{r'}(r, t; R^\varepsilon(\varepsilon\eta, \tau), \tau) \right] d\eta d\tau$$

and \dots are smooth terms since their integrands are smooth if $x \in \Omega_a$.

To estimate I , write I as

$$I = \int_0^t d\tau \int_{-\frac{a}{4\varepsilon}}^{\frac{a}{4\varepsilon}} \Psi'(\eta) d\eta \int_{R^\varepsilon(\varepsilon\eta, \tau)}^{R^\varepsilon(\varepsilon\eta, t)} \tilde{G}_{r', r'}(r, t; r', \tau) dr'.$$

Then, for every $x_1, x_2 \in \Omega_a$, one has the estimate

$$\begin{aligned} & |I(x_1, t) - I(x_2, t)| \\ & \leq \int_0^t d\tau \int_{-\frac{a}{4\varepsilon}}^{\frac{a}{4\varepsilon}} \Psi'(\eta) d\eta \left| \int_{R^\varepsilon(\varepsilon\eta, \tau)}^{R^\varepsilon(\varepsilon\eta, t)} \|\tilde{G}_{r', r'}(\cdot, t, r', \tau)\|_{C^\alpha(\Omega_a)} |x_1 - x_2|^\alpha dr' \right| \\ & \leq \int_0^t d\tau \int_{-\frac{a}{4\varepsilon}}^{\frac{a}{4\varepsilon}} \Psi'(\eta) d\eta |R^\varepsilon(\varepsilon\eta, t) - R^\varepsilon(\varepsilon\eta, \tau)| C_a (t - \tau)^{-\frac{3}{2} - \frac{\alpha}{2}} |x_1 - x_2|^\alpha \\ & \leq \int_0^t d\tau \int_{-\frac{a}{4\varepsilon}}^{\frac{a}{4\varepsilon}} \Psi'(\eta) d\eta C_\beta |t - \tau|^\beta (t - \tau)^{-\frac{3}{2} - \frac{\alpha}{2}} |x_1 - x_2|^\alpha \quad (\text{by Lemma 5.3}) \\ & \leq C_{\beta, a} t^{\beta - \frac{1+\alpha}{2}} |x_1 - x_2|^\alpha \end{aligned}$$

for all $\beta \in (\frac{1+\alpha}{2}, 1)$. Similarly, one can show that

$$|I(x, t_1) - I(x, t_2)| \leq C_{\alpha, a} |t_1 - t_2|^{\alpha/2}, \quad \forall x \in \Omega_a, 0 \leq t_1 \leq t_2 \leq T_a,$$

so that

$$\|I\|_{C^{\alpha, \alpha/2}(Q_t^\varepsilon)} \leq C_{\alpha, a}.$$

Therefore, the function $C(x, t)$ is uniformly (in ε) Hölder continuous.

Finally, we estimate A . By writing it as

$$\begin{aligned} A &= \int_{r_1}^{r_2} \tilde{G}(r, t; r', 0) \left[\Psi\left(\frac{Z^\varepsilon(r', t)}{\varepsilon}\right) - \Psi\left(\frac{Z^\varepsilon(r', 0)}{\varepsilon}\right) \right] dr' \\ &= \int_{Z^\varepsilon(r_1, t)/\varepsilon}^{Z^\varepsilon(r_2, t)/\varepsilon} \tilde{G}(r, t; R^\varepsilon(\varepsilon\eta, t), 0) \Psi(\eta) \frac{\varepsilon}{Z^\varepsilon_r(R^\varepsilon(\varepsilon\eta, t), t)} d\eta \\ &\quad - \int_{Z^\varepsilon(r_1, 0)/\varepsilon}^{Z^\varepsilon(r_2, 0)/\varepsilon} \tilde{G}(r, t; R^\varepsilon(\varepsilon\eta, 0), 0) \Psi(\eta) \varepsilon d\eta, \end{aligned}$$

One can use the same method as in estimating $C(x, t)$ to conclude that

$$\|A\|_{C^{\alpha, \alpha/2}(\Omega_a \times [0, T_a])} \leq C_{\alpha, a}.$$

This completes the proof of Theorem 7.1.

