

On the interior regularity for degenerate elliptic equations

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Abstract

We discuss the concept and motivations of quasiderivatives and give an example constructed by random time change, Girsanov's theorem and Levy's theorem. Then we use this probabilistic technique to investigate the regularity of the probabilistic solution of the Dirichlet problem for degenerate elliptic equations, from linear cases to fully nonlinear cases. In each Dirichlet problem we consider, the probabilistic solution is the unique solution in our setting.

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Chapter 1

Introduction

1.1 Overview

My thesis project started from the Dirichlet problem for linear degenerate elliptic partial differential equations

$$\begin{cases} Lu(x) - c(x)u(x) + f(x) = 0 & \text{in } D \\ u = g & \text{on } \partial D, \end{cases} \quad (1.1)$$

where $Lu := a^{ij}(x)u_{x^i x^j} + b^i(x)u_{x^i}$, summation convention is understood, and D is a bounded smooth domain in \mathbb{R}^d .

Due to the possible degeneracy, many theorems in non-degenerate cases don't hold anymore. For example, if L is uniformly non-degenerate elliptic, i.e. $a^{ij}\xi^i\xi^j \geq \delta|\xi|^2, \forall \xi \in \mathbb{R}^d$, where $\delta > 0$, it is well-known that the interior regularity of u is better than that of the boundary data g and the terms a, b, c, f . However, for degenerate elliptic operators, i.e. $a^{ij}\xi^i\xi^j \geq 0, \forall \xi \in \mathbb{R}^d$, counterexamples suggest that in general, the best regularity property we may expect is that when a, b, c, f, g are of class C^k , u enjoys the same C^k -regularity in D .

The existence, uniqueness and regularity problems for linear degenerate elliptic equations have been extensively studied by a variety of methods. See, for example, Fichera [3], Kohn-Nirenberg [8], Leventorskiĭ [21], and Oleinik-Radkevich [24], in which analysis techniques for PDEs are used. For probabilistic approaches, we refer to Dong [1], Freidlin [4, 5], Friedman [6, 7], Krylov [15, 18, 19, 20], Stroock-Varadhan [27], to name

a few.

My approach is probabilistic. The probabilistic solution of (1.1) is known as

$$u(x) = E \left[g(x_\tau(x)) e^{-\int_0^\tau c(x_t(x)) dt} + \int_0^\tau f(x_t(x)) e^{-\int_0^t c(x_s(x)) ds} dt \right], \quad (1.2)$$

where $x_t(x)$ is the solution to the Itô stochastic equation

$$x_t = x + \int_0^t \sigma(x_s) dw_s + \int_0^t b(x_s) ds, \quad (1.3)$$

where σ satisfies $(1/2)\sigma\sigma^* = a$, and τ is the first exit time of $x_t(x)$ from D .

Any solution of (1.1) in the class of $C^2(D) \cap C(\bar{D})$ satisfies (1.2) via Itô's formula. However, in general, u given by (1.2) doesn't necessarily have first and second derivatives in L , and the differential operator is understood in a generalized sense. We are interested in understanding under what conditions, u defined by (1.2) is twice differentiable and is the unique solution for (1.1) in an appropriate sense.

The probabilistic scheme used in my research is “quasiderivative”, which was introduced by N. V. Krylov in the late 1980s. Heuristically, by differentiating both sides of (1.2), we get a probabilistic representation of the directional derivative of u which involves the derivative of the stochastic process $x_t(x)$. Thanks to the probabilistic techniques such as random time change, underlying probability measure and driving Wiener process, the derivative of $x_t(x)$ can be replaced by a more general stochastic process called quasiderivative, which contains several parameters, and therefore enjoys certain freedom. It turns out that we may steer the quasiderivative so that it is tangent to the boundary when $x_t(x)$ hits it. As a result, the derivative of u along the quasiderivative becomes the derivative of the boundary data g , and estimating the derivative of u is reduced to estimating the moment of the quasiderivative.

My settings and results are as follow. Let D be a $C^{3,1}$ bounded domain in \mathbb{R}^d described by a $C^{3,1}$ function ψ which is non-singular on ∂D , i.e.

$$D := \{x \in \mathbb{R}^d : \psi(x) > 0\}, \quad |\psi_x| \geq 1 \text{ on } \partial D.$$

We also assume that

$$L\psi = a^{ij}(x)\psi_{x^i x^j} + b^i(x)\psi_{x^i} \leq -1 \text{ in } D. \quad (1.4)$$

In (1.2) and (1.3), assume that w_t is a d_1 -dimensional Wiener process, σ , b , c , f , g are of class $C^{0,1}(\bar{D})$, and c is non-negative. Under these assumptions, (1.3) has a unique solution, and u is well defined by (1.2) (see Chapter 3 or [29] for the well-definedness).

Let \mathfrak{B} denote the set of all skew-symmetric $d_1 \times d_1$ matrices. For any positive constant λ , define

$$D_\lambda = \{x \in D : \psi(x) > \lambda\}.$$

For $\xi, \eta \in \mathbb{R}^d$, set

$$u_{(\xi)} = u_{x^i} \xi^i, \quad u_{(\xi)(\eta)} = u_{x^i x^j} \xi^i \eta^j.$$

Assumption 1.1.1. *There exists a positive constant δ , such that*

$$(an, n) > \delta \text{ on } \partial D,$$

where n is the unit normal vector.

Assumption 1.1.2. *There exist functions*

- $\rho(x) : D \rightarrow \mathbb{R}^d$, bounded in D_λ for all $\lambda > 0$;
- $Q(x, y) : D \times \mathbb{R}^d \rightarrow \mathfrak{B}$, bounded with respect to x in D_λ for all $\lambda > 0$, $y \in \mathbb{R}^d$ and linear in y ;
- $M(x) : D \rightarrow \mathbb{R}$, bounded in D_λ for all $\lambda > 0$;

such that for any $x \in D$ and $|y| = 1$,

$$\begin{aligned} & \|\sigma_{(y)}(x) + (\rho(x), y)\sigma(x) + \sigma(x)Q(x, y)\|^2 + \\ & 2(y, b_{(y)}(x) + 2(\rho(x), y)b(x)) \leq c(x) + M(x)(a(x)y, y). \end{aligned}$$

Theorem 1.1.1 (W. Z. [29]). *If Assumption 1.1.1 and Assumption 1.1.2 are satisfied, then $u \in C_{loc}^{0,1}(D)$, and for any $\xi \in \mathbb{R}^d$,*

$$|u_{(\xi)}| \leq N \left(|\xi| + \frac{|\psi(\xi)|}{\psi^{\frac{1}{2}}} \right) (|f|_{0,1,D} + |g|_{0,1,D}) \text{ a.e. in } D, \quad (1.5)$$

where $N = N(|\sigma|_{0,1,D}, |b|_{0,1,D}, |c|_{0,1,D}, |\psi|_{3,1,D}, d, d_1, \delta)$.

Theorem 1.1.2 (W. Z. [29]). *If Assumption 1.1.1 and Assumption 1.1.2 are satisfied, and $\sigma, b, c, f, g \in C^{1,1}(\bar{D})$, then $u \in C_{loc}^{1,1}(D)$, and for any $\xi \in \mathbb{R}^d$,*

$$|u_{(\xi)(\xi)}| \leq N \left(|\xi|^2 + \frac{|\psi(\xi)|^2}{\psi} \right) (|f|_{1,1,D} + |g|_{1,1,D}) \text{ a.e. in } D, \quad (1.6)$$

where $N = N(|\sigma|_{1,1,D}, |b|_{1,1,D}, |c|_{1,1,D}, |\psi|_{3,1,D}, d, d_1, \delta)$.

Furthermore, u is the unique solution in $C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$ of

$$\begin{cases} Lu(x) - c(x)u(x) + f(x) = 0 & \text{a.e. in } D, \\ u = g & \text{on } \partial D. \end{cases}$$

Theorems 1.1.1 and 1.1.2 extend the results in the literature mentioned above in several directions. First, compared to the operators considered in [1, 3, 13, 15, 16, 19, 20, 27], the elliptic operator L in (1.1) is the general linear elliptic operator, and c and f are non-trivial. Also, we estimate the derivatives up to the second order, not just the first order. Second, the regularity assumptions on the boundary data are weaker. More precisely, we obtain the $C^{k,1}$ -interior regularity when g is only assumed to be in the class of $C^{k,1}(\bar{D})$, where $k = 0, 1$. Under these assumptions, since the derivatives of u may blow up near the boundary, the interior regularity results obtained are optimal. The derivative estimates (1.5) and (1.6) are new. Third, not only are Assumptions 1.1.1 and 1.1.2 sufficient, but Assumption 1.1.2 is also necessary, see Remark 3.1.3. Due to the presence of ρ, Q, M , the interior condition, Assumption 1.1.2, enjoys certain invariance properties, and is weaker than the corresponding interior condition in most of the former results.

For the nonlinear cases, we consider the Dirichlet problem for the Bellman equation

$$\begin{cases} \sup_{\alpha \in A} [L^\alpha v(x) - c(\alpha, x)v(x) + f(\alpha, x)] = 0 & \text{in } D \\ v = g & \text{on } \partial D, \end{cases} \quad (1.7)$$

where $L^\alpha v = a^{ij}(\alpha, x)v_{x^i x^j} + b^i(\alpha, x)v_{x^i}$. This problem is of definite interest from view points of theory of partial differential equations and stochastic optimal control.

On the one hand, it is known that under appropriate conditions the Dirichlet problem for the fully nonlinear elliptic equation of second order

$$\begin{cases} F(v_{x^i x^j}(x), v_{x^i}(x), v(x), x) = 0 & \text{in } D \\ v = g & \text{on } \partial D \end{cases} \quad (1.8)$$

can be rewritten as a Bellman equation in the form of (1.7).

On the other hand, from Bellman's dynamic programming principle, (1.7) is satisfied by the value function

$$v(x) = \sup_{\alpha \in \mathfrak{A}} E \left[g(x_{\tau}^{\alpha,x}) e^{-\phi_{\tau}^{\alpha,x}} + \int_0^{\tau} f^{\alpha_t}(x_t^{\alpha,x}) e^{-\phi_t^{\alpha,x}} dt \right], \quad (1.9)$$

$$\text{with } \phi_t^{\alpha,x} = \int_0^t c^{\alpha_s}(x_s^{\alpha,x}) ds,$$

in the optimal control of diffusion processes

$$x_t^{\alpha,x} = x + \int_0^t \sigma^{\alpha_s}(x_s^{\alpha,x}) dw_s + \int_0^t b^{\alpha_s}(x_s^{\alpha,x}) ds. \quad (1.10)$$

The derivation is rigorous if we know a priori that the value function $v \in C^2(D) \cap C(\bar{D})$. However, in general, v defined by (1.9) is not sufficiently smooth, or even continuous, so v in (1.9) is known as a probabilistic solution, or viscosity solution, to (1.7). We are interested in understanding under what conditions, v given by (1.9) is twice differentiable and is the unique solution of (1.7) in an appropriate sense.

In my submitted paper [28], a sufficient condition on the behavior of the controlled diffusion processes is given, under which the value function v given by (1.9) has first and second generalized derivatives and is the unique solution of (1.7). Moreover, since the derivatives of v may not be uniformly bounded up to the boundary of the domain under our setting, we also estimate first and second derivatives.

Let me point out the main difficulties on this problem. First, the differential equation in (1.7) is fully nonlinear. As a result, the martingale properties satisfied in the corresponding linear case doesn't hold anymore. Second, the diffusion term is degenerate. We should try as hard as possible to avoid non-degeneracy assumptions since most applications of stochastic optimal control concern degenerate problems. Third, the stochastic control problem we consider is time-homogeneous with state constraint. As a consequence, the control is with random and infinite time horizon and non-trivial terminal payoff.

My work is closely related to, and in some sense extends, the results obtained by M. V. Safonov [25, 26], P.-L. Lions [22, 23] and N. V. Krylov [14]. In [25] and [26], the domain D is two-dimensional, and the arguments are based on the fact that the controlled processes are in a plane region. In [22, 23], the regularity results are proved by

combinations of probabilistic and PDE arguments, which heavily rely on the assumption that the discount coefficient c^α is sufficiently large to bound the first derivatives of σ^α and b^α . In [14], the boundary data g is assumed to be of class C^4 and it is proved that under certain assumptions, the value function has generalized first and second derivatives which are bounded to the boundary.

My settings and results are as follow. The domain D and the barrier function ψ satisfy the same assumptions as those in the linear case, except that (1.4) is replaced by

$$\sup_{\alpha \in A} L^\alpha \psi \leq -1 \text{ in } D.$$

For any $\alpha \in A$, assume that $\sigma^\alpha, b^\alpha, c^\alpha, f^\alpha, g \in C^{0,1}(\bar{D})$ and $c^\alpha \geq 0$. Let

$$\mu(x, \xi) := \inf_{\zeta: (\xi, \zeta)=1} \sup_{\alpha \in A} a^{ij}(\alpha, x) \zeta^i \zeta^j, \quad \mu(x) := \inf_{|\zeta|=1} \sup_{\alpha \in A} a^{ij}(\alpha, x) \zeta^i \zeta^j.$$

Theorem 1.1.3 (W. Z. [28]). *If for each $\alpha \in A$, Assumption 1.1.1 and Assumption 1.1.2 are satisfied, and*

$$|\psi|_{3,1,D}, |\sigma^\alpha|_{0,1,D}, |b^\alpha|_{0,1,D}, |c^\alpha|_{0,1,D}, |f^\alpha|_{0,1,D}, |g|_{0,1,D} \leq K,$$

then $v \in C_{loc}^{0,1}(D)$, and for any $\xi \in \mathbb{R}^d$,

$$|v_{(\xi)}| \leq N \left(|\xi| + \frac{|\psi_{(\xi)}|}{\psi^{\frac{1}{2}}} \right) \text{ a.e. in } D, \quad (1.11)$$

where $N = N(K, d, d_1, \delta)$.

Theorem 1.1.4 (W. Z. [28]). *If for each $\alpha \in A$, Assumption 1.1.1 and Assumption 1.1.2 are satisfied, and $\sigma^\alpha, b^\alpha, c^\alpha, g \in C^{1,1}(\bar{D})$, $f^\alpha \in C^{1,1}(D)$, satisfying*

$$|\psi|_{3,1,D}, |\sigma^\alpha|_{1,1,D}, |b^\alpha|_{1,1,D}, |c^\alpha|_{1,1,D}, |f^\alpha|_{0,1,D}, |g|_{1,1,D} \leq K,$$

$$f_{xx}^\alpha + KI \geq 0, \text{ a.e. in } D,$$

then for any $\xi \in \mathbb{R}^d$,

$$v_{(\xi)}(\xi) \geq -N \left(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi} \right), \text{ a.e. in } D, \quad (1.12)$$

$$v_{(\xi)}(\xi) \leq \mu(x, \xi/|\xi|)^{-1} N \frac{|\xi|^2}{\psi}, \text{ a.e. in } D^\xi, \quad (1.13)$$

where $D^\xi := \{x \in D : \mu(x, \xi) > 0\}$ and $N = N(K, d, d_1, \delta)$.

If $\mu(x) > 0$ in D , then $v \in C_{loc}^{1,1}(D)$. In addition, v is the unique solution in $C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$ of the Dirichlet problem

$$\begin{cases} \sup_{\alpha \in A} [L^\alpha v(x) - c(\alpha, x)v(x) + f(\alpha, x)] = 0 & \text{a.e. in } D \\ v = g & \text{on } \partial D. \end{cases}$$

1.2 Organization of the thesis

This thesis is organized as follows: Chapter 2 is a review of quasiderivatives. In Chapter 3, we investigate the interior regularity for linear degenerate elliptic equations. It is based on [29], in which we obtain Theorems 1.1.1 and 1.1.2. In Chapter 4, we investigate the interior regularity for fully nonlinear degenerate elliptic equations. It is based on [28], in which we obtain Theorems 1.1.3 and 1.1.4.

1.3 Notation

To conclude this section, we introduce the notation: Above we have already defined $C^k(\bar{D})$, $k = 1$ or 2 , as the space of bounded continuous and k -times continuously differentiable functions in \bar{D} with finite norm given by

$$|g|_{1,D} = |g|_{0,D} + |g_x|_{0,D}, \quad |g|_{2,D} = |g|_{1,D} + |g_{xx}|_{0,D},$$

respectively, where

$$|g|_{0,D} = \sup_{x \in D} |g(x)|,$$

g_x is the gradient vector of g , and g_{xx} is the Hessian matrix of g . For $\beta \in (0, 1]$, the Hölder spaces $C^{k,\beta}(\bar{D})$ are defined as the subspaces of $C^k(\bar{D})$ consisting of functions with finite norm

$$|g|_{k,\beta,D} = |g|_{k,D} + [g]_{\beta,D}, \quad \text{where } [g]_{\beta,D} = \sup_{x,y \in D} \frac{|g(x) - g(y)|}{|x - y|^\beta}.$$

\mathbb{R}^d is the d -dimensional Euclidean space with $x = (x^1, x^2, \dots, x^d)$ representing a typical point in \mathbb{R}^d , and $(x, y) = \sum_{i=1}^d x^i y^i$ is the inner product for $x, y \in \mathbb{R}^d$. For $x, y, z \in \mathbb{R}^d$,

set

$$u_{(y)} = \sum_{i=1}^d u_{x^i} y^i, \quad u_{(y)(z)} = \sum_{i,j=1}^d u_{x^i x^j} y^i z^j, \quad u_{(y)}^2 = (u_{(y)})^2.$$

For any matrix $\sigma = (\sigma^{ij})$,

$$\|\sigma\|^2 := \text{tr} \sigma \sigma^* = \sum_{i,j} (\sigma^{ij})^2.$$

For any $s, t \in \mathbb{R}$, we define

$$s \wedge t = \min(s, t), \quad s \vee t = \max(s, t).$$

Constants K, N and λ appearing in inequalities are usually not indexed. They may differ even in the same chain of inequalities.

Chapter 2

Quasiderivatives

The concept of quasiderivative was first introduced by N. V. Krylov in [13] (1988), in which this probabilistic technique is applied to find weaker and more flexible conditions on σ , b and c such that u in (1.2) is twice continuously differentiable in manifolds without boundary. Since then, this technique has been applied to investigate the interior smoothness of solutions of various elliptic and parabolic partial differential equations. The first derivatives of various linear elliptic and parabolic PDEs have been estimated under various conditions in Krylov [15] (1992), [16] (1993) and [19] (2004), where each case was treated by its particular choice of quasiderivatives. In Krylov [20] (2004), a unified method is presented, also based on quasiderivatives, while σ and b are assumed to be constant. As far as the applications to nonlinear equations, for example, in Krylov [14] (1989), derivative estimates are obtained when controlled diffusion processes and consequently fully nonlinear elliptic equations are considered.

2.1 Definition of quasiderivatives

In what follows, we consider the Itô stochastic equation

$$x_t = x + \int_0^t \sigma^i(x_s) dw_s^i + \int_0^t b(x_s) ds \quad (2.1)$$

on a given complete probability space (Ω, \mathcal{F}, P) , where $x \in \mathbb{R}^d$, σ^i and b are (nonrandom) \mathbb{R}^d -valued functions with bounded domain D in \mathbb{R}^d , defined for $i = 1, \dots, d_1$ with d_1 possibly different from d , and $w_t := (w_t^1, \dots, w_t^{d_1})$ is a d_1 -dimensional Wiener process

with respect to a given increasing filtration $\{\mathcal{F}_t, t \geq 0\}$ of σ -algebras $\mathcal{F}_t \subset \mathcal{F}$, such that \mathcal{F}_t contain all P -null sets. We denote by σ the $d \times d_1$ matrix composed of the column-vectors σ^i , $i = 1, \dots, d_1$. We also assume that σ and b are twice continuously differentiable in \mathbb{R}^d . Based on the assumptions above, for any $x \in D$, it is known that equation (2.1) has a unique solution $x_t(x)$ on $[0, \tau(x))$, where

$$\tau(x) = \inf\{t \geq 0 : x_t(x) \notin D\} \quad (\inf\{\emptyset\} := \infty).$$

Definition 2.1.1. *We write*

$$u \in \mathcal{M}^k(D, \sigma, b)$$

if u is a real-valued k times continuously differentiable function given on \bar{D} such that the process $u(x_t(x))$ is a local $\{\mathcal{F}_t\}$ -martingale on $[0, \tau(x))$ for any $x \in D$.

We abbreviate $\mathcal{M}^k(D, \sigma, b)$ by $\mathcal{M}^k(D)$, or simply \mathcal{M}^k when this will cause no confusion.

Definition 2.1.2. *Let $x \in D$, and let τ be a stopping time, $\tau \leq \tau(x)$. Assume that $\xi \in \mathbb{R}^d$, ξ_t and ξ_t^0 are adapted continuous processes defined on $[0, \tau] \cap [0, \infty)$ with values in \mathbb{R}^d and \mathbb{R} , respectively, such that $\xi_0 = \xi$.*

*We say that ξ_t is a **first quasiderivative** of $x_t(y)$ in the direction of ξ at point x on $[0, \tau)$ if for any $u \in \mathcal{M}^1(D, \sigma, b)$ the following process*

$$u_{(\xi_t)}(x_t(x)) + \xi_t^0 u(x_t(x)) \tag{2.2}$$

*is a local martingale on $[0, \tau)$. In this case the process ξ_t^0 is called a **first adjoint process** for ξ_t . If $\tau = \tau(x)$ we simply say that ξ_t is a first quasiderivative of $x_t(y)$ in D in the direction of ξ at x .*

Definition 2.1.3. *Under the assumptions of Definition 2.1.2, additionally assume that $\eta \in \mathbb{R}^d$, η_t and η_t^0 are adapted continuous processes defined on $[0, \tau] \cap [0, \infty)$ with values in \mathbb{R}^d and \mathbb{R} , respectively, such that $\eta_0 = \eta$.*

*We say that η_t is a **second quasiderivative** of $x_t(y)$ associated with ξ_t in the direction of η at point x on $[0, \tau)$ if for any $u \in \mathcal{M}^2(D, \sigma, b)$ the following process*

$$u_{(\xi_t)(\xi_t)}(x_t(x)) + u_{(\eta_t)}(x_t(x)) + 2\xi_t^0 u_{(\xi_t)}(x_t(x)) + \eta_t^0 u(x_t(x)), \tag{2.3}$$

where ξ_t and ξ_t^0 are first quasiderivative and first adjoint process.

is a local martingale on $[0, \tau)$. In this case the process η_t^0 is called a **second adjoint process** for η_t . If $\tau = \tau(x)$ we simply say that η_t is a second quasiderivative of $x_t(y)$ associated with ξ_t in D in the direction of η at x .

To explain the idea of this probabilistic technique, let us consider

$$u(x) = Eg(x_\tau(x)),$$

that is, we temporarily let $f = c = 0$ in (1.2). Based on the definitions above, if $u \in C^2(\bar{D})$, then the strong Markov property of $x_t(x)$ implies that $u \in \mathcal{M}^2(D)$, and the usual first and second “derivatives” with respect to x of $x_t(x)$, which are defined as the solutions of the Itô equations

$$\xi_t = \xi + \int_0^t \sigma_{(\xi_s)}^k(x_s) dw_s^k + \int_0^t b_{(\xi_s)}(x_s) ds$$

$$\eta_t = \eta + \int_0^t \left[\sigma_{(\xi_s)(\eta_s)}^k(x_s) + \sigma_{(\eta_s)}^k(x_s) \right] dw_s^k + \int_0^t \left[b_{(\xi_s)(\eta_s)}(x_s) + b_{(\eta_s)}(x_s) \right] ds$$

are first and second quasiderivatives with zero adjoint processes. This means, the concept “quasiderivative” is a generalization of the usual “derivative”.

Now we additionally assume that the domain D is of class C^2 with ∂D bounded, $\tau(x) < \infty$ (a.s.), and g is twice continuously differentiable on ∂D .

If the process (2.2) is a uniformly integrable martingale on $[0, \tau(x))$ and $\xi_{\tau(x)}$ is tangent to ∂D at $x_{\tau(x)}(x)$ (a.s.), then we have

$$u_{(\xi)}(x) = E[u_{(\xi_\tau)}(x_\tau) + \xi_\tau^0 u(x_\tau)] = E[g_{(\xi_\tau)}(x_\tau) + \xi_\tau^0 g(x_\tau)].$$

This shows how we can apply first quasiderivatives to get interior estimates of $u_{(\xi)}$ through $|g|_{1,D}$ or $|g|_{1,\partial D}$.

As far as second derivatives are concerned, first notice that

$$4u_{(\xi)(\zeta)}(x) = u_{(\xi+\zeta)(\xi+\zeta)}(x) - u_{(\xi-\zeta)(\xi-\zeta)}(x).$$

So to estimate $u_{(\xi)(\zeta)}(x), \forall \xi, \zeta \in \mathbb{R}^d$, it suffices to estimate $u_{(\xi)(\xi)}(x), \forall \xi \in \mathbb{R}^d$.

Again, if the process (2.3) is a uniformly integrable martingale on $[0, \tau(x))$, $\xi_{\tau(x)}$ and $\eta_{\tau(x)}$ are tangent to ∂D at $x_{\tau(x)}(x)$ (a.s.), then by letting $\eta = 0$, we have

$$u_{(\xi)(\xi)}(x) = u_{(\xi)(\xi)}(x) + u_{(\eta)}(x)$$

$$\begin{aligned}
&= E[u_{(\xi_\tau)(\xi_\tau)}(x_\tau) + u_{(\eta_\tau)}(x_\tau) + 2\xi_\tau^0 u_{(\xi_\tau)}(x_\tau) + \eta_\tau^0 u(x_\tau)] \\
&= E[g_{(\xi_\tau)(\xi_\tau)}(x_\tau) + u_{(n(x_\tau))}(x_\tau) \cdot \xi_\tau^T D^2 h_{x_\tau}(0) \xi_\tau + g_{(\eta_\tau)}(x_\tau) \\
&\quad + 2\xi_\tau^0 g_{(\xi_\tau)}(x_\tau) + \eta_\tau^0 g(x_\tau)],
\end{aligned}$$

where $n(x)$ is the unit inward normal at $x \in \partial D$ and $h_p : T_p(\partial D) \rightarrow \mathbb{R}$ is a local representation of ∂D as a graph over tangent space of ∂D at p . (Notice that it is different from the first order case that generally $u_{(\xi_\tau)(\xi_\tau)}(x_\tau) \neq g_{(\xi_\tau)(\xi_\tau)}(x_\tau)$.) Since D is of class C^2 and ∂D is bounded,

$$\xi_\tau^T D^2 h_{x_\tau}(0) \xi_\tau \leq N |\xi_\tau|^2,$$

where N is constant. This shows how we can apply second quasiderivatives to get interior estimates of $u_{(\xi)(\xi)}$ through $|g|_{2,D}$, or even $|g|_{2,\partial D}$, provided that $u_{(n(y))}(y)$ can be estimated on ∂D in terms of $|g|_{2,D}$ or $|g|_{2,\partial D}$.

It is also worth mentioning that $\eta_{\tau(x)}$ need not be tangent to ∂D at $x_{\tau(x)}$, provided that we can control the moments of $\eta_{t \wedge \tau}$ and estimate the normal derivative of u , because we can write $\eta_{\tau(x)}$ as the sum of the tangential component and the normal component.

2.2 Examples of quasiderivatives

The discussion above motivates us on attempting to construct as many quasiderivatives as possible.

Theorem 2.2.1. *Let $r_t, \hat{r}_t, \pi_t, \hat{\pi}_t, P_t, \hat{P}_t$ be jointly measurable adapted processes with values in $\mathbb{R}, \mathbb{R}, \mathbb{R}^{d_1}, \mathbb{R}^{d_1}, \text{Skew}(d_1, \mathbb{R}), \text{Skew}(d_1, \mathbb{R})$, respectively, where $\text{Skew}(d_1, \mathbb{R})$ denotes the set of all $d_1 \times d_1$ skew-symmetric real matrices. Assume that*

$$\int_0^T (|r_t|^4 + |\hat{r}_t|^2 + |\pi_t|^4 + |\hat{\pi}_t|^2 + |P_t|^4 + |\hat{P}_t|^2) dt < \infty$$

for any $T \in [0, \infty)$. For $x \in D$, $\xi \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^d$, on the time interval $[0, \infty)$, define the process ξ_t and η_t as solutions of the following (linear) equations:

$$\xi_t = \xi + \int_0^t [\sigma_{(\xi_s)} + r_s \sigma + \sigma P_s] dw_s + \int_0^t [b_{(\xi_s)} + 2r_s b - \sigma \pi_s] ds, \quad (2.4)$$

$$\begin{aligned}
\eta_t = \eta + \int_0^t & \left[\sigma_{(\eta_s)} + \hat{r}_s \sigma + \sigma \hat{P}_s + \sigma_{(\xi_s)(\xi_s)} + 2r_s \sigma_{(\xi_s)} \right. \\
& \left. + 2\sigma_{(\xi_s)} P_s + 2r_s \sigma P_s - r_s^2 \sigma + \sigma P_s^2 \right] dw_s \\
& + \int_0^t \left[b_{(\eta_s)} + 2\hat{r}_s b - \sigma \hat{\pi}_s + b_{(\xi_s)(\xi_s)} + 4r_s b_{(\xi_s)} \right. \\
& \left. - 2\sigma_{(\xi_s)} \pi_s - 2r_s \sigma \pi_s - 2\sigma P_s \pi_s \right] ds,
\end{aligned} \tag{2.5}$$

where in σ, b and their derivatives we dropped the argument $x_s(x)$. Also define:

$$\xi_t^0 = \int_0^t \pi_s dw_s, \tag{2.6}$$

$$\eta_t^0 = (\xi_t^0)^2 - \langle \xi^0 \rangle_t + \int_0^t \hat{\pi}_s dw_s. \tag{2.7}$$

Then ξ_t is a first quasiderivative of $x_t(y)$ in D in the direction of ξ at x and ξ_t^0 is its first adjoint process, and η_t is a second quasiderivative of $x_t(y)$ associated with ξ_t in D in the direction of η at x and η_t^0 is its second adjoint process.

Remark 2.2.1. Equations (2.4) and (2.5) give the most general forms of the first and second quasiderivatives known so far. On one hand, they contain various auxiliary processes, $r_t, \pi_t, P_t, \hat{r}_t, \hat{\pi}_t, \hat{P}_t$, which supply us fruitful quasiderivatives for our applications. On the other hand, in specific applications, many of the auxiliary processes are defined to be zero (processes), which make the equations (2.4) and (2.5) shorter.

Proof. Mimic the proof of Theorem 3.2.1 in [19] by replacing $y_t(\varepsilon, x)$ as the solution to the Itô equation

$$\begin{aligned}
dy_t = & \sqrt{1 + 2\varepsilon r_t + \varepsilon^2 \hat{r}_t} \sigma(y_t) e^{\varepsilon P_t} e^{\frac{1}{2} \varepsilon^2 \hat{P}_t} dw_t + \left[(1 + 2\varepsilon r_t + \varepsilon^2 \hat{r}_t) b(y_t) \right. \\
& \left. - \sqrt{1 + 2\varepsilon r_t + \varepsilon^2 \hat{r}_t} \sigma(y_t) e^{\varepsilon P_t} e^{\frac{1}{2} \varepsilon^2 \hat{P}_t} (\varepsilon \pi_t + \frac{1}{2} \varepsilon^2 \hat{\pi}_t) \right] dt
\end{aligned}$$

with initial condition $y = x + \varepsilon \xi + \frac{1}{2} \varepsilon^2 \eta$, and then differentiating the local martingale

$$u(y_t(\varepsilon, x)) \exp \left(\int_0^t (\varepsilon \pi_s + \frac{1}{2} \varepsilon^2 \hat{\pi}_s) dw_s - \frac{1}{2} \int_0^t |\varepsilon \pi_s + \frac{1}{2} \varepsilon^2 \hat{\pi}_s|^2 ds \right)$$

twice which turns out to be a local martingale also.

□

Before ending this section, we introduce two local martingales to be used in the next two chapters.

