Contracting Convex Torus by its Harmonic Mean Curvature Flow

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Abstract

We consider the problem of contracting convex torus in Hyperbolic space by the harmonic mean of the principal curvatures. The shape of the torus is studied theoretically and numerically as each point on the torus moves towards the axis with a speed equal to the harmonic mean curvature. Due to the contrasting behavior $\lambda_1 \to 0, \lambda_2 \to \infty$ of the principal curvatures of contracting torus, HMCF of torus is expected to be uniformly parabolic in $\lambda_1$-direction but degenerating in $\lambda_2$-direction. For the theoretical part, we assume the torus is axially symmetric and obtain estimates of the gradient function and the harmonic mean curvature using the parabolic maximum principle. The main result is that $\lambda_1 \approx e^{-t}, \lambda_2 \approx e^t, \lambda_1 \lambda_2 \approx 1$. We verify that HMCF is indeed uniformly parabolic and the shape in the limit is close to torus. We employ numerical methods to explore the case of general torus and provide numerical evidence that the torus does not evolve into a round shape if the initial surface has a low frequency component in $\lambda_2$-direction.
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In this thesis we consider deforming surfaces immersed in three dimensional hyperbolic manifolds with the speed function given by the curvature of the surface. The surface we consider is a torus whose axis is a closed geodesic of the hyperbolic manifold and each point on the surface moves inward direction at the speed equal to the harmonic mean of the principal curvatures of the surface. We are interested in studying the shape of the surface as it contracts to the axis. The motion of the surface can be described by a family of immersions \( \Phi : M^2 \times [0, T) \rightarrow N^3 \) where \( M^2 = T^2 \) is a torus \( S^1 \times S^1 \) and \( N^3 = H^3/\Gamma \) is a three dimensional hyperbolic space modulo some discrete group action \( \Gamma \). The family of immersions satisfy following equations:

\[
\frac{\partial \Phi}{\partial t}(x, t) = -F(x, t) \cdot N(x, t)
\]

\[
\Phi(x, 0) = \Phi_0(x)
\]

for all \((x, t) \in T^2 \times [0, T)\) where \( \Phi_0 : T^2 \rightarrow H^3/\Gamma \) is an immersion, \( F = \lambda_1\lambda_2/(\lambda_1 + \lambda_2) \) is the harmonic mean of the principal curvatures \( \lambda_1, \lambda_2 (\lambda_1 < \lambda_2) \) of \( \Phi(T^2, t) \) at \( x \), and \( N(x, t) \) is the outward unit normal vector of \( \Phi(T^2, t) \) at \( x \).

Numerous authors have studied the curvature flow problem of this kind under different hypersurfaces \( M \), ambient manifolds \( N \) and the speed functions \( F \). The work by Andrews \([2, 3]\) is one of the first papers that consider this problem with a large class of speed functions, including harmonic mean curvature flow, in Euclidean space and Riemannian manifolds. They show that if the initial immersion of a compact manifold \( M \) without boundary is strictly convex, i.e. all the principal curvatures are positive, then the hypersurface contracts to a point and its shape becomes spherical in the limit \( t \rightarrow T \). When the ambient space is a general Riemannian manifold with sectional curvature bounded below by \(-c\) \( (c > 0) \) they assume the principal curvatures of initial surface is greater than \( \sqrt{c} \) in order to overcome the background geometry. The problem (HMCF) we consider does not satisfy this condition since it can be shown that the small principal curvature tends to 0 uniformly. Moreover, the evolving torus we consider contracts to a line not to a point. We point out that the negative curvature of the background space is used in crucial way to obtain strict convexity of torus and the analysis that follows. Recently, \([6]\) studied general flows focusing on three dimensional hyperbolic space \( H^3 \) as its ambient space and showed that certain surfaces evolve to a round point.

An important feature of (HMCF) in PDE perspective is that it is nonuniformly parabolic. The parabolicity stays strong in one direction but degenerates as time progresses in the other direction as can be seen from following estimates: \( \partial F/\partial \lambda_1 = \lambda_2^2/(\lambda_1 + \lambda_2)^2 \rightarrow 1 \) and
\[ \frac{\partial F}{\partial \lambda_2} = \frac{\lambda_2^2}{(\lambda_1 + \lambda_2)^2} \to 0 \] whenever \( \frac{\lambda_2}{\lambda_1} \to \infty \). We will show that under harmonic mean curvature flow the curvature estimate \( \frac{\lambda_2}{\lambda_1} \to \infty \) holds on torus in hyperbolic manifolds whose axis is a closed geodesic and more generally on torus generated by revolving a graph about its axis. Also note that although the parabolicity is degenerating, it does not become degenerate in finite time.

In Chapter 2 of this thesis, we first consider the case when the initial torus is axially symmetric. Under this assumption, only \( \lambda_1 \) depends on the (spatial) second derivative, thus the parabolicity is determined by \( \lambda_1 \) alone and \( \text{(HMCF)} \) is shown to be uniformly parabolic. The main result of this section is the curvature estimates \( \lambda_1 \approx e^{-t}, \lambda_2 \approx e^{t} \) and \( \lambda_1 \lambda_2 \approx 1 \). The challenge in studying cylindrical surfaces is to show that two principal curvatures move in opposite directions, \( \lambda_1 \to 0 \) and \( \lambda_2 \to \infty \), but their product is equivalent to a constant. We adopt a local coordinate around the closed geodesic developed in [8] on rotationally symmetric spaces and perform similar analysis in order to bound the gradient function. Harmonic mean curvature flow of surfaces generated by revolving a graph about the axis in Euclidean space is also studied in [11].

The issues regarding the degeneration of parabolicity in \( \lambda_2 \) direction is explored in Chapter 3 of this thesis. There have been several studies where a curvature flow that effectively regularizes the initial hypersurface to spherical shape loses its smoothing property when the speed function is raised to a small power and sphere is no longer the only limit shape. The well known result on curve shortening flow by [14] is that a closed embedded convex curve shrinking by its curvature \( k \) evolves to circular shape asymptotically. However, if the speed function is replaced by \( k^\alpha \) where \( \alpha \in (0, 1) \) then the asymptotic shape of the curve depends on \( \alpha \). If \( 1/3 < \alpha < 1 \), circle is the only asymptotic shape [4]. On the other hand, if \( \alpha = 1/3 \), the limiting shape is an ellipse [17] and if \( 0 < \alpha < 1/3 \), the isoperimetric ratios of the evolving curve approach infinity [5]. In another study [1] where they evolve an embedded sphere with speed equal to a power \( \alpha \in (0, 1) \) of the harmonic mean curvature, it is proved that there exist self-similar solutions of the flow other than the shrinking sphere. A self-similar solution is a solution of the flow that scales the initial immersion by a time dependent factor, i.e. the solution can be expressed as \( \phi(x, t) = \psi(t) \cdot \phi_0(x) \) where \( \phi_0 \) is the initial immersion and \( \psi(t) \to 0 \).

Although we are interested in knowing whether certain surfaces would fail to converge to round cylindrical shape as the flow loses its regularizing effect in \( \lambda_2 \) direction, it is very difficult to imagine what the expected result would be because of the contrasting behavior of the principal curvatures and the complexity of computation involved in the analysis. Thus,
we conduct a numerical study of (HMCF) in order to gain insights on the right set-up of the problem to address the issue of nonconvergence and the expected outcome. We adopt spectral methods to solve (HMCF) numerically since it outperform any other numerical schemes when the solution is known to be $C^\infty$ and periodic [9]. We find that if a perfect torus is perturbed only in the direction perpendicular to the axis at a low frequency it fails to converge to round cylindrical shape. However, it does converge if the perturbation is of high frequency or the surface is perturbed in both directions. Based on numerical estimates of the derivatives of graph function, we identify a simple quantity involving the second derivatives of graph function that captures the asymptotic behavior of the curvature functions. Then we examine the eigenvalues of the linearized harmonic mean curvature flow to demonstrate that convergent surfaces attain eigenvalues larger than the ones that do not converge.
Chapter 1

Theoretical Analysis of Axially Symmetric Torus

1.1 Introduction

We consider the contraction of a convex torus embedded in Hyperbolic 3-manifolds to a closed geodesic using the harmonic mean of the principal curvatures. Each point on the torus whose axis is a closed geodesic moves in the normal direction pointing to its axis with a speed equal to the harmonic mean curvature. Let $\Sigma^2 = S^1 \times S^1$ be a two dimensional torus, $N^3$ a Hyperbolic 3-manifold containing a closed geodesic and $\Phi_0: \Sigma^2 \rightarrow N^3$ a smooth initial immersion of $\Sigma^2$ into $N^3$ centered at a closed geodesic. The evolution process is described by one parameter family of immersions $\Phi: \Sigma \times [0, T) \rightarrow N$ satisfying

$$\frac{\partial \Phi(p,t)}{\partial t} = -F(p,t) \cdot N(p,t)$$

$$\Phi(p,0) = \Phi_0(p).$$

Here, $F = \lambda_1 \lambda_2 / (\lambda_1 + \lambda_2)$ is the harmonic mean curvature of $\Sigma_t := \Phi(\Sigma, t)$ where $\lambda_1, \lambda_2$ are the principal curvatures and $N$ is the outward unit normal vector of $\Sigma_t$.

Andrews studied HMCF of strictly convex compact hypersurface without boundary in Euclidean [2] and Riemannian manifolds [3], showing that the evolving hypersurface converges to a round point in finite time. Other authors studied HMCF of hypersurfaces in Euclidean space under various curvature conditions ([10], [11], [12], [13]) and showed that the evolving hypersurface converges, when it does, to a round point. In this paper, we are interested in
surfaces converging to a closed geodesic, not a point, in Hyperbolic 3-manifolds by HMCF. Examples of hypersurfaces in Hyperbolic manifolds converging to a totally geodesic submanifold by HMCF were constructed in \cite{15}. However, only hypersurfaces at a constant distance from totally geodesic submanifolds were considered in \cite{15}, so the curvature flow problem reduced to analyzing simple ODEs. This paper generalizes parts of the results in \cite{15} to axially symmetric surfaces. Recently in \cite{7}, weakly convex hypersurfaces in Euclidean space containing cylindrical regions were shown to shrink to a line segment when the hypersurface is deformed by certain curvature function. However, curvatures of the evolving surface were not analyzed in that paper.

In this paper, we will obtain curvature estimates of an axially symmetric torus contracting to a closed geodesic in Hyperbolic 3-manifold by HMCF. Analyzing the principal curvatures of a torus presents a novel problem since as the torus approaches the axis we expect the small principal curvature to converge zero and the large principal curvature to approach infinity. And the product of the principal curvatures is expected to be more or less constant since it equals 1 (See (1.1)) on a perfect torus whose axis is a closed geodesic. This kind of curvature estimate is different from the estimates obtained for spherical hypersurfaces in Theorem 4.1 of \cite{3} and Theorem 5.1 of \cite{16} where they prove that the ratio of principal curvatures are uniformly bounded. We need to estimate each principal curvature separately to show that they exhibit contrasting dynamics but the product should remain bounded throughout the evolution process.

We will consider a torus $\Sigma^2$ embedded into a Hyperbolic 3-manifold $N^3$ such that it is axially symmetric about a closed geodesic $\gamma: S^1 \to N^3$. Let $r: S^1 \to [0, R]$ be a generating function defined on $\gamma$. An axially symmetric torus can be constructed by revolving the graph of the generating function about the closed geodesic.

**Theorem 1** (Main Theorem). Let $\Sigma_0$ be an axially symmetric torus around a closed geodesic $\gamma$ in a Hyperbolic 3-manifold $N$, generated by revolving a graph of $r: S^1 \to \mathbb{R}^+$ about $\gamma$. Assume $\Sigma_0$ is strictly convex and $\max_{x \in \Sigma_0} F(x) < 1/2$ where $F(x)$ is the harmonic mean curvature at $x \in \Sigma_0$. Then, the solution of the HMCF with initial surface $\Sigma_0$ exists for all $t \in [0, \infty)$ and remains strictly convex. The evolving surface converges to the closed geodesic exponentially fast and the principal curvatures satisfy $\lambda_1 \approx e^{-t}, \lambda_2 \approx e^t$ and $\lambda_1 \lambda_2 \approx 1$.

**Notation.** Uniform constants are denoted by $C_i$. The same symbol $C$ might imply different constants from line to line. The approximation symbol $f \approx g$ denotes that there exist $C_1, C_2 > 0$ such that $C_1 g \leq f \leq C_2 g$. 
Remarks. (1) The reason we impose the curvature condition $\max_{\Sigma_0} F < 1/2$ is because for perfectly symmetric torus $0 < F(r) < 1/2$ for all $r \in (0, \infty)$, i.e. perfectly symmetric torus of all radius satisfies the condition. This can be easily seen as follows. Since the principal curvatures of perfect torus are $\lambda_1 = \tanh r$ and $\lambda_2 = \coth r$ by the Riccati equation $\lambda_i' + (\lambda_i)^2 = 1$, the harmonic mean curvature

$$F = \frac{1}{(\coth r)^{-1} + (\tanh r)^{-1}} = \frac{1}{\coth r + \tanh r}.$$  

Then,

$$\frac{dF}{dr} = \frac{1}{\sinh^2 r + \cosh^2 r} > 0 \quad \text{for all } r$$

But

$$\lim_{r \to 0} \coth r = \infty, \quad \lim_{r \to 0} \tanh r = 0$$

$$\lim_{r \to \infty} \coth r = 1, \quad \lim_{r \to \infty} \tanh r = 1.$$  

Therefore

$$\lim_{r \to 0} F = 0, \quad \lim_{r \to \infty} F = \frac{1}{2}, \quad 0 < F(r) < \frac{1}{2}.$$  

(2) The HMCF of perfectly symmetric torus whose axis is a closed geodesic in a hyperbolic manifold was considered in Theorem 3 of [15]. They showed that the radius $r(t)$ of the evolving torus satisfies

$$r(t) = \frac{1}{2} \sinh^{-1} \left( e^{-t} \sinh 2r_0 \right) \approx e^{-t}$$

where $r_0$ is the radius of the initial torus. Since the principal curvatures of perfect torus are $\lambda_1 = \tanh r$ and $\lambda_2 = \coth r$, we obtain the asymptotic estimates of both principal curvatures:

$$\lambda_1 \approx e^{-t}, \quad \lambda_2 \approx e^t, \quad \lambda_1 \lambda_2 = 1. \quad (1.1)$$  

The main theorem of this paper shows that the principal curvatures of axially symmetric torus contracting to a closed geodesic under HMCF retain the curvature estimates (1.1) of evolving perfectly symmetric torus.
The paper is organized as follows. In Section 2, we derive essential geometric quantities available on axially symmetric spaces. In Section 3, we prove the short and long time existence of HMCF of axially symmetric torus and discuss the preservation of convexity of the surface. We derive the evolution equations of important geometric quantities in Section 4. In Section 5, we prove that the evolving surfaces remains as a graph throughout the deformation process and also prove that $\lambda_2 \approx e^t$. Along the way, we obtain the optimal estimate $\lambda_1 \approx e^{-t}$ and conclude $\lambda_1 \lambda_2 \approx 1$.