8. Convergence to the sharp interface.

In this section, we shall complete the proof of Theorem 4.1; i.e., we shall show that u^ε tends to the solution of the sharp interface problem as $\varepsilon \searrow 0$, as long as the interface of the solution of the sharp interface problem does not touch the external walls.

By the estimates obtained in Theorem 6.1, Theorem 6.2, and Theorem 7.1, there exists, for every sequence $\{\varepsilon_j\}_{j=1}^\infty$ satisfying $\varepsilon_j \searrow 0$ as $j \rightarrow \infty$, a subsequence, which, for simplicity, we still denote by $\{\varepsilon_j\}_{j=1}^\infty$, such that for all $\alpha \in (0, 1)$ and $a > 0$ sufficient small,

$$u^{\varepsilon_j}(x, t) \longrightarrow u(x, t) \quad \text{uniformly in } C^{\alpha, \alpha/2}(Q_{T_a}), \quad (8.1)$$

$$Z^{\varepsilon_j}(x, t) \longrightarrow Z(x, t) \quad \text{uniformly in } C^{1+\alpha, (1+\alpha)/2}(Q_T), \quad (8.2)$$

$$R^{\varepsilon_j}(0, t) \longrightarrow S(t) \quad \text{uniformly in } C^{(1+\alpha)/2}([0, T]) \quad (8.3)$$

for some $u \in C^{\alpha, \alpha/2}(Q_{T_a})$, $Z \in C^{1+\alpha, (1+\alpha)/2}(Q_T)$, and $S \in C^{(1+\alpha)/2}([0, T])$, where T_a is defined in (7.1).

In the following, we shall assume that $a > 0$ is a fixed small constant.

Note that the interface for the solution $(u^\varepsilon, \varphi^\varepsilon)$ of (4.1)–(4.5) is given by

$$\Gamma^\varepsilon := \{(x, t) \in Q_T : Z^\varepsilon(x, t) = 0\} = \{(x, t) \in Q_T : |x| = R^\varepsilon(0, t)\},$$

so that (8.3) indicates the interface Γ^{ε_j} convergences to $\Gamma := \{(x, t) \in Q_T : |x| = S(t)\}$ as $\varepsilon_j \searrow 0$.

To prove the main theorem (Theorem 4.1), we need show that (u, Γ) is the unique solution to the sharp interface problem (4.15)–(4.17). This will be done in Theorem 8.1 and Theorem 8.2 below.

LEMMA 8.1. *Let Z, S be as in (8.2), (8.3). Then*

$$Z(x, t) = |x| - S(t), \quad (x, t) \in Q_T, \quad (8.4)$$

$$\lim_{j \rightarrow \infty} \varphi^{\varepsilon_j}(x, t) = \begin{cases} 1 & \text{if } |x| > R(t), t \in [0, T], \\ -1 & \text{if } |x| < R(t), t \in [0, T]. \end{cases} \quad (8.5)$$

The assertion (8.4) is a consequence of (5.14) and the uniform convergence of R^ε in (8.3) whereas (8.5) follows from the representation $\varphi^\varepsilon = \Psi(Z^\varepsilon/\varepsilon)$, the uniform convergence of Z^ε in (8.2), and equation (8.4).

The following theorem concerning the motion of the interface is a key feature of the equation (4.1), the Cahn–Allen equation [2, 5, 12, 13, 15, 23–26].

THEOREM 8.1. *The function S is of $C^{1+\alpha}([0, T_a])$ and satisfies*

$$\dot{S}(t) = -\frac{N-1}{S(t)} - \beta u(S(t), t), \quad t \in [0, T_a], \quad (8.6)$$

$$S(0) = r_0 \quad (8.7)$$

where β is a constant defined in (8.19) below.

Proof. We need only show (8.6) since (8.7) follows from the equation $Z^\varepsilon(x, 0) = r - r_0$.