Theorem 2.2.2. *Let c , f , g and u be real-valued twice continuously differentiable functions in D . Suppose that u satisfies (1.1). Take the processes $r_t, \hat{r}_t, \pi_t, \hat{\pi}_t, P_t, \hat{P}_t, \xi_t, \eta_t, \xi_t^0, \eta_t^0$ from Theorem 2.2.1. Then for any $x \in D$, the processes*

$$X_t := e^{-\phi_t} \left[u_{(\xi_t)}(x_t) + \tilde{\xi}_t^0 u(x_t) \right] + \int_0^t e^{-\phi_s} \left[f_{(\xi_s)}(x_s) + (2r_s + \tilde{\xi}_s^0) f(x_s) \right] ds, \quad (2.8)$$

$$\begin{aligned} Y_t := e^{-\phi_t} & \left[u_{(\xi_t)(\xi_t)}(x_t) + u_{(\eta_t)}(x_t) + 2\tilde{\xi}_t^0 u_{(\xi_t)}(x_t) + \tilde{\eta}_t^0 u(x_t) \right] \\ & + \int_0^t e^{-\phi_s} \left[f_{(\xi_s)(\xi_s)}(x_s) + f_{(\eta_s)}(x_s) + (4r_s + 2\tilde{\xi}_s^0) f_{(\xi_s)}(x_s) \right. \\ & \left. + (2\hat{r}_s + 4\tilde{\xi}_s^0 r_s + \tilde{\eta}_s^0) f(x_s) \right] ds, \end{aligned} \quad (2.9)$$

with

$$\begin{aligned} \phi_t &:= \int_0^t c(x_s) ds, \\ \xi_t^{d+1} &:= - \int_0^t c_{(\xi_s)}(x_s) + 2r_s c(x_s) ds, \\ \tilde{\xi}_t^0 &:= \xi_t^0 + \xi_t^{d+1}, \\ \eta_t^{d+1} &:= - \int_0^t c_{(\xi_s)(\xi_s)}(x_s) + c_{(\eta_s)}(x_s) + 4r_s c_{(\xi_s)}(x_s) + 2\hat{r}_s c(x_s) ds, \\ \tilde{\eta}_t^0 &:= \eta_t^0 + 2\tilde{\xi}_t^0 \xi_t^{d+1} + (\xi_t^{d+1})^2 + \eta_t^{d+1}, \end{aligned}$$

are local martingales on $[0, \tau_D(x))$. (We keep writing x_t in place of $x_t(x)$ and drop this argument in many places.)

Proof. Introduce two additional equations

$$x_t^{d+1} = - \int_0^t c(x_s) ds, \quad x_t^{d+2} = \int_0^t \exp(x_s^{d+1}) f(x_s) dt.$$

For $\bar{x} = (x, x^{d+1}, x^{d+2}) \in D \times \mathbb{R} \times \mathbb{R}$, define

$$\bar{u}(\bar{x}) = \exp(x^{d+1}) u(x) + x^{d+2}.$$

Itô's formula and the assumption that $a^{ij}(x)u_{x^i x^j} + b^i(x)u_{x^i} - c(x)u + f(x) = 0$ in D imply that $\bar{u}(\bar{x}_t(x, 0, 0))$ is a local martingale on $[0, \tau_D(x))$. That means, $\bar{u}(\bar{x}_t) \in \mathcal{M}^2$.

According to definitions 2.1.2 and 2.1.3,

$$\bar{u}_{(\bar{\xi}_t)}(\bar{x}_t) + \xi_t^0 \bar{u}(\bar{x}_t) \text{ and } \bar{u}_{(\bar{\eta}_t)}(\bar{x}_t) + \bar{u}_{(\bar{\xi}_t)(\bar{\xi}_t)}(\bar{x}_t) + 2\xi_t^0 \bar{u}_{(\bar{\xi}_t)}(\bar{x}_t) + \eta_t^0 \bar{u}(\bar{x}_t)$$

are local martingales on $[0, \tau_D(x))$, where $\bar{\xi}_t = (\xi_t, \xi_t^{d+1}, \xi_t^{d+2})$ and $\bar{\eta}_t = (\eta_t, \eta_t^{d+1}, \eta_t^{d+2})$ are first and second quasiderivatives of $\bar{x}_t((x, 0, 0))$ in the directions of $\bar{\xi} = (\xi, 0, 0)$ and $\bar{\eta} = (\eta, 0, 0)$, respectively.

Direct computation leads to

$$\begin{aligned} \bar{u}_{(\bar{\xi}_t)}(\bar{x}_t) &= \exp(x_t^{d+1}) [u_{(\xi_t)}(x_t) + \xi_t^{d+1} u(x_t)] + \xi_t^{d+2}, \\ \bar{u}_{(\bar{\xi}_t)(\bar{\xi}_t)}(\bar{x}_t) &= \exp(x_t^{d+1}) [u_{(\xi_t)(\xi_t)}(x_t) + 2\xi_t^{d+1} u_{(\xi_t)}(x_t) + (\xi_t^{d+1})^2 u(x_t)], \\ \bar{u}_{(\bar{\eta}_t)}(\bar{x}_t) &= \exp(x_t^{d+1}) [u_{(\eta_t)}(x_t) + \eta_t^{d+1} u(x_t)] + \eta_t^{d+2}, \end{aligned}$$

with

$$\begin{aligned} \xi_t^{d+1} &= - \int_0^t c_{(\xi_s)}(x_s) + 2r_s c(x_s) ds, \\ \xi_t^{d+2} &= \int_0^t \exp(x_s^{d+1}) [f_{(\xi_s)}(x_s) + (\xi_s^{d+1} + 2r_s) f(x_s)] ds, \\ \eta_t^{d+1} &= - \int_0^t c_{(\xi_s)(\xi_s)}(x_s) + c_{(\eta_s)}(x_s) + 4r_s c_{(\xi_s)}(x_s) + 2\hat{r}_s c(x_s) ds, \\ \eta_t^{d+2} &= \int_0^t \exp(x_s^{d+1}) [f_{(\xi_s)(\xi_s)}(x_s) + f_{(\eta_s)}(x_s) + (2\xi_s^{d+1} + 4r_s) f_{(\xi_s)}(x_s) \\ &\quad + ((\xi_s^{d+1})^2 + \eta_s^{d+1} + 4r_s \xi_s^{d+1} + 2\hat{r}_s) f(x_s)] ds. \end{aligned}$$

It remains to notice that ξ_t^0 and η_t^0 are local martingales, so by Lemma II.8.5(c) in [17]

$$\xi_t^0 x_t^{d+2} - \int_0^t \xi_s^0 dx_s^{d+2}, \xi_t^0 \xi_t^{d+2} - \int_0^t \xi_s^0 d\xi_s^{d+2} \text{ and } \eta_t^0 x_t^{d+2} - \int_0^t \eta_s^0 dx_s^{d+2}$$

are local martingales. □

Chapter 3

Interior regularity for linear degenerate elliptic equations

In this chapter, we use quasiderivatives to investigate the regularity of (1.2), which is the probabilistic solution of (1.1).

3.1 Main theorem

Let σ , b and c in (1.2) and (1.3) be twice continuously differentiable in \mathbb{R}^d , and c be non-negative. Let $D \in C^{3,1}$ be a bounded domain in \mathbb{R}^d , then there exists a function $\psi \in C^{3,1}$ satisfying

$$\psi > 0 \text{ in } D, \quad \psi = 0 \text{ and } |\psi_x| \geq 1 \text{ on } \partial D.$$

We also assume that

$$L\psi := a^{ij}(x)\psi_{x^i x^j} + b^i(x)\psi_{x^i} \leq -1 \text{ in } D. \quad (3.1)$$

$$|\sigma^{ij}|_{2,D} + |b^i|_{2,D} + |c|_{2,D} + |\psi|_{4,D} \leq K_0, \quad (3.2)$$

with constant $K_0 \in [1, \infty)$.

Let \mathfrak{B} be the set of all skew-symmetric $d_1 \times d_1$ matrices. For any positive constant λ , define

$$D_\lambda = \{x \in D : \psi(x) > \lambda\}.$$

Assumption 3.1.1. (*non-degeneracy along the normal to the boundary*)

$$(an, n) > 0 \text{ on } \partial D,$$

where n is the unit normal vector.

Assumption 3.1.2. (*interior condition to control the moments of quasiderivatives, weaker than the non-degeneracy*) There exist functions

- $\rho(x) : D \rightarrow \mathbb{R}^d$, bounded in D_λ for all $\lambda > 0$;
- $Q(x, y) : D \times \mathbb{R}^d \rightarrow \mathfrak{B}$, bounded with respect to x in D_λ for all $\lambda > 0, y \in \mathbb{R}^d$ and linear in y ;
- $M(x) : D \rightarrow \mathbb{R}$, bounded in D_λ for all $\lambda > 0$;

such that for any $x \in D$ and $|y| = 1$,

$$\begin{aligned} & \|\sigma_{(y)}(x) + (\rho(x), y)\sigma(x) + \sigma(x)Q(x, y)\|^2 + \\ & 2(y, b_{(y)}(x) + 2(\rho(x), y)b(x)) \leq c(x) + M(x)(a(x)y, y). \end{aligned} \quad (3.3)$$

Our main result is the following:

Theorem 3.1.1. *Define u by (1.2), in which $x_t(x)$ is the solution of (1.3). Suppose that Assumption 3.1.1 and Assumption 3.1.2 are satisfied.*

1. *If $f, g \in C^{0,1}(\bar{D})$, then $u \in C_{loc}^{0,1}(D)$, and for any $\xi \in \mathbb{R}^d$,*

$$|u_{(\xi)}| \leq N \left(|\xi| + \frac{|\psi_{(\xi)}|}{\psi^{\frac{1}{2}}} \right) (|f|_{0,1,D} + |g|_{0,1,D}) \text{ a.e. in } D, \quad (3.4)$$

where $N = N(K_0, d, d_1, D)$.

2. *If $f, g \in C^{1,1}(\bar{D})$, then $u \in C_{loc}^{1,1}(D)$, and for any $\xi \in \mathbb{R}^d$,*

$$|u_{(\xi)(\xi)}| \leq N \left(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi} \right) (|f|_{1,1,D} + |g|_{1,1,D}) \text{ a.e. in } D, \quad (3.5)$$

where $N = N(K_0, d, d_1, D)$. Furthermore, u is the unique solution in $C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$ of

$$\begin{cases} Lu(x) - c(x)u(x) + f(x) = 0 & \text{a.e. in } D \\ u = g & \text{on } \partial D. \end{cases} \quad (3.6)$$

Remark 3.1.1. The “a.e.” in (3.4) and (3.5) are uniform in the choice of ξ .

Remark 3.1.2. The author doesn't know whether the estimates (3.4) and (3.5) are sharp.

Remark 3.1.3. We give two examples to show that Assumption 3.1.2 is necessary and how to take advantage of the parameters ρ, Q, M in (3.3), respectively. They are almost the same as Remark V.8.6 and Example VI.1.7 in [17]. See Example V.8.3, Remark V.8.6, Example VI.1.2 and Example VI.1.7 in [17] for more details.

In the first example, we take $d = d_1 = 1$ and $D = (-2, 2)$. Let $\sigma(x) = x, b(x) = \beta x$ in $[-2, 2]$ and $c(x) = \nu, f(x) = 0$ in $[-1, 1]$, where $\nu > 0, \beta \in \mathbb{R}$ are constants. Extend $c(x)$ and $f(x)$ outside $[-1, 1]$ in such a way that $c(x) \geq \nu, f(x) > 0$, and c, f are smooth on $[-2, 2]$, bounded and have bounded derivatives up to second order. Let $g(x) = 0$ on $\partial D = \{-2, 2\}$. Define

$$\tau_1(x) = \inf\{t \geq 0 : |x_t(x)| \geq 1\}, \quad \tau_2(x) = \inf\{t \geq 0 : |x_t(x)| \geq 2\}.$$

Based on our construction, for all $t \in [0, \tau_2(x)]$ (a.s.),

$$x_t(x) = xe^{w_t + (\beta - 1/2)t}.$$

It follows that for any $x \in (0, 1]$, $x_t(x)$ takes the value 1 at time $\tau_1(x)$ almost surely. Similarly, for any $x \in [-1, 0)$, $x_{\tau_1(x)}(x) = -1$ (a.s.). Also, note that

$$Ee^{-\nu\tau_1(x)} = x^\kappa, \quad \text{with } \kappa = [(\beta - 1/2)^2 + 2\nu]^{1/2} - \beta + 1/2.$$

Hence

$$u(x) = \begin{cases} Ee^{-\nu\tau_1(x)}u(x_{\tau_1(x)}(x)) = u(1)x^\kappa & \text{if } x \in (0, 1], \\ Ee^{-\nu\tau_1(x)}u(x_{\tau_1(x)}(x)) = u(-1)|x|^\kappa & \text{if } x \in [-1, 0), \\ 0 & \text{if } x = 0. \end{cases} \quad (3.7)$$

Notice that $u(1) > 0, u(-1) > 0$, so $u(x)$ has Lipschitz continuous derivatives if and only if $\kappa \geq 2$. It is equivalent to $1 + 2\beta \leq \nu$, which is exactly (3.3) in which ρ, Q, M are vanishing. This example shows that Assumption 3.1.2 is necessary.

Next, we discuss an advantage of the parameters ρ, Q, M in (3.3). More precisely, we show that with the help of these parameters, from some local information, Assumption

3.1.2 holds. Assume that $d = d_1 = 1$ for the sake of simplicity, and for each $x \in D$ where $\sigma(x) = b(x) = 0$, we have

$$|\sigma'(x)|^2 + 2b'(x) < c(x). \quad (3.8)$$

We claim that Assumption 3.1.2 hold. Indeed, we observe that for

$$\rho(x) = -nb(x), \quad Q(x, y) = nb(x)y, \quad M(x) = n,$$

the inequality (3.3) becomes

$$|\sigma'(x)|^2 + 2b'(x) \leq c(x) + n\sigma^2(x) + 4nb^2(x). \quad (3.9)$$

Suppose that there exists D_λ , for any $n \in \{1, 2, \dots\}$, there exists a point x_n at which the inequality converse to (3.9) holds. Then we can exact from the sequence $(\sigma(x_n), \sigma'(x_n), b(x_n), b'(x_n), c(x_n))$ a subsequence that converges to $(\sigma(x_0), \sigma'(x_0), b(x_0), b'(x_0), c(x_0))$ for some $x_0 \in \bar{D}_\lambda$. It follows from (3.2) that

$$n\sigma^2(x_n) + 4nb^2(x_n) < |\sigma'(x_n)|^2 + 2b'(x_n) \leq K_0, \forall n.$$

Therefore, $\sigma(x_0) = b(x_0) = 0$ and

$$|\sigma'(x_0)|^2 + 2b'(x_0) \geq c(x_0)$$

It is a contradiction to (3.8), so for any λ , there exists n_λ , such that the inequality (3.9) holds in D_λ for n_λ . As a consequence, Assumption 3.1.2 is indeed satisfied.

3.2 Auxiliary results

The following two remarks are reductions of Theorem 3.1.1.

Remark 3.2.1. Without loss of generality, we may assume that $c \geq 1$ and replace condition (4.10) by

$$\begin{aligned} & \left\| \sigma_{(y)}(x) + (\rho(x), y)\sigma(x) + \sigma(x)Q(x, y) \right\|^2 + \\ & 2(y, b_{(y)}(x) + 2(\rho(x), y)b(x)) \leq c(x) - 1 + M(x)(a(x)y, y). \end{aligned} \quad (3.10)$$

Indeed, letting $\tilde{u} = \frac{u}{\psi + 1}$ in D , we have

$$u_{x^i} = (\psi + 1)\tilde{u}_{x^i} + \psi_{x^i}\tilde{u}, \quad u_{x^i x^j} = (\psi + 1)\tilde{u}_{x^i x^j} + \psi_{x^j}\tilde{u}_{x^i} + \psi_{x^i}\tilde{u}_{x^j} + \psi_{x^i x^j}\tilde{u}$$

Hence (1.1) turns into

$$\begin{cases} \tilde{a}^{ij}(x)\tilde{u}_{x^i x^j} + \tilde{b}^i(x)\tilde{u}_{x^i} - \tilde{c}(x)\tilde{u} + f(x) = 0, & \text{in } D \\ \tilde{u} = \tilde{g} := g/(1 + \psi), & \text{on } \partial D \end{cases}$$

with

$$\tilde{a}^{ij} = (\psi + 1)a^{ij}, \quad \tilde{b}^i = 2a^{ij}\psi_{x^j} + (\psi + 1)b^i, \quad \tilde{c} = -L\psi + (1 + \psi)c.$$

Notice that $\tilde{\sigma}^{ij} = \sqrt{\psi + 1}\sigma^{ij}$. So a direct computation implies that

$$|\tilde{\sigma}^{ij}|_{2,D} + |\tilde{b}^i|_{2,D} + |\tilde{c}|_{2,D} + |\psi|_{4,D} \leq (d^2 + 2d + 2)K_0^3,$$

which plays the same role as (3.2).

Since $L\psi \leq -1$ and $c \geq 0$, $\tilde{c} \geq 1$.

We also have $(\tilde{a}_n, n) > 0$ on ∂D . Under the substitutions on σ , b and c , by inequality (4.10), we have

$$\begin{aligned} & \frac{1}{\psi + 1} \left\| \tilde{\sigma}_{(y)}(x) - \frac{1}{2} \frac{\psi(y)}{\psi + 1} \tilde{\sigma}(x) + (\rho(x), y) \tilde{\sigma}(x) + \tilde{\sigma}(x) Q(x, y) \right\|^2 \\ & \quad + \frac{2}{\psi + 1} \left(y, \tilde{b}_{(y)}(x) - \frac{\psi(y)}{\psi + 1} \tilde{b}(x) + 2(\rho(x), y) \tilde{b}(x) \right) \\ & \leq \frac{\tilde{c}(x) + L\psi}{\psi + 1} + M(x)(a(x)y, y) + 2 \left(y, \left(\frac{2a\psi_x}{\psi + 1} \right)_{(y)} + 2(\rho(x), y) \frac{2a\psi_x}{\psi + 1} \right). \end{aligned}$$

Collecting similar terms and noticing that $L\psi \leq -1$, we get

$$\begin{aligned} & \left\| \tilde{\sigma}_{(y)}(x) + (\tilde{\rho}(x), y) \tilde{\sigma}(x) + \tilde{\sigma}(x) Q(x, y) \right\|^2 + 2 \left(y, \tilde{b}_{(y)}(x) + 2(\tilde{\rho}(x), y) \tilde{b}(x) \right) \\ & \leq \tilde{c}(x) - 1 + \tilde{M}(x)(\tilde{a}(x)y, y) + 4(\tilde{a}_{(y)}(x)\psi_x, y), \end{aligned}$$

with

$$\tilde{\rho}(x) := \rho(x) - \frac{\psi_x}{2(\psi + 1)},$$

and $\tilde{M}(x)$ is in terms of $M(x)$, K_0 and $|\rho(x)|$.

The term $4(\tilde{a}_{(y)}\psi_x, y)$ can not be bounded by $\tilde{M}(x)(\tilde{a}(x)y, y)$. However, notice that

$$\tilde{a}_{(y)}(x) = \tilde{\sigma}(x)\tilde{\sigma}_{(y)}^*(x).$$

So $\tilde{M}(x)(\tilde{a}(x)y, y) + 4(\tilde{a}_{(y)}\psi_x, y)$ can be rewritten in the form of

$$\left(\tilde{\sigma}(x) \left(\frac{\tilde{M}(x)}{2} \tilde{\sigma}^*(x)y + 4\tilde{\sigma}_{(y)}^*(x)\psi_x \right), y \right),$$

which can play the same role as that of $M(x)(a(x)y, y)$, which, in the proof, will be rewritten in the form of

$$\left(\sigma(x) \cdot \frac{M(x)}{2} \sigma^*(x)y, y \right).$$

A direct computation shows that if \tilde{u} satisfies estimates (3.4) and (3.5), we have the same estimates for u .

Remark 3.2.2. Without loss of generality, we may assume that $u \in C^1(D)$ and $f, g \in C^1(\bar{D})$ when investigating first derivatives of u , and $u \in C^2(D)$ and $f, g \in C^2(\bar{D})$ when investigating second derivatives of u .

Let us take the first situation for example, in which u, f, g can be assumed to be of class C^1 . The second situation can be discussed by almost the same argument.

We define the process $x_t^\varepsilon(x)$ to be the solution to the equation

$$x_t = x_0 + \int_0^t \sigma(x_s) dw_s + \int_0^t \varepsilon I d\tilde{w}_s + \int_0^t b(x_s) ds$$

where \tilde{w}_t is a d -dimensional Wiener process independent of w_t and I is the identity matrix of size $d \times d$, and we define $\tau^\varepsilon(x)$ to be the first exit time of $x_t^\varepsilon(x)$ from D , then for the function

$$u^\varepsilon(x) := E \left[g(x_{\tau^\varepsilon}^\varepsilon(x)) e^{-\phi_{\tau^\varepsilon}^\varepsilon(x)} + \int_0^{\tau^\varepsilon(x)} f(x_t^\varepsilon(x)) e^{-\phi_t^\varepsilon} dt \right],$$

$$\text{with } \phi_t^\varepsilon := \int_0^t c(x_s^\varepsilon(x)) ds,$$

the relation $u^\varepsilon \rightarrow u$ holds as $\varepsilon \rightarrow 0$. Indeed, notice that

$$\begin{aligned} & E |g(x_{\tau^\varepsilon}^\varepsilon(x)) - g(x_\tau(x))| \\ & \leq KE (|x_{\tau^\varepsilon \wedge \tau}^\varepsilon(x) - x_{\tau \wedge \tau}^\varepsilon(x)| + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau) + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau)^{1/2}), \end{aligned}$$

$$\begin{aligned} & E |e^{-\phi_{\tau^\varepsilon}^\varepsilon} - e^{-\phi_\tau}| \leq E e^{-\tau^\varepsilon \wedge \tau} |\phi_{\tau^\varepsilon}^\varepsilon - \phi_\tau| \\ & \leq KE e^{-\tau^\varepsilon \wedge \tau} (\tau^\varepsilon \wedge \tau \cdot \sup_{t \leq \tau^\varepsilon \wedge \tau} |x_t^\varepsilon(x) - x_t(x)| + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau) + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau)^{1/2}) \\ & \leq KE (\sup_{t \leq \tau^\varepsilon \wedge \tau} |x_t^\varepsilon(x) - x_t(x)| + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau) + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau)^{1/2}), \end{aligned}$$

$$E \left| \int_0^{\tau^\varepsilon} f(x_t^\varepsilon(x)) e^{-\phi_t^\varepsilon} dt - \int_0^\tau f(x_t(x)) e^{-\phi_t} dt \right|$$

$$\begin{aligned}
&\leq E \int_0^{\tau^\varepsilon \wedge \tau} |f(x_t^\varepsilon(x))e^{-\phi_t^\varepsilon} - f(x_t(x))e^{-\phi_t}| dt \\
&\quad + KE(\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau) + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau)^{1/2}) \\
&\leq E \int_0^{\tau^\varepsilon \wedge \tau} K(|x_t^\varepsilon(x) - x_t(x)| + t \cdot \sup_{s \leq t} |x_s^\varepsilon(x) - x_s(x)|) e^{-t} dt \\
&\quad + KE(\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau)^{1/2}) \\
&\leq KE(\sup_{t \leq \tau^\varepsilon \wedge \tau} |x_t^\varepsilon(x) - x_t(x)| + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau) + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau)^{1/2}),
\end{aligned}$$

where K is a constant depending on $|g|_{0,1,D}, |f|_{0,1,D}$ and K_0 . It follows that

$$\begin{aligned}
|u^\varepsilon(x) - u(x)| &\leq KE(\sup_{t \leq \tau^\varepsilon \wedge \tau} |x_t^\varepsilon(x) - x_t(x)| + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau) + (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau)^{1/2}) \\
&\leq K \left(E \sup_{t \leq \tau^\varepsilon \wedge \tau \wedge T} |x_t^\varepsilon(x) - x_t(x)| + KP(\tau > T) + EI_1 + EI_2 + \sqrt{EI_1} + \sqrt{EI_2} \right),
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau) I_{\tau > \tau^\varepsilon} = (\tau - \tau^\varepsilon) I_{\tau > \tau^\varepsilon}, \\
I_2 &= (\tau^\varepsilon \vee \tau - \tau^\varepsilon \wedge \tau) I_{\tau < \tau^\varepsilon} = (\tau^\varepsilon - \tau) I_{\tau < \tau^\varepsilon}.
\end{aligned}$$

It remains to notice that

$$E \sup_{t \leq \tau^\varepsilon \wedge \tau \wedge T} |x_t(x) - x_t^\varepsilon(x)| \leq e^{KT} \varepsilon \rightarrow 0, \text{ as } \varepsilon \rightarrow 0,$$

$$P(\tau > T) \leq \frac{E\tau}{T} \leq \frac{1}{T} E \int_0^\tau (-L\psi(x_t(x))) dt = \frac{\psi(x) - \psi(x_\tau(x))}{T} \leq \frac{K_0}{T},$$

and

$$\begin{aligned}
E(\tau - \tau^\varepsilon) I_{\tau > \tau^\varepsilon} &= E \int_{\tau \wedge \tau^\varepsilon}^\tau 1 dt \\
&\leq -E \int_{\tau \wedge \tau^\varepsilon}^\tau L\psi(x_t(x)) dt \\
&= -E \left(\psi(x_\tau(x)) - \psi(x_{\tau^\varepsilon}(x)) \right) I_{\tau^\varepsilon < \tau} \\
&= E\psi(x_{\tau^\varepsilon}(x)) I_{\tau^\varepsilon < \tau} \\
&= E \left(\psi(x_{\tau^\varepsilon}(x)) - \psi(x_{\tau^\varepsilon}^\varepsilon(x)) \right) I_{\tau^\varepsilon < \tau} \\
&\leq E \left(\psi(x_{\tau^\varepsilon}(x)) - \psi(x_{\tau^\varepsilon}^\varepsilon(x)) \right) I_{\tau^\varepsilon < \tau \leq T} + 2K_0 P(\tau > T) \\
&\leq K_0 E \sup_{t \leq \tau^\varepsilon \wedge \tau \wedge T} |x_t(x) - x_t^\varepsilon(x)| + \frac{2K_0^2}{T}
\end{aligned}$$

$$\begin{aligned}
E(\tau^\varepsilon - \tau)I_{\tau < \tau^\varepsilon} &\leq -2E \int_{\tau \wedge \tau^\varepsilon}^{\tau^\varepsilon} L^\varepsilon \psi(x_t^\varepsilon(x)) dt \\
&\leq \dots \leq K_0 E \sup_{t \leq \tau^\varepsilon \wedge \tau \wedge T} |x_t(x) - x_t^\varepsilon(x)| + \frac{2K_0^2}{T}.
\end{aligned}$$

Hence by first letting $\varepsilon \downarrow 0$ and then $T \uparrow \infty$, we conclude that

$$|u^\varepsilon(x) - u(x)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Moreover, for small ε the condition (3.1) holds for 2ψ , taken instead of ψ and L^ε associated to the process $x_t^\varepsilon(x)$. The matrix σ^ε corresponding to the process $x_t^\varepsilon(x)$ is obtained by attaching the identity matrix, multiplied by ε , to the right of the original matrix σ . In this connection we modify $P(x, y)$ by adding zero entries on the right and below to form a $(d_1 + d) \times (d_1 + d)$ matrix. Then the condition (3.10) corresponding to the process $x_t^\varepsilon(x)$ will differ from the original condition by the fact that the term $\varepsilon^2(\rho(x), y)^2 d$ appears on the left, and $\frac{1}{2}M(x)\varepsilon^2$ on the right. From this it is clear that the condition (3.10) for the process $x_t^\varepsilon(x)$ (for all ε) also holds when $M(x)$ is replaced by $M(x) + 2|\rho(x)|^2 d$.

Finally, from analysis of PDE, we know that for $\varepsilon \neq 0$ the nondegenerate elliptic equation $L^\varepsilon w = 0$ in D with the boundary condition $w = g$ on ∂D has a solution that is continuous in \bar{D} and twice continuously differentiable in D , and $u^\varepsilon = w$ in D by Itô's formula. From this it follows that it suffices to prove the theorem for small $\varepsilon \neq 0$, the process $x_t^\varepsilon(x)$, and a function u^ε that is continuously differentiable in D . Of course, we must be sure that the constants N in (3.4) is chosen to be independent of ε , which is true as we can see in the proof of the theorem. Observing further that for each fixed $\varepsilon \neq 0$ the functions f and g can be uniformly approximated in \bar{D} by infinitely differentiable functions, in such a way that the last factor in (3.4) increases by at most a factor of two when f and g are replaced by the approximating functions, while for the latter the function w (i.e., u^ε) has continuous and bounded first derivatives in \bar{D} , we conclude that we may assume u has continuous first derivatives in D and $f, g \in C^1(\bar{D})$ when investigating first derivatives of u .

Before proving the theorem, let us prove four lemmas. In Lemma 3.2.1 we estimate the first exit time. It is a well-known result, but we still prove for the sake of completeness. Lemma 3.2.2 concerns the estimate of the first derivative along the normal to

the boundary, to be used when estimating the second derivatives. In Lemma 3.2.3 and Lemma 3.2.4, we construct two supermartingales, which will play the roles of barriers near the boundary and in the interior of the domain, respectively.

Lemma 3.2.1. *Let $\tau_{D_0}(x)$ be the first exit time of $x_t(x)$ from D_0 , which is a sub-domain of D containing x . Then we have*

$$E\tau_{D_0}(x) \leq E\tau_D(x) \leq \psi(x) \leq |\psi|_{0,D},$$

$$E\tau_{D_0}^2(x) \leq E\tau_D^2(x) \leq 2|\psi|_{0,D}\psi(x) \leq 2|\psi|_{0,D}^2.$$

Proof. The fact that $D_0 \subset D$ implies $E\tau_{D_0}(x) \leq E\tau_D(x)$ and $E\tau_{D_0}^2(x) \leq E\tau_D^2(x)$. Now we abbreviate $\tau_D(x)$ by $\tau(x)$, or simply τ when this will cause no confusion. By (3.1) and Itô's formula, we have

$$\begin{aligned} E\tau &= E \int_0^\tau 1 dt \leq -E \int_0^\tau L\psi dt = \psi(x) - E\psi(x_\tau) = \psi(x), \\ E\tau^2 &= 2E \int_0^\infty (\tau - t) I_{\tau > t} dt = 2E \int_0^\infty I_{\tau > t} E\tau(x_t) dt \\ &\leq 2 \sup_{y \in D} E\tau(y) \cdot E \int_0^\infty I_{\tau > t} dt = 2 \sup_{y \in D} E\tau(y) \cdot E\tau \leq 2|\psi|_{0,D}\psi(x). \end{aligned}$$

□

Lemma 3.2.2. *If $f, g \in C^2(\bar{D})$, and $u \in C^1(\bar{D})$, then for any $y \in \partial D$ we have*

$$|u_{(n)}(y)| \leq K(|g|_{2,D} + |f|_{0,D}), \quad (3.11)$$

where n is the unit inward normal on ∂D and the constant K depends only on K_0 .

Proof. Fix a $y \in \partial D$, and choose $\varepsilon_0 > 0$ so that $y + \varepsilon n \in D$ as long as $0 < \varepsilon \leq \varepsilon_0$. Also, fix an $\varepsilon \in (0, \varepsilon_0]$ and let $x := y + \varepsilon n$. By Itô's formula,

$$d(g(x_t)e^{-\phi t}) = e^{-\phi t} g_{(\sigma^k)}(x_t) dw_t^k + e^{-\phi t} (Lg(x_t) - c(x_t)g(x_t)) dt,$$

$$d(u(x_t)e^{-\phi t}) = e^{-\phi t} u_{(\sigma^k)}(x_t) dw_t^k + e^{-\phi t} f(x_t) dt.$$

Notice that

$$E \int_0^\infty \left(e^{-\phi t} g_{(\sigma^k)}(x_t) \right)^2 I_{t \leq \tau} dt \leq N |g|_{1,D}^2 E\tau < \infty,$$

$$E \int_0^\infty \left(e^{-\phi t} u_{(\sigma^k)}(x_t) \right)^2 I_{t \leq \tau} dt \leq N |u|_{1,D}^2 E\tau < \infty.$$

The Wald identities hold:

$$E \int_0^\tau e^{-\phi t} g_{(\sigma^k)}(x_t) dw_t^k = 0, \quad E \int_0^\tau e^{-\phi t} u_{(\sigma^k)}(x_t) dw_t^k = 0.$$

Thus

$$\begin{aligned} u(x) &= g(x) + E \int_0^\tau e^{-\phi t} (Lg(x_t(x)) - c(x_t(x))g(x_t(x))) dt \\ &\quad + E \int_0^\tau f(x_t(x)) e^{-\phi t} dt \\ &\leq g(x) + (|Lg|_{0,D} + |c|_{0,D}|g|_{0,D} + |f|_{0,D}) E\tau \\ &\leq g(x) + K(|g|_{2,D} + |f|_{0,D}) \psi(x). \end{aligned}$$

Notice that $u(y) = g(y)$ and $\psi(y) = 0$. So we have

$$\frac{u(y + \varepsilon n) - u(y)}{\varepsilon} \leq \frac{g(y + \varepsilon n) - g(y)}{\varepsilon} + K(|g|_{2,D} + |f|_{0,D}) \frac{\psi(y + \varepsilon n) - \psi(y)}{\varepsilon}.$$

Letting $\varepsilon \downarrow 0$, we get

$$u_{(n)}(y) \leq K(|g|_{2,D} + |f|_{0,D}).$$

Replacing u with $-u$ yields the same estimate of $(-u)_{(n)}$ from above, which is an estimate of $u_{(n)}$ from below. Combining the estimates from above and from below leads to (4.46) and proves the lemma. \square

For constants δ and λ , such that $0 < \delta < \lambda^2 < \lambda < 1$, define

$$\begin{aligned} D_\delta &= \{x \in D : \delta < \psi\}, \\ D^\lambda &= \{x \in D : \psi < \lambda\}, \\ D_\delta^\lambda &= \{x \in D : \delta < \psi < \lambda\}, \\ D_{\lambda^2} &= \{x \in D : \lambda^2 < \psi\}. \end{aligned}$$

Lemma 3.2.3. *Introduce*

$$\varphi(x) = \lambda^2 + \psi \left(1 - \frac{1}{4\lambda} \psi\right), \quad B_1(x, \xi) = [\lambda + \sqrt{\psi}(1 + \sqrt{\psi})] |\xi|^2 + K_1 \varphi^{\frac{3}{2}} \frac{\psi^2(\xi)}{\psi},$$

where $K_1 \in [1, \infty)$ is a constant depending only on K_0 .