1.2 Axially Symmetric Space

In this section, we will use the orthonormal frames to derive geometric quantities defined on axially symmetric surfaces. Similar computation was carried out in [8] for general rotationally symmetric spaces. In the neighborhood of the closed geodesic, the hyperbolic metric can be expressed in Fermi coordinates as

$$ds^2 = dr^2 + h(r)^2d\theta^2 + b(r)^2dz^2$$

where $r$ is the distance from the axis, $\theta$ is the angular unit of the circle perpendicular to the axis, $z$ is the position along the axis and $b(r) = \cosh r$, $h(r) = \sinh r$. We have the following orthonormal frames in $(n + 1)$-dimensional rotationally symmetric space.

$$E_0 \equiv E_r = \frac{\partial}{\partial r}, \quad E_1 \equiv E_z = \frac{1}{b(r)} \frac{\partial}{\partial z}, \quad E_i = \frac{1}{h(r)} e_i, \quad i = 2, ..., n$$

where $e_i$ is an orthonormal frame of $S^{n-1}$ with the standard metric. Its dual orthonormal coframe is given by

$$\theta^r = dr, \quad \theta^z = b(r)dz, \quad \theta^i = h(r)e_i, \quad i = 2, ..., n.$$ 

In these frames, the Cartan connection form $\omega^b_a$ defined by $d\theta^b = -\sum_{a=0}^{n} \omega^b_a \wedge \theta^a$ are

$$\omega^z_\theta = \frac{b'(r)}{b(r)}\theta^z, \quad \omega^\theta_r = \frac{h'(r)}{h(r)}\theta^r, \quad \omega^z_i = 0, \quad \omega^z_j = S^j_i$$

where $S^j_i$ is the Cartan connection form on $S^{n-1}$. The covariant derivatives of the orthonormal frames can be computed from the equation $\nabla_X E_a = \sum_{b=0}^{n} \omega^b_a(X) E_b$ and their results are given below. We denote the covariant derivative defined on the ambient manifold by $\nabla$ and the
covariant derivative on the hypersurface by $\nabla$. The symbol $'$ denotes derivative with respect to $r$ and subscripts of $r$ mean derivative with respect to $z$.

$$\nabla_{E_i} E_r = 0, \quad \nabla_{E_r} E_r = \frac{b'(r)}{b(r)} E_z, \quad \nabla_{E_r} E_r = \frac{h'(r)}{h(r)} E_i$$

$$\nabla_{E_i} E_z = 0, \quad \nabla_{E_r} E_z = -\frac{b'(r)}{b(r)} E_r, \quad \nabla_{E_r} E_z = 0,$$  

(1.2)

$$\nabla_{E_i} E_i = 0, \quad \nabla_{E_i} E_i = 0, \quad \nabla_{E_i} E_j = -\frac{h'(r)}{h(r)} \delta_{ij} E_r + S^k \omega^k_j (E_i) E_k$$

for $i = 2, \ldots, n$. For a hypersurface constructed by revolving the graph of a generating function $r: S^1 \to \mathbb{R}^+$, the tangent vector $\sigma$ of the generating curve and the unit normal vector $N$ of the hypersurface are given by

$$\sigma = \frac{1}{\sqrt{r_z^2 + b^2}} \left( r_z E_r + b E_z \right), \quad N = \frac{1}{\sqrt{r_z^2 + b^2}} \left( b E_r - r_z E_z \right).$$

(1.3)

The principal curvatures of the hypersurface in the direction of $\sigma$ and $E_i$ are

$$\lambda_1 = \langle \nabla_{\sigma} \sigma, N \rangle = \frac{1}{\sqrt{r_z^2 + b^2}} \left( \frac{-r_z b + r_z b'}{r_z^2 + b^2} + b' \right),$$

(1.4)

$$\lambda_i = \langle \nabla_{E_i} E_i, N \rangle = \frac{b}{\sqrt{r_z^2 + b^2}} \frac{h'}{h} = \frac{u}{h} \frac{b}{h}, \quad i = 2, \ldots, n,$$

(1.5)

respectively. Note the hypersurfaces of revolution is generated by a graph if

$$u \doteq \langle E_r, N \rangle = \frac{b}{\sqrt{r_z^2 + b^2}}$$

(1.6)

is $> 0$ or equivalently $v \doteq u^{-1} < \infty$. Note that $u \leq 1$ by its definition.

### 1.3 Short and Long Time Existence and Preserving Convexity

In this section, we first prove the short time existence of HMCF of axially symmetric torus and review the long time existence and preservation of convexity proved in [15]. Let $W^j_i = h_{ik} g^{kj}$ be the Weingarten map of $\Sigma_t$ where $h_{ij}$ is the second fundamental form and $g_{ij}$ is the induced metric on $\Sigma_t$. We can view the harmonic mean curvature function as $F(W^j_i) = f(\lambda(W^j_i))$ where $\lambda(W^j_i) = (\lambda_1, \lambda_2)$ is the set of eigenvalues of $W^j_i$ and $f(\lambda_1, \lambda_2) = \lambda_1 \lambda_2 / (\lambda_1 + \lambda_2)$. Let
us first discuss the short time existence of HMCF when the flow equation is casted in terms of the graph function. If we express (HMCF) in terms of the graph function using
\[ \left\langle \frac{\partial \phi}{\partial t}, N \right\rangle = F, \]
we obtain
\[ \frac{\partial r}{\partial t} = -r_{zz} - 2 \tanh r \cdot r_z^2 - \sinh r \cdot \cosh r - 2 \tanh r \cdot r_{zz} - (2 \tanh^2 r + 1) r_z^2 - \sinh^2 r - \cosh^2 r, \]  
(1.7)
for all \((z, t) \in S^1 \times [0, T)\). Since the initial surface is assumed to be strictly convex, from (1.4) and (1.5) we find that at \(t = 0\)
\[ \tilde{\lambda}_1 := -r_{zz} + 2 \tanh r \cdot r_z^2 + \sinh r \cosh r > 0. \]  
(1.8)
We consider positive solutions
\[ r > 0. \]  
(1.9)
We define \(C^\alpha(S^1)\) to be the set of standard Holder continuous functions on \(S^1\) and \(C^{2+\alpha}(S^1)\) to be a space of functions \(g\) on \(S^1\) such that
\[ g, g_z, g_{zz} \in C^\alpha(S^1). \]
We set \(Q_\tau = S^1 \times [0, \tau]\) for some \(\tau > 0\) and define \(C^{2+\alpha}(Q_\tau)\) to be a space functions \(g\) on \(Q_\tau\) such that
\[ g_t, g, g_z, g_{zz} \in C^\alpha(S^1). \]
\[ \text{Lemma 2. Let } r_0 \in C^{2+\alpha}(S^1). \text{ There exists some } t_0 > 0 \text{ such that a unique solution } r \in C^{2+\alpha}(S^1 \times [0, t_0]) \text{ solves (1.7).} \]
\[ \text{Proof. Let } M : C^{2+\alpha}(Q_\tau) \to C^\alpha(Q_\tau) \text{ be a fully nonlinear operator defined by} \]
\[ M(r) = r_t - F(z, t, r, r_z, r_{zz}) \]
where \(F(z, t, r, r_z, r_{zz}) = -\frac{r_{zz} - 2 \tanh r \cdot r_z^2 - \sinh r \cdot \cosh r}{\tanh r \cdot r_{zz} - (2 \tanh^2 r + 1) r_z^2 - \sinh^2 r - \cosh^2 r}. \)
Consider the linearization of $M$ around a function $r \in C^{2+\alpha}(Q,\tau)$ such that $\|r - r_0\| < \delta$ for some $\delta > 0$. If we choose $\delta$ small enough, any such $r$ will satisfy conditions (1.8) and (1.9) since the initial condition $r_0 \in C^{2+\alpha}(S^1)$ satisfies those conditions. Then, the linearized equation around the function $r$

\[
\frac{\partial  \tilde{r}}{\partial t} = DF(r)(\tilde{r}) = \alpha(r, rz, r_{zz})\tilde{r}_{zz} + \beta(r, rz, r_{zz})\tilde{r}_z + \gamma(r, rz, r_{zz})\tilde{r} \tag{1.10}
\]

where

$$
\alpha = \frac{-r^2_z - \cosh^2 r}{(\tanh r \cdot \lambda_1 + r^2_z + \cosh^2 r)^2}
$$

$$
\beta = \frac{(4\tanh^2 r - 4\tanh^2 r - 2)rz\lambda_1 + 4\tanh r \cdot rz(r^2_z + \cosh^2 r)}{(\tanh r \cdot \lambda_1 + r^2_z + \cosh^2 r)^2}
$$

$$
\gamma = \left[ \left( \frac{r_{zz}}{\cosh^2 r} - \frac{2\tanh r}{\cosh^2 r} \frac{r^2_z}{\cosh^2 r} - 3 \sinh r \cosh r + \sinh^2 r \tanh r \right) \lambda_1 \\
+ \left( \frac{2r^2_z}{\cosh^2 r} + \cosh^2 r + \sinh^2 r \right)(r^2_z + \cosh^2 r) \right] / (\tanh r \cdot \lambda_1 + r^2_z + \cosh^2 r)^2.
$$

satisfy

$$
\inf_{Q,\tau} \alpha(r, rz, r_{zz}) > \mu > 0 \quad \text{for some } \mu \text{ and } \alpha, \beta, \gamma \in C^\alpha(Q,\tau).
$$

By standard theory for linear parabolic PDEs, the linearized equation (1.10) with the initial condition $\tilde{r}_0 \in C^{2+\alpha}(S^1)$ has a unique solution $\tilde{r} \in C^{2+\alpha}(Q,\tau)$. Applying inverse function theorem for Banach spaces (See [?] Theorm 8.5), we conclude that there exists $t_0 > 0$ such that (1.7) has a unique solution $r \in C^{2+\alpha}(Q_{t_0})$.

**Remark.** The fully nonlinear equation (1.7) is, in fact, uniformly parabolic due to $C^1$ and $C^2$ estimates of $r$ (Corollary [15]).

In Theorem 6 of [15], it is proved that the solution of (HMCF) exists for infinite time and the evolving surface remains strictly convex. We will restate the theorem dividing it into two
parts: the first stating the lower bound of HMC and the second stating its upper bound. We will
give the entire proof of the second part since some estimates used in the proof will be improved
in Section 4 in order to obtain the asymptotically optimal upper bound for HMC.

**Theorem 3.** Let $N^3$ be a Hyperbolic manifold. If the initial surface is strictly convex, then
$F(x, t) \geq (\min_{M_0} F)e^{-t}$ as long as the solution of HMCF exists, i.e. the surface remains
strictly convex.


**Theorem 4.** Let $N^3$ be a Hyperbolic manifold. Assume that the initial hypersurface is strictly
convex and $\max_{\Sigma_0} F < 1/2$. Then, the solution of HMCF exists for infinite time and $\max_{\Sigma_t} F \leq Ce^{-t/2}$ for some constant $C$ for all $t \in [0, \infty)$.

**Remark.** Note that $f \leq \lambda_1 \leq 2f$ if $\lambda_i > 0$. Therefore, the theorem implies that $\max_{\Sigma_t} \lambda_1 \leq C e^{-t/2}$ where $\lambda_1$ is the smallest principal curvature. Together with Theorem 3 we obtain $C_1 e^{-t} \leq F \leq C_2 e^{-t/2}$.

**Proof.** We find the upper bound for $F$ by analyzing the evolution equation of $F$. We denote
$\mathcal{L} = \frac{\partial F}{\partial h_i} \nabla_i \nabla^j$ which is an elliptic operator as long as the hypersurface is strictly convex.

\[
\frac{\partial F}{\partial t} = \mathcal{L}(F) + F\langle \hat{F}, W^2 \rangle + F\langle \hat{F}^{ij}, R_{i0j0} \rangle
\]

\[
= \mathcal{L}(F) + \sum_i F \frac{\partial f}{\partial \lambda_i} (\lambda_i^2 + R_{i0i0})
\]

\[
\leq \mathcal{L}(F) + \sum_i F^3 - \sum_i F \frac{\partial f}{\partial \lambda_i}
\]

\[
= \mathcal{L}(F) + 2F^3 - F^3 \sum_i \lambda_i^{-2}
\]

\[
\leq \mathcal{L}(F) + 2F^3 - \frac{1}{2}F
\]

By the maximum principle, we can solve the following ODE and obtain an upper bound for $F(x, t)$.

\[
\frac{d\tilde{F}}{dt} = 2\tilde{F}^3 - \frac{1}{2}\tilde{F}
\]

\[
\tilde{F}(0) = \max_{x \in M} F(x, 0)
\]
Since the solution of the ODE is $\tilde{F}(t)^{-2} = (\tilde{F}(0)^{-2} - 4)e^t + 4$, we have $F(x,t) \leq \tilde{F}(t)$ for all $x \in M$ as long as the solution of HMCF exists. For the proof of infinite time existence, see Theorem 6 of [15].

Since disjoint surfaces remain disjoint under HMCF by the maximum principle, given a torus whose axis is a closed geodesic, two perfect tori enclosing it from inside and outside, which are called barriers, will remain disjoint throughout the flow, thus the radius of the evolving torus is comparable to the radii of the barriers.

**Lemma 5.** Let $r$ be the generating function of axially symmetric torus evolving by HMCF. Then there exist $C_1$ and $C_2$ such that $C_1 e^{-t} \leq r(x,t) \leq C_2 e^{-t}$ for all $x \in \Sigma_t$ as long as the solution of HMCF exists.

### 1.4 Evolution Equations

In order to show that the surface of revolution remains as a graph over the closed geodesic, it is sufficient to prove that $v$ remains uniformly bounded for all time. To this end, we first derive the evolution equations of $r$ (Lemma 6) and $v$ (Lemma 8). From now on we will only consider the case $n = 2$.

**Lemma 6.** The generating function satisfies following evolution equation.

$$
\left( \begin{array}{l} \frac{\partial}{\partial t} - \mathcal{L} \end{array} \right) r = \left( \lambda_1 \frac{\partial f}{\partial \lambda_1} - f \right) u - \frac{\partial f}{\partial \lambda_1} \frac{\lambda_1}{b} u^2 - \frac{\partial f}{\partial \lambda_2} \frac{\lambda_1}{h} (1 - u^2).
$$

**Proof.** Let us compute $\frac{\partial r}{\partial t}$ and $\mathcal{L} r$.

$$
\frac{\partial r}{\partial t} = \left\langle \frac{\partial}{\partial t}, \frac{\partial X}{\partial t} \right\rangle = -fu.
$$

We choose a geodesic coordinate $\partial_1 = \sigma, \partial_2 = E_2$ at a fixed point such that $g_{ij} = \delta_{ij}$ and $h_{ij} = \lambda_i \delta_{ij}, i, j = 1, 2$. Since $\nabla_\sigma \sigma = 0$ and $E_2(r) = 0$,

$$
\mathcal{L} r = \dot{f} (\nabla_k \nabla^l r) = \frac{\partial f}{\partial \lambda_1} \nabla_\sigma \nabla r + \frac{\partial f}{\partial \lambda_2} \nabla_2 E_2 r = \frac{\partial f}{\partial \lambda_1} \sigma \sigma(r) - \frac{\partial f}{\partial \lambda_2} (\nabla_2 E_2 r).
$$

Let us first compute the term $\sigma \sigma(r)$. By (1.2)-(1.6),

$$
\sigma(r) = \left\langle \sigma, \frac{\partial}{\partial r} \right\rangle = \frac{rz}{\sqrt{r_z^2 + b^2}} = -\langle N, E_z \rangle
$$
and
\[
\sigma(r) = -\sigma(N, E_z) \\
= -\langle \nabla_{\sigma} N, E_z \rangle - \langle N, \nabla_{\sigma} E_z \rangle \\
= -\langle \lambda_1 \sigma, E_z \rangle - \left( N, -\frac{b'}{\sqrt{r_z^2 + b^2}} E_r \right) \\
= -\lambda_1 u + \frac{b'}{b} u^2. \tag{1.11}
\]

On the other hand,
\[
-(\nabla E_z) E_r = -\langle \nabla E_z, \sigma \rangle \sigma(r) = \frac{h'}{h}(1 - u^2). \tag{1.12}
\]

Combining (1.11) and (1.12), we obtain
\[
\mathcal{L}r = \frac{\partial f}{\partial \lambda_1} \left( -\lambda_1 u + \frac{b'}{b} u^2 \right) + \frac{\partial f}{\partial \lambda_2} \frac{h'}{h}(1 - u^2) \\
\text{and this finishes the proof of the lemma.} \]

It is straightforward to derive the evolution equation of \( \phi(r) \) for a smooth function \( \phi: \mathbb{R} \rightarrow \mathbb{R} \).

