THE CONVERGENCE OF SEMIDISCRETE METHODS FOR A SYSTEM OF REACTION-DIFFUSION EQUATIONS

By

Ling Ma

IMA Preprint Series # 723

November 1990

THE CONVERGENCE OF SEMIDISCRETE METHODS FOR A SYSTEM OF REACTION-DIFFUSION EQUATIONS

By

Ling Ma

IMA Preprint Series # 723

November 1990

THE CONVERGENCE OF SEMIDISCRETE METHODS FOR A SYSTEM OF REACTION-DIFFUSION EQUATIONS*

LING MA†

Abstract. This paper proves the convergence of a semidiscrete scheme for a system of reaction diffusion equations. The global existence and dissipativity results of the system are also obtained.

Key words. maximum principles, semigroups, reaction diffusion equations.

AMS(MOS) subject classifications. 35K55, 65N15

1. Introduction. We consider a system of reaction diffusion equations:

$$\frac{\partial \alpha}{\partial t} = d\Delta\alpha - (f(\beta) + \kappa_1)\alpha \tag{1.1}$$

$$\frac{\partial \beta}{\partial t} = d\Delta\beta + (f(\beta) + \kappa_1)\alpha - \kappa_2\beta \tag{1.2}$$

in the minimally smooth bounded spatial domain $\Omega \subset R^n$ with the boundary conditions:

$$\alpha|_{\partial\Omega} = \zeta, \qquad \beta|_{\partial\Omega} = \eta \tag{1.3}$$

where κ_1, κ_2, d are positive constants and ζ, η satisfy

$$\Delta\zeta = \Delta\eta = 0 \quad in \quad \Omega, \quad \zeta, \eta \in \mathcal{C}^2(\bar{\Omega}), \quad nonnegative \quad on \quad \partial\Omega. \tag{1.4}$$

Furthermore f satisfies:

$$f \in \mathcal{C}^1(R, R) \quad and \quad f(x) \geq 0, \forall x \in R. \tag{1.5}$$

In the case of $\Omega = (-1, 1)$, $f(\beta) = \beta^2$ this system describes autocatalysis of of chemical species A, B in isothermal chemical reactions, where α and β are their nondimensionalized concentrations of A and B respectively. d is the nondimensionalized diffusion coefficient. A detailed background is fully discussed in Scott[7].

Using numerical integration methods such as the orthogonal collocation technique Brindley, Kaas-Peterson, Merkin and Scott [2] obtained some results concerning the long time behaviour of the system in the above-mentioned case. In this note we shall prove the convergence of a semidiscrete scheme for the case $n = 1$. As a preliminary result we shall prove the global existence and dissipativity of the system in more spatial dimensions. This is done in section 2. In section 3 we propose our scheme and convergence is proved in section 4.

---

*This research was supported in part by a grant from the Applied Mathematics and Computational Mathematics Program/DARPA.

†School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455.

Acknowledgement. The author gratefully thanks Professor Mitchell Luskin and Professor George Sell for their help and advice and Professor Bernardo Cockburn for instructive conversations.

2. Global Existence and Dissipativity. In this section we consider the following system of reaciton diffusion equations which is equivalent to (1.1)-(1.3):

$$\frac{\partial u}{\partial t} = d\Delta u - (f(\beta)+\kappa_1)\alpha \tag{2.1}$$
$$\frac{\partial v}{\partial t} = d\Delta v + (f(\beta)+\kappa_1)\alpha - \kappa_2\beta \; in \; \Omega$$

where

$$\alpha = u + \zeta \qquad \beta = v + \eta \tag{2.3}$$

with ζ and η satisfy (1.4), with boundary conditions:

$$u|_{\partial\Omega} = 0, \qquad v|_{\partial\Omega} = 0 \tag{2.4}$$

and initial conditions:

$$u(x,0) = \varphi(x), \qquad v(x,0) = \psi(x). \tag{2.5}$$

Our theorem concerning the global existence is as follows:

THEOREM 1. *Assume that $\varphi, \psi \in W^{2,p} \cap W_0^{1,p}$ for some $p > n$ and $\varphi + \zeta \geq 0, \psi + \eta \geq 0$ in Ω. Then the initial-boundary value problem (2.1)-(2.5) has a pair of unique classical solution: $u(x,t), v(x,t)$ for t >0 and $(u+\zeta)(x,t) \geq 0$, and $(v+\eta)(x,t) \geq 0$ for all $t > 0$.*

Proof of Theorem 1. We adopt the approach of semigroups of linear operators. Since the method is standard we will only sketch the proof. Details can be constructed following Henry[4] or Pazy[5].

We start by considering the closed linear operator $A = -d\Delta$ with the domain $D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ in the pivoting space $L^p(\Omega)$. It is well known that $-A$ generates an analytic semigroup on L^p denoted by e^{-At} and that the fractional powers of A are well defined. We denote the domain of A^α by X^α for $\alpha > 0$. Endowed with the norm $\|A^\alpha u\|_{L^p}$ X^α is a Banach space.