In case u is Lipschitz in x (therefore the solution of (8.6), (8.7) is unique), one can directly use the method developed by Chen in [12, 13] to prove the theorem. Since up until now we only have Hölder estimate for u^ε , we need some modifications to the method developed in [13].

The idea of the proof is to construct, for any $t_0 \in [0, T_a)$ and $\delta > 0$, a supersolution $\varphi^{\varepsilon, \delta, t_0}$ to (4.1) in

$$Q_{t_0, T_a} := \Omega \times [t_0, T_a],$$

where the interface of $\varphi^{\varepsilon, \delta, t_0}$ (the zero level set of $\varphi^{\varepsilon, \delta, t_0}$) is located at $|x| = S^{\varepsilon, \delta, t_0}(t)$ and $S^{\varepsilon, \delta, t_0}(t)$ is a solution to the ODE

$$\frac{d}{dt} S^{\varepsilon, \delta, t_0}(t) = -\frac{N-1}{S^{\varepsilon, \delta, t_0}(t)} - \beta \widetilde{u}^\varepsilon(S^{\varepsilon, \delta, t_0}(t), t) - \delta, \quad t \in [t_0, T], \quad (8.8)$$

$$S^{\varepsilon, \delta, t_0}(t_0) = S(t_0) - 2\delta \quad (8.9)$$

where $\widetilde{u}^\varepsilon$ is a mollifier of u^ε defined in (8.10) below. By first letting $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$, we can conclude that S is a supersolution of (8.6). After using a similar argument to conclude that S is a subsolution of (8.6), one obtains (8.6).

In the following, we shall identify ε_j with ε .

We start by modifying the function u^ε . Let \hat{u}^ε be a radially symmetric $C^{3/4,3/8}$ extension of u^ε in $\mathbf{R}^N \times \mathbf{R}^1$ and $\zeta(x, t)$ be a nonnegative smooth function supported in the unit ball and of unit mass. We define the mollifier $\widetilde{u}^\varepsilon$ of u^ε by

$$\widetilde{u}^\varepsilon(x, t) := \frac{1}{\varepsilon^{\frac{N+2}{2}}} \int_{\mathbf{R}^N} dy \int_{\mathbf{R}^1} d\tau \zeta\left(\frac{y-x}{\varepsilon^{1/2}}, \frac{\tau-t}{\varepsilon}\right) \hat{u}^\varepsilon(y, \tau). \quad (8.10)$$

One can show directly, by using the $C^{3/4,3/8}$ estimate for u^ε (Theorem 7.1), that $\widetilde{u}^\varepsilon$ is radially symmetric and satisfies

$$|\widetilde{u}^\varepsilon(x, t) - u^\varepsilon(x, t)| \leq \sup_{|y-x|+|t-\tau|^{1/2} \leq \varepsilon^{1/2}} |\hat{u}^\varepsilon(y, \tau) - \hat{u}^\varepsilon(x, t)| \leq C\varepsilon^{3/8}, \quad (8.11)$$

$$\|\varepsilon^{1/2} \widetilde{u}^\varepsilon_r, \varepsilon \widetilde{u}^\varepsilon_{rr}, \varepsilon \widetilde{u}^\varepsilon_t\|_{C^0(Q_{T_a})} \leq C, \quad (8.12)$$

$$\|\widetilde{u}^\varepsilon\|_{C^{3/4,3/8}(Q_{T_a})} \leq C \quad (8.13)$$

for some constant C independent of ε .

Next, we define the constant β which appeared in (8.8). Set

$$F(\varphi, \lambda, \mu) := (1 - \varphi^2)(2\varphi + \lambda(1 - \varphi^2)) + \mu. \quad (8.14)$$