**Lemma 7.** Following evolution equation is satisfied by \( \phi \circ r: \Sigma \times [0, \infty) \rightarrow \mathbb{R} \).
\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) \phi(r) = \phi' \left( \lambda_1 \frac{\partial f}{\partial \lambda_1} - f \right) u - \frac{\partial f}{\partial \lambda_1} \frac{b'}{b} u^2 - \frac{\partial f}{\partial \lambda_2} \frac{h'}{h}(1 - u^2) - \phi'' \frac{\partial f}{\partial \lambda_1}(1 - u^2)
\]

**Proof.** We computed following: \( \phi'' \hat{F}^{kl} \nabla_k r \nabla_l r = \phi'' \frac{\partial f}{\partial \lambda_1} (\sigma(r))^2 = \phi'' \frac{\partial f}{\partial \lambda_1} (E_z, N)^2 = \phi'' \frac{\partial f}{\partial \lambda_1}(1 - u^2) \).

**Lemma 8.** The gradient function \( v = u^{-1} \) satisfies following evolution equation.
\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) v = \frac{2}{v} \hat{F}^{kl} \nabla_k v \nabla_l v - \frac{\partial f}{\partial \lambda_1} \left( \frac{b'}{b} \right)' (v - v^{-1}) - \frac{\partial f}{\partial \lambda_1} v \left( v^{-1} \frac{b'}{b} - \lambda_1 \right)^2 \\
+ \left( \frac{\partial f}{\partial \lambda_2} \frac{b'h'}{bh} - \frac{\partial f}{\partial \lambda_2} \frac{h'}{h} + \frac{\partial f}{\partial \lambda_1} \left( \frac{b'}{b} \right)^2 \right) v - \frac{\partial f}{\partial \lambda_2} \lambda_2 \frac{b'}{b} \\
+ \left( \frac{\partial f}{\partial \lambda_2} \frac{h'}{h} - \frac{\partial f}{\partial \lambda_1} \left( \frac{b'}{b} \right)^2 \right) v^{-1}
\]
Proof. Let us first compute \( \dot{F}^{kl} \nabla_k \nabla^l u \) by choosing the geodesic coordinate at a fixed point as before.

\[
\dot{F}^{kl} \nabla_k \nabla^l u = \frac{\partial f}{\partial \lambda_1} \sigma \sigma(u) - \frac{\partial f}{\partial \lambda_2} (\nabla_{E_2} E_2) u
\]

From (1.2) and (1.3), we get \( \nabla_{E_2} E_2 = u \frac{b'}{b} E_z \). Substituting it into following, we obtain

\[
\sigma(u) = \sigma \langle E_r, N \rangle = (\nabla_{E_r} E_r, N) + \langle E_r, \nabla_\sigma N \rangle = \left( u \frac{b'}{b} - \lambda_1 \right) \langle E_z, N \rangle. \tag{1.13}
\]

As a preparation for calculating \( \sigma \sigma(u) \), we first observe following. By (1.3) and (1.13),

\[
\sigma \left( u \frac{b'}{b} \right) = \left[ \left( u \frac{b'}{b} - \lambda_1 \right) \frac{b'}{b} - u \left( \frac{b'}{b} \right)' \right] \langle E_z, N \rangle.
\]

From (1.2) and (1.3), we see \( \nabla_\sigma E_z = -u \frac{b'}{b} E_r \) and get

\[
\sigma \langle E_z, N \rangle = \left( -u \frac{b'}{b} E_r, N \right) + \langle E_z, \lambda_1 \sigma \rangle = -u \left( u \frac{b'}{b} - \lambda_1 \right).
\]

Then,

\[
\sigma \sigma(u) = \left[ \left( u \frac{b'}{b} - \lambda_1 \right) \frac{b'}{b} - u \left( \frac{b'}{b} \right)' \right] (1 - u^2) - \sigma(\lambda_1) \langle E_z, N \rangle - u \left( u \frac{b'}{b} - \lambda_1 \right)^2
\]

where we used that \( \langle E_z, N \rangle^2 = 1 - u^2 \). By (1.2) and (1.13), it is straightforward to compute

\[
(\nabla_{E_2} E_2) u = \langle \nabla_{E_2} E_2, \sigma \rangle \sigma(u) = \frac{h'}{h} \left( u \frac{b'}{b} - \lambda_1 \right) (1 - u^2).
\]

We finally obtain

\[
\dot{F}^{kl} \nabla_k \nabla^l u = \frac{\partial f}{\partial \lambda_1} \left[ \left( u \frac{b'}{b} - \lambda_1 \right) \frac{b'}{b} - u \left( \frac{b'}{b} \right)' \right] (1 - u^2) - \sigma(\lambda_1) \langle E_z, N \rangle - u \left( u \frac{b'}{b} - \lambda_1 \right)^2 - \frac{\partial f}{\partial \lambda_2} \frac{h'}{h} \left( u \frac{b'}{b} - \lambda_1 \right) (1 - u^2). \tag{1.14}
\]

In order to compute \( \partial u / \partial t \), we will use following identities:

\[
\frac{\partial N}{\partial t} = \nabla F, \quad \frac{\partial E_r}{\partial t} = -F \nabla_N E_r = -F \langle E_z, N \rangle \frac{b'}{b} E_z.
\]
Then,
\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \langle N, E_r \rangle \\
= \sigma(F)\langle \sigma, E_r \rangle - F(1 - u^2) \frac{b'}{b} \\
= -\frac{\partial f}{\partial \lambda_1} \sigma(\lambda_1) \langle E_z, N \rangle - \frac{\partial f}{\partial \lambda_2} \sigma(\lambda_2) \langle E_z, N \rangle - F(1 - u^2) \frac{b'}{b} \\
= -\frac{\partial f}{\partial \lambda_1} \sigma(\lambda_1) \langle E_z, N \rangle - \frac{\partial f}{\partial \lambda_2} \left[ \left( \frac{b'}{b} - \lambda_1 \right) \frac{h'}{h} - u \left( \frac{h'}{h} \right) \right] (1 - u^2) - F(1 - u^2) \frac{b'}{b}
\]

(1.15)

where we used \( \langle \sigma, E_r \rangle = -\langle E_z, N \rangle \) in the third equation and in the last equation we substituted
\[
\sigma(\lambda_2) = \sigma \left( \frac{h'}{h} \right) = \left[ \left( \frac{b'}{b} - \lambda_1 \right) \frac{h'}{h} - u \left( \frac{h'}{h} \right) \right] \langle E_z, N \rangle.
\]

From (1.14) and (1.15), we derive
\[
\left( \frac{\partial}{\partial t} - \hat{F}^{kl} \nabla_k \nabla_l \right) u = \frac{\partial f}{\partial \lambda_1} u (1 - u^2) \left( \frac{h'}{h} \right) - F(1 - u^2) \frac{b'}{b} \\
- \frac{\partial f}{\partial \lambda_1} \left[ \left( \frac{b'}{b} - \lambda_1 \right) \frac{b'}{b} - u \left( \frac{b'}{b} \right) \right] (1 - u^2) + \frac{\partial f}{\partial \lambda_1} u \left( \frac{b'}{b} - \lambda_1 \right)^2
\]

By the definition of \( v = u^{-1} \), we have \( \hat{F}^{kl} \nabla_k \nabla_l u = -\frac{1}{v^2} \hat{F}^{kl} \nabla_k \nabla_l v + \frac{2}{v} \hat{F}^{kl} \nabla_k v \nabla_l v \) and \( \frac{\partial u}{\partial t} = -\frac{1}{v^2} \frac{\partial v}{\partial t} \).

\[
\left( \frac{\partial}{\partial t} - \hat{F}^{kl} \nabla_k \nabla_l \right) v = -2 \frac{v}{v} \hat{F}^{kl} \nabla_k v \nabla_l v - \frac{\partial f}{\partial \lambda_1} u \left( \frac{h'}{h} \right) (v - v^{-1}) - \frac{\partial f}{\partial \lambda_1} v \left( v - 1 \right) + \frac{\partial f}{\partial \lambda_2} \frac{b'}{b} \left( v - 1 \right)^2 + \frac{\partial f}{\partial \lambda_2} \left( v - 1 \right)^2 + \frac{\partial f}{\partial \lambda_2} \left( v - 1 \right) (v^2 - 1)
\]

(1.16)

Combining \( v^2 \)-terms in the second line of (1.16) and applying Euler’s identity \( \frac{\partial f}{\partial \lambda_1} + \frac{\partial f}{\partial \lambda_2} = f \) and (1.5), the \( v^2 \)-term can be reduced to a linear term as follows:
\[
\left( \frac{f b'}{b} - \frac{\partial f}{\partial \lambda_1} \frac{b'}{b} \lambda_1 \right) v^2 = \left( \frac{f b'}{b} - \frac{\partial f}{\partial \lambda_2} \lambda_2 \right) v^2 = \frac{b'}{b} \frac{\partial f}{\partial \lambda_2} \lambda_2 v^2 = \frac{b'}{b} \frac{\partial f}{\partial \lambda_2} \frac{h'}{h} v
\]

We then obtain the evolution equation of \( v \) as stated in the lemma.
1.5 Preserving the Property of being a Graph and Curvature Estimates

In this section, we study the solution of HMCF of axially symmetric torus centered at a closed geodesic satisfying the hypothesis of Theorem 1, i.e. the initial surface is strictly convex and \( \max_{\Sigma_0} F < 1/2 \). Since we will prove many technical estimates, we take this opportunity to outline the overall argument of this section. The main goal of this section is to prove that the evolving surface stays as a graph as it converges to the closed geodesic. As discussed in the Section 3, this is equivalent to showing that \( v = u^{-1} \) is uniformly bounded for all time (Theorem 13).

However, we cannot prove the uniform boundedness of \( v \) directly using its evolution equation, so the first step is to obtain a weak estimate: \( vh \leq C \) where \( h(r) = \sinh r \) (Theorem 10). This estimate is weaker than \( v < C \) since the graph function \( r \), thus \( \sinh r \), decays to 0 by the barrier argument Lemma (5). We can then deduce by (1.5) that \( \lambda_2 / \lambda_1 \approx e^t \) as \( t \to \infty \) (Corollary 2.14). Equipped with this new estimate for \( \lambda_2 / \lambda_1 \), we revisit the proof of Theorem 4 and obtain the optimal asymptotic upper bound of the HMC (Theorem 12): \( \lambda_1 \approx e^{-t} \). Finally, we can prove that the gradient function \( v \) is uniformly bounded (Theorem 13) and deduce that \( \lambda_2 \approx e^t \) thanks to the formula (1.5) for \( \lambda_2 \) available on axially symmetric surfaces. We then conclude in Corollary 14 that the principal curvatures of axially symmetric torus behave like those of perfect torus evolving under HMCF as stated in (1.1).

We first consider evolution equations of \( \phi(r)v \) where \( \phi: \mathbb{R} \to \mathbb{R} \) is a test function to be chosen later.
**Lemma 9.** The evolution equation for $\phi(r)v$ is given by

\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) \phi v
= \phi \left( -\frac{\partial f}{\partial \lambda_1} \left( \frac{b'}{b} \right)' (v - v^{-1}) - \frac{\partial f}{\partial \lambda_1} v \left( v^{-1} b' - \lambda_1 \right)^2 - \frac{\partial f}{\partial \lambda_2} \lambda_2 \frac{b'}{b} \right)
+ \left[ \frac{\partial f}{\partial \lambda_2} \left( \frac{h'}{h} \right)' - \frac{\partial f}{\partial \lambda_1} \left( \frac{b'}{b} \right)^2 \right] v^{-1} - f' + \frac{\partial f}{\partial \lambda_2} \frac{h'}{h} (v - v^{-1}) \phi' - \phi'' \frac{\partial f}{\partial \lambda_1} (v - v^{-1}) - \frac{2 F^{kl} \nabla_k (\phi v) \nabla_l v}{v}
\]

**Proof.** Substitute Lemmas 7 and 8 to

\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) \phi v = \phi \left( \frac{\partial}{\partial t} - \mathcal{L} \right) v + v \left( \frac{\partial}{\partial t} - \mathcal{L} \right) \phi - 2 F^{kl} \nabla_k \phi \nabla_l v.
\]

\[
\square
\]

Note that all the terms in the second line of Lemma 9 are nonpositive.

**Theorem 10.** $hv \leq C$ on $M \times [0, \infty)$ where $h(r) = \sinh r$.

**Proof.** Substitute $\phi = h$ to Lemma 9 and ignore all the terms in the second line of the equation since they are nonpositive.

\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) hv \leq \left[ -\frac{\partial f}{\partial \lambda_2} \left( \frac{h'}{h} \right)' + \frac{\partial f}{\partial \lambda_2} \frac{h'}{h} + \frac{\partial f}{\partial \lambda_1} \left( \frac{b'}{b} \right)^2 \right] hv - \frac{\partial f}{\partial \lambda_1} \left( v^{-1} b' - \lambda_1 \right) h'
- \frac{\partial f}{\partial \lambda_2} \frac{h'}{h} (v - v^{-1}) h' - h'' \frac{\partial f}{\partial \lambda_1} (v - v^{-1}) - \frac{2 F^{kl} \nabla_k (hv) \nabla_l v}{v}
\]

\[
= \left[ -h' \frac{\partial f}{\partial \lambda_1} \frac{b'}{b} + h h'' \frac{\partial f}{\partial \lambda_1} + h^2 \frac{\partial f}{\partial \lambda_2} \left( \frac{h'}{h} \right)' \right] (hv)^{-1} + h' \frac{\partial f}{\partial \lambda_1} \lambda_1 - \frac{2 F^{kl} \nabla_k (hv) \nabla_l v}{v}
\]

There exist positive constants $C_0, C_1,$ and $C_2$ such that

\[
- \frac{1}{\cosh^2 r} \frac{\partial f}{\partial \lambda_1} \leq -C_0, \quad \cosh^2 r \frac{\partial f}{\partial \lambda_2} \leq C_1, \quad \cosh r \frac{\partial f}{\partial \lambda_1} \lambda_1 \leq C_2
\]
by Theorem 4, Lemma 5 and
\[ \frac{1}{2} \leq \frac{\partial f}{\partial \lambda_1} \leq 1, \quad 0 \leq \frac{\partial f}{\partial \lambda_2} \leq 1, \quad \text{if } \lambda_i > 0. \] (1.17)

The evolution equation becomes
\[ \left( \frac{\partial}{\partial t} - L \right) vh \leq -C_0 \cdot vh + C_1 (vh)^{-1} + C_2 - \frac{2}{v} F^{\hat{k}\hat{l}} \nabla_k (hv) \nabla_l v \]
and we can apply the maximum principle to obtain a uniform upper bound for $hv$:
\[ \max_{\Sigma_t} hv \leq \max \left\{ \frac{1}{2C_0} \left( C_2 + \sqrt{C_2^2 + 4C_0C_1} \right), \max_{\Sigma_0} hv \right\}. \]

**Corollary 11.** $\lambda_2 > C_1$ and $\lambda_2/\lambda_1 \geq C_2 e^{t/2}$ on $\Sigma_t$ for all $t \in [0, \infty)$.

