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Abstract. In a celebrated work (see 1), Mañé showed that, given a compact set X with Hausdorff dimension d in a Banach space B, there exists a projection P of rank N, for any $N \geq 2d+1$, such that P restricted to X is injective. Here, we prove a stronger result when X has fractal dimension d and B is finite dimensional. Namely, that there exists an orthogonal projection P_0 such that not only P_0 is injective on X but also its inverse is Hölder continuous when restricted to $P_0^{-1}X$.

1. INTRODUCTION

A large class of dissipative evolution equations, have a compact attractor to which all solutions converge (see 2 and 3). Moreover, it is possible to show that the attractor has finite Hausdorff dimension (see 2 and 3). In the more particular case, where the equations arise from fluid mechanics, like 2D Navier-Stokes equations, Kuramoto-Sivashinsky equations, etc., the attractor, that lies in a suitable Hilbert space, has also finite fractoral dimension (see 4 and 2). More recently, some of these equations have also been shown to admit an exponentially attracting set of finite fractoral dimension, called an inertial set (see 5 and 6). In order to study the dynamics on these sets one is obliged to project them onto a finite dimensional space. The motivation to study projections on sets of finite fractal dimension comes from these considerations. On the one hand, we will prove that any projection on a set of finite fractal dimension can be perturbed so that its inverse will exist and moreover will be Hölder continuous as long as the underlying spaces are finite dimensional. On the other hand, we will show that for a specific class of subsets of a Hilbert space, the Hölder exponent of the inverse tends to zero as the fractal dimension of the set goes to ∞, even if its Hausdorff dimension remain zero.

2. FRACTAL DIMENSION

Let H be a separable Hilbert space and $\{e_n : n = 1, 2, \dots\}$ be an orthonormal basis for H. For a compact subset X of H, let $N_\epsilon(X)$ denote the minimum number of ϵ-balls necessary to cover X, then the fractal dimension of X, denoted by $d_F(X)$, is defined by the limit

$$d_F(X) = \overline{\lim_{\epsilon\to 0}} \frac{\log N_\epsilon(X)}{\log(1/\epsilon)}\,. \tag{1}$$

In contrast with the less stringent Hausdorff dimension, the fractal dimension is also a fine measure of geometric properties (see 7). To illustrate this, let us consider the set $X_0 = \{(\log n)^{-1}e_n : n = 2, \dots\} \cup \{0\}$ and the sets $X_\alpha = \{n^{-\alpha}e_n : n = 1, 2, \dots\} \cup \{0\}$ for $\alpha > 0$. Clearly, these sets are compact and being countable, have zero Hausdorff dimension. On the other hand, it can be shown that $d_F(X_0) = +\infty$, whereas $d_F(X_\alpha) = 1/\alpha$.

Let X be a compact subset of H with Hausdorff dimension d, let P be a projection on H of rank N with any fixed $N \geq 2d+1$. Then given $\delta > 0$, there exists a projection P_0, of the same rank as P, that is injective when restricted to X and $\|P - P_0\| < \delta$ (see 1). This is the Mañé's theorem for the Hilbert space setting. (Actually it is slightly stronger.) It is natural to ask when the inverse of P_0, on P_0X, may satisfy further continuity properties such as

Hölder continuity. Unfortunately, this is not the case as the following example shows. Let us take $X = X_0$, then $d = 0$ and $N = 1$ and we are looking for a rank one projection P_0 of the form $y \otimes y$ with $|y|_H = 1$. Clearly, if P_0^{-1} is Hölder continuous on $P_0 X$, then $|(\log n)^{-1} e_n|_H^2 \leq c^2 |P_0((\log n)^{-1} e_n)|_H^{2\theta}$ for every $n = 2, \dots$. If $y = \sum_{n=1}^{\infty} a_n e_n$, then this condition implies that $\sum_{n=2}^{\infty} |a_n|^2 \geq c^{-(2/\theta)} \sum_{n=2}^{\infty} (\log n)^{2(1-1/\theta)}$, which is a contradiction to $|y|_H = 1$, since $\theta \leq 1$. However, one may hope that this situation may not persist for the sets of finite fractal dimension.

THEOREM 1. *Let Y be a compact subset of $\mathbb{R}$, with $d_F(Y) < D < M - 1$. Let P_0 be an orthogonal projection on $\mathbb{R}^M$ of rank $N > D$. Then for every $\delta > 0$, and for every $\theta > 0$ such that $\theta < (1 - \frac{D}{N}) \cdots (1 - \frac{D}{M-1})$ there exists an orthogonal projection $P = P(\delta, \theta)$ of rank N such that $\|P - P_0\| < \delta$ and $|y| \leq c|Py|^\theta$ for every y in Y.*