Then, there exists a constant $\mu_0 > 0$ such that for every $\lambda \in [-1, 1]$ and $\mu \in [0, \mu_0]$ the algebraic equation, for φ ,

$$F(\varphi, \lambda, \mu) = 0, \quad (8.15)$$

has exactly three solutions: $h^-(\lambda, \mu)$, $h^0(\lambda, \mu)$, and $h^+(\lambda, \mu)$, and they satisfy

$$\begin{aligned} h^-(\lambda, \mu) &< h^0(\lambda, \mu) < h^+(\lambda, \mu), \\ h^-(\lambda, \mu) &< 0 < h^+(\lambda, \mu), \\ h^\pm(\lambda, \mu) &\geq \pm 1 + b\mu \end{aligned} \quad (8.16)$$

for some constant $b > 0$. By a result of Aronson and Weinberger [3], there exists a unique solution $(\Lambda(\lambda, \mu), Q(\lambda, \mu, \rho))$ to the nonlinear eigenvalue problem

$$Q_{\rho\rho} - \Lambda Q_\rho + F(Q, \lambda, \mu) = 0, \quad (8.17)$$

$$Q(\lambda, \mu, \pm\infty) = h^\pm(\lambda, \mu), \quad Q(\lambda, \mu, 0) = 0 \quad (8.18)$$

for any $\lambda \in [-1, 1]$ and $\mu \in [0, \mu_0]$. We define the constant β by

$$\beta := \frac{\partial \Lambda}{\partial \lambda}(0, 0). \quad (8.19)$$

Some properties of the solution $(\Lambda(\lambda, \mu), Q(\lambda, \mu, \rho))$ are stated in the following lemma which has been proven in the appendix of [17]

LEMMA 8.2. *There exist positive constants c and A such that for any $\lambda \in [-1, 1]$ and $\mu \in [0, \mu_0]$, the solution (Λ, Q) to (8.17), (8.18) satisfies*

$$Q_\rho > 0, \quad \forall \rho \in R^1, \quad (8.20)$$

$$\sup_{\rho \in R^1} |Q_\rho, \rho Q_\rho, Q_\lambda, Q_{\lambda\rho}, Q_{\lambda\lambda}, \Lambda_\lambda, \Lambda_\mu, \Lambda_{\lambda\lambda}, \Lambda_{\lambda\mu}, \Lambda_{\mu\mu}| \leq A, \quad (8.21)$$

$$Q(\lambda, \mu, \rho) \geq h^+(\lambda, \mu) - Ae^{-c\rho}, \quad \forall \rho > 0. \quad (8.22)$$

Since the right-hand side of (8.8) is smooth (as a function of $S^{\varepsilon, \delta, t_0}$ and t), the ODE system (8.8), (8.9) has a unique (local) solution $S^{\varepsilon, \delta, t_0}$ and the solution exists as long as it remains in the interval (r_1, r_2) . According to the definition of T_a in (7.1), we can assume, without loss of generality, that $S^{\varepsilon, \delta, t_0}$ exists in $[0, T_a]$ and

$$r_1 - a/2 \leq S^{\varepsilon, \delta, t_0}(t) < r_2, \quad \forall t \in [0, T_a]. \quad (8.23)$$

We now define $\varphi^{\varepsilon, \delta, t_0}$ by

$$\varphi^{\varepsilon, \delta, t_0} := Q\left(\varepsilon \widetilde{u}^\varepsilon(x, t), \varepsilon^{9/8}, \frac{|x| - S^{\varepsilon, \delta, t_0}(t)}{\varepsilon}\right), \quad (x, t) \in Q_{t_0, T_a}. \quad (8.24)$$

Since for small enough ε , we have

$$|\varepsilon \widetilde{u}^\varepsilon| \leq 1 \quad \text{and} \quad \varepsilon^{9/8} \leq \mu_0,$$

the first two arguments for the function Q in (8.24) are in the range of its definition, so that $\varphi^{\varepsilon, \delta, t_0}$ is well-defined.

To complete the proof of Theorem 8.1, we need the following Lemma.