**Proof.** The large principal curvature $\lambda_2$ has a uniform lower bound due to Theorem 10 to (1.5).
It follows that the ratio $\lambda_2/\lambda_1$ tends to infinity at a rate $e^{t/2}$ since $\lambda_1 \leq C e^{-t/2}$ from Theorem 4.

We will use the growth estimate of the ratio $\lambda_2/\lambda_1$ to improve the proof of Theorem 4 and squeeze out the optimal upper bound of the harmonic mean curvature $F$. As we shall see below, the ODE associated to the evolution equation of $F$ now has a time dependent coefficient due to the use of growth estimate $\lambda_2/\lambda_1 > C e^{t/2}$. Therefore, we need to analyze the solution of a nonautonomous ODE in order to establish the optimal upper bound of $F$.

**Theorem 12.** There exist $T > 0$ and $C_1, C_2 > 0$ such that for all $t \geq T$
\[ C_1 e^{-t} \leq F \leq C_2 e^{-t}. \]

**Proof.** Since Theorem 3 provides the lower bound, it is enough to prove the upper bound. We
analyze the evolution equation of the harmonic mean curvature $F$ from Theorem 4 again.

\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) F = F \langle \dot{F}, W^2 \rangle + F \langle \dot{F}^{ij}, R_{i0j0} \rangle
\]

\[
= \sum_i F \frac{\partial f}{\partial \lambda_i} (\lambda_i^2 + R_{i0i0})
\]

\[
= \sum_i F^3 - \sum_i F \frac{\partial f}{\partial \lambda_i}
\]

\[
= 2F^3 - F \left( F^2 \sum_i \lambda_i^{-2} \right)
\]

\[
\leq 2F^3 - \delta(t) F
\]

where

\[
\delta(t) = \max \left\{ \frac{1}{2}, 1 - Ce^{-t/2} \right\}
\]

was obtained by observing

\[
F^2 \sum_{i=1}^{2} \lambda_i^{-2} = \left( \lambda_1^{-2} + \lambda_2^{-2} \right) / \left( \lambda_1^{-1} + \lambda_2^{-1} \right)^2 \geq 1/2 \text{ if } \lambda_i > 0 \text{ and }
\]

\[
F^2 \sum_{i=1}^{2} \lambda_i^{-2} \geq 1 - 2 \left( \frac{\lambda_2}{\lambda_1} \right)^{-1} \geq 1 - Ce^{-t/2}
\]

due to Corollary 2.14. By the maximum principle, $F(x, t) \leq \psi(t)$ for all $(x, t) \in \Sigma \times [0, \infty)$ where $\psi(t)$ is the solution of following nonautonomous ODE:

\[
\frac{d\psi}{dt} = -2\psi \left( \delta(t)/2 - \psi^2 \right),
\]

\[
\psi(0) = \max_{\Sigma_0} F.
\]  

(1.18)

Since we are interested in the asymptotic decay rate of the harmonic mean curvature, we will find decay rate of $\psi(t)$ for $t \in [T, \infty)$ for large $T$ by comparing the solution of (1.18) with the solutions of following ODEs. Note that due to the initial condition $\max_{\Sigma_0} F < 1/2$ it is not hard to see that $\psi(t) \to 0$ as $t \to \infty$, thus we can choose large $T$ such that $\psi(T) = \epsilon$ for any given $\epsilon > 0$. Consider following ODEs:

\[
\frac{d\tilde{\psi}}{dt} = -2\tilde{\psi} \left( \delta/2 - \tilde{\psi}^2 \right),
\]  

(1.19)

\[
\frac{d\hat{\psi}}{dt} = -2\hat{\psi} \left( \delta/2 - \hat{\psi}^2 \right),
\]  

(1.20)
on the time interval \([T, \infty)\) with conditions \(\bar{\psi}(T) = \hat{\psi}(T) = \epsilon\).

**Claim I**  \(\psi \leq \bar{\psi} \leq \hat{\psi}\) for all \(t \in [T, \infty)\).

**Proof of Claim I.** Since \(\psi(T) = \epsilon\) and \(\psi\) is nonincreasing for all \(t \in [T, \infty)\), from (1.18) and (1.19)

\[
\frac{d}{dt} \left( \log \psi - \log \bar{\psi} \right) = 2(\psi^2 - \bar{\psi}^2) \leq 0.
\]

Hence, \(\psi \leq \bar{\psi}\) on \([T, \infty)\).

Using this result, we see that from (1.18) and (1.20),

\[
\frac{d}{dt} \left( \log \psi - \log \hat{\psi} \right) = 2(\psi^2 - \bar{\psi}^2) \leq 0.
\]

Hence \(\psi \leq \hat{\psi}\) on \([T, \infty)\). Finally, from (1.19) and (1.20), we have

\[
\frac{d}{dt} \left( \log \hat{\psi} - \log \bar{\psi} \right) = 2(\bar{\psi}^2 - \epsilon^2) \leq 0
\]

since \(\bar{\psi}(T) = \epsilon\) and \(\bar{\psi}\) is nonincreasing. Hence, \(\hat{\psi} \leq \bar{\psi}\) on \([T, \infty)\).

**Claim II**  \(\hat{\psi}(t) \leq C_3 e^{-t}\) for all \(t \geq T\).

**Proof of Claim II.** Let us find the exact solutions of (1.19) and (1.20). Noting that \(\delta(t) = 1 - Ce^{-t/2}\) for \(t \in [T, \infty)\) when \(T\) is large, the solution of (1.19) is

\[
\bar{\psi}(t) = \bar{\psi}(T) \exp \left[ (-1 + 2\epsilon^2)t - 2Ce^{-t/2} + C_1 \right]
\]

where \(C_1 = (1 - 2\epsilon^2)T + 2Ce^{-T/2}\).

Next, substituting (1.21) into (1.20) and integrating in time, we obtain

\[
\log \frac{\hat{\psi}}{\bar{\psi}(T)} = \int_T^t \left( -1 + Ce^{-t/2} + 2\bar{\psi}^2 \right) dt
\]

\[
= -t - 2Ce^{-t/2} + T + 2Ce^{-T/2} + 2 \int_T^t \bar{\psi}^2 dt
\]

But,

\[
\int_T^t \bar{\psi}^2 dt = \bar{\psi}(T)^2 \int_T^t \exp \left[ 2(-1 + 2\epsilon^2)t - 4Ce^{-t/2} + 2C_1 \right] dt \leq C_2
\]

Hence,

\[
\hat{\psi}(t) \leq C_3 e^{-t}.
\]

This finishes the proof of Claim I and II. Now, by the maximum principle and Claim I and II,

\[
\max_{x \in \Sigma_t} F \leq \psi(t) \leq \hat{\psi}(t) \leq C e^{-t}
\]

for all \(t \geq T\).
We are now in a position to prove that \( v \) is uniformly bounded.

**Theorem 13.** There exists a constant \( C > 0 \) such that \( v(x, t) \leq C \) for all \( (x, t) \in \Sigma \times [0, \infty) \).

**Proof.** Define a test function \( \phi(r) = e^{\mu r^{1+\alpha}} \) where \( \mu \) is a positive number to be chosen and \( \alpha \in (0, 1) \) can be any number. Note that the asymptotic behavior \( \phi \to 1, \phi' \to 0, \) and \( \phi'' \to \infty \) as \( r \to 0 \) becomes important when it comes to obtaining the desired estimates for the reaction terms in the evolution equation of \( \phi(r)v \). In particular,

\[
\phi''(r) = \mu(1 + \alpha) \alpha r^{-1+\alpha} \phi + \left( \mu(1 + \alpha) r^\alpha \right)^2 \phi \geq \mu(1 + \alpha) \alpha \max_{\Sigma_0} r^{-1+\alpha} \tag{1.22}
\]

is useful since by choosing \( \mu \) large, \( \phi'' \) can be made greater than any large number but it never becomes infinity in finite time. From Lemma 9,

\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) \phi v \\
\leq - \left[ \frac{\partial f}{\partial \lambda_2} \left( \frac{b'}{h} \right)' + \frac{\partial f}{\partial \lambda_2} \frac{b'}{hb} + \frac{\partial f}{\partial \lambda_1} \left( \frac{b'}{b} \right)^2 \right] \phi v - \frac{\partial f}{\partial \lambda_1} \left( \frac{v - b'}{b} - \lambda_1 \right) \phi' - \phi'' \frac{\partial f}{\partial \lambda_1} (v - v^{-1}) \\
- \frac{2}{v} F^{kl} \nabla_k (\phi v) \nabla_l v
\]

\[
\leq \phi'' \left[ \frac{1}{\phi''} \frac{\partial f}{\partial \lambda_2} \sinh^2 r + \frac{1}{\phi''} \frac{\partial f}{\partial \lambda_2} + \frac{1}{\phi''} \frac{\partial f}{\partial \lambda_1} \left( \frac{\sinh r}{\cosh r} \right)^2 - \frac{1}{\phi''} \frac{\partial f}{\partial \lambda_1} \right] \phi v \tag{1.23}
\]

\[
+ \left[ - \frac{\phi''}{\phi''} \frac{\partial f}{\partial \lambda_1} \sinh r + \phi'' \frac{\partial f}{\partial \lambda_1} \left( \phi v \right)^{-1} + \frac{\phi''}{\phi''} \frac{\partial f}{\partial \lambda_1} \right] \left( \frac{\sinh r}{\cosh r} \right)^2 - \frac{1}{\phi''} \frac{\partial f}{\partial \lambda_1} \lambda_1 \right) - \frac{2}{v} F^{kl} \nabla_k (\phi v) \nabla_l v. \tag{1.24}
\]

Let us first examine (1.23), the coefficient of \( \phi v \). Since \( F \approx e^{-t} \) by Theorem 12, \( \sinh r \approx e^{-t} \) by Lemma 5 and \( \lambda_2 > C \) by Corollary 5.3, the first term

\[
\frac{\partial f}{\partial \lambda_2} \frac{1}{\sinh^2 r} = \frac{f^2 \lambda_2}{\sinh^2 r} \leq C. \tag{1.25}
\]

By Lemma 5 (1.17), (1.22) and (1.25), we see that the first three terms can be made arbitrarily small if we choose a large \( \mu \). On the other hand, the last term in (1.23) is strictly negative since we can find a constant \( C_0 > 0 \) such that \( \frac{1}{\phi''} \frac{\partial f}{\partial \lambda_1} > C_0 \), thus there is a constant \( C_1 > 0 \) such that

\[
\frac{1}{\phi''} \frac{\partial F}{\partial \lambda_2} \frac{1}{\sinh^2 r} + \frac{1}{\phi''} \frac{\partial F}{\partial \lambda_2} + \frac{1}{\phi''} \frac{\partial F}{\partial \lambda_1} \left( \frac{\sinh r}{\cosh r} \right)^2 - \frac{1}{\phi''} \frac{\partial f}{\partial \lambda_1} \leq -C_1.
\]
Using similar argument, we see that the rest of the terms in (1.24) can be uniformly bounded above, so the evolution equation becomes
\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) \phi v \leq \phi'' \left( -C_1 \cdot \phi v + C_2 (\phi v)^{-1} + C_3 \right) - \frac{2}{v} F_{kl} \nabla_k (\phi v) \nabla_l v
\]
and we can apply the maximum principle to conclude that on \( \Sigma \times [0, \infty) \)
\[
v \leq \phi v \leq \max \left\{ \max_{\Sigma_0} \phi v, \frac{C_3 + \sqrt{C_3^2 + 4C_1C_2}}{2C_1} \right\}.
\]

Due to the formula (1.5) for \( \lambda_2 \) available on axially symmetric surfaces, the uniform boundedness of \( v \) implies that \( \lambda_2 \approx 1/ \sinh r \approx e^t \). Together with the asymptotic estimate for \( \lambda_1 \) from Theorem 12, we have shown that the principal curvatures of axially symmetric torus evolving by HMCF have the same asymptotic curvature estimates as the perfect torus shrinking under HMCF as stated in (1.1).

**Corollary 14.** \( \lambda_1 \approx e^{-t}, \lambda_2 \approx e^t, \) and \( \lambda_1 \lambda_2 \approx 1 \) on \( \Sigma \times [0, \infty) \).

Note that uniform boundedness of \( v \) implies that \( |r_z| \) is uniformly bounded. In fact, more can be said about \( |r_z| \) and \( |r_{zz}| \) if we apply the results of Theorems 12 and 13 to the formula (1.4) for \( \lambda_1 \). Moreover, we can deduce a better estimate for \( \lambda_2 \).

**Corollary 15.** We have
\[
\max_{z \in S^1} |r_{zz}| \leq C_1 e^{-t}, \quad \max_{z \in S^1} |r_z| \leq C_2 e^{-t}
\]
and
\[
\max_{z \in S^1} |v - 1| \to 0, \quad \frac{\lambda_2}{\coth r} \to 1 \quad \text{as} \quad t \to \infty.
\]

**Proof.** Solving for \( r_{zz} \) in (1.4), we obtain
\[
r_{zz} = \left[ -\lambda_1 \left( r_z^2 + b^2 \right)^{3/2} + (2r_z^2 + b^2) b' \right] / b.
\]

Using that \( |r_z| \) is uniformly bounded and both \( \lambda_1 \) and \( b' = \sinh r \) decrease at a rate \( e^{-t} \),
\[
|r_{zz}| \leq \left( \frac{r_z^2 + b^2}{b} \right)^{3/2} \lambda_1 + \frac{2r_z^2 + b^2}{b} b' \leq Ce^{-t}.
\]
Since $r$ is a function defined on $S^1$, the derivative $r_z$ cannot have a sign, i.e., at each time $t$ there is $z_0(t)$ such that $r_z(z_0(t), t) = 0$. Then,

$$\max_{z \in S^1} |r_z(z, t)| = \max_{S^1} |r_z(z, t) - r_z(z_0(t), t)| \leq \max_{S^1} \int_{z_0(t)}^{z} |r_{zz}(s, t)| ds \leq C e^{-t}.$$

Now, by the definition of $v$ we see that $v \to 1$ uniformly in space and time and from the formula (1.5) for $\lambda_2$ we obtain uniform convergence $\lambda_2 \to \coth r$. \qed
Chapter 2

Numerical Analysis of General Torus

2.1 Introduction

In this chapter we study the harmonic mean curvature flow of general torus that is not necessarily axially symmetric and provide numerical evidence and supporting analysis that general torus may not evolve into a round cylindrical shape contrary to axially symmetric torus. The reason we expect this difference is that HMCF of torus is nonuniformly parabolic due to the contrasting behavior of the principal curvatures: $\lambda_1 \to 0$ and $\lambda_2 \to \infty$. The derivative of harmonic mean curvature $F$, which is the speed of motion, with respect to the principal curvatures then satisfy

\[
\frac{\partial F}{\partial \lambda_1} = \frac{\lambda_2^2}{(\lambda_1 + \lambda_2)^2} \to 1 \quad \text{and} \quad \frac{\partial F}{\partial \lambda_2} = \frac{\lambda_1^2}{(\lambda_1 + \lambda_2)^2} \to 0.
\]

This implies that the speed of motion is sensitive to changes in $\lambda_1$, thus it is likely to regularize variations in $\lambda_1$, but the flow becomes blind to changes in $\lambda_2$. We will provide numerical evidence that torus with large variations in $\lambda_2$ indeed does not become a round cylindrical. However, we also demonstrate that when $\lambda_1$ and $\lambda_2$ are varied simultaneously general torus can still evolve into perfect torus despite the large variations in $\lambda_2$. The numerical analysis of the eigenvalues of the linearized HMCF shows that convergent surfaces attain eigenvalues that are larger than the eigenvalues of nonconvergent surfaces.

