NAVIER-STOKES EQUATIONS IN THIN 3D DOMAINS:
GLOBAL REGULARITY OF SOLUTIONS I

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NAVIER-STOKES EQUATIONS IN THIN 3D DOMAINS:  
GLOBAL REGULARITY OF SOLUTIONS I* 

GENEVIEVE RAUGEL† AND GEORGE R. SELL‡

Abstract. We examine the Navier-Stokes equations (NS) on a thin 3-dimensional domain \( \Omega_\varepsilon = Q_2 \times (0, \varepsilon) \), where \( Q_2 \) is a suitable bounded domain in \( \mathbb{R}^2 \) and \( \varepsilon \) is a small, positive, real parameter. We consider these equations with various homogeneous boundary conditions, especially spatially periodic boundary conditions. We show that there are large sets \( \mathcal{R}(\varepsilon) \) in \( H^1(\Omega_\varepsilon) \) and \( \mathcal{S}(\varepsilon) \) in \( W^{1,\infty}(0, \infty), L^2(\Omega_\varepsilon) \) such that if \( U_0 \in \mathcal{R}(\varepsilon) \) and \( F \in \mathcal{S}(\varepsilon) \), then (NS) has a strong solution \( U(t) \) that remains in \( H^1(\Omega_\varepsilon) \) for all \( t \geq 0 \) and in \( H^2(\Omega_\varepsilon) \) for all \( t > 0 \). We show that the set of strong solutions of (NS) has a local attractor \( \mathfrak{A}_\varepsilon \) in \( H^1(\Omega_\varepsilon) \), which is compact in \( H^2(\Omega_\varepsilon) \). This local attractor \( \mathfrak{A}_\varepsilon \) is the global attractor for all the weak solutions (in the sense of Leray) of (NS). We also show that, under reasonable assumptions, \( \mathfrak{A}_\varepsilon \) is upper semicontinuous at \( \varepsilon = 0 \).

Key words. attractors, global regularity, Navier-Stokes equations, three-dimensional space

AMS(MOS) subject classifications.

TABLE OF CONTENTS

1. Introduction
2. Notation: Statement of Theorems
3. \( H^1 \)-Regularity: Theorem 1
4. \( H^2 \)-Regularity: Theorems 2, 3
5. Reduced 3-Dimensional Theory: Theorems 10, 11, 12
6. Properties of Attractors: Theorems 4, 5, 6, 7, 8, 9
7. Remarks on Other Boundary Conditions: Theorems 13, 14, 15
8. Appendix: Proofs of Auxiliary Estimates

1. Introduction

The modern mathematical theory of fluid dynamics began over 50 years ago when Leray (1933, 1934a, 1934b) published his pioneering works on the Navier-Stokes equations. These equations describe the time evolution of solutions of mathematical models of viscous incompressible fluid flows. Because of this basic role in the modelling of fluid flows, there is considerable interest in developing a good mathematical theory of the behavior of the solutions of the Navier-Stokes equations. Since the solutions of these equations depend on

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both space and time, one is especially interested in the phenomenon of the time evolution of the spatial variations of the solutions. This phenomenon, which is described with more precision later, is referred to as the \textit{regularity} of solutions, and it is the primary focus of the theory we present in this paper.

The Navier-Stokes equations on a bounded region $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, are given by

\begin{equation}
    U_t - \nu \Delta U + (U \cdot \nabla)U + \nabla P = F, \\
    \nabla \cdot U = 0,
\end{equation}

where $\nabla$ is the gradient operator and $\Delta$ is the Laplacian. In this paper we treat the case where $\Omega = \Omega_\epsilon$ is a thin 3-dimensional domain, i.e., $\Omega_\epsilon = Q_2 \times (0, \epsilon)$, where $Q_2$ is a suitable bounded region in $\mathbb{R}^2$ and $\epsilon$ is a small positive parameter. In particular we will study (1.1) with periodic boundary conditions where $Q_2 = (0, \ell_1) \times (0, \ell_2)$, and $\ell_1$ and $\ell_2$ are positive.

Recall that the Navier-Stokes equations (1.1) on $\Omega$ can be written in the abstract form

\begin{equation}
    U' + \nu AU + B(U, U) = P_n F,
\end{equation}

where $P_n$ is the orthogonal projection of $L^2(\Omega, \mathbb{R}^n)$ onto the space of divergence-free vector fields, $AU = -P_n \Delta U$, and $B(U, V) = P_n (U \cdot \nabla) V$. We will be interested in solutions of (1.2) under the assumption that the initial data $U_0$ satisfy:

\begin{equation}
    U_0 \in D(A^{\frac{1}{2}}),
\end{equation}

where $D(A^{\frac{1}{2}})$ is the domain of $A^{\frac{1}{2}}$, see Témam (1977, 1983) and Constantin and Foias (1988). We also assume that the forcing function $F = F(t)$ satisfies

\begin{equation}
    F(\cdot) \in W^{1, \infty}([0, \infty), L^2(\Omega)).
\end{equation}

In the case of periodic boundary conditions one has $D(A^{\frac{1}{2}}) \subset H^1_{\text{per}}(\Omega)$. We will also assume, in this case, that

$$
\int_{\Omega} U_0 \, dy = \int_{\Omega} F \, dy = 0.
$$

The phrase \textit{global regularity} of solutions, or existence of \textit{strong solutions}, refers to the property that when $U_0$ and $F$ satisfy (1.3) and (1.4), then (1.2) has a solution $U(t)$ that satisfies $U(0) = U_0$ and $U \in C^0([0, \infty), H^1(\Omega))$. The \textit{principal outstanding problem} for the 3-dimensional Navier-Stokes equations (3DNS) is to determine whether or not (1.2) has a global regular solution for every $U_0$ and $F$ satisfying (1.3) and (1.4).

The study of the regularity of solutions, both in 2-dimensions and 3-dimensions, has attracted widespread interest beginning with Leray (1933, 1934a, 1934b). We are unable to give a complete history of this study here, but special mention should be made of the important contributions of Hopf (1951), Serrin (1962), Fujita and Kato (1964), Masuda
(1967), Komatsu (1980), and Caffarelli, Kohn, and Nirenberg (1982). Additional references can be found in Gigi (1988). Before describing our results on the global regularity of solutions of the 3DNS, let us review some aspects of the classical theory of regularity of these solutions.

For the 3DNS it is known that for every $U_0$ and $F$ satisfying (1.3) and (1.4) there is a $T$, which depends on $U_0$ and $F$, $0 < T \leq \infty$, such that (1.2) has a unique strong, or regular, solution $U(t)$ that satisfies $U \in C^0 ([0,T), H^1(\Omega)) \cap L^2_{loc} ([0,T), D(A))$ and $U' \in L^2_{loc} ([0,T), L^2(\Omega))$. Furthermore, if the data $U_0$ and $F$ are small, then (as is known and as we show in Section 2.11) one has $T = \infty$, i.e., (1.2) has a globally regular solution for small data. Other than several theorems which establish the global regularity of solutions for small data, it is essentially unknown whether there are any other initial conditions $U_0$ and $F$ for which (1.2) has a globally regular solution, see Constantin and Foias (1988), Ladyzhenskaya (1969), Lions (1969), Téman (1977, 1983, 1988), and von Wahl (1985).

The theory of global regularity of solutions of the 2-dimensional Navier-Stokes equations (2DNS) is quite different. In this case there exists a globally regular solution of (1.2) for all $U_0$ and $F$ satisfying (1.3) and (1.4). Furthermore, one has $U(t) \in H^2(\Omega)$ for all $t > 0$, and there exist positive constants $K$ and $L_1, L_2$ where $L_1$ and $L_2$ do not depend on $U_0$, such that

\begin{equation}
\|U(t)\|_{H^1(\Omega)} \leq K, \quad 0 \leq t < \infty,
\end{equation}

and

\begin{equation}
\limsup_{t \to \infty} \|U(t)\|_{H^1(\Omega)} \leq L_i, \quad i = 1, 2.
\end{equation}

These classical results can be found in Ladyzhenskaya (1972), as well as in Constantin and Foias (1988) and Téman (1977, 1983, 1988). Because of the relevance of (1.5) and (1.6) for the 3-dimensional theory presented here, proofs of these relations are included in Section 5 below.

As a result of (1.5) and (1.6), it follows that in 2-dimensions, when $F$ is time independent, then (1.2) has a global attractor $\mathcal{A}$, and $\mathcal{A}$ is a compact set in $H^1(\Omega)$ and compact in $H^2(\Omega)$, see Ladyzhenskaya (1972), Hale (1988), and Téman (1988). This means that $\mathcal{A}$ is a Lyapunov stable attracting invariant set\(^1\) in $H^1(\Omega)$. If $F$ is time-varying, but has some compactness property (for example, $F(t)$ is Bohr almost periodic in $t$), then by using the theory of skew-product flows (see Sacker and Sell (1977, 1990) and Section 2.11) one can show that (1.2) generates a global attractor in $H^1(\Omega) \times \mathcal{F}$, where $\mathcal{F}$ is a compact, positively invariant subset of $W^{1,\infty} ([0,\infty), L^2(\Omega))$.

It is the existence of the global attractor for the 2DNS which is the raison d'être of our study of the 3DNS on thin domains. Because a thin 3-dimensional domain is somehow close

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\(^1\)It is also the case that the global attractor $\mathcal{A}$ for the 2DNS has finite dimension, see Mallet-Paret (1976). The dimensionality of $\mathcal{A}$ has been widely studied, see Téman (1988) for references to this literature.
to a 2-dimensional domain, it is natural to ask whether one can use the good properties of the 2DNS to study the global regularity of the 3DNS. As we shall see, the theory presented here gives an affirmative answer to this question.

The idea of exploiting the existence of a global attractor of an evolutionary equation on a n-dimensional domain to obtain better information for a corresponding equation on a thin (n + 1)-dimensional domain has already been used in Hale and Raugel (1989a, 1989b, 1990).

The process of exploiting the fact that $\Omega_\varepsilon$ is close to $Q_2$, when $\varepsilon$ is small, is far from being trivial. The main reason for the complication is due to the fact that the 3DNS on $\Omega_\varepsilon$ is a singular perturbation of the 2DNS on $Q_2$. The regularization of this singular perturbation is done in two steps, and it follows the methods introduced in Hale and Raugel (1989a, 1989b) for reaction diffusion equations and damped wave equations on thin domains. First one maps $\Omega_\varepsilon$ onto $Q_3 = Q_2 \times (0, 1)$ by means of dilation. The Navier-Stokes equations (1.1) on $\Omega_\varepsilon$ are then transformed to the dilated Navier-Stokes equations on $Q_3$, see (2.4) below. This dilation alone does not remove the singular perturbation because some of the differential operators in (2.4) contain coefficients with $\varepsilon^{-1}$, or $\varepsilon^{-2}$, and $\varepsilon$ is small. Nevertheless, since the domain is now fixed to be $Q_3$ for all $\varepsilon > 0$, this opens up the possibility of using other techniques from the theory of partial differential equations to regularize the singular perturbation in (2.4). The second step is accomplished by introducing the orthogonal projection $v = Mu$ where

$$v(y_1, y_2) = \frac{1}{\varepsilon} \int_0^\varepsilon u(y_1, y_2, s) ds.$$

By applying $M$ and $(I - M)$ to the dilated Navier-Stokes evolutionary equation (2.5) one finds an equivalent system (2.23). What we effectively show in this paper is that the system (2.23) is a regular perturbation of the 2DNS when $\varepsilon$ is small.

In Raugel and Sell (1989), we described the above method and announced some of our existence results. Since that time, we noticed that a more accurate Sobolev inequality given in Hale and Raugel (1989b) (also see the Appendix, Section 8) can be used to improve the existence theorem in $H^1(\Omega_\varepsilon)$ in a significant way.

The following theorem is the principal result in this paper:

**Theorem A.** Consider the 3DNS (1.1) on $\Omega_\varepsilon$ with periodic boundary conditions. There is an $\varepsilon_0 = \varepsilon_0(\nu, \lambda_1) > 0$ such that for every $\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$, there are large sets $R(\varepsilon)$ and $S(\varepsilon)$, where

$$R(\varepsilon) \subset \{ U \in H^1(\Omega_\varepsilon) : \nabla \cdot U = 0, \int_{\Omega_\varepsilon} U dy = 0 \},$$

$$S(\varepsilon) \subset \{ F \in W^{1, \infty}([0, \infty), L^2(\Omega_\varepsilon)) : \int_{\Omega_\varepsilon} F dy = 0 \},$$

4
such that if \( U_0 \in \mathcal{R}(\varepsilon) \) and \( F \in \mathcal{S}(\varepsilon) \), then (1.2) has a strong solution \( U(t) \) with \( U(0) = U_0 \), defined for all \( t \geq 0 \), and

\[
\|U(t)\|_{H^1(\Omega_\varepsilon)}^2 \leq \hat{K}_1 < \infty,
\]

where \( \hat{K}_1 \) depends on \( U_0 \) and \( F \). Furthermore there exist constants \( \hat{L}_1 \) and \( \hat{L}_2 \), which do not depend on \( U_0 \) and which satisfy

\[
\limsup_{t \to -\infty} \|U(t)\|_{H^1(\Omega_\varepsilon)} \leq \hat{L}_1, \quad \limsup_{t \to -\infty} \|U(t)\|_{H^2(\Omega_\varepsilon)} \leq \hat{L}_2.
\]

The proof of Theorem A, including a clarification of the significance of the assertion that \( \mathcal{R}(\varepsilon) \) and \( \mathcal{S}(\varepsilon) \) are large sets, will be incorporated into the \( H^1 \) and \( H^2 \) Regularity Theorems, which are discussed in the next three sections. It is a consequence of these Regularity Theorems that the set of strong solutions of the Navier-Stokes evolutionary equation has a local attractor \( \mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon(F) \) whenever \( F \) satisfies some compactness property and \( F \in \mathcal{S}(\varepsilon) \). The basin of attraction of \( \mathcal{A}_\varepsilon \) contains the set \( \mathcal{R}(\varepsilon) \times H^+(F) \), see Section 2.11. Moreover, we show that for \( F \in \mathcal{S}(\varepsilon) \), the set \( \mathcal{A}_\varepsilon \) is the global attractor for the Leray solutions of the 3DNS on \( \Omega_\varepsilon \), i.e., those weak solutions that satisfy the energy inequality (3.35). Furthermore, when \( F \) is time-independent, \( \mathcal{A}_\varepsilon \) is a compact set in \( H^2(\Omega_\varepsilon) \). We also show that, under reasonable assumptions, \( \mathcal{A}_\varepsilon \) is upper semicontinuous at \( \varepsilon = 0 \).

In the next section we shall introduce the notation used in this paper, and we will state the main theorems proved herein. The proofs of the regularity theorems will be given in Sections 3 and 4, and the theory of the reduced 3DNS is presented in Section 5. The reduced 3DNS describe solutions of the 3DNS which depend only on two spatial coordinates. In Section 6 we study the attractor \( \mathcal{A}_\varepsilon \) for the 3DNS and we show that, under reasonable hypotheses, \( \mathcal{A}_\varepsilon \) is close to the global attractor \( \mathcal{A}_0 \) for the reduced 3DNS.

The theory of the 3DNS on thin domains, which we present in Sections 2 through 6, will be formulated in the context of spatially periodic boundary conditions. However the methods we use are valid in other settings. In Section 7 we will show how the theory presented here can be extended to cover the Navier-Stokes equations with other homogeneous boundary conditions. In a forthcoming paper we will present the theory of global regularity for the Navier-Stokes equations with inhomogeneous boundary conditions on thin 3-dimensional domains, and we will consider other types of thin domains.

We express our sincere appreciation to Ciprian Foias, Jack Hale, and Roger Témam for their helpful suggestions on this paper. We are especially grateful to Ciprian Foias for observing that our arguments show that \( \mathcal{A}_\varepsilon \) is the global attractor for the Leray solutions of the 3DNS.

2. Notation: Statement of Theorems

The Navier-Stokes equations on a bounded region \( \Omega \) in \( \mathbb{R}^n \), \( n = 2, 3 \), are given by

\[
U_t - \nu \Delta U + (U \cdot \nabla)U + \nabla P = F,
\]

(2.1)

\[
\nabla \cdot U = 0,
\]
where $\nabla$ is the gradient operator and $\Delta$ is the Laplacian. In this paper we will be especially interested in the case where $n = 3$ and $\Omega$ is a thin domain of the form $\Omega_\varepsilon = Q_2 \times (0, \varepsilon)$, where $Q_2$ is a suitable bounded domain in $\mathbb{R}^2$ and $\varepsilon$ is a small positive number. In particular, we will assume that $Q_2 = (0, \ell_1) \times (0, \ell_2)$, where $\ell_1$, $\ell_2$, and $\varepsilon$ are positive. We will assume that $\varepsilon \leq \ell_2 \leq \ell_1$ and $0 < \varepsilon \leq 1$, and that the solutions $U$ of (2.1) satisfy the periodic boundary conditions:

\[
\begin{align*}
U(y + \ell_i e_i, t) &= U(y, t), & i &= 1, 2, \\
U(y + \varepsilon e_3, t) &= U(y, t),
\end{align*}
\]

where $\{e_1, e_2, e_3\}$ is the natural basis in $\mathbb{R}^3$. In addition we will require that $F$ and the initial data $U_0$ satisfy

\[
\int_{\Omega_\varepsilon} Fdy = \int_{\Omega_\varepsilon} U_0dy = 0.
\]

It then follows that any solution $U$ of (2.1) with $U(0) = U_0$ will also satisfy $\int_{\Omega_\varepsilon} Udy = 0$ for $t > 0$. Set $Q_3 = Q_2 \times (0, 1)$, and define $a = (a_1, a_2, a_3)$, where $a_i = \ell_i^{-1}$, $i = 1, 2, 3$, and $\ell_3 = 1$. The change of variables $(y_1, y_2, y_3) \mapsto (x_1, x_2, x_3)$ where $x_i = y_i$, $i = 1, 2$, and $x_3 = \varepsilon^{-1} y_3$ maps $\Omega_\varepsilon$ onto $Q_3$.


2.1 Dilated Navier-Stokes Equations. The linear operator $J_\varepsilon$ given by $U = J_\varepsilon u$, where

\[
U(y_1, y_2, y_3) = u(y_1, y_2, \varepsilon^{-1} y_3),
\]

sets up a one-to-one correspondence between measurable functions on $\Omega_\varepsilon$ and measurable functions on $Q_3$. Furthermore one has $J_\varepsilon (W^{k,p}(Q_3)) = W^{k,p}(\Omega_\varepsilon)$ for any Sobolev space $W^{k,p}$. We will need the following identity:

\[
\|U\|_{L^p(\Omega_\varepsilon)}^p = \varepsilon \|u\|_{L^p(Q_3)}^p, \quad 1 \leq p < \infty,
\]

where $U = J_\varepsilon u$. We shall use capital Roman letters to denote function on $\Omega_\varepsilon$ and lower case Roman letters for functions on $Q_3$.

We want to let $\varepsilon$ vary in our study of the solutions of (2.1). Rather than studying a fixed equation on a variable domain, it is more convenient to fix the domain and permit the equation to vary. Therefore we shall follow the construction in Hale and Rauge (1989a). In particular, by using the operator $J_\varepsilon$, the Navier-Stokes equations (2.1) are transformed to the following system on $Q_3$:}

\[
\begin{align*}
u_t - \nu \Delta_\varepsilon u + (u \cdot \nabla_\varepsilon)u + \nabla_\varepsilon p &= f, \\
\nabla_\varepsilon \cdot u &= 0,
\end{align*}
\]
where $\nabla_\epsilon = (D_1, D_2, \epsilon^{-1} D_3)$, $\Delta_\epsilon = D_1^2 + D_2^2 + \epsilon^{-2} D_3^2$, $D_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$, $u = J_\epsilon^{-1} U$, $f = J_\epsilon^{-1} F$ and $p = J_\epsilon^{-1} P$. We will refer to (2.4) as the dilated Navier-Stokes equations on $Q_3$. Because of the terms $\epsilon^{-1} D_3$ and $\epsilon^{-2} D_3^2$ where $\epsilon$ is small, the system (2.4) is a singular perturbation of the two-dimensional Navier-Stokes equations.

2.2 Abstract Formulation. The next step is to reformulate the initial value problem for (2.4) as an abstract nonlinear evolutionary equation on a suitable Hilbert space $H_\epsilon$. The approach we use is an adaptation of the theory presented in Témam (1983).

Let $L^2(Q_3) = L^2(Q_3, R^3)$ denote the collection of all functions $u : Q_3 \to R^3$ with the property that
\[
\int_{Q_3} |u|^2 dx = \int_{Q_3} u \cdot u dx < \infty,
\]

and let
\[
||u|| \overset{\text{def}}{=} ||u||_{L^2(Q_3)} = \left( \int_{Q_3} |u|^2 dx \right)^{1/2}
\]
denote the usual norm. For $m = 0, 1, 2, \ldots$, the Sobolev spaces $H^m_p(Q_3) = H^m_p(Q_3, R^3)$ consist of those functions in $H^m_{loc}(R^3, R^3)$ which are periodic in space, i.e.,
\[
u(x + \ell_i e_i) = u(x), \quad i = 1, 2, 3.
\]

One then has $H^0_p(Q_3) = L^2(Q_3)$. Also the norm on $H^m_p(Q_3)$ is generated by the inner product
\[
(u, v)_m = \sum_{|\alpha| \leq m} \int_{Q_3} D^\alpha u \cdot D^\alpha v dx.
\]

Let $H_\epsilon = H_\epsilon(Q_3)$ denote the closure in $L^2(Q_3)$ of those smooth functions $u$ that are periodic on $Q_3$ and satisfy:
\[
\int_{Q_3} u dx = 0 \quad \text{and} \quad \nabla_\epsilon \cdot u \overset{\text{def}}{=} D_1 u_1 + D_2 u_2 + \epsilon^{-1} D_3 u_3 = 0.
\]

Let $P_\epsilon$ denote the orthogonal projection of $L^2(Q_3)$ onto $H_\epsilon$. By applying $P_\epsilon$ to (2.4) and using the fact that $P_\epsilon(\nabla_\epsilon p) = 0$, we obtain the following abstract nonlinear evolutionary equation on $H_\epsilon$:
\[
(2.5) \quad u' + \nu A_\epsilon u + B_\epsilon(u, u) = P_\epsilon f,
\]

where $\frac{\partial}{\partial t} u = u'$, $u = P_\epsilon u \in H_\epsilon$, $A_\epsilon u = -P_\epsilon \Delta_\epsilon u$ (with the periodic boundary conditions) and the bilinear form $B_\epsilon$ satisfies $B_\epsilon(u, v) = P_\epsilon(u \cdot \nabla_\epsilon)v$. We shall refer to (2.5) as the dilated Navier-Stokes evolutionary equation. We define $V^m_\epsilon$ for $m = 0, 1, 2, \ldots$, by
\[
V^m_\epsilon = H_\epsilon \cap H^m_p(Q_3).
\]
Thus $V^0_\epsilon = H_\epsilon$. Also $A_\epsilon$ is a self-adjoint operator with compact resolvent, and one has $D(A_\epsilon) = V_\epsilon^2$ and $D(A_\epsilon^{1/2}) = V_\epsilon^1$. Furthermore, $A_\epsilon$ satisfies

$$\lambda_1 \|u\|^2 \leq \|A_\epsilon^{1/2}u\|^2, \quad u \in D(A_\epsilon^{1/2}),$$

where $\lambda_1 > 0$ is the smallest eigenvalue of $A_\epsilon$. Since $0 < \epsilon < \ell_2 \leq \ell_1$ one has $\lambda_1 = 4\pi^2 a_1^2 = 4\pi^2 \ell_1^{-2}$.

The evolutionary equation (2.5) does not contain the pressure term $\nabla_\epsilon p$ because $P_\epsilon \nabla_\epsilon p = 0$. In order to recover the pressure term we apply $(I - P_\epsilon)$ to (2.4) to obtain

$$\begin{align*}
(1 - P_\epsilon)(u_t - \nu \Delta_\epsilon u + (u \cdot \nabla)(1)u) + \nabla_\epsilon p = (I - P_\epsilon)f.
\end{align*}$$

Since $\nabla_\epsilon \cdot u = 0$ one has $(I - P_\epsilon)u_t = 0$ and, in the case of periodic boundary conditions, $(I - P_\epsilon)\Delta_\epsilon u = 0$. Consequently, (2.7) becomes

$$\begin{align*}
(I - P_\epsilon)(u \cdot \nabla)(1)u + \nabla_\epsilon p = (I - P_\epsilon)f,
\end{align*}$$

where $\nabla_\epsilon \cdot u = 0$. If $u \in V_\epsilon^1$ and $f \in L^2(Q_3)$ are known, then the only unknown term in (2.8) is the pressure $p$. One can solve for $p$ by classical techniques, see Constantin and Foias (1988) and Témam (1977, 1983).

We will assume the forcing term $f$ in (2.4) to be a time-varying function in the space $L^{\infty}((0, \infty), L^2(Q_3))$ and we define the norm $\|f\|_\infty$ by

$$\|f\|_\infty \overset{\text{def}}{=} \text{ess sup}_{0 < t < \infty} \|f(t)\|_{L^2(Q_3)}.$$

For some applications, we will assume that $f \in W^{1,\infty}((0, \infty), L^2(Q_3))$, in which case the function $f$ is absolutely continuous and the mapping $t \to f(t)$ is uniformly continuous.

We shall say that $u(t)$ is a **strong solution** of (2.5) on an interval $[0, T)$, where $0 < T \leq \infty$, if for every $\tau$, $0 < \tau < T$, one has

$$\begin{align*}
u(\cdot) \in C^0([0, \tau], V^1_\epsilon) \cap L^2((0, \tau), V^2_\epsilon).
\end{align*}$$

Recall that if $u(t)$ is a strong solution of (2.5) on $[0, T)$, then it is uniquely determined, see Témam (1977, 1983) or Constantin and Foias (1988). Furthermore, if $u(t)$ is a solution of (2.5) on an interval $[0, T)$, where $0 < T \leq \infty$, and satisfies $u(\cdot) \in C^0([0, \tau], V^1_\epsilon)$ for every $\tau$, $0 < \tau < T$, then $u(\cdot) \in C^0((0, \tau], V^2_\epsilon)$. (See Section 4.)

A strong solution $u(t)$ on an interval $[0, T)$ is said to be **maximally defined** if $u(t)$ does not have a proper extension to a strong solution of (2.5) on a larger interval. Recall that if $u(t)$ is a maximally defined strong solution of (2.5) on an interval $[0, T)$ and if $T < \infty$, then one has

$$\begin{align*}
\|A^{1/2}_\epsilon u(t)\| \to \infty, \quad \text{as } t \to T^-,
\end{align*}$$

see Témam (1977, 1983) or Constantin and Foias (1988).
2.3 Fourier Series. The spaces $V^m_{\varepsilon}$ can also be described in terms of the Fourier series expansion for functions $u \in L^2(Q_3)$. For $k$ in the integer lattice $Z^3$, we define

$$ka \overset{\text{def}}{=} (k_1 a_1, k_2 a_2, k_3 a_3).$$

Then the Fourier series expansion for $u \in L^2(Q_3)$ is given by

$$(2.11) \quad u(x) = \sum_{k \in Z^3} c^k e^{2\pi i k \cdot x}$$

where $c^k \in C^3$, $c^k = -c^k$, and

$$c^k = a_1 a_2 a_3 \int_{Q_3} u(x) e^{-2\pi i k \cdot x} dx, \quad k \in Z^3.$$

Consequently, one has $u \in V^0_{\varepsilon} = H_{\varepsilon}$ if and only if $c^0 = 0$ and

$$(2.12) \quad k_1 a_1 c^k_1 + k_2 a_2 c^k_2 + e^{-1} k_3 a_3 c^k_3 = 0, \quad \text{for all} \quad k \in Z^3.$$

Similarly one has $u \in V^m_{\varepsilon}$ for $m \geq 0$ if and only if $c^0 = 0$, condition (2.12) holds, and

$$\sum_{k \in Z^3} |k|^{2m} |c^k|^2 < \infty,$$

where $|k|^2 = k_1^2 + k_2^2 + k_3^2$. Furthermore, it follows from the Parseval equality that

$$(2.13) \quad \|u\|^2_{L^2_p(Q_3)} = (a_1 a_2 a_3)^{-1} \sum_{|\alpha| \leq m} \sum_{k \in Z^3} |(2\pi)^{\alpha} (k a)^{\alpha} \cdot c^k|^2, \quad u \in H^m_{p}(Q_3),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in N^3$, $N = \{0, 1, 2, \ldots\}$ and

$$(k a)^{\alpha} \cdot c^k = (k_1 a_1)^{\alpha_1} c^k_1 + (k_2 a_2)^{\alpha_2} c^k_2 + (k_3 a_3)^{\alpha_3} c^k_3.$$  

The eigenvalues of $A_{\varepsilon}$ are given by

$$\lambda = 4\pi^2 \left[ (k_1 a_1)^2 + (k_2 a_2)^2 + e^{-2}(k_3 a_3)^2 \right],$$

where $k \in Z^3 - \{(0,0,0)\}$. If $u \in V^2_{\varepsilon} = D(A_{\varepsilon})$ then one has

$$A_{\varepsilon} u = 4\pi^2 \sum_{k \in Z^3} \left[ (k_1 a_1)^2 + (k_2 a_2)^2 + e^{-2}(k_3 a_3)^2 \right] c^k e^{2\pi i k \cdot x}.$$
By using the Fourier series representation, it is easily verified that if \( u \in V^2_\varepsilon \), then 
\( \nabla_\varepsilon \cdot \Delta_\varepsilon u = 0 \). This implies that \( A_\varepsilon u = -\Delta_\varepsilon u \) for all \( u \in D(A_\varepsilon) \).

