Duality and gauge theories at finite density

Michael Ogilvie
Washington University, St. Louis

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The sign problem

- The phase diagram of QCD in the T-\(\mu\) plane is a key goal in the study of QCD.

- The partition function for gauge theories at finite density naturally appears as a sum over complex weights. This kind of problem occurs in many areas of physics and is known as the sign problem.

- Several lattice approaches (reweighting, imaginary chemical potential, complex Langevin) have been studied extensively, but are not fully satisfactory (de Forcrand, 2010; Gupta, 2011; Aarts, 2012)
The sign problem: a simple example

QCD with heavy quarks at finite density

Quarks are favored over antiquarks in the path integral

\[
S_{\text{eff}} = \int d^{d+1}x \left[ \frac{1}{4g^2} (F_{\mu\nu}^a)^2 - h_F \left( e^{\beta \mu} \text{Tr}_F(P) + e^{-\beta \mu} \text{Tr}_F(P^+) \right) \right]
\]

The Polyakov loop operator represents the insertion of a static quark

\[
P(\vec{x}) = \mathcal{P} \exp \left[ i \int_0^\beta dt A_4 (\vec{x}, t) \right]
\]

\[
h_F \sim \exp (-\beta M_q)
\]
Finite density and CT symmetry

A key feature of finite density is CT symmetry

\[
S_{\text{eff}} = \int d^{d+1}x \left[ \frac{1}{4g^2} (F_{\mu\nu}^a)^2 - h_F \left( e^{\beta\mu} Tr_F(P) + e^{-\beta\mu} Tr_F(P^+) \right) \right]
\]

\( C : A_\mu \rightarrow -A_\mu \)

\( \mathcal{T} : i \rightarrow -i \)

\( \text{CT} : P \rightarrow P \quad P(\vec{x}) = \mathcal{P} \exp \left[ i \int_0^\beta dt A_4(\vec{x}, t) \right] \)

Whenever a model has an antilinear symmetry like CT, we can say it has generalized PT symmetry.
Finite density and CT symmetry 2

$$Z = Tr \left[ \exp^{-\beta H_L} \right]$$

$$H_L = H - \mu \int d^d x \, j^0$$

$H_L$ is Hermitian

$$Z = Tr \left[ \exp^{-L H_\beta} \right]$$

$$H_\beta = H - i \mu \int d^d x \, j^d$$

$H_\beta$ has generalized PT symmetry!

$C : j^d \rightarrow -j^d$

$T : i \rightarrow -i$
PT symmetry and non-Hermitian models

- Origin: $ix^3$ potential has only real energy eigenvalues (Bender and Boettcher, 1998). Model itself derived from Lee-Yang theory.

- Suppose an antilinear operator $PT$ commutes with another operator $H$: then the eigenvalues of $H$ are either real or half of a complex-conjugate pair.

- If $H$ has such a symmetry, then $Z$ is always real, but not necessarily positive.

$$H = p^2 + ix^3$$

$$H \left| \psi \right\rangle = E \left| \psi \right\rangle$$

$$HPT \left| \psi \right\rangle = PT H \left| \psi \right\rangle = PT E \left| \psi \right\rangle = E^* PT \left| \psi \right\rangle$$

$$Z = \sum_r e^{-\beta E_r} + \sum_p \left( e^{-\beta E_p} + e^{-\beta E_p^*} \right)$$
Classification of Phases via $\mathcal{PT}$ Symmetry

Example: $d=1 \ Z(3)$ spin chain with complex action (Meisinger, mco & Wiser, 2010)

- Region I: PT symmetry is unbroken, and all eigenvalues are real. Behavior of correlation functions similar to a Hermitian system.

- Region II: PT symmetry is broken by a one or more pairs of excited states becoming complex. Thermodynamic properties are unaffected, but oscillatory behaviors appears in correlation functions.