It is known (see [4], pp.39-40)that for $\alpha \in (\frac{1}{2}, 1)$

$$X^\alpha \hookrightarrow W^{1,p}(\Omega) \hookrightarrow C^\mu(\bar{\Omega}) \quad for \quad \mu = 1 - n/p \tag{2.6}$$

Fix $\alpha : \alpha \in (\frac{1}{2}, 1)$ then $\varphi, \psi \in X^1 \hookrightarrow X^\alpha$.

For $u, v \in X^\alpha$ define

$$F_1(u,v)(x) = -\,(f(v+\eta)+\kappa_1)(u+\zeta)$$
$$F_2(u,v)(x) = (f(v+\eta)+\kappa_1)(u+\zeta)-\kappa_2(v+\eta)$$

Then F_1, F_2 are locally Hölder continuous on $X^\alpha \times X^\alpha$ thanks to (2.6) and (1.5). The local solution (u,v) to (2.1)-(2.5) are constructed through the mild solution argument. First we show that

$$u(t) = e^{-At}\varphi + \int_0^t e^{-A(t-s)} F_1(u(s), v(s))\, ds \tag{2.7}$$

$$v(t) = e^{-At}\psi + \int_0^t e^{-A(t-s)} F_2(u(s), v(s))\, ds \tag{2.8}$$

has a unique pair of solution $(u,v) \in C([0,T] : X^\alpha \times X^\alpha)$ for some $T > 0$. Since $\varphi, \psi \in D(A)$ It can be shown that (see [4], pp.71)

$$u, v \in C^\theta([0,T]; D(A^\alpha)) \quad for \quad some \quad \theta > 0$$

Then (u,v) is the unique pair of solution for the ordinary differential equation in $L^P(\Omega)$ (see [4] pp.53):

$$\frac{du}{dt} = -Au + F_1(u,v)$$
$$\frac{dv}{dt} = -Av + F_2(u,v)$$

with the initial conditions: $u(0) = \varphi, v(0) = \psi$. Next we show that (see [4], pp.71) for any $\gamma \in (0,1)$, $t \to \frac{du}{dt}(t) \in X^\gamma$ is locally Hölder continuous in (0,T].

It follows that $Au = \frac{du}{dt} - F_1(u,v)$ is Hölder continuous in Ω and therefore $u(x,t)$ similarly $v(x,t)$ are classical solutions to (2.1)-(2.5) by Schauder regularity theory.

To obtain global existence in time, since F_1, F_2 maps bounded sets of $X^\alpha \times X^\alpha$ into bounded sets in L^p, it suffices to prove certain a priori estimates which constitutes the following lemma:

LEMMA 1. *Assume the conditions of theorem 1 are satisfied. Then there exists a constant c which depends only on intial and boundary data such that:*

$$\|u\|_{L^p} < c, \quad \|Au\|_{L^p} < c$$
$$\|v\|_{L^p} < c, \quad \|Av\|_{L^p} < c \quad \forall t \geq 0$$

In this note c will alway denote an constant depends only on the initial and boundary data.

Proof of Lemma 1. $\alpha = u + \zeta$, $\beta = v + \eta$ satisfy (1.1)-(1.3) with the initial values:

$$\alpha(x, 0) = \varphi + \zeta, \qquad \beta(x, 0) = \psi + \eta.$$

The nonnegativity of the function f enables us to apply maximum principles (See [3]) to (1.1), we have then α can not attain its negative minimum nor positive maximum in Ω therefore $\alpha \geq 0$ in Ω. It follows from (1.2) that

$$\frac{\partial \beta}{\partial t} \geq d\Delta\beta - \kappa_2 \beta.$$

The maximum principle again gives us that $\beta \geq 0$. Now add (1.2) to (1.1) we have then

$$\frac{\partial(\alpha + \beta)}{\partial t} = d\Delta(\alpha + \beta) - \kappa_2\beta \leq d\Delta(\alpha + \beta).$$

We have then $\alpha + \beta$ is nonnegative and bounded above by its initial and boundary data. It follows that $\alpha(., t), \beta(., t)$ therefore $u(., t), v(., t)$ are bounded in $L^\infty(\Omega)$ for all $t > 0$. It follows then that $\|u\|_{L^p}, \|v\|_{L^p}$ are bounded for all time $t \geq 0$.

Now by virtue of (2.9) we have then

$$\|F_i(u(t), v(t)) - F_i(u(s), v(s))\|_{L^p(\Omega)} \leq c|t - s|^\theta \quad i = 1, 2$$

for $t, s \in (0, T)$ where c is a constant depending only the inital and boundary data.

Since

$$\begin{aligned} Au(t) = &Ae^{-At}\phi + (1 - e^{-At})F_1(u(t), v(t)) \\ &+ \int_0^t Ae^{-A(t-s)} F_1(u(s), v(s)) - F_1(u(t), v(t))\, ds, \end{aligned}$$

recall that $\|e^{-At}\|_{op} \leq Me^{-\mu t}$, and $\|Ae^{-At}\|_{op} \leq Mt^{-1}e^{-\mu t}$ for some $M > 0$ we have therefore

$$\begin{aligned} \|Au(t)\|_{L^p(\Omega)} \leq &Me^{-\mu t}\|A\phi\|_{L^p(\Omega)} + (1 + e^{-\mu t})M \\ &+ M \int_0^t (t - s)^{-1} e^{-\mu(t-s)}(t - s)^\theta\, ds \\ \leq &\; c. \end{aligned}$$

Similarly $\|Av(t)\| \leq c$. The proof of Lemma 1 therefore Theorem 1 is complete. $\square$

Our next lemma will be used in section 4.