The Navier-Stokes equations (2.1) on \( \Omega_\varepsilon \) can be written in the abstract form

\[
(2.14) \quad U' + \nu A U + B(U, U) = P_3 F,
\]

where \( P_3 \) is the orthogonal projection of \( L^2(\Omega_\varepsilon) \) onto the space of divergence-free vector fields, \( A U = -P_3 \Delta U \) and \( B(U, V) = P_3 (U \cdot \nabla) V \), cf. Témam (1983). One can use the operator \( J_\varepsilon \) given by (2.2) to compare the solutions of (2.14) with those of (2.5). For example, if \( U = J_\varepsilon u \), where \( u \) is given by (2.11), then \( U \) has the Fourier expansion

\[
U(y) = \sum_{k \in \mathbb{Z}^3} c^k \varepsilon^2 \pi i (k_1 a_1, k_2 a_2, \varepsilon^{-1} k_3 a_3) \cdot (y_1, y_2, y_3).
\]

The following identities are easily verified:

\[
\frac{\partial}{\partial y_i} J_\varepsilon u = J_\varepsilon e^{-i} \frac{\partial}{\partial x_i} u, \quad i = 1, 2, 3, \quad \text{for } u \in W^{1,p}(Q_3)
\]

where \( \{1\} = \{2\} = 0, \{3\} = 1 \). Also one has

\[
\Delta J_\varepsilon u = J_\varepsilon \Delta_\varepsilon u, \quad AJ_\varepsilon u = J_\varepsilon A_\varepsilon u, \quad \text{for } u \in D(A_\varepsilon).
\]

As a consequence of (2.3) one then obtains

\[
(2.15) \quad \begin{cases}
\| \frac{\partial}{\partial y_i} U \|_{L^2(\Omega_\varepsilon)}^2 = \varepsilon \| e^{-i} \frac{\partial}{\partial x_i} u \|_{L^2(Q_3)}^2, & u \in H^1(Q_3), \\
\| A^* U \|_{L^2(\Omega_\varepsilon)}^2 = \varepsilon \| A_\varepsilon^* u \|_{L^2(Q_3)}^2, & u \in D(A_\varepsilon^*),
\end{cases}
\]

If \( u \) is given by (2.11) and belongs to \( D(A_\varepsilon^{\frac{1}{2}}) \), we have

\[
(2.16) \quad \| A_\varepsilon^{\frac{1}{2}} u \|^2 = (A_\varepsilon u, u)_0 = 4\pi^2 (a_1 a_2 a_3)^{-1} \sum_{k \in \mathbb{Z}^3} (k_1^2 a_1^2 + k_2^2 a_2^2 + \varepsilon^{-2} k_3^2 a_3^2)|c^k|^2.
\]

Moreover, if \( u \) belongs to \( D(A_\varepsilon) \), one has

\[
\| A_\varepsilon u \|^2 = 16\pi^4 (a_1 a_2 a_3)^{-1} \sum_{k \in \mathbb{Z}^3} (k_1^2 a_1^2 + k_2^2 a_2^2 + \varepsilon^{-2} k_3^2 a_3^2)^2 |c^k|^2.
\]

From the Parseval equality (2.13), we conclude that there exist two positive constants \( C_6 \) and \( C_7 \), which are independent of \( \varepsilon \), such that

\[
(2.17) \quad C_6 (\| u \|_{H^1(Q_3)} + \varepsilon^{-1} \| D_3 u \|_{L^2(Q_3)}) \leq \| A_\varepsilon^{\frac{1}{2}} u \|_{L^2(Q_3)} \leq C_7 (\| u \|_{H^1(Q_3)} + \varepsilon^{-1} \| D_3 u \|_{L^2(Q_3)}).
\]
and
\[ C_6(\|u\|_{H^2(Q_3)} + \epsilon^{-1}\|D_3u\|_{H^1(Q_3)} + \epsilon^{-2}\|D_3^2u\|_{L^2(Q_3)}) \]
\[ \leq \|A_\epsilon u\|_{L^2(Q_3)} \]
\[ \leq C_7(\|u\|_{H^2(Q_3)} + \epsilon^{-1}\|D_3u\|_{H^1(Q_3)} + \epsilon^{-2}\|D_3^2u\|_{L^2(Q_3)}). \]
(2.18)

From (2.15) and (2.18), we deduce that \(U = J_\epsilon u\) satisfies
\[ C_6\|U\|_{H^2(\Omega_\epsilon)} \leq \|AU\|_{L^2(\Omega_\epsilon)} \leq 3C_7\|U\|_{H^2(\Omega_\epsilon)}. \]
(2.19)

2.4 The Projection \(M\). For any \(u \in L^2(Q_3)\) we define \(v = Mu\) by
\[ v(x_1, x_2) = \int_0^1 u(x_1, x_2, s)ds, \]
and set \(w = (I - M)u\). Since \(w = (I - M)u\), one has \(Mw = 0\), and \(M\) is an orthogonal projection on \(L^2(Q_3)\) which satisfies
\[ MD_i u = D_i Mu, \quad i = 1, 2, \quad \text{for all} \ u \in W^{1,1}(Q_3), \]
\[ MD_3 u = D_3 Mu = 0, \quad \text{for all} \ u \in H^1_p(Q_3), \]
and, therefore
\[ \nabla_\epsilon \cdot Mu = M\nabla_\epsilon \cdot u, \quad \text{for all} \ u \in H^1_p(Q_3), \]
as well as
\[ A_\epsilon Mu = MA_\epsilon u, \quad \text{for all} \ u \in D(A_\epsilon). \]
(2.20)

As a consequence of all these properties, we conclude that \(M(V_m^\epsilon) \subset V_m^\epsilon\) and that \(M\) is an orthogonal projection in \(V_m^\epsilon\) for all integers \(m \geq 0\). In particular, we have
\[ \|A_r^\epsilon u\|^2 = \|A_r^\epsilon v\|^2 + \|A_r^\epsilon w\|^2, \quad r = 0, \frac{1}{2}, 1. \]
(2.21)

In terms of the Fourier series
\[ w(x) = \sum_{k \in \mathbb{Z}^2} c^k e^{2\pi i k \cdot x}, \]
one has \(Mw = 0\) if and only if
\[ c^{(k_1, k_2, 0)} = 0, \quad \text{for all} \quad (k_1, k_2) \in \mathbb{Z}^2. \]
To put it in another way, if \( c^k \) is any nonzero Fourier coefficient for \( w \), where \( Mw = 0 \), then \( k = (k_1, k_2, k_3) \) satisfies \( k_3 \neq 0 \). Consequently, one has the Poincaré inequality:

\[
\|w\|^2 \leq C_3^2 \varepsilon^2 \|A^{1/2}_\varepsilon w\|^2, \quad w \in V^1_\varepsilon, \quad Mw = 0,
\]

where \( C_3 \) does not depend on \( \varepsilon \). Indeed, from the Parseval equality (2.13) with \( m = 0 \) and from (2.16) one obtains

\[
\|A^{1/2}_\varepsilon w\|^2_{L^2(Q_3)} = 4\pi^2 (a_1 a_2 a_3)^{-1} \sum_{k \in Z^3} (k_1^2 a_1^2 + k_2^2 a_2^2 + \varepsilon^{-2} k_3^2 a_3^2) |c^k|^2
\]

\[
= 4\pi^2 (a_1 a_2 a_3)^{-1} \sum_{k_3 \neq 0} (k_1^2 a_1^2 + k_2^2 a_2^2 + \varepsilon^{-2} k_3^2 a_3^2) |c^k|^2
\]

\[
\geq 4\pi^2 \varepsilon^{-2} (a_1 a_2 a_3)^{-1} \sum_{k \in Z^3} |c^k|^2 = 4\pi^2 \varepsilon^{-2} \|w\|^2_{L^2(Q_3)},
\]

which completes the proof of (2.22). As we shall see, (2.22) plays a critical role in the theory presented here.

2.5 The \( v \) and \( w \) Equations. We now apply the projections \( M \) and \( (I - M) \) to the equation (2.5) where \( v = Mu \) and \( w = (I - M)u \). Since one has \( MB_\varepsilon(v, v) = B_\varepsilon(v, v) \), it follows from (2.20) that one obtains the system:

\[
\begin{cases}
  v' + \nu A_\varepsilon v + B_\varepsilon(v, v) = MP_\varepsilon f - M (B_\varepsilon(v, w) + B_\varepsilon(w, v) + B_\varepsilon (w, w)) \\
  w' + \nu A_\varepsilon w = (I - M)P_\varepsilon f - (I - M) (B_\varepsilon(v, w) + B_\varepsilon(w, v) + B_\varepsilon (w, w)).
\end{cases}
\]

(2.23)

Since \( v \) does not depend on \( x_3 \), one has \( A_\varepsilon v = D_1^2 v + D_2^2 v \), i.e., \( A_\varepsilon v \) is independent of \( \varepsilon \). Similarly \( B_\varepsilon(v, v) \) is independent of \( \varepsilon \). The initial condition \( u(0) = u_0 = v_0 + w_0 \) also splits into a \( v \) and \( w \) component. We will be studying solutions \( (v, w) = (v(t), w(t)) \) that satisfy \( v(0) = v_0 \) and \( w(0) = w_0 \), where \( Mv_0 = v_0 \) and \( Mw_0 = 0 \).

2.6 Reduced 3D Navier-Stokes Evolutionary Equation. The system (2.23) has an invariant set which occurs when

\[
(I - M)P_\varepsilon f = 0, \quad w_0 = 0,
\]

i.e., both the forcing term \( P_\varepsilon f \) and the initial condition \( u_0 \) depend only on \( x_1 \) and \( x_2 \). In this case \( w(t) \equiv 0 \) for all \( t \geq 0 \) and \( \bar{v} = v(t) \) satisfies

\[
\bar{v}' + \nu A_\varepsilon \bar{v} + B_\varepsilon (\bar{v}, \bar{v}) = MP_\varepsilon f
\]

(2.24)

with \( \bar{v}(0) = v_0 \). We refer to (2.24) as the reduced 3D Navier-Stokes evolutionary equation. Note that \( \bar{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3) \) is a three dimensional vector field on \( Q_3 \), and \( \bar{v} \) does not depend on \( x_3 \).
The reduced 3D Navier-Stokes evolutionary equation incorporates the 2DNS on $Q_2$. In order to see this, we let $L^2(Q_2, R^2)$ denote the $L^2$ space of 2-dimensional vector fields $m = (m_1, m_2)$ which depend on $(x_1, x_2) \in Q_2$. Let $P_2$ denote the orthogonal projection of $L^2(Q_2, R^2)$ onto $H_\varepsilon(Q_2)$, where $H_\varepsilon(Q_2)$ is the closure in $L^2(Q_2, R^2)$ of those smooth functions $m$ that are periodic on $Q_2$ and satisfy $\int_{Q_2} m \, dx = 0$ and $(D_1 m_1 + D_2 m_2) = 0$. One then has

$$P_\varepsilon \begin{pmatrix} \bar{v}_1 D_1 + \bar{v}_2 D_2 \\ \bar{v}_1 D_1 + \bar{v}_2 D_2 \end{pmatrix} = P_2 \begin{pmatrix} \bar{v}_1 D_1 + \bar{v}_2 D_2 \bar{v}_1 \\ \bar{v}_1 D_1 + \bar{v}_2 D_2 \bar{v}_2 \end{pmatrix}$$

and

$$P_\varepsilon \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = P_2 \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

where $g = (g_1, g_2, g_3) \in ML^2(Q_2, R^2)$. Furthermore $\bar{v}$ is a solution of the reduced 3D Navier-Stokes evolutionary equation (2.24) if and only if $m = (\bar{v}_1, \bar{v}_2)$ is a solution of the 2D Navier-Stokes evolutionary equation:

$$\frac{d}{dt} m - \nu(D_1^2 + D_2^2)m + P_2(m \cdot \nabla)m = (g_1, g_2),$$

and $\bar{v}_3$ is a solution of the linear equation

$$\frac{d}{dt} \bar{v}_3 - \nu(D_1^2 + D_2^2)\bar{v}_3 + (\bar{v}_1 D_1 + \bar{v}_2 D_2)\bar{v}_3 = g_3,$$

where $g = (g_1, g_2, g_3) = MP_\varepsilon f$.

If we want to emphasize that the terms in (2.24) do not depend on $\varepsilon$ or $x_3$, we introduce the following notation. For $i = 1, 2, 3$, we set $V_i^0 = M V_i^1$, $H_0 = V_0^0 = MH_\varepsilon(Q_3)$. We denote by $A_0$ the restriction of $A_\varepsilon$ to $V_0^0$. If $\bar{v}$ is in $V_0^0$, then $A_0 \bar{v} = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \bar{v}$. We also set $B_0(\bar{v}, \bar{v}) = P_\varepsilon(\bar{v} \cdot \nabla)\bar{v}$ if $\bar{v}$ is in $V_0^1$. Note that $B_0(\bar{v}, \bar{v}) = B_\varepsilon(\bar{v}, \bar{v})$. The reduced 3D Navier-Stokes evolutionary equation (2.24) now becomes

$$\bar{v}' + \nu A_0 \bar{v} + B_0(\bar{v}, \bar{v}) = g$$

with $\bar{v}(0) = \bar{v}_0$ in $V_0^1$ and $g = MP_\varepsilon f$.

2.7 The Trilinear Form $b_\varepsilon$. The trilinear form $b_\varepsilon(u, v, w)$ is defined by

$$b_\varepsilon(u, v, w) = \sum_{i,j=1}^3 \int_{Q_3} \epsilon^{-(i)} u_i(D_j v_j) w_j \, dx,$$
provided the integrals are all defined. We define \( \{1\} = \{2\} = 0 \) and \( \{3\} = 1 \). If \( u, v, w \in V_\varepsilon^1 \), then (2.25) is well-defined, and since \( P_\varepsilon \) is an orthogonal projection with \( P_\varepsilon w = w \), one has

\[
< B_\varepsilon(u, v), w > = < P_\varepsilon(u \cdot \nabla_\varepsilon)v, w > = < (u \cdot \nabla_\varepsilon)v, P_\varepsilon w > = < (u \cdot \nabla_\varepsilon)v, w > = b_\varepsilon(u, v, w),
\]

Note that

\[
(2.26) \quad b_\varepsilon(v^1, w, v^2) = -b_\varepsilon(v^1, v^2, w) = 0, \quad b_\varepsilon(w, v^1, v^2) = 0,
\]

whenever \( Mv^i = v^i, i = 1, 2 \), and \( Mw = 0 \).

**2.8 Statement of Regularity Theorems.** Theorem A, which is stated in Section 1, gives a sufficient condition for the nonlinear evolutionary equation (2.5) to have a strong solution \( u(t) \) that remains in \( V_\varepsilon^1 \) for all \( t \geq 0 \) and in \( V_\varepsilon^2 \) for all \( t > 0 \). In a moment we shall define the sets \( \mathcal{R}(\varepsilon) \) and \( \mathcal{S}(\varepsilon) \), and in Section 2.10 we shall explain why these are large sets. The key to this is the following Hypothesis \( H(a, b) \):

We shall say that the bounded monotone functions \( \eta_i(\varepsilon) \) defined for \( 0 < \varepsilon \leq 1, i = 1, 2, 3, 4 \), and constants \( r \) and \( p \) satisfy Hypothesis \( H(a, b) \), where \( a \) and \( b \) are positive, provided:

1. \( p \geq -1, r > -2 \).
2. \( \varepsilon^4 \eta_i^{-1} \to 0 \) as \( \varepsilon \to 0, i = 1, 2 \).
3. \( \varepsilon^4 \eta_i^{-1} \to 0 \) as \( \varepsilon \to 0, i = 3, 4 \).
4. \( \varepsilon^4 Q(\varepsilon) \) is bounded for \( 0 < \varepsilon \leq 1 \), where
   \[
   Q(\varepsilon) = | \log(2C_5^2 \nu^{-2} \varepsilon^{2+r-p} \eta_3^{-2} \eta_3^2) |.
   \]
5. Let \( a > 0 \) be fixed. Then one has
   \[
   \left\{ \begin{array}{l}
   \varepsilon^8 \eta^{-2} \exp(a \eta^{-4}) \to 0, \\
   \eta^{-2} \to \infty
   \end{array} \right.
   \]
   as \( \varepsilon \to 0^+ \), where
   \[
   (2.27) \quad \eta^{-2} \overset{\text{def}}{=} \max (4\eta_1^{-2} + k_1^2 \eta_3^{-4} + k_2^2 \varepsilon^{2+r} \eta_4^{-2}, 1)
   \]
   and \( \varepsilon^8 \exp(2a\eta_2^{-4}) \) is bounded for \( 0 < \varepsilon \leq 1 \). (The constants \( k_1 \) and \( k_2 \) are defined in Lemma 3.1).
6. Let \( b > 0 \) be fixed. Then for any \( \lambda, 0 < \lambda < 1 \), there is an \( \varepsilon_4 = \varepsilon_4(b, \lambda) > 0 \) such that one has
   \[
   \eta_2^{-2} \exp(b \eta_2^{-4}) \leq \lambda (4\eta_1^{-2} + k_1^2 \eta_3^{-4}), \quad 0 < \varepsilon < \varepsilon_4.
   \]
7. The function \( \varepsilon^{4+2r} \eta_4^{-4} (\log \eta^{-4} + 1) \) is bounded as \( \varepsilon \to 0^+ \).
REMARK 1. The choice of the \( \eta_i \)s may depend on the parameters \( p, r, a \) and \( b \).

REMARK 2. Here is an example where Hypothesis H(a,b) is satisfied for any given choice of \( a > 0 \) and \( b > 0 \). We begin with statement (5). If \( \eta \) satisfies (2.28), then

\[
1 \leq \eta^{-4} \leq (48 \eta_1^{-4} + 3k_1^4 \eta_3^{-8} + 3k_2^4 \epsilon^{2(2+r)} \eta_4^{-4}).
\]

Next fix \( p \geq -1 \) and set

\[
\begin{cases}
    r = -2 + \delta, & \delta > 0 \\
    \eta_4^{-1} = -\log \epsilon \\
    \eta_3^{-2} = \eta_1^{-1}.
\end{cases}
\]

If \( \eta_1^{-2} = (-\log \epsilon^a)^\frac{1}{2} \), where \( a > 0 \), then conditions (1), (2), and (3) hold. Furthermore (4) and (7) are valid. In addition, one has

\[
\eta^{-4} \leq (48 + 3k_1^4) (-\log \epsilon^a) + 3k_2^4 \epsilon^{2(2+r)} (-\log \epsilon)^4.
\]

Consequently (2.27) is valid provided

\[
e^{\frac{5}{8}} (-\log \epsilon^a)^\frac{1}{2} \epsilon^{-a(48+3k_1^4)} \to 0, \quad \text{as } \epsilon \to 0^+.
\]

In other words, if \( a \) is chosen such that \( \frac{5}{8} > (48+3k_1^4)a \alpha \), then (2.27) is satisfied. Likewise, by choosing \( \eta_2^{-2} \) so that

\[
\eta_2^{-2} = (\log(-\log \epsilon)^\beta)^\frac{1}{2}, \quad \text{where } \beta > 0 \text{ and } 2b\beta < 1,
\]

we see that statement (6) is valid. (\( \eta_2^{-2} = \log(\log(-\log \epsilon)) \) is another possible choice of \( \eta_2^{-2} \).) Also note that this example satisfies (2.55) below, provided \(-1 \leq p < 0 \).

In order to prove Theorem A we shall analyze the dilated Navier-Stokes evolutionary equation on \( Q_3 \). This analysis consists of two major steps. The first step, which we call the \( H^1 \)-Regularity Theorem, is to show that there is a constant \( K_1 \) such that \( \|A_{\epsilon}^\frac{1}{2} u(t)\|^2 \leq K_1^2 \) for all \( t \geq 0 \), and that there is a constant \( L_5 \), which does not depend on the initial data, such that \( \limsup_{t \to -\infty} \|A_{\epsilon}^\frac{1}{2} u(t)\|^2 \leq L_5^2 \). The second step, which we call the \( H^2 \)-Regularity Theorem, is to show that \( u(t) \in D(A_{\epsilon}) \) for all \( t > 0 \), and that there is a constant \( L_6 \), which does not depend on the initial data, such that \( \limsup_{t \to -\infty} \|A_{\epsilon} u(t)\|^2 \leq L_6^2 \).

THEOREM 1: \( H^1 \)-REGULARITY. Let \( \eta_i, i = 1, 2, 3, 4, r \) and \( p \) satisfying Hypothesis H(a,b), where \( a \) and \( b \) are sufficiently large. Then there exist \( \epsilon_0 > 0, k_2 > 0 \), a continuous function \( \Gamma \in C([0, \infty), R) \), and for all \( \epsilon, 0 < \epsilon \leq \epsilon_0 \), there exists a \( \tilde{T}_1 = \tilde{T}_1(\epsilon) > 0 \) such that whenever \( u_0 \in D(A_{\epsilon}^\frac{1}{2}) \), \( f \in L^\infty((0, \infty), L^2(Q_3)) \) satisfy

\[
(2.29) \quad \left\{ \begin{array}{l}
    \|A_{\epsilon}^\frac{1}{2} u_0\|^2 \leq \eta_1^{-2}, \\
    \|M P_{\epsilon} f\|^2_{L^\infty} \leq \eta_2^{-2} \\
    \|A_{\epsilon}^\frac{1}{2} w_0\|^2 \leq e^p \eta_3^{-2}, \\
    \|(I - M) P_{\epsilon} f\|^2_{L^\infty} \leq e^r \eta_4^{-2}.
\end{array} \right.
\]
then (2.5) has a solution \( u \) that belongs to \( C^0([0, \infty), V^1_\epsilon) \cap L^\infty([0, \infty), V^1_\epsilon) \), i.e., one has

\[
\|A^\frac{1}{2}_\epsilon u(t)\|^2 \leq K^2_1, \quad t \geq 0,
\]

where \( K_1 \) depends on \( \nu, \lambda_1 \) and \( \eta_i, i = 1, 2, 3, 4, \) but not on \( t \geq 0 \). Moreover, the components of \( u = v + w \) satisfy

\[
\|A^\frac{1}{2}_\epsilon v(t)\|^2 \leq \Gamma(\eta_2^{-2}), \quad t \geq \hat{T}_1,
\]

where \( \Gamma \) is given by (3.84), and

\[
\|A^\frac{1}{2}_\epsilon w(t)\|^2 \leq \max(\epsilon^2, k^2_2 \epsilon^{2+r} \eta_4^{-2}), \quad t \geq \hat{T}_1.
\]

Remarks. 1. The principal objective in any study of the global regularity of solutions of the 3DNS is to show that the conclusions of Theorem 1 apply for all \( u_0 \in V^1_\epsilon \) and \( P_\epsilon f \in L^\infty((0, \infty), L^2(Q_3)) \). Since the techniques developed in this paper seem to fall short of achieving this goal, we seek, instead, to find the largest possible \( u_0 \) and \( f \) (see (2.29)) for which we can prove global regularity.

2. It follows from (2.21), (2.31) and (2.32) that

\[
\|A^\frac{1}{2}_\epsilon u(t)\|^2 \leq L^2_3, \quad t \geq \hat{T}_1,
\]

where \( L^2_3 = \Gamma(\eta_2^{-2}) + \max(\epsilon^2, k^2_2 \epsilon^{2+r} \eta_4^{-2}) \). Since \( L^2_3 \) does not depend on \( \eta_1, \eta_3 \) or \( \rho \), it is independent of the initial condition \( u_0 \). Furthermore, if \( \eta_2^{-2} \) and \( \epsilon^{2+r} \eta_4^{-2} \) are bounded for \( 0 < \epsilon \leq 1 \), it follows that \( L^2_3 \) can be chosen independent of \( \epsilon \).

Theorem 2: \( H^2 \)-Regularity. Let \( r, p, \) and \( \eta_i, i = 1, 2, 3, 4, \) satisfy Hypothesis \( H(a,b) \), where \( a \) and \( b \) are sufficiently large. If \( u_0 \in V^1_\epsilon \) and

\[
P_\epsilon f \in C^0([0, \infty), H^2_\epsilon) \cap L^\infty((0, \infty), H^1_\epsilon) \cap W^{1,\infty}((0, \infty), D(A^{-\frac{1}{2}}_\epsilon))
\]

satisfy (2.29), then for \( 0 < \epsilon \leq \epsilon_0 \), where \( \epsilon_0 \) is given by Theorem 1, the solution \( u(t) \) of (2.5) belongs to \( C^0([0, \infty), V^2_\epsilon) \). Furthermore there exist three positive constants \( K_2, K_3 \) and \( K_4 \), which depend on \( \nu, \lambda_1, \eta_i, i = 1, 2, 3, 4, \) and \( K_1 \), where \( K_1 \) is given by (2.30), such that

\[
\begin{align*}
\|A u(t)\|^2 &\leq K^2_2 + K^2_3 \|A^{-\frac{1}{2}}_\epsilon P_\epsilon f'\|_{\infty}^2 + K^2_4 t^{-1}, \quad \text{for } 0 < t \leq 1, \\
\|A u(t)\|^2 &\leq K^2_2 + K^2_3 \|A^{-\frac{1}{2}}_\epsilon P_\epsilon f'\|_{\infty}^2, \quad \text{for } t \geq 1.
\end{align*}
\]

Moreover there is a positive continuous function \( \Gamma_2 \) on \([0, \infty)\) given by (4.22), such that

\[
\|A u(t)\|^2 \leq L^2_6, \quad t \geq \hat{T}_1 + 1,
\]
where $\hat{T}_1$ is given by Theorem 1, $L_6^2 = \Gamma_2(L_5^2)$, and $L_5^2$ is given by (2.33).

If, in addition, $u_0$ belongs to $\mathcal{D}(A_{\varepsilon})$, then the solution $u$ of (2.5) belongs to the space $C^0([0, \infty); V_\varepsilon^1)$, and one has

$$
(2.37) \quad \|A_{\varepsilon}u(t)\|^2 \leq K_5^2 + K_6^2 \|A_{\varepsilon}u_0\|^2 + K_T^2 \|A_{\varepsilon}^{-\frac{1}{2}}P_{\varepsilon}f'\|_\infty^2, \quad 0 \leq t \leq 1,
$$

where $K_5, K_6$ and $K_T$ are positive constants depending on $\nu, \lambda_1, \eta_1, i = 1, 2, 3, 4,$ and $K_1$.

Let $B_\varepsilon^0, B_\varepsilon^1,$ and $B_\varepsilon^2$ denote the following bounded sets in $V_\varepsilon^1 = D(A_{\varepsilon}^{\frac{1}{2}})$:

$$
(2.38) \quad B_\varepsilon^0 \overset{\text{def}}{=} \{u = v + w : \|A_{\varepsilon}^{\frac{1}{2}}v\|^2 \leq \eta_1^{-2}, \|A_{\varepsilon}^{\frac{1}{2}}w\|^2 \leq \varepsilon^2 \eta_3^{-2}\},
$$

$$
(2.39) \quad B_\varepsilon^1 \overset{\text{def}}{=} \{u = v + w : \|A_{\varepsilon}^{\frac{1}{2}}v\|^2 \leq 4\eta_1^{-2} + k_1^2 \eta_3^{-4}, \|A_{\varepsilon}^{\frac{1}{2}}w\|^2 \leq k_2^2 \varepsilon^2 + r \eta_4^{-2}\},
$$

$$
(2.40) \quad B_\varepsilon^2 \overset{\text{def}}{=} \bigcup_{t \geq 0} S_\varepsilon(P_{\varepsilon}f, t)(B_\varepsilon^0 \cup B_\varepsilon^1),
$$

where $u(t) = S_\varepsilon(P_{\varepsilon}f, t)u_0$ is the strong solution of the equation (2.5) with initial data $u_0$. Due to the Lemmas 3.1 and 3.2 below, $B_\varepsilon^2$ is well defined and is a bounded set in $V_\varepsilon^1$.

**Remarks 1.** Since $L_2^2$ does not depend on the initial condition $u_0$, it follows from (2.36) that the bound $L_6^2$ does not depend on $u_0$. Furthermore, if $f$ is chosen so that $\|P_{\varepsilon}f\|_\infty$ and $\|A_{\varepsilon}^{-\frac{1}{2}}P_{\varepsilon}f'\|_\infty$ are bounded for $0 < \varepsilon \leq 1$, then $L_6^2$ can be chosen to be independent of $\varepsilon$.

2. One can obtain other $H^2$-regularity results if one assumes that $f$ has more spatial regularity, e.g., $P_{\varepsilon}f \in L^\infty((0, \infty); V_1^1)$ instead of satisfying (2.34). (See Foias, Guillopé and Témam (1981)).

3. If in addition to the hypotheses of Theorem 2, $f$ belongs to $W^{1,\infty}([0, \infty), H_\varepsilon)$, then from (2.6) and (2.22) one finds that

$$
(2.41) \quad \|A_{\varepsilon}^{-\frac{1}{2}}P_{\varepsilon}f'\|_\infty^2 \leq \lambda_1^{-1} \|MP_{\varepsilon}f'\|_\infty^2 + C_5^2 \varepsilon^2 \|(I - M)P_{\varepsilon}f'\|_\infty^2,
$$

which can be used in (2.35), (2.36), and (2.37).

**2.9 Small Data Regularity.** As mentioned in the Introduction, it is known that the 3DNS has a globally regular solution whenever the data of the problem are small. The global regularity with small data, which is valid for any reasonable bounded 3-dimensional region, is a simple consequence of the Stable Manifold Theorem. We emphasize that our theorems, which are valid for thin 3-dimensional regions, are not consequences of the small data arguments. Before showing this though, it will be useful to recall one of these small data arguments at this time.\(^2\)

\(^2\)There are several approaches to proving the global regularity of solutions with small data. For all practical purposes, these arguments all lead to the same conclusions described here.
The argument we give here will be for the Navier-Stokes evolutionary equation (2.14) on an arbitrary bounded region \( \Omega \) in \( \mathbb{R}^3 \). We will not exploit, at this time, the assumption that \( \Omega = \Omega_e \) is a thin domain, an assumption which is of special interest elsewhere in this paper. We assume here, for simplicity, that \( F \in L^2(\Omega) \) does not depend on time.

We will use the standard 3D estimate for \( B(U^1, U^2) \):

\[
|\langle B(U^1, U^2), U^3 \rangle| \leq C_8 \| A^{\frac{1}{2}} U^1 \|_{L^2(\Omega)} \| A^{\frac{1}{2}} U^2 \|_{L^2(\Omega)} \| A U \|_{L^2(\Omega)} \| U^3 \|_{L^2(\Omega)},
\]

see Témam (1977, 1983) or Constantin and Foias (1988). The constant \( C_8 = C_8(\Omega) \) depends on \( \Omega \).

By taking the scalar product of (2.14) with \( A U \) we find that

\[
\frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^2 + \nu \| A U \|_{L^2(\Omega)}^2 \leq \| P_3 F \|_{L^2(\Omega)} \| A U \|_{L^2(\Omega)} + C_8 \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^\frac{3}{2} \| A U \|_{L^2(\Omega)}^\frac{3}{2} \leq \frac{\nu}{2} \| A U \|_{L^2(\Omega)}^2 + \frac{1}{\nu} \| P_3 F \|_{L^2(\Omega)}^2 + \frac{27}{4\nu^3} C_8^4 \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^6.
\]

We then get

\[
\frac{d}{dt} \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^2 + \lambda_1 \nu \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^2 \leq \frac{2}{\nu} \| P_3 F \|_{L^2(\Omega)}^2 + \frac{27}{2\nu^3} C_8^4 \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^6,
\]

which in turn, implies that

\[
(2.42) \quad \frac{d}{dt} \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^2 + \lambda_1 \nu \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^2 \leq \frac{2}{\nu} \| P_3 F \|_{L^2(\Omega)}^2 + \frac{27}{2\nu^3} C_8^4 \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^6.
\]

Now set \( R_0^2 = \| A^{\frac{1}{2}} U_0 \|_{L^2(\Omega)}^2 + \| P_3 F \|_{L^2(\Omega)}^2 \) and \( N > \max(1, 4\lambda_1^{-1} \nu^{-2}) \). Since \( R_0^2 \geq \| A^{\frac{1}{2}} U_0 \|_{L^2(\Omega)}^2 \) and \( N > 1 \), it follows from Lemma 3.0 below that there is a \( T_N, 0 < T_N \leq \infty \) such that

\[
(2.43) \quad \| A^{\frac{1}{2}} U(t) \|_{L^2(\Omega)}^2 \leq NR_0^2, \quad 0 < t < T_N.
\]

We assume, without loss of generality, that \( (0, T^N) \) is the maximal time interval for which (2.43) is valid.