PROOF. Without loss of generality, we can assume that $Y \subseteq B(0,1)$. Let $r_n = 1/2^n$ and consider the sets $Y_n = \{y \in Y : r_n < |y| \leq 2r_n\}$. Since $d_F(Y) < D$, there exists a constant $c > 1$ such that for every $\epsilon > 0$, $N_\epsilon(Y) \leq c(1/\epsilon)^D$. Now we set $\rho_n = (2^{1-\theta} \eta r_n)^{1/\theta}$ when $0 < \eta$, $\theta < 1$, and choose an open covering of Y_n with ρ_n-balls centered at a_j^n in Y_n, i.e.

$$Y_n \subset \cup \{B(a_j^n \; \rho_n) \, : \, a_j^n \in Y_n \, , \, j = 1, 2, \dots , m_n\} \tag{2}$$

with $m_n \leq c\rho_n^{-D}$. If for a given projection P, we have

$$(I - P)\mathbb{R}^M \cap B(a_j^n; 2\rho_n) \neq \phi \quad \text{for every} \quad n \quad \text{and for} \quad j = 1, 2, \dots , m_n \, , \tag{3}$$

then it is easy to see that $|Py|^\theta \geq \eta |y|$, for every y in Y. Clearly, our goal is to perturb P_0 just enough to realize (3), which we will achieve by induction on the corank of P_0. So we start with a projection P_0 of corank 1, and look for a nearby projection P necessarily of corank 1, that is $Q_0 = I - P_0 = y_0 \otimes y_0$ and $Q = I - P = y \otimes y$ where y and y_0 are unit vectors. In such a case, (3) simplifies to the condition

$$\mathbb{R} y_0 \cap B(a_j^n; 2\rho_n) = \phi \quad \text{for every} \quad n \quad \text{and for} \quad j = 1, 2, \dots , m_n \, . \tag{4}$$

In order to choose y satisfying (4) and also $|y - y_0| < \delta/2$, hence ensuring $\|P - P_0\| < \delta$, we project all the balls $B(a_j^n; \rho_n)$ onto the surface of the sphere $S(0;1)$ in $\mathbb{R}^M$ and estimate the area of the surface $\sum$ thus obtained. To compute the area of $\sum$ let us isolate one of those projected balls and set $\rho = 2\rho_m$, $a = a_j^n$ and $r = r_n$. Also let $A_\alpha(a, \rho)$ denote the projection of $B(a, \rho)$ onto $S(0; \alpha) = \alpha S(0; 1)$. Then if μ_{M-1} denote the $(M-1)$-volume measure and $\omega_{M-1} = \mu_{M-1}(S(0;1))$, by simple proportion we obtain

$$\mu_{M-1}(A_1(a, \rho)) \leq r^{1-M} \mu_{M-1}(A_{|a|}(a, \rho)) \leq r^{1-M} \omega_{M-1} \rho^{M-1} \tag{5}$$

where we also used the fact that the spherical sector passing through a has $(M-1)$-volume less than the whole surface of the ρ-ball centered at a. Let μ denote the $(M-1)$-volume of $\sum$, then using (5) and the definitions of ρ_n, η and the estimate on m_n, we obtain

$$\mu \leq c\omega_{M-1} (2^{1-\theta} \eta)^{\frac{M-D-1}{\theta}} \sum_{n=1}^{\infty} r_n^{(M-1)(\frac{1}{\theta} - 1) - \frac{D}{\theta}} \, . \tag{6}$$

So if $\lambda = (M-1)(\frac{1}{\theta} - 1) - \frac{D}{\theta} > 0$ from (6) we deduce that

$$\mu \leq c(2^{1-\theta} \eta)^{\frac{M-D-1}{\theta}} (2^\lambda - 1)^{-1} \omega_{M-1} = \epsilon \omega_{M-1} \, . \tag{7}$$

Thus, it follows from (7) that if

$$\eta < \eta_0 = \left[(\frac{\delta}{2})^{M-1} (1 - \frac{\delta}{4})^{\frac{M-2}{2}} \frac{(2^\lambda - 1)}{c} \right]^{\frac{\theta}{M-D-1}} 2^{\theta - 1}$$

then

$$\mu \le (\frac{\delta}{2})^{M-1} (1 - \frac{\delta^2}{4})^{\frac{M-1}{2}} \omega_{M-1} .$$