LEMMA 2. *Assume that in addition to the hypothesis of the Theorem 1 that $\Delta\varphi, \Delta\psi \in W^{2,p} \cap W_0^{1,p}$. Then $\|u(t)\|_{W^{4,p}}, \|v(t)\|_{W^{4,p}}$ remain bounded for all $t > 0$.*

Proof. As in Lemma 1, we have $\|Au\|_{L^p}$, $\|Av\|_{L^p}$ are bounded for all time t, the same argument will then gives us the result observing that

$$\mathcal{D}(A^2) \hookrightarrow W^{4,p}. \quad \square$$

The rest of this section will be devoted to the study of the long time behavior of (1.1)-(1.4). First we have the following dissipativity result:

THEOREM 2. *Assume that the conditions of the theorem 1 are satisfied. Then:*

$$\limsup_{t\to\infty} \|\alpha(.,t)\|_{L^\infty(\Omega)} \leq M$$
$$\limsup_{t\to\infty} \|\beta(.,t)\|_{L^\infty(\Omega)} \leq M$$

where M is a positive constant independent of the initial data φ, ψ if $\varphi + \zeta \geq 0, \psi + \eta \geq 0$ in Ω.

Proof. Consider γ such that:

$$\frac{\partial\gamma}{\partial t} = d\Delta\gamma + \kappa_2\beta \tag{2.10}$$

with boundary conditions: $\gamma|_{\partial\Omega} = M_0 - (\zeta + \eta)$ where M_0 is a positive constant large enough such that $M_0 - (\zeta + \eta) \geq 0$ and initial condition $\gamma(x, 0) = \gamma_0$ where γ_0 is smooth and pointwise nonnegative on Ω.

We have from maximum principle that $\gamma \geq 0$ on Ω.

Take the sum of (2.6) with (1.1), (1.2) we have that

$$\frac{\partial(\alpha + \beta + \gamma)}{\partial t} = d\Delta(\alpha + \beta + \gamma),$$
$$(\alpha + \beta + \gamma)|_{\partial\Omega} = M_0.$$

It follows from the semigroup theory that

$$\|(\alpha + \beta + \gamma)(., t) - M_0\|_{L^\infty} \longrightarrow 0.$$

The result therefore follows. $\square$

REMARK. *In the case of homogeneous boundary conditions (Dirichlet or Neumann type) we have*

$$\lim_{t\to\infty} \|\alpha(.,t)\|_{L^2(\Omega)} = 0, \quad \lim_{t\to\infty} \|\beta(.,t)\|_{L^2(\Omega)} = 0.$$

Proof. Take $L^2(\Omega)$ inner product of (1.1) with α we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\alpha\|^2_{L^2(\Omega)} + (\nabla\alpha, \nabla\alpha) &= (-(f(\beta)+\kappa_1)\alpha, \alpha) \\ &\le -\kappa_1\|\alpha\|^2_{L^2(\Omega)} \end{aligned}$$

Therefore

$$\lim_{t\to\infty} \|\alpha\|^2_{L^2(\Omega)} = 0.$$

Add (1.1) and (1.2) we have

$$\frac{\partial(\alpha+\beta)}{\partial t} = d\Delta(\alpha+\beta) - \kappa_2\beta = d\Delta(\alpha+\beta) - \kappa_2(\alpha+\beta) + \kappa_2\alpha$$

Since $-A_0 = d\Delta - \kappa_2$ generates an analytic semigroup on $L^2(\Omega)$ therfore

$$\beta(.,t) = e^{-A_0 t}(\alpha+\beta)(.,0) + \int_0^t e^{-A_0(t-\tau)}\kappa_2\alpha(\tau)\,d\tau - \alpha(.,t).$$

Since $\|e^{-A_0 t}\| \le e^{-\lambda t}$ for some $\lambda > 0$ the result for $\beta(.,t)$ therefore follows. □

3. Approximation Problem. We consider an approximation problem for (1.1)-(1.4) in the spetial case of one spatial dimension and $\Omega = (-1, 1)$. We consider the boundary conditions as in [2]:

(3.1) $$\alpha = \alpha_l \quad at \quad x = -1, \qquad \alpha = \alpha_r \quad at \quad x = 1;$$

(3.2) $$\beta = \beta_l \quad at \quad x = -1, \qquad \beta = \beta_r \quad at \quad x = 1.$$

The initial conditions are:

(3.3) $$\alpha(x,0) = \alpha_0(x), \quad \beta(x,0) = \beta_0(x)$$

where α_0, β_0 satisfy the boundary conditions (3.1)-(3.2). First let us observe that we applied maximum principles to get global existence in time for (1.1)-(1.4). We would like to have a simidiscrete approximation problem to (1.1)-(1.2), (3.1)-(3.2) and compatible initial conditions where the discrete version of the maximum principle applies. This motivates our choice of the following discrete problem.