Next assume that

\[
(2.44) \quad \frac{27C_8^4}{2\nu^3} N^2 R_0^4 \leq \frac{\lambda_1 \nu}{2}.
\]

Inequality (2.44) is the precise assumption that the data for (2.14) are small. Because of (2.43) and (2.44), it follows from (2.42) that

\[
(2.45) \quad \frac{d}{dt} \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^2 + \frac{\lambda_1 \nu}{2} \| A^{\frac{1}{2}} U \|_{L^2(\Omega)}^2 \leq \frac{2}{\nu} \| P_3 F \|_{L^2(\Omega)}^2.
\]

By applying the Gronwall inequality to (2.45) we get

\[
\| A^{\frac{1}{2}} U(t) \|_{L^2(\Omega)}^2 \leq \exp \left( -\frac{\lambda_1 \nu}{2} t \right) \| A^{\frac{1}{2}} U_0 \|_{L^2(\Omega)}^2 + \frac{4}{\lambda_1 \nu^2} \| P_3 F \|_{L^2(\Omega)}^2 < NR_0^2
\]

for \( 0 < t \leq T^N \). Consequently (2.10) implies that \( T^N = \infty \), which completes the proof of global regularity of solutions for small data.
REMARK. In the case of a thin domain $\Omega_{\epsilon}$, one has $C_3 = C \epsilon^{-\frac{1}{4}}$, where $C$ is independent of $\epsilon$. As a result, inequality (2.44) can be rewritten as

$$
\|A^{\frac{1}{2}} U_0\|^2_{L^2(\Omega_{\epsilon})} + \|P_3 F\|^2_{L^2(\Omega_{\epsilon})} \leq C^* \epsilon,
$$

where $C^*$ depends on $\nu$ and $\lambda_1$, but not on $\epsilon$.

2.10 Large Data Regularity. We now show that the inequalities (2.29) describe large data conditions on both $Q_3$ and the thin domain $\Omega_{\epsilon}$. The inequalities (2.29) describe the norms of the data for (2.5) in the space $L^2(Q_3)$. We now set $p = r = -1$, and assume that $\eta_i(\epsilon) \to 0$ as $\epsilon \to 0^+$ for $1 \leq i \leq 4$. By using the mapping $J_\epsilon$ together with (2.3), (2.15) and (2.21) one finds that

$$
\|A^{\frac{1}{2}} U_0\|^2_{L^2(\Omega_{\epsilon})} = \|A^{\frac{1}{2}} V_0\|^2_{L^2(\Omega_{\epsilon})} + \|A^{\frac{1}{2}} W_0\|^2_{L^2(\Omega_{\epsilon})}
$$

(2.47)

$$
= \epsilon \|A^{\frac{1}{2}} v_0\|^2_{L^2(Q_3)} + \epsilon \|A^{\frac{1}{2}} w_0\|^2_{L^2(Q_3)}
\leq \epsilon \eta_1^{-2} + \eta_3^{-2}.
$$

Similarly one has

$$
\|P_3 F\|^2_{L^2(\Omega_{\epsilon})} = \epsilon \|P_3 f\|^2_{L^2(Q_3)} \leq \epsilon \eta_2^{-2} + \eta_4^{-2}.
$$

Inequalities (2.47) and (2.48) imply that

$$
\|A^{\frac{1}{2}} U_0\|^2_{L^2(\Omega_{\epsilon})} + \|P_3 F\|^2_{L^2(\Omega_{\epsilon})} \leq \epsilon (\eta_1^{-2} + \eta_2^{-2}) + \eta_3^{-2} + \eta_4^{-2}.
$$

(2.49)

Assume that we choose $\eta_3(\epsilon) = \eta_4(\epsilon) = 1$. Then, even in this case, condition (2.49) is much weaker than condition (2.46) since we allow $\eta_1^{-2}(\epsilon)$ and $\eta_2^{-2}(\epsilon)$ to go to $\infty$ as $\epsilon \to 0^+$. To put it another way, assume that $f = 0$ and let $U_1 = U_1(y_1, y_2)$ satisfy $U_1 \in H^2(Q_2, R^3)$, with periodic boundary conditions, divergence free, and $\|U_1\|_{L^2(Q_2)} = 1$. Then

$$
U_0(y_1, y_2, y_3) = \eta_1^{-1}(\epsilon) U_1(y_1, y_2)
$$

will satisfy (2.49), but not (2.46), for small $\epsilon$, whenever $\eta_1^{-1}(\epsilon) \to \infty$ as $\epsilon \to 0^+$.

For $0 < \epsilon \leq \epsilon$ we define $R_1(\epsilon)$ to be the collection of $v_0 \in MV_{\epsilon}^1$ such that

$$
\|A^{\frac{1}{2}} v_0\|^2_{L^2(Q_3)} = \|v_0\|^2_{V_{\epsilon}^1} \leq \eta_1^{-2},
$$

and $R_2(\epsilon)$ to be the collection of $w_0 \in (I - M)V_{\epsilon}^1$ such that

$$
\|A^{\frac{1}{2}} v_0\|^2_{L^2(Q_3)} = \|w_0\|^2_{V_{\epsilon}^1} \leq \epsilon^{-1} \eta_3^{-2}.
$$
Set $R(\epsilon) = R_1(\epsilon) + R_2(\epsilon)$, and let $R_1(\epsilon) = J_\epsilon R_1(\epsilon), R_2(\epsilon) = J_\epsilon R_2(\epsilon)$ and $R(\epsilon) = J_\epsilon R(\epsilon)$ denote the corresponding sets in $H^1(\Omega_\epsilon)$. The sets $R_1(\epsilon)$ and $R_2(\epsilon)$ are bounded sets in $MV^1_\epsilon$ and $(I - M)V^1_\epsilon$ with $V^1_\epsilon$-radius being $\eta_1^{-1}$ and $\epsilon^{-\frac{1}{2}}\eta_3^{-1}$, respectively. From (2.3) we see that $V_0 = J_\epsilon v_0 \in R_1(\epsilon)$ and $W_0 = J_\epsilon w_0 \in R_2(\epsilon)$ if and only if

\[ \|A^\frac{1}{2}V_0\|_{L^2(\Omega_\epsilon)}^2 \leq \eta_1^{-2}, \quad \text{and} \quad \|A^\frac{1}{2}W_0\|_{L^2(\Omega_\epsilon)}^2 \leq \eta_3^{-2}. \]

It then follows from (2.17) that $R_1(\epsilon)$ and $R_2(\epsilon)$ contain bounded sets in $MH^1(\Omega_\epsilon)$ and $(I - M)H^1(\Omega_\epsilon)$ with $H^1(\Omega_\epsilon)$-radius being $C\epsilon^{\frac{1}{2}}\eta_1^{-1}$ and $C\eta_3^{-1}$, respectively. The example constructed in Section 2.8 gives information on the size of these radii as $\epsilon \to 0^+$. The point to note in this example is that $\eta_i^{-1} = (-\log \epsilon^a)^{\frac{1}{2}}$. The assertion in Theorem A that $R(\epsilon)$ is large is a heuristic formulation of the fact that $\eta_i^{-1} \to \infty$ as $\epsilon \to 0^+$, $i = 1, 3$.

Similarly we define $S(\epsilon)$ to be the collection of $f \in L^\infty((0, \infty), L^2(Q_3))$ that satisfy

\[ \|MP_\epsilon f(t)\|_{L^2(Q_3)}^2 \leq \eta_2^{-2}, \quad \text{and} \quad \|(I - M)P_\epsilon f(t)\|_{L^2(Q_3)}^2 \leq \epsilon r\eta_4^{-2} \]

for $0 < t < \infty$, and set $S(\epsilon) = J_\epsilon S(\epsilon)$, where $r = -2 + \delta$, say $0 < \delta \leq \frac{1}{2}$. One then has $F = J_\epsilon f \in S(\epsilon)$ if and only if

\[ \|MP_3 F(t)\|_{L^2(\Omega_\epsilon)}^2 \leq \eta_2^{-2}, \quad \text{and} \quad \|(I - M)P_3 F(t)\|_{L^2(\Omega_\epsilon)}^2 \leq \epsilon^{r+1}\eta_4^{-2}. \]

Once again, the assertion that $S(\epsilon)$ is large is a heuristic formulation of the fact that $\eta_i^{-1} \to \infty$ as $\epsilon \to 0^+$, $i = 2, 4$. The example in Section 2.8 shows that one can choose $\eta_4^{-1} = (-\log \epsilon)$.

If the initial condition $u_0$ belongs to $D(A_\epsilon)$, as in the case of Theorem 2, then one has $v_0, w_0 \in D(A_\epsilon)$, and (2.6) and (2.22) imply that

\[
\left\{
\begin{array}{l}
\|A^\frac{1}{2}v_0\|_{L^2(Q_3)}^2 \leq \lambda_1^{-1}\|A_\epsilon v_0\|_{L^2(Q_3)}^2 \\
\|A^\frac{1}{2}w_0\|_{L^2(Q_3)}^2 \leq C^2_8 \epsilon^2\|A_\epsilon w_0\|_{L^2(Q_3)}^2.
\end{array}
\right.
\]

This means that $u_0 = v_0 + w_0$ will satisfy (2.29) provided one has

\[(2.50)\]

\[
\left\{
\begin{array}{l}
\|A_\epsilon v_0\|_{L^2(Q_3)}^2 \leq \lambda_1 \eta_1^{-2} \\
\|A_\epsilon w_0\|_{L^2(Q_3)}^2 \leq C_5^2 \epsilon^{-2}\eta_3^{-2},
\end{array}
\right.
\]

where we fix $p = -1$. By using the mapping $J_\epsilon$ and (2.3), we see that (2.50) can be written in the equivalent form

\[
\left\{
\begin{array}{l}
\|A_\epsilon V_0\|_{L^2(\Omega_\epsilon)}^2 \leq \lambda_1 \eta_1^{-2} \\
\|A_\epsilon W_0\|_{L^2(\Omega_\epsilon)}^2 \leq C_5^{-2} \epsilon^{-1}\eta_3^{-2}.
\end{array}
\right.
\]

Thus $R_2(\epsilon)$ contains a bounded set in $(I - M)H^2(\Omega_\epsilon)$ of $H^2(\Omega_\epsilon)$-radius $\geq C\epsilon^{-\frac{4}{3}}$, see (2.19).
2.11 Skew-Product Dynamics. We will review here some aspects of the theory of skew-product flows in order to describe the (local) attractors for the Navier-Stokes equations with a time-varying forcing function \( f \). We will formulate this general theory for the Navier-Stokes equations on an arbitrary bounded domain \( \Omega \) in \( \mathbb{R}^n \), where \( n = 2,3 \). For simplicity\(^3\) we will consider forcing functions \( f \) in the space

\[
(2.51) \quad f \in W(\Omega) \overset{\text{def}}{=} C^0(R,L^2(\Omega)) \cap L^\infty(R,L^2(\Omega)),
\]

where \( L^2(\Omega) = L^2(\Omega,\mathbb{R}^n) \). For any linear operator \( T \) on \( L^2(\Omega) \) we let

\[
TW(\Omega) = C^0(R,TL^2(\Omega)) \cap L^\infty(R,TL^2(\Omega)).
\]

A metrizable topology is introduced in the space \( W(\Omega) \) by defining sequential convergence \( f_n \to f \) to mean that for any compact set \( K \subset \mathbb{R} \) one has

\[
\sup_{t \in K} \| f_n(t) - f(t) \|_{L^2(\Omega)} \to 0, \quad \text{as} \ n \to \infty.
\]

We will denote the associated metric by \( \text{dist}_{W(\Omega)} \).

For any \( f \) satisfying (2.51) and any \( \tau \in \mathbb{R} \) we define the translate \( f_\tau(t) \overset{\text{def}}{=} f(\tau + t) \). Note that \( f_\tau \in W(\Omega) \), and the mapping \( (f,\tau) \to f_\tau \) is a continuous mapping of \( W(\Omega) \times \mathbb{R} \) onto itself. This means that \( f_\tau \) defines a (two-sided) flow on \( W(\Omega) \). For each \( f \) satisfying (2.51) we define the positive hull \( H^+(f) \) as

\[
H^+(f) = \text{Closure}_{W(\Omega)} \{ f_\tau : \tau \geq 0 \},
\]

and the hull \( H(f) \) as

\[
H(f) = \text{Closure}_{W(\Omega)} \{ f_\tau : \tau \in \mathbb{R} \}.
\]

If \( f \in W(\Omega) \), then \( H^+(f) \) and \( H(f) \) lie in \( W(\Omega) \). The omega-limit set \( \omega(f) \) is defined by

\[
\omega(f) = \bigcap_{\tau \geq 0} H^+(f_\tau).
\]

Note that \( \omega(f) \) is an invariant set in \( W(\Omega) \).

Without further assumptions on the forcing function \( f \), the omega-limit set \( \omega(f) \) can be empty. However, if \( H^+(f) \) is a compact set, then so is \( H^+(f_\tau) \) for every \( \tau \geq 0 \). Since \( H^+(f_\tau) \subset H^+(f) \) for \( \tau \geq 0 \), we see that if \( H^+(f) \) is compact, then the omega-limit set \( \omega(f) \) is nonempty and compact.

The question of the compactness of \( H^+(f) \) can be resolved by using the Ascoli-Arzelá Theorem, see Sell (1967a, 1967b). In particular, if there is a compact set \( K \subset L^2(\Omega) \) such

\(^3\)By using other topologies one can describe attractors when the forcing function \( f \) is discontinuous. See Miller and Sell (1970), or Sacker and Sell (1977) for details.
that \( f(t) \in \mathcal{X} \) for all \( t \geq 0 \), and the mapping \( t \to f(t) \) is a uniformly continuous mapping of \([0, \infty)\) into \( L^2(\Omega) \), then \( H^+(f) \) is a nonempty compact set; furthermore the omega-limit set \( \omega(f) \) is nonempty, compact, and invariant. We list here five examples of functions \( f \) for which \( H^+(f) \) is compact:

1. \( f \in W^{1,\infty}([0, \infty), L^2(\Omega)) \), and there is a compact set \( \mathcal{X} \subseteq L^2(\Omega) \) such that \( f(t) \in \mathcal{X} \) for all \( t \geq 0 \).
2. \( f \in W^{1,\infty}([0, \infty), H^1(\Omega)) \).
3. \( f(t) \) is continuous and Bohr almost-periodic, or periodic, in \( t \).
4. \( f = g + h \), where \( g \) and \( h \) satisfy (2.51), \( \|h(t)\|_{L^2(\Omega)} \to 0 \) as \( t \to \infty \), and \( g(t) \) is Bohr almost-periodic, or periodic, in \( t \).
5. \( f \in L^2(\Omega) \) is independent of \( t \).

The evaluation mapping \( Ev : W(\Omega) \to L^2(\Omega) \) given by \( Ev(f) = f(0) \) is a continuous mapping of \( W(\Omega) \) in \( L^2(\Omega) \). Therefore if \( H^+(f) \) is a compact set in \( W(\Omega) \), then
\[
Ev(H^+(f)) = \{g(0) : g \in H^+(f)\}
\]
is a compact set in \( L^2(\Omega) \), and one has \( g(t) \in Ev(H^+(f)) \) for all \( t \geq 0 \) and \( g \in H^+(f) \).

Let \( H \) be a Banach space with a norm \( \| \cdot \|_H \) and let \( \mathcal{F} \) be any positively invariant subset of \( W(\Omega) \). Let \( \mathcal{O} \) be an open set in \( H \times \mathcal{F} \times [0, \infty) \) with
\[
\{(u, f, 0) : (u, f) \in H \times \mathcal{F}\} \subset \mathcal{O},
\]
and let \( \pi : \mathcal{O} \to H \times \mathcal{F} \) be a mapping of the form
\[
(2.52) \quad \pi(u, f, \tau) = (S(f, \tau)u, f_{\tau}), \quad (u, f, \tau) \in \mathcal{O}.
\]

For each \((u, f) \in H \times \mathcal{F}\), let \( I_{(u, f)} = [0, \tau) \), where \( \tau = \tau(u, f) \), denote the maximal time interval for which \((u, f, t) \in \mathcal{O} \) for \( 0 \leq t < \tau \). We say that \( \pi \) is a skew-product semiflow on \( H \times \mathcal{F} \) if the following properties are satisfied:

1. \( S(f, 0)u = u \), for all \((u, f) \in H \times \mathcal{F}\).
2. Whenever \( t \in I_{(u, f)} \) and \( s \in I_{(S(f, t)u, f_t)} \), then \( (t + s) \in I_{(u, f)} \) and one has
\[
S(f_t, s)S(f, t)u = S(f, t + s)u.
\]
3. The mapping \((u, f, t) \to \pi(u, f, t)\) is continuous in \((u, f) \in H \times \mathcal{F}\) for \( t \) fixed, and continuous in \( t \) for \((u, f) \) fixed.
4. If \((u, f) \in H \times \mathcal{F}\) and \( \tau(u, f) < \infty \), then one has
\[
\limsup_{t \to \tau^-} \|S(f, t)u\|_H = \infty.
\]
It is a consequence of (2) that one has
\[ \pi(u, f, t + s) = \pi(\pi(u, f, t), s). \]

Let \( \mathcal{K} \) be a subset of \( H \times \mathcal{T} \) and assume that \( t \) satisfies \( 0 \leq t < \tau(u, f) \) for all \( (u, f) \in \mathcal{K} \). For this \( t \) we define \( \pi(\mathcal{K}, t) \) to be the collection of all \( \pi(u, f, t) \) with \( (u, f) \in \mathcal{K} \). A subset \( \mathcal{K} \) in \( H \times \mathcal{K} \) is said to be invariant for \( \pi \) if one has \( \tau(u, f) = \infty \) for all \( (u, f) \in \mathcal{K} \) and \( \pi(\mathcal{K}, t) = \mathcal{K} \) for all \( t \geq 0 \). If \( \mathcal{K} \) is any subset of \( H \times \mathcal{T} \) with \( \tau(u, f) = \infty \) for all \( (u, f) \in \mathcal{K} \), we define the omega limit set of \( \mathcal{K} \) as \( \omega(\mathcal{K}) \) where
\[ \omega(\mathcal{K}) = \bigcap_{\tau \geq 0} \text{ Closure}_{H \times \mathcal{T}} \left( \bigcup_{t \geq \tau} \pi(\mathcal{K}, t) \right). \]

In the case of the dilated Navier-Stokes evolutionary equation on \( Q_3 \) with periodic boundary conditions, we set
\[ (2.53) \quad S(F, t)U_0 = U(t), \quad 0 \leq t < \tau(U_0, F), \]
where \( U(t) \) is the maximally defined strong solution on \( [0, \tau(U_0, F)) \) that satisfies \( U(0) = U_0, U_0 \in V^1_\varepsilon \) and \( F \in W(Q_3) \). One can show that
\[ 0 \overset{\text{def}}{=} \{ (U, F, t) \in V^1_\varepsilon \times W(Q_3) \times [0, \infty) : 0 \leq t < \tau(U, F) \} \]
is an open set, and the mapping \( \pi \) defined by (2.52) and (2.53) generates a skew-product semiflow on \( V^1_\varepsilon \times W(Q_3) \).

The same construction generates a skew-product semiflow for the Navier-Stokes equations on any reasonable bounded domain \( \Omega \) in \( \mathbb{R}^n \), for \( n = 2, 3 \), and under other homogeneous boundary conditions, see Constantin and Foias (1988). In this case, the semiflow is on the space \( V^1 \times W(\Omega) \), where \( V^1 = D(A^{1/2}) \). For the 2DNS, one has \( \tau(u, f) = \infty \), for all \( (u, f) \in V^1 \times W(\Omega) \), i.e., \( \pi \) is a global semiflow in this case. Furthermore by using the Leray solutions of the 2DNS instead of the strong solutions, this global semiflow extends to a global semiflow on \( H \times W(\Omega) \), where \( H = P_2 L^2(\Omega) \), see Constantin and Foias (1988).

For the Navier-Stokes equations we will be studying the semiflow generated by (2.53) on \( V^1_\varepsilon \times \mathcal{T} \), where \( \mathcal{T} \) is a compact, invariant set in \( W(Q_3) \). For example, with equation (2.5) one might assume either \( \mathcal{T} = H^+(f) \) to be compact, or \( \mathcal{T} = H^+(P_0 f) \) to be compact. Either assumption leads to a reasonable dynamical theory for (2.5). The stronger condition that \( H^+(f) \) be compact is important primarily in the study of the original system (2.4) where \( (I - P_0)f \) is used. Similarly for the reduced 3D Navier-Stokes evolutionary system (2.24), one gets a good dynamical theory by assuming that any one of the following three sets to be compact: \( H^+(f), H^+(P_0 f), \) or \( H^+(MP_0 f) \).

If the forcing term \( f \) is time-independent and in \( L^2(\Omega) \), then the hull \( H(f) \) consists of a single point \( \{ f \} \), and the Navier-Stokes equations generate (local) semi flows on appropriate Hilbert spaces. For the 3DNS equations the strong solutions \( S(t)u_0 \) generate a semi flow on the Hilbert space \( H^1(\Omega) \cap P_3(\Omega) \). The weak solution of the 2DNS generate a semi flow on \( P_2(L^2(\Omega)) \).
2.12 Local and Global Attractors. We will continue to use the notation introduced in Section 2.11. Let \( \pi \) be the skew-product semiflow on \( H \times F \) given by (2.52), where \( F \) is a positively invariant subset of \( W(Q_3) \). A subset \( A \) in \( H \times F \) is said to be a (local) attractor for \( \pi \) if \( A \) is compact, invariant and one has \( A = \omega(U) \), where \( U \) is some open neighborhood of \( A \) in \( H \times F \). The basin of attraction \( B(A) \) is defined to be the collection of all \((u,f) \in H \times F\) with the property that

\[
\text{dist}_{H \times F}(\pi(u,f,t), A) \rightarrow 0, \quad \text{as } t \rightarrow \infty.
\]

If it happens that \( A \) is an attractor with \( B(A) = H \times F \), then \( A \) is said to be a global attractor for \( \pi \).

As is well-known, see Hale (1988) for example, a compact, invariant set \( A \) is a local attractor for \( \pi \) if and only if the following two conditions hold: (1) \( A \) is an attracting set for \( \pi \), and (2) \( A \) is Lyapunov-stable. Recall that \( A \) is an attracting set for \( \pi \) whenever there is an open neighborhood \( U \) of \( A \) with the property that (2.54) is valid for all \((u,f) \in U\). Also \( A \) is Lyapunov stable if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that whenever one has

\[
\text{dist}_{H \times F}((u,f), A) < \delta,
\]

then it follows that

\[
\text{dist}_{H \times F}(\pi(u,f,t), A) < \epsilon, \quad \text{for all } t \geq 0.
\]

We note that the flow generated by the strong solutions of the 2DNS always has a global attractor, provided the positive hull \( H^+(f) \) of the forcing function \( f \) is compact. As we shall see below, the reduced 3DNS also has a global attractor, when \( H^+(f) \) is compact. The theorems, which we describe in the next section, effectively state that when \( \epsilon \) is small the full 3DNS has a local attractor \( A_\epsilon \), and that \( A_\epsilon \) has a large basin of attraction.

2.13 Statement of Theorems about Attractors. In this section we assume that \( f \in W(Q_3) \) is chosen so that \( H^+(f) \) is compact, see Section 2.11. This includes the case where \( f \in L^2(Q_3) \) is time-independent. We assume that Hypothesis \( H(a,b) \) is satisfied, where \( a \) and \( b \) are sufficiently large. Let \( \mathcal{B}_0^0, \mathcal{B}_1^1, \) and \( \mathcal{B}_2^2 \) be given by (2.38), (2.39), and (2.40). It is an immediate consequence of Theorems 1 and 2 that for \( u_0 \in \mathcal{B}_0^0 \cup \mathcal{B}_1^1 \) the solution \( S_\epsilon(f,t)u_0 \) lies in a bounded set in \( V_\epsilon^2 \) for \( t \geq \hat{T}_1 \). Therefore \( S_\epsilon(f,t)u_0 \) lies in a compact set in \( V_\epsilon^1 \) for \( t \geq \hat{T}_1 \). As a matter of fact, we are able to prove the following compactness result:

**Theorem 3.** Let \( \eta_i, i = 1, 2, 3, 4, r \) and \( p \) satisfy Hypothesis \( H(a,b) \), where \( a \) and \( b \) are sufficiently large. Assume that \( f \in W(Q_3) \) is chosen so that \( P_\epsilon f \in W^{1,\infty}((0,\infty), H_\epsilon) \), \( H^+(f) \) is compact and that (2.29) is satisfied. Let \( S_\epsilon(f,t)u_0 \) denote the strong solution of
(2.5) with initial data \( u_0 \in V^1_\epsilon \). Then for any \( \tau > 0 \) there is a compact subset \( \mathcal{K}(\tau) \) of \( V^2_\epsilon \) such that
\[
S_\epsilon(f, t)(B^0_\epsilon \cup B^1_\epsilon) \subset \mathcal{K}(\tau), \quad t \geq \tau.
\]

The proof of Theorem 3 is given in Section 4. If we do not assume \( H^+(f) \) to be compact in Theorem 3, then we can only prove that, for \( t > 0 \), \( S_\epsilon(t)(B^0_\epsilon \cup B^1_\epsilon) \) belongs to a compact set \( \mathcal{K}(t) \) which may depend on \( t \).

The following theorems are proved in Section 6. Let \( u(t) = S_\epsilon(P_\epsilon f, t)u_0 \) denote the strong solution of the equation (2.5) with initial data \( u_0 \) in \( V^1_\epsilon \) and let \( \pi_\epsilon(u_0, P_\epsilon f, \tau) = (S_\epsilon(P_\epsilon f, \tau)u_0, (P_\epsilon f)\tau) \) denote the skew-product flow generated by the strong solutions of the dilated 3D Navier-Stokes evolutionary equation (2.5).

**Theorem 4.** Let \( \eta_i \), \( i = 1, 2, 3, 4, r \) and \( p \) satisfy Hypothesis \( H(a,b) \), where \( a \) and \( b \) are sufficiently large. Assume that \( f \in W(Q_3) \) is chosen so that \( P_\epsilon f \in W^{1, \infty}((0, \infty), H_\epsilon) \), \( H^+(f) \) is compact, and one has
\[
\|MP_\epsilon f\|_\infty^2 \leq \eta_2^{-2}, \quad \|(I - M)P_\epsilon f\|_\infty^2 \leq \epsilon^r \eta_4^{-2}.
\]
Let \( \epsilon_0 > 0 \) be given by Theorem 1. Then, for \( 0 < \epsilon \leq \epsilon_0 \), the skew-product flow \( \pi_\epsilon \) generated by the strong solutions of the dilated 3D Navier-Stokes evolutionary equation (2.5) has a unique, maximal compact (local) attractor \( \mathfrak{A}_\epsilon \) included in \( B^2_\epsilon \times \omega(P_\epsilon f) \), which attracts \( B^2_\epsilon \times H^+(P_\epsilon f) \) in the space \( V^1_\epsilon \times P_\epsilon W(Q_3) \). Furthermore, the basin of attraction \( B(\mathfrak{A}_\epsilon) \) contains the set \( B^2_\epsilon \times H^+(P_\epsilon f) \), where \( B^2_\epsilon \) is given by (2.40), and
\[
\mathfrak{A}_\epsilon \subset \{ u = v + w : \|A^{1/2}_\epsilon v\|^2 \leq \Gamma(\eta_2^{-2}), \|A^{1/2}_\epsilon w\|^2 \leq k^2_2 \epsilon^{2 + r} \eta_4^{-2} \} \times \omega(P_\epsilon f).
\]
Moreover, \( \mathfrak{A}_\epsilon \) is bounded and compact in \( V^2_\epsilon \times \omega(P_\epsilon f) \) and attracts the bounded set \( (B^2_\epsilon \cap V^2_\epsilon) \times H^+(P_\epsilon f) \) in the space \( V^2_\epsilon \times P_\epsilon W(Q_3) \).

In the next theorem we show that, under an added condition on \( \eta_i \), \( i = 1, 2, 3, 4, \) see (2.55) below, the attractor \( \mathfrak{A}_\epsilon \) is the global attractor for the Leray solutions of the dilated 3D Navier-Stokes evolutionary equation (2.5), i.e., the weak solutions that satisfy the energy inequality (3.35), see Foias and Témam (1987).\(^4\) Note that the example given in Remark 2 prior to the statement of Theorem 1 satisfies (2.55).

**Theorem 5.** Let the hypotheses of Theorem 4 be satisfied. Assume in addition that for every \( \lambda > 0 \) there is an \( \epsilon_{10} = \epsilon_{10}(\lambda) > 0 \) such that
\[
\eta_2^{-2} + \epsilon^{2 + r} \eta_4^{-2} \leq \lambda \min(\eta_1^{-2}, \epsilon^p \eta_3^{-2}), \quad 0 < \epsilon \leq \min(\epsilon_0, \epsilon_{10}).
\]
\(^4\)Recall that the 3DNS can have weak solutions that do not satisfy the energy inequality (3.35), see Témam (1983).
Then for every Leray solution \( u(t) \) of the dilated 3D Navier-Stokes evolutionary equation, there is a \( t_0, 0 \leq t_0 < \infty \), such that

\[
u(t_0) \in \mathcal{B}_\epsilon^0, \quad 0 < \epsilon \leq \min(\epsilon_0, \epsilon_{10}),\]

where \( \epsilon_{10} = \epsilon_{10}(\lambda) \) and \( \lambda^{-1} > 2\nu^{-2}\max(\lambda_1^{-1}, C_\delta^2) \). In particular, \( u(t) \) is a regular solution of (2.5) for \( t \geq t_0 \), and the attractor \( \mathfrak{A}_\epsilon \) given in Theorem 4 is the global attractor for the Leray solutions of (2.5), provided \( 0 < \epsilon \leq \min(\epsilon_0, \epsilon_{10}) \).

Let us now consider the reduced 3D Navier-Stokes evolutionary equation (2.24), and let us denote by \( S_0(g, t)\overline{v}_0 \) the strong solution of (2.24) with initial data \( \overline{v}_0 \) in \( MV^1 \), where \( g = MP_{\epsilon}f \). We denote by \( \pi_0(\overline{v}_0, g, \tau) = (S_0(g, \tau)\overline{v}_0, g, \tau) \) the skew-product semiflow generated by the strong solutions of (2.24). As noted in Section 2.6, the terms in (2.24) do not depend on \( x_3 \) and \( \epsilon \). We have the following result:

**Theorem 6.** Assume that \( f \in W(Q_3) \) is chosen so that \( P_{\epsilon}f \in W^{1,\infty}((0, \infty); H_\epsilon) \) and \( H^+(f) \) is compact. Then \( \pi_0 \) admits a global attractor \( \mathfrak{A}_0 = \mathfrak{A}_0(g) \) in \( MV^1 \times H^+(g) \). Furthermore, if \( \|MP_{\epsilon}f\|_\infty \leq \eta^{-2}_2 \), where \( \eta_2 \) is given by Hypothesis \( H(a, b) \), then

\[
(2.56) \quad \mathfrak{A}_0(g) \subset \{ u = v + w : v \in MV^1, \|A^\frac{1}{2}_\tau v\|^2 \leq \Gamma(\eta^{-2}_2), w = 0 \} \times \omega(g).
\]

If in addition, one has

\[
(I - M)P_{\epsilon}f = 0,
\]

then the attractors \( \mathfrak{A}_\epsilon \) and \( \mathfrak{A}_0 \) coincide for \( 0 < \epsilon \leq \epsilon_0 \), where \( \epsilon_0 \) is given by Theorem 1.

If \( (I - M)P_{\epsilon}f \neq 0 \), a comparison of the two attractors \( \mathfrak{A}_\epsilon \) and \( \mathfrak{A}_0(g) \) is more difficult. Nevertheless, we are able to derive some important results establishing the upper semiconitnuity of \( \mathfrak{A}_\epsilon \) at \( \epsilon = 0 \).

