

**A NEW SPLITTING METHOD FOR SCALAR
CONSERVATION LAWS WITH STIFF SOURCE TERMS**

By

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A New Splitting Method for Scaler Conservation Laws with Stiff Source Terms

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Abstract

In this paper we present a new splitting method for numerical integration of conservation laws with stiff source terms. We show that splitting could be interpreted as integrating the source terms along characteristic. We present a splitting scheme with arbitrary high order of accuracy based on piecewise linear approximation of the characteristic lines.

1 Wrong Wave Speed

In this paper we study the problems associated with numerical schemes for scaler conservation laws with source terms:

$$u_t + f(u)_x = u_t + a(u)u_x = g(u) \quad (1)$$

In particular we are interested in problems in which the forcing function $g(u)$ has a faster scale than the convection. For example consider

$$u_t + u_x = cu \quad (2)$$

Where the c is much larger than one.

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To illustrate a point let us discretize this equation in the following manner:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) + c\Delta t u_j^n$$

Where $u_j^n = u(x_j, t_n) = u(\Delta x j, \Delta t n)$.

The scheme for the convection is the standard first order upwind and the source term is represented in a natural way. But we wish to emphasize that for large values of c this is the wrong scheme. Since this is a linear equation we need to consider only the the initial condition, $u_0^0 = 1$ and $u_j^0 = 0$ for j different from zero. If we denote the operator $E u_j = u_{j-1}$, then we have an explicit formula for u_j^n .

$$u_j^n = \left(1 - \frac{\Delta t}{\Delta x} + c\Delta t + \frac{\Delta t}{\Delta x} E\right)^n u_j^0$$

Now if we have $\frac{\Delta t}{\Delta x} = 1$ then we have

$$u_j^n = (c\Delta t + E)^n u_j^0$$

It is enough to consider one iteration,

$$u_j^1 = (c\Delta t + E)u_j^0 = c\Delta t u_j^0 + u_{j-1}^0$$

$$u_0^1 = c\Delta t \quad u_1^1 = 1$$

which is the wrong solution. The problem comes from the fact that we are not integrating the source along the characteristics. Figure 1 shows the calculation done for $c = 10$ and periodic boundary conditions. The solid line is the initial condition and the broken line is the calculated solution at $t = 1$ multiplied by e^{-10} . Note how the solution lags behind.

One can remedy the above scheme by using

$$u_j^{n+1} = \left(u_j^n - \frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)\right)(1 + c\Delta t)$$

Then we get the following formula for u_j^n ,

$$u_j^n = (1 + c\Delta t)^n \left(1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x} E\right)^n u_j^0$$

For $\frac{\Delta t}{\Delta x} = 1$ we get,

$$u_j^n = (c\Delta t + 1)^n u_{j-n}^0$$

which is the right solution.

We wish to extend this trick to general cases. Let us observe that we can view the above scheme as equivalent to solving,

$$u_t + u_x = 0 \quad 0 \leq t \leq \Delta t$$

and then solving

$$u_t = cu \quad 0 \leq t \leq \Delta t$$

This is also equivalent to tracking the characteristics back and using the value of u at time $t = 0$ as the initial value for the equation $u_t = cu$.

First let us review the solution of the equation

$$u_t + a(u)u_x = u_t + f(u)_x = g(u) \quad u(0, x) = q(x) \quad (3)$$

The above is equivalent to solving [1]

$$\frac{d}{dt}x = a(u) \quad x(0) = x_0 \quad \frac{d}{dt}u = g(u) \quad u(0) = q(x_0) \quad (4)$$

Let the solution of the second equation be given as the evolution operator, $S(t)$, then

$$u(t) = S(t)q(x_0).$$

Then define

$$F(x, x_0, t) = x - x_0 - \int_0^t a(S(\tau)q(x_0))d\tau$$

Then we have

$$\frac{\partial F}{\partial x_0} = -1 - \int_0^t a'(S(\tau)q(x_0)) \frac{d}{dx_0}(S(\tau)q(x_0))d\tau$$

Now if the $\frac{d}{dx_0}(S(\tau)q(x_0))$ is bounded and t is small, then by implicit function theorem we can solve for x_0 in terms of x, t . If we let $x_0 = H(x, t)$, then we have $u(x, t) = S(t)q(x_0) = S(t)q(H(x, t))$.

