Properties of Cut Polytopes

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Abstract

In this paper we begin by identifying some general properties shared by all cut polytopes which can be used to analyze the general cut polytope in detail. We then characterize the face vectors and diameter of the cut polytopes of all chordless graphs. We finish with a brief summary of *cd*-indices and their possible application to broad classes of cut polytopes.

Contents

1 Introduction

A convex polytope is the generalization of convex polygons into a general number of dimensions. These objects are very well studied by mathematicians. This makes them very useful in problems which can be reduced to linear optimization problems such as the min/max cut problem. The class of polytopes called cut polytopes describe the set of cuts of graphs, which not only allows us to optimize cuts using linear programming, but also lets us make general statements about cuts in graphs and their values.

Overall, the set of cut polytopes or related classes of polytopes tend to appear often (see [8], [7] and [1]), but while their facet-defining inequalities have been very well studied, many properties of the cut polytope have not. The purpose of this paper is to begin a characterization of cut-polytopes.

In section 2 of this paper, we will define a variety of concepts and notation that will be used later in the paper. Section 3 covers the main properties of the cut polytope. We break this up into three smaller discussions; section 3.1 covers some simple but powerful properties exhibited by all cut polytopes, section 3.2 examines the cut polytopes of graphs without C_4 minors, section 3.3 provides a characterization of the cut polytopes of chordless cycles, and section 3.4 closes with an analysis of diameter. Finally, section 4 concludes this paper with a brief discussion of the face lattices of polytopes and how research into face lattices might bear on the characterization of cut polytopes.

2 Definitions and Notation

This paper concerns the properties of cut polytopes in $\mathbb{R}^d$: a class of polytopes which can be used to describe the cuts of graphs. Before we can begin, some discussion of notation is in order.

2.1 Polytopes

In this paper, we will often refer to the **square-length** of a vector. Most of the vectors we deal with will be in $\{0,1\}^d$, so the square length of a vector counts the number components of the vector equal to 1.

we treat a **convex set** $V \subseteq \mathbb{R}^d$ as a set such that for all points $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $\lambda \in [0,1]$, $\lambda\boldsymbol{v}_1 + (1-\lambda)\boldsymbol{v}_2 \in V$. Equivalently, for any two points in V, the line segment connecting them is in V as well.

Let $\text{conv}(V)$ denote the **convex hull** of V, which is the intersection of all convex sets containing all points in V [15]. It is fairly easy to show that for a set of n points $V \subset \mathbb{R}^d$,

$$\text{conv}(V) = \left\{ \sum_{k=1}^{n} \lambda_k \boldsymbol{v}_k \;\middle|\; \sum_{k=1}^{n} \lambda_k = 1, \lambda_k \geq 0 \text{ and } \boldsymbol{v}_k \in V \text{ for } k \in \{1, \ldots, n\} \right\}$$

A **convex polytope** is simply the convex hull of a finite set of points. Since we do not consider concave polytopes, all polytopes in this discussion will be convex.

For any vector $\boldsymbol{x} \in \mathbb{R}^d$, we denote the kth component of $\boldsymbol{x}$ to be x_k. Furthermore, for any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, we say $\boldsymbol{x} \leq \boldsymbol{y}$ if and only if for each $k \leq d$, $x_k \leq y_k$. A linear inequality $\boldsymbol{c} \cdot \boldsymbol{x} \leq c_0$ is called **valid** with respect to a polytope P, if it is true for all $\boldsymbol{x} \in P$.

Every polytope can be represented two different ways. The **$\mathcal{V}$-representation** of a polytope P is the smallest set of points V such that $\text{conv}(V) = P$. We call this set of points the **vertices** of P. The **$\mathcal{H}$-representation** of a polytope P is a set of valid inequalities that completely define and are defined by P. It is a nontrivial result that these two representations are equivalent [15].

Given a polytope P, a **face** F of P is a subset of P such that there exists a valid inequality that holds with equality over F and is a strict inequality over $P \setminus F$.

The **dimension** of a polytope is the lowest dimension of any affine plane containing it. Any dimensions we calculate will be fairly self evident in this paper.

The dimension of a face is determined the same way. For a d-dimensional polytope (called a d-polytope), its $(d-1)$-faces are called **facets**. The vertices of a polytope are its 0-dimensional faces. We call the 1-dimensional faces of a polytope its **edges**, and any two vertices connected by an edge are **adjacent**. Finally, we say an edge is incident to a vertex if that vertex is one of the edge's two endpoints.

A **poset** is any set equipped with a comparison that is reflexive, antisymmetric and transitive, but not necessarily complete. A **graded poset** is a poset such that there exists a **rank function $\boldsymbol{\rho}$** such that if $x \geq y$ then $\rho(x) \geq \rho(y)$. The converse is false only if x and y are incomparable.

For any polytope P, we can construct a vector such that its kth component is the number of k-faces on P for $k \in \{0, \ldots, d\}$. This vector is called the **f-vector** of P. Similarly, we can create a graded poset of the faces of P ordered by inclusion. The rank of each element F is $\dim(F) + 1$. This poset is called the **face lattice** of P. Two polytopes are considered combinatorially equivalent if they have congruent face lattices.

For any two vertices $\boldsymbol{x}$,$\boldsymbol{y}$ of a polytope P, we say that the **distance** between the vertices, $\mathrm{dist}(\boldsymbol{x}, \boldsymbol{y})$, is the number of edges in the smallest path between them. The **diameter** of P, $\mathrm{diam}(P)$, is the maximum distance between any two vertices in the polytope. A polytope is **neighborly** if it has diameter 1.

A d-**simplex** (Δ_d) is the combinatorially unique d-polytope of $d+1$ vertices. A d-**cube** is any polytope combinatorially equivalent to $\mathrm{conv}(\{0,1\}^d)$.

2.2 Graphs

A **graph** G is a set of vertices V and a set of edges E connecting them. We usually say $G = (V, E)$ when introducing a graph.

Given two sets A and B, we say the **union** of the sets, $A \cup B$, is the set of all elements in either set. The **intersection** of the sets, $A \cap B$ is the set of elements in both sets, and the **symmetric difference** of the sets, $A \triangle B$, is the set of elements in exactly one of A or B.

A **path** of a graph is a sequence of vertices in the graph where each vertex is adjacent to the previous and the next vertex such that every vertex in the sequence is distinct. A **cycle** is similar to a path except the first and last vertex in the would-be path are the same. A **chord** is a path connecting two vertices in the same cycle that has no edges in common with the cycle.

The **minor** of a graph is any graph that can be obtained by deleting edges and isolated vertices as well as contracting edges. If a graph has a C_4 minor, it must have a C_n subgraph for $n \geq 4$.

We say a graph is **connected** if for any two vertices in the graph, there exists a path connecting them. Assume all graphs are connected unless otherwise stated.

Let the n-**cycle**, C_n, be the graph consisting only of the cycle of n edges. Also, let the **complete graph** of n vertices, K_n, be the graph consisting of n vertices with all pairs of vertices adjacent to each other. Finally, let the **complete bipartite graph** of n and m vertices, $B_{n,m}$, be the graph obtained from two sets of nonadjacent vertices of size n and m such that every vertex in one set is adjacent to all vertices in the other set.

Define the n-**sum** of two graphs to be the operation that takes the union of two graphs with K_{n+1} subgraphs assuming the K_{n+1} subgraphs are the intersection of the graphs. This sum is denoted by the $\#_k$ operator.