Let us consider a sequence of positive numbers \( \epsilon_n \to 0 \) when \( n \to \infty \). We introduce a sequence of functions \( f_n \) in \( W(Q_3) \cap W^{1,\infty}((0, \infty), L^2(Q_3)) \) such that \( f_n \to f_0 \) in \( W(Q_3) \), where \( f_0 \in MW(Q_3) \cap W^{1,\infty}((0, \infty); ML^2(Q_3)) \). We set \( g_n = P_{\epsilon_n}f_n \) and \( g_0 = MP_{\epsilon_0}f_0 \). According to the comments made in Section 2.6, \( P_{\epsilon_n}f_0(t) \) belongs to \( MH_{\epsilon_n} \), for every \( t \), and consequently

\[
(2.57) \quad g_0 = P_{\epsilon_n}f_0 = \left( P_2 \left( \begin{array}{c} f_{01} \\ f_{02} \\ f_{03} \end{array} \right) \right)
\]

where \( f_0 = (f_{01}, f_{02}, f_{03}) \). It follows from the above convergence hypothesis and from (2.57) that

\[
\lim_{n \to \infty} \|g_n - g_0\|_\infty = 0.
\]
We consider next the reduced 3D Navier-Stokes evolutionary equation

\[(2.58) \quad \ddot{u}' + \nu A_0 \ddot{u} + B_0(\ddot{u}, \dot{u}) = g_0\]

with initial data $\ddot{u}(0) = \ddot{u}_0$ in $V^1_0$, and we let $S_0(g_0, t)\ddot{u}_0$ denote the strong solutions of (2.58) with initial data $\ddot{u}_0$ in $V^1_0$. As a consequence of Theorem 6, the skew-product semiflow $\pi_0(\ddot{u}_0, g_0, \tau) = (S_0(g_0, \tau)\ddot{u}_0, g_0, \tau)$ admits a global compact attractor $\mathcal{A}_0 = \mathcal{A}_0(g_0)$ in $V^1_0 \times \omega(g_0)$, which is also the global compact attractor in $V^2_0 \times \omega(g_0)$.

Let $E$ be a subset of $V^1_\epsilon \times W(Q_3)$. For any $\delta > 0$, we denote by $N_{V^1_\epsilon \times W(Q_3)}(E, \delta)$ the $\delta$-neighborhood of $E$ in $V^1_\epsilon \times W(Q_3)$. We will prove the following result:

**Theorem 7.** Let $\eta_i$, $i = 1, 2, 3, 4, r$ and $p$ satisfy Hypothesis $H(a,b)$, where $a$ and $b$ are sufficiently large, and assume that

\[(2.59) \quad \epsilon^{4+2r} \eta_4^{-4}(\epsilon) \to 0, \quad \text{as } \epsilon \to 0^+.

Let $\epsilon_n$ be a sequence of positive numbers with $\epsilon_n \to 0$ as $n \to \infty$. Let $\mathcal{F}$ be any positively invariant, compact subset of $W(Q_3) \cap W^{1,\infty}((0, \infty), L^2(Q_3))$, and let $f_n$ be a sequence of functions in $\mathcal{F}$ that satisfies

\[(2.60) \quad \lim_{n \to \infty} \|f_n - f_0\|_\infty = 0,

where $f_0 \in M\mathcal{F}$. Assume further that

\[\|MP_{\epsilon_n}f_n\|_2^2 \leq \eta_2^{-2}, \quad \|(I - M)P_{\epsilon_n}f_n\|_2^2 \leq \epsilon_n^r \eta_4^{-2}.

Then the attractors $\mathcal{A}_{\epsilon_n}$, given in Theorem 4, are upper semicontinuous in $V^1_\epsilon \times \mathcal{F}$ at $\epsilon = 0$, i.e., for any $\delta > 0$, there is an $n_0 > 0$ such that

$\mathcal{A}_{\epsilon_n} \subset N_{V^1_\epsilon \times W(Q_3)}(\mathcal{A}_0(g_0), \delta)$

for $n \geq n_0$, where $g_0 = MP_{\epsilon_n}f_0 = P_{\epsilon_n}f_0$.

Theorem 7 has some interesting extensions. The following result, which we formulate in the case where the forcing terms $f_n$ are independent of time $t$, allows for the possibility that $f_n$ can be chosen so that

\[\|(I - M)P_{\epsilon_n}f_n\|_2^2 \to \infty, \quad \text{as } n \to \infty.

If $f$ is independent of $t$, the mapping $S_\epsilon(t) \overset{\text{def}}{=} S_\epsilon(P_{\epsilon}f, t)$ is a (local) $C^0$-semigroup on $V^1_\epsilon$. We then deduce from Theorem 4 that $S_\epsilon(t)$ admits a unique, maximal compact (local) attractor $\mathcal{A}_\epsilon$ included in $\mathcal{B}_\epsilon^2$ which attracts $\mathcal{B}_\epsilon^2$ in the space $V^1_\epsilon$. Actually, we have

$\mathcal{A}_\epsilon = \mathcal{A}_\epsilon \times \{P_{\epsilon}f\}.$

Likewise, $S_0(t) \overset{\text{def}}{=} S_0(g, t)$ is a $C^0$-semigroup on $V^1_0 = MV^1_\epsilon$, where $g = MP_{\epsilon}f$. We deduce from Theorem 6 that $S_0(t)$ admits a global, compact attractor $\mathcal{A}_0 = \mathcal{A}_0(g)$ in $V^1_0$; and we have

$\mathcal{A}_0 = \mathcal{A}_0 \times \{g\}.$

The following result is proved in Section 6.
THEOREM 8. Let $\eta_i, i = 1, 2, 3, 4, r$ and $p$ satisfy Hypothesis $H(a, b)$, where $a$ and $b$ are sufficiently large, and assume that (2.59) holds. Let $\epsilon_n$ be a sequence of positive numbers with $\epsilon_n \to 0$ as $n \to \infty$. Let $f_n$ be a sequence in $L^2(Q_3)$ that satisfies
\[
\lim_{n \to \infty} \| M\epsilon_n f_n - g_0 \| = 0.
\]
for some $g_0 \in H_0$. Assume further that
\[
\| M\epsilon_n f_n \|^2 \leq \eta_2^{-2}, \quad \| (I - M)\epsilon_n f_n \|^2 \leq \epsilon_n \eta_4^{-2}.
\]
Then the attractors $\tilde{A}_{\epsilon_n}$ of (2.5) with forcing term $P_{\epsilon_n} f_n$ are upper semicontinuous at $\epsilon = 0$ in $V_{\epsilon_n}^1$, i.e.,
\[
(2.61) \quad \sup_{u_n \in \tilde{A}_{\epsilon_n}} \inf_{v \in \tilde{A}_0} \| A_{\epsilon_n}^\frac{1}{2}(u_n - v) \| \to 0, \quad \text{as } \epsilon = \epsilon_n \to 0,
\]
where $\tilde{A}_0 = \tilde{A}_0(g_0)$ is the global attractor of (2.58).

Using the fact that $\mathcal{A}_0 = \mathcal{A}_0(g_0)$ is also the global compact attractor of the skew-product semiflow $\pi_0(\cdot, g_0, \tau)$ in $V_0^2 \times H^+(g_0)$, one also obtains the result

THEOREM 9. Assume that the hypotheses of Theorem 7 hold and that
\[
(2.62) \quad \lim_{n \to \infty} \| f'_n - f'_0 \|_{\infty} = 0.
\]
Then the attractors $A_{\epsilon_n}$ are also upper semicontinuous at $\epsilon = 0$ in $V_{\epsilon}^2 \times W(Q_3)$, i.e., for any $\delta > 0$, there exists an integer $n_1 > 0$ such that, for $n \geq n_1$,
\[
(2.63) \quad A_{\epsilon_n} \subset \mathcal{N}_{V_{\epsilon_n}^2 \times W(Q_3)}(A_0(g_0), \delta).
\]

REMARK. We now give an example where the condition (2.60) is satisfied. Let $F = F(t, y)$ be given, where $F \in C^0(R, W^{1, \infty}(Q_3)) \cap L^\infty(R, W^{1, \infty}(Q_3))$. As in Section 2.1, we introduce the mapping $f_\epsilon$ by setting $F = J_\epsilon f_\epsilon$
\[
f_\epsilon(t, x_1, x_2, x_3) = F(t, x_1, x_2, \epsilon x_3), \quad (x_1, x_2, x_3) \in Q_3.
\]
Next set $f_0(t, x_1, x_2, x_3) = F(t, x_1, x_2, 0)$. By applying the integral Taylor formula, we obtain
\[
f_\epsilon(t, x) - f_0(t, x) = F(t, x_1, x_2, \epsilon x_3) - F(t, x_1, x_2, 0) = \epsilon \int_0^1 \frac{\partial F}{\partial x_3}(t, x_1, x_2, sx_3)x_3 ds,
\]
\[
28
\]
and therefore
\[ \|f_\varepsilon - f_0\|_{C^0([0,\infty);L^\infty(Q_3))} \leq \varepsilon \|D_3 F\|_{C^0([0,\infty);L^\infty(Q_3))}. \]

In particular, we have
\[ \|f_\varepsilon - f_0\|_{\infty} \leq C\varepsilon. \]

If in addition, \( F \) belongs to \( C^1(R, W^{1,\infty}(Q_3)) \cap L^\infty(R, W^{1,\infty}(Q_3)) \), then the condition (2.62) is also satisfied; more precisely, we have
\[ \|f'_\varepsilon - f'_0\|_{\infty} \leq \tilde{C}\varepsilon. \]

### 3. \( H^1 \)-Regularity: Theorem 1

In this section we shall prove Theorem 1, the \( H^1 \)-Regularity Theorem, which gives the global regularity of the Navier-Stokes equations on \( \Omega_\varepsilon \) in the Sobolev space \( H^1 \). We assume that the forcing function \( f \) is in \( L^\infty \left((0,\infty), L^2(Q_3)\right) \) and that the initial condition \( u_0 \) satisfies \( u_0 \in D(A_\varepsilon^{\frac{1}{2}}) \). Also we assume that one has

\[
\begin{cases}
\|A_\varepsilon^{\frac{1}{2}} v_0\|^2 \leq \eta_1^{-2}, \\
\|MP_\varepsilon f\|_\infty^2 \leq \eta_2^{-2}, \\
\|A_\varepsilon^{\frac{1}{2}} w_0\|^2 \leq \varepsilon^p \eta_3^{-2}, \\
\|(I - M)P_\varepsilon f\|_\infty^2 \leq \varepsilon^p \eta_4^{-2},
\end{cases}
\]

where \( \eta_i(\varepsilon) \) is bounded and monotone for \( 0 < \varepsilon \leq 1, i = 1, 2, 3, 4 \). (We are primarily interested in the case where \( \eta_i(\varepsilon) \to 0 \) as \( \varepsilon \to 0, i = 1, 2, 3, 4 \).) Throughout this section we shall let \( D_1, D_2, \ldots \) denote positive functions of the viscosity \( \nu \) and \( \lambda_1 \), the first eigenvalue of \( A_\varepsilon \). These functions will not depend on \( \varepsilon \) for \( 0 < \varepsilon \leq 1 \).

The proof of the \( H^1 \)-Regularity Theorem is done in two steps. In the first lemma, which is the Short Time Argument, we show that the \( u(t) \)-term becomes small very rapidly. The second lemma is referred to as the Long Time Argument. This is the induction step needed for the proof of the \( H^1 \)-Regularity Theorem. We show that if \( u \) is initially small and \( v \) is of reasonable size, then the strong solution of (2.5) exists on an interval \([0,2T_0]\), where \( T_0 = T_0(\varepsilon) \) is finite but large. We also show that \( v(t) \) remains small on this interval and that the size of \( v(t) \) is controllable on the half-interval \([T_0,2T_0]\). This permits one to set up the induction argument for the main theorem.

We will use the following auxiliary estimates concerning the trilinear form \( b_\varepsilon \): If \( v^1, v^2, v^3 \in R(M) \), then these functions depend only on \( x_1 \) and \( x_2 \), and one has

\[
|b_\varepsilon(v^1, v^2, v^3)| \leq C_1 \|v^1\|^\frac{1}{2} \|A_\varepsilon^{\frac{1}{2}} v^1\|^\frac{1}{2} \|A_\varepsilon^{\frac{1}{2}} v^2\|^\frac{3}{2} \|A_\varepsilon v^3\|^\frac{1}{2} \|v^3\|.
\]

The proof of (3.2) is accomplished by using 2D Sobolev imbeddings, see Téman (1983) or Constantin and Foias (1988) for details. If one has \( v \in R(M) \) and \( Mw^1 = Mw^2 = Mw = 0 \),
then the following hold

\[
\begin{align*}
&|\vec{b}_\varepsilon(w^1, w^2, u)| \leq C_2 \varepsilon^{\frac{1}{2}} \|A_\varepsilon^{\frac{1}{2}} w^1\| \|A_\varepsilon^{\frac{1}{2}} w^2\| \frac{1}{2} \|A_\varepsilon w^2\| \frac{1}{2} \|u\| \\
&|\vec{b}_\varepsilon(w, u^2, u^3)| \leq C_3 \varepsilon^{\frac{3}{2}} \|A_\varepsilon^{\frac{1}{2}} w\| \|A_\varepsilon^{\frac{3}{2}} w\| \frac{1}{2} \|A_\varepsilon w\| \|u^2\| \|u^3\| \\
&|\vec{b}_\varepsilon(v, w, u)| \leq C_4 \varepsilon^{\frac{3}{4}} \|A_\varepsilon^{\frac{1}{2}} v\| \|A_\varepsilon^{\frac{3}{2}} w\| \frac{1}{2} \|A_\varepsilon w\| \|u\|.
\end{align*}
\]

(3.3)

The proof of (3.3) is given in the Appendix, Section 8. It is important to note that the constants $C_1$, $C_2$, $C_3$, and $C_4$ appearing above do not depend on $\varepsilon$, for $0 < \varepsilon \leq 1$.

We shall also use below the Young inequality:

\[
ab \leq \frac{c p a^p}{p} + \frac{b^q}{q c^q} = \delta a^p + c_\delta b^q
\]

(3.4)

where $a, b, c, \delta$, and $c_\delta$ are positive, $1 \leq p, q, p^{-1} + q^{-1} = 1$, as well as

\[
(a + b)^3 \leq 4(a^3 + b^3), \quad a, b \geq 0.
\]

(3.5)

The proof of the following result can be easily derived from the theory presented in Constantin and Foias (1988), Témam (1977, Chapter III, Lemma 1.2 and Theorem 3.11), Témam (1983, Section 3, Theorem 3.2), as well as the other references cited above.

**Lemma 3.0.** Let $u_0 \in D(A_\varepsilon^{\frac{1}{2}})$ and $f \in L^\infty(0, T; H_\varepsilon)$. Then there exists a time $T_*$, $0 < T_* \leq \infty$, such that there exists a unique solution $u$ of (2.5) on $(0, T_*)$. Moreover $u$ satisfies: $u \in C^0([0, T_*]; V_\varepsilon^1) \cap L^2(0, T_*; V_\varepsilon^2)$, and $u_t \in L^2(0, T_*; H_\varepsilon)$. Assume furthermore that $R_\varepsilon^3 \geq \|A_\varepsilon^{\frac{1}{2}} u_0\|^2$ and $N > 1$. Then there exists a positive time $T^N$, $0 < T^N \leq T_*$, such that $\|A_\varepsilon^{\frac{1}{2}} u(t)\|^2 \leq N R_\varepsilon^3$ for $0 \leq t < T^N$.

**3.1 The Short Time Argument.** We shall say that **Hypothesis H1** is satisfied if one has:

1. $p \geq -1, \ r > -2$.
2. $\varepsilon^{\frac{1}{2}} \eta_i^{-1} \rightarrow 0$ as $\varepsilon \rightarrow 0$, $i = 1, 2$.
3. $\varepsilon^{\frac{1}{2}} \eta_i^{-1} \rightarrow 0$ as $\varepsilon \rightarrow 0$, $i = 3, 4$.
4. $\varepsilon^{\frac{1}{2}} Q(\varepsilon)$ is bounded for $0 < \varepsilon \leq 1$, where

\[
(3.6) \quad Q(\varepsilon) = \log(2 C^2_\delta \nu^{-2} \varepsilon^{2+r-p} \eta_4^{-2} \eta_3^2).
\]

Here $\eta_i(\varepsilon)$ denote bounded monotone functions defined for $0 < \varepsilon \leq 1$, $i = 1, 2, 3, 4$, and $r$ and $p$ are constants.

We now prove the following result.
Lemma 3.1. Assume that Hypothesis H1 is satisfied and that (3.1) is valid. Then there are positive constants \( k_1 \) and \( k_2 \) and an \( \epsilon_1 > 0 \) such that for all \( 0 < \epsilon \leq \epsilon_1 \) there exists a time \( T_1 = T_1(\epsilon) > 0 \) such that \( u(t) \in D(A_0^\frac{1}{2}) \) for \( 0 \leq t \leq T_1 \) and

\[
\begin{align*}
(3.7) \quad \|A_0^\frac{1}{2}v(T_1)\|^2 & \leq 4\eta_1^{-2} + k_1^2 \eta_3^{-4} \\
\|A_0^\frac{1}{2}w(T_1)\|^2 & \leq k_2^2 \epsilon^2 + \eta_4^{-2}.
\end{align*}
\]

Proof. Define \( R_0^2 \) by

\[
(3.8) \quad R_0^2 \text{ def } = \eta_1^{-2} + \epsilon^2 \eta_3^{-2} + \eta_3^{-4} + \eta_2^{-2} + \epsilon^5 \eta_4^{-2}.
\]

Since \( R_0^2 \geq \|A_0^\frac{1}{2}u_0\|^2 \), it follows from Lemma 3.0 that for any \( N > 1 \), there is a time \( T^N > 0 \) such that

\[
(3.9) \quad \|A_0^\frac{1}{2}u(t)\|^2 \leq NR_0^2, \quad 0 \leq t < T^N.
\]

Without a loss of generality we let \([0, T^N]\) denote the maximal time interval for which \( (3.9) \) is valid. If \( T^N < \infty \), then one must have

\[
(3.10) \quad \|A_0^\frac{1}{2}u(T^N)\|^2 = NR_0^2.
\]

For the remainder of the proof of this lemma we restrict our attention to \( t \in (0, T^N) \).

The equation satisfied by \( w = (I - M)u \) in (2.23) is

\[
(3.11) \quad \frac{dw}{dt} + \nu A_\epsilon w = (I - M)P_\epsilon f - (I - M)(B_\epsilon(w, v) + B_\epsilon(v, w) + B_\epsilon(w, w))
\]

since \((I - M)B_\epsilon(v, v) = 0\). By taking the scalar product of (3.11) with \( A_\epsilon w \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|A_\epsilon^\frac{1}{2} w\|^2 + \nu \|A_\epsilon w\|^2 \leq \|((I - M)P_\epsilon f, A_\epsilon w)\| + |b_\epsilon(w, v, A_\epsilon w)| + |b_\epsilon(v, w, A_\epsilon w)| + |b_\epsilon(w, w, A_\epsilon w)|
\]

By using (3.3) and the Young inequality (3.4) we obtain

\[
(3.12) \quad \frac{1}{2} \frac{d}{dt} \|A_\epsilon^\frac{1}{2} w\|^2 + \nu \|A_\epsilon w\|^2 \leq \frac{\nu}{2} \|A_\epsilon w\|^2 + \frac{1}{2} \nu \|((I - M)P_\epsilon f\|^2 + C_3 \epsilon^\frac{5}{12} \|A_0^\frac{1}{2} v\| \|A_0^\frac{1}{2} w\| \|A_\epsilon w\| \|A_\epsilon w\| \frac{\nu}{2} + C_4 \epsilon^\frac{1}{12} \|A_0^\frac{1}{2} w\| \|A_0^\frac{1}{2} w\| \|A_\epsilon w\| \|A_\epsilon w\| \frac{\nu}{2} + C_2 \epsilon^\frac{1}{2} \|A_\epsilon w\| \|A_\epsilon w\| \|A_\epsilon w\| \frac{\nu}{2}.
\]
Since $Mw = 0$ one can use (2.22) together with (3.12) to find
\[
\frac{d}{dt} \|A^{\frac{1}{2}}v\|^2 + \nu \|A_{\epsilon}w\|^2 \leq \frac{1}{\nu} \|(I - M)P_{\epsilon}f\|^2_{\infty} + 2C_5^{\frac{3}{2}}C_3^{\frac{1}{2}}\|A^{\frac{1}{2}}v\|\|A_{\epsilon}w\|^2 \\
+ 2C_5^{\frac{1}{2}}C_4^{\frac{3}{2}}\|A^{\frac{1}{2}}v\|\|A_{\epsilon}w\|^2 + 2C_5^{\frac{1}{2}}C_2^{\frac{1}{2}}\|A^{\frac{1}{2}}w\|\|A_{\epsilon}w\|^2.
\]

From the Pythagorean relation (2.21) one obtains
\[
(3.13) \quad \frac{d}{dt} \|A^{\frac{1}{2}}w\|^2 + \left(\nu - D_1\epsilon^{\frac{3}{2}}\|A^{\frac{1}{2}}u\|\right) \|A_{\epsilon}w\|^2 \leq \frac{1}{\nu} \|(I - M)P_{\epsilon}f\|^2_{\infty},
\]
where $D_1 = 2(C_5^{\frac{3}{2}}C_3 + C_5^{\frac{1}{2}}C_4 + C_5^{\frac{1}{2}}C_2)$. From Hypothesis H1 we see that for $0 \leq t < T^N$ one has
\[
(3.14) \quad D_1\epsilon^{\frac{3}{2}}\|A^{\frac{1}{2}}u\| \leq D_1\epsilon^{\frac{3}{2}}N^{\frac{1}{2}}R_0 \leq D_1N^{\frac{1}{2}}\epsilon^{\frac{3}{2}}(\eta_1^{-1} + \epsilon^{\frac{3}{2}}\eta_3^{-1} + \eta_3^{-2} + \eta_2^{-1} + \epsilon^{\frac{3}{2}}\eta_4^{-1})
\]
which goes to 0, as $\epsilon \to 0^+$. Consequently there is a positive number $\epsilon_2 = \epsilon_2(N)$ that satisfies
\[
(3.15) \quad D_1N^{\frac{1}{2}}\epsilon^{\frac{3}{2}}R_0 \leq \frac{\nu}{2}, \quad 0 \leq \epsilon \leq \epsilon_2.
\]
(Later $N$ will be fixed, and it will depend only on $\nu$ and $\lambda_1$.) For $0 < \epsilon \leq \epsilon_2$ and $0 \leq t < T^N$ it follows from (3.13), (3.14), and (3.15) that
\[
(3.16) \quad \frac{d}{dt} \|A^{\frac{1}{2}}w\|^2 + \frac{\nu}{2} \|A_{\epsilon}w\|^2 \leq \frac{1}{\nu} \|(I - M)P_{\epsilon}f\|^2_{\infty},
\]
and from (2.22)
\[
(3.17) \quad \frac{d}{dt} \|A^{\frac{1}{2}}w\|^2 + \nu C_5^{-2}\epsilon^{-2}\frac{\nu C_5^{-2}\epsilon^{-2}}{2} \|A^{\frac{1}{2}}w\|^2 \leq \frac{1}{\nu} \|(I - M)P_{\epsilon}f\|^2_{\infty}.
\]

We then apply the Gronwall inequality to (3.17) to obtain
\[
(3.18) \quad \|A^{\frac{1}{2}}w(t)\|^2 \leq \exp\left(-\nu C_5^{-2}\epsilon^{-2}\frac{\nu C_5^{-2}\epsilon^{-2}}{2} t\right) \|A^{\frac{1}{2}}w_0\|^2 + \frac{2C_5^{2}\epsilon^{2}}{\nu^2} \|(I - M)P_{\epsilon}f\|^2_{\infty},
\]
for $0 \leq t < T^N$ and $0 < \epsilon \leq \epsilon_2$. By integrating (3.16) we also obtain
\[
(3.19) \quad \int_0^t \|A_{\epsilon}w(s)\|^2 ds \leq \frac{2t}{\nu^2} \|(I - M)P_{\epsilon}f\|^2_{\infty} + \frac{2}{\nu} \|A^{\frac{1}{2}}w_0\|^2
\]
for $0 \leq t < T^N$ and $0 < \epsilon \leq \epsilon_2$. 

32
For the remainder of the proof we shall restrict our attention to $0 < \epsilon \leq \epsilon_2$. We will need an estimate of $\int_0^t \| A_\epsilon^\frac{1}{2} w \|^3 \| A_\epsilon w \| ds$. From (3.5) and (3.18) we obtain

$$\int_0^t \| A_\epsilon^\frac{1}{2} w \|^6 ds \leq 4 \int_0^t \left[ \exp \left( \frac{-3\nu C_5^2 \epsilon^2}{2} s \right) \| A_\epsilon^\frac{1}{2} w_0 \|^6 + \frac{8C_5^6 \epsilon^6}{\nu^6} \|(I - M)P_{\epsilon f}\|_\infty \right] ds \leq 4 \left( \frac{2C_5^2 \epsilon^2}{3\nu} \| A_\epsilon^\frac{1}{2} w_0 \|^6 + \frac{8C_5^6 \epsilon^6}{\nu^6} \|(I - M)P_{\epsilon f}\|_\infty \right).$$  

(3.20)

By using the Schwarz inequality with (3.19) and (3.20) we next obtain

$$\int_0^t \| A_\epsilon^\frac{1}{2} w \|^3 \| A_\epsilon w \| ds \leq \left( \int_0^t \| A_\epsilon^\frac{1}{2} w \|^6 ds \right)^\frac{1}{2} \left( \int_0^t \| A_\epsilon w \|^2 ds \right)^\frac{1}{2} \leq 4C_5\nu^{-1} \epsilon \left( \frac{1}{\sqrt{3}} \| A_\epsilon^\frac{1}{2} w_0 \|^3 + \frac{2C_5^2 \epsilon^2 t}{\nu} \|(I - M)P_{\epsilon f}\|_\infty \right) \times \left( \| A_\epsilon^\frac{1}{2} w_0 \| + \frac{t^{\frac{1}{2}}}{\nu^\frac{1}{2}} \|(I - M)P_{\epsilon f}\|_\infty \right).$$  

(3.21)

Let us return to inequality (3.18). Note that there is time $T_1 = T_1(\epsilon) > 0$ such that

$$\epsilon^2 \eta_3^{-2} \exp \left( \frac{-\nu C_5^{-2} \epsilon^2}{2} T_1 \right) = \frac{2C_5^2 \epsilon^2}{\nu^2} \epsilon^\nu \eta_4^{-2}.$$  

Indeed this time $T_1$ is given by

$$T_1 \overset{\text{def}}{=} 2C_5^2 \epsilon^2 \nu^{-1} Q(\epsilon),$$  

(3.22)

where $Q(\epsilon)$ is given by (3.6). It follows from (3.18) that if $T_1 < T_1^N$, then one has

$$\| A_\epsilon^\frac{1}{2} w(t) \|^2 \leq k_2^2 \epsilon^2 + r \eta_4^{-2}, \quad T_1 \leq t < T_1^N,$$

(3.23)

where $k_2^2 = 4C_5^2 \nu^{-2}$.