In D^λ , if we construct first and second quasiderivatives by (2.4) and (2.5), in which

$$r(x, \xi) := \rho(x, \xi) + \frac{\psi(\xi)}{\psi}, \quad r_t := r(x_t, \xi_t),$$

$$\text{where } \rho(x, \xi) := -\frac{1}{A} \sum_{k=1}^{d_1} \psi_{(\sigma^k)}(\psi_{(\sigma^k)})(\xi), \quad \text{with } A := \sum_{k=1}^{d_1} \psi_{(\sigma^k)}^2;$$

$$\hat{r}(x, \xi) := \frac{\psi^2(\xi)}{\psi^2}, \quad \hat{r}_t := \hat{r}(x_t, \xi_t);$$

$$\pi^k(x, \xi) := \frac{2\psi_{(\sigma^k)}\psi(\xi)}{\varphi\psi}, \quad k = 1, \dots, d_1, \quad \pi_t := \pi(x_t, \xi_t);$$

$$P^{ik}(x, \xi) := \frac{1}{A} [\psi_{(\sigma^k)}(\psi_{(\sigma^i)})(\xi) - \psi_{(\sigma^i)}(\psi_{(\sigma^k)})(\xi)], \quad i, k = 1, \dots, d_1, \quad P_t := P(x_t, \xi_t);$$

$$\hat{\pi}_t^k = \hat{P}_t^{ik} = 0, \quad \forall i, k = 1, \dots, d_1, \forall t \in [0, \infty).$$

Then for sufficiently small λ , when $x_0 \in D_\delta^\lambda$, $\xi_0 \in \mathbb{R}^d$ and $\eta_0 = 0$, we have

1. $B_1(x_t, \xi_t)$ and $\sqrt{B_1(x_t, \xi_t)}$ are local supermartingales on $[0, \tau_1^\delta]$, where $\tau_1^\delta = \tau_{D_\delta^\lambda}(x_0)$;

2. $E \int_0^{\tau_1^\delta} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} dt \leq NB_1(x_0, \xi_0)$;

3. $E \sup_{t \leq \tau_1^\delta} |\xi_t|^2 \leq NB_1(x_0, \xi_0)$;

4. $E |\eta_{\tau_1^\delta}| \leq E \sup_{t \leq \tau_1^\delta} |\eta_t| \leq NB_1(x_0, \xi_0)$;

5. $E \left(\int_0^{\tau_1^\delta} |\eta_t|^2 dt \right)^{\frac{1}{2}} \leq NB_1(x_0, \xi_0)$;

where N is a constant depending on K_0 and λ .

Proof. Throughout the proof, keep in mind that the constant K depend only on K_0 , while the constants $N \in [1, \infty)$ and $\lambda_0 \in (0, 1)$ depend on K_0 and λ .

First, notice that, on ∂D , we have

$$A = \sum_{k=1}^{d_1} \psi_{(\sigma^k)}^2 = 2(a\psi_x, \psi_x) = 2|\psi_x|(an, n) \geq 2\delta,$$

where the constant $\delta > 0$, because of the compactness of ∂D . Replacing ψ by $\psi/2\delta$ if needed, we may, therefore, assume that $A \geq 1$.

By Itô's formula, for $t < \tau_1^\delta$, we have

$$d\psi_{(\xi_t)} = [(\psi_{(\sigma^i)})_{(\xi_t)} + r_t\psi_{(\sigma^i)} + \psi_{(\sigma^k)}P_t^{ki}]dw_t^i + [(L\psi)_{(\xi_t)} + 2r_tL\psi - \psi_{(\sigma^i)}\pi_t^i]dt.$$

A crucial fact about this equation is that owing to our choice of r and P

$$(\psi_{(\sigma^i)})_{(\xi_t)} + r_t\psi_{(\sigma^i)} + \psi_{(\sigma^k)}P_t^{ki} = \frac{\psi_{(\xi_t)}}{\psi}\psi_{(\sigma^i)}.$$

Thus

$$d\psi_{(\xi_t)} = \frac{\psi_{(\xi_t)}}{\psi}\psi_{(\sigma^i)}dw_t^i + [(L\psi)_{(\xi_t)} + 2r_tL\psi - \psi_{(\sigma^i)}\pi_t^i]dt. \quad (3.12)$$

Let

$$\bar{\sigma} := \sigma_{(\xi)} + r\sigma + \sigma P, \quad \bar{b} := b_{(\xi)} + 2rb.$$

We have

$$\|\bar{\sigma}\| \leq K(|\xi| + \frac{|\psi_{(\xi)}|}{\psi}), \quad (3.13)$$

$$|\bar{b}| \leq K(|\xi| + \frac{|\psi_{(\xi)}|}{\psi}). \quad (3.14)$$

By Itô's formula,

$$dB_1(x_t, \xi_t) = \Gamma_1(x_t, \xi_t)dt + \Lambda_1^k(x_t, \xi_t)dw_t^k \quad (3.15)$$

with

$$\Gamma_1(x, \xi) = I_1 + I_2 + \dots + I_{13}$$

where

$$I_1 = \lambda[2(\xi, \bar{b}) + \|\bar{\sigma}\|^2] \leq \lambda K(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi^2}) \leq K\lambda^{\frac{5}{2}}\frac{|\xi|^2}{\psi^{\frac{3}{2}}} + K\varphi^{\frac{1}{2}}\frac{\psi_{(\xi)}^2}{\psi^2},$$

here we apply (3.13), (3.14) and $\lambda \leq \varphi^{\frac{1}{2}}$,

$$\begin{aligned} I_2 &= -\lambda 2(\xi, \sigma^k)\pi^k \leq \frac{K\lambda|\xi||\psi_{(\xi)}|}{\varphi\psi} \leq \frac{\lambda^2|\xi|^2}{32 \cdot 2^{\frac{3}{2}}\varphi^{\frac{5}{2}}} + \frac{K\varphi^{\frac{1}{2}}\psi_{(\xi)}^2}{\psi^2} \\ &\leq \frac{|\xi|^2}{32 \cdot 2^{\frac{3}{2}}\varphi^{\frac{3}{2}}} + \frac{K\varphi^{\frac{1}{2}}\psi_{(\xi)}^2}{\psi^2} \leq \frac{|\xi|^2}{32\psi^{\frac{3}{2}}} + K\varphi^{\frac{1}{2}}\frac{\psi_{(\xi)}^2}{\psi^2}, \end{aligned}$$

here we apply $\lambda^2 \leq \varphi$, and then observe that $\psi \leq 2\varphi$,

$$I_3 = \sqrt{\psi}(1 + \sqrt{\psi})2(\xi, \bar{b}) \leq \sqrt{\psi}K|\xi|(|\xi| + \frac{|\psi_{(\xi)}|}{\psi}) \leq K\lambda\frac{|\xi|^2}{\psi^{\frac{3}{2}}},$$

here we apply (3.14),

$$\begin{aligned} I_4 &= -\sqrt{\psi}(1 + \sqrt{\psi})2(\xi, \sigma^k)\pi^k \leq \frac{K\sqrt{\psi}|\xi||\psi(\xi)|}{\varphi\psi} \leq \frac{\psi|\xi|^2}{32 \cdot 2^{\frac{5}{2}}\varphi^{\frac{5}{2}}} + \frac{K\varphi^{\frac{1}{2}}\psi^2(\xi)}{\psi^2} \\ &\leq \frac{|\xi|^2}{32\psi^{\frac{3}{2}}} + K\varphi^{\frac{1}{2}}\frac{\psi^2(\xi)}{\psi^2}, \end{aligned}$$

here we observe that $\psi \leq 2\varphi$,

$$I_5 = \sqrt{\psi}(1 + \sqrt{\psi})\|\bar{\sigma}\|^2 \leq K\sqrt{\psi}(|\xi|^2 + \frac{\psi^2(\xi)}{\psi^2}) \leq K\lambda^2\frac{|\xi|^2}{\psi^{\frac{3}{2}}} + K\varphi^{\frac{1}{2}}\frac{\psi^2(\xi)}{\psi^2},$$

here we apply (3.13),

$$I_6 = (1 + 2\sqrt{\psi})|\xi|^2\left[\frac{L\psi}{2\sqrt{\psi}} - \frac{A}{8\psi^{\frac{3}{2}}}\right] \leq -\frac{|\xi|^2}{8\psi^{\frac{3}{2}}},$$

$$I_7 = \frac{A}{4\psi}|\xi|^2 \leq K\frac{|\xi|^2}{\psi} \leq K\sqrt{\lambda}\frac{|\xi|^2}{\psi^{\frac{3}{2}}},$$

$$I_8 = (1 + 2\sqrt{\psi})\frac{\psi(\sigma^k)}{\sqrt{\psi}}(\xi, \bar{\sigma}^k) \leq K\frac{|\xi|}{\sqrt{\psi}}(|\xi| + \frac{|\psi(\xi)|}{\psi}) \leq K\lambda\frac{|\xi|^2}{\psi^{\frac{3}{2}}} + \frac{|\xi|^2}{32\psi^{\frac{3}{2}}} + K\varphi^{\frac{1}{2}}\frac{\psi^2(\xi)}{\psi^2},$$

here we apply (3.13) and $\psi \leq 2\varphi$,

$$I_9 = K_1\frac{3}{2}\varphi^{\frac{1}{2}}\left[\left(1 - \frac{\psi}{2\lambda}\right)L\psi - \frac{A}{4\lambda}\right]\frac{\psi^2(\xi)}{\psi} + K_1\varphi^{\frac{3}{2}}3\frac{\psi^2(\xi)}{\psi^2}L\psi \leq 0,$$

$$I_{10} = K_1\frac{3}{8}\varphi^{-\frac{1}{2}}\left(1 - \frac{\psi}{2\lambda}\right)^2A\frac{\psi^2(\xi)}{\psi} \leq K_1\frac{3}{8}\frac{\psi}{\varphi^{\frac{1}{2}}}A\frac{\psi^2(\xi)}{\psi^2} \leq K_1\frac{3}{4}\varphi^{\frac{1}{2}}A\frac{\psi^2(\xi)}{\psi^2},$$

here we use $\psi \leq 2\varphi$,

$$\begin{aligned} I_{11} &= K_1\varphi^{\frac{3}{2}}2\frac{\psi(\xi)}{\psi}[(L\psi)_{(\xi)} + 2\rho L\psi] \leq K_1K\varphi^{\frac{3}{2}}\frac{|\psi(\xi)|}{\psi}|\xi| \leq K_1K\lambda\varphi^{\frac{1}{2}}\frac{|\psi(\xi)|}{\psi}|\xi| \\ &\leq K_1\lambda\varphi^{\frac{1}{2}}\frac{\psi^2(\xi)}{\psi^2} + K_1K\lambda^3\frac{|\xi|^2}{\psi^{\frac{3}{2}}}, \end{aligned}$$

here we first notice that $\varphi \leq 2\lambda$, and then apply $\psi \leq 2\varphi$,

$$I_{12} = -K_1\varphi^{\frac{3}{2}}\frac{4}{\varphi}A\frac{\psi^2(\xi)}{\psi^2} = -K_14\varphi^{\frac{1}{2}}A\frac{\psi^2(\xi)}{\psi^2},$$

$$I_{13} = K_1\frac{3}{2}\varphi^{\frac{1}{2}}\left(1 - \frac{\psi}{2\lambda}\right)A\frac{\psi^2(\xi)}{\psi^2} \leq K_1\frac{3}{2}\varphi^{\frac{1}{2}}A\frac{\psi^2(\xi)}{\psi^2}.$$

Collecting our estimates above we see that, when $x \in D_\delta^\lambda$,

$$\begin{aligned} \Gamma_1(x, \xi) &\leq \left[K(\lambda^{\frac{5}{2}} + \sqrt{\lambda}) + K_1 K \lambda^3 + \left(\frac{3}{32} - \frac{1}{8} \right) \right] \frac{|\xi|^2}{\psi^{\frac{3}{2}}} \\ &\quad + \left[K + K_1 \lambda + K_1 A \left(\frac{3}{4} + \frac{3}{2} - 4 \right) \right] \varphi^{\frac{1}{2}} \frac{\psi^2(\xi)}{\psi^2}. \end{aligned}$$

Recall that K and K_1 depend only on K_0 . By first choosing K_1 such that $K_1 \geq K$, then letting λ be sufficiently small, we get

$$\Gamma_1(x, \xi) \leq -\frac{1}{64} \frac{|\xi|^2}{\psi^{\frac{3}{2}}} - \frac{1}{2} \varphi^{\frac{1}{2}} \frac{\psi^2(\xi)}{\psi^2} \leq -\frac{1}{64\lambda^{\frac{3}{2}}} |\xi|^2 - \frac{\lambda}{2} \frac{\psi^2(\xi)}{\psi^2} \leq 0. \quad (3.16)$$

It follows that $B_1(x_t, \xi_t)$ is a local supermartingale on $[0, \tau_1^\delta]$.

Also, notice that $f(x) = \sqrt{x}$ is concave, so $\sqrt{B_1(x_t, \xi_t)}$ is a local supermartingale on $[0, \tau_1^\delta]$. Thus (1) is proved.

From (3.16), there exists a sufficiently small positive λ_0 , such that

$$\Gamma_1(x, \xi) + \lambda_0 \left(|\xi|^2 + \frac{\psi^2(\xi)}{\psi^2} \right) \leq 0, \forall x \in D_\delta^\lambda.$$

Therefore,

$$\begin{aligned} \lambda_0 E \int_0^{\tau_1^\delta} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} dt &\leq -E \int_0^{\tau_1^\delta} \Gamma_1(x_t, \xi_t) \\ &= B_1(x_0, \xi_0) - EB_1(x_{\tau_1^\delta}, \xi_{\tau_1^\delta}) \leq B_1(x_0, \xi_0), \end{aligned}$$

which proves (2).

Since

$$|\xi_t|^2 = |\xi_0|^2 + \int_0^t 2(\xi_s, \bar{b}) + \|\bar{\sigma}\|^2 ds + \int_0^t 2(\xi_s, \bar{\sigma}) dw_s,$$

by Burkholder-Davis-Gundy inequality, for $\tau_n = \tau_1^\delta \wedge \inf\{t \geq 0 : |\xi_t| \geq n\}$, we have,

$$\begin{aligned} E \sup_{t \leq \tau_n} |\xi_t|^2 &\leq |\xi_0|^2 + \int_0^{\tau_n} \left(2|\xi_t| \cdot |\bar{b}| + \|\bar{\sigma}\|^2 \right) dt + 6E \left(\int_0^{\tau_n} |(\xi_t, \bar{\sigma})|^2 \right)^{\frac{1}{2}} \\ &\leq |\xi_0|^2 + NE \int_0^{\tau_n} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} dt + E \left(\int_0^{\tau_n} N |\xi_t|^2 \left(|\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} \right) dt \right)^{\frac{1}{2}} \\ &\leq NB_1(x_0, \xi_0) + E \left[\sup_{t \leq \tau_n} |\xi_t| \cdot \left(\int_0^{\tau_n} N \left(|\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} \right) dt \right)^{\frac{1}{2}} \right] \end{aligned}$$

$$\begin{aligned}
&\leq NB_1(x_0, \xi_0) + \frac{1}{2}E \sup_{t \leq \tau_n} |\xi_t|^2 + \frac{1}{2}E \left(\int_0^{\tau_n} N(|\xi_t|^2 + \frac{\psi_{(\xi_t)}^2}{\psi^2}) dt \right) \\
&\leq NB_1(x_0, \xi_0) + \frac{1}{2}E \sup_{t \leq \tau_n} |\xi_t|^2,
\end{aligned}$$

which implies that

$$E \sup_{t \leq \tau_n} |\xi_t|^2 \leq NB_1(x_0, \xi_0).$$

Now (3) is obtained by letting $n \rightarrow \infty$.

Now we estimate the moments of second quasiderivative η_t . Based on our definition, we have

$$d\eta_t = [\sigma(\eta_t) + G(x_t, \xi_t)]dw_t + [b(\eta_t) + H(x_t, \xi_t)]dt,$$

with

$$\begin{aligned}
G(x, \xi) &= \sigma_{(\xi)}(\xi) + 2r\sigma_{(\xi)} + (2\sigma_{(\xi)} + 2r\sigma + \sigma P)P + (\hat{r} - r^2)\sigma, \\
H(x, \xi) &= b_{(\xi)}(\xi) + 4rb_{(\xi)} + 2\frac{\psi_{(\xi)}^2}{\psi^2}b.
\end{aligned}$$

Therefore, we have the estimates

$$\|G\| \leq N|\xi|(|\xi| + \frac{|\psi_{(\xi)}|}{\psi}), \quad |H| \leq N(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi^2}).$$

Itô's formula implies

$$d(|\eta_t|^2 e^{2\varphi}) = \theta(x_t, \xi_t, \eta_t)dt + \mu^k(x_t, \xi_t, \eta_t)dw_t^k,$$

where

$$\begin{aligned}
\theta(x, \xi, \eta) &= e^{2\varphi} \left\{ 2|\eta|^2 \left[\left(1 - \frac{\psi}{2\lambda}\right)L\psi - \frac{A}{4\lambda} + \left(1 - \frac{\psi}{2\lambda}\right)^2 A \right] + \|\sigma(\eta) + G(x, \xi)\|^2 \right. \\
&\quad \left. + 2(\eta, b(\eta) + H(x, \xi)) + 2(\eta, \sigma(\eta) + G(x, \xi)) \left[\left(1 - \frac{\psi}{2\lambda}\right)\psi_{(\sigma^k)} \right] \right\}.
\end{aligned}$$

It is not hard to see that, for any $x \in D_\delta^\lambda$,

$$\theta(x, \xi, \eta) \leq e^{2\varphi} \left\{ 2\left(1 - \frac{1}{4\lambda}\right)A|\eta|^2 + N \left[|\eta|^2 + |\xi|^2 \left(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi^2} \right) + |\eta| \left(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi^2} \right) \right] \right\}.$$

So for sufficiently small λ , we have

$$\theta(x, \xi, \eta) + \lambda_0|\eta|^2 \leq Ne^{2\varphi} (|\xi|^2 + |\eta|) \left(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi^2} \right).$$

Then for any bounded stopping time γ with respect to $\{\mathcal{F}_t\}$, we have

$$E(e^{2\varphi}|\eta_\gamma|^2) + \lambda_0 E \int_0^\gamma |\eta_t|^2 dt \leq E \int_0^\gamma N e^{2\varphi} (|\xi_t|^2 + |\eta_t|) \left(|\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} \right) dt.$$

Let $\tau_n = \tau_1^\delta \wedge \inf\{t \geq 0 : e^\varphi |\eta_t| \geq n\}$. Recall that $\eta_0 = 0$. By Theorem III.6.8 in [17], we have

$$\begin{aligned} & E \sup_{t \leq \tau_n} (e^\varphi |\eta_t|) \\ & \leq 3E \left(\int_0^{\tau_n} N e^{2\varphi} (|\xi_t|^2 + |\eta_t|) \left(|\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} \right) dt \right)^{\frac{1}{2}} \\ & \leq E \left[\left(\int_0^{\tau_n} 9N e^{2\varphi} |\xi_t|^2 \left(|\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} \right) dt \right)^{\frac{1}{2}} + \left(\int_0^{\tau_n} 9N e^{2\varphi} |\eta_t| \left(|\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} \right) dt \right)^{\frac{1}{2}} \right] \\ & \leq E \left[N \sup_{t \leq \tau_n} |\xi_t| \cdot \left(\int_0^{\tau_n} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} dt \right)^{\frac{1}{2}} + \sup_{t \leq \tau_n} \sqrt{e^\varphi |\eta_t|} \cdot \left(\int_0^{\tau_n} N (|\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2}) dt \right)^{\frac{1}{2}} \right] \\ & \leq NE \sup_{t \leq \tau_n} |\xi_t|^2 + NE \int_0^{\tau_n} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} dt + \frac{1}{2} E \sup_{t \leq \tau_n} (e^\varphi |\eta_t|) \\ & \leq \frac{1}{2} E \sup_{t \leq \tau_n} (e^\varphi |\eta_t|) + NB_1(x_0, \xi_0), \end{aligned}$$

which implies that

$$\begin{aligned} E \sup_{t \leq \tau_n} |\eta_t| & \leq E \sup_{t \leq \tau_n} (e^\varphi |\eta_t|) \leq NB_1(x_0, \xi_0), \\ E \left(\int_0^{\tau_n} |\eta_t|^2 dt \right)^{\frac{1}{2}} & \leq NB_1(x_0, \xi_0). \end{aligned}$$

Letting $n \rightarrow \infty$, we conclude that (4) and (5) are true. \square

Lemma 3.2.4. *Introduce*

$$B_2(x, \xi) = \lambda^{\frac{3}{4}} |\xi|^2.$$

If we construct first and second quasiderivatives by (2.4) and (2.5), in which

$$\begin{aligned} r(x, y) & := (\rho(x), y), \quad r_t := r(x_t, \xi_t), \quad \hat{r}_t := r(x_t, \eta_t), \\ \pi(x, y) & := \frac{M(x)}{2} \sigma^*(x) y, \quad \pi_t := \pi(x_t, \xi_t), \quad \hat{\pi}_t := \pi(x_t, \eta_t), \\ P(x, y) & := Q(x, y), \quad P_t := P(x_t, \xi_t), \quad \hat{P}_t := P(x_t, \eta_t). \end{aligned}$$

Then for sufficiently small λ , when $x_0 \in D_{\lambda^2}$, $\xi_0 \in \mathbb{R}^d$ and $\eta_0 = 0$, we have

1. $e^{-\phi_t} \mathbf{B}_2(x_t, \xi_t)$ and $\sqrt{e^{-\phi_t} \mathbf{B}_2(x_t, \xi_t)}$ are local supermartingales on $[0, \tau_2)$, where $\tau_2 = \tau_{D_{\lambda^2}}(x)$;
2. $E \int_0^{\tau_2} e^{-\phi_t} |\xi_t|^2 dt \leq N \mathbf{B}_2(x_0, \xi_0)$;
3. $E \sup_{t \leq \tau_2} e^{-\phi_t} |\xi_t|^2 \leq N \mathbf{B}_2(x_0, \xi_0)$;
4. $E e^{-\phi_{\tau_2}} |\eta_{\tau_2}| \leq E \sup_{t \leq \tau_2} e^{-\phi_t} |\eta_t| \leq N \mathbf{B}_2(x_0, \xi_0)$;
5. $E \left(\int_0^{\tau_2} e^{-2\phi_t} |\eta_t|^2 dt \right)^{\frac{1}{2}} \leq N \mathbf{B}_2(x_0, \xi_0)$;
6. The above inequalities are still all true if we replace ϕ_t by $\phi_t - \frac{1}{2}t$. More precisely, we have

$$E \int_0^{\tau_2} e^{-\phi_t + \frac{1}{2}t} |\xi_t|^2 dt \leq N \mathbf{B}_2(x_0, \xi_0), \quad E \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2}t} |\xi_t|^2 \leq N \mathbf{B}_2(x_0, \xi_0),$$

$$E \left(\int_0^{\tau_2} e^{-2\phi_t + t} |\eta_t|^2 dt \right)^{\frac{1}{2}} \leq N \mathbf{B}_2(x_0, \xi_0), \quad E \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2}t} |\eta_t| \leq N \mathbf{B}_2(x_0, \xi_0),$$

where N is constant depending on K_0 and λ .

Proof. First of all, replacing K_0 by

$$\max \left\{ K_0, \sup_{x \in D_{\lambda^2}} |\rho(x)|, \sup_{x \in D_{\lambda^2}, y \in \mathbb{R}^d} \frac{\|Q(x, y)\|}{|y|}, \sup_{x \in D_{\lambda^2}} M(x) \right\},$$

we may assume that

$$\sup_{x \in D_{\lambda^2}} |\rho(x)| \leq K_0, \quad \sup_{x \in D_{\lambda^2}} \|Q(x, y)\| \leq K_0 |y|, \quad \forall y \in \mathbb{R}^d, \quad \sup_{x \in D_{\lambda^2}} M(x) \leq K_0.$$

By Itô's formula, for $t < \tau_2$, we have

$$d|\xi_t|^2 = \Lambda_2^k(x_t, \xi_t) dw_t^k + \Gamma_2(x_t, \xi_t) dt,$$

where

$$\Lambda_2(x, \xi) = 2(\xi_t, \sigma(\xi_t) + r_t \sigma + \sigma P_t),$$

$$\Gamma_2(x, \xi) = [2(\xi, b(\xi) + 2rb - \sigma\pi) + \|\sigma(\xi) + r\sigma + \sigma P\|^2] \leq (c-1)|\xi|^2.$$

So

$$\begin{aligned} d(e^{-\phi t}|\xi_t|^2) &= e^{-\phi t} [\Gamma_2(x_t, \xi_t) - c(x_t)|\xi_t|^2] dt + dm_t \\ &\leq -e^{-\phi t}|\xi_t|^2 dt + dm_t, \end{aligned} \quad (3.17)$$

where m_t is a local martingale.

Thus $e^{-\phi t}B_2(x_t, \xi_t)$ is a local supermartingale on $[0, \tau_2)$.

Also, notice that $f(x) = \sqrt{x}$ is concave, so $\sqrt{e^{-\phi t}B_2(x_t, \xi_t)}$ is a local supermartingale on $[0, \tau_2]$. (1) is proved.

From (3.17), we also have

$$E \int_0^{\tau_2} e^{-\phi t} |\xi_t|^2 dt = B_2(x_0, \xi_0) - E e^{-\phi \tau_2} B_2(x_{\tau_2}, \xi_{\tau_2}) \leq B_2(x_0, \xi_0),$$

which proves (2).

Since

$$e^{-\phi t} |\xi_t|^2 = |\xi_0|^2 + \int_0^t e^{-\phi s} [2(\xi_s, \bar{b}) + \|\bar{\sigma}\|^2 - c|\xi_s|^2] ds + \int_0^t e^{-\phi s} 2(\xi_s, \bar{\sigma}) dw_s,$$

by Burkholder-Davis-Gundy inequality, for $\tau_n = \tau_2 \wedge \inf\{t \geq 0 : |\xi_t| \geq n\}$, we have,

$$\begin{aligned} E \sup_{t \leq \tau_n} e^{-\phi t} |\xi_t|^2 &\leq |\xi_0|^2 + \int_0^{\tau_n} e^{-\phi t} [2|\xi_t| \cdot |\bar{b}| + \|\bar{\sigma}\|^2 + c|\xi_t|^2] dt \\ &\quad + 12E \left(\int_0^{\tau_n} e^{-2\phi t} |(\xi_t, \bar{\sigma})|^2 dt \right)^{\frac{1}{2}} \\ &\leq |\xi_0|^2 + NE \int_0^{\tau_n} e^{-\phi t} |\xi_t|^2 dt + E \left(\int_0^{\tau_n} N e^{-2\phi t} |\xi_t|^4 dt \right)^{\frac{1}{2}} \\ &\leq NB_2(x_0, \xi_0) + E \left[\sup_{t \leq \tau_n} e^{-\frac{1}{2}\phi t} |\xi_t| \cdot \left(\int_0^{\tau_n} N e^{-\phi t} |\xi_t|^2 dt \right)^{\frac{1}{2}} \right] \\ &\leq NB_2(x_0, \xi_0) + \frac{1}{2} E \sup_{t \leq \tau_n} e^{-\phi t} |\xi_t|^2 + \frac{1}{2} E \left(\int_0^{\tau_n} N e^{-\phi t} |\xi_t|^2 dt \right) \\ &\leq NB_2(x_0, \xi_0) + \frac{1}{2} E \sup_{t \leq \tau_n} e^{-\phi t} |\xi_t|^2, \end{aligned}$$

which implies that

$$E \sup_{t \leq \tau_n} e^{-\phi t} |\xi_t|^2 \leq NB_2(x_0, \xi_0).$$

So (3) is true by letting $n \rightarrow \infty$.

Now we estimate the moments of the second quasiderivative η_t . Based on our definition, we have

$$d\eta_t = [\tilde{\sigma} + G]dw_t + [\tilde{b} + H]dt,$$

where

$$\begin{aligned}\tilde{\sigma} &= \bar{\sigma}(x, \eta) = \sigma_{(\eta)} + \hat{r}\sigma + \sigma\hat{P}, \\ \tilde{b} &= \bar{b}(x, \eta) = b_{(\eta)} + 2\hat{r}b - \sigma\hat{\pi}, \\ G &= G(x, \xi) = \sigma_{(\xi)}(\xi) + 2r\sigma_{(\xi)} - r^2\sigma + (2\sigma_{(\xi)} + 2r\sigma + \sigma P)P, \\ H &= H(x, \xi) = b_{(\xi)}(\xi) + 4rb_{(\xi)} - 2(\sigma_{(\xi)} + r\sigma - \sigma P)\pi.\end{aligned}$$

From the expressions above, we have the estimates

$$\|G\| \leq N|\xi|^2, \quad |H| \leq N|\xi|^2.$$

Hence Itô's formula implies

$$d(e^{-2\phi t}|\eta_t|^2) = e^{-2\phi t} [2(\eta_t, \tilde{b} + H) + \|\tilde{\sigma} + G\|^2 - 2c|\eta_t|^2] dt + 2e^{-2\phi t}(\eta_t, \tilde{\sigma} + G)dw_t^k.$$

Notice that

$$\begin{aligned}& 2(\eta, \tilde{b} + H) + \|\tilde{\sigma} + G\|^2 - 2c|\eta|^2 \\ &= 2(\eta, \tilde{b}) + \|\tilde{\sigma}\|^2 - 2c|\eta|^2 + 2(\eta, H) + |H|^2 + 2(\tilde{\sigma}^k, G^k) \\ &\leq (c-1)|\eta|^2 - 2c|\eta|^2 + |\eta|^2 + N|\xi|^4 \\ &\leq -|\eta|^2 + N|\xi|^4.\end{aligned}$$

So for any bounded stopping time γ with respect to $\{\mathcal{F}_t\}$, we have

$$Ee^{-2\phi\gamma}|\eta_\gamma|^2 + E \int_0^\gamma e^{-2\phi t}|\eta_t|^2 dt \leq E \int_0^\gamma Ne^{-2\phi t}|\xi_t|^4 dt.$$

Recall that $\eta_0 = 0$. By Theorem III.6.8 in [17], we have

$$\begin{aligned}E \sup_{t \leq \tau_2} e^{-\phi t}|\eta_t| &\leq 3E \left(\int_0^{\tau_2} Ne^{-2\phi t}|\xi_t|^4 dt \right)^{\frac{1}{2}} \\ &\leq E \left[\sup_{t \leq \tau_2} e^{-\frac{1}{2}\phi t}|\xi_t| \cdot \left(\int_0^{\tau_2} 9Ne^{-\phi t}|\xi_t|^2 dt \right)^{\frac{1}{2}} \right] \\ &\leq \frac{1}{2}E \sup_{t \leq \tau_2} e^{-\phi t}|\xi_t|^2 + \frac{1}{2}E \int_0^{\tau_2} 9Ne^{-\phi t}|\xi_t|^2 dt \\ &\leq NB_2(x_0, \xi_0), \\ E \left(\int_0^{\tau_2} e^{-2\phi t}|\eta_t|^2 dt \right)^{\frac{1}{2}} &\leq 3E \left(\int_0^{\tau_2} Ne^{-2\phi t}|\xi_t|^4 dt \right)^{\frac{1}{2}} \leq NB_2(x_0, \xi_0),\end{aligned}$$

which implies that (4) and (5) are true.