LEMMA 8.3. *For every $\delta > 0$, there exists a constant $\varepsilon_\delta > 0$ such that $\forall \varepsilon \in (0, \varepsilon_\delta]$, the function $\varphi^{\varepsilon, \delta, t_0}$ defined in (8.24) satisfies*

$$\varphi^{\varepsilon, \delta, t_0}(x, t) \geq \varphi^\varepsilon(x, t), \quad (x, t) \in Q_{t_0, T}. \quad (8.25)$$

Consequently,

$$S^{\varepsilon, \delta, t_0}(t) \leq R^\varepsilon(0, t), \quad t \in [t_0, T]. \quad (8.26)$$

We continue the proof of theorem of 8.1. Since the Hölder norm of the right-hand side of (8.8) is bounded independent of ε , the $C^{1+\alpha}([t_0, T_a])$ norm of $S^{\varepsilon, \delta, t_0}$ is also bounded, so that the set $\{S^{\varepsilon, \delta, t_0}\}_{0 < \delta \leq \delta_0, 0 < \varepsilon \leq \varepsilon_\delta}$ is equicontinuous in $C^1[t_0, T_a]$. Therefore,

$$S_{t_0}(t) := \overline{\lim}_{\delta \rightarrow 0^+} \overline{\lim}_{\varepsilon \rightarrow 0^+} S^{\varepsilon, \delta, t_0}(t)$$

exists, and S_{t_0} is in $C^{1+\alpha}[t_0, T_a]$ and satisfies

$$\dot{S}_{t_0}(t_0) = -\frac{N-1}{S(t_0)} - \beta u(S(t_0), t_0).$$

Hence,

$$\begin{aligned} \dot{S}^-(t_0) &:= \liminf_{h \rightarrow 0^+} \frac{S(t_0 + h) - S(t_0)}{h} \\ &= \liminf_{h \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{R^\varepsilon(0, t_0 + h) - S(t_0)}{h} \quad (\text{by (8.3)}) \\ &\geq \liminf_{h \rightarrow 0^+} \overline{\lim}_{\delta \rightarrow 0^+} \overline{\lim}_{\varepsilon \rightarrow 0^+} \frac{S^{\varepsilon, \delta, t_0}(t_0 + h) - S(t_0)}{h} \quad (\text{by (8.26)}) \\ &= \dot{S}_{t_0}(t_0) = -\frac{N-1}{S(t_0)} - \beta u(S(t_0), t_0). \end{aligned} \quad (8.27)$$

That is, S is a supersolution of (8.6). Similarly, we can show S is a subsolution of (8.6), and therefore, S is a solution of (8.6).

To complete the proof of Theorem 8.1, it remains to prove Lemma 8.3. To do this, we need an auxiliary lemma.

LEMMA 8.4. *There exists a positive constant $\varepsilon_\delta > 0$ such that for all $\varepsilon \in (0, \varepsilon_\delta]$, one has*

$$\delta\varepsilon + \varepsilon b \widetilde{u}^\varepsilon - \Lambda(\varepsilon \widetilde{u}^\varepsilon, \varepsilon^{9/8}) \geq 0, \quad (x, t) \in Q_T. \quad (8.28)$$

Proof. By Taylor's expansion, one has

$$\begin{aligned} \Lambda(\varepsilon \widetilde{u}^\varepsilon, \varepsilon^{9/8}) &= \Lambda(0, 0) + \frac{\partial \Lambda}{\partial \lambda}(0, 0) \varepsilon \widetilde{u}^\varepsilon + \frac{\partial \Lambda}{\partial \mu}(0, 0) \varepsilon^{9/8} + O(|\varepsilon \widetilde{u}^\varepsilon|^2 + |\varepsilon^{9/8}|^2) \\ &= \varepsilon \beta \widetilde{u}^\varepsilon + O(\varepsilon^{9/8}) \end{aligned}$$

since $\Lambda(0, 0) = 0$ and $\widetilde{u}^\varepsilon$ is bounded. The inequality (8.28) thus holds for ε sufficient small.

Proof of Lemma 8.3. We need only prove (8.25) since (8.26) follows from the fact that $R^\varepsilon(0, t)$ and $S^{\varepsilon, \delta, t_0}(t)$ are the zero level sets of φ^ε and $\varphi^{\varepsilon, \delta, t_0}$ respectively.