Spectral methods is employed to numerically solve fully nonlinear parabolic PDE. The reason we choose spectral methods is that it outperforms any other numerical methods when the solution is $C^\infty$ and periodic [9]. Convex surfaces in Riemannian manifold evolving under HMCF is known to be $C^\infty$ if the initial surface is so [3] and torus can be viewed as a periodic surface, so spectral methods is a good choice for our problem.
This chapter is organized as follows. In Section 2, we derive the HMCF of torus when it can be expressed as a graph over the axis. In Section 3, we present the numerical scheme and the numerical results. The choice of initial values of the numerical solutions is inspired by examining the solutions of linearized HMCF. In Section 4, we first obtain estimates on the derivatives of the solution based on the numerical results and use them to derive asymptotic formulas for the curvature functions and analyze the eigenvalues of the linearized HMCF.

2.2 HMCF Equation

2.2.1 Derivation of HMCF Equation

Let $\gamma$ be a simple closed geodesic contained in hyperbolic 3-manifold. We consider an embedding of a torus into the hyperbolic space whose axis is the closed geodesic $\gamma$. With respect to Fermi coordinates, hyperbolic metric can locally be expressed as

$$g = dr^2 + h^2(r) d\theta^2 + f^2(r) dz^2$$

where $r$ is the distance to $\gamma$, $\theta$ is the angular coordinate of the unit circle perpendicular to $\gamma$, and $z$ is the position on $\gamma$, and $h = \sinh r$ and $f = \cosh r$ are warping functions. We will consider an embedded torus that can be expressed as a graph over the axis $\gamma$. Abusing notation, let $r: S^1 \times S^1 \to \mathbb{R}$ be a smooth positive function. The embedding map $\Phi: S^1 \times S^1 \to H^3/\Gamma$ is defined by $\Phi((z, \theta)) = (r(z, \theta), \theta, z)$ in Fermi coordinates. The local tangent vectors obtained from this parametrization are

$$Z_0 = \Phi_*(\partial_z) = f E_z + r_z E_r$$
$$Z_1 = \Phi_*(\partial_{\theta}) = r_\theta E_r + rh E_\theta$$

where $E_r = \partial_r$, $E_\theta = h^{-1}\partial_\theta$, $E_z = f^{-1}\partial_z$. The subscript of $r$ denotes derivatives and $'$ denote derivative with respect to $r$. Let $\tilde{N}$ be the normal vector of the surface pointing inward. We can find $a_0, a_1$ by solving the equations $0 = \langle \tilde{N}, Z_0 \rangle = \langle \tilde{N}, Z_1 \rangle$:

$$\tilde{N} = -E_r + a_0 Z_0 + a_1 Z_1$$
$$a_0 = h^2 r^2 r_z / D$$
$$a_1 = r_\theta (r_\theta^2 - r_z^2 + h^2 r^2) / D$$

where $D = f^2 (r_\theta^2 + h^2 r^2) + (r_z h r)^2$. 

(2.1)
To find the second fundamental form $h_{ij}$ of the $\Phi(S^1 \times S^1)$ we first compute $\nabla Z_0 Z_0, \nabla Z_0 Z_1$ and $\nabla Z_1 Z_1$:

$$
\nabla Z_0 Z_0 = (-ff' + rz)E_r + 2f'rzE_z \\
\nabla Z_0 Z_1 = r_zE_r + r_\theta f'E_z + (rzh + rz h')E_\theta \\
\nabla Z_1 Z_1 = r_\theta r(2h' + h)E_\theta + (rr_\theta - r^2 hh')E_r.
$$

Letting $N = -\tilde{N}/|\tilde{N}|$ be the outward unit normal vector of the surface,

$$
\begin{align*}
    h_{00} &= \langle N, \nabla Z_0 Z_0 \rangle = \frac{1}{|N|}((-1 + a_0 r_z + a_1 r_\theta)(-ff' + rz) + 2a_0 f' rz) \\
    h_{01} &= \langle N, \nabla Z_1 Z_0 \rangle = \frac{1}{|N|}((-1 + a_0 r_z + a_1 r_\theta)r_z + a_0 r_\theta f') \\
    h_{11} &= \langle N, \nabla Z_1 Z_1 \rangle = \frac{1}{|N|}(a_1 r^2 r_\theta h(2h' + h) + (-1 + a_0 r_z + a_1 r_\theta)(rr_\theta - r^2 hh')).
\end{align*}
$$

The induced metric $g_{ij}$ on the surface and its inverse are

$$
\begin{align*}
    g_{00} &= \langle Z_0, Z_0 \rangle = f^2 + r_z^2 \\
    g_{01} &= \langle Z_0, Z_1 \rangle = r_z r_\theta \\
    g_{11} &= \langle Z_1, Z_1 \rangle = r_\theta^2 + (rh)^2,
\end{align*}
$$

and

$$
\begin{align*}
    g^{00} &= (r_\theta^2 + (rh)^2)/D \\
    g^{01} &= -r_z r_\theta /D \\
    g^{11} &= (f^2 + r_z^2)/D.
\end{align*}
$$

The gradient function $v$ is defined as $v = \langle N, E_r \rangle^{-1} = |\tilde{N}|/(1 - a_0 r_z - a_1 r_\theta)$.

When $r = r(\theta)$ depends only on $\theta$, we have

$$
\begin{align*}
    \lambda_1 &= \frac{1}{|\tilde{N}|} \frac{f'}{f} (1 - (r_\theta/f)^2) \\
    \lambda_2 &= \frac{1}{|\tilde{N}|} \frac{h^{-1}}{1 + (r_\theta/rh)^2} \left( (-1 + (r_\theta/f)^2) \left( \frac{r_\theta h}{rh} + h + h' \right) + h + 2h' \right)
\end{align*}
$$

where $|\tilde{N}| = \sqrt{(1 - (r_\theta/f)^2)^2 + (r_\theta rh/f)^2}$. Using $v = |\tilde{N}|/(1 - (r_\theta/f)^2)$, define

$$
\begin{align*}
    \tilde{\lambda}_1 &= v \lambda_1 = \frac{f'}{f} \\
    \tilde{\lambda}_2 &= v \lambda_2 = \frac{h^{-1}}{1 + (r_\theta/rh)^2} \left( -\frac{r_\theta h}{rh} - h - h' + \frac{h + 2h'}{1 - (r_\theta/f)^2} \right).
\end{align*}
$$
When \( r = r(z) \) depends only on \( z \), we have

\[
\lambda_1 = \frac{-r_{zz} + 2r_z^2 h/f + fh}{r_z^2 + f^2} |\tilde{N}| \\
\lambda_2 = \frac{h'}{h} |\tilde{N}|.
\]  

(2.7)

where \( |\tilde{N}| = f/\sqrt{r_z^2 + f^2} \). Using \( v = |\tilde{N}|^{-1} \), define

\[
\tilde{\lambda}_1 = v \lambda_1 = \frac{-r_{zz} + 2r_z^2 h/f + fh}{r_z^2 + f^2} \\
\tilde{\lambda}_2 = v \lambda_2 = \frac{h'}{h}.
\]  

(2.8)

Recall the definition of Gauss curvature, mean curvature, and harmonic mean curvature:

\[
K = \lambda_1 \lambda_2 = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{h_{00} h_{11} - h_{01}^2}{g_{00} g_{11} - g_{01}^2} \\
H = \lambda_1 + \lambda_2 = \text{tr}(h_{ij}) = g^{ij} h_{ij} = g^{00} h_{00} + g^{11} h_{11} + 2 g^{01} h_{01}
\]  

(2.9)

When the torus is evolved by HMCF, the graph function \( r \) in general satisfies following PDE:

\[
\frac{\partial r}{\partial t} = -\lambda_1 \lambda_2 v \\
\]  

(2.10)

which can be written more concisely in the form

\[
\frac{\partial r}{\partial t} = -\frac{\tilde{\lambda}_1 \tilde{\lambda}_2}{\lambda_1 + \lambda_2}.
\]  

(2.11)

Substituting (2.9) into (2.10), we solve following PDE with periodic initial value numerically:

\[
\frac{\partial r}{\partial t} = G(r, r_z, r_{z\theta}, r_{zz}, r_{z\theta}, r_{z\theta\theta}) \quad \text{in } [0, 2\pi] \times [0, 2\pi], \quad \text{(HMCF)}
\]

\[
r(0, z, \theta) = r_0(z, \theta) \quad \text{for } (z, \theta) \in [0, 2\pi] \times [0, 2\pi],
\]

where \( r_0 \) is a periodic function in each coordinate.

### 2.2.2 Linearization of HMCF

We consider a linear nonuniformly parabolic equation as a model for linearized HMCF. This linear model captures the characteristics of HMCF of torus that the parabolicity degenerates
in the direction of larger principal curvature only. The explicit solutions of the linear PDE
give an idea about how solutions of nonuniformly parabolic PDEs behave. We find that the
oscillations in the direction of strong parabolicity decays exponentially but oscillations in the
degenerating direction fails to converge. When the solution has oscillations in both directions
then it successfully converges to 0. Consider following PDE:
\[ u_t = u_{xx} + a(t)u_{yy} \quad \text{on } \Omega = [0, 2\pi] \times [0, 2\pi], \]
\[ u(0, x, y) = \phi(x, y) \quad \text{for } (x, y) \in \Omega, \]

where \( a(t) = e^{-kt} \) and \( \phi \) is periodic in each coordinate.

Let us find the solution of (2.12) with different initial values. Expressing the solution by its
Fourier series \( u = \sum_{k,l} \hat{u}_{k,l}(t)e^{-i(kx+ly)} \) and substituting it into (2.12), we obtain the solution
of (2.12) by solving following system of ODEs:
\[ \frac{d\hat{u}_{k,l}}{dt} = -(k^2 + a(t)l^2)\hat{u}_{k,l} \quad \text{for all } k, l. \]

**Example 1.** If \( \phi = \sin(mx) \), then the solution to (2.12) is \( u = e^{-mt}\sin(mx) \): the solution
decays to zero exponentially fast.

Since \( \phi = \sin(mx) = (e^{imx} - e^{-imx})/2i \), \( \hat{\phi}_{k,l} = 1/2i \) if \( (k, l) = (\pm m, 0) \) and \( = 0 \) if
\((k, l) \neq (\pm m, 0)\). We only need to solve (2.13) when \( (k, l) = (\pm m, 0) \) with initial conditions
\( \hat{u}_{k,l}(0) = 1/2i \) for \( (k, l) = (\pm m, 0) \). We find that \( \hat{u}_{k,0}(t) = e^{-k^2t}/2i \) for \( k = \pm m \)
and combining them \( u = \hat{u}_{m,0}e^{imx} - \hat{u}_{-m,0}e^{-imx} \) yields the solution.

**Example 2.** If \( \phi = \sin(ny) \), then the solution to (2.12) is \( u = e^{-nt}\sin(ny) \).

The solution does not converge to 0 because of the fast decay of \( a(t) \).

Similar to Example 1, \( \hat{\phi}_{k,l} = 1/2i \) if \( (k, l) = (0, \pm n) \) and \( = 0 \) otherwise. We only need to solve
(2.13) when \( (k, l) = (0, \pm n) \) with initial conditions \( \hat{u}_{k,l}(0) = 1/2i \) for \( (k, l) = (0, \pm n) \) and obtain
the solution.

**Example 3.** If \( \phi = \sin(mx)\sin(ny) \), we obtain
\[ u = \exp \left( -m^2t - n^2 \int_0^t a \right) \sin(mx) \sin(ny). \]
Note that the solution decays to 0 due to the nontrivial frequency in the $x$-direction.

Now, let us see that the linearization of HMCF of torus is nonuniformly parabolic if following cylindrical curvature estimate is assumed:

\[
\min_{\Phi} \frac{\lambda_2}{\lambda_1} \to \infty \quad \text{as } t \to \infty. \tag{2.14}
\]

Assuming the evolving surface is strictly convex, i.e. \( \lambda_2 \geq \lambda_1 > 0 \) and (2.14) holds, we have

\[
\frac{1}{4} \leq \frac{\partial F}{\partial \lambda_1} = \frac{1}{(1 + \frac{\lambda_1}{\lambda_2})^2} < 1
\]

\[
\frac{\partial F}{\partial \lambda_2} = \frac{1}{(1 + \frac{\lambda_2}{\lambda_2})^2} \to 0^+ \quad \text{as } t \to \infty. \tag{2.15}
\]

where $F$ is the harmonic mean curvature of the evolving surface. This observation implies that when the initial immersion evolves into cylindrical shape the speed of motion, $F$, is always sensitive to changes in $\lambda_1$ but becomes blind to changes in $\lambda_2$. This suggests that HMCF of torus is not so effective in regularizing the surface in direction of larger principal curvature. We emphasize that this phenomenon arises due to the geometry of underlying surface but is not characteristics of HMCF. For example, it is proved in [2] that HMCF of spherical hypersurface is uniformly parabolic due to different curvature estimate: $\lambda_i/\lambda_j < C$ for all $i, j$.

Now we derive linearized HMCF when the initial surface is expressed as a graph $r$ depending on $z$ or $\theta$ only and verify that the linearized HMCF can be either nondegenerate or degenerating. Let $r_{\epsilon}(z, t)$ be a family of function satisfying the equation (HMCF) and $\eta(z, t) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} r_{\epsilon}(z, t)$. First consider when $r$ depends only on $z$. Using equation (2.11) and the formulas (2.8)

\[
\frac{dG}{d\epsilon} \bigg|_{\epsilon=0} = \frac{\partial G}{\partial \lambda_1} \frac{d\tilde{\lambda}_1}{d\epsilon} \bigg|_{\epsilon=0} + \frac{\partial G}{\partial \lambda_2} \frac{d\tilde{\lambda}_2}{d\epsilon} \bigg|_{\epsilon=0}
\]

\[
= \frac{\partial G}{\partial \lambda_1} \frac{\partial \tilde{\lambda}_1}{\partial X} \eta_{zz} + \left( \frac{\partial G}{\partial \lambda_1} \frac{\partial \tilde{\lambda}}{\partial p} + \frac{\partial G}{\partial \lambda_2} \frac{\partial \tilde{\lambda}_2}{\partial p} \right) \eta_z + \left( \frac{\partial G}{\partial \lambda_1} \frac{\partial \tilde{\lambda}}{\partial u} + \frac{\partial G}{\partial \lambda_2} \frac{\partial \tilde{\lambda}_2}{\partial u} \right) \eta \tag{2.16}
\]

Thus, the linearized equation is uniformly parabolic since the coefficient of the second order term

\[
\frac{\partial G}{\partial \lambda_1} \frac{\partial \tilde{\lambda}_1}{\partial X} = \frac{\partial G}{\partial \lambda_1} \left( \frac{1}{\sqrt{\gamma_z^2 + f^2}} \right) > c_0 > 0
\]
by the gradient estimate $\max_{\Phi_t} |r_z| < c_1$ from Corollary 15. 