2.3 Cut Polytopes

Given a connected graph $G = (V, E)$, a set of edges $C \in E$ is a **cut** if $G \setminus C$ is no longer connected, and every edge $e \in C$ connects two disjoint components of $G \setminus C$. We can come up with a second equivalent definition here by noting that each cut partitions the vertices of G into two sets. Then a set C of edges is a cut if and only if there exists a partition of the vertices into two sets such that C is the set of edges connecting the two vertex sets.

If $G = (V, E)$ is a graph, let C be a cut of G. Then $\boldsymbol{x}^C$ is a **cut vector** if

$$x_e^C = \begin{cases} 1 & \text{if } e \in C \\ 0 & \text{if } e \notin C \end{cases}$$

Finally, let G be a graph. Then the **cut polytope** of G is the convex hull of all cut vectors of G. The operation $\text{cut}(G)$ returns the cut polytope of G.

3 Characteristics of Cut Polytopes

For the remainder of the paper, we will focus on the major properties of cut polytopes that we were able to find in the literature as well as in our own research.

3.1 Basic properties

We begin with the most simple properties of cut polytopes.

Definition 3.1.1. *A 0/1 d-polytope is a d-polytope such that all of its vertices are elements of* $\{0, 1\}^d$.

It follows immediately from the definition of a cut polytope that all cut polytopes are 0/1 polytopes. Throughout this paper we will find several consequences of this, for example:

Conjecture 3.1.2. *(Hirsch Conjecture): Given a d-polytope P with n vertices, the diameter of P is less than or equal to $n - d$.*

This conjecture was actually disproven in 2010 by Francisco Santos in [11], but it is true for 0/1 polytopes with additional condition that the diameter of a 0/1 polytope is less than or equal to its dimension [15], so we already have a lower bound on the diameter of a given cut polytope. Some other useful properties are derived below.

Proposition 3.1.3. *If $P \subset \mathbb{R}^d$ is a cut polytope, then dim(P) $= d$. In otherwords, P is full-dimensional.*

Proof. Barahona, Grötschel and Mahjoub show this in [2] $\square$

The next proposition simplifies the problem of finding the vertices of a cut polytope:

Proposition 3.1.4. *Let G be a graph with d edges and $\mathcal{C}$ be the set of all cuts of G. Then the mapping $f : \mathcal{C} \to \mathbb{R}^d$ defined by $f(C) = \boldsymbol{x}^C$ is a bijection between the cuts of G and the vertices of $P =$ cut(G).*

Proof. Since f is clearly a bijection between the set of cuts and the set of cut vectors, it will suffice to show that all cut vectors are vertices of P, but since P is a 0/1 polytope, it has the property the vertices of P are exactly the points in the set $\{0,1\}^d \cap P$. $\square$

Because of this bijection, we will often treat cuts and cut-vectors interchangeably. For instance, when we mention the "adjacency" of two cuts, we mean the adjacency of the cut vectors in the cut polytope. This result can be used to introduce a rather nice property about the vertices of a cut polytope:

Theorem 3.1.5. *If G is a graph with n vertices then cut(G) has 2^{n-1} vertices.*

Proof. We use the second definition of a cut to identify a unique partition of the vertices of G given a cut C. Let $v \in V$ be a vertex in G. Place v in one set of our partition for C and assume $v_0 \in V$ is adjacent to v. Then if the edge connecting them is in C, v_0 belongs in the opposite set as v. Otherwise v_0 belongs in the same set. In this way we can determine exactly which set each of v's neighbors belong.

Because G is connected, we can trace a path from v to any vertex v_1. In this way, since each vertex along a path is adjacent to the next, we can determine exactly to which set each vertex on the path belongs. Thus we can determine to which set every vertex in G belongs.

Since every partition corresponds to a unique cut by definition, we have shown a one-to-one correspondence between partitions of V and cuts of G. Since there are 2^{n-1} partitions of V, by Proposition 3.1.4, there must also be 2^{n-1} vertices in cut(G). $\square$

Proposition 3.1.6. *For a graph $G = (V, E)$, an edge set $C \subseteq E$ is a cut if and only if every chordless cycle in G contains an even number of elements of C.*

Proof. We can reuse the path tracing technique we used to prove Theorem 3.1.5. Suppose we partition the vertices of G into the red set and the blue set. Suppose G_i is a cycle in G with an odd number of edges in C. Place an arbitrary vertex $v_r \in G_i$ in the red set. Trace the cycle, and at each point determine where the vertex belongs much as we did in the proof of Theorem 3.1.5. The set in which the vertices belong only switches when two vertices are adjacent to an edge in C, but there an odd number of elements of C in G_i. As a result, by the time we get back to v_r, the vertices have switched colors an odd number of times, so v_r is in the blue set, but we defined v_r to be in the red set. This is a contradiction, so C cannot be a cut of G.

In the other direction, suppose every cycle in G contains an even number of edges in C. Then since we assume G is connected, choose v_r and place it in the red set. Now we can trace every point in the set. Every time we get back to v_r, we must have followed a cycle with an even number of elements in C, so the vertex color has changed an even number of times and v_r is in the red set. Similarly, if we visit any vertex multiple times, we must have done so via a cycle, so its color remains consistent. Since we do not run into any contradictions, we have constructed a partition of the vertices corresponding to C. C is a cut. □

We now look at the symmetry of cut polytopes. They exhibit a very high level of symmetry, and we finally have the tools to prove the following theorem which clearly illustrates this symmetry.

Theorem 3.1.7. *Let P be a cut polytope, and let $\boldsymbol{v}$ and $\boldsymbol{w}$ be vertices of P. Then there exists a reflection R such that $R(\boldsymbol{v}) = \boldsymbol{w}$, and $R(P) = P$.*

Proof. This theorem claims cut polytopes "look" the same from the perspective of all vertices. If true, then all combinatorial properties of a single vertex apply to all other vertices as well.

We begin with a lemma that may seem unrelated at first, but will prove quite useful later on in this proof.

Lemma 1. *For any graph, the symmetric difference between two cuts is a cut.*

Proof. Let C_0 and C_1 be cuts of a graph G. Let G_0 be a cycle in G. Then by Proposition 3.1.6, it suffices to show that $(C_0 \triangle C_1)$ has an even number of elements in G_0. In otherwords, $|G_0 \cap (C_0 \triangle C_1)|$ is even. To see this, let $C_0 \cap G_0 = D_0$ and $C_1 \cap G_1 = D_1$. Then,

$$G_0 \cap (C_0 \triangle C_1) = (C_0 \cap G_0) \triangle (C_1 \cap G_0) = D_0 \triangle D_1$$

Using inclusion-exclusion,

$$|D_0 \triangle D_1| = |D_0| + |D_1| - 2|D_0 \cap D_1|$$

but by Proposition 3.1.6, both $|D_0|$ and $|D_1|$ are even, so $|G_0 \cap (C_0 \triangle C_1)|$ must be even. $\square$

Now, we begin with a weaker version of the theorem using the fact that the all cut polytopes contain the origin as a vertex.