By means of a comparison principle for semilinear parabolic equations, one can prove (8.25) provided that one can show the following:

$$\varphi^\varepsilon(x, t_0) \leq \varphi^{\varepsilon, \delta, t_0}(x, t_0), \quad x \in \Omega, \quad (8.29)$$

$$\varphi^\varepsilon(x, t) \leq \varphi^{\varepsilon, \delta, t_0}(x, t), \quad (x, t) \in \partial\Omega \times [t_0, T_a], \quad (8.30)$$

$$\mathcal{L}\varphi^{\varepsilon, \delta, t_0} := \varphi_t^{\varepsilon, \delta, t_0} - \Delta\varphi^{\varepsilon, \delta, t_0} - \frac{1}{\varepsilon^2} F(\varphi^{\varepsilon, \delta, t_0}, \varepsilon u^\varepsilon, 0) \geq 0, \quad (x, t) \in Q_{t_0, T_a}. \quad (8.31)$$

To prove (8.29), consider two cases:

- (i) $|x| < S(t_0) - \delta$;
- (ii) $|x| \geq S(t_0) - \delta$.

In case (i), one has the bound

$$Z^\varepsilon(x, t) = Z^\varepsilon(x, t) - Z^\varepsilon(R^\varepsilon(0, t), t) = Z_r^\varepsilon(\xi, t)(|x| - R^\varepsilon(0, t)) \leq -\frac{\delta}{4}$$

by the mean value theorem, the estimate $1/2 < Z_r^\varepsilon < 2$ (Lemma 5.1) and the fact

$$|R^\varepsilon(0, t) - S(t)| \leq \delta/2 \quad (8.32)$$

if ε (actually ε_j) is small enough. Therefore, one has

$$\begin{aligned} \varphi^\varepsilon(x, t_0) &= \Psi\left(\frac{Z^\varepsilon(x, t_0)}{\varepsilon}\right) \\ &\leq \Psi\left(-\frac{\delta}{4\varepsilon}\right) \leq -1 + 2e^{-\frac{\delta}{4\varepsilon}} \\ &\leq -1 + b\varepsilon^{9/8} \leq h^-(\varepsilon\widetilde{u}^\varepsilon, \varepsilon^{9/8}) \quad (\text{by (8.16)}) \\ &\leq \varphi^{\varepsilon, \delta, t_0}(x, t_0). \end{aligned} \quad (8.33)$$

In case (ii), we can use the initial condition for $S^{\varepsilon, \delta, t_0}$ in (8.8) to conclude that

$$\begin{aligned} \varphi^{\varepsilon, \delta, t_0}(x, t_0) &= Q\left(\frac{|x| - S(t_0) + 2\delta}{\varepsilon}, \varepsilon\widetilde{u}^\varepsilon, \varepsilon^{9/8}\right) \\ &\geq Q\left(\frac{\delta}{\varepsilon}, \varepsilon\widetilde{u}^\varepsilon, \varepsilon^{9/8}\right) \\ &\geq h^+(\varepsilon\widetilde{u}^\varepsilon, \varepsilon^{9/8}) - Ae^{-c\delta/\varepsilon} \\ &\geq 1 + b\varepsilon^{9/8} - Ae^{-c\delta/\varepsilon} \\ &\geq 1 \geq \varphi^\varepsilon(x, t_0) \end{aligned} \quad (8.34)$$

where (8.22) and (8.16) have been used. Combining (8.33) with (8.34), inequality (8.29) follows.

Similarly, we can show that (8.30) holds by using (8.23) and the definition of T_a .