Finally, rewritting $c-1$ by $(c-\frac{1}{2})-\frac{1}{2}$ and repeating the argument above, we conclude that (6) is true. \square

3.3 Proof of Theorem 3.1.1

Now we are ready to prove the theorem.

Proof of (3.4). Denote $\tau_{D_\delta^\lambda}(x)$ and $\tau_{D_{\lambda^2}}(x)$ by τ_1^δ and τ_2 , respectively.

From (1.2) we immediately have

$$|u|_{0,D} \leq |g|_{0,D} + |f|_{0,D} E \int_0^\tau e^{-t} dt \leq |g|_{0,D} + |f|_{0,D}. \quad (3.18)$$

When $x_0 \in D_\delta^\lambda$, by Theorem 2.2.2, we have

$$u_{(\xi_0)}(x_0) = X_0 = EX_{\tau_1^\delta}.$$

So from (2.8) and (3.18),

$$\begin{aligned} |u_{(\xi_0)}(x_0)| &\leq E \left| u_{(\xi_{\tau_1^\delta})}(x_{\tau_1^\delta}) + (\xi_{\tau_1^\delta}^0 + \xi_{\tau_1^\delta}^{d+1}) u(x_{\tau_1^\delta}) \right| \\ &\quad + |f|_{1,D} E \int_0^{\tau_1^\delta} e^{-s} \left(|\xi_s| + 2r_s + |\xi_s^0| + |\xi_s^{d+1}| \right) ds \\ &\leq E \left| u_{(\xi_{\tau_1^\delta})}(x_{\tau_1^\delta}) \right| + \left(|g|_{0,D} + |f|_{0,D} \right) \left(E |\xi_{\tau_1^\delta}^0| + E |\xi_{\tau_1^\delta}^{d+1}| \right) \\ &\quad + |f|_{1,D} \left(E \int_0^{\tau_1^\delta} |\xi_s| + 2r_s ds + E \sup_{t \leq \tau_1^\delta} |\xi_t^0| + E \sup_{t \leq \tau_1^\delta} |\xi_t^{d+1}| \right). \end{aligned}$$

By Lemma 3.2.3, Davis inequality and Hölder inequality,

$$\begin{aligned} E |u_{(\xi_{\tau_1^\delta})}(x_{\tau_1^\delta})| &\leq \sup_{x \in \partial D_\delta^\lambda} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} \cdot E \sqrt{B_1(x_{\tau_1^\delta}, \xi_{\tau_1^\delta})} \\ &\leq \sup_{x \in \partial D_\delta^\lambda} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} \cdot \sqrt{B_1(x_0, \xi_0)}, \\ E |\xi_{\tau_1^\delta}^0| &\leq E \sup_{t \leq \tau_1^\delta} |\xi_t^0| \leq 3E \langle \xi^0 \rangle_{\tau_1^\delta}^{\frac{1}{2}} \leq 3 \left(E \langle \xi^0 \rangle_{\tau_1^\delta} \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq N \left(E \int_0^{\tau_1^\delta} \frac{\psi_{(\xi_s)}^2}{\psi^2} ds \right)^{\frac{1}{2}} \leq N \sqrt{\mathbf{B}_1(x_0, \xi_0)}, \\
E|\xi_{\tau_1^\delta}^{d+1}| &\leq E \sup_{t \leq \tau_1^\delta} |\xi_t^{d+1}| \leq NE \int_0^{\tau_1^\delta} |\xi_s| + \frac{|\psi_{(\xi_s)}|}{\psi} ds \\
&\leq NE \int_0^\infty I_{s \leq \tau_1^\delta} \cdot I_{s \leq \tau_1^\delta} \left(|\xi_s| + \frac{|\psi_{(\xi_s)}|}{\psi} \right) ds \\
&\leq N \left(E\tau_1^\delta \right)^{\frac{1}{2}} \left(E \int_0^{\tau_1^\delta} |\xi_s|^2 + \frac{\psi_{(\xi_s)}^2}{\psi^2} ds \right)^{\frac{1}{2}} \\
&\leq N \sqrt{\mathbf{B}_1(x_0, \xi_0)}, \\
E \int_0^{\tau_1^\delta} |\xi_s| + 2r_s ds &\leq NE \int_0^{\tau_1^\delta} |\xi_s| + \frac{|\psi_{(\xi_s)}|}{\psi} ds \leq N \sqrt{\mathbf{B}_1(x_0, \xi_0)}.
\end{aligned}$$

Collecting all estimates above, we conclude that

$$|u_{(\xi_0)}(x_0)| \leq \sup_{x \in \partial D_\delta^\lambda} \frac{|u_{(\xi)}(x)|}{\sqrt{\mathbf{B}_1(x, \xi)}} \cdot \sqrt{\mathbf{B}_1(x_0, \xi_0)} + N(|g|_{0,D} + |f|_{1,D}) \sqrt{\mathbf{B}_1(x_0, \xi_0)}.$$

So for any $x_0 \in D_\delta^\lambda$, $\xi_0 \in \mathbb{R}^d \setminus \{0\}$, we have

$$\frac{|u_{(\xi_0)}(x_0)|}{\sqrt{\mathbf{B}_1(x_0, \xi_0)}} \leq \sup_{x \in \partial D_\delta^\lambda} \frac{|u_{(\xi)}(x)|}{\sqrt{\mathbf{B}_1(x, \xi)}} + N_1, \quad (3.19)$$

with

$$N_1 = N(|g|_{1,D} + |f|_{1,D}). \quad (3.20)$$

Similarly, when $x_0 \in D_{\lambda^2}$, by Theorem 2.2.2, we have

$$u_{(\xi_0)}(x_0) = X_0 = EX_{\tau_2}.$$

Again, from (2.8) and (3.18),

$$\begin{aligned}
|u_{(\xi_0)}(x_0)| &\leq E e^{-\phi_{\tau_2}} \left| u_{(\xi_{\tau_2})}(x_{\tau_2}) + (\xi_{\tau_2}^0 + \xi_{\tau_2}^{d+1}) u(x_{\tau_1^\delta}) \right| \\
&\quad + |f|_{1,D} E \int_0^{\tau_2} e^{-\phi_s} \left(|\xi_s| + 2r_s + |\xi_s^0| + |\xi_s^{d+1}| \right) ds \\
&\leq E e^{-\frac{1}{2}\phi_{\tau_2}} \left| u_{(\xi_{\tau_2})}(x_{\tau_2}) \right| + \left(|g|_{0,D} + |f|_{0,D} \right) \left(E e^{-\frac{1}{2}\phi_{\tau_2}} |\xi_{\tau_2}^0| + E e^{-\phi_{\tau_2}} |\xi_{\tau_2}^{d+1}| \right) \\
&\quad + |f|_{1,D} \left(E \int_0^{\tau_2} e^{-\phi_s} (|\xi_s| + 2r_s) ds + 4E \sup_{t \leq \tau_2} e^{-\frac{3}{4}\phi_t} |\xi_t^{d+1}| + 2E \sup_{s \leq \tau_2} e^{-\frac{1}{2}\phi_s} |\xi_s^0| \right).
\end{aligned}$$

By Lemma 3.2.4, Davis inequality and Hölder inequality,

$$\begin{aligned}
Ee^{-\frac{1}{2}\phi_{\tau_2}}|u_{(\xi_{\tau_2})}(x_{\tau_2})| &\leq \sup_{x \in \partial D_{\lambda^2}} \frac{|u_{(\xi)}(x)|}{\sqrt{B_2(x, \xi)}} \cdot E\sqrt{e^{-\phi_{\tau_2}}B_2(x_{\tau_2}, \xi_{\tau_2})} \\
&\leq \sup_{x \in \partial D_{\lambda^2}} \frac{|u_{(\xi)}(x)|}{\sqrt{B_2(x, \xi)}} \cdot \sqrt{B_2(x_0, \xi_0)}, \\
Ee^{-\frac{1}{2}\phi_{\tau_2}}|\xi_{\tau_2}^0| &\leq E \sup_{s \leq \tau_2} e^{-\frac{1}{2}\phi_s} |\xi_s^0| = E \sup_{s \leq \tau_2} \left| \int_0^s e^{-\frac{1}{2}\phi_s} \pi_r dw_r \right| \\
&\leq 3E \left(\int_0^{\tau_2} e^{-\phi_r} |\pi_r|^2 dr \right)^{\frac{1}{2}} \\
&\leq NE \left(\int_0^{\tau_2} e^{-\phi_r} |\xi_r|^2 dr \right)^{\frac{1}{2}} \\
&\leq N\sqrt{B_2(x_0, \xi_0)}, \\
Ee^{-\phi_{\tau_2}}|\xi_{\tau_2}^{d+1}| &\leq E \sup_{t \leq \tau_2} e^{-\frac{3}{4}\phi_t} |\xi_t^{d+1}| \leq NE \int_0^{\tau_2} e^{-\frac{3}{4}\phi_s} |\xi_s| ds \\
&\leq NE \left(\int_0^{\tau_2} e^{-\frac{1}{2}\phi_s} ds \right)^{\frac{1}{2}} \left(\int_0^{\tau_2} e^{-\phi_s} |\xi_s|^2 ds \right)^{\frac{1}{2}} \\
&\leq N \left(E \int_0^{\tau_2} e^{-\phi_s} |\xi_s|^2 ds \right)^{\frac{1}{2}} \\
&\leq N\sqrt{B_2(x_0, \xi_0)}, \\
E \int_0^{\tau_2} e^{-\phi_s} (|\xi_s| + 2r_s) ds &\leq NE \int_0^{\tau_2} e^{-\phi_s} |\xi_s| ds \leq N\sqrt{B_2(x_0, \xi_0)}.
\end{aligned}$$

Collecting all estimates above, we conclude that

$$|u_{(\xi_0)}(x_0)| \leq \sup_{x \in \partial D_{\lambda^2}} \frac{|u_{(\xi)}(x)|}{\sqrt{B_2(x, \xi)}} \cdot \sqrt{B_2(x_0, \xi_0)} + N(|g|_{0,D} + |f|_{1,D})\sqrt{B_2(x_0, \xi_0)}.$$

So for any $x_0 \in D_{\lambda^2}$, $\xi_0 \in \mathbb{R}^d \setminus \{0\}$, we have

$$\frac{|u_{(\xi_0)}(x_0)|}{\sqrt{B_2(x_0, \xi_0)}} \leq \sup_{x \in \partial D_{\lambda^2}} \frac{|u_{(\xi)}(x)|}{\sqrt{B_2(x, \xi)}} + N_1, \quad (3.21)$$

with N_1 defined by (3.20).

Notice that

$$B_1(x, \xi) \begin{cases} \geq \sqrt{\psi}(1 + \sqrt{\psi})|\xi|^2 \geq \lambda^{\frac{1}{2}}|\xi|^2 & \text{on } \{\psi = \lambda\} \\ \leq \lambda(2 + \lambda)|\xi|^2 + K_1(2\lambda^2)^{\frac{3}{2}} \frac{\psi^2(\xi)}{\lambda^2} \leq K\lambda|\xi|^2 & \text{on } \{\psi = \lambda^2\}. \end{cases}$$

Recall that K doesn't depend on λ . So for sufficiently small λ , we have

$$B_1(x, \xi) \geq 4B_2(x, \xi) \text{ when } \psi = \lambda, \quad 4B_1(x, \xi) \leq B_2(x, \xi) \text{ when } \psi = \lambda^2.$$

Then on $\{x \in D : \psi(x) = \lambda\}$, we have

$$\begin{aligned} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} &\leq \frac{1}{2} \frac{|u_\xi(x)|}{\sqrt{B_2(x, \xi)}} \\ &\leq \frac{1}{2} \left(\sup_{\{\psi=\lambda^2\}} \frac{|u_\xi(x)|}{\sqrt{B_2(x, \xi)}} + N_1 \right) \\ &\leq \frac{1}{4} \sup_{\{\psi=\lambda^2\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + \frac{N_1}{2} \\ &\leq \frac{1}{4} \left(\sup_{\{\psi=\lambda\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + \sup_{\{\psi=\delta\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + N_1 \right) + \frac{N_1}{2} \\ &= \frac{1}{4} \sup_{\{\psi=\lambda\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + \frac{1}{4} \sup_{\{\psi=\delta\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + \frac{3N_1}{4}, \end{aligned}$$

which implies that

$$\sup_{\{\psi=\lambda\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} \leq \frac{1}{3} \sup_{\{\psi=\delta\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + N_1. \quad (3.22)$$

Meanwhile, on $\{x \in D : \psi(x) = \lambda^2\}$, we have

$$\begin{aligned} \frac{|u_\xi(x)|}{\sqrt{B_2(x, \xi)}} &\leq \frac{1}{2} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} \\ &\leq \frac{1}{2} \left(\sup_{\{\psi=\lambda\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + \sup_{\{\psi=\delta\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + N_1 \right) \\ &\leq \frac{1}{2} \left(\frac{1}{3} \sup_{\{\psi=\delta\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + N_1 \right) + \frac{1}{2} \sup_{\{\psi=\delta\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + \frac{N_1}{2} \\ &= \frac{2}{3} \sup_{\{\psi=\delta\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + N_1. \end{aligned}$$

Therefore,

$$\sup_{\{\psi=\lambda^2\}} \frac{|u_\xi(x)|}{\sqrt{B_2(x, \xi)}} \leq \frac{2}{3} \sup_{\{\psi=\delta\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + N_1. \quad (3.23)$$

Combining (3.19) and (3.22), we get, for any $x \in D_\delta^\lambda$, $\xi \in \mathbb{R}^d \setminus \{0\}$,

$$\frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} \leq \frac{4}{3} \sup_{\{\psi=\delta\}} \frac{|u_\xi(x)|}{\sqrt{B_1(x, \xi)}} + 2N_1. \quad (3.24)$$

Combining (3.21) and (3.23), we get, for any $x \in D_{\lambda^2}$, $\xi \in \mathbb{R}^d \setminus \{0\}$,

$$\frac{|u_{(\xi)}(x)|}{\sqrt{B_2(x, \xi)}} \leq \frac{2}{3} \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)}(x)|}{\sqrt{B_1(x, \xi)}} + 2N_1. \quad (3.25)$$

Thus it remains to estimate

$$\overline{\lim}_{\delta \downarrow 0} \left(\sup_{\{\psi=\delta\}} \frac{|u_{(\xi)}(x)|}{\sqrt{B_1(x, \xi)}} \right).$$

Notice that for each δ , there exist $x(\delta) \in \{\psi = \delta\}$ and $\xi(\delta) \in \{\xi : |\xi| = 1\}$, such that

$$\sup_{\{\psi=\delta\}} \frac{|u_{(\xi)}(x)|}{\sqrt{B_1(x, \xi)}} = \frac{|u_{(\xi(\delta))}(x(\delta))|}{\sqrt{B_1(x(\delta), \xi(\delta))}}.$$

A subsequence of $(x(\delta), \xi(\delta))$ converges to some (y, ζ) , such that $y \in \partial D$ and $|\zeta| = 1$.

If $\psi_{(\zeta)}(y) \neq 0$, then $B_1(x(\delta), \xi(\delta)) \rightarrow \infty$ as $\delta \downarrow 0$. In this case,

$$\overline{\lim}_{\delta \downarrow 0} \left(\sup_{\{\psi=\delta\}} \frac{|u_{(\xi)}(x)|}{\sqrt{B_1(x, \xi)}} \right) = \overline{\lim}_{\delta \downarrow 0} \frac{|u_{(\xi(\delta))}(x(\delta))|}{\sqrt{B_1(x(\delta), \xi(\delta))}} = 0.$$

If $\psi_{(\zeta)}(y) = 0$, then ζ is tangential to ∂D at y . In this case,

$$\begin{aligned} \overline{\lim}_{\delta \downarrow 0} \left(\sup_{\{\psi=\delta\}} \frac{|u_{(\xi)}(x)|}{\sqrt{B_1(x, \xi)}} \right) &= \overline{\lim}_{\delta \downarrow 0} \frac{|u_{(\xi(\delta))}(x(\delta))|}{\sqrt{B_1(x(\delta), \xi(\delta))}} \\ &= \frac{|g_{(\zeta)}(y)|}{\sqrt{\lambda}} \leq N \sup_{\partial D} |g_x|. \end{aligned} \quad (3.26)$$

From (3.24), (3.25) and (3.26), we have

$$\begin{aligned} \frac{|u_{(\xi)}(x)|}{\sqrt{B_1(x, \xi)}} &\leq N(|f|_{1,D} + |g|_{1,D}), \text{ when } x \in D^\lambda; \\ \frac{|u_{(\xi)}(x)|}{\sqrt{B_2(x, \xi)}} &\leq N(|f|_{1,D} + |g|_{1,D}), \text{ when } x \in D_{\lambda^2}. \end{aligned}$$

Notice that $D^\lambda \cup D_{\lambda^2} = D$, and

$$\begin{aligned} \sqrt{B_1(x, \xi)} &\leq N(|\xi| + \frac{|\psi_{(\xi)}|}{\psi^{\frac{1}{2}}}), \text{ when } x \in D^\lambda; \\ \sqrt{B_2(x, \xi)} &\leq N(|\xi| + \frac{|\psi_{(\xi)}|}{\psi^{\frac{1}{2}}}), \text{ when } x \in D_{\lambda^2}. \end{aligned}$$

We conclude that, for any $x \in D$ and $\xi \in \mathbb{R}^d$,

$$|u_{(\xi)}(x)| \leq N(|\xi| + \frac{|\psi(\xi)|}{\psi^{\frac{1}{2}}})(|f|_{1,D} + |g|_{1,D}).$$

The inequality (3.4) is proved. \square

The proof of (3.5) is similar.

Proof of (3.5). When $x_0 \in D_\delta^\lambda$, by Theorem 2.2.2, we have

$$u_{(\xi_0)(\xi_0)}(x_0) = u_{(\xi_0)(\xi_0)}(x_0) + u_{(\eta_0)}(x_0) = Y_0 = EY_{\tau_1^\delta}.$$

From (2.9) and (3.18),

$$\begin{aligned} |u_{(\xi_0)(\xi_0)}(x_0)| &\leq E|u_{(\xi_{\tau_1^\delta})(\xi_{\tau_1^\delta})}(x_{\tau_1^\delta})| + \sup_{x \in \partial D_\delta^\lambda, |\zeta|=1} |u_{(\zeta)}(x)| \cdot Ee^{-\tau_1^\delta} \left(|\eta_{\tau_1^\delta}| + 2|\tilde{\xi}_{\tau_1^\delta}^0| |\xi_{\tau_1^\delta}| \right) \\ &+ \left(|g|_{0,D} + |f|_{0,D} \right) Ee^{-\tau_1^\delta} |\tilde{\eta}_{\tau_1^\delta}^0| + |f|_{2,D} E \int_0^{\tau_1^\delta} e^{-s} \left[|\xi_s|^2 + |\eta_s| \right. \\ &\left. + (4r_s + 2|\tilde{\xi}_s^0|) |\xi_s| + 2\hat{r}_s + 4|\tilde{\xi}_s^0| r_s + |\tilde{\eta}_s^0| \right] ds. \end{aligned}$$

Recall that in this case,

$$\tilde{\xi}_t^0 = \xi_t^0 + \xi_t^{d+1}, \quad \tilde{\eta}_t^0 = 2\xi_t^0 \xi_t^{d+1} + (\xi_t^{d+1})^2 + \eta_t^{d+1}.$$

It follows that

$$\begin{aligned} |u_{(\xi_0)(\xi_0)}(x_0)| &\leq E|u_{(\xi_{\tau_1^\delta})(\xi_{\tau_1^\delta})}(x_{\tau_1^\delta})| + N \left(|g|_{0,D} + |f|_{0,D} + \sup_{x \in \partial D_\delta^\lambda, |\zeta|=1} |u_{(\zeta)}(x)| \right) \\ &\cdot Ee^{-\tau_1^\delta} \left(|\eta_{\tau_1^\delta}| + |\xi_{\tau_1^\delta}|^2 + |\xi_{\tau_1^\delta}^0|^2 + |\xi_{\tau_1^\delta}^{d+1}|^2 + |\eta_{\tau_1^\delta}^{d+1}| \right) \\ &+ N|f|_{2,D} E \int_0^{\tau_1^\delta} e^{-s} \left[|\xi_s|^2 + |\eta_s| + |\xi_s^0|^2 + |\xi_s^{d+1}|^2 + |\eta_s^{d+1}| + r_s^2 + \hat{r}_s \right] ds \\ &\leq E|u_{(\xi_{\tau_1^\delta})(\xi_{\tau_1^\delta})}(x_{\tau_1^\delta})| + N \left(|g|_{0,D} + |f|_{2,D} + \sup_{x \in \partial D_\delta^\lambda, |\zeta|=1} |u_{(\zeta)}(x)| \right) \\ &\cdot \left(E \sup_{t \leq \tau_1^\delta} |\eta_t| + E \sup_{t \leq \tau_1^\delta} |\xi_t|^2 + E \sup_{t \leq \tau_1^\delta} |\xi_t^0|^2 + E \sup_{t \leq \tau_1^\delta} e^{-\frac{1}{2}t} |\xi_t^{d+1}|^2 \right. \\ &\left. + E \sup_{t \leq \tau_1^\delta} e^{-\frac{1}{2}t} |\eta_t^{d+1}| + E \int_0^{\tau_1^\delta} r_s^2 + \hat{r}_s ds \right). \end{aligned}$$

By Lemma 3.2.3, Davis inequality and Hölder inequality,

$$\begin{aligned}
E|u_{(\xi_{\tau_1^\delta})(\xi_{\tau_1^\delta})}(x_{\tau_1^\delta})| &\leq \sup_{x \in \partial D_\delta^\lambda} \frac{|u_{(\xi)}(\xi)(x)|}{B_1(x, \xi)} \cdot EB_1(x_{\tau_1^\delta}, \xi_{\tau_1^\delta}) \\
&\leq \sup_{x \in \partial D_\delta^\lambda} \frac{|u_{(\xi)}(\xi)(x)|}{B_1(x, \xi)} \cdot B_1(x_0, \xi_0), \\
E \sup_{t \leq \tau_1^\delta} |\eta_t| &\leq NB_1(x_0, \xi_0), \\
E \sup_{t \leq \tau_1^\delta} |\xi_t|^2 &\leq NB_1(x_0, \xi_0), \\
E \sup_{t \leq \tau_1^\delta} |\xi_t^0|^2 &\leq 4E\langle \xi^0 \rangle_{\tau_1^\delta} \leq NE \int_0^{\tau_1^\delta} \frac{\psi^2(\xi_t)}{\psi^2} dt \leq NB_1(x_0, \xi_0), \\
E \sup_{t \leq \tau_1^\delta} e^{-\frac{1}{2}t} |\xi_t^{d+1}|^2 &\leq NE \sup_{t \leq \tau_1^\delta} e^{-\frac{1}{2}t} \left(\int_0^t |\xi_s| + \frac{|\psi(\xi_s)|}{\psi} ds \right)^2 \\
&\leq NE \sup_{t \leq \tau_1^\delta} e^{-\frac{1}{2}t} \int_0^t |\xi_s|^2 + \frac{\psi^2(\xi_s)}{\psi^2} ds \\
&\leq NE \int_0^{\tau_1^\delta} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} dt \\
&\leq NB_1(x_0, \xi_0), \\
E \sup_{t \leq \tau_1^\delta} e^{-\frac{1}{2}t} |\eta_t^{d+1}| &\leq NE \sup_{t \leq \tau_1^\delta} e^{-\frac{1}{2}t} \int_0^t |\xi_s|^2 + \frac{\psi^2(\xi_s)}{\psi^2} + |\eta_s| ds \\
&\leq NE \sup_{t \leq \tau_1^\delta} e^{-\frac{1}{2}t} \left[\int_0^t |\xi_s|^2 + \frac{\psi^2(\xi_s)}{\psi^2} ds + \sqrt{t} \left(\int_0^t |\eta_s|^2 ds \right)^{\frac{1}{2}} \right] \\
&\leq N \left[E \int_0^{\tau_1^\delta} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} dt + E \left(\int_0^{\tau_1^\delta} |\eta_t|^2 dt \right)^2 \right] \\
&\leq NB_1(x_0, \xi_0), \\
E \int_0^{\tau_1^\delta} r_s^2 + \hat{r}_s ds &\leq NE \int_0^{\tau_1^\delta} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} dt \leq NB_1(x_0, \xi_0).
\end{aligned}$$

Collecting all estimates above, we conclude that

$$\begin{aligned}
|u_{(\xi_0)(\xi_0)}(x_0)| &\leq \sup_{x \in \partial D_\delta^\lambda} \frac{|u_{(\xi)}(\xi)(x)|}{B_1(x, \xi)} \cdot B_1(x_0, \xi_0) \\
&\quad + N \left(|g|_{0,D} + |f|_{2,D} + \sup_{x \in \partial D_\delta^\lambda, |\zeta|=1} |u_{(\zeta)}(x)| \right) B_1(x_0, \xi_0).
\end{aligned}$$

So for any $x_0 \in D_\delta^\lambda$, $\xi_0 \in \mathbb{R}^d \setminus \{0\}$, we have

$$\frac{|u_{(\xi_0)(\xi_0)}(x_0)|}{B_1(x_0, \xi_0)} \leq \sup_{x \in \partial D_\delta^\lambda} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + N_2, \quad (3.27)$$

with

$$N_2 = N \left(|g|_{2,D} + |f|_{2,D} + \sup_{x \in \partial D_\delta^\lambda, |\zeta|=1} |u_{(\zeta)}(x)| \right). \quad (3.28)$$

When $x_0 \in D_{\lambda^2}$, by Theorem 2.2.2, we have

$$u_{(\xi_0)(\xi_0)}(x_0) = u_{(\xi_0)(\xi_0)}(x_0) + u_{(\eta_0)}(x_0) = Y_0 = EY_{\tau_2}.$$

Again, from (2.9) and (3.18),

$$\begin{aligned} |u_{(\xi_0)(\xi_0)}(x_0)| &\leq Ee^{-\phi_{\tau_2}} |u_{(\xi_{\tau_2})(\xi_{\tau_2})}(x_{\tau_2})| + \sup_{x \in \partial D_{\lambda^2}, |\zeta|=1} |u_{(\zeta)}(x)| \cdot Ee^{-\phi_{\tau_2}} \left(|\eta_{\tau_2}| + 2|\tilde{\xi}_{\tau_2}^0| |\xi_{\tau_2}| \right) \\ &+ \left(|g|_{0,D} + |f|_{0,D} \right) Ee^{-\phi_{\tau_2}} |\tilde{\eta}_{\tau_2}^0| + |f|_{2,D} E \int_0^{\tau_2} e^{-\phi_s} \left[|\xi_s|^2 + |\eta_s| \right. \\ &\left. + (4r_s + 2|\tilde{\xi}_s^0|) |\xi_s| + 2\hat{r}_s + 4|\tilde{\xi}_s^0| r_s + |\tilde{\eta}_s^0| \right] ds. \end{aligned}$$

Recall that in this case,

$$\tilde{\xi}_t^0 = \xi_t^0 + \xi_t^{d+1}, \quad \tilde{\eta}_t^0 = \eta_t^0 + 2\xi_t^0 \xi_t^{d+1} + (\xi_t^{d+1})^2 + \eta_t^{d+1}.$$

Also, notice that by (3.4)

$$\sup_{x \in \partial D_{\lambda^2}, |\zeta|=1} |u_{(\zeta)}(x)| \leq N \left(1 + \frac{|\psi|_{1,D}}{\lambda^2} \right) \left(|f|_{1,D} + |g|_{1,D} \right) \leq N \left(|f|_{1,D} + |g|_{1,D} \right).$$

Therefore,

$$\begin{aligned} |u_{(\xi_0)(\xi_0)}(x_0)| &\leq Ee^{-\phi_{\tau_2}} |u_{(\xi_{\tau_2})(\xi_{\tau_2})}(x_{\tau_2})| + N \left(|g|_{1,D} + |f|_{1,D} \right) \\ &\cdot Ee^{-\phi_{\tau_2}} \left(|\eta_{\tau_2}| + |\xi_{\tau_2}|^2 + |\xi_{\tau_2}^0|^2 + |\xi_{\tau_2}^{d+1}|^2 + |\eta_{\tau_2}^{d+1}| + |\eta_{\tau_2}^0| \right) \\ &+ N|f|_{2,D} E \int_0^{\tau_2} e^{-\phi_s} \left[|\xi_s|^2 + |\eta_s| + |\xi_s^0|^2 + |\xi_s^{d+1}|^2 + |\eta_s^{d+1}| + |\eta_s^0| + r_s^2 + \hat{r}_s \right] ds \\ &\leq Ee^{-\phi_{\tau_2}} |u_{(\xi_{\tau_2})(\xi_{\tau_2})}(x_{\tau_2})| + N \left(|g|_{1,D} + |f|_{2,D} \right) \\ &\cdot \left(E \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2}t} |\eta_t| + E \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2}t} |\xi_t|^2 + E \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2}t} |\xi_t^0|^2 \right. \\ &\quad \left. + E \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{4}t} |\xi_t^{d+1}|^2 + E \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{4}t} |\eta_t^{d+1}| \right) \end{aligned}$$

$$+ E \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{2}t} |\eta_t^0| + E \int_0^{\tau_2} e^{-\phi s} (r_s^2 + \hat{r}_s) ds).$$

By Lemma 3.2.4, Davis inequality and Hölder inequality,

$$\begin{aligned} E e^{-\phi \tau_2} |u_{(\xi_{\tau_2})(\xi_{\tau_2})}(x_{\tau_2})| &\leq \sup_{x \in \partial D_{\lambda^2}} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} \cdot E e^{-\phi \tau_2} B_2(x_{\tau_2}, \xi_{\tau_2}) \\ &\leq \sup_{x \in \partial D_{\lambda^2}} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} \cdot B_2(x_0, \xi_0), \end{aligned}$$

$$E \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{2}t} |\eta_t| \leq N B_2(x_0, \xi_0),$$

$$E \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{2}t} |\xi_t|^2 \leq N B_2(x_0, \xi_0),$$

$$\begin{aligned} E \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{2}t} |\xi_t^0|^2 &= E \sup_{t \leq \tau_2} \left| \int_0^t e^{-\frac{1}{2}\phi t + \frac{1}{4}t} \pi_s dw_s \right|^2 \\ &\leq 4E \int_0^{\tau_2} e^{-\phi t + \frac{1}{2}t} |\pi_t|^2 dt \\ &\leq NE \int_0^{\tau_2} e^{-\phi t + \frac{1}{2}t} |\xi_t|^2 dt \\ &\leq N B_2(x_0, \xi_0), \end{aligned}$$

$$\begin{aligned} E \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{4}t} |\xi_t^{d+1}|^2 &\leq NE \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{4}t} \left(\int_0^t |\xi_s| ds \right)^2 \\ &\leq NE \sup_{t \leq \tau_2} e^{-\frac{1}{4}t} \left(\int_0^t e^{-\frac{1}{2}\phi s + \frac{1}{4}s} |\xi_s| ds \right)^2 \\ &\leq NE \sup_{t \leq \tau_2} e^{-\frac{1}{4}t} \cdot t \int_0^t e^{-\phi s + \frac{1}{2}s} |\xi_s|^2 ds \\ &\leq NE \int_0^{\tau_2} e^{-\phi s + \frac{1}{2}s} |\xi_s|^2 ds \\ &\leq N B_2(x_0, \xi_0), \end{aligned}$$

$$\begin{aligned} E \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{4}t} |\eta_t^{d+1}| &\leq NE \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{4}t} \int_0^t |\xi_s|^2 + |\eta_s| ds \\ &\leq NE \int_0^{\tau_2} e^{-\phi s + \frac{1}{4}s} |\xi_s|^2 ds + NE \sup_{t \leq \tau_2} e^{-\frac{1}{4}t} \int_0^t e^{-\phi s + \frac{1}{2}s} |\eta_s| ds \\ &\leq N B_2(x_0, \xi_0) + NE \sup_{t \leq \tau_2} e^{-\frac{1}{4}t} \cdot \sqrt{t} \left(\int_0^t e^{-2\phi s + s} |\eta_s|^2 ds \right)^{\frac{1}{2}} \\ &\leq N B_2(x_0, \xi_0) + NE \left(\int_0^{\tau_2} e^{-2\phi s + s} |\eta_s|^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq NB_2(x_0, \xi_0), \\
E \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2}t} |\eta_t^0| &\leq E \sup_{t \leq \tau_2} \left(\left| \int_0^t e^{-\frac{1}{2}\phi_t + \frac{1}{4}t} \pi_s dw_s \right|^2 + \int_0^t e^{-\phi_t + \frac{1}{2}t} |\pi_s|^2 ds \right. \\
&\quad \left. + \left| \int_0^t e^{-\phi_t + \frac{1}{2}t} \hat{\pi}_s dw_s \right| \right) \\
&\leq 5E \int_0^{\tau_2} e^{-\phi_s + \frac{1}{2}s} |\pi_s|^2 ds + 3E \left(\int_0^{\tau_2} e^{-2\phi_t + t} |\hat{\pi}_s|^2 \right)^{\frac{1}{2}} ds \\
&\leq NB_2(x_0, \xi_0), \\
E \int_0^{\tau_2} e^{-\phi_s} (r_s^2 + \hat{r}_s) ds &\leq NE \int_0^{\tau_2} e^{-\phi_s} (|\xi_s|^2 + |\eta_s|) ds \\
&\leq NB_2(x_0, \xi_0) + N \left(\int_0^{\tau_2} e^{-s} ds \right)^{\frac{1}{2}} \left(\int_0^{\tau_2} e^{-2\phi_s + s} |\eta_s|^2 ds \right)^{\frac{1}{2}} \\
&\leq NB_2(x_0, \xi_0).
\end{aligned}$$

Collecting all estimates above, we conclude that

$$|u_{(\xi_0)(\xi_0)}(x_0)| \leq \sup_{x \in \partial D_{\lambda^2}} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} \cdot B_2(x_0, \xi_0) + N(|g|_{1,D} + |f|_{2,D})B_2(x_0, \xi_0).$$

So for any $x_0 \in D_{\lambda^2}$, $\xi_0 \in \mathbb{R}^d \setminus \{0\}$, we have

$$\frac{|u_{(\xi_0)(\xi_0)}(x_0)|}{B_2(x_0, \xi_0)} \leq \sup_{x \in \partial D_{\lambda^2}} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} + N_2, \quad (3.29)$$

with N_2 defined by (3.28).