Lemma 2. *Let $\boldsymbol{v}$ be a vertex in $P =$cut(G) corresponding to a cut $C_{\boldsymbol{v}}$. Then there exists a reflection $R^{\boldsymbol{v}}$ that preserves P, maps $\boldsymbol{v}$ to the origin and is equivalent to a mapping $\sigma_{C_{\boldsymbol{v}}} : 2^E \to 2^E$ that permutes the cuts of G.*

Proof. We utilize the identity below:

$$I \triangle J = I \triangle K \iff J = k$$

By proposition 3.1.6 and the identity above, if we define $\sigma_{C_{\boldsymbol{v}}}$ by

$$\sigma_{C_{\boldsymbol{v}}}(I) = C_{\boldsymbol{v}} \triangle I$$

then $\sigma_{C_{\boldsymbol{v}}}$ is a permutation of the cuts of G.

Define

$$R^{\boldsymbol{v}}(\boldsymbol{w}) = \boldsymbol{x}^{\sigma_{C_{\boldsymbol{v}}}(C_{\boldsymbol{w}})}$$

we can equivalently state $R^{\boldsymbol{v}}$ as follows:

$$R^{\boldsymbol{v}}_e(\boldsymbol{w}) = \begin{cases} 1 - w_e & \text{if } e \in C_{\boldsymbol{v}} \\ w_e & \text{if } e \notin C_{\boldsymbol{v}} \end{cases}$$

So $R^{\boldsymbol{v}}(\boldsymbol{v}) = \mathbf{0}$. All that remains to be proven is that $R^{\boldsymbol{v}}$ is a reflection. To do this, let

$$H = \left\{ \boldsymbol{x} \in \mathbb{R}^d \;\middle|\; v_e x_e = \frac{1}{2} v_e \text{ for all } e \in E \right\} = \left\{ \boldsymbol{x} \in \mathbb{R}^d \;\middle|\; R^{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{x} \right\}$$

Let $\boldsymbol{y} \in \mathbb{R}^d$ be arbitrarily chosen, and pick $\boldsymbol{z}$ such that

$$z_e = \begin{cases} \frac{1}{2} & \text{for } e \in C_{\boldsymbol{v}} \\ y_e & \text{for } e \notin C_{\boldsymbol{v}} \end{cases}$$

Then $\boldsymbol{z} \in H$. Also for any $\boldsymbol{a}$ and $\boldsymbol{b} \in H$, $(\boldsymbol{z} - \boldsymbol{y}) \cdot (\boldsymbol{a} - \boldsymbol{b})$ is defined by

$$(z_e - y_e)(a_e - b_e) = \begin{cases} \left(\frac{1}{2} - y_e\right)\left(\frac{1}{2} - \frac{1}{2}\right) & \text{for } e \in C_{\boldsymbol{v}} \\ (y_e - y_e)(a_e - b_e) & \text{for } e \notin C_{\boldsymbol{v}} \end{cases} = 0$$

So $\boldsymbol{z}$ is the unique point on H such that $\boldsymbol{z} - \boldsymbol{y}$ is orthogonal to H. Thus, the reflection of $\boldsymbol{y}$ over H is given by $\boldsymbol{y} + 2(\boldsymbol{z} - \boldsymbol{y}) = 2\boldsymbol{z} - \boldsymbol{y}$. But,

$$2z_e - y_e = \begin{cases} 1 - y_e & \text{for } e \in C_{\boldsymbol{v}} \\ y_e & \text{for } e \notin C_{\boldsymbol{v}} \end{cases} = R_e^{\boldsymbol{v}}(\boldsymbol{y})$$

So $R^{\boldsymbol{v}}(\boldsymbol{y})$ is the reflection of $\boldsymbol{y}$ over H. □

Let $R = R^{\boldsymbol{v}} \circ R^{\boldsymbol{w}}$. Then,

$$R(\boldsymbol{x}^I) = R^{\boldsymbol{v}}(\boldsymbol{x}^{C_{\boldsymbol{w}} \triangle I}) = \boldsymbol{x}^{C_{\boldsymbol{v}} \triangle (C_{\boldsymbol{w}} \triangle I)} = \boldsymbol{x}^{(C_{\boldsymbol{v}} \triangle C_{\boldsymbol{w}}) \triangle I} = R^{\boldsymbol{v} \triangle \boldsymbol{w}}(\boldsymbol{x}^I)$$

Which means R is once again a reflection.

$$R(\boldsymbol{w}) = R^{\boldsymbol{v}}(\boldsymbol{0}) = \boldsymbol{v}$$

And,

$$R(\boldsymbol{v}) = \boldsymbol{x}^{C_{\boldsymbol{v}} \triangle (C_{\boldsymbol{v}} \triangle C_{\boldsymbol{w}})} = \boldsymbol{x}^{C_{\boldsymbol{w}}} = \boldsymbol{w}$$

So R is our reflection that swaps $\boldsymbol{v}$ and $\boldsymbol{w}$ while preserving P. □

Theorem 3.1.7 will prove useful. This theorem allows us to figure out the combinatorial properties of cut polytopes from the perspective of the origin and extrapolate those results to the other vertices of the graph.

Definition 3.1.8. *A cut I is called a minimal cut if no cuts are properly contained in I.*

Now we can get a very simple discription of which cuts are adjacent.

Proposition 3.1.9. *Let I and J be two cuts in G. Then $\boldsymbol{x}^I$ and $\boldsymbol{x}^J$ in P are adjacent if and only if $I \triangle J$ is a minimal cut in G.*

Proof. First assume I is the empty cut.

If $I \triangle J$ is a minimal cut, then since $I \triangle J = J$, J is also a minimal cut.

Let $\boldsymbol{c} = \boldsymbol{x}^J - \mathbf{1}$. Then $\boldsymbol{c} \cdot \boldsymbol{v} = 0$ for $\boldsymbol{v} \in \{0,1\}^d$ if and only if $C_{\boldsymbol{v}} \subseteq J$. Thus the equation $\boldsymbol{c} \cdot \boldsymbol{v} = 0$ only has the origin and $\boldsymbol{x}^J$ as solutions among all the vertices of P. For the remaining vertices, $\boldsymbol{c} \cdot \boldsymbol{x}^C = -|C \setminus J| < 0$.

This means $\boldsymbol{c} \cdot \boldsymbol{x} \leq 0$ is a valid inequality with equality only for $\boldsymbol{x}^J$ and $\mathbf{0}$, so J is adjacent to I.

Now suppose $\boldsymbol{x}^J$ is adjacent to the origin. Then there exists a valid inequality $\boldsymbol{c} \cdot \boldsymbol{v} \leq c_0$ such that $\boldsymbol{c} \cdot \boldsymbol{x}^J = \boldsymbol{c} \cdot \mathbf{0} = 0 = c_0$. So

$$\sum_{e \in J} c_e = 0$$

Now suppose there exists a cut $K \subset J$ that is not empty. Then $K \triangle J = J \setminus K$ is also a cut by Lemma 1 in Theorem 3.1.7. Furthermore, $K \cup (K \triangle J) = J$, and K and $K \triangle J$ are disjoint:

$$\boldsymbol{c} \cdot \boldsymbol{x}^K + \boldsymbol{c} \cdot \boldsymbol{x}^{K \triangle J} = \sum_{i \in K} c_i + \sum_{j \in K \triangle J} c_j = \sum_{k \in J} c_k = 0$$

So either $\boldsymbol{c} \cdot \boldsymbol{x}^K \geq 0$ or $\boldsymbol{c} \cdot \boldsymbol{x}^{K \triangle J} \geq 0$ meaning the origin and $\boldsymbol{x}^J$ are not adjacent.