The next step is to return to (2.23) and the equation satisfied by $v = Mu$:

$$\frac{dv}{dt} + \nu A_\epsilon v = MP_{\epsilon f} - MB_\epsilon(v, v) - MB_\epsilon(v, w) - MB_\epsilon(w, v) - MB_\epsilon(w, w).$$  

(3.24)

By taking the scalar product of (3.24) with $A_\epsilon v$ we obtain

$$\frac{1}{2} \frac{d}{dt} \| A_\epsilon^\frac{1}{2} v \|^2 + \nu \| A_\epsilon v \|^2 \leq \| MP_{\epsilon f} A_\epsilon v \| - b_\epsilon(v, v, A_\epsilon v) - b_\epsilon(w, w, A_\epsilon v)$$

(3.25)
since \( b_\epsilon(v, w, A_\epsilon v) = b_\epsilon(w, v, A_\epsilon v) = 0 \) from (2.26). By using the Young inequality (3.4) with (3.2) and (3.3) we obtain

\[
\frac{1}{2} \frac{d}{dt} \| A_\epsilon^{\frac{1}{2}} v \|^2 + \nu \| A_\epsilon v \|^2 \leq \frac{\nu}{2} \| A_\epsilon v \|^2 + \frac{1}{2\nu} \| MP_\epsilon f \|_\infty^2 + C_1 \| v \|^{\frac{3}{2}} \| A_\epsilon^{\frac{1}{2}} v \| \| A_\epsilon v \|^{\frac{3}{2}} \\
+ C_2 \epsilon^{\frac{3}{2}} \| A_\epsilon^{\frac{1}{2}} w \|^{\frac{3}{2}} \| A_\epsilon w \|^{\frac{3}{2}} \| A_\epsilon v \|,
\]

which implies that

\[
\frac{d}{dt} \| A_\epsilon^{\frac{1}{2}} v \|^2 + \nu \| A_\epsilon v \|^2 \leq \frac{1}{\nu} \| MP_\epsilon f \|_\infty^2 + \frac{\nu}{2} \| A_\epsilon v \|^2 + \frac{27}{2\nu^3} C_1^4 \| v \|^{2} \| A_\epsilon^{\frac{3}{2}} v \|^{4} \\
+ \frac{\nu}{2} \| A_\epsilon v \|^2 + \frac{2 C_2^2 \epsilon}{\nu} \| A_\epsilon^{\frac{1}{2}} w \|^{3} \| A_\epsilon w \|.
\]

Consequently

\[
\frac{d}{dt} \| A_\epsilon^{\frac{1}{2}} v \|^2 \leq \left( \frac{27}{2\nu^3} C_1^4 \| v \|^2 \| A_\epsilon^{\frac{1}{2}} v \|^2 \right) \| A_\epsilon^{\frac{1}{2}} v \|^2 + \frac{1}{\nu} \| MP_\epsilon f \|_\infty^2 + \frac{2 C_2^2 \epsilon}{\nu} \| A_\epsilon^{\frac{1}{2}} w \|^{3} \| A_\epsilon w \|.
\]

By using the Gronwall inequality with (3.28), one finds that

\[
\| A_\epsilon^{\frac{1}{2}} v(t) \|^2 \leq e^{G(t)} \left( \| A_\epsilon^{\frac{1}{2}} v_0 \|^2 + H(t) \right), \quad t \geq 0,
\]

where

\[
H(t) = \int_0^t h(s) ds, \quad G(t) = \int_0^t g(s) ds
\]

\[
h(t) = \frac{1}{\nu} \| MP_\epsilon f \|_\infty^2 + \frac{2 C_2^2 \epsilon}{\nu} \| A_\epsilon^{\frac{1}{2}} w \|^{3} \| A_\epsilon w \|
\]

\[
g(t) = \frac{27}{2\nu^3} C_1^4 \| v \|^2 \| A_\epsilon^{\frac{1}{2}} v \|^2.
\]

Restricting to \( t \leq \min(T_1, T^N) \) and using (3.21) we see that

\[
\frac{2 C_2^2 \epsilon}{\nu} \int_0^t \| A_\epsilon^{\frac{1}{2}} w \|^{3} \| A_\epsilon w \| ds \leq D_2 \epsilon^2 \left( \| A_\epsilon^{\frac{1}{2}} w_0 \| + t^{\frac{1}{4}} \| (I - M)P_\epsilon f \|_\infty \right) \\
\times \left( \| A_\epsilon^{\frac{1}{2}} w_0 \|^{3} + \epsilon^2 t^{\frac{1}{4}} \| (I - M)P_\epsilon f \|_\infty^3 \right)
\]

where \( D_2 = 8 C_2^2 C_5 \nu^{-2} \max(3^{-\frac{1}{4}}, 2 C_3^2 \nu^{-\frac{3}{4}}) \max(1, \nu^{-\frac{1}{4}}) \). By using (3.1) and (3.33) one obtains

\[
H(t) \leq T_1 \nu^{-\frac{1}{2}} \eta_2^{-2} + D_2 \epsilon^2 \left( \epsilon^{\frac{3}{2}} \eta_3^{-3} + T_1^{\frac{1}{2}} \epsilon^2 + \eta_4^{-3} \right) \left( \epsilon^{\frac{3}{2}} \eta_3^{-1} + T_1^{\frac{1}{2}} \epsilon^{\frac{3}{2}} \eta_4^{-1} \right)
\]

34


for $0 \leq t \leq \min(T_1, T^N)$. Consequently from (3.22), the fact that $p \geq -1$, $r > -2$ and the Young inequality (3.4) one deduces that

$$H(t) \leq E_1(\epsilon) + \frac{7}{4} D_2 \eta_3^{-4}, \quad 0 \leq t \leq \min(T_1, T^N),$$

where

$$E_1(\epsilon) \overset{\text{def}}{=} D_3 \left( \epsilon^2 Q \eta_2^{-2} + \epsilon^2 Q \eta_4^{-4} + \epsilon^2 Q \eta_4^{-4} + \epsilon^\frac{3}{2} Q \eta_3^{-1} \eta_4^{-3} \right)$$

and

$$D_3 \overset{\text{def}}{=} \max \left( 2C_5^2 \nu^{-2}, 2D_2 C_5^2 \nu^{-1}, \sqrt{2}D_2 C_5 \nu^{-\frac{1}{2}}, D_2 C_5^4 \nu^{-2} \right).$$

By using Hypothesis H1 we see that

$$E_1(\epsilon) \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0^+.$$

It follows from (3.1) and (3.29) that

$$\|A^{\frac{1}{2}}\nu(t)\|^2 \leq e^{G(t)} \left( \eta_1^{-2} + E_1(\epsilon) + \frac{7}{4} D_2 \eta_3^{-4} \right), \quad 0 \leq t \leq \min(T_1, T^N).$$

The next objective is to show that $G(t)$ is small. By taking the scalar product of (2.5) with $u$ and using the fact that $b(\nu, u, u) = 0$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|A^{\frac{1}{2}} u\|^2 \leq \|Pf\| \|u\| = \|A^{\frac{1}{2}} Pf, A^{\frac{1}{2}} u\|

\leq \frac{1}{2} \nu \|A^{\frac{1}{2}} u\|^2 + \frac{1}{2\nu} \|A^{-\frac{1}{2}} Pf\|^2,$$

which implies that

$$\frac{d}{dt} \|u\|^2 + \nu \|A^{\frac{1}{2}} u\|^2 \leq \frac{1}{\nu} \|A^{-\frac{1}{2}} Pf\|^2 \leq \frac{1}{\nu} \|A^{-\frac{1}{2}} [M + (I - M)] Pf\|^2

\leq \frac{2}{\nu} (\|A^{\frac{1}{2}} M Pf\|^2 + \|A^{-\frac{1}{2}} (I - M) Pf\|^2).$$

By using (2.6), (2.22), and the Gronwall inequality one finds

$$\|v(t)\|^2 \leq \|u(t)\|^2 \leq \|u_0\|^2 + 2\lambda_1^{-2} \nu^{-2} \|MPf\|^2 + 2C_5^2 \lambda_1^{-1} \nu^{-2} \epsilon^2 \|(I - M) Pf\|^2.$$

Since $v_0, w_0 \in D(A^{\frac{1}{2}})$ one has from (2.6) and (2.22) that

$$\|u_0\|^2 = \|v_0\|^2 + \|w_0\|^2 \leq \lambda_1^{-1} \|A^{\frac{1}{2}} v_0\|^2 + C_5^2 \epsilon^2 \|A^{\frac{1}{2}} w_0\|^2.$$
Now putting (3.36) and (3.37) together we obtain

\[ (3.38) \quad \| v(t) \|_2^2 \leq D_4 \left( \| A_\epsilon^{\frac{1}{2}} v_0 \|_2^2 + \epsilon^2 \| A_\epsilon^{\frac{1}{2}} w_0 \|_2^2 + \| MP_{\epsilon f} \|_\infty^2 + \epsilon^2 \| (I - M) P_{\epsilon f} \|_\infty^2 \right) \]

where \( D_4 = \max(\lambda_1^{-1}, C_5^2, 2\lambda_1^{-2} \nu^{-2}, 2C_5^2 \lambda_1^{-1} \nu^{-2}) \). From (3.9) one has

\[ (3.39) \quad \| A_\epsilon^{\frac{1}{2}} v(t) \|_2^2 \leq \| A_\epsilon^{\frac{1}{2}} u(t) \|_2^2 \leq NR_0^2, \quad 0 \leq t < T_N. \]

Next we use (3.1), (3.22), (3.38), and (3.39) to observe that

\[ (3.40) \quad G(t) \leq E_2(\epsilon), \quad 0 \leq t \leq \min(T_1, T_N), \]

where

\[ (3.41) \quad E_2(\epsilon) \overset{\text{def}}{=} \epsilon^2 QD_5 \left( \eta_1^{-2} + \epsilon^{2+p_1} \eta_3^{-2} + \eta_2^{-2} + \epsilon^{2+r} \eta_4^{-2} \right) NR_0^2, \]

and \( D_5 = 27C_4^4 C_5^2 \nu^{-4} D_4 \). After using (3.8) and expanding, one observes that the right-hand side of (3.41) contains 13 terms of the form

\[ c\epsilon^{b_0} \eta_i^{-2} \eta_j^{-2} \epsilon^{\frac{1}{4}} Q(\epsilon), \]

3 terms of the form

\[ c\epsilon^{b} \eta_i^{-2} \eta_j^{-2} \epsilon^{\frac{1}{4}} Q(\epsilon), \]

and 4 terms of the form

\[ c\epsilon^{b_1} \eta_i^{-2} \eta_j^{-2} \epsilon^{\frac{1}{4}} Q(\epsilon), \]

where \( b_0 \geq 1, b_1 \geq \frac{2}{5}, i, j = 1, 2, 3, 4 \) and where \( c \) denotes positive constants which are bounded as \( \epsilon \to 0^+ \). By using Hypothesis H1, it is a straightforward verification to see that each of these terms goes to 0 as \( \epsilon \to 0 \). In other words, \( E_2(\epsilon) \to 0 \) as \( \epsilon \to 0^+ \). By combining (3.34) and (3.40) one obtains

\[ (3.42) \quad \| A_\epsilon^{\frac{1}{2}} v(t) \|_2^2 \leq \epsilon E_2(\epsilon) \left( \eta_1^{-2} + E_1(\epsilon) + \frac{7}{4} D_2 \eta_3^{-4} \right), \quad 0 \leq t \leq \min(T_1, T_N), \]

provided \( 0 < \epsilon \leq \epsilon_2 \).

Now set \( N \overset{\text{def}}{=} 1 + \max(4, \frac{7}{2} D_2) \), where \( D_2 \) is given above, and choose \( \epsilon_3 \) so that \( 0 < \epsilon_3 \leq \epsilon_2(N) \) and

\[ (3.43) \quad \epsilon E_2(\epsilon) \leq 2, \quad E_1(\epsilon) \leq \eta_1^{-2}, \quad 2C_5^2 \epsilon^{\frac{3}{5}} \leq \nu^2 \]

for \( 0 < \epsilon \leq \epsilon_3 \).
We claim that for $0 < \epsilon \leq \epsilon_3$ one has $T_1 < T^N$. To prove this we assume on the contrary that $T^N \leq T_1 < \infty$. Then (3.1) and (3.18) imply that
\[
\|A_{\epsilon}^{\frac{1}{2}} w(T^N)\|^2 \leq \|A_{\epsilon}^{\frac{1}{2}} w_0\|^2 + \frac{1}{2} k_2^2 \epsilon^2 \|(I - M)P_{\epsilon}f\|_{\infty}^2 \leq \epsilon^p \eta_3^{-2} + \frac{1}{2} k_2^2 \epsilon^{2 + r} \eta_4^{-2},
\]
where $k_2^2 \equiv 4 C_2^2 \nu^{-2}$. Since $r > -2$ we have $2 + r \geq \frac{2}{3} + \frac{5}{3}$, and consequently (3.43) implies that
\[
(3.44) \quad \|A_{\frac{1}{2}}^{\frac{1}{2}} w(T^N)\|^2 \leq \epsilon^p \eta_3^{-2} + \epsilon^5 \eta_4^{-2}, \quad 0 < \epsilon \leq \epsilon_3.
\]
On the other hand, (3.42) and (3.43) imply that for $0 < \epsilon \leq \epsilon_3$ one has
\[
(3.45) \quad \|A_{\frac{1}{2}}^{\frac{1}{2}} v(t)\|^2 \leq 2 \left(2 \eta_1^{-2} + \frac{7}{4} D_2 \eta_3^{-4}\right), \quad 0 \leq t \leq \min(T_1, T^N).
\]
By adding (3.44) and (3.45) we obtain
\[
\|A_{\frac{1}{2}}^{\frac{1}{2}} u(T^N)\|^2 \leq \left(4 \eta_1^{-2} + \frac{7}{2} D_2 \eta_3^{-4} + \epsilon^p \eta_3^{-2} + \epsilon^5 \eta_4^{-2}\right) < NR_0^2, \quad 0 < \epsilon \leq \epsilon_3,
\]
which contradicts (3.10). Hence one has $T_1 < T^N$ for $0 < \epsilon \leq \epsilon_3$.

Finally we set $k_1^2 \equiv \frac{7}{2} D_2$ and $\epsilon_1 \equiv \epsilon_3$. It then follows from (3.45) that
\[
\|A_{\frac{1}{2}}^{\frac{1}{2}} v(T_1)\|^2 \leq 4 \eta_1^{-2} + k_1^2 \eta_3^{-4},
\]
and from (3.23) that
\[
\|A_{\frac{1}{2}}^{\frac{1}{2}} w(T_1)\|^2 \leq k_2^2 \epsilon^{2 + r} \eta_4^{-2},
\]
which completes the proof of Lemma 3.1. 

REMARK. The proof of Lemma 3.1 still works if we take $r = -2$. However the result of Lemma 3.1 is interesting only in the case where $\epsilon^{2 + r} \eta_4^{-2}$ is bounded as $\epsilon \to 0^+$, see also Hypothesis H2(a,b) given below. If $\eta_4^{-2} \to \infty$ as $\epsilon \to 0^+$, this implies that $r$ must satisfy $r > -2$. In Theorems 7, 8, and 9 we will impose a stronger requirement, viz. that $\epsilon^{2 + r} \eta_4^{-2} \to 0$ as $\epsilon \to 0^+$. This is the reason why we impose the requirement that $r > -2$ in Hypothesis H1.

3.2 Strategy of Proof. The argument of Lemma 3.1 can of course be repeated with the new initial conditions satisfying (3.7) instead of (3.1). By making the realistic assumption that
\[
(3.46) \quad k_2^2 \epsilon^{2 + r} \eta_4^{-2} \leq \epsilon^p \eta_3^{-2}
\]
one needs only to replace $\eta_1^{-2}$ with $(4\eta_1^{-2} + k_1^2\eta_3^{-4})$, and the entire argument carries through. Unfortunately, this is not a good strategy, because one is forced to choose a smaller value for $\varepsilon_2$, and thereby a smaller value for $\varepsilon_1$. It is important to take advantage of the fact that $k_2^2\varepsilon^{2+r}\eta_4^{-2}$ can be made small instead of using the crude bound (3.46). As a result of Lemma 3.1 we can now assume the initial condition $\|A_\varepsilon^{1/2} w_0\|^2$ to be small for $0 < \varepsilon \leq \varepsilon_1$.

For the Long Time Argument we begin by assuming $\|A_\varepsilon^{1/2} w_0\|^2$ is small, i.e., $\|A_\varepsilon^{1/2} w_0\|^2 \leq k_2^2\varepsilon^{2+r}\eta_4^{-2}$, and $\|A_\varepsilon^{1/2} v_0\|^2 \leq 4\eta_1^{-2} + k_1^2\eta_3^{-4}$. In the course of the argument we show that if $\varepsilon$ is sufficiently small then the dilated 3D Navier-Stokes evolutionary equation (2.5) has a strong solution on a suitable interval $[0, 2T_0]$ where $T_0 = T_0(\varepsilon)$ is finite but large. We also show that $\|A_\varepsilon^{1/2} w(t)\|^2 \leq k_2^2\varepsilon^{2+r}\eta_4^{-2}$ and $\|A_\varepsilon^{1/2} v(t)\|^2 \leq \frac{1}{2}(4\eta_1^{-2} + k_1^2\eta_3^{-4})$ on the half-interval $t \in [T_0, 2T_0]$. This, of course, permits one to prove the $H^1$-Regularity Theorem by using the Long Time Argument with induction.

The principal reason why the strategy behind the Long Time Argument is effective is that the solutions of the full 3DNS are close to solutions of the reduced 3D problem since $\|A_\varepsilon^{1/2} w(t)\|^2$ remains small on the interval $[0, 2T_0]$. As is known, and as is explained in Section 5, any solution $\tilde{v}(t)$ of the reduced 3D Navier-Stokes evolutionary equation satisfies

$$\limsup_{t \to -\infty} \|A_\varepsilon^{1/2} \tilde{v}(t)\|^2 \leq L_1,$$

where $L_1$ depends on $\|MP_{\varepsilon f}\|_\infty$ and does not depend on the initial data $v_0$.

### 3.3 Long-Time Argument.

In this section we continue the analysis of the Short-Time Argument. The terms $\eta_i(\varepsilon)$, $1 \leq i \leq 4$, $r$ and $p$ will be assumed to satisfy Hypothesis H1. In addition we will assume that the following Hypothesis H2(a,b) is satisfied, where $a$ and $b$ are sufficiently large:

1. Let $a > 0$ be fixed. Then one has

$$\begin{align*}
e \delta \eta^{-2} \exp(a\eta^{-4}) \to 0, \\
\eta^{-2} \to \infty
\end{align*}$$

as $\varepsilon \to 0^+$, where

$$\eta^{-2} \overset{\text{def}}{=} \max(4\eta_1^{-2} + k_1^2\eta_3^{-4} + k_2^2\varepsilon^{2+r}\eta_4^{-2}, 1),$$

and $\epsilon^{\delta} \exp(2a\eta_2^{-4})$ is bounded for $0 < \varepsilon \leq 1$. (The constants $k_1$ and $k_2$ are given in Lemma 3.1.)

2. Let $b > 0$ be fixed. Then for any $\lambda$, $0 < \lambda < 1$, there is an $\varepsilon_4 = \varepsilon_4(b, \lambda) > 0$ such that one has

$$\eta_2^{-2} \exp(b\eta_2^{-4}) \leq \lambda(4\eta_1^{-2} + k_1^2\eta_3^{-4}), \quad 0 < \varepsilon < \varepsilon_4.$$

3. The function $\varepsilon^{4+2r}\eta_4^{-4}(\log \eta^{-4} + 1)$ is bounded as $\varepsilon \to 0^+$.

Our objective now is to prove the following result:
Lemma 3.2. Assume that both Hypotheses H1 and H2(a,b) are satisfied, where a and b are sufficiently large. Then there is an \( \epsilon_0 > 0 \) such that for every \( \epsilon, 0 < \epsilon \leq \epsilon_0 \), there is a time \( T_0 = T_0(\epsilon) > 0 \) with the property that whenever the initial conditions

\[
\begin{align*}
\| A^{\frac{1}{2}} v_0 \|^2 &\leq 4\eta_1^{-2} + k_1^2 \eta_3^{-4}, & ||MP_\epsilon f||^2_\infty &\leq \eta_2^{-2} \\
\| A^{\frac{1}{2}} w_0 \|^2 &\leq k_2^2 \epsilon^2 + \eta_4^{-2}, & ||(I - M)P_\epsilon f||^2_\infty &\leq \epsilon \eta_4^{-2}
\end{align*}
\]

are satisfied, then the solution \( u(t) \) of (2.5) satisfies \( u(t) \in D(A^{\frac{1}{2}}) \) for \( 0 \leq t \leq 2T_0 \) and

\[
\begin{align*}
\| A^{\frac{1}{2}} v(t) \|^2 &\leq \frac{1}{2}(4\eta_1^{-2} + k_1^2 \eta_3^{-4}), \\
\| A^{\frac{1}{2}} w(t) \|^2 &\leq k_2^2 \epsilon^2 + \eta_4^{-2}
\end{align*}
\]

for \( T_0 \leq t \leq 2T_0 \).

Proof. The proof begins as in Lemma 3.1. For any positive numbers \( d_1 \) and \( d_2 \) we define \( R_0^2 = R_0^2(\epsilon, d_1, d_2) \) by

\[
R_0^2 = 1 + (\eta^{-2} + \eta_2^{-2} + d_1)(1 + \exp(d_2 \eta_4^{-4}) \exp(2d_2 \eta_4^{-4})).
\]

The values of \( d_1 \) and \( d_2 \) will be fixed later. Note that \( R_0^2 \geq \eta^{-2} \geq \| A^{\frac{1}{2}} u_0 \|^2 \). Therefore it follows from Lemma 3.0 that for any \( N > 1 \), there is a time \( T_N, 0 < T_N \leq \infty \), such that

\[
\| A^{\frac{1}{2}} u(t) \|^2 \leq NR_0^2, \quad 0 \leq t < T_N.
\]

Without loss of generality we let \([0, T_N) \) denote the maximal time interval for which (3.50) is valid. Therefore if \( T_N < \infty \) one must have

\[
\| A^{\frac{1}{2}} u(T_N) \|^2 = NR_0^2.
\]

By taking the scalar product of the \( w \)-equation (3.11) with \( A_\epsilon w \), we then obtain (3.13) with the same value for \( D_1 \). For \( 0 \leq t < T_N \) one has

\[
D_1^2 \epsilon^4 \| A^{\frac{1}{2}} u \|^2 \leq D_1^2 N \epsilon^{\frac{8}{3}} R_0^2,
\]

where \( R_0^2 \) is given by (3.49). From Hypotheses H1 and H2(a,b), the right-side term in (3.52) goes to 0 as \( \epsilon \to 0+ \), provided \( a \geq d_2 \) and \( b \geq 2d_2 \). Consequently there is an \( \epsilon_5 = \epsilon_5(N, d_1, d_2), 0 < \epsilon_5 \leq \epsilon_1 \), where \( \epsilon_1 \) is given by Lemma 3.1, such that

\[
D_1 N^{\frac{1}{2}} \epsilon^{\frac{8}{3}} R_0 \leq \frac{\nu}{2}, \quad 0 < \epsilon \leq \epsilon_5.
\]

39
As a result, (3.16), (3.17), (3.18) and (3.19) are valid for $0 \leq t < T^N$ and $0 \leq \epsilon \leq \epsilon_5$. We now restrict to $0 < \epsilon \leq \epsilon_5$ for the remainder of the argument. By using (3.47) we see that for $0 < \epsilon \leq \epsilon_5$ inequality (3.18) now assumes the form

$$
(3.53) \quad \|A^\frac{1}{2}_\epsilon w(t)\| \leq \left[ k_2^2 \exp \left( \frac{-\nu C_5^{-2}\epsilon^{-2}}{2} t \right) + \frac{1}{2} k_2^2 \right] \epsilon^{2+r} \eta_4^{-2} \leq \frac{3}{2} k_2^2 \epsilon^{2+r} \eta_4^{-2}
$$

for $0 \leq t < T^N$. By using (3.47) once again (3.19) becomes

$$
\int_0^t \|A_\epsilon w(s)\|^2 ds \leq \frac{2}{\nu} \|A^\frac{1}{2}_\epsilon w_0\|^2 + \frac{2t}{\nu} \|(I - M)P_\epsilon f\|_\infty^2 \\
\leq D_5^2(\epsilon^2 + t)\epsilon^r \eta_4^{-2}
$$

for $0 \leq t < T^N$, where $D_5^2 = \max(2k_2^2 \nu^{-1}, 2\nu^{-2})$. In addition, by integrating (3.16) and using (3.53) we have for $0 < \epsilon \leq \epsilon_5 \leq 1$

$$
\int_{t-1}^t \|A_\epsilon w\|^2 ds \leq \frac{2}{\nu} \|A^\frac{1}{2}_\epsilon w(t - 1)\|^2 + \frac{2}{\nu} \|(I - M)P_\epsilon f\|_\infty^2 \\
\leq D_{10}^2 \epsilon^r \eta_4^{-2},
$$

for $1 \leq t < T^N$, where $D_{10}^2 = 3k_2^2 \nu^{-1} + 2\nu^{-2}$. It follows from (3.5) and (3.53) that

$$
\int_0^t \|A^\frac{1}{2}_\epsilon w\|^6 ds \leq 4 \int_0^t \left[ k_2^6 \exp \left( \frac{-3\nu C_5^{-2}\epsilon^{-2}}{2} s \right) + \frac{1}{8} k_2^6 \right] \epsilon^{6+3r} \eta_4^{-6} ds \\
\leq D_{11}^2(\epsilon^2 + t)\epsilon^{6+3r} \eta_4^{-6}
$$

for $0 \leq t < T^N$, where $D_{11}^2 = 4k_2^6 \max(2C_5^2(3\nu)^{-1}, \frac{1}{8})$. Using $0 < \epsilon \leq 1$ one has

$$
\int_{t-1}^t \|A^\frac{1}{2}_\epsilon w\|^6 ds \leq 2D_{11}^2 \epsilon^{6+3r} \eta_4^{-6}
$$

for $1 \leq t < T^N$. Using the argument in (3.21) one then obtains

$$
(3.54) \quad \int_0^t \|A^\frac{1}{2}_\epsilon w\|^3 \|A_\epsilon w\| ds \leq D_9 D_{11}(\epsilon^2 + t)\epsilon^{3+2r} \eta_4^{-4}, \quad 0 \leq t < T^N,
$$

and, since $0 < \epsilon \leq 1$,

$$
(3.55) \quad \int_{t-1}^t \|A^\frac{1}{2}_\epsilon w\|^3 \|A_\epsilon w\| ds \leq 2D_{10} D_{11} \epsilon^{3+2r} \eta_4^{-4}, \quad 1 \leq t < T^N.
$$
Next we return to the v-equation (3.24). By taking the scalar product of (3.24) with v and using $b_\epsilon(v,v,v) = 0$ together with (2.26) and (3.3) we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 + \lambda_1 \nu \|v\|^2 \leq \frac{1}{2} \frac{d}{dt} \|v\|^2 + \nu \|A^{\frac{1}{2}}_\epsilon v\|^2$$

$$\leq \left( \|MP_\epsilon f\|_{\infty} + C_2 \epsilon \|A^{\frac{1}{2}}_\epsilon w\| \|A_\epsilon w\|^{\frac{3}{2}} \right) \|v\|$$

$$\leq \lambda_1^{-\frac{1}{2}} \left( \|MP_\epsilon f\|_{\infty} + C_2 \epsilon \|A^{\frac{1}{2}}_\epsilon w\| \|A_\epsilon w\|^{\frac{3}{2}} \right) \|A^{\frac{1}{2}}_\epsilon v\|.$$

By using the Young inequality we get

$$\frac{d}{dt} \|v\|^2 + \lambda_1 \nu \|v\|^2 \leq \frac{1}{\lambda_1 \nu} \left( \|MP_\epsilon f\|^2_{\infty} + C_2^2 \epsilon \|A^{\frac{1}{2}}_\epsilon w\|^3 \|A_\epsilon w\| \right)$$

and

$$\frac{d}{dt} \|v\|^2 + \nu \|A^{\frac{1}{2}}_\epsilon v\|^2 \leq \frac{1}{\lambda_1 \nu} \left( \|MP_\epsilon f\|^2_{\infty} + C_2^2 \epsilon \|A^{\frac{1}{2}}_\epsilon w\|^3 \|A_\epsilon w\| \right).$$

By using the Gronwall inequality for (3.56), one finds that

$$\|v(t)\|^2 \leq e^{-\nu \lambda_1 t} \|v_0\|^2 + \lambda_1^{-1} \nu^{-2} \|MP_\epsilon f\|^2_{\infty}$$

$$+ \lambda_1^{-1} \nu^{-1} C_2^2 \epsilon \int_0^t \|A^{\frac{1}{2}}_\epsilon w\|^3 \|A_\epsilon w\| ds$$

for $0 \leq t < T^N$. Next by using (2.6), (3.47), and (3.54) we find

$$\|v(t)\|^2 \leq D_{12} \gamma(\epsilon,t), \quad 0 \leq t < T^N,$$

where $D_{12} = \max(\lambda_1^{-1}, \lambda_1^{-2} \nu^{-2}, \lambda_1^{-1} \nu^{-1} C_2^2 D_9 D_{11})$ and

$$\gamma = \gamma(\epsilon,t) \overset{\text{def}}{=} (e^{-\nu \lambda_1 t} \eta^{-2} + \eta_2^{-2} + [\epsilon^2 + t] \epsilon^{4+2r} \eta^{-4}).$$

Similarly by integrating (3.57) and using (3.47) and (3.54) we obtain

$$\int_0^t \|A^{\frac{1}{2}}_\epsilon v\|^2 ds \leq \nu^{-1} \|v_0\|^2 + \lambda_1^{-1} \nu^{-2} t \|MP_\epsilon f\|^2_{\infty}$$

$$+ \lambda_1^{-1} \nu^{-2} C_2^2 \epsilon \int_0^t \|A^{\frac{1}{2}}_\epsilon w\|^3 \|A_\epsilon w\| ds$$

$$\leq \lambda_1^{-1} \nu^{-1} (4 \eta_1^{-2} + k^2_3 \eta^{-4}) + \lambda_1^{-1} \nu^{-2} t \eta_2^{-2}$$

$$+ \lambda_1^{-1} \nu^{-2} C_2^2 D_9 D_{11} (\epsilon^2 + t) \epsilon^{4+2r} \eta^{-4},$$

41
for $0 \leq t < T^N$. It follows that

$$
\int_0^t ||A_x^{1/2}v||^2 ds \leq D_{13} \gamma(\epsilon, t), \quad 0 \leq t < \min(1, T^N),
$$

where $D_{13} = \max(\lambda_1^{-1}\nu^{-1} e^{\nu\lambda_1}, \lambda_1^{-1}\nu^{-2}, \lambda_1^{-1}\nu^{-2}C_2^2D_9D_{11})$. Furthermore by integrating (3.57) once again we find

$$
\int_{t-1}^t ||A_x^{1/2}v||^2 ds \leq \nu^{-1}||v(t-1)||^2 + \lambda_1^{-1}\nu^{-2}||MP\epsilon f||^2_\infty
$$

$$
+ \lambda_1^{-1}\nu^{-2}C_2^2\epsilon \int_{t-1}^t ||A_x^{1/2}w||^3 ||A_xw|| ds.
$$

From (3.47), (3.55), (3.59) and (3.60) we get

$$
\int_{t-1}^t ||A_x^{1/2}v||^2 ds \leq D_{14} \gamma(\epsilon, t), \quad 1 \leq t < T^N,
$$

where

$$
D_{14} = \nu^{-1} \max(D_{12} + \lambda_1^{-1}\nu^{-1}, D_{12}e^{\nu\lambda_1} + 2D_{10}D_{11}\lambda_1^{-1}\nu^{-1}C_2^2).
$$

By combining (3.59), (3.60), (3.62) and (3.64) we find

$$
\int_0^t ||v||^2 ||A_x^{1/2}v||^2 ds \leq \sup_{0 \leq s \leq t} ||v(s)||^2 \int_0^t ||A_x^{1/2}v||^2 ds
$$

$$
\leq e^{\nu\lambda_1}D_{12}D_{13}\gamma(\epsilon, t)^2
$$

for $0 \leq t < \min(1, T^N)$, and

$$
\int_{t-1}^t ||v||^2 ||A_x^{1/2}v||^2 ds \leq e^{\nu\lambda_1}D_{12}D_{14}\gamma(\epsilon, t)^2, \quad 1 \leq t < T^N.
$$

Next by taking the scalar product of (3.24) with $A_xv$ we obtain (3.25), (3.26) (3.27) and (3.28). For $0 \leq t < \min(1, T^N)$ we apply the Gronwall inequality to (3.28) to obtain

$$
||A_x^{1/2}v(t)||^2 \leq e^{G(t)} \left(||A_x^{1/2}v_0||^2 + H(t)\right) \quad 0 \leq t < \min(1, T^N),
$$

where $H(t)$ and $G(t)$ are given by (3.30), (3.31), and (3.32). From (3.47), (3.54) and (3.65) we find

$$
||A_x^{1/2}v(t)||^2 \leq D_{16}\gamma(\epsilon, t) \exp(D_{17}\gamma(\epsilon, t)^2), \quad 0 \leq t < \min(1, T^N),
$$

where $D_{16} = \max(e^{\nu\lambda_1}, \nu^{-1}, 2\nu^{-1}C_2^2D_9D_{11})$ and $D_{17} = 2^{-1}27\nu^{-3}C_1^4e^{\nu\lambda_1}D_{12}D_{13}$.  

42
For $1 \leq t < T^N$ we use the uniform Gronwall inequality, see Foias et al (1987),\textsuperscript{5} on (3.28) to obtain

\[(3.68) \quad \|A_{\xi}^{\frac{1}{2}} v(t)\|^2 \leq \left( \int_{t-1}^{t} \|A_{\xi}^{\frac{1}{2}} v\|^2 ds + \int_{t-1}^{t} h(s) ds \right) \exp \left( \int_{t-1}^{t} g(s) ds \right),\]

where $h$ and $g$ are given by (3.31) and (3.32). For $t \geq 1$ we use (3.55), (3.66), and (3.64) to derive an inequality similar to (3.67). This can be combined with (3.67) to obtain

\[(3.69) \quad \|A_{\xi}^{\frac{1}{2}} v(t)\|^2 \leq D_{18} \gamma(\epsilon, t) \exp \left( D_{19} \gamma(\epsilon, t)^2 \right), \quad 0 \leq t < T^N,\]

where

\[
\begin{align*}
D_{18} &= \max(D_{16}, D_{14} + \max(\nu^{-1}, 4C_{2}^{2}D_{10}D_{11}\nu^{-1})) \\
D_{19} &= 27C_{4}^{2}\nu^{2}\lambda^{2}2^{-1}\nu^{-3}\max(D_{13}, D_{14})D_{12}.
\end{align*}
\]

The next step is to define $T_0 = T_0(\epsilon)$. Since $\eta^{-4} \to \infty$ as $\epsilon \to 0^+$, there is no loss in generality in assuming that $2D_{19}\eta^{-4} > 1$.