Finally we verify (8.31). We compute the identity

$$\begin{aligned} \mathcal{L}\varphi^{\varepsilon, \delta, t_0} &= -\frac{1}{\varepsilon}Q_\rho S^{\varepsilon, \delta, t_0}_t + \varepsilon Q_{\lambda\widetilde{u}^\varepsilon}_t \\ &\quad - \left[\frac{1}{\varepsilon^2}Q_{\rho\rho} + \frac{1}{\varepsilon} \frac{N-1}{|x|}Q_\rho + 2Q_{\rho\lambda}\widetilde{u}^\varepsilon_r + \varepsilon Q_{\lambda\Delta\widetilde{u}^\varepsilon} + \varepsilon^2 Q_{\lambda\lambda}\widetilde{u}^{\varepsilon^2}_r \right] \\ &\quad - \frac{1}{\varepsilon^2} \left[F(Q, \varepsilon u^\varepsilon, 0) + F(Q, \varepsilon\widetilde{u}^\varepsilon, \varepsilon^{9/8}) - F(Q, \varepsilon\widetilde{u}^{\varepsilon^\varepsilon}, \varepsilon^{9/8}) \right]. \end{aligned}$$

Using the equation for Q [(8.17)] and the definition of F [(8.14)], one obtains

$$\begin{aligned}
\mathcal{L}\varphi^{\varepsilon,\delta,t_0} &= \frac{Q_\rho}{\varepsilon} \left[-\frac{d}{dt} S^{\varepsilon,\delta,t_0} - \frac{\Lambda(\varepsilon\widetilde{u}^\varepsilon, \varepsilon^{9/8})}{\varepsilon} - \frac{N-1}{|x|} \right] \\
&\quad - \left[Q_\lambda(\varepsilon\Delta\widetilde{u}^\varepsilon - \varepsilon\widetilde{u}^\varepsilon_t) + 2Q_{\rho\lambda}\widetilde{u}^\varepsilon_r + \varepsilon^2 Q_{\lambda\lambda}\widetilde{u}^{\varepsilon^2}_r \right] \\
&\quad - \frac{1}{\varepsilon^2} \left[\varepsilon(u^\varepsilon - \widetilde{u}^\varepsilon)(1-Q^2)^2 - \varepsilon^{9/8} \right] \\
&:= I + II + III.
\end{aligned} \tag{8.35}$$

One can estimate II and III as

$$II \geq -C\varepsilon^{-1/2}, \tag{8.36}$$

$$III \geq -C\varepsilon^{-1/2} + \varepsilon^{-7/8} \tag{8.37}$$

by using (8.11), (8.12) and (8.21).

Since $S^{\varepsilon,\delta,t_0}$ satisfies the ODE (8.8), we can write I as

$$\begin{aligned}
I &= \frac{Q_\rho}{\varepsilon} \left[\frac{N-1}{S^{\varepsilon,\delta,t_0}(t)} + \beta\widetilde{u}^\varepsilon(S^{\varepsilon,\delta,t_0}(t), t) + \delta - \frac{\Lambda(\varepsilon\widetilde{u}^\varepsilon, \varepsilon^{9/8})}{\varepsilon} - \frac{N-1}{|x|} \right] \\
&= \frac{Q_\rho}{\varepsilon} \left[\frac{\delta\varepsilon + \varepsilon\beta\widetilde{u}^\varepsilon - \Lambda(\varepsilon\widetilde{u}^\varepsilon, \varepsilon^{9/8})}{\varepsilon} + \beta(\widetilde{u}^\varepsilon(S^{\varepsilon,\delta,t_0}(t), t) - \widetilde{u}^\varepsilon(x, t)) + (N-1)\frac{|x| - S^{\varepsilon,\delta,t_0}(t)}{|x|S^{\varepsilon,\delta,t_0}(t)} \right] \\
&\geq -\frac{Q_\rho}{\varepsilon} \left[0 + C\beta|S^{\varepsilon,\delta,t_0}(t) - |x||^{3/4} + C(N-1)|x| - S^{\varepsilon,\delta,t_0}(t) \right] \\
&\geq -C\varepsilon^{-1/4} \sup_{\rho \in \mathbb{R}^1} |\rho^{3/4} Q_\rho| \geq -C\varepsilon^{-1/4},
\end{aligned} \tag{8.38}$$

where Lemma 8.4, (8.20), (8.13), and (8.21) have been used.