Then on $\{x \in D : \psi(x) = \lambda\}$, we have

$$\begin{aligned}
\frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} &\leq \frac{1}{4} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} \\
&\leq \frac{1}{4} \left(\sup_{\{\psi=\lambda^2\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} + N_2 \right) \\
&\leq \frac{1}{16} \sup_{\{\psi=\lambda^2\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{N_2}{4} \\
&\leq \frac{1}{16} \left(\sup_{\{\psi=\lambda\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + N_2 \right) + \frac{N_2}{4} \\
&= \frac{1}{16} \sup_{\{\psi=\lambda\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{1}{16} \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{5N_2}{16},
\end{aligned}$$

which implies that

$$\sup_{\{\psi=\lambda\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} \leq \frac{1}{15} \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{N_2}{3}. \quad (3.30)$$

Meanwhile, on $\{x \in D : \psi(x) = \lambda^2\}$, we have

$$\begin{aligned} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} &\leq \frac{1}{4} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} \\ &\leq \frac{1}{4} \left(\sup_{\{\psi=\lambda\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + N_2 \right) \\ &\leq \frac{1}{4} \left(\frac{1}{15} \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{N_2}{3} \right) + \frac{1}{4} \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{N_2}{4} \\ &= \frac{4}{15} \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{N_2}{3}. \end{aligned}$$

Therefore,

$$\sup_{\{\psi=\lambda^2\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} \leq \frac{4}{15} \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{N_2}{3}. \quad (3.31)$$

Combining (3.27) and (3.30), we get, for any $x \in D_\delta^\lambda$, $\xi \in \mathbb{R}^d \setminus \{0\}$,

$$\frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} \leq \frac{16}{15} \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{4N_2}{3}. \quad (3.32)$$

Combining (3.29) and (3.31), we get, for any $x \in D_{\lambda^2}$, $\xi \in \mathbb{R}^d \setminus \{0\}$,

$$\frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} \leq \frac{4}{15} \sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{4N_2}{3}. \quad (3.33)$$

Thus it remains to estimate

$$\overline{\lim}_{\delta \downarrow 0} \left(\sup_{\{\psi=\delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} \right) \text{ and } \overline{\lim}_{\delta \downarrow 0} \sup_{x \in \partial D_\delta^\lambda, |\zeta|=1} |u_{(\zeta)}(x)|.$$

First, notice that

$$\begin{aligned} \overline{\lim}_{\delta \downarrow 0} \sup_{x \in \partial D_\delta^\lambda, |\zeta|=1} |u_{(\zeta)}(x)| &\leq \sup_{x \in \partial D, |\zeta|=1} |u_{(\zeta)}(x)| + \sup_{x \in \{\psi=\lambda\}, |\zeta|=1} |u_{(\zeta)}(x)| \\ &\leq \sup_{x \in \partial D, |l|=1, l \parallel \partial D} |u_{(l)}(x)| + \sup_{x \in \partial D, |n|=1, n \perp \partial D} |u_{(n)}(x)| \end{aligned}$$

$$+ \sup_{x \in \{\psi = \lambda\}, |\zeta| = 1} |u_{(\zeta)}(x)|,$$

Apply Lemma 4.3.1 and first derivative estimate (3.4), we get

$$\begin{aligned} \overline{\lim}_{\delta \downarrow 0} \sup_{x \in \partial D_\delta^\lambda, |\zeta| = 1} |u_{(\zeta)}(x)| &\leq \sup_{x \in \partial D, |l| = 1, l \perp \partial D} |g_{(l)}(x)| + N(|g|_{2,D} + |f|_{0,D}) \\ &\quad + N\left(1 + \frac{|\psi|_{1,D}}{\lambda}\right)(|g|_{1,D} + |f|_{1,D}) \\ &\leq N(|g|_{2,D} + |f|_{1,D}). \end{aligned}$$

Second, notice that for each δ , there exist $x(\delta) \in \{\psi = \delta\}$ and $\xi(\delta) \in \{\xi : |\xi| = 1\}$, such that

$$\sup_{\{\psi = \delta\}} \frac{|u_{(\xi)}(x)|}{B_1(x, \xi)} = \frac{|u_{(\xi(\delta))}(\xi(\delta))(x(\delta))|}{B_1(x(\delta), \xi(\delta))}.$$

A subsequence of $(x(\delta), \xi(\delta))$ converges to some (y, ζ) , such that $y \in \partial D$ and $|\zeta| = 1$.

If $\psi_{(\zeta)}(y) \neq 0$, then $B_1(x(\delta), \xi(\delta)) \rightarrow \infty$ as $\delta \downarrow 0$. In this case,

$$\overline{\lim}_{\delta \downarrow 0} \left(\sup_{\{\psi = \delta\}} \frac{|u_{(\xi)}(x)|}{B_1(x, \xi)} \right) = \overline{\lim}_{\delta \downarrow 0} \frac{|u_{(\xi(\delta))}(\xi(\delta))(x(\delta))|}{B_1(x(\delta), \xi(\delta))} = 0.$$

If $\psi_{(\zeta)}(y) = 0$, then ζ is tangential to ∂D at y . In this case,

$$\overline{\lim}_{\delta \downarrow 0} \left(\sup_{\{\psi = \delta\}} \frac{|u_{(\xi)}(x)|}{B_1(x, \xi)} \right) = \overline{\lim}_{\delta \downarrow 0} \frac{|u_{(\xi(\delta))}(\xi(\delta))(x(\delta))|}{B_1(x(\delta), \xi(\delta))} = \frac{|g_{(\zeta)}(y)| + K|u_{(n)}(y)|}{\lambda}.$$

By Lemma (4.3.1), we have

$$\frac{|g_{(\zeta)}(y)| + K|u_{(n)}(y)|}{\lambda} \leq N(|g|_{2,D} + |f|_{0,D}).$$

Therefore, we have

$$\begin{aligned} \frac{|u_{(\xi)}(x)|}{B_1(x, \xi)} &\leq N(|f|_{2,D} + |g|_{2,D}), \text{ when } x \in D^\lambda; \\ \frac{|u_{(\xi)}(x)|}{B_2(x, \xi)} &\leq N(|f|_{2,D} + |g|_{2,D}), \text{ when } x \in D_{\lambda^2}. \end{aligned}$$

It follows that, for any $x \in D$ and $\xi \in \mathbb{R}^d$,

$$|u_{(\xi)}(x)| \leq N(|\xi|^2 + \frac{\psi^2(\xi)}{\psi})(|f|_{2,D} + |g|_{2,D}).$$

The inequality (3.5) is proved. □

Proof of the existence and uniqueness of (3.6). The fact that u given by (1.2) satisfies (3.6) follows from Theorem 1.3 in [12].

To proof the uniqueness, assume that $u_1, u_2 \in C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$ are solutions of (3.6). Let $\Lambda = |u_1|_{0,D} \vee |u_2|_{0,D}$. For constants δ and ε satisfying $0 < \delta < \varepsilon < 1$, define

$$\Psi(x, t) = \varepsilon(1 + \psi(x))\Lambda e^{-\delta t}, \quad U(x, t) = u(x)e^{-\varepsilon t} \text{ in } \bar{D} \times (0, \infty),$$

$$F[U] = U_t + LU - cU + f \text{ in } D \times (0, \infty).$$

Notice that a.e. in D , we have

$$F[U_1 - \Psi] = -\varepsilon e^{-\varepsilon t} u_1 + \delta \Psi - \varepsilon \Lambda e^{-\delta t} L\psi + c\Psi \geq \varepsilon \Lambda (e^{-\delta t} - e^{-\varepsilon t}) \geq 0,$$

$$F[U_2 + \Psi] = \varepsilon e^{-\varepsilon t} u_2 - \delta \Psi + \varepsilon \Lambda e^{-\delta t} L\psi - c\Psi \leq \varepsilon \Lambda (e^{-\varepsilon t} - e^{-\delta t}) \leq 0.$$

On $\partial D \times (0, \infty)$, we have

$$U_1 - U_2 - 2\Psi = -2\Psi \leq 0.$$

On $\bar{D} \times T$, where $T = T(\varepsilon, \delta)$ is a sufficiently large constant, we have

$$U_1 - U_2 - 2\Psi = (u_1 - u_2)e^{-\varepsilon T} - 2\varepsilon(1 + \psi)\Lambda e^{-\delta T} \leq 2\Lambda(e^{-\varepsilon T} - \varepsilon e^{-\delta T}) \leq 0.$$

Applying Theorem 1.1 in [9], we get

$$U_1 - U_2 - 2\Psi \leq 0 \text{ a.e. in } \bar{D} \times (0, T).$$

It follows that

$$u_1 - u_2 \leq 2\varepsilon(1 + \psi)\Lambda e \rightarrow 0, \text{ as } \varepsilon \rightarrow 0, \text{ a.e. in } D.$$

Similarly, $u_2 - u_1 \leq 0$ a.e. in D . The uniqueness is proved. \square

Remark 3.3.1. *Based on our proof, if we replace $\sigma(x), b(x), c(x), f(x)$ and $g(x)$ in (1.3) and (1.2) by $\sigma(\omega, t, x), b(\omega, t, x), c(\omega, t, x), f(\omega, t, x)$ and $g(\omega, t, x)$, defined on $\Omega \times [0, \infty) \times D$, under appropriate measurable assumptions, the first and second derivative estimates (3.4) and (3.5) are still true.*

Chapter 4

Interior regularity for fully nonlinear degenerate elliptic equations

In this chapter, we study the smoothness of the value function in the time-homogeneous optimal control of degenerate diffusion processes with state constraint. The corresponding dynamic programming equation is the degenerate Bellman equation with boundary condition.

4.1 Main theorem

Assume that (Ω, \mathcal{F}, P) is a complete probability space and $\{\mathcal{F}_t; t \leq 0\}$ an increasing filtration of σ -algebras $\mathcal{F}_t \subset \mathcal{F}$ which are complete with respect to \mathcal{F}, P . Let $(w_t, \mathcal{F}_t; t \geq 0)$ be a d_1 -dimensional Wiener process on (Ω, \mathcal{F}, P) .

Let A be a separable metric space. Suppose that the following have been defined for each $\alpha \in A$ and $x \in \mathbb{R}^d$: a $d \times d_1$ matrix $\sigma^\alpha(x)$, a d -dimensional vector $b^\alpha(x)$ and real scalars $c^\alpha(x) \geq 0$ and $f^\alpha(x)$. We assume that σ, b, c and f are Borel measurable on $A \times \mathbb{R}^d$, and $g(x)$ is a Borel measurable function on \mathbb{R}^d . We also assume that $\sigma^\alpha, b^\alpha, c^\alpha$ and their first and second derivatives are all continuous in x uniformly with respect to α .

Let $D \in C^{3,1}$ be a bounded domain in \mathbb{R}^d , then there exists a function $\psi \in C^{3,1}$ satisfying

$$\psi > 0 \text{ in } D, \quad \psi = 0 \text{ and } |\psi_x| \geq 1 \text{ on } \partial D.$$

Additionally, we assume that

$$\sup_{\alpha \in A} L^\alpha \psi \leq -1 \text{ in } D,$$

with

$$L^\alpha := (a^\alpha)^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + (b^\alpha)^i(x) \frac{\partial}{\partial x^i},$$

where $a = 1/2(\sigma\sigma^*)$. We also assume that

$$|(\sigma^\alpha)^{ij}|_{2,D} + |(b^\alpha)^i|_{2,D} + |c^\alpha|_{2,D} + |\psi|_{4,D} \leq K_0, \quad (4.1)$$

$$\forall \alpha \in A, 1 \leq i \leq d, 1 \leq j \leq d_1,$$

with $K_0 \in [1, \infty)$, not depending on α .

By \mathfrak{A} , we denote the set of all functions $\alpha_r(\omega)$ on $\Omega \times [0, \infty)$ which are \mathcal{F}_r -adapted and measurable in (ω, r) with values in A .

For $\alpha \in \mathfrak{A}$ and $x \in D$, we consider the Itô equation

$$x_t^{\alpha,x} = x + \int_0^t \sigma^{\alpha_s}(x_s^{\alpha,x}) dw_s + \int_0^t b^{\alpha_s}(x_s^{\alpha,x}) ds. \quad (4.2)$$

The solution of this equation is known to exist and to be unique by our assumptions on σ^α and b^α .

Let $\tau^{\alpha,x}$ be the first exit time of $x_t^{\alpha,x}$ from D :

$$\tau^{\alpha,x} = \inf\{t \leq 0 : x_t^{\alpha,x} \notin D\}.$$

For any $t \geq 0$, we define

$$\phi_t^{\alpha,x} = \int_0^t c^{\alpha_s}(x_s^{\alpha,x}) ds.$$

Set

$$v(x) = \sup_{\alpha \in \mathfrak{A}} v^\alpha(x), \quad (4.3)$$

with

$$v^\alpha(x) = E_x^\alpha \left[g(x_\tau) e^{-\phi_\tau} + \int_0^\tau f^{\alpha_s}(x_s) e^{-\phi_s} ds \right], \quad (4.4)$$

where we use common abbreviated notation, according to which we put the indices α and x beside the expectation sign instead of explicitly exhibiting them inside the expectation sign for every object that can carry all or part of them. Namely,

$$\begin{aligned} & E_x^\alpha \left[g(x_\tau) e^{-\phi_\tau} + \int_0^\tau f^{\alpha_s}(x_s) e^{-\phi_s} ds \right] \\ &= E \left[g(x_{\tau^{\alpha,x}}) e^{-\phi_{\tau^{\alpha,x}}} + \int_0^{\tau^{\alpha,x}} f^{\alpha_s}(x_s^{\alpha,x}) e^{-\phi_s^{\alpha,x}} ds \right]. \end{aligned}$$

The value function $v(x)$ given by (4.3) and (4.4) is the probabilistic solution of the Dirichlet problem for the Bellman equation:

$$\begin{cases} \sup_{\alpha \in A} [L^\alpha v - c^\alpha v + f^\alpha] = 0 & \text{in } D \\ v = g & \text{on } \partial D. \end{cases} \quad (4.5)$$

Define

$$\mu(x, \xi) := \inf_{\zeta: (\xi, \zeta)=1} \sup_{\alpha \in A} a^{ij}(\alpha, x) \zeta^i \zeta^j, \quad (4.6)$$

$$\mu(x) := \inf_{|\zeta|=1} \sup_{\alpha \in A} a^{ij}(\alpha, x) \zeta^i \zeta^j. \quad (4.7)$$

The condition $\mu(x, \xi) > 0$ means that $v_{(\xi)(\xi)}(x)$ is actually “present” in the Bellman equation in (4.5). More precisely, for any fixed $x \in D$ and $\xi \in \mathbb{R}^d \setminus \{0\}$, $\mu(x, \xi) > 0$ if and only if there exists a control $\alpha \in A$ such that the corresponding diffusion matrix $a^\alpha(x)$ is non-degenerate in the direction ξ . For example, consider the linear equation

$$u_{x^1 x^1} + 2u_{x^1 x^2} + u_{x^2 x^2} = 0. \quad (4.8)$$

By (4.6), here

$$\mu(x, \xi) = \inf_{(\xi, \zeta)=1} (\zeta^1 + \zeta^2)^2.$$

$\mu(x, \xi) > 0$ if and only if $\xi \parallel \xi_0 = (1, 1)$. So only $u_{(\xi_0)(\xi_0)}$ is “present” in (4.8). In fact, the equation (4.8) can be rewritten as

$$u_{(\xi_0)(\xi_0)} = 0,$$

so that no other second-order derivatives is actually “present” in the equation, even though $u_{x^1 x^1}$ and $u_{x^2 x^2}$ exist explicitly in (4.8).

Also, it is not hard to see that

$$\mu(x) = \inf_{|\xi|=1} \mu(x, \xi).$$

Note that we have $\mu(x) > 0$ at a point x if and only if for any $\xi \neq 0$, there exists a control $\alpha \in \mathfrak{A}$, such that the corresponding diffusion term $a^\alpha(x)$ is non-degenerate in the direct of ξ .

Let \mathfrak{B} be the set of all skew-symmetric $d_1 \times d_1$ matrices. For any positive constant λ , define

$$D_\lambda = \{x \in D : \psi(x) > \lambda\}.$$

Our main result is the following:

Theorem 4.1.1. *Suppose that*

1. *(uniform non-degeneracy along the normal to the boundary) There exists a positive constant δ_0 , such that*

$$(a^\alpha n, n) \geq \delta_0 \text{ on } \partial D, \forall \alpha \in A, \quad (4.9)$$

where n is the unit normal vector.

2. *(interior condition to control the moments of quasiderivatives, weaker than the non-degeneracy) There exist a function $\rho^\alpha(x) : A \times D \rightarrow \mathbb{R}^d$, bounded on every set in the form of $A \times D_\lambda$ for all $\lambda > 0$, a function $Q^\alpha(x, y) : A \times D \times \mathbb{R}^d \rightarrow \mathfrak{B}$, bounded with respect to (α, x) on every set in the form of $A \times D_\lambda$ for all $\lambda > 0, y \in \mathbb{R}^d$ and linear in y , and a function $M^\alpha(x) : A \times D \rightarrow \mathbb{R}$, bounded on every set in the form of $A \times D_\lambda$ for all $\lambda > 0$, such that for any $\alpha \in A, x \in D$ and $|y| = 1$,*

$$\begin{aligned} & \left\| \sigma_{(y)}^\alpha(x) + (\rho^\alpha(x), y) \sigma^\alpha(x) + \sigma^\alpha(x) Q^\alpha(x, y) \right\|^2 + \\ & 2(y, b_{(y)}^\alpha(x) + 2(\rho^\alpha(x), y) b^\alpha(x)) \leq c^\alpha(x) + M^\alpha(x) (a^\alpha(x) y, y). \end{aligned} \quad (4.10)$$

Then we have

1. *If for any $\alpha \in A, f^\alpha, g \in C^{0,1}(\bar{D})$, satisfying*

$$\sup_{\alpha \in A} |f^\alpha|_{0,1,D} + |g|_{0,1,D} \leq K_0,$$

then $v \in C^{0,1}(D)$, and for any $\xi \in \mathbb{R}^d$,

$$|v_{(\xi)}(x)| \leq N \left(|\xi| + \frac{|\psi_{(\xi)}|}{\psi^{\frac{1}{2}}} \right), \text{ a.e. in } D, \quad (4.11)$$

where the constant N depends only on d , d_1 and K_0 .

2. If for any $\alpha \in A$, $f^\alpha \in C^{0,1}(\bar{D})$, $g \in C^{1,1}(\bar{D})$, satisfying

$$\sup_{\alpha \in A} |f^\alpha|_{0,1,D} + |g|_{1,1,D} \leq K_0,$$

and f_{xx}^α exists almost everywhere in D , satisfying

$$f_{xx}^\alpha + K_0 I \geq 0, \text{ a.e. in } D,$$

then for any $\xi \in \mathbb{R}^d$,

$$v_{(\xi)(\xi)}(x) \geq -N \left(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi} \right), \text{ a.e. in } D, \quad (4.12)$$

$$v_{(\xi)(\xi)}(x) \leq \mu(x, \xi/|\xi|)^{-1} N \frac{|\xi|^2}{\psi}, \text{ a.e. in } D(\xi), \quad (4.13)$$

where $D(\xi) := \{x \in D : \mu(x, \xi) > 0\}$, and the constant N depends only on d , d_1 and K_0 .

If $\mu(x) > 0$ in D , then $v \in C_{loc}^{1,1}(D)$. In addition, v given by (4.4) is the unique solution in $C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$ of

$$\begin{cases} \sup_{\alpha \in A} [L^\alpha v(x) - c(\alpha, x)v(x) + f(\alpha, x)] = 0 & \text{a.e. in } D \\ v = g & \text{on } \partial D. \end{cases} \quad (4.14)$$

We emphasize that the constants N in (4.11), (4.12) and (4.13) are independent of ρ^α , Q^α and M^α in (3.10).

Remark 4.1.1. *The author doesn't know whether the estimates (4.11), (4.12) and (4.13) are sharp.*

4.2 Auxiliary convergence results

Let U be a connected open subset in \mathbb{R}^d . Assume that, for any $\alpha \in \mathfrak{A}, \omega \in \Omega, t \geq 0$, and $x \in U$, we are given a $d \times d_1$ matrix $\kappa_t^\alpha(x)$ and a d -dimensional vector $\nu_t^\alpha(x)$. We assume that κ_t^α and ν_t^α are continuous in x for any α, ω, t , measurable in (ω, t) for any α, x , and \mathcal{F}_t -measurable in ω for any α, t, x . Assume that for any $\alpha \in \mathfrak{A}$, the Itô equation

$$d\zeta_t^{\alpha, \zeta} = \kappa_t^\alpha(\zeta_t^{\alpha, \zeta})dw_t + \nu_t^\alpha(\zeta_t^{\alpha, \zeta})dt \quad (4.15)$$

has a unique solution.

We suppose that for an $\epsilon_0 \in (0, 1]$ and for each $\epsilon \in [0, \epsilon_0]$, we are given

$$\kappa_t^\alpha(\epsilon) = \kappa_t^\alpha(x, \epsilon), \quad \nu_t^\alpha(\epsilon) = \nu_t^\alpha(x, \epsilon)$$

having the same meaning and satisfying the same assumptions as those of κ_t^α and ν_t^α . Assume that for any $\alpha \in \mathfrak{A}$, the Itô equation (4.15) corresponding to $\kappa_t^\alpha(\epsilon)$ and $\nu_t^\alpha(\epsilon)$ with initial condition $\zeta(\epsilon) \in U$

$$d\zeta_t^{\alpha, \zeta(\epsilon)}(\epsilon) = \kappa_t^\alpha(\zeta_t^{\alpha, \zeta(\epsilon)}(\epsilon), \epsilon)dw_t + \nu_t^\alpha(\zeta_t^{\alpha, \zeta(\epsilon)}(\epsilon), \epsilon)dt \quad (4.16)$$

has a unique solution denoted by $\zeta_t^{\alpha, \zeta(\epsilon)}(\epsilon)$.

Lemma 4.2.1. *Let $q \in [2, \infty)$, $\theta \in (0, 1)$, $M \in (0, \infty)$ be constants and M_t^α be a \mathcal{F}_t -adapted nonnegative process for any $\alpha \in \mathfrak{A}$.*

1. *If for any $\alpha \in \mathfrak{A}, t \geq 0, x \in U$,*

$$\|\kappa_t^\alpha(x)\| + |\nu_t^\alpha(x)| \leq M|x| + M_t^\alpha, \quad (4.17)$$

then for any bounded stopping times $\gamma^\alpha \leq \tau_U^{\alpha, \zeta}, \forall \alpha$

$$\begin{aligned} & \sup_{\alpha \in \mathfrak{A}} E_\zeta^\alpha \sup_{t \leq \gamma} e^{-Nt} |\zeta_t|^q \\ & \leq |\zeta|^q + (2q-1) \sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^\gamma M_t^q e^{-Nt} dt, \end{aligned} \quad (4.18)$$

$$\begin{aligned} & \sup_{\alpha \in \mathfrak{A}} E_\zeta^\alpha \sup_{t \leq \gamma} e^{-Nt} |\zeta_t|^{q\theta} \\ & \leq \frac{2-\theta}{1-\theta} \left(|\zeta|^{q\theta} + (2q-1)^\theta \sup_{\alpha \in \mathfrak{A}} E^\alpha \left(\int_0^\gamma M_t^q e^{-Nt} dt \right)^\theta \right), \end{aligned} \quad (4.19)$$

where $N = N(q, M)$.

2. If for any $\alpha \in A, t \geq 0, x \in U$, and some $\epsilon \in [0, \epsilon_0]$,

$$\|\kappa_t^\alpha(x) - \kappa_t^\alpha(y, \epsilon)\| + |\nu_t^\alpha(x) - \nu_t^\alpha(y, \epsilon)| \leq M|x - y| + \epsilon M_t^\alpha, \quad (4.20)$$

then for any bounded stopping times $\gamma^\alpha \leq \tau_U^{\alpha, \zeta} \wedge \tau_U^{\alpha, \zeta(\epsilon)}(\epsilon)$, $\forall \alpha$

$$\begin{aligned} & \sup_{\alpha \in \mathfrak{A}} E \sup_{t \leq \gamma^\alpha} e^{-Nt} |\zeta_t^{\alpha, \zeta(\epsilon)}(\epsilon) - \zeta_t^{\alpha, \zeta}|^q \\ & \leq |\zeta(\epsilon) - \zeta|^q + \epsilon^q (2q - 1) \sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^\gamma M_t^q e^{-Nt} dt, \end{aligned} \quad (4.21)$$

$$\begin{aligned} & \sup_{\alpha \in \mathfrak{A}} E \sup_{t \leq \gamma^\alpha} e^{-Nt} |\zeta_t^{\alpha, \zeta(\epsilon)}(\epsilon) - \zeta_t^{\alpha, \zeta}|^{q\theta} \\ & \leq \frac{2 - \theta}{1 - \theta} \left(|\zeta(\epsilon) - \zeta|^{q\theta} + \epsilon^{q\theta} (2q - 1)^\theta \sup_{\alpha \in \mathfrak{A}} E^\alpha \left(\int_0^\gamma M_t^q e^{-Nt} dt \right)^\theta \right), \end{aligned} \quad (4.22)$$

where $N = N(q, M)$.

Remark 4.2.1. Observe that $q\theta$ covers $(0, \infty)$.

Proof. It suffices to prove the uncontrolled version of (4.18), (4.19), (4.21) and (4.22), so we drop the index α in what follows for simplicity of notation. We also denote $\zeta_t = \zeta_t^{\alpha, \zeta}$, $\zeta_t(\epsilon) = \zeta_t^{\alpha, \zeta(\epsilon)}(\epsilon)$.

Also, choosing a localizing sequence of stopping times $\gamma_n \uparrow \infty$ such that $\int_0^{t \wedge \gamma_n} M_s^q e^{-Ns} ds$ are bounded for every n , we see, in view of the Monotone Convergence Theorem, that it will suffice to consider the case in which $\int_0^t M_s^q e^{-Ns} ds$ are bounded with respect to (ω, t) .

By Itô's formula, we have

$$\begin{aligned} de^{-Nt} |\zeta_t|^q &= e^{-Nt} \left[q |\zeta_t|^{q-2} (\zeta_t, \nu_t(\zeta_t)) + \frac{q}{2} |\zeta_t|^{q-2} \|\kappa_t(\zeta_t)\|^2 \right. \\ & \quad \left. + \frac{q(q-2)}{2} |\zeta_t|^{q-4} |\kappa_t^*(\zeta_t) \zeta_t|^2 - N |\zeta_t|^q \right] dt + dm_t, \end{aligned}$$

where m_t is a local martingale starting at zero. By (4.17) and Young's inequality

$$\begin{aligned} q |\zeta_t|^{q-2} (\zeta_t, \nu_t(\zeta_t)) &\leq (qM + q - 1) |\zeta_t|^q + M_t^q \\ \frac{q}{2} |\zeta_t|^{q-2} \|\kappa_t(\zeta_t)\|^2 &\leq q |\zeta_t|^{q-2} (M^2 |\zeta_t|^2 + M_t^2) \leq (qM^2 + q - 2) |\zeta_t|^q + 2M_t^q \\ \frac{q(q-2)}{2} |\zeta_t|^{q-4} |\kappa_t^*(\zeta_t) \zeta_t|^2 &\leq (q-2) [(qM^2 + q - 2) |\zeta_t|^q + 2M_t^q] \end{aligned}$$

So for sufficiently large constant $N = N(q, M)$, we have

$$e^{-Nt}|\zeta_t|^q \leq |\zeta|^q + (2q-1) \int_0^t M_t^q e^{-Nt} dt.$$

Applying Lemma 7.3(i) in [18], we get

$$E \sup_{t \leq \gamma} e^{-Nt} |\zeta_t|^q \leq |\zeta|^q + (2q-1) E \int_0^\gamma M_t^q e^{-Nt} dt.$$

Due to Lemma 7.3(ii) in [18], we conclude that

$$\begin{aligned} E \sup_{t \leq \gamma} e^{-Nt} |\zeta_t|^{q\theta} &\leq \frac{2-\theta}{1-\theta} E \left(|\zeta|^q + (2q-1) \int_0^\gamma M_t^q e^{-Nt} dt \right)^\theta \\ &\leq \frac{2-\theta}{1-\theta} \left(|\zeta|^{q\theta} + (2q-1)^\theta E \left(\int_0^\gamma M_t^q e^{-Nt} dt \right)^\theta \right). \end{aligned}$$

Similarly, by Itô's formula,

$$\begin{aligned} &d \left(e^{-Nt} |\zeta_t(\epsilon) - \zeta_t|^q \right) \\ &= e^{-Nt} \left[q |\zeta_t(\epsilon) - \zeta_t|^{q-2} \left(\zeta_t(\epsilon) - \zeta_t, \nu_t(\zeta_t(\epsilon), \epsilon) - \nu_t(\zeta_t) \right) \right. \\ &\quad + \frac{q}{2} |\zeta_t(\epsilon) - \zeta_t|^{q-2} \|\kappa_t(\zeta_t(\epsilon), \epsilon) - \kappa_t(\zeta_t)\|^2 \\ &\quad + \frac{q(q-2)}{2} |\zeta_t(\epsilon) - \zeta_t|^{q-4} \left| (\kappa_t^*(\zeta_t(\epsilon), \epsilon) - \kappa_t^*(\zeta_t)) (\zeta_t(\epsilon) - \zeta_t) \right|^2 \\ &\quad \left. - N |\zeta_t(\epsilon) - \zeta_t|^q \right] dt + dm_t, \end{aligned}$$

where m_t is a local martingale starting at zero. By (4.20), we have

$$\|\kappa_t(\zeta_t(\epsilon), \epsilon) - \kappa_t(\zeta_t)\| + |\nu_t(\zeta_t(\epsilon), \epsilon) - \nu_t(\zeta_t)| \leq M |\zeta_t(\epsilon) - \zeta_t| + \epsilon M_t,$$

which can play the same role as (4.17). So (4.21) and (4.22) can be proved by mimicking the argument for proving (4.18) and (4.19). □

Next, we introduce the quasiderivatives to be used in the proof of the main theorem and apply Lemmas 4.2.1 to estimate moments of these quasiderivatives.