Similarly, we can derive the linearized HMCF when the graph depends only on $\theta$ using equation (2.11) and the formulas (2.6).

\[
\eta_t = \frac{\partial G}{\partial \tilde{\lambda}_2} \frac{\partial X}{\partial \eta} \eta_{t\theta} + \left( \frac{\partial G}{\partial \tilde{\lambda}_1} \frac{\partial \tilde{\lambda}_1}{\partial p} + \frac{\partial G}{\partial \tilde{\lambda}_2} \frac{\partial \tilde{\lambda}_2}{\partial p} \right) \eta_{\theta} + \left( \frac{\partial G}{\partial \tilde{\lambda}_1} \frac{\partial \tilde{\lambda}_1}{\partial u} + \frac{\partial G}{\partial \tilde{\lambda}_2} \frac{\partial \tilde{\lambda}_2}{\partial u} \right) \eta_{t\theta} + \left( \frac{\partial G}{\partial \tilde{\lambda}_1} \frac{\partial \tilde{\lambda}_1}{\partial u} + \frac{\partial G}{\partial \tilde{\lambda}_2} \frac{\partial \tilde{\lambda}_2}{\partial u} \right) \eta_{\theta}
\]

\[\eta_{t\theta} := \mathcal{L}_\theta(\eta)\]

We can verify numerically (Figure 2.1) that

\[
\frac{\partial G}{\partial \tilde{\lambda}_2} \frac{\partial \tilde{\lambda}_2}{\partial X} = \frac{\partial G}{\partial \tilde{\lambda}_2} \frac{1}{rh^2} \frac{1}{1 + (r_\theta/rh)^2}
\]

approaches 0 as time progresses since the term $1/rh^2$ is suppressed by the strong decay of $\partial G/\partial \tilde{\lambda}_2$.

### 2.3 Numerical Analysis

In this section we present numerical solutions of the HMCF that are convergent, nonconvergent and partially convergent. We are interested in finding what type of initial surfaces converge or does not converge to perfect torus asymptotically. Let $r(z, \theta)$ for $0 \leq z, \theta \leq 2\pi$ be the distance from the axis to the surface. The initial surface viewed as a graph over the axis is defined as a
perturbation of perfect torus:

\[ r(z, \theta) = R + \epsilon_1 \sin(m_1 z) \sin(n_1 \theta) + \epsilon_2 \sin(m_2 z) \sin(n_2 \theta) \]  

(2.18)

where \( \epsilon_1, \epsilon_2 \) and \( R \) are positive constants and \( m_1, n_1, m_2 \) and \( n_2 \) are positive integers. This type of initial condition is motivated by the solutions to linear nonuniformly parabolic PDEs discussed in Section 3.2.2. They are also the simplest linear combination of the Fourier basis \( \{e^{ikz}e^{il\theta}\}_{k,l \in \mathbb{Z}} \) that satisfy the periodic initial condition.

The dynamics of \( \lambda_1 \) and \( \lambda_2 \) will be captured by harmonic mean curvature \( F = \lambda_1 \lambda_2 / (\lambda_1 + \lambda_2) \) and mean curvature \( H = \lambda_1 + \lambda_2 \), respectively, because when the surface is strictly convex, i.e. \( \lambda_1, \lambda_2 > 0 \) we have \( F \approx \lambda_1 \) by \( \lambda_1 / 2 \leq F \leq \lambda_1 \) and when the surface evolves like a torus, i.e. \( \lambda_1 \to 0 \) and \( \lambda_2 \to \infty \), we have \( H \approx \lambda_2 \). To see if \( \lambda_2 \) converges uniformly to the same value as they tend to \( \infty \) we will look at the ratio \( \max_{\theta} H / \min_{\theta} H \). The product \( \lambda_1 \lambda_2 \) of two principal curvatures will be examined and it is expected to converge 1 if the surface evolves into a perfect torus. Since these curvature functions are coordinate invariant, unlike the principal curvatures, it makes them into better geometric quantities to be studied.

2.3.1 Numerical Scheme

Given the complexity of equation (HMCF), it is very difficult to obtain rigorous results for its numerical solution. However, we take advantage of the fact that the solution of equation (HMCF) is periodic and surfaces evolving under HMCF is smooth if the initial surface is smooth. Spectral methods is known to achieve spectral accuracy in its approximations and it outperforms any other numerical methods when the solution is periodic and smooth. Therefore, we adopt spectral methods to solve (HMCF) numerically and examine its solution under various initial conditions. Following is the standard algorithm to solve PDEs using spectral methods.

Step I Take fourier transform of (HMCF) and consider finite number of frequency modes in the range \( -N/2 \leq k, l < N/2 \) where \( N = 48 \). Finding the numerical solution of (HMCF) turns into solving following systems ODEs:

\[ \frac{\partial \hat{r}_{kl}}{\partial t} = \hat{G}_{kl} \quad \text{for} \quad -N/2 \leq k, l < N/2 \]  

(2.19)

where \( \hat{r}_{kl} \) and \( \hat{G}_{kl} \) are the fourier coefficients of \( r \) and \( G \), respectively.
Since $G$ is a highly nonlinear function of the solution and its derivatives, it is not possible to find its Fourier coefficients analytically. Instead we employ a standard method used in spectral methods to obtain its Fourier coefficients. The idea is to first obtain all the Fourier coefficients of the derivatives of the solution, then take the inverse Fourier transform of each of them to recover their approximated value in physical space. The nonlinear terms in $G$ is computed using those data. Finally we take the Fourier transform of $G$ to find its Fourier coefficients. In order to remove aliasing effect introduced while computing the nonlinearities, the Fourier coefficients obtained from Step II(i) below need to be zero-padded. See next section for more discussion on choosing the size of zero-padding to remove aliasing effects.

**Step II** Given the data $\{\hat{r}_{kl}^n, -N/2 \leq k, l < N/2\}$ at the $n$th time step, approximate $\hat{G}_{kl}^n$ as follows.

(i) Find Fourier coefficients of the derivatives of $r$ directly from $\{\hat{r}_{kl}^n\}$:

$$
\hat{(r_z)_n}_{kl}, \quad \hat{(r_\theta)_n}_{kl}, \quad \hat{(r_{\theta\theta})_n}_{kl}, \quad \hat{(r_{\theta\theta})_n}_{kl}, \quad \hat{(r_{z\theta})_n}_{kl}.
$$

(ii) Add $4N$ zeros to the end of every Fourier coefficient matrices obtained in (i).

(iii) Take the IFT of each zero-padded Fourier coefficient matrices from (ii) and denote them by

$$
(r_z)_p^n, \quad (r_\theta)_p^n, \quad (r_{zz})_p^n, \quad (r_{\theta\theta})_p^n, \quad (r_{z\theta})_p^n.
$$

(iv) Evaluate $G$ by substituting (2.20) into (HMCF) and denote it by $G_p^n$. It can now accommodate higher frequencies since the discretization of the domain becomes finer when Fourier coefficient matrix is enlarged by adding zeros.

**Step III** Solve (2.19) by RK45-method repeating Step II whenever $\hat{G}_{kl}^n$ needs to be calculated at each time step.

### 2.3.2 Dealiasing

In Step II of the numerical scheme, we transform the data from Fourier space to physical space, evaluate the nonlinear functions and take inverse DFT to find the Fourier coefficients of $G$ of
It is standard procedure to add zero-padding to the initial data in Fourier space in order to remove aliasing effect introduced while taking the DFT of the nonlinear functions. The size of zero-padding can be determined if we know the degree of algebraic nonlinearities. For the sake of completeness, we discuss the method of dealiasing in detail [9] and argue why the zero-padding size $M = 4N$ is large enough for (2.10) when $r = r(\theta)$.

Given Fourier coefficients $\{\hat{u}_m\}, \{\hat{v}_n\}, -N/2 \leq m, n \leq N/2 - 1$, we construct the corresponding functions in physical space as follows.

\[
\begin{align*}
  u_j &= \sum_{m=-N/2}^{N/2-1} \hat{u}_m e^{imx_j} \\
  u_j &= \sum_{n=-N/2}^{N/2-1} \hat{v}_n e^{inx_j}
\end{align*}
\]

where $x_j = 2\pi j/N$ and $0 \leq j \leq N - 1$. Our goal is to find the Fourier coefficients of a nonlinear function

\[
s_j = u_jv_j \quad 0 \leq j \leq N - 1
\]

by taking DFT.

\[
\hat{s}_k = \frac{1}{N} \sum_{j=0}^{N-1} e^{-ikx_j} s_j
\]

\[
= \frac{1}{N} \sum_{j=0}^{N-1} e^{-ikx_j} \left( \sum_{m=-N/2}^{N/2-1} \hat{u}_m e^{imx_j} \right) \left( \sum_{n=-N/2}^{N/2-1} \hat{v}_n e^{inx_j} \right)
\]

\[
= \frac{1}{N} \sum_{m,n} \hat{u}_m \hat{v}_n \sum_{j=0}^{N-1} e^{i(-k+m+n)x_j}
\]

\[
= \sum_{m+n=k} \hat{u}_m \hat{v}_n + \sum_{m+n=k\pm N} \hat{u}_m \hat{v}_n
\]

The purpose of fattening the Fourier coefficients of $u$ and $v$ by adding zeros is to remove the second term in the last line which is responsible for the aliasing effect and keep the first term as it is.

Let $M > N$. Define the zero-padded Fourier coefficients $\{\hat{u}_m\}, -M/2 \leq m \leq M/2 - 1$ by

\[
\hat{u}_m = \begin{cases} 
  \hat{u}_m & \text{if } -N/2 \leq m \leq N/2 - 1 \\
  0 & \text{otherwise}
\end{cases}
\]
and define \( \{ \hat{v} \} \) similarly. Let \( \bar{u}_j, \bar{v}_j \) and \( \bar{s}_j, -M/2 \leq j \leq M/2 - 1 \) be the corresponding functions as above. Then

\[
\hat{\bar{s}}_k = \sum_{m+n=k} \hat{\bar{u}}_m \hat{\bar{v}}_n + \sum_{m+n=k\pm M} \hat{\bar{u}}_m \hat{\bar{v}}_n.
\]

\( M \) has to be chosen large enough so that the second sum is zero for all \( |k| \leq N/2 \). It has to be at least

\[
\frac{N}{2} + \frac{N}{2} \leq -\frac{N}{2} + M,
\]

thus

\[
M \geq \frac{3N}{2}.
\]

If the nonlinear function were a product of \( \alpha \) functions, then

\[
M \geq \frac{(1 + \alpha)N}{2}.
\]

Although it is not possible to read off the algebraic nonlinearity of (HMCF) from the equation, we can examine the lower order terms of its Taylor expansion and get a sense of the degree of its algebraic nonlinearity. We obtain the Taylor expansion of R.H.S of (HMCF) when \( r = r(\theta) \) under a set of assumptions: \( \tilde{\lambda}_1/\tilde{\lambda}_2 \ll 1, r_\theta \ll 1 \) and \( r_\theta \theta/r_h \ll 1 \). These assumptions are consistent with the numerical results. For more detail about the assumptions, see Discussion.

\[
G = \frac{\tilde{\lambda}_1 \tilde{\lambda}_2}{\lambda_1 + \lambda_2} = \frac{\tilde{\lambda}_1}{1 + \lambda_1/\lambda_2}
\]

\[
= \tilde{\lambda}_1 \left( 1 - \frac{\tilde{\lambda}_1}{\lambda_2} + \left( \frac{\tilde{\lambda}_1}{\lambda_2} \right)^2 - \left( \frac{\tilde{\lambda}_1}{\lambda_2} \right)^3 + \ldots \right)
\]

\[
\approx \tilde{\lambda}_1 \left( 1 - \frac{\tilde{\lambda}_1}{\lambda_2} \right).
\]

In order to find the Taylor expansion of the second term, we first rearrange the formula (2.6) to get

\[
\tilde{\lambda}_2^{-1} = \frac{h(1 + (r_\theta/r_h)^2)(1 - (r_\theta/f)^2)}{h' - \frac{r_\theta}{r_h}(1 - (r_\theta/f)^2) + (r_\theta/f)^2(h + h')}
\]

\[
\approx h(1 + (r_\theta/r_h)^2)(1 - (r_\theta/f)^2)
\]

\[
\cdot \frac{1}{(h')^2} \left( h' + \frac{r_\theta}{r_h}(1 - (r_\theta/f)^2) - (r_\theta/f)^2(h + h') \right)
\]
Then
\[ \frac{\lambda_1^2}{\lambda_2} = \frac{h^3}{f^4} \left( 1 + \left( \frac{r_\theta}{rh} \right)^2 \right) \left( 1 - \left( \frac{r_\theta}{f} \right)^2 \right) \]
\[ \cdot \left( h' + \frac{r_{\theta\theta}}{rh} \left( 1 - \left( \frac{r_\theta}{f} \right)^2 \right) - \left( \frac{r_\theta}{f} \right)^2 (h + h') \right) \]
\[ \approx \frac{h^3}{f^4} \left( h' + \frac{r_{\theta\theta}}{rh} - \left( \frac{r_\theta}{f} \right)^2 (h + h') \right) \]
\[ \approx h^3 h' + \frac{h^2 r_{\theta\theta}}{r} - h^3 r_{\theta\theta}^2 (h + h') \]

where in the last line we approximated \( f \approx 1 \). We see that the degree of algebraic nonlinearity of each term is less than 6. Thus, the zero-padding size \( M = 4N \) would be large enough.

### 2.3.3 Numerical Results

We first present numerical results regarding the barriers and harmonic mean curvature since both showed qualitatively the same results in all the simulations considered below. Barrier is a family of perfect tori evolving under HMCF and surrounds the surface from outside (or inside) without intersection at the initial time. Throughout the evolving process, the barrier does not intersect the evolving surface by the parabolic maximum principle. Figure 1 shows that numerical solution is in agreement with the theory. Each line in Figure 1 shows the minimum distance from the evolving surface to the barrier at each time of outer and inner barriers. Positive (negative) sign means that the barrier lies outside (inside) the torus. The fact that both outer and inner barriers do not change their signs shows that barrier and the surface remain disjoint. It is also found that HMC, which is the speed of motion, agrees with the theoretical result obtained in Chapter 2. It is proved there that \(-t + c_1 \leq \log F \leq -t + c_2\) for axially symmetric surfaces and the numerical results show that \( \log F \) decreases at a rate \(-t\). This is the rate at which perfect torus evolves under HMCF and the correct behavior of barrier is possible due to the agreement of speed of barriers and evolving surface.