2 Splitting Sources

In this section we present the general splitting scheme for equations of type,

$$u_t + a(u)u_x = u_t + f(u)_x = g(u) \quad u(0, x) = q(x), \quad 0 \leq t \leq \Delta t \quad (5)$$

In the last section we reviewed how the above equation is equivalent to the following system and showed existence of solution for small time.

$$\frac{d}{dt}x = a(u) \quad x(0) = x_0, \quad \frac{d}{dt}u = g(u) \quad u(0) = q(x_0), \quad 0 \leq t \leq \Delta t \quad (6)$$

Now let us approximate this system of ODE and then use the results for the original equation.

$$x = x_0 + \int_0^{\Delta t} a(u(t))dt \quad u(t) = S(\Delta t)h(x_0) \quad (7)$$

We solve it via

$$x = x_0 + \Delta t \sum_{n=1}^N \beta_n a(S(\gamma_n \Delta t)q(x_0))$$

when

$$\sum \beta_n = 1, \gamma_n = \sum_{j=1}^n \alpha_j$$

Now we take β_n and α_j such that the first formula is an approximation of the integral to any order. For example one can use Gaussian formulae. [2] Then this is equivalent to solving these equations in this order,

$$\begin{aligned} u_t &= g(u) \quad 0 \leq t \leq \alpha_1 \Delta t \\ u_t + a(u)u_x &= 0 \quad 0 \leq t \leq \beta_1 \Delta t \\ u_t &= g(u) \quad 0 \leq t \leq \alpha_2 \Delta t \\ u_t + a(u)u_x &= 0 \quad 0 \leq t \leq \beta_2 \Delta t \\ &\vdots \\ u_t &= g(u) \quad 0 \leq t \leq \alpha_j \Delta t \\ u_t + a(u)u_x &= 0 \quad 0 \leq t \leq \beta_n \Delta t \end{aligned}$$

Using the solution of each stage as the initial data for the next stage. Next we prove that in this case the error is of order Δt^{2N} .

Theorem 1 *Let $u(x, t)$ be the exact solution and $\bar{u}(x, t)$ the approximate solution obtained by splitting, then we have,*

$$|u(x, t) - \bar{u}(x, t)| \leq Ct^{2N}$$

Let us define,

$$x = H(x_0, t) = x_0 + \int_0^t a(u(\tau))d\tau$$

$$\bar{x} = h(x_0, t) = x_0 + t \sum_{n=1}^N \beta_n a(S(\gamma_n t)q(x_0))$$

Then we have

$$x_0 = H^{-1}(x, t), \bar{x}_0 = h^{-1}(x, t)$$

By using a quadrature rule we can have,

$$|H(x_0, t) - h(x_0, t)| \leq C_a t^{2N}$$

Where C_a is a constant depending on $a(S(t)h(x_0))$, then we have

$$u(x, t) = S(t)q(x_0) = S(t)q(H^{-1}(x, t))$$

$$\bar{u}(x, t) = S(t)q(\bar{x}_0) = S(t)q(h^{-1}(x, t))$$

Assume that we have constants, $|S|$ and $|q|$, such that

$$|S(t)x - S(t)y| \leq |S||x - y|$$

$$|q(x) - q(y)| \leq |q||x - y|$$

Then we have,

$$|u(x, t) - \bar{u}(x, t)| = |S(t)q(x_0) - S(t)q(\bar{x}_0)| \leq |S||q||x_0 - \bar{x}_0|$$