For any $\alpha \in \mathfrak{A}$, let $r_t^\alpha, \hat{r}_t^\alpha, \pi_t^\alpha, \hat{\pi}_t^\alpha, P_t^\alpha, \hat{P}_t^\alpha$ be jointly measurable adapted processes with values in $\mathbb{R}, \mathbb{R}, \mathbb{R}^{d_1}, \mathbb{R}^{d_1}, \text{Skew}(d_1, \mathbb{R}), \text{Skew}(d_1, \mathbb{R})$, respectively, where $\text{Skew}(d_1, \mathbb{R})$

denotes the set of all $d_1 \times d_1$ skew-symmetric real matrices. Let ϵ be a small positive constant. For each $\alpha \in \mathfrak{A}$, $x, y, z \in D$, $\xi, \eta \in \mathbb{R}^d$, we consider the Itô equation (4.2) and the following four other Itô equations:

$$dy_t^{\alpha,y}(\epsilon) = \sqrt{1 + 2\epsilon r_t^\alpha} \sigma^{\alpha t}(y_t^{\alpha,y}(\epsilon)) e^{\epsilon P_t^\alpha} dw_t + \left[(1 + 2\epsilon r_t^\alpha) b^{\alpha t}(y_t^{\alpha,y}(\epsilon)) - \sqrt{1 + 2\epsilon r_t^\alpha} \sigma^{\alpha t}(y_t^{\alpha,y}(\epsilon)) e^{\epsilon P_t^\alpha} \epsilon \pi_t^\alpha \right] dt, \quad (4.23)$$

$$dz_t^{\alpha,z}(\epsilon) = \sqrt{1 + 2\epsilon r_t^\alpha + \epsilon^2 \hat{r}_t^\alpha} \sigma^{\alpha t}(z_t^{\alpha,z}(\epsilon)) e^{\epsilon P_t^\alpha} e^{\frac{\epsilon^2}{2} \hat{P}_t^\alpha} dw_t + \left[(1 + 2\epsilon r_t^\alpha + \epsilon^2 \hat{r}_t^\alpha) b^{\alpha t}(z_t^{\alpha,z}(\epsilon)) - \sqrt{1 + 2\epsilon r_t^\alpha + \epsilon^2 \hat{r}_t^\alpha} \sigma^{\alpha t}(z_t^{\alpha,z}(\epsilon)) e^{\epsilon P_t^\alpha} e^{\frac{\epsilon^2}{2} \hat{P}_t^\alpha} (\epsilon \pi_t^\alpha + \frac{\epsilon^2}{2} \hat{\pi}_t^\alpha) \right] dt, \quad (4.24)$$

$$d\xi_t^{\alpha,\xi} = \left[\sigma_{(\xi_t^{\alpha,\xi})}^{\alpha t} + r_t^\alpha \sigma^{\alpha t} + \sigma^{\alpha t} P_t^\alpha \right] dw_t + \left[b_{(\xi_t^{\alpha,\xi})}^{\alpha t} + 2r_t^\alpha b^{\alpha t} - \sigma^{\alpha t} \pi_t^\alpha \right] dt, \quad (4.25)$$

$$d\eta_t^{\alpha,\eta} = \left[\sigma_{(\eta_t^{\alpha,\eta})}^{\alpha t} + \hat{r}_t^\alpha \sigma^{\alpha t} + \sigma^{\alpha t} \hat{P}_t^\alpha + \sigma_{(\xi_t^{\alpha,\xi})(\xi_t^{\alpha,\xi})}^{\alpha t} + 2r_t^\alpha \sigma_{(\xi_t^{\alpha,\xi})}^{\alpha t} + 2\sigma_{(\xi_t^{\alpha,\xi})}^{\alpha t} P_t^\alpha + 2r_t^\alpha \sigma^{\alpha t} P_t^\alpha - (r_t^\alpha)^2 \sigma^{\alpha t} + \sigma^{\alpha t} (P_t^\alpha)^2 \right] dw_t + \left[b_{(\eta_t^{\alpha,\eta})}^{\alpha t} + 2\hat{r}_t^\alpha b^{\alpha t} - \sigma^{\alpha t} \hat{\pi}_t^\alpha + b_{(\xi_t^{\alpha,\xi})(\xi_t^{\alpha,\xi})}^{\alpha t} + 4r_t^\alpha b_{(\xi_t^{\alpha,\xi})}^{\alpha t} - 2\sigma_{(\xi_t^{\alpha,\xi})}^{\alpha t} \pi_t^\alpha - 2r_t^\alpha \sigma^{\alpha t} \pi_t^\alpha - 2\sigma^{\alpha t} P_t^\alpha \pi_t^\alpha \right] dt, \quad (4.26)$$

where σ^α and b^α satisfy (4.1) and we drop the arguments $x_t^{\alpha,x}$ in $\sigma^{\alpha t}$ and $b^{\alpha t}$ and their derivatives in (4.25) and (4.26).

Let $\bar{\tau}_D^{\alpha,y}(\epsilon)$ be the first exit time of $y_t^{\alpha,y}(\epsilon)$ from D , and $\hat{\tau}_D^{\alpha,z}(\epsilon)$ be the first exit time of $z_t^{\alpha,z}(\epsilon)$ from D .

It is known that if

$$\int_0^T (|r_t^\alpha|^2 + |\pi_t^\alpha|^2 + |P_t^\alpha|^2) dt < \infty, \quad (4.27)$$

$$\forall T \in [0, \infty), \forall \alpha \in \mathfrak{A},$$

then (4.23) and (4.25) have unique solutions on $[0, \bar{\tau}_D^{\alpha,y}(\epsilon))$ and $[0, \tau_D^{\alpha,x})$, respectively. If

$$\int_0^T (|\hat{r}_t^\alpha|^2 + |\hat{\pi}_t^\alpha|^2 + |\hat{P}_t^\alpha|^2 + |r_t^\alpha|^4 + |\pi_t^\alpha|^4 + |P_t^\alpha|^4) dt < \infty, \quad (4.28)$$

$$\forall T \in [0, \infty), \forall \alpha \in \mathfrak{A},$$

then (4.24) and (4.26) have unique solutions on $[0, \hat{\tau}_D^{\alpha, z}(\epsilon))$ and $[0, \tau_D^{\alpha, x})$, respectively. It is shown in Theorem 2.2.2 that for each $\alpha \in \mathfrak{A}$, $\xi_t^{\alpha, \xi}$ is a first quasiderivative of $x_t^{\alpha, x}$ in D in the direction of ξ at x and

$$\xi_t^{0, \alpha} = \int_0^t \pi_s^\alpha dw_s \quad (4.29)$$

is its first adjoint process, and $\eta_t^{\alpha, \eta}$ is a second quasiderivative of $x_t^{\alpha, x}$ associated with $\xi_t^{\alpha, \xi}$ in D in the direction of η at x and

$$\eta_t^{0, \alpha} = (\xi_t^{0, \alpha})^2 - \langle \xi^{0, \alpha} \rangle_t + \int_0^t \hat{\pi}_s^\alpha dw_s \quad (4.30)$$

is its second adjoint process.

Theorem 4.2.1. *Given constants $p \in (0, \infty)$, $p' \in [0, p)$, $T \in [1, \infty)$, $x \in D$, $\xi \in \mathbb{R}^d$. Suppose (4.27) is satisfied. Assume that there exists a constant $K \in [1, \infty)$ and for any $\alpha \in \mathfrak{A}$, an adapted nonnegative process K_t^α , such that*

$$|r_t^\alpha| + |\pi_t^\alpha| + |P_t^\alpha| \leq K |\xi_t^{\alpha, \xi}| + K_t^\alpha, \forall \alpha. \quad (4.31)$$

1. *Given stopping times $\gamma^\alpha \leq \tau_D^{\alpha, x}$, $\alpha \in \mathfrak{A}$, if*

$$\sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^{\gamma \wedge T} K_t^{2 \vee p} dt < \infty, \quad (4.32)$$

then we have

$$\sup_{\alpha \in \mathfrak{A}} E_\xi^\alpha \sup_{t \leq \gamma \wedge T} |\xi_t|^p < \infty. \quad (4.33)$$

2. *Let the constant ϵ_0 be sufficiently small so that $B(x, \epsilon_0 |\xi|) \subset D$. For any $\epsilon \in [0, \epsilon_0]$, given stopping times $\gamma^\alpha(\epsilon) \leq \tau_D^{\alpha, x} \wedge \bar{\tau}_D^{\alpha, x + \epsilon \xi}(\epsilon)$, $\alpha \in \mathfrak{A}$, if*

$$\sup_{\epsilon \in [0, \epsilon_0]} \sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^{\gamma(\epsilon) \wedge T} K_t^{2(2 \vee p)} dt < \infty, \quad (4.34)$$

then we have

$$\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathfrak{A}} \sup_{t \leq \gamma^\alpha(\epsilon) \wedge T} \frac{|y_t^{\alpha, x + \epsilon \xi}(\epsilon) - x_t^{\alpha, x}|^p}{\epsilon^{p'}} = 0, \quad (4.35)$$

$$\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathfrak{A}} \sup_{t \leq \gamma^\alpha(\epsilon) \wedge T} \left| \frac{y_t^{\alpha, x + \epsilon \xi}(\epsilon) - x_t^{\alpha, x}}{\epsilon} - \xi_t^{\alpha, \xi} \right|^p = 0. \quad (4.36)$$

Proof. In the proof, we drop the superscripts α , α_t , etc., when this will not cause confusion.

To prove (1) we consider the Itô equation (4.15) in which $\zeta_t^{\alpha,\zeta} = \xi_t^{\alpha,\xi}$. By conditions (4.1) and (4.31), we have

$$\|\sigma_{(\xi_t)} + r_t\sigma + \sigma P_t\| + |b_{(\xi_t^{\alpha,\xi})} + 2r_t b - \sigma\pi_t| \leq M|\xi_t| + M_t, \forall \alpha,$$

where $M = N(K, K_0)$, $M_t^\alpha = N(K_0)K_t^\alpha$. Applying Lemma 4.2.1(1), we have

$$\sup_{\alpha \in \mathfrak{A}} E_\xi^\alpha \sup_{t \leq \gamma \wedge T} |\xi_t|^p \leq \begin{cases} e^{NT} (|\xi|^p + (2p-1) \sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^{\gamma \wedge T} M_t^p dt) & \text{if } p \geq 2 \\ e^{NT \frac{4-p}{2-p}} (|\xi|^p + 3^{\frac{p}{2}} (\sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^{\gamma \wedge T} M_t^2 dt)^{\frac{p}{2}}) & \text{if } p < 2. \end{cases}$$

To prove (2) we first consider the Itô equations (4.15) and (4.16) in which

$$\zeta_t^{\alpha,\zeta} = x_t^{\alpha,x}, \quad \zeta_t^{\alpha,\zeta(\epsilon)}(\epsilon) = y_t^{\alpha,x+\epsilon\xi}(\epsilon).$$

Notice that

$$\begin{aligned} \|\kappa_t(y, \epsilon) - \kappa_t(x)\| &= \|\sqrt{1+2\epsilon r_t}\sigma(y)e^{\epsilon P_t} - \sigma(x)\| \\ &\leq |\sqrt{1+2\epsilon r_t} - 1| \|\sigma(y)e^{\epsilon P_t}\| + \|\sigma(y)\| \|e^{\epsilon P_t} - I_{d_1 \times d_1}\| \\ &\quad + \|\sigma(y) - \sigma(x)\| \\ &\leq 2\epsilon|r_t|K_0 + K_0\epsilon e^{\epsilon P_t} + K_0|y-x| \\ &\leq M|y-x| + \epsilon M_t, \\ |\nu_t(y, \epsilon) - \nu_t(x)| &= |(1+2\epsilon r_t)b(y) - \sqrt{1+2\epsilon r_t}\sigma(y)e^{\epsilon P_t}\epsilon\pi_t - b(x)| \\ &\leq 2\epsilon|r_t|K_0 + (1+\epsilon|r_t|)K_0\epsilon|\pi_t| + K_0|y-x| \\ &\leq M|y-x| + \epsilon M_t, \end{aligned}$$

where $M = K_0$, $M_t^\alpha = N(K, K_0)(|\xi_t^{\alpha,\xi}| + (K_t^\alpha)^2 \vee 1)$. Applying Lemma 4.2.1(2), we have

$$\begin{aligned} &\sup_{\alpha \in \mathfrak{A}} E \sup_{t \leq \gamma^\alpha(\epsilon) \wedge T} |y_t^{\alpha,x+\epsilon\xi}(\epsilon) - x_t^{\alpha,x}|^p \\ &\leq \begin{cases} \epsilon^p e^{NT} (|\xi|^p + (2p-1) \sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^{\gamma(\epsilon) \wedge T} M_t^p dt) & \text{if } p \geq 2 \\ \epsilon^p e^{NT \frac{4-p}{2-p}} (|\xi|^p + 3^{\frac{p}{2}} (\sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^{\gamma(\epsilon) \wedge T} M_t^2 dt)^{\frac{p}{2}}) & \text{if } p < 2. \end{cases} \end{aligned}$$

Due to (4.34) and (4.33), we have

$$\sup_{[0,\epsilon_0]} \sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^{\gamma(\epsilon) \wedge T} M_t^{2 \vee p} dt < \infty,$$

which completes the proof of (4.35).

Next, we first consider the Itô equations (4.15) and (4.16) in which

$$\zeta_t^{\alpha, \zeta} = \xi_t^{\alpha, \xi}, \quad \zeta_t^{\alpha, \zeta(\epsilon)}(\epsilon) = \xi_t^{\alpha, \xi}(\epsilon) := \frac{y_t^{\alpha, x + \epsilon \xi}(\epsilon) - x_t^{\alpha, x}}{\epsilon}.$$

Observe that

$$\begin{aligned} \left\| \frac{\sigma(y_t(\epsilon)) - \sigma(x_t)}{\epsilon} - \sigma_{(\xi_t)}(x_t) \right\| &= \left\| \sigma_{(\xi_t(\epsilon))}(y_t^*(\epsilon)) - \sigma_{(\xi_t)}(x_t) \right\| \\ &= \left\| \sigma_{(\xi_t(\epsilon))}(y_t^*(\epsilon)) - \sigma_{(\xi_t(\epsilon))}(x_t) \right\| + \left\| \sigma_{(\xi_t(\epsilon))}(x_t) - \sigma_{(\xi_t)}(x_t) \right\| \\ &\leq K_0 \frac{|y_t(\epsilon) - x_t|^2}{\epsilon} + K_0 |\xi_t(\epsilon) - \xi_t|, \\ \left| \frac{b(y_t(\epsilon)) - b(x_t)}{\epsilon} - b_{(\xi_t)}(x_t) \right| &\leq K_0 \frac{|y_t(\epsilon) - x_t|^2}{\epsilon} + K_0 |\xi_t(\epsilon) - \xi_t|, \\ \left| \frac{\sqrt{1 + 2\epsilon r_t} - 1}{\epsilon} - r_t \right| &= \left| r_t \left(\frac{2}{\sqrt{1 + 2\epsilon r_t} + 1} - 1 \right) \right| \\ &= \left| \frac{-2\epsilon r_t^2}{(1 + \sqrt{1 + 2\epsilon r_t})^2} \right| \leq 2\epsilon |r_t|^2, \\ \left\| \frac{e^{\epsilon P_t} - 1}{\epsilon} - P_t \right\| &= \frac{\epsilon}{2} \|P_t^2 e^{\epsilon' P_t}\| \leq \frac{\epsilon}{2} \|P_t\|^2. \end{aligned}$$

The equation (4.36) can be proved by mimicking the proof of (4.35). \square

Theorem 4.2.2. *Given constants $p \in (0, \infty)$, $p' \in [0, p)$, $T \in [1, \infty)$, $x \in D$, $\xi \in \mathbb{R}^d$, $\eta \in \mathbb{R}^d$. Suppose (4.28) is satisfied. Assume that there exists a constant $K \in [1, \infty)$ and for any $\alpha \in \mathfrak{A}$, an adapted nonnegative process K_t^α , such that*

$$|\hat{r}_t^\alpha| + |\hat{\pi}_t^\alpha| + |\hat{P}_t^\alpha| + |r_t^\alpha|^2 + |\pi_t^\alpha|^2 + |P_t^\alpha|^2 \leq K(|\eta_t^{\alpha, \eta}| + |\xi_t^{\alpha, \xi}|^2) + K_t^\alpha, \forall \alpha. \quad (4.37)$$

1. *Given stopping times $\gamma^\alpha \leq \tau_D^{\alpha, x}$, $\alpha \in \mathfrak{A}$, if (4.32) holds, then we have (4.33) and*

$$\sup_{\alpha \in \mathfrak{A}} E_\eta^\alpha \sup_{t \leq \gamma \wedge T} |\eta_t|^p < \infty. \quad (4.38)$$

2. *Let the constant ϵ_0 be sufficiently small so that $B(x, \epsilon_0|\xi|) \subset D$. For any $\epsilon \in [0, \epsilon_0]$, let*

$$x(\epsilon) = x + \epsilon \xi + \frac{\epsilon^2}{2} \eta.$$

If (4.34) holds for given stopping times $\gamma_2^\alpha(\epsilon)$ satisfying

$$\gamma_2^\alpha(\epsilon) \leq \tau_D^{\alpha, x} \wedge \hat{\tau}_D^{\alpha, x(\epsilon)}(\epsilon), \alpha \in \mathfrak{A},$$

then we have

$$\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathfrak{A}} \sup_{t \leq \gamma_2^\alpha(\epsilon) \wedge T} \frac{|z_t^{\alpha, x(\epsilon)}(\epsilon) - x_t^{\alpha, x}|^p}{\epsilon^{p'}} = 0, \quad (4.39)$$

$$\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathfrak{A}} \sup_{t \leq \gamma_2^\alpha(-\epsilon) \wedge T} \frac{|z_t^{\alpha, x(-\epsilon)}(-\epsilon) - x_t^{\alpha, x}|^p}{\epsilon^{p'}} = 0, \quad (4.40)$$

$$\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathfrak{A}} \sup_{t \leq \gamma_2^\alpha(\epsilon) \wedge T} \left| \frac{z_t^{\alpha, x(\epsilon)}(\epsilon) - x_t^{\alpha, x}}{\epsilon} - \xi_t^{\alpha, \xi} \right|^p = 0. \quad (4.41)$$

If (4.34) holds for given stopping times $\gamma_3^\alpha(\epsilon)$ satisfying

$$\gamma_3^\alpha(\epsilon) \leq \tau_D^{\alpha, x} \wedge \hat{\tau}_D^{\alpha, x(\epsilon)}(\epsilon) \wedge \hat{\tau}_D^{\alpha, x(-\epsilon)}(-\epsilon), \alpha \in \mathfrak{A},$$

then we have

$$\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathfrak{A}} \sup_{t \leq \gamma_3^\alpha(\epsilon) \wedge T} \left| \frac{z_t^{\alpha, x(\epsilon)}(\epsilon) - 2x_t^{\alpha, x} + z_t^{\alpha, x(-\epsilon)}(-\epsilon)}{\epsilon} - \eta_t^{\alpha, \eta} \right|^p = 0. \quad (4.42)$$

Proof. Again, we drop superscripts α , x_t , etc., when this will cause no confusion.

The inequality (4.38) can be proved by observing that (4.37) and (4.32) imply that

$$\sup_{\alpha \in \mathfrak{A}} E_\xi^\alpha \sup_{t \leq \gamma \wedge T} |\xi_t|^{2p} < \infty$$

and then mimicking the proof of (4.33).

The equations (4.39) and (4.41) are obtained by repeating the proof of (4.35) and (4.36). The equation (4.40) is obvious once we get (4.39).

To proof (4.42), we observe that, for example,

$$\begin{aligned} & \frac{\sigma(z_t(\epsilon)) - 2\sigma(x_t) + \sigma(z_t(-\epsilon))}{\epsilon^2} \\ &= \frac{1}{\epsilon^2} [\sigma_{(z_t(\epsilon)-x_t)}(x_t) + \frac{1}{2}\sigma_{(z_t(\epsilon)-x_t)(z_t(\epsilon)-x_t)}(\bar{z}_t(\epsilon)) \\ & \quad + \sigma_{(z_t(-\epsilon)-x_t)}(x_t) + \frac{1}{2}\sigma_{(z_t(-\epsilon)-x_t)(z_t(-\epsilon)-x_t)}(\bar{z}_t(-\epsilon))] \\ &= \sigma_{(\eta_t(\epsilon))} + \frac{1}{2} [\sigma_{(\xi_t(\epsilon))(\xi_t(\epsilon))}(\bar{z}_t(\epsilon)) + \sigma_{(\xi_t(-\epsilon))(\xi_t(-\epsilon))}(\bar{z}_t(-\epsilon))], \end{aligned}$$

where

$$\eta_t(\epsilon) = \frac{z_t(\epsilon) - 2x_t + z_t(-\epsilon)}{\epsilon^2}, \quad \xi_t(\epsilon) = \frac{z_t(\epsilon) - x_t}{\epsilon},$$

$\bar{z}_t(\epsilon)$ is a point on the straight line segment with endpoints x_t and $z_t(\epsilon)$, and $\bar{z}_t(-\epsilon)$ is a point on the straight line segment with endpoints x_t and $z_t(-\epsilon)$.

It follows that

$$\begin{aligned} & \left\| \frac{\sigma(z_t(\epsilon)) - 2\sigma(x_t) + \sigma(z_t(-\epsilon))}{\epsilon^2} - \sigma_{(\eta_t)}(x_t) - \sigma_{(\xi_t)(\xi_t)}(x_t) \right\| \\ & \leq K_0 \left| \frac{z_t(\epsilon) - 2x_t + z_t(-\epsilon)}{\epsilon^2} - \eta_t \right| + K_0 \frac{|z_t(\epsilon) - x_t|^3}{\epsilon^2} + K_0 \frac{|z_t(-\epsilon) - x_t|^3}{\epsilon^2} \\ & \quad + K_0 \left| \frac{z_t(\epsilon) - x_t}{\epsilon} - \xi_t \right|^2 + K_0 \left| \frac{z_t(-\epsilon) - x_t}{\epsilon} - \xi_t \right|^2. \end{aligned}$$

It remains to mimic the proof of (4.36). \square

We end up this section by showing a convergence result about the stopping times which will be applied in the next section.

Theorem 4.2.3. *Let δ be a positive constant such that $D_\delta = \{x \in D : \psi > \delta\}$ is nonempty, and δ_1, δ_2 be positive constants satisfying $\delta_1 < \delta_2$. Let $D_{\delta_1}^{\delta_2} = \{x \in D : \delta_1 < \psi < \delta_2\}$. Then for any $x \in D$, if (4.35) holds with*

$$\gamma^\alpha(\epsilon) = \tau_D^{\alpha,x} \wedge \bar{\tau}_D^{\alpha,x+\epsilon\xi}(\epsilon),$$

for $p = 1, p' = 0$ and $\forall T \in [1, \infty)$, then we have

$$\limsup_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E(\tau_D^{\alpha,x} - \tau_D^{\alpha,x} \wedge \bar{\tau}_D^{\alpha,x+\epsilon\xi}(\epsilon)) = 0. \quad (4.43)$$

For any $x \in D$, if (4.39) and (4.40) hold with

$$\gamma^\alpha(\epsilon) = \tau_D^{\alpha,x} \wedge \hat{\tau}_D^{\alpha,x(\epsilon)}(\epsilon) \text{ and } \gamma^\alpha(-\epsilon) = \tau_D^{\alpha,x} \wedge \hat{\tau}_D^{\alpha,x(-\epsilon)}(-\epsilon),$$

respectively, for $p = 1, p' = 0$ and $\forall T \in [1, \infty)$, then we have

$$\limsup_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E(\tau_D^{\alpha,x} - \tau_D^{\alpha,x} \wedge \hat{\tau}_D^{\alpha,x(\epsilon)}(\epsilon) \wedge \hat{\tau}_D^{\alpha,x(-\epsilon)}(-\epsilon)) = 0. \quad (4.44)$$

The statement still holds when replacing D by D_δ or $D_{\delta_1}^{\delta_2}$, provided that δ_2 is sufficiently small.

Proof. We drop the subscript D and the argument ϵ for simplicity of notation. Notice that, for any $\alpha \in \mathfrak{A}$,

$$E(\tau^{\alpha,x} - \gamma^\alpha) = E \int_{\gamma^\alpha}^{\tau^{\alpha,x}} 1 dt$$

$$\begin{aligned}
&\leq -E \int_{\gamma^\alpha}^{\tau^{\alpha,x}} L^\alpha \psi(x_t^{\alpha,x}) dt \\
&= -E \left(\psi(x_{\bar{\tau}^\alpha}^{\alpha,x}) - \psi(x_{\gamma^\alpha}^{\alpha,x}) \right) I_{\gamma^\alpha < \tau^{\alpha,x}} \\
&= E \psi(x_{\bar{\tau}^\alpha, x+\epsilon\xi}^{\alpha,x}) I_{\bar{\tau}^\alpha, x+\epsilon\xi < \tau^{\alpha,x}} \\
&= E \left(\psi(x_{\bar{\tau}^\alpha, x+\epsilon\xi}^{\alpha,x}) - \psi(y_{\bar{\tau}^\alpha, x+\epsilon\xi}^{\alpha, x+\epsilon\xi}) \right) I_{\bar{\tau}^\alpha, x+\epsilon\xi < \tau^{\alpha,x}} \\
&\leq E \left(\psi(x_{\bar{\tau}^\alpha, x+\epsilon\xi}^{\alpha,x}) - \psi(y_{\bar{\tau}^\alpha, x+\epsilon\xi}^{\alpha, x+\epsilon\xi}) \right) I_{\bar{\tau}^\alpha, x+\epsilon\xi < \tau^{\alpha,x} \leq T} + 2K_0 P_x^\alpha(\tau > T).
\end{aligned}$$

Due to (4.35), we have

$$\begin{aligned}
&\overline{\lim}_{\epsilon \downarrow 0} \left(\sup_\alpha E \left(\psi(x_{\bar{\tau}^\alpha, x+\epsilon\xi}^{\alpha,x}) - \psi(y_{\bar{\tau}^\alpha, x+\epsilon\xi}^{\alpha, x+\epsilon\xi}) \right) I_{\bar{\tau}^\alpha, x+\epsilon\xi < \tau^{\alpha,x} \leq T} \right) \\
&\leq \sup_D |\psi_x| \cdot \overline{\lim}_{\epsilon \downarrow 0} \left(\sup_\alpha E \left| x_{\bar{\tau}^\alpha, x+\epsilon\xi}^{\alpha,x} - y_{\bar{\tau}^\alpha, x+\epsilon\xi}^{\alpha, x+\epsilon\xi} \right| I_{\bar{\tau}^\alpha, x+\epsilon\xi < \tau^{\alpha,x} \leq T} \right) \\
&= 0.
\end{aligned}$$

Also, notice that for any $\alpha \in \mathfrak{A}$, $T \in [1, \infty)$,

$$P_x^\alpha(\tau > T) \leq \frac{1}{T} E_x^\alpha \tau \leq \frac{1}{T} E_x^\alpha \int_0^\tau \left(-L^\alpha \psi(x_t) \right) dt = \frac{1}{T} \left(\psi(x) - \psi(x_{\tau^\alpha}^{\alpha,x}) \right) \leq \frac{K_0}{T}.$$

It turns out that

$$\overline{\lim}_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E(\tau_D^{\alpha,x} - \tau_D^{\alpha,x} \wedge \bar{\tau}_D^{\alpha,x+\epsilon\xi}(\epsilon)) \leq \frac{2K_0^2}{T} \rightarrow 0, \text{ as } T \uparrow \infty.$$

To prove (4.43), we just need to notice that for any stopping times τ, γ_1, γ_2

$$\tau - \tau \wedge \gamma_1 \wedge \gamma_2 = (\tau - \tau \wedge \gamma_1) I_{\gamma_1 < \gamma_2} + (\tau - \tau \wedge \gamma_2) I_{\gamma_1 \geq \gamma_2}.$$

By noticing that

$$\psi - \delta = 0 \text{ on } \partial D_\delta, \quad \psi - \delta > 0, \quad \sup_{\alpha \in \mathfrak{A}} L^\alpha(\psi - \delta) = \sup_{\alpha \in \mathfrak{A}} L^\alpha \psi \leq -1 \text{ in } D_\delta,$$

we see that the statement is true in the subdomain D_δ .

Similarly, notice that

$$(\psi - \delta_1)(\delta_2 - \psi) = 0 \text{ on } \partial D_{\delta_1}^{\delta_2}, \quad (\psi - \delta_1)(\delta_2 - \psi) > 0 \text{ in } D_{\delta_1}^{\delta_2},$$

$$L^\alpha((\psi - \delta_1)(\delta_2 - \psi)) = (\delta_1 + \delta_2 - 2\psi) L^\alpha \psi - 2(a^\alpha \psi_x, \psi_x)$$

$$\leq (\delta_1 + \delta_2) |L^\alpha \psi| - 2|\psi_x^* \sigma^\alpha|^2 \text{ in } D_{\delta_1}^{\delta_2}, \forall \alpha \in \mathfrak{A}.$$

On ∂D it holds that $\psi_x = |\psi_x|n$, where $n(x)$ is the unit inward normal vector at $x \in \partial D$. So due to (4.9) and the compactness of ∂D ,

$$|\psi_x^* \sigma^\alpha|^2 = 2|\psi_x|^2 (a^\alpha n, n) \geq 2|\psi_x|^2 \delta_0 \geq 2\delta'_0 \text{ on } \partial D,$$

where δ'_0 is a positive constant. By continuity

$$|\psi_x^* \sigma^\alpha|^2 \geq \delta'_0 \text{ in } D_{\delta_1}^{\delta_2},$$

if δ_1 and δ_2 are sufficiently small. It turns out that

$$\sup_{\alpha \in \mathfrak{A}} L^\alpha \frac{(\psi - \delta_1)(\delta_2 - \psi)}{\delta'_0} \leq -1,$$

when δ_1 and δ_2 are sufficiently small. So the statement is still true in the subdomain $D_{\delta_1}^{\delta_2}$ when δ_1, δ_2 are sufficiently small. \square

4.3 Proof of Theorem 4.1.1

Before proving the main theorem, we state two remarks and one lemma. Remarks 4.3.1 and 4.3.2 are about two reductions of the problem, and Lemma 4.3.1 will be used when estimating the second derivatives. They are nonlinear counterparts of Remarks 3.2.1 and 3.2.2 and Lemma 3.2.2, and there is no essential change when extending them from linear case to nonlinear case.

Remark 4.3.1. *Without loss of generality, we may assume that $c^\alpha \geq 1, \forall \alpha \in \mathfrak{A}$, and replace condition (4.10) by*

$$\begin{aligned} & \left\| \sigma_{(y)}^\alpha(x) + (\rho^\alpha(x), y) \sigma^\alpha(x) + \sigma^\alpha(x) Q^\alpha(x, y) \right\|^2 + \\ & 2(y, b_{(y)}^\alpha(x) + 2(\rho^\alpha(x), y) b^\alpha(x)) \leq c^\alpha(x) - 1 + M^\alpha(x) (a^\alpha(x) y, y). \end{aligned} \quad (4.45)$$

Remark 4.3.2. *Without loss of generality, we may assume that $v \in C^1(D)$ and $f^\alpha, g \in C^1(\bar{D})$ when investigating first derivatives of v , and $v \in C^2(D)$ and $f^\alpha, g \in C^2(\bar{D})$ when investigating second derivatives of v .*

Lemma 4.3.1. *If $f^\alpha, g \in C^2(\bar{D})$, and $v \in C^1(\bar{D})$, then for any $y \in \partial D$ we have*

$$|v_{(n)}(y)| \leq K(|g|_{2,D} + \sup_{\alpha \in A} |f^\alpha|_{0,D}), \quad (4.46)$$

where n is the unit inward normal on ∂D and the constant K depends only on K_0 .

Let δ and λ be constants satisfying $0 < \delta < \lambda^2 < \lambda < 1$ and that the three sets defined below are nonempty:

$$\begin{aligned} D_\delta &:= \{x \in D : \delta < \psi(x)\} \\ D_\delta^\lambda &:= \{x \in D : \delta < \psi(x) < \lambda\} \\ D_{\lambda^2} &:= \{x \in D : \lambda^2 < \psi(x)\} \end{aligned}$$

For each $\alpha \in \mathfrak{A}$, we use the same quasiderivatives and barrier functions constructed in Chapter 3. Their properties are collected in the following two lemmas.

Lemma 4.3.2. *In D_δ^λ , introduce*

$$\varphi(x) = \lambda^2 + \psi(1 - \frac{1}{4\lambda}\psi), \quad B_1(x, \xi) = [\lambda + \sqrt{\psi}(1 + \sqrt{\psi})]|\xi|^2 + K_1\varphi^{\frac{3}{2}}\frac{\psi^2(\xi)}{\psi},$$

where $K_1 \in [1, \infty)$ is a constant only depending on K_0 .