We then define $T_0$ by requiring

\[(3.70) \quad 2D_{19}e^{-2\nu\lambda_{1}t}\eta^{-4} \leq 1, \quad T_0 \leq t,\]

that is, set

\[(3.71) \quad T_0 \overset{\text{def}}{=} \frac{1}{2\nu\lambda_{1}} \log(2D_{19}\eta^{-4}).\]

Also define

\[(3.72) \quad E_{3}(\epsilon) \overset{\text{def}}{=} (\epsilon^{2} + 2T_{0}(\epsilon))e^{4+2r}\eta^{-4}.\]

It follows from Hypothesis H2(a,b) that there are constants $D_{20}$ and $D_{21}$ such that

\[(3.73) \quad E_{3}(\epsilon) \leq D_{20}, \quad \frac{3}{2}k_{2}^{2}\epsilon^{2+r}\eta^{-2} \leq D_{21},\]

\textsuperscript{5}Let $y, g, h$ be nonnegative locally integrable functions on $(0, \infty)$, where $y$ is absolutely continuous on $(0, \infty)$, and which satisfy

\[y' \leq g y + h, \quad 0 < t < \infty.\]

Then one has

\[y(t) \leq \left( \frac{1}{t - \tau} \int_{t-\tau}^{t} y(s) ds + \int_{t-\tau}^{t} h(s) ds \right) \exp \left( \int_{t-\tau}^{t} g(s) ds \right), \quad 0 < t < \infty,\]

where $\tau = \max(0, t - 1)$.  

43
for $0 < \epsilon \leq 1$. The term $R_0^2 = R_0^2(\epsilon, d_1, d_2)$ is now fixed so that

(3.74) \[ R_0^2 = R_0^2(\epsilon, D_{20}, 2D_{19}), \]

and the term $N$ is fixed so that

(3.75) \[ N > \max(1, D_{21}, D_{22}), \]

where $D_{22} = D_{18} \exp(4D_{19}D_{20}^2)$. Finally we fix $\epsilon_5 = \epsilon_5(N, D_{20}, 2D_{19})$ for these choices of $N$, $d_1$ and $d_2$. Furthermore, we require that the constants $a$ and $b$ in Hypothesis H2(a,b) satisfy $a \geq 2D_{19}$ and $b \geq 4D_{19}$.

Let us return to the function $\gamma$, which we will write as

\[ \gamma = \gamma(\epsilon, t) = e^{-\nu \lambda_1 t} \eta^{-2} + \beta, \]

where

\[ \beta = \beta(\epsilon, t) \overset{\text{def}}{=} \eta_2^{-2} + (\epsilon^2 + t)\epsilon^{4+2r} \eta_4^{-4}. \]

Note that

(3.76) \[ \gamma^2 \leq 2e^{-2\nu \lambda_1 t} \eta^{-4} + 2\beta^2, \]

and from (3.72) and (3.73) we have

(3.77) \[ \beta(\epsilon, t) \leq \eta_2^{-2} + D_{20}, \quad 0 \leq t \leq 2T_0. \]

From (3.70) we see that

(3.78) \[ D_{18} \gamma \exp(D_{19} \gamma^2) \leq D_{22}(e^{-\nu \lambda_1 t} \eta^{-2} + \eta_2^{-2} + D_{20}) \exp(4D_{19} \eta_2^{-4} + 1) \]

for $T_0 \leq t \leq 2T_0$.

From (3.53) and (3.73) we find

(3.79) \[ \|A_\epsilon^{\frac{1}{2}} w(t)\|^2 \leq D_{21}, \quad 0 \leq t < T^N. \]

By using (3.69), (3.76) and (3.77) we find that

(3.80) \[ \|A_\epsilon^{\frac{1}{2}} v(t)\|^2 \leq D_{22}(e^{-\nu \lambda_1 t} \eta^{-2} + \eta_2^{-2} + D_{20}) \exp(2D_{19} \eta^{-4}) \exp(4D_{19} \eta_2^{-4}) \]

for $0 \leq t < \min(2T_0, T^N)$. 

44
We claim that $2T_0 \leq T^N$. In order to prove this, we assume on the contrary that $T^N < 2T_0$. From the Pythagorean relation (2.21) and (3.79) and (3.80) it follows that

$$\|A^\frac{1}{2}_\epsilon u(T^N)\| \leq D_{21} + D_{22}(\eta^{-2} + \eta_2^{-2} + D_{20}) \exp(2D_{19} \eta^{-4}) \exp(4D_{19} \eta_2^{-4}).$$

From the definition of $R_0^2$ in (3.49) and (3.74) and the characterization of $N$ in (3.75) we obtain

$$\|A^\frac{1}{2}_\epsilon u(T^N)\|^2 < NR_0^2,$$

which contradicts (3.51). Hence one has $2T_0 \leq T^N$.

Finally we turn to the verification of (3.48). For this purpose we restrict $t$ to the interval $[T_0, 2T_0]$. From (3.53) we have

$$\|A^\frac{1}{2}_\epsilon w(t)\|^2 \leq \left[ k_2^2 \exp \left( \frac{-\nu C_5^{-2} \epsilon^{-2}}{2} T_0 \right) + \frac{1}{2} k_2^2 \right] \epsilon^{2+r} \eta_4^{-2}, \quad T_0 \leq t \leq 2T_0.$$

From the definition of $T_0$ in (3.71), we see that there is an $\epsilon_6, 0 < \epsilon_6 \leq \epsilon_5$, such that

$$2 \nu^{-1} C_5^2 \epsilon^2 \log 2 \leq T_0(\epsilon), \quad 0 < \epsilon \leq \epsilon_6.$$

Now (3.81) implies that $\exp \left( \frac{-\nu C_5^{-2} \epsilon^{-2}}{2} T_0 \right) \leq \frac{1}{2}$, and consequently,

$$\|A^\frac{1}{2}_\epsilon w(t)\|^2 \leq k_2^2 \epsilon^{2+r} \eta_4^{-2}, \quad T_0 \leq t \leq 2T_0,$$

for $0 < \epsilon \leq \epsilon_6$. From (3.69), (3.70) and (3.78) one finds

$$\|A^\frac{1}{2}_\epsilon v(t)\|^2 \leq \Gamma(\eta_2^{-2}), \quad T_0 \leq t \leq 2T_0,$$

where

$$\Gamma(r) \overset{\text{def}}{=} D_{22}(r + D_{24}) \exp(4D_{19} r^2 + 1).$$

and $D_{24} = (2D_{19})^{-\frac{1}{2}} + D_{20}$. From Hypothesis H2(a,b) we see that there is an $\epsilon_7, 0 < \epsilon_7 \leq \epsilon_6$, such that for $0 < \epsilon \leq \epsilon_7$ one has

$$\Gamma(\eta_2^{-2}) \leq \frac{1}{2} (4\eta_1^{-2} + k_1^2 \eta_3^{-4}), \quad 0 < \epsilon \leq \epsilon_7.$$

It then follows that for $0 < \epsilon \leq \epsilon_7$ one has

$$\|A^\frac{1}{2}_\epsilon v(t)\|^2 \leq \frac{1}{2} (4\eta_1^{-2} + k_1^2 \eta_3^{-4}), \quad T_0 \leq t \leq 2T_0.$$

By setting $\epsilon_0 = \epsilon_7$, we complete the proof of the lemma. □

**Proof of Theorem 1.** Since $0 < \epsilon_0 \leq \epsilon_1$, where $\epsilon_1$ and $\epsilon_0$ are given by Lemmas 3.1 and 3.2, the proof of Theorem 1, for $0 < \epsilon \leq \epsilon_0$, now follows by first applying Lemma 3.1 and then using Lemma 3.2 with induction, with $T_1 = T_0 + T_1$. The estimate for $L^2_\epsilon$ appearing in (2.33) follows from the Pythagorean identity (2.21) together with (3.82) and (3.83). The fact that $u(\cdot)$ belongs to $C^0([0, \infty), V^1_\epsilon)$ is now a direct consequence of the local result contained in Lemma 3.0. □
4. $H^2$-Regularity: Theorems 2 and 3

In this section, we will prove Theorems 2 and 3.

Proof of Theorem 2. It is known that if $u_0$ belongs to $V^2_\epsilon = D(A_\epsilon)$ and $P_\epsilon f$ belongs to $C^0([0, \infty), H_\epsilon) \cap W^{1,\infty}((0, \infty), D(A_\epsilon^{3/4}))$, then $u$ is in the space $C^0([0, \infty), V^2_\epsilon)$, and the time-derivative $u'$ belongs to $C^0([0, \infty); H_\epsilon)$ and is the solution of the equation

$$
\frac{du'}{dt} + \nu A_\epsilon u' + B_\epsilon(u', u) + B_\epsilon(u, u') = P_\epsilon f'
$$

with initial condition

$$
u u'(0) = P_\epsilon f(0) - B_\epsilon(u_0, u_0) - \nu A_\epsilon u_0,
$$

see Témam (1982, 1983). On the other hand, if $u_0$ belongs only to $V^1_\epsilon$, we will show that one can choose $t_0 > 0$, arbitrarily close to 0, such that $u(t_0)$ belongs to $D(A_\epsilon)$. It then follows that $u'(t_0)$ belongs to $H_\epsilon$ for every such $t_0$, that $u$ belongs to $C^0([t_0, \infty), V^2_\epsilon)$, and that $u'$ is in the space $C^0([t_0, \infty); H_\epsilon)$ and is the solution of the equation (4.1) on $(t_0, \infty)$. Our main objective here is to prove the estimates (2.35), (2.36) and (2.37) of Theorem 2. The proof will be given in four steps. We will not use here the decomposition $u = v + w$.

Step 1. First we derive an estimate for $\int^t_\tau \|A_\epsilon u(s)\| ds$. Taking the scalar product of (2.5) with $A_\epsilon u$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|A_\epsilon^{3/2} u\|^2 + \nu \|A_\epsilon u\|^2 \leq \frac{1}{\nu} \|P_\epsilon f\|_{\infty}^2 + \frac{\nu}{4} \|A_\epsilon u\|^2 + |b_\epsilon(u, u, A_\epsilon u)|, \quad t \geq 0.
$$

From the inequalities (8.8) and (2.18), we deduce that,

$$
|b_\epsilon(u^1, u^2, u^3)| \leq C_9 \|A_\epsilon^{3/2} u^1\| \|A_\epsilon^{3/2} u^2\| \|A_\epsilon^{3/2} u^3\|,
$$

for any $u^1 \in D(A_\epsilon^{3/2})$, $u^2 \in D(A_\epsilon)$, $u^3 \in H_\epsilon$, where $C_9$ is a positive constant independent on $\epsilon$. This in turn implies that

$$
|b_\epsilon(u, u, A_\epsilon u)| \leq C_9 \|A_\epsilon^{3/2} u\| \|A_\epsilon u\|^{3/2}.
$$

Using (4.4) and the Young inequality (3.4), we deduce from (4.2) that

$$
\frac{d}{dt} \|A_\epsilon^{3/2} u\|^2 + \nu \|A_\epsilon u\|^2 \leq 2 \|P_\epsilon f\|_{\infty}^2 + \frac{27 C_9^4}{2 \nu^3} \|A_\epsilon^{3/2} u\|^6, \quad t \geq 0.
$$

By integrating (4.5) we infer that for $\max(0, t - 1) \leq \tau \leq t$, one has

$$
\int^t_\tau \|A_\epsilon u(s)\|^2 ds \leq \frac{2}{\nu^2} (t - \tau) \|P_\epsilon f\|_{\infty}^2 + D_{25} (t - \tau) \sup_{\tau \leq s \leq t} \|A_\epsilon^{3/2} u(s)\|^6 + \frac{1}{\nu} \|A_\epsilon^{3/2} u(\tau)\|^2,
$$

46
where $D_{25} = 27C_9^42^{-1}\nu^{-4}$. Since the right side of (4.6) is bounded for any $t > 0$, it follows that the integrand on the left side is finite almost everywhere. Therefore there exists arbitrarily small $t_0 > 0$ such that $u(t_0) \in D(A_\epsilon)$.

**STEP 2.** Next we derive an estimate of $\int_\tau^t \|u'(s)||^2 ds$ for $0 \leq \tau \leq t$. First we observe that (2.5) yields the identity

$$\langle u', u' \rangle = \langle P_\epsilon f, u' \rangle - \langle \nu A_\epsilon u, P_\epsilon f - \nu A_\epsilon u - B_\epsilon(u, u) \rangle - \langle B_\epsilon(u, u), u' \rangle,$$

and consequently, one finds that

$$\|u'||^2 \leq \|P_\epsilon f\|\|u'|| + \nu \|P_\epsilon f\|\|A_\epsilon u\| + \nu^2 \|A_\epsilon u||^2 + \nu |b_*(u, u, A_\epsilon u)| + |b_*(u, u, u')|.$$  

By applying the Young inequality (3.4) several times and using (4.2) and (4.4), we obtain

(4.7) $$\|u'||^2 \leq 3(\nu^2 + 1)\|A_\epsilon u||^2 + \nu^4 C_9^4 \|A_\epsilon^\frac{1}{2} u||^6 + 3\|P_\epsilon f\|^2.$$  

As a result one has

(4.8) $$\int_\tau^t \|u'(s)||^2 ds \leq \int_\tau^t \left(3(\nu^2 + 1)\|A_\epsilon u(s)||^2 + 3\|P_\epsilon f\|^2 + \nu^4 C_9^4 \|A_\epsilon^\frac{1}{2} u(s)||^6\right) ds,$$

for $t \geq \tau \geq 0$. The inequalities (4.6) and (4.8) imply that for $\max(0, t - 1) \leq \tau \leq t$, one has

(4.9) $$\int_\tau^t \|u'(s)||^2 ds \leq D_{26}(t - \tau)\|P_\epsilon f\|^2 + D_{27}(t - \tau) \sup_{\tau \leq s \leq t} \|A_\epsilon^\frac{1}{2} u(s)||^6 + D_{28} \|A_\epsilon^\frac{1}{2} u(\tau)||^2,$$

where $D_{26} = 3 + 6(\nu^2 + 1)\nu^{-2}$, $D_{27} = 3(\nu^2 + 1)D_{25} + \nu^4 C_9^4$, and $D_{28} = 3(\nu^2 + 1)\nu^{-1}$.

**STEP 3.** Next we shall derive an estimate of $\|u'(t)||^2$ for $t > 0$. Let $t_0$ be fixed so that $0 \leq t_0 < \frac{1}{2}$, and $t_0$ is close to $0$ with $u(t_0) \in D(A_\epsilon)$. By taking the scalar product of (4.1) with $u'(t)$, we obtain

(4.10) $$\frac{1}{2} \frac{d}{dt} \|u'||^2 + \nu \|A_\epsilon^\frac{1}{2} u'||^2 \leq \frac{1}{\nu} \|A_\epsilon^{-\frac{1}{2}} P_\epsilon f'||^2 + \frac{\nu}{4} \|A_\epsilon^\frac{1}{2} u'||^2 + |b_*(u', u, u')|, \quad t \geq t_0.$$  

However

$$|b_*(u', u, u')| = \left| \sum_{i,j=1}^3 \int_Q u'_i e^{-(i)}(D_i u_j) u'_j dx \right|$$

$$\leq \sum_{i,j=1}^3 \left( \int_Q e^{-2(i)}(D_i u_j)^2 dx \right)^{\frac{1}{2}} \left( \int_Q (u'_i)^2 (u'_j)^2 dx \right)^{\frac{1}{2}}$$

$$\leq C \|A_\epsilon^\frac{1}{2} u\| \left( \sum_{i,j=1}^3 \left( \int_Q (u'_i)^2 dx \right)^{\frac{1}{4}} \left( \int_Q (u'_j)^6 dx \right)^{\frac{1}{4}} \right),$$

47
which gives

\[(4.11) \quad |b_\varepsilon(u', u, u')| \leq C_{10} \|A_\varepsilon^{\frac{1}{2}} u\| \|u'\|^{\frac{3}{2}} \|A_\varepsilon^{\frac{3}{4}} u'\|^{\frac{3}{2}}.\]

Once again by using the Young inequality (3.4), we infer from (4.10) and (4.11) that

\[(4.12) \quad \frac{d}{dt} \|u'(t)\|^2 + \nu \|A_\varepsilon^{\frac{1}{2}} u'(t)\|^2 \leq \frac{2}{\nu} \|A_\varepsilon^{-\frac{1}{2}} P_\varepsilon f'\|_\infty^2 + D_{29} \|A_\varepsilon^{\frac{3}{4}} u(t)\|^4 \|u'(t)\|^2,
\]

for \(t \geq t_0\), where \(D_{29} = 27C_{10}^4 2^{-1} \nu^{-3}\). Next we apply the uniform Gronwall inequality, as in (3.68), on (4.12) to obtain

\[(4.13) \quad \|u'(t)\|^2 \leq \left( \frac{1}{t - \tau_0} \int_{\tau_0}^t \|u'(s)\|^2 \, ds + \int_{\tau_0}^t \frac{2}{\nu} \|A_\varepsilon^{-\frac{1}{2}} P_\varepsilon f'\|_\infty^2 \right) \exp \left( \int_{\tau_0}^t D_{29} \|A_\varepsilon^{\frac{3}{4}} u(s)\|^4 \, ds \right),\]

where \(\tau_0 = \max(t_0, t - 1)\). Therefore, by using (4.9) for \(t_0 \leq t \leq 1\), one has

\[(4.14) \quad \|u'(t)\|^2 \leq \left( \frac{2}{\nu} \|A_\varepsilon^{-\frac{1}{2}} P_\varepsilon f'\|_\infty^2 + D_{26} \|P_\varepsilon f\|_\infty^2 + D_{27} \sup_{t_0 \leq s \leq t} \|A_\varepsilon^{\frac{1}{2}} u(s)\|^6 \right.
\]

\[\left. + \frac{D_{28}}{t - t_0} \|A_\varepsilon^{\frac{1}{2}} u(t_0)\|^2 \right) \times \exp \left( D_{29} \sup_{t_0 \leq s \leq t} \|A_\varepsilon^{\frac{3}{4}} u(s)\|^4 \right).\]

Likewise for \(t \geq 1\) one has \((t - t_0) \geq \frac{1}{2}\) and (4.13), together with (4.9), implies that

\[(4.15) \quad \|u'(t)\|^2 \leq \left( \frac{2}{\nu} \|A_\varepsilon^{-\frac{1}{2}} P_\varepsilon f'\|_\infty^2 + D_{26} \|P_\varepsilon f\|_\infty^2 + D_{27} \sup_{t - 1 \leq s \leq t} \|A_\varepsilon^{\frac{1}{2}} u(s)\|^6 \right.
\]

\[\left. + 2D_{28} \sup_{t - 1 \leq s \leq t} \|A_\varepsilon^{\frac{1}{2}} u(s)\|^2 \right) \times \exp \left( D_{29} \sup_{t - 1 \leq s \leq t} \|A_\varepsilon^{\frac{3}{4}} u(s)\|^4 \right).\]

Let us now assume that \(u_0\) belongs to \(D(A_\varepsilon)\). Thanks to the Gronwall inequality, we deduce from (4.12) that one has

\[(4.16) \quad \|u'(t)\|^2 \leq \left( \|u'(0)\|^2 + \frac{2}{\nu} \|A_\varepsilon^{-\frac{1}{2}} P_\varepsilon f'\|_\infty^2 \right) \times \exp \left( D_{29} \sup_{0 \leq s \leq t} \|A_\varepsilon^{\frac{3}{4}} u(s)\|^4 \right), \quad 0 \leq t \leq 1.\]

However (4.7) implies that

\[\|u'(0)\|^2 \leq 3(\nu^2 + 1) \|A_\varepsilon u_0\|^2 + 3 \|P_\varepsilon f\|^2 + 3 \nu^4 C_0^4 \|A_\varepsilon^{\frac{1}{2}} u_0\|^2.\]

By combining the last two inequalities, we find that for \(t \leq 1\), one has

\[(4.17) \quad \|u'(t)\|^2 \leq \left( 3(\nu^2 + 1) \|A_\varepsilon u_0\|^2 + 3 \nu^4 C_0^4 \|A_\varepsilon^{\frac{1}{2}} u_0\|^2 \right) \times \exp \left( D_{29} \sup_{0 \leq s \leq t} \|A_\varepsilon^{\frac{3}{4}} u(s)\|^4 \right).\]
STEP 4. In this last step we shall verify inequalities (2.35), (2.36), and (2.37). By taking the scalar product of (2.5) with \( A_\varepsilon u \), we obtain

\[
\nu \| A_\varepsilon u \|^2 \leq \| u' \| \| A_\varepsilon u \| + \| P_\varepsilon f \| \| A_\varepsilon u \| + |b_\varepsilon(u, u, A_\varepsilon u)|.
\]

By using the Young inequality (3.4) with (4.4), we find that

\[
\| A_\varepsilon u(t) \|^2 \leq \frac{3}{\nu^2} \| u'(t) \|^2 + \frac{3}{\nu^2} \| P_\varepsilon f \|_\infty^2 + \frac{9^3 C_9^4}{16 \nu^4} \| A_\varepsilon^{1/2} u(t) \|^6, \quad t \geq t_0,
\]

and consequently, \( u(t) \in D(A_\varepsilon) \) for all \( t \geq t_0 \). Since \( t_0 \) can be chosen arbitrarily small, one has \( u(t) \in D(A_\varepsilon) \) for all \( t > 0 \). Inequalities (4.14) and (4.18) then imply that for \( 0 < t_0 \leq t \leq 1 \), one has

\[
\| A_\varepsilon u(t) \|^2 \leq \left( D_{30} \| A_\varepsilon^{-\frac{1}{2}} P_\varepsilon f' \|_\infty^2 + D_{31} \| P_\varepsilon f \|_\infty^2 + D_{32} \sup_{0 \leq s \leq t} \| A_\varepsilon^{1/2} u(s) \|^6
\]

\[
+ \frac{D_{33}}{t - t_0} \| A_\varepsilon^{1/2} u(t_0) \|^2 \right) \times \exp \left( D_{29} \sup_{0 \leq s \leq t} \| A_\varepsilon^{1/2} u(s) \|^4 \right),
\]

where \( D_{30} = 6 \nu^{-3} \), \( D_{31} = 3 \nu^{-2}(D_{26} + 1) \), \( D_{32} = 3 \nu^{-2}D_{27} + 9^3 C_9^4 (2\nu)^{-4} \) and \( D_{33} = 3 \nu^{-2}D_{28} \). Since (4.19) is valid for any \( t_0 \) satisfying \( 0 < t_0 < t \leq 1 \), we can replace \( t_0 \) with its limit value \( t_0 = 0 \) to obtain

\[
\| A_\varepsilon u(t) \|^2 \leq \left( D_{30} \| A_\varepsilon^{-\frac{1}{2}} P_\varepsilon f' \|_\infty^2 + D_{31} \| P_\varepsilon f \|_\infty^2 + D_{32} \sup_{0 \leq s \leq t} \| A_\varepsilon^{1/2} u(s) \|^6
\]

\[
+ D_{33} t^{-1} \| A_\varepsilon^{1/2} u_0 \|^2 \right) \times \exp \left( D_{29} \sup_{0 \leq s \leq t} \| A_\varepsilon^{1/2} u(s) \|^4 \right),
\]

for \( 0 < t \leq 1 \). For \( t \geq 1 \), one obtains from (4.15) instead that

\[
\| A_\varepsilon u(t) \|^2 \leq \left( D_{30} \| A_\varepsilon^{-\frac{1}{2}} P_\varepsilon f' \|_\infty^2 + D_{31} \| P_\varepsilon f \|_\infty^2 + D_{32} \sup_{t-1 \leq s \leq t} \| A_\varepsilon^{1/2} u(s) \|^6
\]

\[
+ 2D_{33} \sup_{t-1 \leq s \leq t} \| A_\varepsilon^{1/2} u(s) \|^2 \right) \times \exp \left( D_{29} \sup_{t-1 \leq s \leq t} \| A_\varepsilon^{1/2} u(s) \|^4 \right).
\]

The quantities \( K_2^2 \), \( K_3^2 \), and \( K_4^2 \) appearing in (2.35) are now readily identified from (4.20), (4.21), and (2.31). In the case that \( t \geq \hat{T}_1 + 1 \), where \( \hat{T}_1 \) is given by Theorem 1, we are able to use the bound (2.33) for \( \| A_\varepsilon^{1/2} u \|^2 \). As a result (4.21) implies that

\[
\| A_\varepsilon u(t) \|^2 \leq L_6^2 \overset{\text{def}}{=} \Gamma_2(L_6^2), \quad t \geq \hat{T}_1 + 1
\]

where

\[
\Gamma_2(\rho) = \left( D_{30} \| A_\varepsilon^{-\frac{1}{2}} P_\varepsilon f' \|_\infty^2 + D_{31} \| P_\varepsilon f \|_\infty^2 + 2D_{33} \rho + D_{32} \rho^3 \right) \exp(D_{29} \rho^2).
\]
Note that since $L^2_\varepsilon$ does not depend on the initial condition $u_0$, it follows from (4.22) that $L^2_\varepsilon$ is independent of $u_0$ as well. This completes the proof of (2.35) and (2.36).

Let us now assume that $u_0$ belongs to $D(A_\varepsilon)$. Then we deduce from (4.18) and (4.17) that, for $t \leq 1$ one has

$$
\|A_\varepsilon u(t)\|^2 \leq \left( D_{30} \|A_\varepsilon^{-\frac{1}{2}} P_\varepsilon f'\|_\infty^2 + D_{34} \|P_\varepsilon f\|_\infty^2 + D_{35} \|A_\varepsilon^{\frac{3}{2}} u(t)\|^6 + D_{36} \|A_\varepsilon^{\frac{1}{2}} u_0\|^2 \\
+ D_{37} \|A_\varepsilon u_0\|^2 \right) \times \exp \left( D_{29} \sup_{0 \leq s \leq t} \|A_\varepsilon^{\frac{1}{4}} u(s)\|^4 \right),
$$

(4.23)

where $D_{34} = 12 \nu^{-2}$, $D_{35} = 9 \nu^4(2\nu)^{-4}$, $D_{36} = 9 \nu^2$ and $D_{37} = 9(\nu^2 + 1)\nu^{-2}$. The quantities $K^2_\varepsilon$, $L^2_\varepsilon$, and $K^2_\varepsilon$ appearing in (2.37) are now readily identified in (4.23). This completes the proof of Theorem 2. $\square$

**Remarks.** 1. Depending on the choice of $\eta_2$, $\eta_4$ and $r$, one could have $L^2_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. If this happens, then one finds that $L^2_\varepsilon \to \infty$ as well. On the other hand, one can easily give conditions whereby both $L^2_\varepsilon$ and $L^2_\varepsilon$ are bounded for $0 < \varepsilon \leq 1$. We will be treating the latter situation in detail in Section 6 wherein we prove the upper semicontinuity of the attractors at $\varepsilon = 0$.

2. The decomposition $u = v + w$, as used in Section 3, together with the arguments used here, may lead to slight improvements in the estimates appearing in Theorem 2.

3. The proof of the Theorem 3, which we give next, is similar to the argument in Babin and Vishik (1989, Theorem 2, Section 6, Chapter 1) for the 2DNS.

**Proof of Theorem 3.** Let $f$ satisfy the hypothesis of Theorem 3, and let $u_0$ be any point in $B^0_\varepsilon \cup B^1_\varepsilon$. Let $\tau > 0$ be fixed. Without loss of generality we will assume that $0 < \tau \leq 1$.

Assume for the moment that there is a compact set $K^0(\tau)$ in $L^2(Q_3)$ such that

$$
A_\varepsilon S_\varepsilon(f, t) u_0 \in K^0(\tau), \quad t \geq \tau, \quad u_0 \in B^0_\varepsilon \cup B^1_\varepsilon,
$$

(4.24)

or equivalently

$$
S_\varepsilon(f, t)(B^0_\varepsilon \cup B^1_\varepsilon) \subset A_\varepsilon^{-1}(K^0(\tau)), \quad t \geq \tau.
$$

The continuity of $A_\varepsilon^{-1}$ assures us that $K(\tau) \overset{\text{def}}{=} A_\varepsilon^{-1}(K^0(\tau))$ is a compact set in $V_\varepsilon^2$. In order to prove (4.24) we will use the fact that if $K_1$, $K_2$ are compact sets in $L^2(Q_3)$, then $K_1 + K_2$ is compact in $L^2(Q_3)$.

Since $H^+(f)$ is assumed to be compact, the sets $Ev(H^+(f))$ and $K_1 \overset{\text{def}}{=} Ev(P_\varepsilon H^+(f))$ are compact sets in $L^2(Q_3)$ that satisfy

$$
P_\varepsilon g(t) \in K_1, \quad \text{for all } g \in H^+(f), \quad t \geq 0,
$$

(4.25)
see Section 2.11. Now the equation (2.5) can be rewritten as

\[(4.26) \quad \nu A_\varepsilon u(t) = P_\varepsilon f(t) - u'(t) - B_\varepsilon(u(t), u(t)), \quad t > 0.\]

Assume next that there are functions \(L_1(\tau)\) and \(L_2(\tau)\), defined for \(\tau > 0\), which depend only on \(\nu, \lambda_1, \) and \(\eta_i, i = 1, 2, 3, 4\), such that

\[(4.27) \quad \|A_\varepsilon^{\frac{1}{2}} u'(t)\|^2 \leq L_1(\tau), \quad t \geq \tau,
\]

and

\[(4.28) \quad \|A_\varepsilon^{\frac{1}{2}} B_\varepsilon(u(t), u(t))\|^2 \leq L_2(\tau), \quad t \geq \tau.\]

In this case there is a set \(K_2(\tau)\), which is bounded in \(V_\varepsilon^2\) and compact in \(H_\varepsilon \subset L^2(Q_3)\), such that

\[(4.29) \quad -(u'(t) + B_\varepsilon(u(t), u(t))) \in K_2(\tau), \quad t \geq \tau.\]

By combining (4.25), (4.26), and (4.29) one has

\[\nu A_\varepsilon u(t) \in K_1 + K_2(\tau), \quad t \geq \tau,\]

which implies (4.24).

From Constantin and Foias (1988) we note that there is a constant \(E_3\) such that\(^6\)

\[(4.30) \quad |b_\varepsilon(u^1, u^2, A_\varepsilon^{\frac{1}{2}} u^3)| \leq E_3\|A_\varepsilon u^1\|\|A_\varepsilon u^2\|\|u^3\|, \quad u^1, u^2 \in V_\varepsilon^2, \quad u^3 \in V_\varepsilon^1.\]

Now (4.30) implies that \(\|A_\varepsilon^{\frac{1}{2}} B_\varepsilon(u, u)\| \leq E_3\|A_\varepsilon u\|^2\), when \(u \in V_\varepsilon^2\). Hence (2.35) implies that (4.28) holds. In order to prove (4.27), we note that from (2.30), (2.35), (2.41), (4.14), and (4.15) one has

\[(4.31) \quad \|u'(t)\|^2 + \|A_\varepsilon u(t)\|^2 \leq K_8^2 \tau^{-1} \quad t \geq \tau,\]

where \(K_8^2\) is a positive constant depending only on \(\nu, \lambda_1, \) and \(\eta_i, i = 1, 2, 3, 4\).

Let us now take the scalar product of (4.1) with \(A_\varepsilon u'\) for \(t > 0\). We obtain

\[\frac{1}{2} \frac{d}{dt} \|A_\varepsilon^{\frac{1}{2}} u'\|^2 + \nu \|A_\varepsilon u'\|^2 \leq \nu^{-1}\|P_\varepsilon f'\|^2 + E_3\|A_\varepsilon u'\|^2 + |b_\varepsilon(u', u, A_\varepsilon u')| + |b_\varepsilon(u, u', A_\varepsilon u')|\]

However, using (4.3) and (4.30) one then obtains

\[(4.32) \quad \frac{d}{dt} \|A_\varepsilon^{\frac{1}{2}} u'(t)\|^2 + \nu \|A_\varepsilon u'(t)\|^2 \leq 2\nu^{-1} \left(\|P_\varepsilon f'\|^2 + 8E_3^2 K_8^2 \tau^{-1}\|A_\varepsilon^{\frac{1}{2}} u'(t)\|^2\right)^\]

\[^6\text{One can show that } E_3 \text{ is independent of } \varepsilon.\]
for \( t \geq \tau \), where \( E_4^{\tau} = (C_0^{\tau} + E_4^2)(\lambda_1^{-1} + 1) \). Now apply the uniform Gronwall inequality to (4.32) to obtain:

\[
\|A_\tau^{\frac{1}{2}}u'(t)\|^2 \leq \left( \frac{1}{t - \tau} \int_{\tau}^{t} \|A_\tau^{\frac{1}{2}}u'(s)\|^2 ds + 2\nu^{-1}\|P_\epsilon f'\|_\infty^2(t - \tau_1) \right) \\
\times \exp \left( 8\nu^{-1}E_4^2K_2^2\tau^{-1}(t - \tau_1) \right)
\]

for \( t \geq \tau \), where \( \tau_1 = \max(\tau, t - 1) \).