**Experiment 1. Initial graph:** \( r = R + \epsilon_1 \sin m_1 z \cdot \sin n_1 \theta \)

The given graph function deals with three types of surface: (i) \( z \)-symmetric \( (m_1 \neq 0, n_1 = 0) \), (ii) \( \theta \)-symmetric \( (m_1 = 0, n_1 \neq 0) \) and (iii) general surface \( (m_1, n_1 \neq 0) \). In (i) and (ii), \( m_1 = 0 \) or \( n_1 = 0 \) implies that the sine function whose frequency component is zero is set to one. The convergence results of (i), (ii), (iii) and other graphs not shown here are summarized...
in Table 2.3. The first row of the table is Case (i), the first column is Case (ii) and the second row is Case (iii).

Case (i): The initial graph, \( r = R + \epsilon_1 \sin m_1 z \), is axially symmetric and the solution remains so. The ratio of the maximum and minimum of mean curvature \( H \) over the evolving surface at each time converges to 1 showing that \( \lambda_2 \) converges uniformly. Although not shown, \( \log \) of the mean curvature is found to grow at a rate \( t \) that agrees with theoretical result obtain in Chapter 2 for axially symmetric surfaces. Gauss curvature \( K \) of the numerical solution converge to 1, gauss curvature of perfect torus, rapidly (Figure 2.4) which is, in fact, a result stronger than the theoretical estimate \( K \approx \text{cst} \) obtained in Chapter 2.

Case (ii): The initial graph, \( r = R + \epsilon_1 \sin n_1 \theta \), is \( \theta \)-symmetric. We find that the convergence result depends on the perturbation frequency. The surface does not converge to perfect torus when the perturbation frequency is low \( n_1 = 2, 4 \) but it converges for high frequency \( \theta = 6 \). Among the low frequencies \( n_1 = 2, 4 \), the curvature functions of the high frequency surface \( n_1 = 4 \) approach very close to perfect torus, on the other hand, the curvature functions of low frequency surface \( n_1 = 2 \) diverges from perfect torus. We see a transition from nonconvergent to convergent solution as the perturbation frequency increases. Although Example 2 of the linear model in Section 3.2.2 never converges to 0, we still see that higher frequency of the initial
value results in solutions closer to 0.

Case (iii): The initial graph, \( r = R + \epsilon_1 \sin m_1 z \sin n_1 \theta \), has nontrivial frequency in both \( z \)- and \( \theta \)-directions. In contrast to Case (ii) which possesses \( \theta \)-frequency only, the presence of non-trivial \( z \)-frequency enables the surface to converge to perfect torus for all sizes of \( \theta \)-frequency. As discussed in Example 3 of the linear model in Section 3.2, nontrivial frequency components in the direction of strong parabolicity helps effectively damp the fluctuations in the solution. In Figure 2.3, all the other curvature functions converge to the perfect torus with no difficulty and the rate of converge improves as the \( z \)-frequency increases.

**Experiment 2. Initial graph:** \( r = R + \epsilon_1 \sin 2 \theta + \epsilon_2 \sin m_2 z \cdot \sin n_2 \theta \)

In the following set of simulations, we examine the convergence issue when the initial graph is the sum of more than one fourier bases. In particular, we are interested in studying what happens to the low \( \theta \)-frequency function, e.g. \( \sin 2 \theta \), which is known to be nonconvergent, when a frequency component that is known to converge, e.g. \( \sin m_2 z \sin n_2 \theta \), is added. We want to see if adding the extra frequency component facilitates regularizing the low \( \theta \)-frequency or it
resists to decay. We find that with the newly added function, \( \sin m_2 z \sin 2\theta \), the curvature functions indeed decays for a short time. However, all the solutions start behaving similarly to the curvature functions of the nonconvergent component \( \sin 2\theta \). In Figure 2.7 we see that after the initial drop both curvature functions diverge showing the same asymptotic behavior regardless of the \( z \)- and \( \theta \)-frequency of the added term.

We can identify what causes the curvature functions to diverge later in time by examining the power spectrum of the evolving surface. It is the power spectrum of the rescaled surface that needs to be examined since the entire power spectrum of the evolving surface vanishes too rapidly to distinguish different frequency components. Figure 2.8 shows the power spectrum of the evolution of the initial surface \( r = R + \epsilon \sin 2\theta + \epsilon \sin 2z \sin 4\theta \). We see that the entire power spectrum is captured by the Fourier coefficients of \( \sin 2\theta \) which is the frequency component that resists to decay and the rest of the Fourier coefficients vanish rapidly. The power spectrum demonstrates that HMCF filters out all of the frequency components except for the low frequency components in purely \( \theta \)-direction, i.e. \( \hat{r}_{0,l} \) where \( l \) is small. In this perspective, HMCF can be viewed as a deformation process that passes only the low \( \theta \)-frequencies and filters out the rest of the frequency components in the spectrum. It is an interesting observation that HMCF of torus filters frequencies in direction selective manner.
2.4 Discussion

Although the numerical simulation presented in previous section provides evidence that perturbations of torus with low θ-frequency does not converge to perfect torus, it does not give any explanation of why it fails to converge. In the following sections, based on some estimates on the solution and its derivatives observed from the numerical results, we identify key terms causing the nonconvergent behavior and find a quantity that distinguishes convergent and nonconvergent solutions.

We will assume following $C^0, C^1$ and $C^2$ estimates of $r$ which are verified to be true for all numerical experiments we conduct. Under these assumptions, the asymptotic formula for the curvature functions can be expressed in simple form.

\[ r = O(e^{-t}), \]  
\[ \frac{r_z}{r_h} = o(1), \quad \frac{r_\theta}{r_h} = O(1) \]  
\[ \frac{r_{zz}}{h} = o(1), \quad \frac{r_{\theta\theta}}{r_h} = O(1), \quad \frac{r_{z\theta}}{r_h} = o(1). \]

where we used the usual big and little O notation and $h(r) = \sinh r$. We will use (C0) and (C1) to derive the asymptotic estimates of (2.1), (2.2) and (2.4), then use (C2) to find the asymptotic expressions for the curvature functions (2.25), (2.27) and (2.28).

**Remark**  (1) The scaling factors appearing in (C1) and (C2) show up as a natural factor when
estimating \( g^{ij} \) (2.23) and the curvature functions \( K \) (2.24) and \( H \) (2.26).

(2) The scaling factor for \( r_{zz} \) is \( h \), unlike the scaling factor \( rh \) for \( r_{\theta\theta} \), which makes it easier to control \( r_{zz}/h \). We will see that the scaled derivatives \( r_\theta/rh \) and \( r_{\theta\theta}/rh \) being bounded (See Figure 4) but not always tending to 0 determine the nonconvergent behavior of the curvature functions.

(3) \( (C0) \) implies \( h(r) = \sinh r = O(e^{-t}) \) and \( f(r) = \cosh r \to 1 \).

### 2.4.1 Curvature Estimates

In the following analysis, we will use the estimates \( (C0), (C1) \) and \( (C2) \) in order to derive a simple expression for the asymptotic value of the curvature functions. The asymptotic formula for the curvature functions will be checked against the numerical results to verify that they are in agreement. We will see that when \( r_{\theta\theta} \) fails to converge to 0 faster than \( rh \) the curvature functions no longer converge to those of perfect torus.

We first derive the asymptotic expressions for the normal vector (2.1), the second fundamental form (2.2) and inverse of the metric (2.4) from which all the curvature functions can be derived. Let’s show that \( \tilde{N} = -E_r + a_0Z_0 + a_1Z_1 \to -E_r \) by demonstrating \( a_0, a_1 \to 0 \). Applying
Figure 2.7: Experiment 2. \( r = R + \varepsilon \sin 2\theta + \varepsilon \sin mz \sin 2\theta \)

(C1) we easily obtain the asymptotic estimate:

\[
a_0 = \frac{h^2 r^2 r_z}{(f^2 + r_z^2)h^2 r^2 + f^2 r_{\theta}^2} = \frac{r_z}{f^2 + r_z^2 + f^2(r_{\theta}/rh)^2} \to 0
\]

\[
a_1 = \frac{r_{\theta}(r_{\theta}^2 - r_z^2 + h^2 r_{\theta}^2)}{(f^2 + r_z^2)h^2 r^2 + f^2 r_{\theta}^2} = \frac{r_{\theta}((r_{\theta}/rh)^2 - (r_z/rh)^2 + 1)}{f^2 + r^2 + f^2(r_{\theta}/rh)^2} \to 0.
\]

In order to find the asymptotic expression for the second fundamental form we identify the terms that decrease most slowly, thus dominate all the other terms. We write the second fundamental form as the sum of singletons and then observe that most of the factors constituting each summand converge to zero. We then associate a degree to each summand by counting the number of vanishing factor it has and the summands with low degree are decreasing slowly, thus they are identified as the dominant terms. Let us describe how the asymptotic formula for \( h_{00} \) is obtained. The other elements of \( h_{ij} \) can be worked out similarly. From (2.2), we have

\[
h_{00} = \frac{1}{|N|} (ff' - r_{zz} + a_0 ff'r_z + a_0 r_z r_{zz} - a_1 f f' r_\theta + a_1 r_{zz} r_\theta)
\]

\[
\approx ff' - r_{zz}
\]

(2.21)

Note that except for the first two terms in the first line of (2.21) each term is a product of two or more vanishing factors. The third term is a product of three vanishing factors \( a_0, f', \) and \( r_z \) and the fourth term is a product of three vanishing factors \( a_0, r_z, \) and \( r_{zz} \) and so on. Thus we identify the first two terms as the asymptotic value of \( h_{00} \). Similarly, the asymptotic expressions
for $h_{11}$ and $h_{01}$ are found to be

\[
h_{11} \approx r^2 h_1 h' - r r_{\theta\theta},
\]

\[
h_{01} \approx -r z \theta.
\]

We obtain the asymptotic formulas for $g^{ij}$ by applying (C0) and (C1):

\[
g^{00} = \frac{r_\theta^2 + (r h)^2}{f^2(r_\theta^2 + r h^2) + r_z^2(r h)^2} = \frac{1}{f^2 + r_\theta^2 (r h)^2} \to 1,
\]

\[
g^{11} = \frac{f^2 + r_z^2}{(f^2 + r_z^2)(r h)^2 + f^2 r_\theta^2} \approx \frac{1}{(r h)^2 + r_\theta^2}.
\]

\[
g^{01} = \frac{-r_z r_\theta}{(f^2 + r_z^2)(r h)^2 + f^2 r_\theta^2} \approx -\frac{r_z}{r h} \cdot \frac{r_\theta}{f^2 + r_z^2 + f^2 (r_\theta/r h)^2} \approx -\frac{r_z}{r h} \cdot \frac{r_\theta}{1 + (r_\theta/r h)^2}.
\]

Now we are ready to find the asymptotic values for the curvature functions. First we need

\[
\det g_{ij} = (f^2 + r_z^2)(r h)^2 + f^2 r_\theta^2 \approx (r h)^2 + r_\theta^2.
\]
where both \((rh)^2\) and \(r_\theta^2\) are chosen to approximate \(\det g_{ij}\) since from numerical results we see that under certain initial conditions (e.g. \(r = R + \epsilon \sin 2\theta\)) both terms decay at at comparable rate.

Substituting (2.21) and (2.22) into (2.9), we obtain

\[
\frac{h_{00} h_{11}}{\det g} \approx \frac{(f f' - r_{zz})(r h h' - r_{\theta\theta})r}{(r h)^2 + r_\theta^2} = \left(f - \frac{r_{zz}}{h}\right) \cdot \frac{h' - r_{\theta\theta}/rh}{1 + (r_\theta/rh)^2},
\]

(2.24)

\[
\frac{h_{01}^2}{\det g} \approx \frac{r_{z\theta}^2}{(r h)^2 + r_\theta^2} = \frac{(r_{z\theta}/rh)^2}{1 + (r_\theta/rh)^2}.
\]

Note that \(h_{01}^2/\det g_{ij} \to 0\) due to (C2), thus we arrive at the asymptotic formula for \(K\):

\[
K = \frac{h_{00} h_{11}}{\det g_{ij}} - \frac{h_{01}^2}{\det g_{ij}} \approx \frac{h_{00} h_{11}}{\det g_{ij}} \approx \frac{h' - r_{\theta\theta}/rh}{1 + (r_\theta/rh)^2} := K_{asy}.
\]

(2.25)

Substituting (2.21), (2.22), and (2.23) into (2.9), we obtain

\[
\begin{align*}
\frac{h_{00} g^{00}}{\det g_{ij}} & \approx (f f' - r_{zz}) \cdot 1 \to 0 \\
\frac{h_{11} g^{11}}{\det g_{ij}} & \approx \frac{r_{z\theta}^2 h h' - r r_{\theta\theta}}{(r h)^2 + r_\theta^2} = \frac{1}{h} \cdot \frac{h' - r_{\theta\theta}/rh}{1 + (r_\theta/rh)^2} \\
\frac{h_{01} g^{01}}{\det g_{ij}} & \approx r_{z\theta} \cdot \frac{r_z}{r h} \cdot \frac{r_\theta}{r h} \cdot \frac{1}{1 + (r_\theta/rh)^2} \to 0,
\end{align*}
\]

(2.26)

thus the asymptotic formula for \(H\) is

\[
H \approx \frac{h_{11} g^{11}}{\det g_{ij}} \approx \frac{1}{h} \cdot \frac{h' - r_{\theta\theta}/rh}{1 + (r_\theta/rh)^2} := H_{asy}.
\]

(2.27)

The asymptotic estimate for harmonic mean curvature can be derived from (2.25) and (2.27):

\[
F = \frac{K}{H} \approx h.
\]

(2.28)

The asymptotic formulas of mean curvature \(K_{asy}\) of (2.27) and Gauss curvature \(K_{asy}\) of (2.25) capture the behavior of both curvature functions quite well when the initial graph is \(r = R + \epsilon \sin 2\theta\) (See Figure 3). The ratio \(\max H/\min H\) fails to converge to one and the ratio \(\max H_{asy}/\min H_{asy}\) of the asymptotic formula for \(H\) indeed approximates the numerical solution very well. The \(\log\) of Gauss curvature does not converge to zero and its asymptotic formula (2.25) is again a good estimate of the numerical solution.

The fact that the asymptotic formulas of the curvature functions approximate their true numerical values well indicates that \(r_{\theta\theta}/rh\) and \(r_\theta/rh\) are the key terms that determine the asymptotic
behavior of $H$ and $K$. In fact, $r_{\theta \theta}/r h$ is essentially the only term that matters since $(r_{\theta}/r h)^2$ becomes quite small compared to 1. Figure 4 shows that $r_{\theta \theta}/r h$ indeed reach plateau around $t = 1.5$, which is approximately the time $\max H / \min H$ and $\log K$ start leveling off.

### 2.4.2 Eigenvalues of Linearized HMC Flow

From the numerical results in Section 3.3.2 and analysis of curvature functions in Section 3.4.1 we found that initial surfaces with low $\theta$-frequency fail to converge to perfect torus due to the weak decay rate of $r_{\theta \theta}$. With the goal of understanding why certain surfaces converge and others don’t, we will compare the eigenvalues of linearized HMCF with different initial conditions and provide numerical evidence that the eigenvalues of convergent surfaces are larger than the eigenvalues of surfaces that do not converge. A similar analysis is conducted in [?] to show a large class of closed curves do not converge to a round point under curve shortening flow where the speed is a small power of curvature. They prove that the eigenvalues of the nonconvergent curves are less than a critical value, thus is unstable.