Finally, drop the assumption that I is the empty cut. Then by Theorem 3.1.7, I and J correspond to adjacent vertices if and only if $R^{\boldsymbol{x}^I}(I) = \emptyset$ and $R^{\boldsymbol{x}^I}(J) = I \triangle J$ are adjacent. So I and J are adjacent if and only if $I \triangle J$ is minimal. $\square$

Corollary 3.1.10. *Two cuts I and J of a connected graph $G = (V, E)$ correspond to adjacent vertices in cut(G) if and only if the graph $H = (V, E \setminus (I \triangle J))$ has exactly two connected components.*

Proof. This was proved by Barahona in [3], but we provide an alternate proof here.

By Proposition 3.1.9, I and J are adjacent if and only if $I \triangle J$ is a minimal cut.

Suppose $C \subset I \triangle J$ is a cut. Then since $C \triangle (I \triangle J) = (I \triangle J) \setminus C$ is a nonempty cut, we can first remove C from E and then remove $(I \triangle J) \setminus C$. Removing the first cut splits G into at least two connected components. Then removing a disjoint nonempty cut splits at least one of the components of G yet again. Thus if I and J are not adjacent, G is split into more than two connected components.

Suppose $D = I \triangle J$. Then since it is a cut, removing D from E will split G into at least two connected components. Suppose D splits G into more than two components. Then we can partition H into three graphs G_1, G_2 and G_3 such that (1) each contains at least one connected component of H, (2) None of the subgraphs are connected to either of the others by any edges.

The edges required to make H connected are comprised of at least two cuts. So D is not minimal. □

We finish this section with a final result relating to the face characteristics of cut polytopes:

Theorem 3.1.11. *A graph G is not contractable to K_5 if and only if cut(G) is defined by the following $\mathcal{H}$-representation:*

$$0 \leq v_e \leq 1 \text{ for each edge that does not belong to a triangle}$$

$$\left(\boldsymbol{x}^F - \boldsymbol{x}^{C \setminus F}\right) \cdot \boldsymbol{v} \leq |F| - 1, \text{ for each chordless cycle } C, F \subseteq C, |F| \text{ odd}$$

Proof. This is Corollary 3.10 in [3]. □

3.2 Graphs without a C_4 minor

We have found that graphs without C_4 minors are very simple to analyze:

Proposition 3.2.1. *A graph G does not have a C_4 minor if and only if it can be expressed as the 0-sum of several copies of K_2 and K_3.*

Proof. We use the fact that any graph has a C_4 minor if and only if it has a C_n subgraph for $n \geq 4$.

Suppose G is expressible as a 0-sum of K_2's and K_3's. Then G does not have any cycles larger than 3 edges, so G does not have a C_4 minor.

Suppose G cannot be expressed as a 0-sum of K_2's and K_3's. Suppose G_0 is a subgraph of G such that it cannot be expressed as the 0-sum of K_2, K_3 or any other graph (For example, if $G = C_5 \#_0 K_2$ then $G_0 = C_5$). Then every edge in G_0 is part of a cycle (otherwise there would be a 0-sum involving K_2). G_0 is therefore composed of 3-cycles, and every 3-cycle in G_0 is attached to the rest of the graph by an edge. Therefore, $C_3\#_1C_3$ is a subgraph of G_0, but $C_3\#_1C_3$ has a C_4 minor. $\square$

Proposition 3.2.1 allows us to analyze this class of graphs in terms of very simple 0-sums.

Definition 3.2.2. *A d-polytope P is simple if every vertex $\boldsymbol{v}$ in P is adjacent to exactly d edges.*

Simple polytopes are considered dual to another class of polytopes called **Simplicial** polytopes. Together, these represent the most thoroughly researched class of polytopes. Their face vectors have been completely classified [15].

Definition 3.2.3. *A 0/1 d-polytope is smooth if for each vertex $\boldsymbol{v}$, the set of edges adjacent to it form a basis of $\mathbb{Z}^d$. [6]*

This class of polytopes has algebraic relevance. Because of this, one of the most common definitions of a smooth polytope is any polytope with a smooth toric variety (these definitions are equivalent by Theorem 2.4.3 in [6]). However, we do not deal with toric varieties in this paper.

Proposition 3.2.4. *A 0/1 full dimensional polytope is smooth if and only if it is simple*

Proof. Suppose P is a smooth 0/1 polytope. Since $\dim(\mathbb{Z}^d) = d$, any basis of $\mathbb{Z}^d$ must have d elements. Thus all smooth polytopes must have d edges incident to each vertex, so all smooth 0/1 polytopes are simple.

Suppose P is a simple 0/1 polytope. Since it is full dimensional, the edges incident to each vertex must span $\mathbb{R}^d$. Furthermore, since it has exactly d edges per vertex, these edges form a basis of $\mathbb{R}^d$. Now, because P is a 0/1 polytope, the components of its edges are all 0's and 1's, so they span $\mathbb{Z}^d$. $\square$

Proposition 3.2.4 can be used to translate results in the literature about smooth polytopes to simple polytopes:

Theorem 3.2.5. *The cut polytope $P = cut(G)$ is simple if and only if G has no C_4 minors.*

Proof. Sullivant and Stermfels [13] prove this for smooth toric varieties in Corollary 2.4. By [6], this implies that $P =$ cut (G) is smooth if and only if G has no C_4 minors. Proposition 3.2.4 shows this is true for simple polytopes as well. $\square$

Because of Theorem 3.2.5, by the end of the section we will have completely classified the face vectors of all simple cut polytopes. The simplest way to do this is to use the f and h polynomials:

Definition 3.2.6. *Given the f-vector of a simple polytope P, the f polynomial is the function $f(x) = \sum_{k=0} f_k x^k$. The h polynomial is the function $h(x) = f(x-1)$. The h-vector is the vector of coefficients of the h polynomial, so h_k is the coefficient of x^k in $h(x)$.*

in [15], Ziegler applies more intuitive definition of h-vector to derive a result similar to the definition above for simplicial polytopes. This definition sets the h-vector equal to the h-vector of its dual polytope using Ziegler's definition.

Definition 3.2.7. *Given two polytopes P_1 and P_2, the product of these polytopes $P = P_1 \times P_2$ is given by the set*

$$P = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \middle| \boldsymbol{x} \in P_1 \text{ and } \boldsymbol{y} \in P_2 \right\}$$

Now there is a very nice way to characterize both the f and h vectors of a polytope product using polynomials:

Proposition 3.2.8. *Let $P = P_1 \times P_2$ where f_1 and f_2 are the f polynomials of P_1 and P_2 respectively. Then the f polynomial of P is $f_1 \cdot f_2$.*

Proof. We can clearly see that P_1 and P_2 are faces of P. Therefore, any face of P must be composed of products of faces of P_1 and P_2. To see this, suppose P_1 is a d_1-polytope, and P_2 is a d_2-polytope. Let $F \subseteq P$ be a face of P that maximized the linear function $\boldsymbol{c} \in \mathbb{R}^{d_1+d_2}$. Let $\boldsymbol{c_1}$ be the projection of $\boldsymbol{c}$ to its first d_1 components and $\boldsymbol{c_2}$ the projection of $\boldsymbol{c}$ to its last d_2 components. Then if F_1 is maximized over $\boldsymbol{c_1}$ and F_2 is maximized over $\boldsymbol{c_2}$ they are clearly faces of P_1 and P_2 respectively. Thus, the hyperplane which is fixed in F_1 intersects F in an area combinatorially equivalent to F_2 and vice versa, so $F = F_1 \times F_2$.