Consequently, the ball $B(y_0, \delta/2)$ intersects $\sum' = S(0;1)\setminus \sum$ which in turn guarantees the existence of y in $\sum'$ such that $|y - y_0| < \delta/2$. Now assume that the theorem is true if the codimension of P_0 is less than or equal to N_0, whenever $1 \le N < M - D$. In the case $N_0 + 1 \ge M - D$, the theorem is already proven. Otherwise, the given projection P_0 has rank $N = M - N_0 - 1$ and $N > D$. Choose a unit vector h_0 such that $P_0 h_0 = 0$ and set $P_0' = P_0 + (h_0 \otimes h_0)$. By the induction assumption for any fixed $\theta'' \in \left(0, \prod_{k=N+1}^{M} (1 - \frac{D}{k})\right)$ and for any $\delta' > 0$, there exists an orthogonal projection P' (of rank $N+1$) such that $||P' - P_0'|| < \delta'$ and for some fixed $\eta' > 0$, $\eta'|y| \le |P'y|^{\theta'}$, for all y in Y. We set $Y' = P'Y$ and apply the first part of the proof to Y' in $P'\mathbb{R}^M \simeq \mathbb{R}^{N+1}$ and to the orthogonal projection P_0'' from $P'\mathbb{R}^M$ onto $P'P_0\mathbb{R}^M$. Note that in such a case, we have $P_0''\mathbb{R}^M = P'P_0\mathbb{R}^M$ and $P\prime\prime_0 P'P_0 = P'P_0$. Here we replace δ by δ' and θ by any fixed θ' on $P'\mathbb{R}^M$ such that $||P'' - P_0''|| < \delta'$ and $\eta''|P'y| \le |P''P'y|^{\theta'}$ for all y in Y and for some fixed $\eta'' > 0$. Next, we set $P = P''P'$. Obviously, P is an orthogonal projection in $\mathbb{R}^M$ of rank N such that

$$\eta''(\eta')^{\theta}|y| \le |Py|^{\theta'\theta''} \quad \text{for all} \quad y \quad \text{in} \quad Y .$$

At the same time,

$$\begin{aligned} ||P - P_0|| &= ||P''P' - P_0|| \le ||P''P' - P'P_0|| + ||(I - P')P_0|| \le ||P''P' - P'P_0|| + \delta' \\ &\le 2\delta' + ||P_0''P' - P'P_0|| = 2\delta' + ||P_0''P'(I - P_0)|| \le 3\delta' + ||P_0''P_0'(I - P_0|| \quad (8) \\ &\le 3\delta' + ||P_0''(h_0 \otimes h_0)|| = 3\delta' + |P_0''h_0| , \end{aligned}$$

where we have utilized the following facts: $(I - P_0')P_0 = 0$; $||P'' - P_0''|| < \delta'$; $P_0''P'P_0 = P'P_0$; $||P' - P_0'|| < \delta'$ and $P_0' = P_0 + (h_0 \otimes h_0)$ in their stated order. To estimate $|P_0''h_0|$, one makes use of the fact that $||P'P_0 - P_0|| < \delta'$ and obtains that $|P_0''h_0| \le \frac{1+\delta'}{1-\delta'}\delta'$. Consequently, if $\delta' < \delta(3+\delta)^{-1}$ then $||P - P_0|| < \delta$ and the induction carries through. ■

THEOREM 2. *(Hölder-Mañé Projections) Let X be a compact subset of $\mathbb{R}^M$ with $d_F(X) < d$ where $M \ge N > 2d$ and let P_0 be an orthogonal projection of $\mathbb{R}^M$ of rank N. Then for any $\delta > 0$ and $\theta \in \left(0, \prod_{k=N+1}^{M}(1 - \frac{D}{k})\right)$, there exists an orthogonal projection $P = P(\delta, \theta)$ of $\mathbb{R}^M$ of rank N such that $||P - P_0|| < \delta$ and $|x_1 - x_2| \le c|P(x_1 - x_2)|^\theta$ for every x_1 and x_2 in X.*

PROOF. Let $Y = X - X = \{x_1 - x_2 \,:\, x_1, x_2 \in X\}$, then $d_F(Y) \le 2d = D$ and the Theorem 1 applies. ■

If we return to the sets X_α considered in the 2nd section, we can also deduce a partial converse to the Theorem.

PROPOSITION. *If a projection P, on H, has a Hölder continuous inverse on PX_α with the Hölder exponent θ then $\theta < 2\alpha/(2\alpha + 1)$. Moreover, every such exponent θ can be achieved by a rank one projection.*

PROOF. The second part of the claim is easy to prove. Let $x_0 = \sum_{n=1}^{\infty} n^{-\gamma} e_n$, with $\gamma = \alpha((1/\theta) - 1)$, then the projection defined by $P = |x_0|_H^{-2}(x_0 \otimes x_0)$ is of rank one and satisfies $|y|_H \leq |x_0|_H^{\theta} |Py|_H^{\theta}$ for every y in X_α. As for the first part, assuming that $|y|_H \leq c|Py|_H^{\theta}$ for every y in X_α and taking $y = n^{-\alpha} e_n$, it follows that $C|Pe_n|_H \geq n^{\alpha(1-1/\theta)}$. Therefore,

$$C \text{ rank } P = C \text{ Trace } P = C \sum_{n=1}^{\infty} |Pe_n|_H^2 \geq \sum_{n=1}^{\infty} n^{2\alpha(1-1/\theta)} .$$

Since, the rank of P is finite, we must have $2\alpha(1/\theta - 1) > 1$. ∎

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