- Region III: PT symmetry is broken by the ground state becoming complex. The system is in a spatially modulated phase ("crystal").

\[ E_j \in \mathbb{R} \ \forall j \]

\[ E_j \neq E_j^* \ j > 0 \]

\[ E_0 \neq E_0^* \]
Example: QCD$_2$ with heavy quarks

In 1+1 dimensions, problems reduces to PT-symmetric quantum mechanics on the gauge group.

Meisinger and mco, 2009
Abelian lattice models with CT symmetry

- Lattice duality maps CT-symmetric models with complex actions into dual models with real actions.

- Explicit duality relations are given for models for spin and gauge models based on $\mathbb{Z}(N)$ and $U(1)$ symmetry groups.

- The dual forms are generalizations of the $\mathbb{Z}(N)$ chiral model and the lattice Frenkel-Kontorova model, respectively.

- For extended regions of parameter space, calculable for each model, duality resolves the sign problem for both analytic methods and computer simulations.

- From this equivalence, a rich set of spatially-modulated phases is found in the strong-coupling region of the original models.
d=2 U(1): derivation

Villain form of XY model with a chemical potential

Heat kernel action
\[
Z = \int_{S^1} [d\theta] \sum_{n_{\nu}} \exp \left[ -\frac{K}{2} \sum_{x,\nu} \left( \partial_{\nu}\theta(x) - i\mu\delta_{\nu2} - 2\pi n_{\nu}(x) \right)^2 \right]
\]

Duality transform of action
\[
Z = \int_{S^1} [d\theta] \prod_{x,\nu} \sum_{p_{\nu}(x) \in \mathbb{Z}} \frac{1}{\sqrt{2\pi K}} e^{-p_{\nu}^2(x)/2K} e^{ip_{\nu}(x)(\nabla_\nu \theta(x) - i\delta_{\nu2}\mu)}
\]

Introduction of dual variables
\[
Z = \sum_{\{m(X)\} \in \mathbb{Z}} \frac{1}{\sqrt{2\pi K}} e^{K - \sum_{x,\nu} \left[ (\nabla_{\nu} m(X))^2/2K + \mu\delta_{\nu2}\epsilon_{\nu\rho} \nabla_{\rho} m(X) \right]}
\]
\[
p_\rho(x) = \epsilon_{\rho\nu} \nabla_{\nu} m(X)
\]

Introduction of dual scalar field
\[
Z = \int_{\mathbb{R}} [d\phi(X)] e^{-\sum_{x,\nu} \left[ (\nabla_{\nu}\phi(X))^2/2K - \mu\nabla_{1}\phi(X) \right]} \sum_{\{m(X)\} \in \mathbb{Z}} e^{2\pi i m(X)\phi(X)}
\]
d=2 U(1): interpretation

\[ Z = \int_R [d\phi(X)] \exp \left[ -\sum_{X,\mu} \frac{1}{2K} (\nabla_\mu \phi(X))^2 - \sum_X \mu \nabla_1 \phi(X) + \sum_X 2y \cos(2\pi \phi(X)) \right] \]

- \( m=1 \) contributions only gives a lattice sine-Gordon model with an extra term: lattice form of Frenkel-Kontorova model.

- For fixed \( X_2 \), derivative term counts the number of kinks on that slice.

- Continuum form equivalent to a massive Thirring model with a chemical potential.

- Frenkel-Kontorova model has rich modulated phase structure.

Chou and Griffiths, 1986
Duality: why it works

• Lattice duality for Abelian systems uses the Fourier transform on the group

• PT symmetry is analogous to reality of the Fourier transform

Ising model:

\[ e^{J\sigma\sigma'} = \cosh(J) [1 + \tanh(J)\sigma\sigma'] \]

\[ H(p, x) = p^2 + ix^3 \]

\[ H^*(p, x) = H(p, -x) \]
\( d=2 \) \( Z(N) \)

- Villain form again using methods of Elitzur et al. (1979)

\[
Z = \sum \sum \exp \left[ -\frac{J}{2} \sum \left( \frac{2\pi}{N} \partial_{\nu} m(x) - i\mu \delta_{\nu,2} - 2\pi n_{\nu}(x) \right)^2 \right]
\]

- Exact duality statements

- Incommensurate phase (IC) for \( J \) small (Ostlund, 1981) that extends the Coulomb phase for \( N>4 \). Li is the ordered phase. \( C_0 \) is the dual ordered phase. \( C_1 \) is a high-density phase.