We discretize I=[-1,1] into (N+1) equilenghth intervals with interior grid points $x_\nu = -1 + \nu h, \nu = 1, 2, ..., N$ where $h = \frac{2}{(N+1)}$. The Laplacian operator is discretized by the 3-point approximation. Define

$$t_u(x,h,t) = \frac{1}{h^2}(u(x-h,t) - 2u(x,t) + u(x+h,t)) - \Delta u(x,t)$$

for sufficiently smooth u.

Suppose that $u(,.t) \in H^4(I)$ a Sobolev imbedding theorem then guarantees that $\frac{\partial^3 u}{\partial x^3}$ is Hölder continous with exponent $\frac{1}{2}$. One has then by mean value theorem that:

$$\|t_u(x,h,t)\|_{L^\infty(I)} \leq ch^{\frac{3}{2}} \tag{3.4}$$

where c depends only on $\|u\|_{H^4}$. If $\frac{\partial^3 u}{\partial x^3}$ is Lipschitz then we would have instead of (3.4)

$$\|t_u(x,h,t)\|_{L^\infty(I)} \leq ch.$$

We may therefore have a semidiscretized problem which consists of $2N$ ordinary differential equations:

$$\frac{d\overrightarrow{\alpha}}{dt} = -\frac{d}{h^2}(A\overrightarrow{\alpha} - c_b^\alpha) - (D_\beta + \kappa_1)\overrightarrow{\alpha} \tag{3.5}$$

$$\frac{d\overrightarrow{\beta}}{dt} = -\frac{d}{h^2}(A\overrightarrow{\beta} - c_b^\beta) + (D_\beta + \kappa_1)\overrightarrow{\alpha} - \kappa_2\overrightarrow{\beta} \tag{3.6}$$

with the initial conditions:

$$\overrightarrow{\alpha}(0) = (\alpha_0(x_1), \alpha_0(x_2), \ldots, \alpha_0(x_N))^T \tag{3.7}$$

$$\overrightarrow{\beta}(0) = (\beta_0(x_1), \beta_0(x_2), \ldots, \beta_0(x_N))^T \tag{3.8}$$

where $D_\beta = diag(f(\beta_1), f(\beta_2), \ldots, f(\beta_N))$ and

$$\overrightarrow{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)^T$$
$$\overrightarrow{\beta} = (\beta_1, \beta_2, \ldots, \beta_N)^T.$$

A is a matrix of order $N \times N$:

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}.$$

and

$$c_b^\alpha = (\alpha_l, 0, \dots, 0, \alpha_r)^T,$$
$$c_b^\beta = (\beta_l, 0, \dots, 0, \beta_r)^T.$$

It is well known that A is a positive definite matrix with eigenvalues:

$$\lambda_k = 2 - 2\cos\left(\frac{k\pi}{N+1}\right), k = 1, 2, \dots, N.$$

4. Convergence Results. In R^N define two discrete inner products and associated norms by

$$(\vec{u}, \vec{v})_{0,h} = h\sum_{\nu=1}^{N} u_\nu v_\nu, \|\vec{u}\|_{0,h}^2 = (\vec{u}, \vec{u})_{0,h}$$
$$(\vec{u}, \vec{v})_{1,h} = \frac{1}{h}(A^{\frac{1}{2}}\vec{u}, A^{\frac{1}{2}}\vec{v})_{0,h}, \|\vec{u}\|_{1,h}^2 = (\vec{u}, \vec{u})_{1,h}$$

For $\vec{u} = (u_1, u_2, \dots, u_N)$, $\vec{v} = (v_1, v_2, \dots, v_N)$ and $h = \dfrac{2}{N+1}$.

We also define

$$\|\vec{u}\|_{\infty,h} = \max_{1\le\nu\le N} |u_\nu|, \quad \|\vec{u}\|_0 = \sqrt{\sum_{\nu=1}^{N} u_\nu^2}.$$

We are now ready to state our main theorem:

THEOREM 3. *Assume that α_0, β_0 satisfy the boundary conditions (3.1)-(3.2) and*

$$\alpha_0, \beta_0 \in H^2; \Delta\alpha_0, \Delta\beta_0 \in H^2 \cap H_0^1; \alpha_0 \ge 0, \beta_0 \ge 0 \quad on \quad I.$$

Let $\alpha(x,t)$, $\beta(x,t)$ be the classical solutions to (1.1)-(1.2), (3.1)-(3.3) and

$$\vec{\alpha}_s(t) = (\alpha(x_1,t), \alpha(x_2,t), \dots, \alpha(x_N,t))^T, \vec{\beta}_s(t) = (\beta(x_1,t), \beta(x_2,t), \dots, \beta(x_N,t))^T,$$

Then:

1) For any integer $N > 0$ (3.5)-(3.8) have a unique pairs of solutions $\vec{\alpha}(t), \vec{\beta}(t)$, defined for all $t > 0$, and

$$\|\vec{\alpha}\|_{\infty,h}, \|\vec{\beta}\|_{\infty,h} \le \max(\alpha_l + \beta_l, \alpha_r + \beta_r, \|\vec{\alpha}_0 + \vec{\beta}_0\|_{\infty,h}); \tag{4.1}$$

2) For any $T > 0$, there exists a constants C which depents on T and the initial and boundary data such that:

$$\sup_{0\le t\le T} \|\overrightarrow{\alpha}(t) - \overrightarrow{\alpha}_s(t)\|_{0,h} \le Ch^{\frac{3}{2}}, \tag{4.2}$$

$$\sup_{0\le t\le T} \|\overrightarrow{\alpha}(t) - \overrightarrow{\alpha}_s(t)\|_{1,h} \le Ch^{\frac{1}{2}}. \tag{4.3}$$

Furthermore we have:

$$\sup_{0\le t\le T} \max_{1\le \nu\le N} |\alpha_\nu(t) - \alpha(x_\nu, t)| \le Ch^{\frac{1}{2}}. \tag{4.4}$$

Similary results hold for $(\overrightarrow{\beta}(t) - \overrightarrow{\beta}_s(t))$.