Now, if F_1 maximizes some linear function $\boldsymbol{c}_1$ over P_1, and F_2 maximizes some linear function $\boldsymbol{c}_2$ over P_2, then $(\boldsymbol{c}_1, \boldsymbol{c}_2)$ maximizes $F_1 \times F_2$ over P. So $F_1 \times F_2$ must be a face of P.

Since F is a face of P if and only if it can be decomposed into a product of faces of P_1

and P_2, we can calculate the kth component of the f-vector of P as follows:

$$f_k = \sum_{i=0}^{k} f_{1,i} f_{2,k-i}$$

so,

$$f(x) = \sum_{k=0}^{d_1+d_2} \sum_{i=0}^{k} f_{1,i} f_{2,k-i} x^k = (f_1 \cdot f_2)(x)$$

□

Corollary 3.2.9. *the h polynomial of a product of polytopes is the product of the h polynomials of each polytope.*

Proof. This is a direct consequence of Proposition 3.2.8. □

Now we get to the most significant result in this section:

Theorem 3.2.10. *The cut polytope of the 0-sum of two graphs is the product of the cut polytopes of each graph.*

Proof. Let $G = G_1 \#_0 G_2$, $P = \mathrm{cut}(G)$, $P_1 = \mathrm{cut}(G_1)$ and $P_2 = \mathrm{cut}(G_2)$. Then it will suffice to show that $\boldsymbol{v} \in \mathrm{vert}(P)$ if and only if $\boldsymbol{v}$ is the cartesian product of a vertex in P_1 and a vertex in P_2. Then as a consequence of Proposition 3.2.8, the vertices of P are also the vertices of $P_1 \times P_2$.

Let C_1 be a cut of G_1 and C_2 be a cut of G_2. Then $C_1 \cup C_2$ corresponds to $\boldsymbol{x}^{C_1} \times \boldsymbol{x}^{C_2}$. To show that this is a cut of G, notice that G_1 and G_2 have only a single vertex in common. Thus we can assign that vertex to an arbitrary set and trace the vertex partitions of both C_1 and C_2, meaning that $C_1 \cup C_2$ is a cut.

Suppose C is not a union of cuts. Furthermore, suppose without loss of generality $C \cap G_1 = D$ is not a cut. Then there exists a cycle in G_1 with an odd number of edges in D. Since G_1 and G_2 only share a vertex, $C \cap G_2$ does not change this, so C cannot be a cut since there is an cycle in G with an odd number of edges in C. □

And now we can generate a complete characterization of the face vectors of simple cut polytopes:

Corollary 3.2.11. *The h polynomial is a bijection between the set of simple cut polytope and all functions of the form $h(x) = (1+x)^a(1+x+x^2+x^3)^b$ where a and b are nonnegative integers.*

Proof. By Corollary 3.2.5, the set of graphs without C_4 minors generates the set of simple cut polytopes. These graphs can be represented as the 0-sum of K_2 and K_3 graphs.

The h-polynomial of K_2 is $(x-1)+2 = x+1$, and the h-polynomial of K_3 is $(x-1)^3+4(x-1)^2+6(x-1)+4 = x^3+x^2+x+1$. Thus, by Corollary 3.2.9 and Proposition 3.2.1, all cut polytopes of graphs without C_4 minors have h polynomials of the form $h(x) = (x+1)^a(x^3+x^2+x+1)^b$ for nonnegative integers a and b. Furthermore, we can construct a set of graphs with the correct cut polytope given a and b. Note, there are multiple graphs with the same cut polytope and therefore same h-vector. Regardless, since both the h-vectors and the cut polytopes are constructed the same way using polytope products, each h-vector maps to a unique cut polytope by the commutativity of polytope products. □

3.3 The Cut Polytope of a Cyclic Graph

To extend our results of the previous section, we first need a better understanding of cyclic graphs and their cut polytopes. Given a path, all sets of edges are cuts. Thus the only restrictions an what edges are cuts come from cycles. Fortunately, we already have all of the tools necessary to analyze these polytopes. Our goal is to characterize the face vectors of all cut polytopes of chordless cycles. To do this we first need a little bit of notation.

Definition 3.3.1. *We say $E_d = cut(C_d)$ is an even d-polytope.*

Proposition 3.3.2. *The set of cuts of a Cyclic Graph are the set of all even subsets of the edges.*

Proof. This is a weaker statement of Proposition 3.1.6. □

Corollary 3.3.3. *The vertex set of E_d is the set of all points in $\{0,1\}^d$ with an even square-length.*

Proof. Each point in $\{0,1\}^d$ with an even number of 1's is the incidence vector of an even set of edges, which by Proposition 3.3.2 is a cut in C_d. Also, every point in $\{0,1\}^d$ with an odd number of 1's is the incidence vector of an odd set of edges of C_d which is not a cut. □

It is very easily verifiable that E_3 is the 3-simplex Δ_3. Using this and Theorem 3.1.11, we can achieve an inductive characterization of the facets of E_d for $d \geq 4$.

Proposition 3.3.4. *All facets of E_d are either simplices or even polytopes.*

Proof. By Theorem 3.1.11, we can represent $2d$ facets of E_d by the equations

$$0 \leq v_e \leq 1 \tag{1}$$

for all edges. Note, these are the equations the define the $\mathcal{H}$-representation of the unit d-cube. Therefore, we have $2d$ facets of E_d inscribed on the facets of the d-cube.

To identify these facets, we look at the set of facets defined by $v_e \geq 0$. These are all of the facets defined by (1) that contain the origin. Fix $e \in E$ and let H be the hyperplane defined by this equation. Then $\text{vert}(E_d) \cap H = \{\boldsymbol{x} \in \{0,1\}^d | \boldsymbol{x}$ has an even square-length and $x_e = 0\}$. However, this is just the set of vertices in E_{d-1}. Since the facets described by equation (1) containing the origin are all E_{d-1}, all of the equations described by (1) are E_{d-1} by Theorem 3.1.7.

Theorem 3.1.11 states that the remaining facets in E_d are described by

$$\left(\boldsymbol{x}^F - \boldsymbol{x}^{C_d \setminus F}\right) \cdot \boldsymbol{v} = |F| - 1 \text{ for } F \subseteq E \text{ and } |F| \text{ odd} \tag{2}$$

Fix F. Then we can see equation (2) is only satisfied for all cuts of size $|F| - 1$ contained in F and by all cuts of size $|F| + 1$ that contain F. There are exactly $|F|$ cuts of the first type and $d - |F|$ cuts of the second type. In other words, for each F, equation (2) defines a $(d-1)$-polytope with d vertices. This is only satisfied by Δ_{d-1}. $\square$

Corollary 3.3.5. *For $d \geq 3$, all faces of E_d are either simplices or even polytopes.*

Proof. We know E_3 is a simplex, so all of its faces are simplices.

Suppose E_{d-1} has only even polytopes and simplices for faces. By Proposition 3.3.4, the facets of E_d are Δ_{d-1} and E_{d-1}. Therefore, the faces of E_d are itself, its facets and the faces of its facets, which are by assumption all either even polytopes or they are simplices.

Thus, by induction, the faces of E_d are all simplices and even polytopes for $d \geq 3$. $\square$

We can also define adjacency in E_d with minimal difficulty.