Now we have,

$$|x_0 - \bar{x}_0| = |h^{-1}(x) - H^{-1}(x)| = |h^{-1}(x) - h^{-1}(h(H^{-1}(x)))| \leq$$

$$C_{h^{-1}}|H(H^{-1}(x)) - h(H^{-1}(x))| \leq C_{h^{-1}}C_a t^{2N}$$

$$|u(x, t) - \bar{u}(x, t)| \leq |S||q|C_{h^{-1}}C_a t^{2N}$$

which proves the theorem. We wish to emphasize that this proof is true in the region that the solution is smooth. When shocks develop everything breaks down. In particular H^{-1} fails to be defined. Also if we have constant coefficient linear equation, then $a(u)$ is a constant and C_a is zero which proves the following corollary.

Corollary 1 *For linear equations with constant coefficient, splitting is exact.*

Let us look at two examples, first consider one point Gaussian. We have $\beta_1 = 1$ and $\gamma_1 = .5$. Then the scheme is,

$$u_t = g(u) \quad 0 \leq t \leq .5\Delta t$$

$$u_t + a(u)u_x = 0 \quad 0 \leq t \leq \Delta t$$

$$u_t = g(u) \quad 0 \leq t \leq .5\Delta t$$

For two point Gaussian we have, $\beta_1 = .5$ and $\beta_2 = .5$ and $\gamma_1 = .21132487$ and $\gamma_2 = .78867513$ Which translates into the following scheme:

$$u_t = g(u) \quad 0 \leq t \leq \gamma_1\Delta t$$

$$u_t + a(u)u_x = 0 \quad 0 \leq t \leq .5\Delta t$$

$$u_t = g(u) \quad \gamma_1\Delta t \leq t \leq \gamma_2\Delta t$$

$$u_t + a(u)u_x = 0 \quad .5\Delta t \leq t \leq \Delta t$$

$$u_t = g(u) \quad \gamma_2\Delta t \leq t \leq \Delta t$$

References

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- [2] Conte, Samuel ,de Boor, Carl, Elementary Numerical Analysis, 1980, pp. 311-319

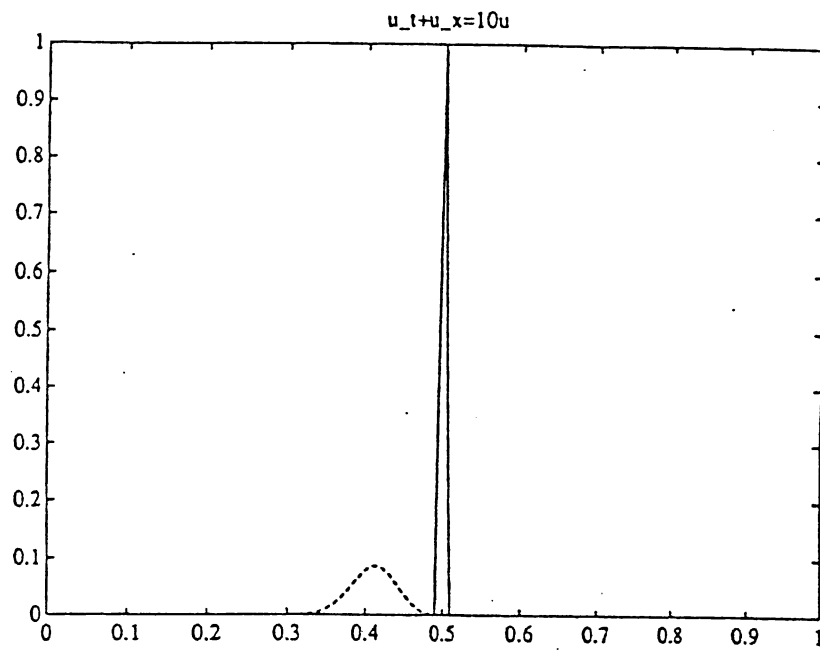


Figure 1: Stiff Source

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