REMARK. *The order of convergence in (4.2)-(4.4) can be replaced by h^2, h, h respectively if the solution has more regularity, e.g., if $\dfrac{\partial^3 u}{\partial x^3}$ is Lipschitz continuous.*

Proof. 1) The global existence and estimates (4.1) of (3.5)-(3.8) follows from the discretized maximun principle. The proof is analogous to the continuous case.

2) Observe that $\overrightarrow{\alpha}_s$, $\overrightarrow{\beta}_s$ satisfy:

$$\frac{d\overrightarrow{\alpha}_s}{dt} = -\frac{d}{h^2}(A\overrightarrow{\alpha}_s - c_b^\alpha) - (D_{\beta_s} + \kappa_1)\overrightarrow{\alpha}_s + G^\alpha \tag{4.5}$$

$$\frac{d\overrightarrow{\beta}_s}{dt} = -\frac{d}{h^2}(A\overrightarrow{\beta}_s - c_b^\beta) + (D_{\beta_s} + \kappa_1)\overrightarrow{\alpha}_s - \kappa_2\beta + G^\beta \tag{4.6}$$

and

$$\overrightarrow{\alpha}_s(0) = \overrightarrow{\alpha}(0), \quad \overrightarrow{\beta}_s(0) = \overrightarrow{\beta}(0)$$

where $D_{\beta_s} = diag(f(\beta(x_1)), f(\beta(x_2)), \ldots, f(\beta(x_N)))$ and

$$G^\alpha = (t_\alpha(x_1, h, t), t_\alpha(x_2, h, t), \ldots, t_\alpha(x_N, h, t)), \tag{4.7}$$

$$G^\beta = (t_\beta(x_1, h, t), t_\beta(x_2, h, t), \ldots, t_\beta(x_N, h, t)). \tag{4.8}$$

t_α, t_β are defined in (3.4).

Let $\varepsilon_\alpha(t) = \overrightarrow{\alpha}(t) - \overrightarrow{\alpha}_s(t)$, $\varepsilon_\beta(t) = \overrightarrow{\beta}(t) - \overrightarrow{\beta}_s(t)$, We have then

$$\frac{d\varepsilon_\alpha}{dt} = -\frac{d}{h^2}A\varepsilon_\alpha - \kappa_1\varepsilon_\alpha - G_\alpha + D_{\beta_s}\overrightarrow{\alpha}_s - D_\beta\overrightarrow{\alpha} \tag{4.9}$$

$$\frac{d(\varepsilon_\alpha + \varepsilon_\beta)}{dt} = -\frac{d}{h^2}A(\varepsilon_\alpha + \varepsilon_\beta) - \kappa_2\varepsilon_\beta - G \tag{4.10}$$

where $G = G^\alpha + G^\beta$, ε_α, ε_β also satisfy the initial conditons:

$$\varepsilon_\alpha(0) = 0, (\varepsilon_\alpha + \varepsilon_\beta)(0) = 0. \tag{4.11}$$

(4.10)-(4.11) yield then

$$\varepsilon_\beta(t) = \int_0^t e^{-(\frac{d}{h^2}A+\kappa_2)(t-\tau)}(\kappa_2\varepsilon_\alpha(\tau) - G)\, d\tau - \varepsilon_\alpha(t). \tag{4.12}$$

Since every component of $\beta_s(t)$, $\beta(t)$ are bounded by the initial and boundary data and $f \in C^1(R, R)$

(4.13)

$$\begin{aligned} D_{\beta_s}\overrightarrow{\alpha}_s - D_\beta\alpha &= -D_{\beta_s}\varepsilon_\alpha - diag(f(\beta(x_1)) \\ &\quad - f(\beta_1), f(\beta(x_2)) - f(\beta_2), \ldots, f(\beta(x_N)) - f(\beta_N))\overrightarrow{\alpha} \\ &= -D_{\beta_s}\varepsilon_\alpha + D\varepsilon_\beta, \end{aligned}$$

where D is a diagonal matrix whose diagonal elements are bounded and the bounds depend only on the initial and boundary value data.