Proposition 3.3.6. *Two cuts I and J are adjacent in C_d if and only if $I \triangle J$ has exactly two elements.*

Proof. By Proposition 3.3.2, the set all of cuts of C_d is the set of all even subsets of the edge set of C_d. Thus a nonempty cut is minimal in C_d if and only if it contains exactly two elements. The rest of the proof follows directly from Proposition 3.1.9. $\square$

Corollary 3.3.7. *E_d has diameter $\lfloor \frac{d}{2} \rfloor$.*

Proof. By Theorem 3.1.7, it suffices to show that the greatest distance between the origin and any vertex in E_d is $\lfloor \frac{d}{2} \rfloor$. To travel from any point $\boldsymbol{v}$ back to the origin, Proposition 3.3.6 says we can proceed by changing exactly two components of $\boldsymbol{v}$. So, the shortest path from $\boldsymbol{v}$ to the origin would subtract two from the square length of the vertex for every edge travelled. Therefore, the distance between $\boldsymbol{v}$ and the origin is half the square-length of $\boldsymbol{v}$. Since the square length of all vertices in E_d is even, the diameter is half the largest even number less than or equal to d, or in otherwords, the diameter is $\lfloor \frac{d}{2} \rfloor$. □

We only need two more theorems before we're ready to directly compute the f-vector for E_d.

Theorem 3.3.8. *For $d \geq 4$ and $3 < k < d$, exactly $\binom{d}{k}2^{d-k}$ k-faces of E_d are combinatorially equivalent to E_k.*

Proof. In the proof of Proposition 3.3.4, we proved that every E_{d-1} facet of E_d is inscribed in a facet of the unit d-cube, and that every facet of the unit d-cube has an even facet inscribed. Suppose for $k > 3$, the set of E_{k+1} faces of E_d are mapped bijectively to the set of $(k+1)$-faces of the unit d-cube by inclusion. Then all of the E_k facets of each E_{k+1} face will be inscribed in each facet of the $(k+1)$-cube inscribed around the face. But then every k-face of the d-cube is circumscribed around a E_k face, and every E_k face is inscribed in a k-face of the d-cube.

By induction, the number of k-faces of E_d that are even is equal to the number of k-faces of a d-cube. To derive this number, notice that each k-face of a d-cube is the set of all points in $[0,1]^d$ such that $d-k$ components of these points stay constant and the other k components vary freely. There are 2^{d-k} choice to choose the arrangement of the constant components, and there are $\binom{d}{d-k}$ ways to choose which components are fixed over the k-face. There are $2^{d-k}\binom{d}{d-k} = 2^{d-k}\binom{d}{k}$ ways for this to happen. Thus there are $2^{d-k}\binom{d}{k}$ even k-faces of E_d.

Note: this proof doesn't quite fail for the case of $k = 3$, but the issue is complicated because E_3 is a simplex. □

And finally,

Theorem 3.3.9. *There are $\frac{d2^{d-1}}{k+1}\binom{d-1}{k}$ simplex k-faces of E_d when $k > 1$ and $k \neq 3$. There are $2^{d-3}\left(4\binom{d}{4} + \binom{d}{3}\right)$ simplex k-faces of E_d when $k = 3$.*

Proof. Let us select a set of $k+1$ pairwise adjacent cuts. Suppose one of the cuts is the empty cut. Then all other cuts must have 2 edges by Proposition 3.3.6. For any of the other two cuts to both be adjacent and have 2 edges, they must have exactly one edge in common. If $k \neq 3$, then all of the chosen cuts that are not empty must have exactly one edge in common. If $k = 3$, then we have the case of three cuts sharing three edges (any two nonempty cuts have an edge in common, but all three nonempty chosen cuts have no edges in common).

So, when $k \neq 3$, we choose our constant edge. This gives us d choices. Let this edge be e. Then we choose our k cuts of size 2 and all containing e. This gives us a total of $d\binom{d-1}{k}$ pairwise adjacent sets of $k+1$ cuts. Look at the set of vertices V represented by these cuts. Let $S = \{f \in E | f \neq e, v_f = 1 \text{ for some } \boldsymbol{v} \in V\}$. Finally, let

$$\delta(f) = \begin{cases} 1 & \text{if } f = e \\ -1 & \text{if } f \in S \\ -2 & \text{otherwise} \end{cases}$$

Then the affine plane

$$\sum_{f \in E} \delta(f) x_f = 0$$

is valid and holds with equality for $\boldsymbol{v} \in V$ so V is the set of vertices of a simplex k-face of E_d.

If $k = 3$, then the choice of vertices above is a valid option. In addition, we may characterize some of the possible choices as a set of three edges, each vertex of the simplex other than the origin is contains two of those edges. Then there are $\binom{d}{3}$ such simplices. Let $S = \{e_1, e_2, e_3\}$ be the set containing those three edges, and let

$$\delta(f) = \begin{cases} 0 & \text{if } f \in S \\ -1 & \text{otherwise} \end{cases}$$

Then once again, the affine plane defined by

$$\sum_{f \in E} \delta(f) x_f = 0$$

is maximized over those vertices. In conclusion, we have $d\binom{d-1}{k}$ k simplex faces containing the origin. There are 2^{d-1} vertices by Theorem 3.1.5, so by Theorem 3.1.7, there are a total of $\frac{d2^{d-1}}{k+1}\binom{d-1}{k}$ k simplex faces when $k \neq 3$ and $k > 1$, and $\frac{d2^{d-1}}{4}\binom{d-1}{3} + \frac{2^{d-1}}{4}\binom{d}{3} = 2^{d-3}\left(4\binom{d}{4} + \binom{d}{3}\right)$ simplex faces when $k = 3$. ☐

Corollary 3.3.10. *In the f-vector of E_d,*

$$f_k = \begin{cases} 2^{d-1} & \text{if } k = 0 \\ 2^{d-2}\binom{d}{2} & \text{if } k = 1 \\ \frac{d2^{d-1}}{k+1}\binom{d-1}{k} & \text{if } k = 2 \\ \frac{d2^{d-1}}{k+1}\binom{d-1}{k} + 2^{d-k}\binom{d}{k} & \text{otherwise} \end{cases}$$

Proof. This is just a direct application of Proposition 3.3.5 and Theorems 3.3.8 and 3.3.9. ☐

So we now have characterized the f-vectors of all simple cut polytopes. In addition we have the f-vectors of all even cut polytopes. Using 3.2.10, we can now quite easily find the cut vector of any graph for which all cycles are chordless.

3.4 Neighborliness and Diameter

We've spent a lot of time analyzing the face vectors of polytopes, but one of the most important properties of cut polytopes is their diameter. This puts a lower bound on the number of edges that need to be hopped in the simplex algorithm to reach the optimal solution. The easiest way to calculate this is to split the graph into 0-sums. To do this, we need to know how adjacency and diameter work with respect to polynomial products.