The eigenvalues of the linearized HMCF can be understood at the conceptual level as follows. The desirable outcome of HMCF of torus is to smooth out the perturbation of the surface and evolve it into surface with constant curvature during the course of deformation process. To achieve this goal the points on the torus with large radius need to move fast to catch up with
the average distance and those with small radius need to slow down. In other words, the speed of motion has to increase or decrease depending on whether the distance to the axis is large or small. This simple observation leads to considering the derivative of speed of motion with respect to the distance to the axis, which can be viewed as the eigenvalue of linearized HMCF as we shall see below. Note that although this derivative is not globally well-defined, it makes sense on the intervals where distance function is monotone.

**Theoretical Analysis**

Let $G$ be the HMC operator in (2.10). Since the computation of $dG/dr$ is manageable if $r = r(z)$ or $r = r(\theta)$, we will examine those special cases. Let us first consider when $r = R + \epsilon \sin n\theta$.

\[
\frac{\partial G}{\partial r} = \frac{\partial G}{\partial \lambda_1} \frac{d\lambda_1}{dr} + \frac{\partial G}{\partial \lambda_2} \frac{d\lambda_2}{dr}
\]

\[
= \frac{\partial G}{\partial \lambda_1} \frac{d\tilde{\lambda}_1}{d\theta} / \eta + \frac{\partial G}{\partial \lambda_2} \frac{d\tilde{\lambda}_2}{d\theta} / \eta
\]

\[
= (\mathcal{L}_\theta \eta) / \eta
\]

where $\mathcal{L}_\theta$ is the linearized HMC operator from (2.17) and $\eta = r_\theta$. If we assume $\eta$ is the eigenfunction of the linearized HMC operator, $\partial G/\partial r$ is the corresponding eigenvalue. For the given $r$, it can be verified that $\eta$ is approximately an eigenfunction as we shall see below.
that eigenfunctions of linearized HMCF at $r = \text{cst}$ include $\cos n\theta$, thus the eigenfunctions of the linearized HMCF operator at $r = R + \epsilon \sin n\theta$ is expected to be close to $\cos n\theta$. Denote $G_1 := \frac{\partial G}{\partial \tilde{\lambda}_1} \frac{d\tilde{\lambda}_1}{dr}$ and $G_2 := \frac{\partial G}{\partial \tilde{\lambda}_2} \frac{d\tilde{\lambda}_2}{dr}$ and view them as the contribution of $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ to the eigenvalue, respectively. Now, we compute the coefficients of $L_\theta \eta$.

$$\frac{\partial \tilde{\lambda}_1}{\partial u} = \frac{1}{f^2} \left( 1 - \left( \frac{r_\theta}{f} \right)^2 \right) + 2 \left( \frac{r_\theta f'}{f^2} \right)^2$$

$$\frac{\partial \tilde{\lambda}_1}{\partial p} = -\frac{2 f'}{f^3} r_\theta$$

$$\frac{\partial \tilde{\lambda}_2}{\partial u} = \frac{2(1 + rf/h)}{(1 + (r_\theta/rh)^2)^2 (rh)^3} \left( h' + (-1 + (r_\theta/f)^2) \frac{r_{\theta\theta}}{r h} + (r_\theta/f)^2 (h' + h) \right)$$

$$- \frac{f}{1 + (r_\theta/rh)^2} \frac{1}{h^2} \left( h' + (-1 + (r_\theta/f)^2) \frac{r_{\theta\theta}}{r h} + (r_\theta/f)^2 (h' + h) \right)$$

$$\frac{1}{1 + (r_\theta/rh)^2} \left( -\frac{2 r_{\theta\theta}^2 r_{\theta\theta}}{r h} + (1 - (r_\theta/f)^2) \frac{r_{\theta\theta}}{(rh)^2} (1 + rf/h) \right)$$

$$- \frac{2}{f^3} r_\theta^2 (h + h') + (r_\theta/f)^2 (f/h + 1)$$

$$\frac{\partial \tilde{\lambda}_2}{\partial p} = \frac{-1}{1 + (r_\theta/rh)^2} \frac{2r_\theta}{r^2 h^3} \left( h' + (-1 + (r_\theta/f)^2) \frac{r_{\theta\theta}}{r h} + (r_\theta/f)^2 (h' + h) \right)$$

$$+ \frac{2r_{\theta\theta}^2 r_{\theta\theta}}{1 + (r_\theta/rh)^2} \frac{r_{\theta\theta}}{rh^2 f^2} + \frac{1}{1 + (r_\theta/rh)^2} \frac{2r_\theta}{f^2 h} (h + h')$$

$$\frac{\partial \tilde{\lambda}_2}{\partial X} = \frac{1}{1 + (r_\theta/rh)^2} \frac{-1 + (r_\theta/f)^2}{r h^2}.$$  

Applying the asymptotic estimates of $r, r_\theta$ and $r_{\theta\theta}$ obtained numerically, we obtain

$$\frac{\partial \tilde{\lambda}_1}{\partial u} \approx 1, \quad \frac{\partial \tilde{\lambda}_1}{\partial p} \approx 0$$

$$\frac{\partial \tilde{\lambda}_2}{\partial u} \approx \frac{1}{r h} r_{\theta\theta}, \quad \frac{\partial \tilde{\lambda}_2}{\partial p} \approx \frac{r_\theta r_{\theta\theta}}{r h} \frac{1}{r h}, \quad \frac{\partial \tilde{\lambda}_2}{\partial X} \approx \frac{1}{r h^2},$$
from which we estimate

\[ G_1 = \frac{\partial G}{\partial \tilde{\lambda}_1} \left( \frac{\partial \tilde{\lambda}_1}{\partial u} \eta + \frac{\partial \tilde{\lambda}_1}{\partial p} \eta_\theta \right) / \eta \approx \frac{\partial G}{\partial \tilde{\lambda}_1} \frac{\partial \tilde{\lambda}_1}{\partial u} \approx \frac{\partial G}{\partial \tilde{\lambda}_1}, \tag{2.29} \]

\[ G_2 = \frac{\partial G}{\partial \tilde{\lambda}_2} \left( \frac{\partial \tilde{\lambda}_2}{\partial u} + \frac{\partial \tilde{\lambda}_2}{\partial p} \eta_\theta + \frac{\partial \tilde{\lambda}_2}{\partial X} \eta_\theta \right) / \eta \approx \frac{\partial G}{\partial \tilde{\lambda}_2} \frac{\partial \tilde{\lambda}_2}{\partial u} \approx \frac{\partial G}{\partial \tilde{\lambda}_2} \frac{n^2}{r h^2}. \tag{2.30} \]

From the analysis, we see that the contribution \( G_1 \) to the eigenvalue from \( \tilde{\lambda}_1 \) does not depend on \( \theta \)-frequency \( n \) and is more or less constant. On the other hand, the contribution \( G_2 \) from \( \tilde{\lambda}_2 \) increases with \( \theta \)-frequency \( n \). But note that due to fast decay of \( \partial G / \partial \tilde{\lambda}_2 \) the term \( 1/rh^2 \) is suppressed. As will be shown in next section this dynamics is well captured by the numerical results.

It remains to verify that when the solution of HMCF takes the form \( r = R + \epsilon \sin n\theta, \eta = r_\theta = \cos n\theta \) is approximately an eigenfunction of the linearized HMCF. To this end, we examine the linearization of HMCF at \( r = \text{cst} \) and show \( \cos n\theta \) is an eigenfunction. Letting \( r_\theta = r_{\theta\theta} = 0 \), we get

\[ \frac{\partial \tilde{\lambda}_1}{\partial u} = \frac{1}{f^2}, \quad \frac{\partial \tilde{\lambda}_1}{\partial p} = 0, \]

\[ \frac{\partial \tilde{\lambda}_2}{\partial u} = -\frac{f^2}{h^2}, \quad \frac{\partial \tilde{\lambda}_2}{\partial p} = 0, \quad \frac{\partial \tilde{\lambda}_2}{\partial X} = -\frac{1}{r h^2} \]

\[ \frac{\partial G}{\partial \tilde{\lambda}_1} = \frac{1}{(1 + (h/f)^2)^2}, \quad \frac{\partial G}{\partial \tilde{\lambda}_2} = \frac{1}{(1 + (f/h)^2)^2} \]

and the linearized HMCF is

\[ \mathcal{L}_\eta = \frac{r^{-1}h^2}{(h^2 + f^2)^2} \eta_\theta + \frac{f^{-2}(h^2 - 1)}{(h^2 + f^2)^2} \eta. \]

Thus, \( \cos n\theta \) is eigenfunction with eigenvalue \( (f^2(1 - h^2) + n^2r^{-1}h^2)/(h^2 + f^2)^2 \approx 1 + n^2h \).

Next, consider surfaces \( r = R + \epsilon \sin(mz) \) that depend only on \( z \). Similarly as above we compute

\[ \frac{\partial \tilde{\lambda}_1}{\partial u} = \frac{-2fh}{(r_z^2 + f^2)^2} (-r_{zz} + 2fr_z^2/f + ff') + \frac{1}{r_z^2 + f^2} (2r_z^2/f + h^2 + f^2) \]

\[ \frac{\partial \tilde{\lambda}_1}{\partial p} = \frac{2r_z}{r_z^2 + f^2} (r_{zz} + fh) \]
\[
\frac{\partial \tilde{\lambda}_1}{\partial X} = \frac{-1}{r_z^2 + f^2}
\]

\[
\frac{\partial \tilde{\lambda}_2}{\partial u} = -\frac{1}{h^2}.
\]

Using the numerical estimates of \(r, r_z\), and \(r_{zz}\), we obtain

\[
\frac{\partial \tilde{\lambda}_1}{\partial u} \approx hr_{zz} + h^2 r_z^2 + h^2 + r_z^2 + 1 \approx 1,
\]

\[
\frac{\partial \tilde{\lambda}_1}{\partial p} \approx r_z r_{zz} + hr_z^3 + hr_z \approx r_z r_{zz} + hr_z,
\]

\[
\frac{\partial \tilde{\lambda}_1}{\partial X} \approx 1,
\]

\[
\frac{\partial \tilde{\lambda}_2}{\partial u} = -\frac{1}{h^2}.
\]

Then we estimate

\[
G_1 = \frac{\partial G}{\partial \tilde{\lambda}_1} \left( \frac{\partial \tilde{\lambda}_1}{\partial u} \eta + \frac{\partial \tilde{\lambda}_1}{\partial p} \eta_\theta + \frac{\partial \tilde{\lambda}_1}{\partial X} \eta_{\theta \theta} \right) \bigg/ \eta
\]

\[
\approx \frac{\partial G}{\partial \tilde{\lambda}_1} \left( \frac{\partial \tilde{\lambda}_1}{\partial u} + \frac{\partial \tilde{\lambda}_1}{\partial X} \eta_{\theta \theta} \right) \bigg/ \eta \approx \frac{\partial G}{\partial \tilde{\lambda}_1} (1 + m^2)
\]

\[
G_2 = \frac{\partial G}{\partial \tilde{\lambda}_2} \frac{\partial \tilde{\lambda}_2}{\partial u} = -\frac{\partial G}{\partial \tilde{\lambda}_2} \frac{1}{h^2}.
\]

Thus the contribution from \(\tilde{\lambda}_1\) is \(\approx 1 + m^2\), which increases with \(z\)-frequency \(m\). On the other hand, the contribution from \(\tilde{\lambda}_2\) is suppressed due to fast decay of \(\partial G / \partial \tilde{\lambda}_2\) and as we will see from the numerical result it is negligible.

**Numerical Results**

Next we discuss the dynamics of each component of \(G_i = \frac{\partial G}{\partial \tilde{\lambda}_i} \frac{d \tilde{\lambda}_i}{dr}\) for \(i = 1, 2\) and how they interact to contribute to the eigenvalue \(dG / dr\). When the curvature estimate \(\tilde{\lambda}_2 / \tilde{\lambda}_1 \to \infty\) holds we observed in previous section that \(\partial G / \partial \tilde{\lambda}_1 \to 1, \partial G / \partial \tilde{\lambda}_2 \to 0\). This curvature estimate...
indicates that the contribution from $\partial \tilde{\lambda}_1 / \partial r$ is passed to $dG/dr$ with little obstruction, however, $d\tilde{\lambda}_2 / dr$ is suppressed due to rapid decay of $\partial G / \partial \tilde{\lambda}_2$. On the other hand, the derivative of principal curvatures $\partial \tilde{\lambda}_i / dr$ depends on the frequency of initial conditions. If $r = r(\theta)$, from (2.29) we see that $\partial \tilde{\lambda}_1 / dr$ does not change with the perturbation frequency and from (2.30) we observe that $\partial \tilde{\lambda}_2 / dr$ increases with the frequency. If $r = r(z)$, from (2.33) it is found that $\partial \tilde{\lambda}_1 / dr$ increases as the frequency increases and from (2.31) we see that $\partial \tilde{\lambda}_2 / dr$ is independent of the frequency.

In order to attain large eigenvalue of the linearized HMCF, each $d \tilde{\lambda}_i / dr$ needs to contribute to $dG/dr$ by significant amount after being modulated by $\partial G / \partial \tilde{\lambda}_i$. If $r = r(\theta)$, $d \tilde{\lambda}_1 / dr$ does not get larger by increasing the frequency and its impact is not strong enough to yield large $dG/dr$ despite the fact that $\partial G / \partial \tilde{\lambda}_1$ is close to 1. For $d \tilde{\lambda}_2 / dr$ to contribute to the eigenvalue it has to overcome the strong decay of $\partial G / \partial \tilde{\lambda}_2$, thus it is necessary for the frequency of the perturbation to be high enough. Figure 2.11 shows that $dG/dr$ increases as the perturbation frequency increases and the contribution from $d \tilde{\lambda}_1 / dr$ is comparable to $d \tilde{\lambda}_2 / dr$ at low frequency but it becomes insignificant as the frequency increases. If $r = r(z)$, the contribution from $d \tilde{\lambda}_1 / dr$ is substantial due to nontrivial perturbation frequency and is passed onto $dG/dr$ with little attenuation since $\partial G / \partial \tilde{\lambda}_1$ is close to 1. On the other hand, the degeneration of $\partial G / \partial \tilde{\lambda}_2$ is so fast that the contribution from $d \tilde{\lambda}_2 / dr$ is negligible even though $d \tilde{\lambda}_2 / dr < 0$ and its magnitude grows in time (Figure 2.12).

Comparing $dG/dr$ of both types $r = r(\theta)$ and $r(z)$ that are convergent and nonconvergent, we see that $dG/dr$ of converging surfaces is larger than the nonconverging ones regardless of their initial values. For the graphs $r = r(\theta)$, only the surface with the highest perturbation frequency $n = 6$ is able to overcome the degeneration of $\partial G / \partial \tilde{\lambda}_2$ and converge, however, for the graphs $r = r(z)$, all the surfaces are able to converge thanks to the uniform estimate $\partial G / \partial \tilde{\lambda}_1 \to 1$. 
Figure 2.11: $dG/dr$ when $r = R + \epsilon \sin m\theta$
Figure 2.12: $\frac{dG}{dr}$ when $r = R + \epsilon \sin nz$
References