Ostlund, 1981
d=3 $Z(N)$: duality

Heat kernel action for gauge and spins.

\[
Z = \sum_{m,n,\nu,p, \rho} \exp \left[ -\frac{J}{2} \sum_{x,\nu} \left( \frac{2\pi}{N} \partial_\nu m(x) - \frac{2\pi}{N} p_\nu - i\mu_\nu - 2\pi n_\nu(x) \right)^2 \right] 
\times \exp \left[ -\frac{K}{2} \sum_{x,\nu,\rho} \left( \frac{2\pi}{N} (\partial_\nu p_\rho - \partial_\rho p_\nu) - iG_{\nu\rho} - 2\pi q_{\nu\rho} \right)^2 \right]
\]

$G$ is a background field, but corresponds to an electric field in Minkowski space. This is again a sign problem (Shintani et al. 2006; Alexandru 2008).

Duality swaps between gauge and spin degrees of freedom.

\[
J \to \tilde{J} = \frac{N^2}{4\pi^2 K}
\]

\[
K \to \tilde{K} = \frac{N^2}{4\pi^2 \tilde{J}}
\]

\[
\mu_\nu \to \tilde{\mu}_\nu = -i \frac{2\pi K}{N} \epsilon_{\nu\rho\sigma} G_{\rho\sigma}
\]

\[
G_{\nu\rho} \to \tilde{G}_{\nu\rho} = -i \frac{2\pi J}{N} \epsilon_{\nu\rho\sigma} \mu_\sigma
\]
Duality in $d=3$ maps $\mathbb{Z}(N)$ gauge theory to the chiral $\mathbb{Z}(N)$ spin model

The dual model has a rich set of commensurate modulated phases.

\[ \Delta = KG_{12} \]

**Figure 1.** Schematic representation (not to scale) of the $(T, \Delta)$ phase diagram of the chiral Potts or asymmetric clock model illustrating the unbounded sequences of commensurate phases of character $(3^2)^k$ and $(12^k)$ for $k = 0, 1, 2, \ldots$ Fisher and Yeomans, 1981
General result for \( Z(N) \)

Arbitrary PT-symmetric \( Z(N) \) potential has \( N \) real parameters

\[
V(z) = \sum_{j=0}^{N-1} v_j z^j \quad \quad \quad V_j = V(\omega^j) \quad \quad \quad \omega = \exp(2\pi i/N)
\]

CT invariance \( V_{N-j} = V_j^* \)

Duality is a \( Z(N) \) Fourier transform:

\[
\exp(-\tilde{V}_j) = \sum_{k=0}^{N-1} \omega^{jk} \exp(-V_k)
\]

Dual positive weight region for \( Z(3) \)

\[
v_r = v_1 + v_2 \quad \quad \quad v_i = v_1 - v_2
\]

In the dual positive weight region

\[
\tilde{V}(w) = \sum_{j=0}^{N-1} \tilde{v}_j w^j \quad \quad \quad \text{dual potential is real with chiral phases; N real parameters}
\]

\[
\tilde{v}_j = \tilde{v}_{N-j}^*
\]
Conclusion and Prospects

- Solution of the sign problem: both analytical and simulation methods can be applied to a large class of Abelian models.

- SU(N) deformed to U(1)\(^{N-1}\) can be treated.

- d=4 (Gattringer et al.)

- finite temperature

- Connection to condensed matter physics, e.g., bosonic quantum Hall effect
Solution of the \( \mathcal{PT} \)-symmetric \( Z(3) \) model in \( d=1 \)

Meisinger, Ogilvie and Wiser, 2010