Proposition 3.4.1. *If $P = P_1 \times P_2$ with vertices $\boldsymbol{v_0}$ and $\boldsymbol{v_1}$ in P_1 and $\boldsymbol{w_0}$ and $\boldsymbol{w_1}$ in P_2, then $(\boldsymbol{v_0}, \boldsymbol{w_0})$ and $(\boldsymbol{v_1}, \boldsymbol{w_1})$ are adjacent if and only if*

(1) $\boldsymbol{v_0}$ is adjacent to $\boldsymbol{v_1}$ with $\boldsymbol{w_0} = \boldsymbol{w_1}$ or,

(2) $\boldsymbol{w_0}$ is adjacent to $\boldsymbol{w_1}$ with $\boldsymbol{v_0} = \boldsymbol{v_1}$

Proof. By the proof of Proposition 3.2.8, each edge in P is necessarily the product of a vertex and an edge in P_0 and P_1. ☐

Theorem 3.4.2. *Polytope products on cut polytopes are additive with respect to diameter.*

Proof. Let $P = P_1 \times P_2$ be polytopes with $\text{diam}(P_1) = a$, $\text{diam}(P_2) = b$ and $\text{diam}(P) = c$. Let $\boldsymbol{v_0}$ and $\boldsymbol{v_1}$ be vertices in P_1 and let $\boldsymbol{w_0}$ and $\boldsymbol{w_1}$ be vertices in P_2.

Then, we can make a path from $(\boldsymbol{v_0}, \boldsymbol{w_0})$ to $(\boldsymbol{v_1}, \boldsymbol{w_0})$ to $(\boldsymbol{v_1}, \boldsymbol{w_1})$. Thus,

$$\text{dist}((\boldsymbol{v_0}, \boldsymbol{w_0}), (\boldsymbol{v_1}, \boldsymbol{w_1})) \leq \text{dist}(\boldsymbol{v_0}, \boldsymbol{v_1}) + \text{ dist } (\boldsymbol{w_0}, \boldsymbol{w_1})$$

Since this applies to all vertices,

$$\text{diam} P \leq \text{diam} P_1 + \text{diam} P_2$$

Furthermore, by Proposition 3.4.1, the only way to get from $(\boldsymbol{v_0}, \boldsymbol{w_0})$ to $(\boldsymbol{v_1}, \boldsymbol{w_1})$ is to separately follow a path in P_1 to $\boldsymbol{v_1}$ and in P_2 to $\boldsymbol{w_1}$. Thus

$$\text{dist}((\boldsymbol{v_0}, \boldsymbol{w_0}), (\boldsymbol{v_1}, \boldsymbol{w_1})) = \text{dist}(\boldsymbol{v_0}, \boldsymbol{v_1}) + \text{ dist } (\boldsymbol{w_0}, \boldsymbol{w_1})$$

and

$$\text{diam} P = \text{diam} P_1 + \text{diam} P_2$$

□

Corollary 3.4.3. *Suppose a graph $G = (V, E)$ has no C_4 minors, meaning it can be expressed as a zero sum of a number of copies of K_2 and K_3. Then if the sum contains n elements, diam(G) $= n$.*

Proof. K_2 and K_3 correspond to the line segment and the 3-simplex respectively, both of which have diameter one. Therefore, if the sum contains n components,

$$\text{diam}(G) = \sum_{k=1}^{n} 1 = n$$

□

Next, we can, to a limited extent, see the effect of adding extra edges:

Theorem 3.4.4. *Let $H = (V, F)$ be a connected graph and let $F \subset E$. Then if $G = (V, E)$, diam(cut(G)) $\leq$ diam(cut(H)).*

Proof. Let $P =$cut(G) and $Q =$cut(H). Suppose $\boldsymbol{v}$ and $\boldsymbol{w}$ correspond to the same partition of vertices in G and H respectively, and suppose $\boldsymbol{v}$ is not adjacent to the origin in Q. Then there exists an alternate partition of the vertex set of G which is contained in $C_{\boldsymbol{v}}$. It's clear that as long as H remains connected, this alternate partition of the vertex set must also yield a cut contained in $C_{\boldsymbol{w}}$, so $\boldsymbol{w}$ is not adjacent to the origin.

Thus, adding edges to a graph preserves adjacency in graphs. This implies the inequality we are trying to prove. Be careful to note however that the reverse relation is not true. Removing an edge from K_3 transforms a neighborly simplex to a square which is not neighborly. Furthermore, the inequality is not strict. The graph of C_4 with a single chord and chordless C_4 both have diameter two. $\square$

Proposition 3.4.5. *cut(K_n) is neighborly*

Proof. This has been proven multiple times. See [3] and [14] for two different proofs. $\square$

Proposition 3.4.6. *K_n is the only set of graphs that have neighborly cut polytopes.*

Proof. Let D_n be K_n with a single edge removed. By Theorem 3.4.4, it will suffice to show D_n is not neighborly.

Let the vertices adjacent to the missing edge in D_n be v_1 and v_2. Then $B_{2,n-2}$ is a subgraph of D_n. Let C be the cut which is equal to the edge set of $B_{2,n-2}$. Then C separates v_1 and v_2 from the rest of the vertices of D_n. If we place v_2 in the other set, the resulting cut C' does not consist of any additional edges, but it contains only half the edges of $B_{2,n-2}$. Thus C is not a minimal cut and D_n is not neighborly $\square$

In fact, the cut C described in the proof above is the only cut of D_n which is not minimal. As a result, every vertex is nonadjacent to exactly one other vertex. This creates an interesting pairing.

3.5 Further Inquiries

The author was not able to provide a complete characterization of face vectors or diameters of cut polytopes. One interesting area of study would be the set of graphs reducible to K_5. This topic came up often, because of certain algebraic properties of these graphs, yet they seemed poorly studied.

Another useful study would be the effects of chords in cycles. Such a study could possibly expand the individual results of this paper into a general result for all cut polytopes.

4 The cd-index

Most of this paper has been devoted to face vectors and diameters of cut polytopes. We now discuss the face lattice of cut polytopes.

The face vectors of all simple and simplicial polytopes have been completely characterized as a set of well understood conditions on the h-vectors of these polytopes. Much of this work was based on the Dehn-Summerville equations [15]. In 1985, Billera and Bayer found generalized Dehn-Summerville equations which applied to all Eulerian Posets (of which arbitrary polytope face lattices are a subclass) [5]. Later, Bayer and Klapper invented the *cd*-index [4]. We will follow their construction of this index, cite a few results in the field and discuss possible applications of the *cd*-index in the field of cut polytopes.

4.1 Construction and a few results

We begin as always with a few definitions.

Definition 4.1.1. *We define the f-flag-vector of a lattice $\mathcal{L}$ as the vector of elements with indices of the form $S \subseteq \{1, 2, \ldots, d-1\}$ so that $f_S(\mathcal{L})$ is the number of chains in $\mathcal{L}$ excluding both the maximum and minimum element such that for each chain, the set of ranks of all elements in the chain is S.[10]*

The f-flag-vector encodes much more information about a given polytope than the f-vector alone does. Unfortunately, the cost of this is that the f-flag-vector also grows exponentially with dimension. The f-flag-vector comes with a h-flag-vector as well.

Definition 4.1.2. *The h-flag-vector of a given lattice $\mathcal{L}$ is given by*

$$h_S(\mathcal{L}) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(\mathcal{L})$$

This vector can be encoded in a noncommutative polynomial of two variables.

Definition 4.1.3. *For an Eulerian poset $\mathcal{L}$, let ϕ be a mapping so that for $S \subseteq \{1, \ldots, d\}$*

$$\phi_k(S) = \begin{cases} b & \textit{if } k \in S \\ a & \textit{otherwise} \end{cases}$$

where a and b are noncommutative. Then

$$\Psi_{(a,b)}(\mathcal{L}) = \sum_{S\subseteq\{1,\ldots,d\}} \left(h_S \prod_{k=1}^{d} \phi_k(S) \right)$$

is the ab-index of $\mathcal{L}$.