It remains to estimate the term \( \int_{\tau_1}^{t} \|A_\tau^{\frac{1}{2}}u'(s)\|^2 ds \). Integrating the inequality (4.12) between \( \tau_1 \) and \( t \) and using the estimates (2.30) and (4.31), we obtain:

\[
\frac{1}{t - \tau_1} \int_{\tau_1}^{t} \|A_\tau^{\frac{1}{2}}u'(s)\|^2 ds \leq \frac{2}{\nu^2}\|A_\tau^{-\frac{1}{2}}P_\epsilon f'\|_\infty^2 + \frac{1}{\nu}K_2^2\tau^{-1}(1 + D_29K_1^4).
\]

By combining (4.33) and (4.34), we deduce that (4.27) holds where

\[
L_1(\tau) = \left( \frac{2}{\nu^2}\|A_\tau^{-\frac{1}{2}}P_\epsilon f'\|_\infty^2 + \frac{1}{\nu}K_2^2\tau^{-1}(1 + D_29K_1^4) + \frac{2}{\nu}\|P_\epsilon f'\|_\infty^2 \right) \exp \left( 8\nu^{-1}E_4^2K_2^2\tau^{-1} \right),
\]

which completes the proof of Theorem 3. \( \square \)

5. The Reduced 3-Dimensional Theory: Theorems 10, 11, 12

We return to the study of

\[
\ddot{v}' + \nu A_\epsilon \ddot{v} + B_\epsilon(\ddot{v}, \ddot{v}) = MP_\epsilon f,
\]

the reduced 3D Navier-Stokes evolutionary equation. As explained in Section 2, when \((I - M)P_\epsilon f = 0\), then \(\{w = 0\}\) is a positively invariant set for (2.5), the dilated Navier-Stokes evolutionary equation. Since \(\dot{v}\) and \(MP_\epsilon f\) do not depend on \(x_3\), the terms in (5.1) do not depend on \(\epsilon\). Nevertheless the estimates derived in Sections 3 and 4 are valid for (5.1), see Ladyzhenskaya (1972).

We define \(\alpha \overset{\text{def}}{=} \|A_\tau^{\frac{1}{2}}\ddot{v}_0\|\) and \(\beta \overset{\text{def}}{=} \|MP_\epsilon f\|_\infty\). Let \(L_1, L_2, \ldots\) denote functions of \(\nu, \lambda_1\) and \(\beta\) which are independent of \(\alpha\), and \(M_1, M_2, \ldots\) denote functions of \(\nu, \lambda_1, \alpha\) and \(\beta\). Set \(d_1 = 27C^4_12^{-1}\nu^{-3}\), and let \(D_1, D_2, \ldots\) be defined as in Section 3.

From (3.58) we find that

\[
\|\ddot{v}(t)\|^2 \leq \lambda_1^{-1}e^{-\nu\lambda_1t}\alpha^2 + \lambda_1^{-2}\nu^{-2}\beta^2, \quad t \geq 0.
\]

As a result one has

\[
\|\ddot{v}(t)\|^2 \leq D_1(e^{-\nu\lambda_1t}\alpha^2 + \beta^2), \quad t \geq 0.
\]
Similarly (3.61) and (3.62) become
\[
\int_0^t \| A_{\varepsilon}^{\frac{1}{2}} \tilde{v} \|^2 ds \leq \nu^{-1} \lambda_1^{-1} e^{\nu \lambda_1 t} \alpha^2 + \lambda_1^{-1} \nu^{-2} \beta^2 \\
\leq D_{13} (e^{-\nu \lambda_1 t} \alpha^2 + \beta^2),
\]
for \(0 \leq t \leq 1\), while (3.63) gives
\[
\int_{t-1}^t \| A_{\varepsilon}^{\frac{1}{2}} \tilde{v} \|^2 ds \leq \nu^{-1} \lambda_1^{-1} e^{\nu \lambda_1 t} e^{-\nu \lambda_1 t} \alpha^2 + \lambda_1^{-1} \nu^{-2} \beta^2 \\
\leq D_{14} (e^{-\nu \lambda_1 t} \alpha^2 + \beta^2),
\]
for \(1 \leq t\). Consequently (3.65) and (3.66) become
\[
\int_0^t \| \tilde{v} \|^2 \| A_{\varepsilon}^{\frac{1}{2}} \tilde{v} \|^2 ds \leq e^{\nu \lambda_1} D_{12} D_{13} (e^{-\nu \lambda_1 t} \alpha^2 + \beta^2)^2, \quad 0 \leq t \leq 1,
\]
and
\[
\int_{t-1}^t \| \tilde{v} \|^2 \| A_{\varepsilon}^{\frac{1}{2}} \tilde{v} \|^2 ds \leq e^{\nu \lambda_1} D_{12} D_{14} (e^{-\nu \lambda_1 t} \alpha^2 + \beta^2)^2, \quad 1 \leq t.
\]

The analysis used to derive (3.28) now yields
\[
\frac{d}{dt} \| A_{\varepsilon}^{\frac{1}{2}} \tilde{v} \|^2 + \nu \| A_{\varepsilon} \tilde{v} \|^2 \leq d_1 \| \tilde{v} \|^2 \| A_{\varepsilon}^{\frac{1}{2}} \tilde{v} \|^2 \| A_{\varepsilon}^{\frac{1}{2}} \tilde{v} \|^2 + \frac{1}{\nu} \| MP_{\varepsilon} f \|^2_{\infty}.
\]
For \(0 \leq t < 1\), the Gronwall inequality for (5.4) implies that
\[
\| A_{\varepsilon}^{\frac{1}{2}} \tilde{v}(t) \|^2 \leq e^{G(t)} \left( \| A_{\varepsilon}^{\frac{1}{2}} \tilde{v}_0 \|^2 + \frac{1}{\nu} \| MP_{\varepsilon} f \|^2_{\infty} \right), \quad 0 \leq t < 1,
\]
where
\[
G(t) \leq d_1 \int_0^t \| \tilde{v} \|^2 \| A_{\varepsilon}^{\frac{1}{2}} \tilde{v} \|^2 ds.
\]
As a result, for \(0 \leq t < 1\) one finds
\[
\| A_{\varepsilon}^{\frac{1}{2}} \tilde{v}(t) \|^2 \leq (e^{\nu \lambda_1} e^{-\nu \lambda_1 t} \alpha^2 + \nu^{-1} \beta^2) \exp (d_1 e^{\nu \lambda_1} D_{12} D_{13} (e^{-\nu \lambda_1 t} \alpha^2 + \beta^2)^2),
\]
that is, (3.67) takes on the form
\[
\| A_{\varepsilon}^{\frac{1}{2}} \tilde{v}(t) \|^2 \leq D_{16} (e^{-\nu \lambda_1 t} \alpha^2 + \beta^2) \exp (D_{17} (e^{-\nu \lambda_1 t} \alpha^2 + \beta^2)^2), \quad 0 \leq t \leq 1.
\]
For $t \geq 1$, we apply the uniform Gronwall inequality to (5.4) to obtain
\[
\|A_t^{\frac{1}{2}} \tilde{v}(t)\|^2 \leq \left( \int_{t-1}^{t} \|A_s^{\frac{1}{2}} \tilde{v}\|^2 ds + \frac{1}{\nu} \|MP_{\epsilon} f\|_2^2 \right) \exp \left( d_1 \int_{t-1}^{t} \|\tilde{v}\|^2 \|A_s^{\frac{1}{2}} \tilde{v}\|^2 ds \right),
\]
and thanks to (5.2) and (5.3) we find that
\[
(5.6) \quad \|A_t^{\frac{1}{2}} \tilde{v}(t)\|^2 \leq \left( D_{14}(e^{-\nu \lambda_1 t} \alpha^2 + \beta^2) + \frac{1}{\nu} \beta^2 \right) \exp \left( d_1 e^{\nu \lambda_1 t} D_{12} D_{14}(e^{-\nu \lambda_1 t} \alpha^2 + \beta^2)^2 \right),
\]
for $t \geq 1$. From (5.5) and (5.6), we deduce that
\[
(5.7) \quad \|A_t^{\frac{1}{2}} \tilde{v}(t)\|^2 \leq D_{18}(e^{-\nu \lambda_1 t} \alpha^2 + \beta^2) \exp \left( D_{19}(e^{-\nu \lambda_1 t} \alpha^2 + \beta^2)^2 \right), \quad t \geq 0.
\]
By using the fact that
\[
e^{xD} \leq 1 + xDe^D, \quad 0 \leq x \leq 1, \quad D > 0,
\]
with $x = e^{-\nu \lambda_1 t}$, one can rewrite (5.7) as
\[
(5.8) \quad \|A_t^{\frac{1}{2}} \tilde{v}(t)\|^2 \leq M_1 e^{-\nu \lambda_1 t} + L_1, \quad t \geq 0,
\]
where
\[
(5.9) \quad \begin{cases}
L_1 = D_{18} \beta^2 \exp(D_{19} \beta^4) \\
M_1 = D_{18} \left( \alpha^2 \exp(D_{19} \beta^4) + D_{19}(\alpha^4 + 2\alpha^2 \beta^2)(\alpha^2 + \beta^2) \right) \exp(D_{19}(\alpha^2 + \beta^2)^2).
\end{cases}
\]
In summary, we have shown the following result:

**Theorem 10.** Let $\tilde{v}(t)$ be a solution of the reduce 3D Navier-Stokes evolutionary equation (5.1) with $\tilde{v}_0 \in MD(A_t^{\frac{1}{2}})$. Then there are functions
\[
M_1 = M_1(\|A_t^{\frac{1}{2}} \tilde{v}_0\|^2, \|MP_{\epsilon} f\|_{\infty}^2), \quad L_1 = L_1(\|MP_{\epsilon} f\|_{\infty}^2)
\]
given by (5.9), such that
\[
\|A_t^{\frac{1}{2}} \tilde{v}(t)\|^2 \leq M_1 e^{-\nu \lambda_1 t} + L_1, \quad t \geq 0.
\]

Notice that, by the definition (3.84), one has
\[
(5.10) \quad L_1 \leq \Gamma(\beta^2).
\]
Moreover, there exists a time $\tau_1 > 0$, such that one has $M_1 e^{-\nu \lambda_1 t} \leq L_1$ for $t \geq \tau_1$. Combining this with (5.8) and (5.10) we get
\[
\|A_t^{\frac{1}{2}} \tilde{v}(t)\|^2 \leq 2\Gamma(\beta^2), \quad t \geq \tau_1.
\]
If $\beta^2 \leq 2^{-2}$ and if $0 < \epsilon \leq \epsilon_0$, we then have
\[
\|A_t^{\frac{1}{2}} \tilde{v}(t)\|^2 \leq 2\Gamma(\eta_2^{-2}) \leq (4\eta_1^{-2} + k_1^2 \eta_3^{-4}), \quad t \geq \tau_1,
\]
that is, for $0 < \epsilon \leq \epsilon_0$ and $t \geq \tau_1$, $\tilde{v}(t)$ belongs to $B_1^2$.

Let us denote by $S_0(g, t)$ the mapping generated on $MD(A_t^{\frac{1}{2}})$ by the strong solutions of equation (5.1), where $g = MP_{\epsilon} f$. Arguing as in Section 4, one has the following regularity result.
Theorem 11. If

\[ M\mathcal{P}_f \in C^0([0, \infty); MH_\mathcal{C}) \cap L^{\infty}((0, \infty); MH_\mathcal{C}) \cap W^{1, \infty}((0, \infty); MD(A_{t^3})), \]

then there exist six positive functions \( K_1^* = K_1^*(\|A_{t^3} \bar{v}_0\|, \|MP_\mathcal{C} f\|_\infty) \) such that

\[
\begin{cases}
\|A_{t^3} \bar{v}(t)\|^2 \leq K_1^* + K_2^* \|A_{t^3} MP_\mathcal{C} f\|_\infty^2 + K_3^* t^{-1}, & \text{for } 0 < t \leq 1, \\
\|A_{t^3} \bar{v}(t)\|^2 \leq K_1^* + K_2^* \|A_{t^3} MP_\mathcal{C} f\|_\infty^2, & \text{for } t \geq 1,
\end{cases}
\]

Moreover, if \( \bar{v}_0 \) belongs to \( MD(A_{t^3}) \), then one has

\[
\|A_{t^3} \bar{v}(t)\|^2 \leq K_4^* + K_5^* \|A_{t^3} \bar{v}_0\|^2 + K_6^* \|A_{t^3} MP_\mathcal{C} f\|_\infty^2, \quad 0 \leq t \leq 1.
\]

Moreover, if \( f \) belongs to \( W(Q_3) \), then for any \( \tau > 0, S_0(g, t) \bar{v}_0 \) belongs to a compact set of \( MV^1_{t^2} \) for \( t \geq \tau \), provided \( \bar{v}_0 \) belongs to a bounded set of \( MV^1_{t^3} \). Furthermore, if \( f \) belongs to \( W(Q_3) \cap W^{1, \infty}((0, \infty); L^2(Q_3)) \) and is chosen so that \( H^+(f) \) is compact, then for any bounded set \( B \) of \( MV^1_{t^3} \) and any \( \tau > 0, S_0(g, t)B \) is included in a compact set \( K_0(\tau, B) \) for \( t \geq \tau \).

If \( H^+(f) \) is no longer compact, then, under the above hypotheses, we can prove that, for \( t > 0, S_0(g, t)B \) is included in a compact set \( \mathcal{K}_0(t, B) \) which may depend on \( t \).

Assume now that \( f \in W(Q_3) \cap W^{1, \infty}((0, \infty); L^2(Q_3)) \) is chosen so that \( H^+(f) \) is compact. Due to the Theorems 10 and 11, \( S_0(g, t) \) maps \( MD(A_{t^3}) \) into itself, is bounded dissipative in \( MD(A_{t^3}) \), and for \( t \geq t_1 > 0 \) is completely continuous in \( MD(A_{t^3}) \). Therefore, the skew-product semiflow \( \pi_0(\bar{v}, g, t) = (S_0(g, t)\bar{v}_0, g_t) \) defined in Section 2.11 admits a global compact attractor \( \mathfrak{A}_0(g) \) in \( MD(A_{t^3}) \times H^+(g) \), see Hale (1988, Theorem 2.4.7) for example. Since, by Theorem 11, \( S_0(g, t) \) is also bounded dissipative in \( MD(A_{t^3}) \) and for \( t \geq t_1 > 0 \) completely continuous in \( MD(A_{t^3}) \), \( \mathfrak{A}_0(g) \) is also the global compact attractor in \( MD(A_{t^3}) \times H^+(g) \). By the estimates (5.8), (5.9) and (5.10), we have:

\[
\mathfrak{A}_0(g) \subset \left\{ u = \bar{v} + w : \|A_{t^3} \bar{v}\|^2 \leq L_1 \leq \Gamma(\|MP_\mathcal{C} f\|_\infty^2), w = 0 \right\} \times \omega(g)
\]

\[
\subset B_{t^3} \times \omega(MP_\mathcal{C} f).
\]

We can also obtain estimates of \( \|A_{t^3} \bar{v}(t)\| \) in another way. By integrating (5.4) and using (5.3) and (5.8), we find that

\[
\int_{t-1}^{t} \|A_{t^3} \bar{v}\|^2 ds \leq M_3 e^{-\nu \lambda_1 t} + L_3
\]

where \( L_3 = \nu^{-1}(L_1 + d_1 e^{\nu \lambda_1} D_{12} D_{14} \beta^4 + \nu^{-1} \beta^2) \) and \( M_3 = \nu^{-1} e^{\nu \lambda_1} (M_1 + d_1 D_{12} D_{14} (a^4 + 2a^2 \beta^2)) \). As shown in Foias, et al (1988), the estimate (5.12) permits one to use the uniform Gronwall inequality to estimate \( \|A_{t^3} \bar{v}(t)\|^2 \) for \( t \geq 1 \). As a result one can readily prove the following result:
THEOREM 12. Assume that $M P_{\epsilon} f$ is in $L^\infty((0, \infty); M D(A_{\epsilon}^{1/2}))$ and that $\bar{v}_0$ belongs to $M D(A_{\epsilon})$. Then there are functions

$$M_2 = M_2(\|A_{\epsilon} \bar{v}_0\|, \|A_{\epsilon}^{1/2} M P_{\epsilon} f\|_\infty), \quad L_2 = L_2(\|A_{\epsilon}^{1/2} M P_{\epsilon} f\|_\infty),$$

(where $L_2$ is independent of the initial condition $\bar{v}_0$) such that the solution $\bar{v}(t)$ of (5.1) satisfies:

$$\|A_{\epsilon} \bar{v}(t)\|^2 \leq M_2 e^{-\nu \lambda_1 t} + L_2, \quad t \geq 0.$$

6. Properties of Attractors: Theorems 4, 5, 6, 7, 8, 9

We turn next to the proofs of Theorems 4-9 concerning the attractors for the Navier-Stokes equations. Let $\mathcal{B}_\epsilon^0$, $\mathcal{B}_\epsilon^1$, and $\mathcal{B}_\epsilon^2$ be given by (2.38), (2.39), and (2.40). By Lemmas 3.1 and 3.2, $\mathcal{B}_\epsilon^2$ is well-defined and is a bounded set in $V_\epsilon^1$.

Proof of Theorem 4. Set $\mathcal{U}_\epsilon^2 = \mathcal{B}_\epsilon^2 \times H^+(P_{\epsilon} f)$. For $u_0 \in V_\epsilon^1$ and $f \in W(Q_3) \cap W^{1,\infty}((0, \infty); L^2(Q_3))$ we let $\pi_\epsilon(u_0, P_{\epsilon} f, \tau) = (S_\epsilon(P_{\epsilon} f, \tau)u_0, (P_{\epsilon} f)_\tau)$ denote the skew-product semiflow generated by the strong solutions of the dilated Navier-Stokes evolutionary equation (2.5), see Section 2.11. Let $\mathfrak{A}_\epsilon = \omega(\mathcal{U}_\epsilon^2)$ be the $\omega$-limit set of $\mathcal{U}_\epsilon^2$ in $V_\epsilon^1 \times P_{\epsilon} W(Q_3)$, i.e.,

$$\mathfrak{A}_\epsilon \overset{\text{def}}{=} \bigcap_{\tau \geq 0} \text{Closure}_{V_\epsilon^1 \times P_{\epsilon} W(Q_3)} \left( \bigcup_{t \geq \tau} \pi_\epsilon(\mathcal{U}_\epsilon^2, t) \right).$$

It follows from (2.36) in Theorem 2 that for $\tau \geq \hat{T}_1 + 1$, the set

$$\bigcup_{t \geq \tau} S_\epsilon(P_{\epsilon} f, t) \mathcal{B}_\epsilon^2$$

lies in a bounded set in $V_\epsilon^2$, and thus a compact set in $V_\epsilon^1$. Since $H^+(P_{\epsilon} f)$ is compact, it then follows that

$$\text{Closure}_{V_\epsilon^1 \times P_{\epsilon} W(Q_3)} \left( \bigcup_{t \geq \tau} \pi_\epsilon(\mathcal{U}_\epsilon^2, t) \right)$$

is a nonempty compact set in $V_\epsilon^1 \times H^+(P_{\epsilon} f)$ for each $\tau \geq \hat{T}_1 + 1$. Consequently $\mathfrak{A}_\epsilon$ is a nonempty compact invariant set in $V_\epsilon^1 \times H^+(P_{\epsilon} f)$. Since

$$S_\epsilon(P_{\epsilon} f, t) \mathcal{B}_\epsilon^2 \subset \mathcal{B}_\epsilon^2, \quad t \geq 0,$$

$\mathcal{U}_\epsilon^2$ is a positively invariant neighborhood of $\mathfrak{A}_\epsilon$. Therefore $\mathfrak{A}_\epsilon$ is a local attractor for the strong solutions of (2.5) in $V_\epsilon^1 \times H^+(P_{\epsilon} f)$, and the basin of attraction satisfies $\mathcal{B}_\epsilon^2 \times H^+(P_{\epsilon} f) \subset B(\mathfrak{A}_\epsilon)$. □
Remarks 1. While the basin of attraction \( B(\mathcal{A}_\varepsilon) \) is a large set in \( V_\varepsilon^1 \times H^+ (P_\varepsilon f) \), we do not know whether \( B(\mathcal{A}_\varepsilon) = V_\varepsilon^1 \times H^+ (P_\varepsilon f) \). As a result we do not know whether \( \mathcal{A}_\varepsilon \) is the global attractor of \( \pi_\varepsilon \). The reason for this is that there may exist \( u_0 \in V_\varepsilon^1 \) such that the solution \( S_\varepsilon(P_\varepsilon f, t)u_0 \) is not globally regular. Because of this, the fact that Theorem 5 allows us to conclude that, \( \mathcal{A}_\varepsilon \) is the global attractor in the space of Leray solutions and \( B(\mathcal{A}_\varepsilon) = H_\varepsilon \times H^+ (P_\varepsilon f) \) is all the more surprising.

2. The fact, that the Leray solutions of (2.5) may not be unique, is not a concern from the point of view of the dynamics. One can overcome this problem by using the Bebutov flow, see Sell (1973).

Proof of Theorem 5. For any Leray solution of (2.5) we integrate (3.35) to obtain

\[
\frac{1}{t} \int_0^t \| A_\varepsilon^{1/2} u \|^2 \, ds \leq \frac{\nu^{-1}}{t} \| u_0 \|^2 + 2\nu^{-2} \left( \| A_\varepsilon^{-1/2} M P_\varepsilon f \|_\infty^2 + \| A_\varepsilon^{-1/2} (I - M) P_\varepsilon f \|_\infty^2 \right).
\]

for all \( t > 0 \). From (2.6), (2.22), (3.1), and (2.55) with \( \lambda^{-1} > 2\nu^{-2} \max(\lambda_1^{-1}, C_2^2) \) we obtain

\[
\frac{1}{t} \int_0^t \| A_\varepsilon^{1/2} u \|^2 \, ds \leq \frac{\nu^{-1}}{t} \| u_0 \|^2 + 2\nu^{-2} \left( \lambda_1^{-1} \| M P_\varepsilon f \|_\infty^2 + C_2^2 \varepsilon^2 \| (I - M) P_\varepsilon f \|_\infty^2 \right)
\]

\[
\leq \frac{\nu^{-1}}{t} \| u_0 \|^2 + 2\nu^{-2} \left( \lambda_1^{-1} \eta_2^{-2} + C_2^2 \varepsilon^2 + r \eta_4^{-2} \right)
\]

\[
\leq \frac{\nu^{-1}}{t} \| u_0 \|^2 + k \min(\eta_1^{-2}, \varepsilon^p \eta_3^{-2}),
\]

for all \( t > 0 \), and \( 0 < \varepsilon \leq \varepsilon_{10}(\lambda) \), where \( 0 < k < 1 \). Therefore for

\[
T = \frac{2\nu^{-1} \| u_0 \|^2}{(1 - k) \min(\eta_1^{-2}, \varepsilon^p \eta_3^{-2})}
\]

there is a \( t_0 \), \( 0 \leq t_0 < T \) such that

\[
\| A_\varepsilon^{1/2} u(t_0) \|^2 \leq \min(\eta_1^{-2}, \varepsilon^p \eta_3^{-2}), \quad 0 < \varepsilon \leq \varepsilon_{10}.
\]

For this \( t_0 \), it follows from (2.21) that

\[
\| A_\varepsilon^{1/2} u(t_0) \|^2 \leq \| A_\varepsilon^{1/2} u(t_0) \|^2 \leq \eta_1^{-2}
\]

\[
\| A_\varepsilon^{1/2} w(t_0) \|^2 \leq \| A_\varepsilon^{1/2} u(t_0) \|^2 \leq \varepsilon^p \eta_3^{-2}.
\]

Consequently for \( 0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_{10}) \), where \( \varepsilon_0 \) is given by Theorem 1, one has \( u(t_0) \in \mathcal{B}_\varepsilon^1 \). Theorem 1 then implies that \( u(t) \) is regular for all \( t \geq t_0 \). Since this argument applies for every \( u_0 \in H_\varepsilon \), it follows that for the Leray solutions the basin satisfies \( B(\mathcal{A}_\varepsilon) = H_\varepsilon \times H^+ (f) \). Consequently \( \mathcal{A}_\varepsilon \) is the global attractor for the Leray solutions. \( \square \)
Remark. The concept of a weak attractor for the 3DNS was studied in Foias and Témam (1987). It follows from Theorem 5 that the weak attractor coincides with \( \mathfrak{A}_\varepsilon \) for thin domains.

**Proof of Theorem 6.** The proof of the existence of a global attractor \( \mathfrak{A}_0 \) for the skew-product semiflow \( \pi_0(\cdot, g, t) \) in \( MD(A_0^{\frac{1}{2}}) \times H^+(g) \), where \( g = MP_\varepsilon f \), has been given in Section 5. The proof of (2.56) follows from (5.11) and (5.10).

Assume now that \( (I-M)P_\varepsilon f = 0 \) and that \( 0 < \varepsilon \leq \varepsilon_0 \). Clearly, any solution \( \tilde{\nu}(t) \) of (2.24) with initial data \( \tilde{\nu}_0 \in MD(A_0^{\frac{1}{2}}) \) is a solution of the dilated Navier-Stokes evolutionary equation (2.5). As \( \mathfrak{A}_0(g) \) is included in \( B_1^1 \times \omega(MP_\varepsilon f) \subset B(\mathfrak{A}_\varepsilon) \) and is an invariant set for equation (2.5), it follows that \( \mathfrak{A}_0(g) \subset \mathfrak{A}_\varepsilon \). Let us now show that \( \mathfrak{A}_\varepsilon \subset \mathfrak{A}_0(g) \). Since \( (I-M)P_\varepsilon f = 0 \), inequality (3.53) takes on the form

\[
\| A_\varepsilon^{\frac{1}{2}} w(t) \|^2 \leq k^2 \varepsilon^{2+\eta_4} \exp \left( \frac{-\nu C_5^{-2} \varepsilon^{-2} t}{2} \right), \quad t \geq 0,
\]

provided \( u_0 \in B_1^1 \). This implies that the \( u \)-component of the \( \omega \)-limit set of \( B_1^1 \times H^+(P_\varepsilon f) \) belongs to the set of functions in \( B_1^1 \) which are independent of the variable \( x_3 \), i.e., \( \mathfrak{A}_\varepsilon \subset \mathfrak{A}_0(g) \). \( \square \)

**Proof of Theorems 7 and 8.** We begin with Theorem 7. Let us consider a sequence of positive numbers \( \varepsilon_n \rightarrow 0 \) as \( n \rightarrow \infty \). Let \( \mathcal{F} \) be any positively invariant compact subset of \( W(Q_3) \cap W^{1,\infty}((0, \infty), L^2(Q_3)) \), and let \( f_n \) be a sequence of functions \( f_n \in \mathcal{F} \) that satisfies

\[
\lim_{n \rightarrow \infty} \| f_n - f_0 \|_{\infty} = 0,
\]

where \( f_0 \in MF \). Then each of the positive hulls \( H^+(f_n) \) and \( H^+(f_0) \) are compact sets in \( \mathcal{F} \). We set \( g_n = P_\varepsilon_n f_n \) and \( g_0 = MP_\varepsilon f_0 \). According to the comments made in Section 2.6, \( P_\varepsilon f_0(t) \) belongs to \( MH_\varepsilon \) for every \( t \), and consequently

\[
g_0 = MP_\varepsilon f_0 = P_\varepsilon f_0 = \begin{pmatrix} f_{01} \\ f_{02} \\ f_{03} \end{pmatrix}
\]

where \( f_0 = (f_{01}, f_{02}, f_{03}) \). It follows from (6.1) and (6.2) and the fact that \( P_\varepsilon \) is a projection that

\[
\lim_{n \rightarrow \infty} \| MP_\varepsilon f_n - g_0 \|_{\infty} = 0
\]

and

\[
\lim_{n \rightarrow \infty} \| (I-M)P_\varepsilon f_n \|_{\infty} = 0.
\]
The last two conditions can be written as

\begin{equation}
\lim_{n \to \infty} \| P_n f_n - g_0 \|_\infty = 0. \tag{6.4}
\end{equation}

For every \( n \), we consider the dilated Navier-Stokes evolutionary equation, i.e.,

\begin{equation}
\mathcal{A}^{\epsilon_n} u' + v A^{\epsilon_n} u + B^{\epsilon_n}(u, u) = g_n. \tag{6.5}
\end{equation}

Let \( S^{\epsilon_n}(g_n, t) u_{0n} = u_n(t) = v_n(t) + w_n(t) \) denote the strong solution of the equation (6.5) with initial data \( u_{0n} \) in \( V_{\epsilon_n}^1 \). We also consider the reduced 3D Navier-Stokes evolutionary equation

\begin{equation}
\tilde{\mathcal{A}}^{\epsilon_n} \tilde{v}' + \nu A_0 \tilde{v} + B_0(\tilde{v}, \tilde{v}) = g_0, \tag{6.6}
\end{equation}

with initial data \( \tilde{v}(0) = \tilde{v}_0 \) in \( V_0^1 \). Let \( S_0(g_0, t) \tilde{v}_0 \) denote the strong solution of (6.6) with initial condition \( \tilde{v}_0 \in V_0^1 \). It follows from (6.4), that there exist an integer \( n_1 \) and a positive constant \( E_0 \) such that

\[ \max(\|g_0\|_\infty^2, \|M g_n\|_\infty^2, \|g_n\|_\infty^2) \leq E_0, \quad n \geq n_1. \]

According to Theorem 1 and Lemma 3.2, every solution of (6.5) with initial data in the bounded set \( B_{\epsilon_n}^0 \) ultimately enters into the bounded set \( B_{\epsilon_n}^1 \) as well as the bounded set \( B_{\epsilon_n}^3 \) where

\[ B_{\epsilon_n}^3 \overset{\text{def}}{=} \{ u = v + w \in B_{\epsilon_n}^1 : \| A^{\epsilon_n}_n v \|_2^2 \leq \Gamma(E_0), \| A^{\epsilon_n}_n w \|_2^2 \leq k_\frac{2}{2}^{\epsilon_n^2 + r \eta_4^2} \}. \]

In particular, the (local) attractor \( \mathcal{A}_{\epsilon_n} \) of (6.5), see Theorem 4, is included in \( B_{\epsilon_n}^3 \times \omega(g_n) \), for \( n \geq n_1 \). Likewise, due to the property (5.11), the global attractor \( \mathcal{A}_0 = \mathcal{A}_0(g_0) \) of (6.6) is included in the bounded set \( B_0^3 \times \omega(g_0) \), where

\[ B_0^3 = \{ u = v + w \in V_{\epsilon_n}^1 : \| A^{\epsilon_n}_n v \|_2^2 \leq \Gamma(E_0), w = 0 \}. \]

Note that, for every \( n \), one has \( M B_{\epsilon_n}^3 = B_0^3 \). Now define \( E_1 = \Gamma(E_0) \).