It follows then from (4.9), (4.12), (4.13) that

$$\frac{d\varepsilon_\alpha}{dt} = -(\frac{d}{h^2}A + \kappa_1)\varepsilon_\alpha - D_{\beta_s}\varepsilon_\alpha - G^\alpha + D\left[\int_0^t e^{-(\frac{d}{h^2}(t-\tau)A+\kappa_2)}(\kappa_2\varepsilon_\alpha(\tau) - G)\, d\tau - \varepsilon_\alpha(t)\right].$$

Therefore

$$\begin{aligned} \varepsilon_\alpha(t) = \int_0^t e^{-(\frac{d}{h^2}A+\kappa_1)(t-\tau)}\Big[&-D_{\beta_s}\varepsilon_\alpha - G^\alpha \\ &- D\varepsilon_\alpha + D\int_0^\tau e^{-(\frac{d}{h^2}A+\kappa_2)(\tau-\omega)}(\kappa_2\varepsilon_\alpha(\omega) - G)\, d\omega\Big]\, d\tau. \end{aligned} \tag{4.14}$$

Recall that given $\|\cdot\|_0$ norm on R^N and $T \in \mathcal{L}(R^N, R^N)$, T is sysmetric, $\|T\|$ is the largest of the absolute value of the eigenvalues of T, therefore there exists constants c_1, c_2 which depends only on the initial and boundary data that

$$\|D_{\beta_s} + D\| \leq c_1, \|D\| \leq c_2.$$

Obviously

$$\|e^{(\mu A+\lambda)(t-\tau)}\| \leq e^{\lambda(t-\tau)}$$

if A is positive definite and $\lambda > 0$ and $\mu > 0$, (4.14) then leads to

$$\begin{aligned} \|\varepsilon_\alpha(t)\|_0 \leq \int_0^t e^{-\kappa_1(t-\tau)}\Big[&c_1\|\varepsilon_\alpha(\tau)\|_0 + \|G^\alpha\|_0 \\ &+ c_2\int_0^\tau e^{-\kappa_2(\tau-\omega)}(\kappa_2\|\varepsilon_\alpha(\omega)\|_0 + \|G\|_0)\, d\omega\Big]\, d\tau. \end{aligned} \tag{4.15}$$

Note that from the definition of G, G^α and (3.4), $\|G\|_0 \le c' N^{\frac{1}{2}} h^{\frac{3}{2}} \le c_3 h$. Similarly $\|G^\alpha\|_0 \le c_4 h$ we obtain from (4.15) that after integration by parts

$$\|\varepsilon_\alpha(t)\|_0 \le c_5 \int_0^t e^{-\lambda_0(t-\tau)} \|\varepsilon_\alpha(\tau)\|_0 \, d\tau + c_6 h \tag{4.16}$$

where c_3, c_4, c_5, c_6, λ_0 are some constants independent of h and N. (4.2) therefore follows from Gronwall's inequality and the fact that $\|\varepsilon_\alpha\|_{0,h} = h^{\frac{1}{2}} \|\varepsilon_\alpha\|_0$.

To prove (4.3) apply $A^{\frac{1}{2}}$ to (4.13) and observe that $A^{\frac{1}{2}}$ commutes with $e^{\mu A + \lambda}$ for any real μ, λ, we obtain

$$\begin{aligned} A^{\frac{1}{2}} \varepsilon_\alpha(t) = \int_0^t e^{-(\frac{d}{h^2} A + \kappa_1)(t-\tau)} \Big[-(D_{\beta_s} + D)' A^{\frac{1}{2}} \varepsilon_\alpha - A^{\frac{1}{2}} G^\alpha \\ + D' \int_0^\tau e^{-(\frac{d}{h^2} A + \kappa_2)(\tau - \omega)} (\kappa_2 \varepsilon_\alpha(\omega) - G) \, d\omega \Big] \, d\tau. \end{aligned} \tag{4.17}$$

where $D' = A^{\frac{1}{2}} D A^{-\frac{1}{2}}$, and $(D_{\beta_s} + D)' = A^{\frac{1}{2}} (D_{\beta_s} + D) A^{-\frac{1}{2}}$.

Clearly

$$\|D'\| = \|D\|, \qquad \|(D_{\beta_s} + D)'\| = \|(D_{\beta_s} + D)\|. \tag{4.18}$$

To estimate $\|A^{\frac{1}{2}} G\|_0$ observe

$$\|A^{\frac{1}{2}} G\|_0^2 = (AG, G) = G_1^2 + \sum_{1 < \nu < N} (G_{\nu+1} - G_\nu)^2 + G_N^2$$

Therefore $\|A^{\frac{1}{2}} G\|_0 \le c_7 h$ for some c_7 independent of h and N.

Since

$$\|\varepsilon_\alpha\|_{1,h} = \frac{1}{h^{\frac{1}{2}}} \|A^{\frac{1}{2}} \varepsilon_\alpha\|_0$$

(4.3) follows from similar arguments through (4.17). To complete proof of our theorem let $\varepsilon_\alpha = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ observe that

$$\begin{aligned} |\varepsilon_\nu| &\le \sum_{0 \le k < \nu} |\varepsilon_{k+1} - \varepsilon_k| \qquad (\varepsilon_0 = 0) \\ &\le N^{\frac{1}{2}} \sqrt{\varepsilon_1^2 + \sum_{1 < k < N} (\varepsilon_{k+1} - \varepsilon_k)^2 + \varepsilon_N^2} \\ &\le N^{\frac{1}{2}} \|A^{\frac{1}{2}} \varepsilon_\alpha\|_0 \\ &\le c_8 \|\varepsilon_\alpha\|_{1,h} \end{aligned}$$

for some positive constant c_8. (4.3) therefore follows from (4.2) and above inequality.