And the cd-index is

Definition 4.1.4. *For noncommutative variables c and d and Eulerian poset $\mathcal{L}$,*

$$\Psi(\mathcal{L}) = \Psi_{(c,d)}(\mathcal{L}) = \Psi_{(a+b,ab+ba)}(\mathcal{L})$$

is the cd-index of $\mathcal{L}$.

The cd-index is useful because it encodes the f-flag vector and the generalized Dehn-Sommerville equations into a single polynomial of fibonacci length [4]. Furthermore, it turns out the coefficients of the cd-index are nonnegative [12] integers [4].

In general the cd-index is difficult to analyze because of its complexity, but there are alternate formulations of it that simply its computation.

Theorem 4.1.5. *Let $\mathcal{L}$ be an Eulerian poset and define*

$$\mathcal{L}(a,b) = \{c \in \mathcal{L} | a \leq c \leq b\}$$

as a subposet of $\mathcal{L}$. Then

$$\begin{aligned} 2\Psi(\mathcal{L}) = & \sum_{\substack{\hat{0}<x<\hat{1} \\ \rho(x,\hat{1})=2j-1}} \Psi(\mathcal{L}(\hat{0},x))c(c^2-2d)^{j-1} \\ & - \sum_{\substack{\hat{0}<x<\hat{1} \\ \rho(x,\hat{1})=2j}} \Psi(\mathcal{L}(\hat{0},x))c(c^2-2d)^{j} + \begin{cases} 2(c^2-2d)^{k-1} & \textit{if } \rho(\mathcal{L}) = 2k-1 \\ 0 & \textit{if } \rho(\mathcal{L}) = 2k \end{cases} \end{aligned}$$

Proof. This is just Theorem 1.1 in [12] □

This is an easy formula to work with if $\mathcal{L}$ is the face lattice of a polytope P, since all subposets of the form $\mathcal{L}(\hat{0},x)$ are the face lattices of faces of P.

A paper by Lee also addresses this issue by offering an alternative formula for the *cd*-index (see [9]). Mark Purtill offers a combinatorial interpretation for the *cd*-index of certain classes of polytopes. If this could be extended it would simplify the computation of *cd*-indices of general cut polytopes [10].

Finally, the most interesting set of papers are those that examine transformations of polytopes and their effects on the *cd*-index. Richard Ehrenborg and Harold Fox provide a paper which produces a simple recursive formula to calculate the *cd*-index of a polynomial product given the *cd* indices of each original polynomial. This could be used in conjunction with 3.2.10 to compute a large number of polytopes.

Unfortunately we were unable to derive any specific properties of the *cd*-indices of cut polytopes, but we did use available algorithms to compute the *cd*-indices for all graphs of dimension less than or equal to 6, with the exception of C_6, for future researchers (see appendix).

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A Table of cd-indices for low dimensional cut polytopes

Let L_n be the 0-sum of n K_2's.

Graph	Diameter	Polytope type	cd-Index
K_2	1	line	c
L_2	2	2-cube	$c^2 - 2d$
K_3	1	Δ_3	$c^3 + 2cd + 2dc$
L_3	3	3-cube	$c^3 + 4cd + 6dc$
L_4	4	4-cube	$c^4 + 6c^2d + 16cdc + 14dc^2 + 20d^2$
$K_3\#_0K_2$	2	Prism(Δ_3)	$c^4 + 4c^2d + 8cdc + 6dc^2 + 8d^2$
C_4	2	E_4	$c^4 + 14c^2d + 16cdc + 6dc^2 + 20d^2$
L_5	5	5-cube	$c^5 + 64cd^2 + 80dcd + 100d^2c + 8c^3d + 30c^2dc + 48cdc^2 + 30dc^3$
$K_3\#_0L_2$	3	2-cube $\times\Delta_3$	$c^5 + 32cd^2 + 36dcd + 44d^2c + 6c^3d + 18c^2dc + 24cdc^2 + 14dc^3$
$C_3\#_1C_3$	2	Unknown	$c^5 + 24cd^2 + 20dcd + 24d^2c + 6c^3d + 16c^2dc + 16cdc^2 + 6dc^3$
C_5	2	E_5	$c^5 + 112cd^2 + 80dcd + 132d^2c + 24c^3d + 94c^2dc + 64cdc^2 + 14dc^3$
$C_4\#_0K_2$	3	Prism(E_4)	$c^5 + 80cd^2 + 80dcd + 84d^2c + 16c^3d + 46c^2dc + 40cdc^2 + 14dc^3$
L_6	6	6-cube	$488d^3 + 140c^2d^2 + 320cdcd + 384cd^2c + 236dc^2d + 480dcdc + 356d^2c^2 + 10c^4d + 48c^3dc + 110c^2dc^2 + 128cdc^3 + 62dc^4 + c^6$
$K_3\#_0L_3$	4	$\Delta_3\times$ 3-cube	$224d^3 + 80c^2d^2 + 160cdcd + 192cd^2c + 112dc^2d + 224dcdc + 164d^2c^2 + 8c^4d + 32c^3dc + 62c^2dc^2 + 64cdc^3 + 30dc^4 + c^6$
$K_3\#_0K_3$	2	$\Delta_3 \times \Delta_3$	$104d^3 + 44c^2d^2 + 80cdcd + 96cd^2c + 52dc^2d + 104dcdc + 76d^2c^2 + 6c^4d + 20c^3dc + 34c^2dc^2 + 32cdc^3 + 14dc^4 + c^6$

Graph	Diameter	Polytope type	cd-Index
$C_4\#_0L_2$	4	$E_4\times$2-cube	$488d^3 + 204c^2d^2 + 352cdcd + 416cd^2c + 220dc^2d + 448dcdc + 276d^2c^2 + 18c^4d + 80c^3dc + 134c^2dc^2 + 96cdc^3 + 30dc^4 + c^6$
$(C_3\#_1C_3)\#_0K_2$	3	Unknown	$136d^3 + 68c^2d^2 + 112cdcd + 136cd^2c + 64dc^2d + 132dcdc + 92d^2c^2 + 8c^4d + 30c^3dc + 50c^2dc^2 + 40cdc^3 + 14dc^4 + c^6$
$C_3\#_1C_4$	2	$E_4 \times \Delta_3$	$248d^3 + 156c^2d^2 + 216cdcd + 280cd^2c + 96dc^2d + 232dcdc + 156d^2c^2 + 16c^4d + 68c^3dc + 106c^2dc^2 + 64cdc^3 + 14dc^4 + c^6$
$C_4\#_2C_4$ minus an edge	2	Unknown	$488d^3 + 316c^2d^2 + 432cdcd + 432cd^2c + 188dc^2d + 360dcdc + 188d^2c^2 + 34c^4d + 108c^3dc + 130c^2dc^2 + 72cdc^3 + 14dc^4 + c^6$
$C_5\#_0K_2$	3	Prism(E_5)	$648d^3 + 348c^2d^2 + 480cdcd + 672cd^2c + 236dc^2d + 640dcdc + 420d^2c^2 + 26c^4d + 144c^3dc + 254c^2dc^2 + 144cdc^3 + 30dc^4 + c^6$
K_4	1	Unknown	$104d^3 + 76c^2d^2 + 104cdcd + 96cd^2c + 52dc^2d + 80dcdc + 44d^2c^2 + 14c^4d + 32c^3dc + 34c^2dc^2 + 20cdc^3 + 6dc^4 + c^6$

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