For each α , we define the first and second quasiderivatives by (4.25) and (4.26), in which

$$\begin{aligned} r(x, \xi) &:= \rho(x, \xi) + \frac{\psi(\xi)}{\psi}, \quad r_t := r(x_t, \xi_t), \\ \text{with } \rho(x, \xi) &:= -\frac{1}{\Upsilon} \sum_{k=1}^{d_1} \psi_{(\sigma^k)}(\psi_{(\sigma^k)})(\xi), \quad \Upsilon := \sum_{k=1}^{d_1} \psi_{(\sigma^k)}^2; \\ \hat{r}(x, \xi) &:= \frac{\psi^2(\xi)}{\psi^2}, \quad \hat{r}_t := \hat{r}(x_t, \xi_t); \\ \pi^k(x, \xi) &:= \frac{2\psi_{(\sigma^k)}\psi(\xi)}{\varphi\psi}, \quad k = 1, \dots, d_1, \quad \pi_t := \pi(x_t, \xi_t); \\ P^{ik}(x, \xi) &:= \frac{1}{\Upsilon} [\psi_{(\sigma^k)}(\psi_{(\sigma^i)})(\xi) - \psi_{(\sigma^i)}(\psi_{(\sigma^k)})(\xi)], \quad i, k = 1, \dots, d_1, \quad P_t := P(x_t, \xi_t); \\ \hat{\pi}_t^k &= \hat{P}_t^{ik} = 0, \quad \forall i, k = 1, \dots, d_1, \forall t \in [0, \infty). \end{aligned}$$

where we drop the superscript α or α_t without confusion. Then (4.33), (4.35), (4.36), (4.38), (4.39), (4.40), (4.41) and (4.42) all hold for any constants $p \in (0, \infty)$, $p' \in [0, p)$, $T \in [1, \infty)$, $x \in D_\delta^\lambda$, $\xi, \eta \in \mathbb{R}^d$ and stopping times

$$\begin{aligned}\gamma^\alpha &\leq \tau_{D_\delta^\lambda}^{\alpha, x}, \quad \gamma^\alpha(\epsilon) \leq \tau_{D_\delta^\lambda}^{\alpha, x} \wedge \bar{\tau}_{D_\delta^\lambda}^{\alpha, x + \epsilon \xi}(\epsilon), \quad \gamma_2^\alpha(\epsilon) \leq \tau_{D_\delta^\lambda}^{\alpha, x} \wedge \hat{\tau}_{D_\delta^\lambda}^{\alpha, x(\epsilon)}(\epsilon), \\ \gamma_3^\alpha(\epsilon) &\leq \tau_{D_\delta^\lambda}^{\alpha, x} \wedge \hat{\tau}_{D_\delta^\lambda}^{\alpha, x(\epsilon)}(\epsilon) \wedge \hat{\tau}_{D_\delta^\lambda}^{\alpha, x(-\epsilon)}(-\epsilon),\end{aligned}$$

where $x(\epsilon) = x + \epsilon \xi + \frac{\epsilon^2}{2} \eta$.

When λ is sufficiently small, for $x \in D_\delta^\lambda$, $\xi \in \mathbb{R}^d$ and $\eta = 0$, we have

1. For each $\alpha \in \mathfrak{A}$, $B_1(x_t^{\alpha, x}, \xi_t^{\alpha, \xi})$ and $\sqrt{B_1(x_t^{\alpha, x}, \xi_t^{\alpha, \xi})}$ are local supermartingales on $[0, \tau_1^\delta]$, where $\tau_1^\delta = \tau_{D_\delta^\lambda}^{\alpha, x}$;
2. $\sup_{\alpha \in \mathfrak{A}} E_{x, \xi}^\alpha \int_0^{\tau_1^\delta} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} dt \leq NB_1(x, \xi)$;
3. $\sup_{\alpha \in \mathfrak{A}} E_\xi^\alpha \sup_{t \leq \tau_1^\delta} |\xi_t|^2 \leq NB_1(x, \xi)$;
4. $\sup_{\alpha \in \mathfrak{A}} E_0^\alpha |\eta_{\tau_1^\delta}| \leq \sup_{\alpha \in \mathfrak{A}} E_0^\alpha \sup_{t \leq \tau_1^\delta} |\eta_t| \leq NB_1(x, \xi)$;
5. $\sup_{\alpha \in \mathfrak{A}} E_0^\alpha \left(\int_0^{\tau_1^\delta} |\eta_t|^2 dt \right)^{\frac{1}{2}} \leq NB_1(x, \xi)$;

where N is a constant depending on K_0 and ϵ .

Proof. Notice that $\sup_{\alpha \in \mathfrak{A}} |\Upsilon^\alpha|_{0, D_\delta^\lambda}$ is bounded from below by a positive constant due to (4.9), so conditions (4.31) and (4.37) hold with $K_t^\alpha = 0$.

The properties (1)-(5) are nothing but Lemma 3.2.3, because the constant N there doesn't depend on α . \square

Lemma 4.3.3. *In D_{λ^2} , introduce*

$$B_2(x, \xi) = \lambda^{\frac{3}{4}} |\xi|^2.$$

For each $\alpha \in \mathfrak{A}$, we define the first and second quasiderivatives by (4.25) and (4.26), in which

$$r(x, y) := (\rho(x), y), \quad r_t := r(x_t, \xi_t), \quad \hat{r}_t := r(x_t, \eta_t),$$

$$\begin{aligned}\pi(x, y) &:= \frac{M(x)}{2} \sigma^*(x) y, \quad \pi_t := \pi(x_t, \xi_t), \quad \hat{\pi}_t := \pi(x_t, \eta_t), \\ P(x, y) &:= Q(x, y), \quad P_t := P(x_t, \xi_t), \quad \hat{P}_t := P(x_t, \eta_t).\end{aligned}$$

where $\rho(x)$, $M(x)$ and $Q(x, y)$ are defined in the statement of the main theorem and satisfy (4.10), and again, we drop the superscript α or α_t without confusion. Then (4.33), (4.35), (4.36), (4.38), (4.39), (4.40), (4.41) and (4.42) all hold for any constants $p \in (0, \infty)$, $p' \in [0, p)$, $T \in [1, \infty)$, $x \in D_\delta^\lambda$, $\xi, \zeta \in \mathbb{R}^d$ and stopping times

$$\begin{aligned}\gamma^\alpha &\leq \tau_{D_{\lambda^2}}^{\alpha, x}, \quad \gamma^\alpha(\epsilon) \leq \tau_{D_{\lambda^2}}^{\alpha, x} \wedge \bar{\tau}_{D_{\lambda^2}}^{\alpha, x + \lambda \xi}, \quad \gamma_2^\alpha(\epsilon) \leq \tau_{D_{\lambda^2}}^{\alpha, x} \wedge \hat{\tau}_{D_{\lambda^2}}^{\alpha, x + \epsilon \xi + \frac{\epsilon^2}{2} \eta}(\epsilon), \\ \gamma_3^\alpha(\epsilon) &\leq \tau_{D_{\lambda^2}}^{\alpha, x} \wedge \hat{\tau}_{D_{\lambda^2}}^{\alpha, x + \epsilon \xi + \frac{\epsilon^2}{2} \eta}(\epsilon) \wedge \hat{\tau}_{D_{\lambda^2}}^{\alpha, x - \epsilon \xi + \frac{\epsilon^2}{2} \eta}(-\epsilon),\end{aligned}$$

where $x(\epsilon) = x + \epsilon \xi + \frac{\epsilon^2}{2} \eta$.

Furthermore, for $x \in D_{\lambda^2}$, $\xi \in \mathbb{R}^d$ and $\eta = 0$, we have

1. $e^{-\phi_t^{\alpha, x}} \mathbf{B}_2(x_t^{\alpha, x}, \xi_t^{\alpha, \xi})$ and $\sqrt{e^{-\phi_t^{\alpha, x}} \mathbf{B}_2(x_t^{\alpha, x}, \xi_t^{\alpha, \xi})}$ are local supermartingales on $[0, \tau_2)$, where $\tau_2 = \tau_{D_{\epsilon^2}}^{\alpha, x}$.
2. $\sup_{\alpha \in \mathfrak{A}} E_{x, \xi}^\alpha \int_0^{\tau_2} e^{-\phi_t} |\xi_t|^2 dt \leq N \mathbf{B}_2(x, \xi)$
3. $\sup_{\alpha \in \mathfrak{A}} E_{x, \xi}^\alpha \sup_{t \leq \tau_2} e^{-\phi_t} |\xi_t|^2 \leq N \mathbf{B}_2(x, \xi)$
4. $\sup_{\alpha \in \mathfrak{A}} E_{x, 0}^\alpha e^{-\phi_{\tau_2}} |\eta_{\tau_2}| \leq \sup_{\alpha \in \mathfrak{A}} E_{x, 0}^\alpha \sup_{t \leq \tau_2} e^{-\phi_t} |\eta_t| \leq N \mathbf{B}_2(x, \xi)$
5. $\sup_{\alpha \in \mathfrak{A}} E_{x, 0}^\alpha \left(\int_0^{\tau_2} e^{-2\phi_t} |\eta_t|^2 dt \right)^{\frac{1}{2}} \leq N \mathbf{B}_2(x, \xi)$
6. The above inequalities are still all true if we replace $\phi_t^{\alpha, x}$ by $\phi_t^{\alpha, x} - \frac{1}{2}t$. More precisely, we have

$$\begin{aligned}\sup_{\alpha \in \mathfrak{A}} E_{x, \xi}^\alpha \int_0^{\tau_2} e^{-\phi_t + \frac{1}{2}t} |\xi_t|^2 dt &\leq N \mathbf{B}_2(x, \xi), \quad \sup_{\alpha \in \mathfrak{A}} E_{\xi, 0}^\alpha \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2}t} |\xi_t|^2 \leq N \mathbf{B}_2(x, \xi) \\ \sup_{\alpha \in \mathfrak{A}} E_{x, 0}^\alpha \left(\int_0^{\tau_2} e^{-2\phi_t + t} |\eta_t|^2 dt \right)^{\frac{1}{2}} &\leq N \mathbf{B}_2(x, \xi), \quad \sup_{\alpha \in \mathfrak{A}} E_{x, 0}^\alpha \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2}t} |\eta_t| \leq N \mathbf{B}_2(x, \xi)\end{aligned}$$

where N is constant depending on K_0 and λ .

Proof. The same as Lemma 4.3.2. \square

We split the proof of Theorem 4.1.1 into three parts. Note that in the proof, for simplicity of notation, we may drop the superscripts such as α when it will cause no confusion.

Proof of (4.11). First, we fix an $x \in D_\delta^\lambda$ and a $\xi \in \mathbb{R}^d \setminus \{0\}$. Choose $\epsilon_0 > 0$ sufficiently small, so that $B(x, \epsilon_0|\xi|) := \{y : |y - x| \leq \epsilon_0|\xi|\} \subset D_\delta^\lambda$. For any $\epsilon \in (0, \epsilon_0)$, by Bellman's principle (Theorem 1.1 in [2], in which Q is defined by $D \times [-1, T + 1]$, where T is an arbitrary positive constant), we have, with the stopping time $\gamma^\alpha = \tau_{D_\delta^\lambda}^{\alpha, x + \epsilon\xi} \wedge \tau_{D_\delta^\lambda}^{\alpha, x} \wedge T$,

$$\begin{aligned} \frac{v(x + \epsilon\xi) - v(x)}{\epsilon} &= \frac{1}{\epsilon} \left\{ \sup_{\alpha \in \mathfrak{A}} E_{x + \epsilon\xi}^\alpha \left[v(x_\gamma) e^{-\phi_\gamma} + \int_0^\gamma f^{\alpha_s}(x_s) e^{-\phi_s} ds \right] \right. \\ &\quad \left. - \sup_{\alpha \in \mathfrak{A}} E_x^\alpha \left[v(x_\gamma) e^{-\phi_\gamma} + \int_0^\gamma f^{\alpha_s}(x_s) e^{-\phi_s} ds \right] \right\}. \end{aligned}$$

By Theorem 2.1 in [10] and Lemmas 2.1 and 2.2 in [11],

$$\begin{aligned} &\sup_{\alpha \in \mathfrak{A}} E_{x + \epsilon\xi}^\alpha \left[v(x_\gamma) e^{-\phi_\gamma} + \int_0^\gamma f^{\alpha_s}(x_s) e^{-\phi_s} ds \right] \\ &= \sup_{\alpha \in \mathfrak{A}} E_{x + \epsilon\xi}^\alpha \left[v(y_\gamma(\epsilon)) p_\gamma(\epsilon) e^{-\phi_\gamma(\epsilon)} + \int_0^\gamma (1 + 2\epsilon r_s) f^{\alpha_s}(y_s(\epsilon)) p_s(\epsilon) e^{-\phi_s(\epsilon)} ds \right], \end{aligned}$$

in which $y_t^{\alpha, y}(\epsilon)$ is the solution to the Itô equation (4.23),

$$\phi_t^{\alpha, y}(\epsilon) := \int_0^t (1 + 2\epsilon r_s^\alpha) c^{\alpha_s}(y_s^{\alpha, y}(\epsilon)) ds,$$

and

$$p_t^\alpha(\epsilon) := \exp \left(\int_0^t \epsilon \pi_s^\alpha dw_s - \frac{1}{2} \int_0^t |\epsilon \pi_s^\alpha|^2 ds \right). \quad (4.47)$$

with $\alpha \in \mathfrak{A}$, $r_s^\alpha, \pi_s^\alpha, P_s^\alpha$ defined in Lemma 4.3.2.

Let

$$\begin{aligned} q_t^\alpha(\epsilon) &= \int_0^t (1 + 2\epsilon r_s^\alpha) f^{\alpha_s}(y_s(\epsilon)) p_s(\epsilon) e^{-\phi_s(\epsilon)} ds, \\ \bar{y}_t^{\alpha, y}(\epsilon) &= (y_t^{\alpha, y}(\epsilon), -\phi_t^\alpha(\epsilon), p_t^\alpha(\epsilon), q_t^\alpha(\epsilon)), \\ \bar{x}_t^{\alpha, x} &= (x_t^{\alpha, x}, -\phi_t^\alpha(0), p_t^\alpha(0), q_t^\alpha(0)). \end{aligned}$$

For any $\bar{x} = (x, x^{d+1}, x^{d+2}, x^{d+3}) \in D \times \mathbb{R}^- \times \mathbb{R}^+ \times \mathbb{R}$, introduce

$$V(\bar{x}) = v(x) \exp(x^{d+1}) x^{d+2} + x^{d+3}. \quad (4.48)$$

Then we have

$$\frac{v(x + \epsilon\xi) - v(x)}{\epsilon} = \frac{1}{\epsilon} \left(\sup_{\alpha \in \mathfrak{A}} E_{x+\epsilon\xi}^\alpha V(\bar{y}_\gamma(\epsilon)) - \sup_{\alpha \in \mathfrak{A}} E_x^\alpha V(\bar{x}_\gamma) \right).$$

Since the difference of two supremums is less than the supremum of the differences, and the supremum of a sum is less than the sum of the supremums, we have

$$\begin{aligned} \frac{v(x + \epsilon\xi) - v(x)}{\epsilon} &\leq \sup_{\alpha \in \mathfrak{A}} E \frac{V(\bar{y}_{\gamma^\alpha}^{\alpha, x+\epsilon\xi}(\epsilon)) - V(\bar{x}_{\gamma^\alpha}^{\alpha, x})}{\epsilon} \\ &\leq \sup_{\alpha \in \mathfrak{A}} E \frac{V(\bar{y}_{\gamma^\alpha}^{\alpha, x+\epsilon\xi}(\epsilon)) - V(\bar{x}_{\gamma^\alpha}^{\alpha, x})}{\epsilon} - V_{(\bar{\xi}_{\gamma^\alpha}^{\alpha, \xi})}(\bar{x}_{\gamma^\alpha}^{\alpha, x}) + \sup_{\alpha \in \mathfrak{A}} E V_{(\bar{\xi}_{\gamma^\alpha}^{\alpha, \xi})}(\bar{x}_{\gamma^\alpha}^{\alpha, x}) \\ &:= I_1(\epsilon, T) + I_2(\epsilon, T), \end{aligned}$$

where

$$\bar{\xi}_t^{\alpha, \xi} = (\xi_t^{\alpha, \xi}, \xi_t^{d+1, \alpha}, \xi_t^{d+2, \alpha}, \xi_t^{d+3, \alpha}), \quad (4.49)$$

with

$$\begin{aligned} \xi_t^{d+1, \alpha} &:= - \int_0^t \left[c_{(\xi_s^{\alpha, \xi})}^{\alpha_s}(x_s^{\alpha, x}) + 2r_s^\alpha c^{\alpha_s}(x_s^{\alpha, x}) \right] ds, \\ \xi_t^{d+2, \alpha} &:= \xi_t^{0, \alpha} = \int_0^t \pi_s^\alpha dw_s, \\ \xi_t^{d+3, \alpha} &:= \int_0^t e^{-\phi_s^{\alpha, x}} \left[f_{(\xi_s^{\alpha, \xi})}^{\alpha_s}(x_s^{\alpha, x}) + (2r_s^\alpha + \xi_s^{d+1, \alpha} + \xi_s^{d+2, \alpha}) f^{\alpha_s}(x_s^{\alpha, x}) \right] ds. \end{aligned}$$

We claim that

$$\lim_{\epsilon \downarrow 0} I_1(\epsilon, T) = 0. \quad (4.50)$$

To show it, bearing in mind that for any $h^\alpha(x) \in C^1(\bar{D}_\delta)$, we have, for any $x, y \in D_\delta$ and $\xi \in \mathbb{R}^d$, $r \in \mathbb{R}^d$ and $n \in \mathbb{N}$,

$$\begin{aligned} & \left| \frac{(1 + 2\epsilon r)h^\alpha(y) - h^\alpha(x)}{\epsilon} - h_{(\xi)}^\alpha(x) - 2rh^\alpha(x) \right| \\ &= \left| h_{\left(\frac{y-x}{\epsilon}\right)}^\alpha(y^*) - h_{(\xi)}^\alpha(x) + 2r(h^\alpha(y) - h^\alpha(x)) \right| \\ &\leq \left| (h_x^\alpha(y^*) - h_x^\alpha(x), \frac{y-x}{\epsilon}) \right| + \left| (h_x^\alpha(x), \frac{y-x}{\epsilon} - \xi) \right| + 2K_0(\epsilon r^2 + \frac{|y-x|^2}{\epsilon}), \end{aligned} \quad (4.51)$$

where y^* is a point on the line segment with ending points x and y .

First, by Lemma 4.3.2, for any constants p and p' satisfying $0 \leq p' < p < \infty$, we have

$$\sup_{\alpha \in \mathfrak{A}} E_\xi^\alpha \sup_{t \leq \gamma} |\xi_t|^p < \infty, \quad (4.52)$$

$$\limsup_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E \sup_{t \leq \gamma^\alpha} \frac{|y_t^{\alpha, x + \epsilon \xi}(\epsilon) - x_t^{\alpha, x}|^p}{\epsilon^{p'}} = 0, \quad (4.53)$$

$$\limsup_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E \sup_{t \leq \gamma^\alpha} \left| \frac{y_t^{\alpha, x + \epsilon \xi}(\epsilon) - x_t^{\alpha, x}}{\epsilon} - \xi_t^{\alpha, \xi} \right|^p = 0. \quad (4.54)$$

Second, apply (4.51) to $c^\alpha(x)$ we get

$$\overline{\lim}_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E^\alpha \sup_{t \leq \gamma} \left| \frac{\phi_t(0) - \phi_t(\epsilon)}{\epsilon} - \xi_t^{d+1} \right|^p = 0. \quad (4.55)$$

Third, we notice that

$$\frac{p_t(\epsilon) - p_t(0)}{\epsilon} = \frac{p_t(\epsilon) - 1}{\epsilon} = \int_0^t p_s(\epsilon) \pi_s dw_s.$$

It follows that

$$\begin{aligned} E^\alpha \sup_{t \leq \gamma} \left| \frac{p_t(\epsilon) - 1}{\epsilon} - \xi_t^{d+2} \right|^p &\leq N(p) E^\alpha \left(\int_0^\gamma (p_t(\epsilon) - 1)^2 |\pi_t|^2 dt \right)^{p/2} \\ &\leq \epsilon^p N(p) E^\alpha \left(\sup_{t \leq \gamma} \left| \frac{p_t(\epsilon) - 1}{\epsilon} \right|^{2p} + \int_0^\gamma |\pi_t|^{2p} dt \right) \\ &\leq \epsilon^p N(p) E^\alpha \left(\int_0^\gamma p_t^{2p}(\epsilon) |\pi_t|^{2p} dt + \int_0^\gamma |\pi_t|^{2p} dt \right). \end{aligned}$$

Hence

$$\limsup_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E^\alpha \sup_{t \leq \gamma} \left| \frac{p_t(\epsilon) - p_t(0)}{\epsilon} - \xi_t^{d+2} \right|^p = 0. \quad (4.56)$$

Fourth, bearing in mind that

$$\begin{aligned} &\left| \frac{f(\epsilon)g(\epsilon) - fg}{\epsilon} - f'g - fg' \right| \\ &\leq \left| \frac{f(\epsilon) - f}{\epsilon} - f' \right| |g(\epsilon)| + \left| \frac{g(\epsilon) - g}{\epsilon} - g' \right| |f| + |f'| |g(\epsilon) - g| \\ &\leq \left| \frac{f(\epsilon) - f}{\epsilon} - f' \right| |g(\epsilon)| + \left| \frac{g(\epsilon) - g}{\epsilon} - g' \right| |f| + \epsilon (|f'|^2 + \frac{|g(\epsilon) - g|^2}{\epsilon^2}). \end{aligned}$$

Therefore, to prove

$$\limsup_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E^\alpha \sup_{t \leq \gamma} \left| \frac{q_t(\epsilon) - q_t(0)}{\epsilon} - \xi_t^{d+3} \right|^p = 0, \quad (4.57)$$

it suffices to show that

$$\limsup_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E^\alpha \sup_{t \leq \gamma} \left| \frac{(1 + 2\epsilon r_t) f^{\alpha_t}(y_t(\epsilon)) - f^{\alpha_t}(x_t)}{\epsilon} - f_{(\xi_t)}^{\alpha_t}(x_t) - 2r_t f^{\alpha_t}(x_t) \right|^p = 0,$$

$$\limsup_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E^\alpha \sup_{t \leq \gamma} \left| \frac{e^{-\phi_t(\epsilon)} - e^{-\phi_t(0)}}{\epsilon} + \xi_t^{d+1} e^{-\phi_t(0)} \right|^p = 0.$$

The first equation is true due to (4.51) with $h^\alpha = f^\alpha$. The second one is true by a similar argument.

Finally, observe that for any $\bar{x} = (x, x^{d+1}, 1, x^{d+3})$, $\bar{y} = (y, y^{d+1}, y^{d+2}, y^{d+3})$, $\bar{\xi} = (\xi, \xi^{d+1}, \xi^{d+2}, \xi^{d+3}) \in D \times \mathbb{R}^- \times \mathbb{R}^+ \times \mathbb{R}$, we have

$$\begin{aligned} \frac{V(\bar{y}) - V(\bar{x})}{\epsilon} - V_{(\bar{\xi})}(\bar{x}) &= \frac{v(y)e^{y^{d+1}}y^{d+2} - v(x)e^{x^{d+1}}}{\epsilon} + \frac{y^{d+3} - x^{d+3}}{\epsilon} \\ &\quad - e^{x^{d+1}} [v_{(\xi)}(x) + v(x)(\xi^{d+1} + \xi^{d+2})] - \xi^{d+3}. \end{aligned}$$

It is not hard to see (4.50) is true with (4.54), (4.55), (4.56) and (4.57) in hand.

To estimate $I_2(\epsilon, T)$, we notice that $V_{(\bar{\xi}_t^{\alpha, \epsilon})}(\bar{x}_t^{\alpha, x})$ is exactly X_t^α defined by (2.8), in which u is replaced by v . More precisely,

$$\begin{aligned} V_{(\bar{\xi}_t^{\alpha, \epsilon})}(\bar{x}_t^{\alpha, x}) &= X_t^\alpha := e^{-\phi_t^{\alpha, x}} \left[v_{(\xi_t^{\alpha, \epsilon})}(x_t^{\alpha, x}) + \tilde{\xi}_t^{0, \alpha} v(x_t^{\alpha, x}) \right] \\ &\quad + \int_0^t e^{-\phi_s^{\alpha, x}} \left[f_{(\xi_s^{\alpha, \epsilon})}^{\alpha_s}(x_s^{\alpha, x}) + (2r_s^\alpha + \tilde{\xi}_s^{0, \alpha}) f^{\alpha_s}(x_s^{\alpha, x}) \right] ds, \end{aligned}$$

where

$$\tilde{\xi}_t^{0, \alpha} = \xi_t^{0, \alpha} + \xi_t^{d+1, \alpha}.$$

It follows that

$$I_2(\epsilon, T) = \sup_{\alpha \in \mathfrak{A}} E^\alpha X_\gamma \leq \sup_{\alpha \in \mathfrak{A}} E e^{-\phi_{\gamma^\alpha}^{\alpha, x}} v_{(\xi_{\gamma^\alpha}^{\alpha, \epsilon})}(x_{\gamma^\alpha}^{\alpha, x}) + \sup_{\alpha \in \mathfrak{A}} E \left(X_{\gamma^\alpha}^\alpha - e^{-\phi_{\gamma^\alpha}^{\alpha, x}} v_{(\xi_{\gamma^\alpha}^{\alpha, \epsilon})}(x_{\gamma^\alpha}^{\alpha, x}) \right).$$

It is showed in the proof of (3.4), that for each α ,

$$E \sup_{t \leq \tau_{D_\delta^\lambda}^{\alpha, x}} \left(X_t^\alpha - e^{-\phi_t^{\alpha, x}} v_{(\xi_t^{\alpha, \epsilon})}(x_t^{\alpha, x}) \right) \leq N \sqrt{B_1(x, \xi)},$$

where N is independent of α . So

$$I_2(\epsilon, T) \leq \sup_{\alpha \in \mathfrak{A}} E e^{-\phi_{\gamma^\alpha}^{\alpha, x}} v_{(\xi_{\gamma^\alpha}^{\alpha, \epsilon})}(x_{\gamma^\alpha}^{\alpha, x}) + N \sqrt{B_1(x, \xi)}.$$

We next notice that

$$\begin{aligned}
\sup_{\alpha \in \mathfrak{A}} E v_{(\xi_{\gamma^\alpha}^{\alpha, \xi})}(x_{\gamma^\alpha}^{\alpha, x}) &= \sup_{\alpha \in \mathfrak{A}} E \frac{v_{(\xi_{\gamma^\alpha}^{\alpha, \xi})}(x_{\gamma^\alpha}^{\alpha, x})}{\sqrt{\mathbf{B}_1(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi})}} \cdot \sqrt{\mathbf{B}_1(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi})} \\
&\leq \sup_{\alpha \in \mathfrak{A}} E \left(\frac{v_{(\xi_{\gamma^\alpha}^{\alpha, \xi})}(x_{\gamma^\alpha}^{\alpha, x})}{\sqrt{\mathbf{B}_1(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi})}} - \frac{v_{(\xi_{\gamma^\alpha}^{\alpha, \xi})}(x_{\tau_{D_\delta^\lambda}^{\alpha, x}})}{\sqrt{\mathbf{B}_1(x_{\tau_{D_\delta^\lambda}^{\alpha, x}}, \xi_{\gamma^\alpha}^{\alpha, \xi})}} \right) \cdot \sqrt{\mathbf{B}_1(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi})} \\
&\quad + \sup_{\alpha \in \mathfrak{A}} E \frac{v_{(\xi_{\gamma^\alpha}^{\alpha, \xi})}(x_{\tau_{D_\delta^\lambda}^{\alpha, x}})}{\sqrt{\mathbf{B}_1(x_{\tau_{D_\delta^\lambda}^{\alpha, x}}, \xi_{\gamma^\alpha}^{\alpha, \xi})}} \cdot \sqrt{\mathbf{B}_1(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi})} \\
&:= J_1(h, T) + J_2(h, T).
\end{aligned}$$

Notice that

$$\frac{v_{(\xi)}(x)}{\sqrt{\mathbf{B}_1(x, \xi)}} = \frac{v_{(\xi/|\xi|)}(x)}{\sqrt{\mathbf{B}_1(x, \xi/|\xi|)}}$$

is a continuous function from $D_\delta^\lambda \times S_1$ to \mathbb{R} , where S_1 is the unit sphere in \mathbb{R}^d . By Weierstrass Approximation Theorem, there exists a polynomial $W(x, \xi) : D_\delta^\lambda \times S_1 \rightarrow \mathbb{R}$, such that

$$\sup_{x \in D_\delta^\lambda, \xi \in S_1} \left| \frac{v_{(\xi)}(x)}{\sqrt{\mathbf{B}_1(x, \xi)}} - W(x, \xi) \right| \leq 1.$$

It follows that

$$\begin{aligned}
J_1(h, T) &\leq \sup_{\alpha \in \mathfrak{A}} E |W(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi}) - W(x_{\tau_{D_\delta^\lambda}^{\alpha, x}}, \xi_{\gamma^\alpha}^{\alpha, \xi})| \sqrt{\mathbf{B}_1(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi})} \\
&\quad + 2 \sup_{\alpha \in \mathfrak{A}} E \sqrt{\mathbf{B}_1(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi})} \\
&\leq N \sup_{\alpha \in \mathfrak{A}} E |x_{\gamma^\alpha}^{\alpha, x} - x_{\tau_{D_\delta^\lambda}^{\alpha, x}}| \sqrt{\mathbf{B}_1(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi})} + 2 \sqrt{\mathbf{B}_1(x, \xi)} \\
&\leq N \sqrt{\mathbf{B}_1(x, \xi)} \sup_{\alpha \in \mathfrak{A}} E |x_{\gamma^\alpha}^{\alpha, x} - x_{\tau_{D_\delta^\lambda}^{\alpha, x}}|^2 + \frac{\sup_{\alpha \in \mathfrak{A}} E \mathbf{B}_1(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi})}{\sqrt{\mathbf{B}_1(x, \xi)}} \\
&\quad + 2 \sqrt{\mathbf{B}_1(x, \xi)} \\
&\leq N \sqrt{\mathbf{B}_1(x, \xi)} \left(E |x_{\gamma^\alpha}^{\alpha, x} - x_{\tau_{D_\delta^\lambda}^{\alpha, x} \wedge T}|^2 + E |x_{\tau_{D_\delta^\lambda}^{\alpha, x} \wedge T} - x_{\tau_{D_\delta^\lambda}^{\alpha, x}}|^2 \right) \\
&\quad + 3 \sqrt{\mathbf{B}_1(x, \xi)} \\
&\leq N \sqrt{\mathbf{B}_1(x, \xi)} \left(E |\gamma^\alpha - \tau_{D_\delta^\lambda}^{\alpha, x} \wedge T| + E |\tau_{D_\delta^\lambda}^{\alpha, x} \wedge T - \tau_{D_\delta^\lambda}^{\alpha, x}| \right) + 3 \sqrt{\mathbf{B}_1(x, \xi)}.
\end{aligned}$$

Thus

$$\overline{\lim}_{T \uparrow \infty} \overline{\lim}_{\epsilon \downarrow 0} J_1(\epsilon, T) \leq 3\sqrt{B_1(x, \xi)}.$$

Also, notice that

$$\begin{aligned} J_2(\epsilon, T) &\leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v(\zeta)(y)}{\sqrt{B_1(y, \zeta)}} \sup_{\alpha \in \mathfrak{A}} E \sqrt{B_1(x_{\gamma^\alpha}^{\alpha, x}, \xi_{\gamma^\alpha}^{\alpha, \xi})} \\ &\leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v(\zeta)(y)}{\sqrt{B_1(y, \zeta)}} \cdot \sqrt{B_1(x, \xi)}. \end{aligned}$$

Hence,

$$\overline{\lim}_{T \uparrow \infty} \overline{\lim}_{\epsilon \downarrow 0} I_2(\epsilon, T) \leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v(\zeta)(y)}{\sqrt{B_1(y, \zeta)}} \cdot \sqrt{B_1(x, \xi)} + N \sqrt{B_1(x, \xi)}.$$

We conclude that

$$\frac{v(\xi)(x)}{\sqrt{B_1(x, \xi)}} \leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v(\zeta)(y)}{\sqrt{B_1(y, \zeta)}} + N, \quad \forall x \in D_\delta^\lambda, \xi \in \mathbb{R}^d \setminus \{0\}.$$

Notice that $B_1(x, \xi) = B_1(x, -\xi)$. Replacing ξ by $-\xi$, we have

$$\frac{-v(\xi)(x)}{\sqrt{B_1(x, \xi)}} \leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v(\zeta)(y)}{\sqrt{B_1(y, \zeta)}} + N, \quad \forall x \in D_\delta^\lambda, \xi \in \mathbb{R}^d \setminus \{0\},$$

which implies that

$$\frac{|v(\xi)(x)|}{\sqrt{B_1(x, \xi)}} \leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{|v(\zeta)(y)|}{\sqrt{B_1(y, \zeta)}} + N, \quad \forall x \in D_\delta^\lambda, \xi \in \mathbb{R}^d \setminus \{0\}. \quad (4.58)$$

Repeating the argument above in D_{λ^2} , we have

$$\frac{|v(\xi)(x)|}{\sqrt{B_2(x, \xi)}} \leq \sup_{y \in \partial D_{\lambda^2}, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{|v(\zeta)(y)|}{\sqrt{B_2(y, \zeta)}} + N, \quad \forall x \in D_{\lambda^2}, \xi \in \mathbb{R}^d \setminus \{0\}. \quad (4.59)$$

The inequalities (4.58) and (4.59) are the same as (3.19) and (3.21). So by repeating the argument after (3.21), we get

$$v(\xi)(x) \leq N \left(|\xi| + \frac{|\psi(\xi)(x)|}{\psi^{\frac{1}{2}}(x)} \right), \quad \text{a.e. in } D.$$

(4.11) is proved. \square

Proof of (4.12). The idea is the same as the first order case. Fix $x \in D_\delta^\lambda$, $\xi \in \mathbb{R}^d \setminus \{0\}$ and sufficiently small positive ϵ_0 , so that $B(x, \epsilon_0|\xi|) \subset D_\delta^\lambda$. For each $\alpha \in \mathfrak{A}$, let $\gamma^\alpha := \tau_{D_\delta^\lambda}^\alpha(x + \epsilon\xi) \wedge \tau_{D_\delta^\lambda}^\alpha(x) \wedge \tau_{D_\delta^\lambda}^\alpha(x - \epsilon\xi) \wedge T$, where $T \in [1, \infty)$. We have

$$\begin{aligned} & - \frac{v(x + \epsilon\xi) - 2v(x) + v(x - \epsilon\xi)}{\epsilon^2} \\ &= \frac{1}{\epsilon^2} \left\{ - \sup_{\alpha \in \mathfrak{A}} E_{x+\epsilon\xi}^\alpha \left[v(x_\gamma) e^{-\phi_\gamma} + \int_0^\gamma f^{\alpha_s}(x_s) e^{-\phi_s} ds \right] \right. \\ & \quad + 2 \sup_{\alpha \in \mathfrak{A}} E_x^\alpha \left[v(x_\gamma) e^{-\phi_\gamma} + \int_0^\gamma f^{\alpha_s}(x_s) e^{-\phi_s} ds \right] \\ & \quad \left. - \sup_{\alpha \in \mathfrak{A}} E_{x-\epsilon\xi}^\alpha \left[v(x_\gamma) e^{-\phi_\gamma} + \int_0^\gamma f^{\alpha_s}(x_s) e^{-\phi_s} ds \right] \right\} \\ &= \frac{1}{\epsilon^2} \left\{ - \sup_{\alpha \in \mathfrak{A}} E_{x+\epsilon\xi}^\alpha \left[v(z_\gamma(\epsilon)) q_\gamma(\epsilon) e^{-\phi_\gamma(\epsilon)} + \int_0^\gamma f^{\alpha_s}(z_s(\epsilon)) q_s(\epsilon) e^{-\phi_s(\epsilon)} ds \right] \right. \\ & \quad + 2 \sup_{\alpha \in \mathfrak{A}} E_x^\alpha \left[v(x_\gamma) q_\gamma e^{-\phi_\gamma} + \int_0^\gamma f^{\alpha_s}(x_s) q_s e^{-\phi_s} ds \right] \\ & \quad \left. - \sup_{\alpha \in \mathfrak{A}} E_{x-\epsilon\xi}^\alpha \left[v(z_\gamma(-\epsilon)) q_\gamma(-\epsilon) e^{-\phi_\gamma(-\epsilon)} + \int_0^\gamma f^{\alpha_s}(z_s(-\epsilon)) q_s(-\epsilon) e^{-\phi_s(-\epsilon)} ds \right] \right\}, \end{aligned}$$

in which $z_t^{\alpha,z}(\epsilon)$ is the solution to the Itô equation (4.24),

$$\hat{\phi}_t^{\alpha,z}(\epsilon) := \int_0^t (1 + 2\epsilon r_s^\alpha + \epsilon^2 \hat{r}_s^\alpha) c^{\alpha_s}(z_s^{\alpha,z}(\epsilon)) ds,$$

and

$$\hat{p}_t^\alpha(\epsilon) := \exp \left(\int_0^t (\epsilon \pi_s^\alpha + \frac{\epsilon^2}{2} \hat{\pi}_s^\alpha) dw_s - \frac{1}{2} \int_0^t |\epsilon \pi_s^\alpha + \frac{\epsilon}{2} \hat{\pi}_s^\alpha|^2 ds \right).$$

with $\alpha \in \mathfrak{A}$, $r_s^\alpha, \pi_s^\alpha, P_s^\alpha, \hat{r}_s^\alpha, \hat{\pi}_s^\alpha, \hat{P}_s^\alpha$ defined in lemma (4.3.2).