The results for $(\vec{\beta}(t) - \vec{\beta}_s(t))$ follows from (4.12) and (4.2)-(4.4). □

REFERENCES

[1] R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] J. BRINDLEY, C. KAAS-PETERSEN, J. H. MERKIN AND S. K. SCOTT, *Bi-periodic Behavior in the Diffusive Autocatalator*, Physics Letters A, 128 (1988), pp. 260-265.

[3] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., 1964.

[4] D. HENRY, *Geometric theory os semilinear parabolic equations, Lecture notes in Mathematics 840*, Springer-Verlag, 1981.

[5] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical sciences, 44 (1983).

[6] M. H. PROTTER-H. F. WEINBERGER, *Maximum Principles in differential Equations*, Prentice-Hall, inc., 1967.

[7] S. K. SCOTT, Chem. Eng. Sci., 42 (1987), pp. 307.

Recent IMA Preprints

#	Author/s	Title
637	**Xinfu Chen**	Generation and propagation of the interface for reaction–diffusion equations
638	**Philip Korman**	Dynamics of the Lotka–Volterra systems with diffusion
639	**Harlan W. Stech**	Generic Hopf bifurcation in a class of integro-differential equations
640	**Stephane Laederich**	Periodic solutions of non linear differential difference equations
641	**Peter J. Olver**	Canonical Forms and Integrability of BiHamiltonian Systems
642	**S.A. van Gils, M.P. Krupa and W.F. Langford**	Hopf bifurcation with nonsemisimple 1:1 Resonance
643	**R.D. James and D. Kinderlehrer**	Frustration in ferromagnetic materials
644	**Carlos Rocha**	Properties of the attractor of a scalar parabolic P.D.E.
645	**Debra Lewis**	Lagrangian block diagonalization
646	**Richard C. Churchill and David L. Rod**	On the determination of Ziglin monodromy groups
647	**Xinfu Chen and Avner Friedman**	A nonlocal diffusion equation arising in terminally attached polymer chains
648	**Peter Gritzmann and Victor Klee**	Inner and outer j- Radii of convex bodies in finite-dimensional normed spaces
649	**P. Szmolyan**	Analysis of a singularly perturbed traveling wave problem
650	**Stanley Reiter and Carl P. Simon**	Decentralized dynamic processes for finding equilibrium
651	**Fernando Reitich**	Singular solutions of a transmission problem in plane linear elasticity for wedge-shaped regions
652	**Russell A. Johnson**	Cantor spectrum for the quasi-periodic Schrödinger equation
653	**Wenxiong Liu**	Singular solutions for a convection diffusion equation with absorption
654	**Deborah Brandon and William J. Hrusa**	Global existence of smooth shearing motions of a nonlinear viscoelastic fluid
655	**James F. Reineck**	The connection matrix in Morse–Smale flows II
656	**Claude Baesens, John Guckenheimer, Seunghwan Kim and Robert Mackay**	Simple resonance regions of torus diffeomorphisms
657	**Willard Miller, Jr.**	Lecture notes in radar/sonar: Topics in Harmonic analysis with applications to radar and sonar
658	**Calvin H. Wilcox**	Lecture notes in radar/sonar: Sonar and Radar Echo Structure
659	**Richard E. Blahut**	Lecture notes in radar/sonar: Theory of remote surveillance algorithms
660	**D.V. Anosov**	Hilbert's 21st problem (according to Bolibruch)
661	**Stephane Laederich**	Ray–Singer torsion for complex manifolds and the adiabatic limit
662	**Geneviève Raugel and George R. Sell**	Navier-Stokes equations in thin 3d domains: Global regularity of solutions I
663	**Emanuel Parzen**	Time series, statistics, and information
664	**Andrew Majda and Kevin Lamb**	Simplified equations for low Mach number combustion with strong heat release
665	**Ju. S. Il'yashenko**	Global analysis of the phase portrait for the Kuramoto–Sivashinsky equation
666	**James F. Reineck**	Continuation to gradient flows
667	**Mohamed Sami Elbialy**	Simultaneous binary collisions in the collinear N–body problem
668	**John A. Jacquez and Carl P. Simon**	Aids: The epidemiological significance of two different mean rates of partner-change
669	**Carl P. Simon and John A. Jacquez**	Reproduction numbers and the stability of equilibria of SI models for heterogeneous populations
670	**Matthew Stafford**	Markov partitions for expanding maps of the circle
671	**Ciprian Foias and Edriss S. Titi**	Determining nodes, finite difference schemes and inertial manifolds
672	**M.W. Smiley**	Global attractors and approximate inertial manifolds for abstract dissipative equations
673	**M.W. Smiley**	On the existence of smooth breathers for nonlinear wave equations
674	**Hitay Özbay and Janos Turi**	Robust stabilization of systems governed by singular integro-differential equations
675	**Mary Silber and Edgar Knobloch**	Hopf bifurcation on a square lattice
676	**Christophe Golé**	Ghost circles for twist maps
677	**Christophe Golé**	Ghost tori for monotone maps
678	**Christophe Golé**	Monotone maps of $T^n \times R^n$ and their periodic orbits
679	**E.G. Kalnins and W. Miller, Jr.**	Hypergeometric expansions of Heun polynomials
680	**Victor A. Pliss and George R. Sell**	Perturbations of attractors of differential equations
681	**Avner Friedman and Peter Knabner**	A transport model with micro- and macro-structure
682	**E.G. Kalnins and W. Miller, Jr.**	A note on group contractions and radar ambiguity functions
683	**George R. Sell**	References on dynamical systems
684	**Shui-Nee Chow, Kening Lu and George R. Sell**	Smoothness of inertial manifolds
685	**Shui-Nee Chow, Xiao-Biao Lin and Kening Lu**	Smooth invariant foliations in infinite dimensional spaces