Substituting (8.36)–(8.38) into (8.35), one obtains

$$\mathcal{L}\varphi^{\varepsilon,\delta,t_0} \geq -C\varepsilon^{-1/4} - C\varepsilon^{-1/2} - C\varepsilon^{-1/2} + \varepsilon^{-7/8} \geq 0$$

if ε is sufficient small. This proves (8.31) and completes the proof of Lemma 8.2 and also Theorem 8.1. \square

THEOREM 8.2. *Let*

$$Q_1 := \{(x, t) \in Q_{T_a} : |x| < S(t)\},$$

$$Q_2 := \{(x, t) \in Q_{T_a} : |x| > S(t)\},$$

$$\Gamma := \{(x, t) \in Q_{T_a} : |x| = S(t)\}.$$

Then, the function u defined in (8.1) satisfies

$$u \in C^{1+\alpha, (1+\alpha)/2}(\overline{Q_1}) \cup C^{1+\alpha, (1+\alpha)/2}(\overline{Q_2}), \quad (8.39)$$

$$u_t - \Delta u = 0 \quad \text{in } Q_1 \cup Q_2, \quad (8.40)$$

$$\dot{S}(t) = [u_r]_+^- \quad \text{on } \Gamma. \quad (8.41)$$

Proof. Let $\Gamma(x, t; \xi, \tau)$ be the fundamental solution of the heat operator $\partial_t - \Delta$, and $w(x, t)$ be the function defined by the surface potential

$$w(x, t) := \int_0^t d\tau \int_{|\xi|=S(\tau)} \dot{S}(\tau) \Gamma(x, t; \xi, \tau) dS_\xi.$$

Then, since $\Gamma \in C^{1+\alpha}$, we know [20, Chapt. 5] that

$$w \in C^{1+\alpha, (1+\alpha)/2}(\overline{Q_1}) \cup C^{1+\alpha, (1+\alpha)/2}(\overline{Q_2}), \quad (8.42)$$

$$w_t - \Delta w = 0 \quad \text{in } Q_1 \cup Q_2, \quad (8.43)$$

$$[w_r]_+^- = \dot{S}(t) \quad \text{on } \Gamma. \quad (8.44)$$

Therefore, for every $\zeta(x, t) \in C_0^\infty(Q_{T_a})$, one has

$$\begin{aligned} 0 &= \iint_{Q_{T_a}} [u_t^{\varepsilon_j} - \Delta u^{\varepsilon_j} + \frac{1}{2} \varphi_t^{\varepsilon_j}] \zeta - \iint_{Q_1 \cup Q_2} (w_t - \Delta w) \zeta \\ &= \iint_{Q_{T_a}} (u^{\varepsilon_j} - w)(-\zeta_t - \Delta \zeta) - \frac{1}{2} \iint_{Q_{T_a}} \varphi^{\varepsilon_j} \zeta_t + \int_0^{T_a} \dot{S}(\tau) \zeta(S(\tau), \tau) d\tau \end{aligned}$$

Letting $\varepsilon_j \rightarrow 0$ and using (8.1) and Lemma 8.1, one gets

$$\begin{aligned} 0 &= \iint_{Q_T} (u - w)(-\zeta_t - \Delta \zeta) - \frac{1}{2} \iint_{Q_2} \zeta_t + \frac{1}{2} \iint_{Q_1} \zeta_t + \int_0^T \dot{S}(\tau) \zeta(S(\tau), \tau) d\tau \\ &= \iint_{Q_T} (u - w)(-\zeta_t - \Delta \zeta), \end{aligned}$$

Hence,

$$(u - w)_t - \Delta(u - w) = 0, \quad \forall (x, t) \in Q_{T_a}, \quad (8.45)$$

and therefore

$$u - w \in C^\infty(Q_{T_a}).$$

Theorem 8.2 thus follows from (8.42)–(8.44). \square

Recall that the solution of (4.14)–(4.16) is unique [14], so that (8.1)–(8.3) are valid for all sequence $\varepsilon \rightarrow 0$. Letting $a \rightarrow 0$, one can conclude, from the definition of T_a , that $T_a \rightarrow T^*$, thereby proving Theorem 4.1.

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