By intruducing

$$\begin{aligned} \hat{q}_t^\alpha(\epsilon) &= \int_0^t (1 + 2\epsilon r_s^\alpha + \epsilon^2 \hat{r}_s^\alpha) f^{\alpha_s}(z_s(\epsilon)) \hat{p}_s(\epsilon) e^{-\hat{\phi}_s(\epsilon)} ds, \\ \bar{z}_t^{\alpha,z}(\epsilon) &= (z_t^{\alpha,z}(\epsilon), -\hat{\phi}_t^\alpha(\epsilon), \hat{p}_t^\alpha(\epsilon), \hat{q}_t^\alpha(\epsilon)), \\ \bar{x}_t^{\alpha,x} &= (x_t^{\alpha,x}, -\hat{\phi}_t^\alpha(0), \hat{p}_t^\alpha(0), \hat{q}_t^\alpha(0)), \end{aligned}$$

we get

$$\begin{aligned} & - \frac{v(x + \epsilon\xi) - 2v(x) + v(x - \epsilon\xi)}{\epsilon^2} \\ &= \frac{1}{\epsilon^2} \left(- \sup_{\alpha \in \mathfrak{A}} E_{x+\epsilon\xi}^\alpha V(\bar{z}_\gamma(\epsilon)) + 2 \sup_{\alpha \in \mathfrak{A}} E_x^\alpha V(\bar{x}_\gamma) - \sup_{\alpha \in \mathfrak{A}} E_{x-\epsilon\xi}^\alpha V(\bar{z}_\gamma(-\epsilon)) \right) \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\alpha \in \mathfrak{A}} \frac{-E_{x+\epsilon\xi}^\alpha V(\bar{z}_\gamma(\epsilon)) + 2E_x^\alpha V(\bar{x}_\gamma) - E_{x-\epsilon\xi}^\alpha V(\bar{z}_\gamma(-\epsilon))}{\epsilon^2} \\
&= \sup_{\alpha \in \mathfrak{A}} E \frac{-V(\bar{z}_{\gamma^\alpha}^{\alpha, x+\epsilon\xi}(\epsilon)) + 2V(\bar{x}_{\gamma^\alpha}^{\alpha, x}) - V(\bar{z}_{\gamma^\alpha}^{\alpha, x-\epsilon\xi}(-\epsilon))}{\epsilon^2} \\
&\leq \sup_{\alpha \in \mathfrak{A}} E \left[\frac{-V(\bar{z}_{\gamma^\alpha}^{\alpha, x+\epsilon\xi}(\epsilon)) + 2V(\bar{x}_{\gamma^\alpha}^{\alpha, x}) - V(\bar{z}_{\gamma^\alpha}^{\alpha, x-\epsilon\xi}(-\epsilon))}{\epsilon^2} - V_{(\bar{\eta}_{\gamma^\alpha}^{\alpha, 0})}(\bar{x}_{\gamma^\alpha}^{\alpha, x}) - V_{(\bar{\xi}_{\gamma^\alpha}^{\alpha, \xi})(\bar{\xi}_{\gamma^\alpha}^{\alpha, \xi})}(\bar{x}_{\gamma^\alpha}^{\alpha, x}) \right] \\
&\quad + \sup_{\alpha \in \mathfrak{A}} \left[V_{(\bar{\eta}_{\gamma^\alpha}^{\alpha, 0})}(\bar{x}_{\gamma^\alpha}^{\alpha, x}) + V_{(\bar{\xi}_{\gamma^\alpha}^{\alpha, \xi})(\bar{\xi}_{\gamma^\alpha}^{\alpha, \xi})}(\bar{x}_{\gamma^\alpha}^{\alpha, x}) \right] \\
&:= G_1(\epsilon, T) + G_2(\epsilon, T),
\end{aligned}$$

where V and $\bar{\xi}_t^{\alpha, \xi}$ are defined by (4.48) and (4.49), respectively, and

$$\bar{\eta}_t^{\alpha, \eta} := (\eta_t^{\alpha, \eta}, \eta_t^{d+1, \alpha}, \eta_t^{d+2, \alpha}, \eta_t^{d+3, \alpha}),$$

with

$$\begin{aligned}
\eta_t^{d+1, \alpha} &:= - \int_0^t c_{(\xi_s^{\alpha, \xi})(\xi_s^{\alpha, \xi})}(x_s^{\alpha, x}) + c_{(\eta_s^{\alpha, \eta})}(x_s^{\alpha, s}) + 4r_s^\alpha c_{(\xi_s^{\alpha, \xi})}(x_s^{\alpha, x}) + 2\hat{r}_s^\alpha c(x_s^{\alpha, x}) ds, \\
\eta_t^{d+2, \alpha} &:= \eta_t^{0, \alpha} = \left(\int_0^t \pi_s^\alpha dw_s \right)^2 - \int_0^t |\pi_s^\alpha|^2 ds + \int_0^t \hat{\pi}_s ds, \\
\eta_t^{d+3, \alpha} &:= \int_0^t e^{-\phi_s^{\alpha, x}} \left[f_{(\xi_s^{\alpha, \xi})(\xi_s^{\alpha, \xi})}^{\alpha_s}(x_s^{\alpha, x}) + f_{(\eta_s^{\alpha, \eta})}(x_s^{\alpha, x}) + (2\xi_s^{d+1, \alpha} + 4r_s^\alpha) f_{(\xi_s^{\alpha, \xi})}^{\alpha_s}(x_s^{\alpha, x}) \right. \\
&\quad \left. + ((\xi_s^{d+1, \alpha})^2 + \eta_s^{d+1, \alpha} + 4r_s^\alpha \xi_s^{d+1, \alpha} + 2\hat{r}_s^\alpha) f^{\alpha_s}(x_s^{\alpha, x}) \right] ds.
\end{aligned}$$

We first claim that

$$\lim_{\epsilon \downarrow 0} G_1(\epsilon, T) = 0.$$

The proof is similar as that of (4.50) with the help of the following two second-order counterparts.

First, if $h^\alpha(x) \in C^2(\bar{D}_\delta)$, then for any $x, z, z' \in D_\delta$, $\xi, \eta \in \mathbb{R}^d$, $r, \hat{r} \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{h^\alpha(z) - 2h^\alpha(x) + h^\alpha(z')}{\epsilon^2} \\
&= \frac{1}{\epsilon^2} \left[h_{(z-x)}^\alpha(x) + \frac{1}{2} h_{(z-x)(z-x)}^\alpha(z^*) + h_{(z'-x)}^\alpha(x) + \frac{1}{2} h_{(z'-x)(z'-x)}^\alpha(z^*) \right] \\
&= h_{\left(\frac{z-2x+z'}{\epsilon^2}\right)}^\alpha(x) + \frac{1}{2} \left[h_{\left(\frac{z-x}{\epsilon}\right)\left(\frac{z-x}{\epsilon}\right)}^\alpha(z^*) + h_{\left(\frac{z'-x}{\epsilon}\right)\left(\frac{z'-x}{\epsilon}\right)}^\alpha(z^*) \right],
\end{aligned}$$

where z^* and $z^{*'}$ are on the line segments \overline{xz} and $\overline{xz'}$, respectively. Hence,

$$\begin{aligned}
& \left| \frac{(1 + 2\epsilon r + \epsilon^2 \hat{r})h^\alpha(z) - 2h^\alpha(x) + (1 - 2\epsilon r + \epsilon^2 \hat{r})h^\alpha(z')}{\epsilon^2} \right. \\
& \quad \left. - (h_{(\xi)(\xi)}^\alpha(x) + h_{(\eta)}^\alpha(x) + 4rh_{(\xi)}^\alpha(x) + 2\hat{r}h^\alpha(x)) \right| \\
\leq & \left| \frac{h^\alpha(z) - 2h^\alpha(x) + h^\alpha(z')}{\epsilon^2} - (h_{(\xi)(\xi)}^\alpha(x) + h_{(\eta)}^\alpha(x)) \right| \\
& + 2|r| \left| \frac{h^\alpha(z) - h^\alpha(z')}{\epsilon} - 2h_{(\xi)}^\alpha(x) \right| + |\hat{r}| |h^\alpha(z) + h^\alpha(z') - 2h^\alpha(x)| \\
\leq & |h_{(\frac{z-2x+z'}{2}-\eta)}^\alpha(x)| + \frac{1}{2} \left[|h_{(\frac{z-x}{\epsilon})(\frac{z-x}{\epsilon})}^\alpha(z^*) - h_{(\xi)(\xi)}^\alpha(x)| + |h_{(\frac{z'-x}{\epsilon})(\frac{z'-x}{\epsilon})}^\alpha(z^{*'}) - h_{(\xi)(\xi)}^\alpha(x)| \right] \\
& + 2|r| \left[\left| \frac{h^\alpha(z) - h^\alpha(x)}{\epsilon} - h_{(\xi)}^\alpha(x) \right| + \left| \frac{h^\alpha(z') - h^\alpha(x)}{-\epsilon} - h_{(\xi)}^\alpha(x) \right| \right] \\
& + |\hat{r}| \left[|h^\alpha(z) - h^\alpha(x)| + |h^\alpha(z') - h^\alpha(x)| \right].
\end{aligned}$$

Second, by noticing that

$$\begin{aligned}
\frac{\hat{p}_t(\epsilon) - 2\hat{p}_t(0) + \hat{p}_t(-\epsilon)}{\epsilon^2} &= \int_0^t \left(\frac{\hat{p}_t(\epsilon) - \hat{p}_t(-\epsilon)}{\epsilon} \pi_s + \frac{\hat{p}_t(\epsilon) + \hat{p}_t(-\epsilon)}{2} \hat{\pi}_t \right) dw_s, \\
\eta_t^{d+2} &= \eta_t^0 = 2 \int_0^t \xi_s^0 \pi_s dw_s,
\end{aligned}$$

we have

$$\begin{aligned}
& E^\alpha \sup_{t \leq \gamma} \left| \frac{\hat{p}_t(\epsilon) - 2\hat{p}_t(0) + \hat{p}_t(-\epsilon)}{\epsilon^2} - \eta_t^{d+2} \right|^p \\
& \leq N(p) E^\alpha \left(\int_0^\gamma \left(\frac{\hat{p}_t(\epsilon) - \hat{p}_t(-\epsilon)}{\epsilon} - 2\xi_t^0 \right)^2 |\pi_t|^2 + \left(\frac{\hat{p}_t(\epsilon) + \hat{p}_t(-\epsilon)}{2} - 1 \right)^2 |\hat{\pi}_t|^2 dt \right)^{p/2} \\
& \leq N(p) E^\alpha \left(\epsilon^{-p} \sup_{t \leq \gamma} \left| \frac{\hat{p}_t(\epsilon) - \hat{p}_t(-\epsilon)}{\epsilon} - 2\xi_t^0 \right|^{2p} + \epsilon^{-p} \sup_{t \leq \gamma} \left| \frac{\hat{p}_t(\epsilon) + \hat{p}_t(-\epsilon)}{2} - 1 \right|^{2p} \right. \\
& \quad \left. + \epsilon^p \int_0^\gamma |\pi_t|^{2p} dt + \epsilon^p \int_0^\gamma |\hat{\pi}_t|^{2p} dt \right) \\
& \leq \epsilon^p N(p) E^\alpha \left(\int_0^\gamma \hat{p}_t^{2p}(\epsilon) |\pi_t + \frac{\epsilon}{2} \hat{\pi}_t|^{2p} dt + \int_0^\gamma \hat{p}_t^{2p}(-\epsilon) |\pi_t + \frac{\epsilon}{2} \hat{\pi}_t|^{2p} dt \right. \\
& \quad \left. + \int_0^\gamma |\pi_t + \frac{\epsilon}{2} \hat{\pi}_t|^{2p} dt + \int_0^\gamma |\pi_t|^{2p} dt + \int_0^\gamma |\hat{\pi}_t|^{2p} dt \right).
\end{aligned}$$

Therefore,

$$\limsup_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E^\alpha \sup_{t \leq \gamma} \left| \frac{\hat{p}_t(\epsilon) - 2\hat{p}_t(0) + \hat{p}_t(-\epsilon)}{\epsilon^2} - \eta_t^{d+2} \right|^p = 0.$$

In order to estimate $G_2(\epsilon, T)$, we notice that $V_{(\tilde{\eta}_t^{\alpha,0})(\bar{x}_t^{\alpha,x})} + V_{(\bar{\xi}_t^{\alpha,\xi})(\bar{x}_t^{\alpha,x})}$ is exactly Y_t^α defined by (2.9), in which u is replaced by v , that is

$$\begin{aligned} & V_{(\tilde{\eta}_t^{\alpha,0})(\bar{x}_t^{\alpha,x})} + V_{(\bar{\xi}_t^{\alpha,\xi})(\bar{x}_t^{\alpha,x})} = Y_t^\alpha \\ & := e^{-\phi_t^{\alpha,x}} \left[v_{(\xi_t^{\alpha,\xi})(\xi_t^{\alpha,\xi})}(x_t^{\alpha,x}) + v_{(\eta_t^{\alpha,0})(\eta_t^{\alpha,0})}(x_t^{\alpha,x}) + 2\tilde{\xi}_t^0 v_{(\xi_t^{\alpha,\xi})(\xi_t^{\alpha,\xi})}(x_t^{\alpha,x}) + \tilde{\eta}_t^0 v_{(\eta_t^{\alpha,0})(\eta_t^{\alpha,0})}(x_t^{\alpha,x}) \right] \\ & + \int_0^t e^{-\phi_s^{\alpha,x}} \left[f_{(\xi_s^{\alpha,x})(\xi_s^{\alpha,\xi})}^{\alpha_s}(x_s^{\alpha,x}) + f_{(\eta_s^{\alpha,0})(\eta_s^{\alpha,0})}^{\alpha_s}(x_s^{\alpha,x}) + (4r_s^\alpha + 2\tilde{\xi}_s^0) f_{(\xi_s^{\alpha,\xi})(\xi_s^{\alpha,\xi})}^{\alpha_s}(x_s^{\alpha,x}) \right. \\ & \quad \left. + (2\hat{r}_s^\alpha + 4\tilde{\xi}_s^0 r_s^\alpha + \tilde{\eta}_s^0) f^{\alpha_s}(x_s) \right] ds, \end{aligned}$$

where

$$\tilde{\eta}_t^0 = \eta_t^{d+2} + 2\xi_t^{d+2}\xi_t^{d+1} + (\xi_t^{d+1})^2 + \eta_t^{d+1}.$$

In Chapter 3, it is proved that for each α ,

$$E \sup_{t \leq \tau_{D_\delta^\lambda}^{\alpha,x}} (Y_t^\alpha - e^{-\phi_t^{\alpha,x}} v_{(\xi_t^{\alpha,\xi})(\xi_t^{\alpha,\xi})}(x_t^{\alpha,x})) \leq NB_1(x, \xi),$$

where N is independent of α . So

$$G_2(\epsilon, T) \leq \sup_{\alpha \in \mathfrak{A}} E e^{-\phi_t^{\alpha,x}} v_{(\xi_t^{\alpha,\xi})(\xi_t^{\alpha,\xi})}(x_t^{\alpha,x}) + NB_1(x, \xi).$$

By mimicking the argument in the proof of (4.11), we have

$$\overline{\lim}_{T \uparrow \infty} \overline{\lim}_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E e^{-\phi_t^{\alpha,x}} v_{(\xi_t^{\alpha,\xi})(\xi_t^{\alpha,\xi})}(x_t^{\alpha,x}) \leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v_{(\zeta)(\zeta)}(y)}{B_1(x, \zeta)} + 3B_1(x, \xi).$$

So we conclude that

$$\overline{\lim}_{T \uparrow \infty} \overline{\lim}_{\epsilon \downarrow 0} G_2(\epsilon, T) \leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v_{(\zeta)(\zeta)}(y)}{B_1(y, \zeta)} \cdot B_1(x, \xi) + NB_1(x, \xi),$$

which implies that

$$\frac{v_{(\xi)(\xi)}(x)}{B_1(x, \xi)} \leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v_{(\zeta)(\zeta)}(y)}{B_1(y, \zeta)} + N, \quad \forall x \in D_\delta^\lambda, \xi \in \mathbb{R}^d \setminus \{0\}. \quad (4.60)$$

Repeating the argument above for D_{λ^2} , we have

$$\frac{v_{(\xi)(\xi)}(x)}{B_1(x, \xi)} \leq \sup_{y \in \partial D_{\lambda^2}, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v_{(\zeta)(\zeta)}(y)}{B_1(y, \zeta)} + N, \quad \forall x \in D_{\lambda^2}, \xi \in \mathbb{R}^d \setminus \{0\}. \quad (4.61)$$

Since (4.60) and (4.61) are similar as (3.27) and (3.29), by repeating the argument after (3.29), we get

$$-v_{(\xi)(\xi)}(x) \leq N \left(|\xi|^2 + \frac{\psi_{(\xi)}^2(x)}{\psi(x)} \right), \quad \text{a.e. in } D.$$

The inequality (4.12) is proved. \square

Proof of (4.13). Fix an $x \in D$. For simplicity of the notations we will drop the argument x through the proof below.

From (4.12) we have

$$v_{(\xi)(\xi)} + N \left(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi} \right) \geq 0, \forall \xi \in \mathbb{R}^d.$$

It follows that

$$v_{(\xi)(\xi)} + \frac{N}{\psi} |\xi|^2 \geq 0, \forall \xi \in \mathbb{R}^d.$$

Let

$$V = v_{xx} + \frac{N}{\psi} I + I,$$

where I is the identity matrix of size $d \times d$.

Then we have

$$(V\xi, \xi) \geq |\xi|^2 > 0, \forall \xi \in \mathbb{R}^d \setminus \{0\}.$$

Fix a $\xi \in \mathbb{R}^d$ such that $\mu(\xi) > 0$. Introduce

$$\kappa = \sqrt{V}\xi, \quad \theta = |\kappa|^{-2}\kappa, \quad \zeta = \sqrt{V}\theta.$$

Then

$$\begin{aligned} \text{tr}(a^\alpha V) &= \text{tr}(\sqrt{V}a^\alpha\sqrt{V}) \\ &\geq |\theta|^{-2}(\sqrt{V}a^\alpha\sqrt{V}\theta, \theta) = |\kappa|^2(a^\alpha\zeta, \zeta) = (V\xi, \xi)(a^\alpha\zeta, \zeta). \end{aligned}$$

Taking the supremum and noticing that $(\xi, \zeta) = (\kappa, \theta) = 1$, we get

$$\sup_{\alpha \in A} \text{tr}(a^\alpha V) \geq (V\xi, \xi) \sup_{\alpha \in A} (a^\alpha\zeta, \zeta) \geq (V\xi, \xi)\mu(\xi).$$

It follows that

$$v_{(\xi)(\xi)} \leq (V\xi, \xi) \leq \mu^{-1}(\xi) \sup_{\alpha \in A} \text{tr}(a^\alpha V)$$

$$\leq \mu^{-1}(\xi) \left[\sup_{\alpha \in A} \text{tr}(a^\alpha v_{xx}) + \frac{N}{\psi} \sup_{\alpha \in A} \text{tr}(a^\alpha) \right].$$

Notice that

$$\mu(\xi) = |\xi|^{-2} \mu(\xi/|\xi|),$$

so it remains to estimate $\sup_{\alpha \in A} \text{tr}(a^\alpha v_{xx})$ from above. The equation

$$\sup_{\alpha \in A} [L^\alpha v - c^\alpha v + f^\alpha] = 0$$

implies that

$$L^\alpha v - c^\alpha v + f^\alpha \leq 0, \forall \alpha \in A.$$

Thus

$$\text{tr}(a^\alpha v_{xx}) = (a^\alpha)^{ij} v_{x^i x^j} \leq |(b^\alpha)^i|_{0,D} |v_{x^i}|_{0,D} + |c^\alpha|_{0,D} |v|_{0,D} + |f^\alpha|_{0,D} \leq K.$$

□

Proof of the existence and uniqueness of (4.14). The fact that u given by (4.4) satisfies (4.14) follows from Theorem 1.3 in [12].

To proof the uniqueness, assume that $v_1, v_2 \in C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$ are solutions of (4.14). Let $\Lambda = |v_1|_{0,D} \vee |v_2|_{0,D}$. For constants δ and ε satisfying $0 < \delta < \varepsilon < 1$, define

$$\Psi(x, t) = \varepsilon(1 + \psi(x))\Lambda e^{-\delta t}, \quad V(x, t) = v(x)e^{-\varepsilon t} \text{ in } \bar{D} \times (0, \infty),$$

$$F[V] = \sup_{\alpha \in A} (V_t + L^\alpha V - c^\alpha V + f^\alpha) \text{ in } D \times (0, \infty).$$

Notice that a.e. in D , we have

$$F[V_1 - \Psi] \geq -\varepsilon e^{-\varepsilon t} v_1 + \delta \Psi - \varepsilon \Lambda e^{-\delta t} \sup_{\alpha} L^\alpha \psi + \inf_{\alpha} c^\alpha \Psi \geq \varepsilon \Lambda (e^{-\delta t} - e^{-\varepsilon t}) \geq 0,$$

$$F[V_2 + \Psi] \leq \varepsilon e^{-\varepsilon t} v_2 - \delta \Psi + \varepsilon \Lambda e^{-\delta t} \sup_{\alpha} L^\alpha \psi - \inf_{\alpha} c^\alpha \Psi \leq \varepsilon \Lambda (e^{-\varepsilon t} - e^{-\delta t}) \leq 0.$$

On $\partial D \times (0, \infty)$, we have

$$V_1 - V_2 - 2\Psi = -2\Psi \leq 0.$$

On $\bar{D} \times T$, where $T = T(\varepsilon, \delta)$ is a sufficiently large constant, we have

$$V_1 - V_2 - 2\Psi = (v_1 - v_2)e^{-\varepsilon T} - 2\varepsilon(1 + \psi)\Lambda e^{-\delta T} \leq 2\Lambda(e^{-\varepsilon T} - \varepsilon e^{-\delta T}) \leq 0.$$

Applying Theorem 1.1 in [9], we get

$$V_1 - V_2 - 2\Psi \leq 0 \text{ a.e. in } \bar{D} \times (0, T).$$

It follows that

$$v_1 - v_2 \leq 2\varepsilon(1 + \psi)\Lambda e \rightarrow 0, \text{ as } \varepsilon \rightarrow 0, \text{ a.e. in } D.$$

Similarly, $v_2 - v_1 \leq 0$ a.e. in D . The uniqueness is proved.

□

References

- [1] H. Dong, *About smoothness of solutions of the heat equations in closed, smooth space-time domains*, *Comm. Pure Appl. Math.* **58** (2005), no. 6, 799–820. MR 2142631 (2006d:35095)
- [2] H. Dong and N. V. Krylov, *On time-inhomogeneous controlled diffusion processes in domains*, *Ann. Probab.* **35** (2007), no. 1, 206–227. MR 2303952 (2008e:60248)
- [3] G. Fichera, *Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine*, *Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8)* **5** (1956), 1–30. MR 0089348 (19,658a)
- [4] M. I. Freĭdlin, *The smoothness of the solutions of degenerate elliptic equations*, *Izv. Akad. Nauk SSSR Ser. Mat.* **32** (1968), 1391–1413. MR 0237944 (38 #6221)
- [5] ———, *Functional integration and partial differential equations*, *Annals of Mathematics Studies*, vol. 109, Princeton University Press, Princeton, NJ, 1985. MR 833742 (87g:60066)
- [6] A. Friedman, *Stochastic differential equations and applications. Vol. 1*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975, Probability and Mathematical Statistics, Vol. 28. MR 0494490 (58 #13350a)
- [7] ———, *Stochastic differential equations and applications. Vol. 2*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976, Probability and Mathematical Statistics, Vol. 28. MR 0494491 (58 #13350b)
- [8] J. J. Kohn and L. Nirenberg, *Degenerate elliptic-parabolic equations of second order*, *Comm. Pure Appl. Math.* **20** (1967), 797–872. MR 0234118 (38 #2437)

- [9] N. V. Krylov, *Some new results in the theory of controlled diffusion processes*, Mat. Sb. (N.S.) **109(151)** (1979), no. 1, 146–164, 166. MR 538554 (80j:60104)
- [10] ———, *On controllable diffusion processes with unbounded coefficients*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 4, 734–759, 927. MR 631436 (84b:93070)
- [11] ———, *On the control of a diffusion process until the moment of the first exit from the domain*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 5, 1029–1048, 1199. MR 637615 (83g:49029)
- [12] ———, *On the traditional proof of the Bellman equation for controllable diffusion processes*, Litovsk. Mat. Sb. **21** (1981), no. 1, 59–68. MR 617995 (82h:93087)
- [13] ———, *Moment estimates for the quasiderivatives, with respect to the initial data, of solutions of stochastic equations and their application*, Mat. Sb. (N.S.) **136(178)** (1988), no. 4, 510–529, 591. MR 965890 (89k:60078)
- [14] ———, *Smoothness of the payoff function for a controllable diffusion process in a domain*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 66–96. MR 992979 (90f:93040)
- [15] ———, *On the first quasiderivatives of solutions of Itô stochastic equations*, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), no. 2, 398–426. MR 1180379 (93m:60119)
- [16] ———, *Quasiderivatives for solutions of Itô's stochastic equations and their applications*, Stochastic analysis and related topics (Oslo, 1992), Stochastics Monogr., vol. 8, Gordon and Breach, Montreux, 1993, pp. 1–44. MR 1268004 (95e:60057)
- [17] ———, *Introduction to the theory of diffusion processes*, Translations of Mathematical Monographs, vol. 142, American Mathematical Society, Providence, RI, 1995, Translated from the Russian manuscript by Valim Khidekel and Gennady Pasechnik. MR 1311478 (96k:60196)
- [18] ———, *Adapting some ideas from stochastic control theory to studying the heat equation in closed smooth domains*, Appl. Math. Optim. **46** (2002), no. 2-3, 231–261, Special issue dedicated to the memory of Jacques-Louis Lions. MR 1944761 (2003k:35085)

- [19] ———, *Probabilistic methods of investigating interior smoothness of harmonic functions associated with degenerate elliptic operators*, Pubblicazioni del Centro di Ricerca Matematica Ennio de Giorgi. [Publications of the Ennio de Giorgi Mathematical Research Center], Scuola Normale Superiore, Pisa, 2004. MR 2144644 (2006h:60004)
- [20] ———, *Quasiderivatives and interior smoothness of harmonic functions associated with degenerate diffusion processes*, Electron. J. Probab. **9** (2004), no. 21, 615–633 (electronic). MR 2082053 (2005j:60148)
- [21] S. Leventorskii, *Degenerate elliptic equations*, Mathematics and its Applications, vol. 258, Kluwer Academic Publishers Group, Dordrecht, 1993. MR 1247957 (95b:35079)
- [22] P.-L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness*, Comm. Partial Differential Equations **8** (1983), no. 11, 1229–1276. MR 709162 (85i:49043b)
- [23] ———, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. III. Regularity of the optimal cost function*, Nonlinear partial differential equations and their applications. Collège de France seminar, Vol. V (Paris, 1981/1982), Res. Notes in Math., vol. 93, Pitman, Boston, MA, 1983, pp. 95–205. MR 725360 (85i:49043c)
- [24] O. A. Oleĭnik and E. V. Radkevič, *Second order equations with nonnegative characteristic form*, Plenum Press, New York, 1973, Translated from the Russian by Paul C. Fife. MR 0457908 (56 #16112)
- [25] M. V. Safonov, *The Dirichlet problem for Bellman's equation in a plane domain*, Mat. Sb. (N.S.) **102(144)** (1977), no. 2, 260–279, 327. MR 0445110 (56 #3455)
- [26] ———, *On the Dirichlet problem for Bellman's equation in a plane region*, Mat. Sb. (N.S.) **105(147)** (1978), no. 4, 594–600, 640. MR 496596 (82a:35045)
- [27] D. Stroock and S. R. S. Varadhan, *On degenerate elliptic-parabolic operators of second order and their associated diffusions*, Comm. Pure Appl. Math. **25** (1972), 651–713. MR 0387812 (52 #8651)

- [28] W. Zhou, *A probabilistic approach to interior regularity of fully nonlinear degenerate elliptic equations in smooth domains, submitted*, arXiv:1112.5692.
- [29] ———, *Quasiderivative method for derivative estimates of solutions to degenerate elliptic equations, submitted*, arXiv:1112.5689.