686 **Kening Lu**, A Hartman–Grobman theorem for scalar reaction-diffusion equations
687 **Christophe Golé and Glen R. Hall**, Poincaré's proof of Poincaré's last geometric theorem
688 **Mario Taboada**, Approximate inertial manifolds for parabolic evolutionary equations via Yosida approximations
689 **Peter Rejto and Mario Taboada**, Weighted resolvent estimates for Volterra operators on unbounded intervals
690 **Joel D. Avrin**, Some examples of temperature bounds and concentration decay for a model of solid fuel combustion
691 **Susan Friedlander and Misha M. Vishik**, Lax pair formulation for the Euler equation
692 **H. Scott Dumas**, Ergodization rates for linear flow on the torus
693 **A. Eden, A.J. Milani and B. Nicolaenko**, Finite dimensional exponential attractors for semilinear wave equations with damping
694 **A. Eden, C. Foias, B. Nicolaenko & R. Temam**, Inertial sets for dissipative evolution equations
695 **A. Eden, C. Foias, B. Nicolaenko & R. Temam**, Hölder continuity for the inverse of Mañé's projection
696 **Michel Chipot and Charles Collins**, Numerical approximations in variational problems with potential wells
697 **Huanan Yang**, Nonlinear wave analysis and convergence of MUSCL schemes
698 **László Gerencsér and Zsuzsanna Vágó**, A strong approximation theorem for estimator processes in continuous time
699 **László Gerencsér**, Multiple integrals with respect to L-mixing processes
700 **David Kinderlehrer and Pablo Pedregal**, Weak convergence of integrands and the Young measure representation
701 **Bo Deng**, Symbolic dynamics for chaotic systems
702 **P. Galdi, D.D. Joseph, L. Preziosi, S. Rionero**, Mathematical problems for miscible, incompressible fluids with Korteweg stresses
703 **Charles Collins and Mitchell Luskin**, Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem
704 **Peter Gritzmann and Victor Klee**, Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces
705 **A. Ronald Gallant and George Tauchen**, A nonparametric approach to nonlinear time series analysis: estimation and simulation
706 **H.S. Dumas, J.A. Ellison and A.W. Sáenz**, Axial channeling in perfect crystals, the continuum model and the method of averaging
707 **M.A. Kaashoek and S.M. Verduyn Lunel**, Characteristic matrices and spectral properties of evolutionary systems
708 **Xinfu Chen**, Generation and Propagation of interfaces in reaction diffusion systems
709 **Avner Friedman and Bei Hu**, Homogenization approach to light scattering from polymer-dispersed liquid crystal films
710 **Yoshihisa Morita and Shuichi Jimbo**, ODEs on inertial manifolds for reaction-diffusion systems in a singularly perturbed domain with several thin channels
711 **Wenxiong Liu**, Blow-up behavior for semilinear heat equations:multi-dimensional case
712 **Hi Jun Choe**, Hölder continuity for solutions of certain degenerate parabolic systems
713 **Hi Jun Choe**, Regularity for certain degenerate elliptic double obstacle problems
714 **Fernando Reitich**, On the slow motion of the interface of layered solutions to the scalar Ginzburg–Landau equation
715 **Xinfu Chen and Fernando Reitich**, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling
716 **C.C. Lim, J.M. Pimbley, C. Schmeiser and D.W. Schwendeman**, Rotating waves for semiconductor inverter r
717 **W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya**, Multisummability of formal power series solutions of linear ordinary differential equations
718 **Peter J. Olver and Chehrzad Shakiban**, Dissipative decomposition of partial differential equations
719 **Clark Robinson**, Homoclinic bifurcation to a transitive attractor of Lorenz type, II
720 **Michelle Schatzman**, A simple proof of convergence of the QR algorithm for normal matrices without shifts
721 **Ian M. Anderson, Niky Kamran and Peter J. Olver**, Internal, external and generalized symmetries
722 **C. Foias and J.C. Saut**, Asymptotic integration of Navier–Stokes equations with potential forces. I
723 **Ling Ma**, The convergence of semidiscrete methods for a system of reaction-diffusion equations
724 **Adelina Georgescu**, Models of asymptotic approximation
725 **A. Makagon and H.Salehi**, On bounded and harmonizable solutions on infinite order arma systems
726 **San-Yih Lin and Yan-Shin Chin**, An upwind finite-volume scheme with a triangular mesh for conservation laws
727 **J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart**, On the dynamics of fine structure
728 **KangPing Chen and Daniel D. Joseph**, Lubrication theory and long waves