For \( \tau \in R \), we let \( f_{n, \tau}, g_{n, \tau} \) and \( g_{0, \tau} \), denote the translate of \( f_n \), \( g_n \), and \( g_0 \), see Section 2.11. Then from (6.4) it follows that for every \( \delta > 0 \) there is an integer \( n_2 \geq n_1 \) such that

\[ \| g_{n, \tau} - g_{0, \tau} \|_\infty \leq \frac{\delta}{2}, \quad n \geq n_2, \quad \tau \geq 0. \]

Furthermore, there is a \( T \geq 0 \) such that

\[ \text{dist}_{W(Q_3)}(g_{0, \tau}, \omega(g_0)) \leq \frac{\delta}{2}, \quad \tau \geq T. \]
It then follows that

\[(6.7)\quad \text{dist}_{W(Q_3)}(g_n, \tau, \omega(g_0)) \leq \delta, \quad n \geq n_2, \quad \tau \geq T,\]

which implies that the attractors \(\omega(g_n)\) are upper semicontinuous as \(n \to \infty\).

In the remainder of the argument we shall use the weaker condition (6.3) in place of (6.1). As a result the argument now applies both to Theorems 7 and 8.

We claim that there is an integer \(n_3 \geq n_2\) and two positive constants \(k_3, E_2\), with \(E_2 \geq \max(E_0, E_1)\), such that

\[(6.8)\quad \|A_{\frac{1}{\varepsilon}} v_n(t)\|^2 \leq E_2, \quad \|A_{\frac{1}{\varepsilon}} w_n(t)\|^2 \leq k_3^2 e^{2+\varepsilon} \eta_4^{-2},\]

for \(\varepsilon = \varepsilon_n, n \geq n_3\) and \(t \geq 0\), provided \(u_{0n} \in B_3^2\). Furthermore, one has

\[(6.9)\quad \|A_{\frac{1}{\varepsilon}} \bar{v}(t)\|^2 \leq E_2,\]

for \(t \geq 0\), provided \((\bar{v}_0, 0) \in B_3^3\). Indeed (6.8) and (6.9) are immediate consequences of (3.82), (3.83) and Theorem 10.

Now we want to compare the orbits of the dilated Navier-Stokes equation (6.5) with those of the reduced 3D Navier-Stokes equation (6.6) when \(u_{0n}\) belongs to \(B_3^2\). To this end, we consider the equation satisfied by \(z_n(t) = v_n(t) - \bar{v}(t)\) where \(z_n(0) = 0\) (i.e., \(v_n(0) = \bar{v}(0) = v_{0n}\), \(w_n(0) = w_{0n}\), and \(u_{0n} = v_{0n} + w_{0n}\) belongs to \(B_3^3\). We have

\[(6.10)\quad z'_n + \nu A_{\varepsilon_n} z_n = (Mg_n - g_0) - M(B_{\varepsilon_n}(u_n, u_n) - B_{\varepsilon_n}(\bar{v}, \bar{v})),\]

and \(Mz_n = 0\). Taking the inner product of (6.10) by \(A_{\varepsilon_n} z_n\), we obtain

\[(6.11)\quad \frac{1}{2} \frac{d}{dt} \|A_{\frac{1}{\varepsilon_n}} z_n\|^2 + \nu \|A_{\varepsilon_n} z_n\|^2 \leq \|Mg_n - g_0\|_{\infty} \|A_{\varepsilon_n} z_n\| + \|b_{\varepsilon_n}(v_n, v_n, A_{\varepsilon_n} z_n) - b_{\varepsilon_n}(\bar{v}, \bar{v}, A_{\varepsilon_n} z_n)\| + \|b_{\varepsilon_n}(w_n, w_n, A_{\varepsilon_n} z_n)\|\]

for \(t \geq 0\). However we can write

\[(6.12)\quad \|b_{\varepsilon_n}(v_n, v_n, A_{\varepsilon_n} z_n) - b_{\varepsilon_n}(\bar{v}, \bar{v}, A_{\varepsilon_n} z_n)\| = |b_{\varepsilon_n}(z_n, \bar{v}, A_{\varepsilon_n} z_n) + b_{\varepsilon_n}(v_n, z_n, A_{\varepsilon_n} z_n)|\]

From inequality (8.13) we obtain

\[(6.13)\quad |b_{\varepsilon_n}(z_n, \bar{v}, A_{\varepsilon_n} z_n)| \leq c_{12} \|z_n\|_{L^\infty(Q_3)} \|A_{\frac{1}{\varepsilon_n}} \bar{v}\| \|A_{\varepsilon_n} z_n\|.

Since \(z_n\) does not depend on the variable \(x_3\), we can apply the following Gagliardo-Nirenberg type inequality:

\[(6.14)\quad \|z_n\|_{L^\infty(Q_3)} \leq c \|z_n\|_{H^2(Q_3)}^{\frac{1}{2}} \|z_n\|_{L^2(Q_3)}^{\frac{1}{2}}.

60
see Friedman (1964). The estimates (6.13), (6.14) and (2.18) imply that

\[
(6.15) \quad |b_{\epsilon_n}(z_n, \bar{v}, A_{\epsilon_n} z_n)| \leq C_{10} \|z_n\|^{\frac{1}{2}} \|A_{\epsilon_n}^{\frac{1}{2}} \bar{v}\| \|A_{\epsilon_n} z_n\|^{\frac{3}{2}},
\]

for some constant $C_{10}$. From (6.11), (6.12), (6.15), (3.2), and (3.3), we find that

\[
(6.16) \quad \frac{1}{2} \frac{d}{dt} \|A_{\epsilon_n}^{\frac{1}{2}} z_n\|^2 + \nu \|A_{\epsilon_n} z_n\|^2 \leq \|A_{\epsilon_n} z_n\| \left( \|M g_n - g_0\|_{\infty} + C_{\epsilon_n}^{\frac{1}{2}} \|A_{\epsilon_n}^{\frac{1}{2}} w\|^2 \|A_{\epsilon_n} w\|^{\frac{1}{2}} \right)
+ C_1 \|v_n\|^{\frac{1}{2}} \|A_{\epsilon_n}^{\frac{1}{2}} v_n\|^{\frac{1}{2}} \|A_{\epsilon_n} z_n\|^{\frac{1}{2}} \|A_{\epsilon_n} z_n\|^{\frac{3}{2}} + C_{10} \|z_n\|^{\frac{1}{2}} \|A_{\epsilon_n}^{\frac{1}{2}} \bar{v}\| \|A_{\epsilon_n} z_n\|^{\frac{3}{2}}.
\]

for $t \geq 0$. Using the Young inequality, we derive from (6.16) that one has

\[
\frac{d}{dt} \|A_{\epsilon_n}^{\frac{1}{2}} z_n\|^2 + \nu \|A_{\epsilon_n} z_n\|^2 \leq \frac{4}{\nu} \|M g_n - g_0\|_{\infty}^2 + \frac{4}{\nu} C_{\epsilon_n}^{\frac{1}{2}} \|A_{\epsilon_n}^{\frac{1}{2}} w\|^3 \|A_{\epsilon_n} w\|
+ \frac{108}{\nu^3} \left( C_1^4 \|v_n\|^2 \|A_{\epsilon_n}^{\frac{1}{2}} v_n\|^2 \|A_{\epsilon_n} z_n\|^2 + C_{10}^4 \|z_n\|^2 \|A_{\epsilon_n}^{\frac{1}{2}} \bar{v}\|^4 \right)
\]

for $t \geq 0$, or, by (6.8) and (6.9),

\[
(6.17) \quad \frac{d}{dt} \|A_{\epsilon_n}^{\frac{1}{2}} z_n\|^2 + \nu \|A_{\epsilon_n} z_n\|^2 \leq \frac{4}{\nu} \|M g_n - g_0\|_{\infty}^2 + \frac{4}{\nu} C_{\epsilon_n}^{2} \|A_{\epsilon_n}^{\frac{1}{2}} w\|^3 \|A_{\epsilon_n} w\|
+ D_{24} E_{2}^2 \|A_{\epsilon_n}^{\frac{1}{2}} z_n\|^2,
\]

where $D_{24} = 108 \lambda_1^{-1} \nu^{-3} (C_4 + C_{10})$.

Integrating the inequality (6.17) from 0 to $t$ and using a Gronwall inequality, we deduce that

\[
\|A_{\epsilon_n}^{\frac{1}{2}} z_n(t)\|^2 \leq \frac{4}{\nu} \left( t \|M g_n - g_0\|_{\infty}^2 + C_{\epsilon_n}^{2} \int_0^t \|A_{\epsilon_n}^{\frac{1}{2}} w(s)\|^3 \|A_{\epsilon_n} w(s)\| \, ds \right) \exp(D_{24} E_{2}^2 t).
\]

for $t \geq 0$. Arguing as in the proof of (3.54), we see that there exists a positive constant $D_{25}$ such that

\[
\frac{4}{\nu} C_{\epsilon_n}^{2} \int_0^t \|A_{\epsilon_n}^{\frac{1}{2}} w(s)\|^3 \|A_{\epsilon_n} w(s)\| \, ds \leq D_{25} (1 + t) \epsilon_n^{4 + 2r} \eta_4^{-4}, \quad t \geq 0.
\]

Finally, we obtain

\[
(6.18) \quad \|A_{\epsilon_n}^{\frac{1}{2}} z_n(t)\|^2 \leq \left( \frac{4}{\nu} t \|M g_n - g_0\|^2 + D_{25} (1 + t) \epsilon_n^{4 + 2r} \eta_4^{-4} \right) \exp(D_{24} E_{2}^2 t).
\]

Thanks to hypothesis (2.59) and condition (6.3), we infer, from (6.18) that, for any positive numbers $\delta$ and $T$, there exists an integer $n_4$, $n_4 \geq n_3 \geq 0$, such that

\[
(6.19) \quad \|A_{\epsilon_n}^{\frac{1}{2}} z_n(T)\|^2 \leq \frac{\delta}{3}, \quad n \geq n_4.
\]
Let $\delta$ be a positive number. Since $\mathcal{A}_0$ is the global attractor of (6.6), there exists a positive time $\tau_0 \equiv \tau_0(\delta)$ such that

\begin{equation}
\pi_0(\mathcal{B}_0^3, H^+(g_0), t) \subset \mathcal{N}_{\mathcal{V}_0^1 \times W(Q_3)}(\mathcal{A}_0, \frac{\delta}{3}), \quad t \geq \tau_0,
\end{equation}

where $\mathcal{N}_{\mathcal{V}_0^1 \times W(Q_3)}(\mathcal{A}_0, \alpha)$ denotes the $\alpha$-neighborhood of $\mathcal{A}_0$ in $\mathcal{V}_0^1 \times W(Q_3)$. Using the properties (6.8) and (6.19), as well as the hypothesis (2.59), we see that exists an integer $n_0$, $n_0 \geq n_4$, such that

\begin{equation}
\|A_{e_n}^3(v_0(\tau_0) - \bar{v}(\tau_0))\|^2 + \|A_{e_n}^4 w_n(\tau_0)\|^2 \leq \frac{2\delta}{3}, \quad n \geq n_0,
\end{equation}

where $u_n(t) = v_n(t) + w_n(t) = S_{e_n}(g_n, t)u_{0n}$, $\bar{v}(t) = S_0(g_0, t)Mu_{0n}$, and $u_{0n} \in \mathcal{B}_{e_n}^3$. From (6.7), (6.20) and (6.21), we infer that

\begin{equation}
\pi_{e_n}(\mathcal{B}_{e_n}^3, H^+(g_n), \tau_0) \subset \mathcal{N}_{\mathcal{V}_{e_n}^1 \times W(Q_3)}(\mathcal{A}_0, \delta), \quad n \geq n_0,
\end{equation}

and, in particular,

\begin{equation}
\pi_{e_n}(\mathcal{A}_{e_n}, \tau_0) \subset \mathcal{N}_{\mathcal{V}_{e_n}^1 \times W(Q_3)}(\mathcal{A}_0, \delta), \quad n \geq n_0.
\end{equation}

Due to the invariance property of the attractors $\mathcal{A}_{e_n}$, we at once deduce the upper semi-continuity result (2.61) from (6.22). This completes the proof of Theorems 7 and 8. \[\]

**Proof of Theorem 9.** We shall only give a sketch of the proof of Theorem 9. We keep here the notation of the proof of Theorem 7.

According to Theorems 2 and 11, every solution of (6.5), for $n \geq n_1$, (resp. of (6.6)) with initial data in the bounded set $\mathcal{B}_0^0$ (resp. in any bounded set of $\mathcal{V}_0^1$) ultimately enters into the bounded set $\mathcal{B}_{e_n}^4$ (resp. $\mathcal{B}_0^4$) where

\begin{equation}
\mathcal{B}_{e_n}^4 \overset{\text{def}}{=} \{u \in \mathcal{V}_{e_n}^2 : \|A_{e_n} u\|^2 \leq E_4\}
\end{equation}

(resp. $\mathcal{B}_0^4 = \{v \in \mathcal{V}_0^2 : \|A_0 v\|^2 \leq E_4\}$) where $E_4$ is a positive constant independent of $n$. Note that, for every $n$, $M\mathcal{B}_{e_n}^4 = \mathcal{B}_0^4$. In particular, the (local) attractor $\mathcal{A}_{e_n}$ of (6.5) (resp. the global attractor $\mathcal{A}_0$ of (6.6)) is included in the bounded set $\mathcal{B}_{e_n}^4 \times \omega(g_n)$ (resp. $\mathcal{B}_0^4 \times \omega(g_0)$). Furthermore, due to Theorems 2 and 11, there exist an integer $n_5$, $n_5 \geq n_4$, and a positive constant $E_5$, with $E_5 \geq \max(E_4, E_2)$ such that, for $t \geq 0$,

\[\|A_{e_n} u_n(t)\|^2 \leq E_5, \quad \text{for } n \geq n_5, u_{0n} \in \mathcal{B}_{e_n}^3 \cap \mathcal{B}_{e_n}^4,\]

and

\[\|A_0 \bar{v}(t)\|^2 \leq E_5, \quad \text{for } \bar{v}_0 \in \mathcal{B}_0^3 \cap \mathcal{B}_0^4.\]
Let $\delta$ be a positive number. Since $\mathcal{A}_0$ is the global attractor of (6.6) in $V_0^2 \times W(Q_3)$, there exists a positive time $\tau_1 \equiv \tau_1(\delta)$, with $\tau_1 > 1$ for instance, such that
\begin{equation}
\pi_0(\mathcal{B}_0^3 \cap \mathcal{B}_0^4, H^+(g_0), t) \subset N_{V_0^2 \times W(Q_3)}(\mathcal{A}_0, \delta^3), \quad t \geq \tau_1.
\end{equation}

As in the proof of Theorem 7, due to the properties (6.7) and (6.23), the upper semicontinuity result (2.63) is valid if we show that there exists an integer $n_6$, $n_6 \geq n_5$, such that, for $n \geq n_6$, one has
\begin{equation}
\frac{2\delta}{3}
\end{equation}
where $u_n(t) = S_{\epsilon_n}(g_n, t)u_{0n}, \bar{v}(t) = S_0(g_0, t)\bar{v}_0$ and $u_{0n} \in \mathcal{B}_{\epsilon_n}^3 \cap \mathcal{B}_{\epsilon_n}^4$.

Note that $z_n \equiv u_n - \bar{v}$ is the solution of the equation
\begin{equation}
z_n' + \nu A_{\epsilon_n} z_n = (g_n - g_0) - (B_{\epsilon_n}(u_n, u_n) - B_{\epsilon_n}(\bar{v}, \bar{v})),
\end{equation}
with initial condition $z_n(0) = u_{0n} - Mu_{0n} = w_{0n}$. The proof of the estimate (6.24) follows the lines of the proof of the estimates (2.35), (2.36) of Theorem 2. (See Section 4, Steps 1 to 4). As the proof of (6.24) is rather long and completely similar to the proof of Theorem 2, we omit the details. Let us just point out that, as in Section 4, we use the following auxiliary equation:
\begin{equation}
\frac{d}{dt} z_n' + \nu A_{\epsilon_n} z_n' = (g_n' - g_0) - (B_{\epsilon_n}(u_n', u_n) + B_{\epsilon_n}(u_n, u_n') - B_{\epsilon_n}(\bar{v}', \bar{v}) - B_{\epsilon_n}(\bar{v}, \bar{v}')),
\end{equation}
with initial condition
\begin{equation}
z_n'(0) = (g_n - g_0)(0) - (B_{\epsilon_n}(u_{0n}, u_{0n}) - B_{\epsilon_n}(v_{0n}, v_{0n})) - \nu A_{\epsilon_n}(u_{0n} - v_{0n}).
\end{equation}

7. Remarks on Other Boundary Conditions: Theorems 13, 14, 15

In this section, we assume that $\Omega_\epsilon = Q_2 \times (0, \epsilon)$, where $Q_2$ is a bounded domain in $\mathbb{R}^2$ with a boundary of class $C^s$, $s \geq 2$. The smoothness hypothesis $s \geq 2$ is made to avoid any problem of regularity of the solutions of the corresponding stationary Stokes equation. As in Section 2, we set $Q_3 = Q_2 \times (0, 1)$ and we use the change of variables $(y_1, y_2, y_3) \mapsto (x_1, x_2, x_3)$ where $x_i = y_i$, $i = 1, 2$, and $x_3 = \epsilon^{-1} y_3$. This change of variables sends $\Omega_\epsilon$ onto $Q_3$.

7.1 Mixed Periodic-Dirichlet Boundary Conditions. We are interested here in solutions of the Navier-Stokes evolutionary equation (2.5) that satisfy periodic boundary conditions on $\Gamma_1 = Q_2 \times \{0\} \cup Q_2 \times \{\epsilon\}$ and Dirichlet boundary conditions on $\Gamma_2 = \partial Q_2 \times (0, \epsilon)$. As before we use the operator $J_\epsilon$ of Section 2.1. Let $H_\epsilon$ (respectively $V_\epsilon^1$)
denote the closure in $L^2(Q_3)$ (respectively $H^1(Q_3)$) of those smooth functions $u$ that satisfy periodic boundary conditions on $\Gamma_1$, Dirichlet boundary conditions on $\Gamma_2$ and $\nabla \cdot u = 0$ in $Q_3$. We let denote by $P_{\epsilon}$ the orthogonal projection of $L^2(Q_3)$ onto $H_{\epsilon}$. By applying $P_{\epsilon}$ to (2.4), we obtain (as in Section 2.2) the nonlinear evolutionary equation (2.5) on $H_{\epsilon}$, where $u \in H_{\epsilon}, A_{\epsilon}u = -P_{\epsilon}\Delta u$. We set $V_{\epsilon}^2 = D(A_{\epsilon})$. Using regularity results (see Dauge (1984) and the references therein), one can show that $V_{\epsilon}^2 = V_{\epsilon}^1 \cap H^2(Q_3)$. One also has $V_{\epsilon}^1 = D(A_{\epsilon}^1)$. Using the classical Poincaré inequality, one easily shows that the inequalities (2.17) still hold. Likewise, thanks to the estimates (2.17) and to regularity results in Dauge (1984), one can prove the inequalities (2.18). Like in Section 2.4, we introduce the projection $M$. All the properties given in Section 2.4 are still true. In particular, if $u \in D(A_{\epsilon})$, we have

$$
\begin{align*}
MA_{\epsilon}u = MA_{\epsilon}Mu = A_{\epsilon}Mu \\
(I - M)A_{\epsilon}u = (I - M)A_{\epsilon}(I - M)u
\end{align*}
$$

(7.1)

The crucial estimate (2.22) still holds (see Hale and Raugel (1989A)). The above property (7.1) allows us to write the equation (2.5) as the system (2.23) of two equations in $v = Mu$ and $w = (I - M)u$.

As in Section 2.6, we obtain a reduced 3D Navier-Stokes evolutionary equation which is given by (2.24). The reduced 3D Navier-Stokes evolutionary equation incorporates the 2DNS equation on $Q_2$ with homogeneous Dirichlet boundary conditions. In order to see this, we let $L^2(Q_2,R^3)$ denote the $L^2$-space of 2-dimensional vector fields $m = (m_1,m_2)$ which depend on $(x_1, x_2) \in Q_2$, and let $H(Q_2)$ denote the closure in $L^2(Q_2,R^2)$ of the smooth functions $u$ that satisfy $D_1m_1 + D_2m_2 = 0$ on $Q_2$. Finally we let $P_2$ denote the orthogonal projection of $L^2(Q_2,R^2)$ onto the space $H(Q_2)$. Then $P_{\epsilon}$ and $P_2$ satisfy the relations described in Section 2.6. Furthermore, $\bar{v}$ is a solution of the reduced 3D Navier-Stokes evolutionary equation (2.24) if and only if $m = (\bar{v}_1, \bar{v}_2)$ is a solution of the 2D Navier-Stokes evolutionary equation

$$
\frac{d}{dt} m - \nu P_2(D_1^2 + D_2^2)m + P_2(m \cdot \nabla)m = (g_1, g_2)
$$

and $\bar{v}_3$ is a solution of the linear equation

$$
\frac{d}{dt} \bar{v}_3 - \nu(D_1^2 + D_2^2)\bar{v}_3 + (\bar{v}_1D_1 + \bar{v}_2D_2)\bar{v}_3 = g_3
$$

where $g = (g_1, g_2, g_3) = MP_{\epsilon}f$. With the changes made above in the definitions of the spaces $H_{\epsilon}, V_{\epsilon}^1, V_{\epsilon}^2$ and the operators $P_{\epsilon}$ and $P_2$, all the results given in Sections 2 to 6 (see also Section 8) are still true in the case where we have periodic boundary conditions on $\Gamma_0 \cup \Gamma_1$ and homogeneous Dirichlet boundary conditions on $\Gamma_2$. Moreover the proofs given in Sections 3 to 6 are exactly the same.
7.2 Homogeneous Dirichlet Boundary Conditions. This case is quite different from the cases previous by studied. Here we consider the Navier-Stokes equations (2.1) on \( \Omega_\epsilon \) (resp. (2.4) on \( Q_3 \)) with homogeneous Dirichlet boundary conditions on \( \partial \Omega_\epsilon \) (resp. on \( \partial Q_3 \)). Here we introduce the spaces

\[
H_\epsilon = \{ u \in L^2(Q_3) : \nabla_\epsilon \cdot u = 0, u \cdot n|_{\partial Q_3} = 0 \}
\]

and

\[
V_\epsilon^1 = \{ u \in H_0^1(Q_3) : \nabla_\epsilon \cdot u = 0 \},
\]

and we let denote by \( P_\epsilon \) the orthogonal projection of \( L^2(Q_3) \) onto \( H_\epsilon \). By applying \( P_\epsilon \) to (2.4), we obtain (as in Section 2.2) the dilated Navier-Stokes evolutionary equation (2.5) where \( u = P_\epsilon u \in H_\epsilon, A_\epsilon u = -P_\epsilon \Delta_\epsilon u \) (with homogeneous Dirichlet boundary conditions). One has \( V_\epsilon^1 = D(A_\epsilon^{\frac{1}{2}}) \) and we set \( V_\epsilon^2 = D(A_\epsilon) \). Using the regularity results given in Dauge (1984, 1989), one obtains, that \( V_\epsilon^2 = V_\epsilon^1 \cap H^2(Q_3) \). Using the classical Poincaré inequality, one shows at once that the estimates (2.17) still hold. Arguing as in Hale and Raugel (1989A; Corollary 2.8) one shows that

\[
\| A_\epsilon^i u \| \leq C_{11} \epsilon \| A_\epsilon^{i+\frac{1}{2}} u \|, \quad \text{for } i = 0, 1,
\]

where \( C_{11} \) is a positive constant that does not depend on \( \epsilon \). Using the inequality (7.2) several times and the regularity results of Dauge (1984, 1989), one proves that

\[
C_6(\| u \|_{H^2(Q)} + \epsilon^{-1} \| D_3 u \| + \epsilon^{-1} \| D_1 D_3 u \| + \epsilon^{-1} \| D_2 D_3 u \| + \epsilon^{-2} \| D_3^2 u \|) \leq \epsilon^{-1} \| A_\epsilon u \|,
\]

and

\[
\| A_\epsilon u \| \leq C_7(\| u \|_{H^2(Q_3)} + \epsilon^{-1} \| D_3 u \|_{H^1(Q_3)} + \epsilon^{-2} \| D_3^2 u \|).
\]

The inequalities (7.3) and (8.20) imply that

\[
| b_\epsilon(u^1, u^2, u^3) | \leq C_{12} \| A_\epsilon^{\frac{1}{2}} u^1 \| \| A_\epsilon^{\frac{1}{2}} u^2 \| \| A_\epsilon^{\frac{1}{2}} u^3 \|,
\]

for \( u^1 \in D(A_\epsilon^{\frac{1}{2}}), u^2 \in D(A_\epsilon), \) and \( u^3 \in H_\epsilon \). We now state the following results which do not use the decomposition \( u = Mu + (I - M)u \). We assume that \( 0 < \epsilon \leq 1 \).

**Theorem 13.** Let \( p \) and \( r \) be two real numbers satisfying \(-1 < p < 0 \) and \( r > -3 \), and let \( \bar{C}_1 \) and \( \bar{C}_2 \) be two positive constants. Then there exists \( \epsilon_0 > 0 \) such that, for \( 0 < \epsilon \leq \epsilon_0 \), whenever \( u_0 \in D(A_\epsilon^{\frac{1}{2}}), f \in L^\infty((0, \infty), L^2(Q_3)) \) satisfy

\[
\| A_\epsilon^{\frac{1}{2}} u_0 \|_2 \leq \bar{C}_1 \epsilon^p, \quad \| P_\epsilon f \|_\infty^2 \leq \bar{C}_2 \epsilon^r,
\]

65
then (2.5) has a solution $u$ that belongs to $C^0([0, \infty), V^1_\varepsilon)$ and we have
\[
\| A^\frac{1}{2}_{\varepsilon} u(t) \|^2 \leq \exp \left( -\nu \frac{C_{11}^{-2} \varepsilon^{-2} t}{2} \right) \tilde{C}_1 e^p + 2C_{11}^2 \tilde{C}_2 \nu^{-2} \varepsilon^{2+r}, \quad t \geq 0.
\]

Proof. \footnote{This proof of Theorem 7.1 is, in fact, a small data argument.} We set
\[
R_0^2 = \tilde{C}_1 e^p + 2C_{11}^2 \tilde{C}_2 \nu^{-2} \varepsilon^{2+r}.
\]
Since $R_0^2 \geq \| A^\frac{1}{2}_{\varepsilon} u_0 \|^2$, it follows from Lemma 3.0 that there is a time $T^0 > 0$ such that
\[
(7.5) \quad \| A^\frac{1}{2}_{\varepsilon} u(t) \|^2 \leq 2R_0^2, \quad 0 \leq t < T^0.
\]
Without loss of generality, we let $[0, T^0)$ denote the maximal time interval for which (7.5) is valid. If $T^0 < \infty$, then we must have
\[
(7.6) \quad \| A^\frac{1}{2}_{\varepsilon} u(T^0) \|^2 = 2R_0^2.
\]
By taking the scalar product of (2.5) with $A_{\varepsilon} u$ and using (7.2) and (7.4), we obtain, for $0 \leq t \leq T^0$,
\[
(7.7) \quad \frac{d}{dt} \| A^\frac{1}{2}_{\varepsilon} u \|^2 + \nu \| A_{\varepsilon} u \|^2 \leq \frac{1}{\nu} \| P_{\varepsilon} f \|_\infty^2 + 2C_{12} C_{11}^\frac{1}{2} \varepsilon^{\frac{1}{2}} \| A^\frac{1}{2}_{\varepsilon} u \| \| A_{\varepsilon} u \|^2.
\]
For $0 \leq t \leq T^0$, we have
\[
2C_{12} C_{11}^\frac{1}{2} \varepsilon^{\frac{1}{2}} \| A^\frac{1}{2}_{\varepsilon} u \| \leq 2\sqrt{2} C_{12} C_{11}^\frac{1}{2} \varepsilon^{\frac{1}{2}} \tilde{C}_1^\frac{1}{2} \varepsilon^{\frac{1}{2}} (\tilde{C}_1^\frac{1}{2} \varepsilon^{\frac{1}{2}} + \sqrt{2} C_{11} \tilde{C}_2^\frac{1}{2} \nu^{-1} \varepsilon^{1+\frac{r}{2}}),
\]
which goes to 0 as $\varepsilon \to 0^+$. Consequently there is a positive number $\varepsilon_0$ such that
\[
(7.8) \quad 2\sqrt{2} C_{12} C_{11}^\frac{1}{2} \varepsilon^{\frac{1}{2}} R_0 \leq \frac{\nu}{2}.
\]
For $0 < \varepsilon \leq \varepsilon_0$, we deduce from (7.7), (7.8) and (7.2) that
\[
\frac{d}{dt} \| A^\frac{1}{2}_{\varepsilon} u \|^2 + \frac{\nu \varepsilon^{-2}}{2C_{11}^2} \| A^\frac{1}{2}_{\varepsilon} u \|^2 \leq \frac{1}{\nu} \| P_{\varepsilon} f \|_\infty^2, \quad 0 \leq t \leq T^0,
\]
which, by the Gronwall inequality, implies that
\[
\| A^\frac{1}{2}_{\varepsilon} u \|^2 \leq \exp \left( -\nu \frac{C_{11}^{-2} \varepsilon^{-2} t}{2} \right) \| A^\frac{1}{2}_{\varepsilon} u_0 \|^2 + \frac{2C_{11}^2 \varepsilon^2}{\nu^2} \| P_{\varepsilon} f \|_\infty^2,
\]
or
\[
(7.9) \quad \| A^\frac{1}{2}_{\varepsilon} u \|^2 \leq \exp \left( -\nu \frac{C_{11}^{-2} \varepsilon^{-2} t}{2} \right) \tilde{C}_1 e^p + \frac{2C_{11}^2 \tilde{C}_2}{\nu^2} \varepsilon^{2+r},
\]
for $0 \leq t \leq T^0$. From (7.9), it follows that
\[
\| A^\frac{1}{2}_{\varepsilon} u(T^0) \|^2 \leq R_0^2 < 2R_0^2,
\]
which contradicts (7.6). Therefore $T^0 = \infty$. \qed
REMARK. Like in Section 4 (see Theorem 2), one can show that, under the hypotheses of Theorem 13, if moreover $P_\epsilon f$ belongs to $C^0([0, \infty), H_\epsilon) \cap W^{1, \infty}((0, \infty), D(A_\epsilon^{\frac{1}{2}}))$, then the solution $u(t)$ of (2.5) belongs to $C^0((0, \infty), V_\epsilon^2)$. Let $S_\epsilon(P_\epsilon f, t)u_0$ denote the strong solution of (2.5) with initial data $u_0 \in V_\epsilon^1$ and let $\mathcal{B}_\epsilon^1 = \{u \in V_\epsilon^1; \|A_\epsilon^{\frac{1}{2}}u\|^2 \leq \tilde{C}_1 \epsilon^p + 2C_{11}^2 \tilde{C}_2 \nu^{-2} \epsilon^{2+r}\}$. As in Section 4 (see Theorem 2) one can show that, under the assumptions of Theorem 13, if in addition, $f \in W(Q_3)$ is chosen so that $P_\epsilon f \in W^{1, \infty}((0, \infty); H_\epsilon)$ and $H^+(f)$ is compact, then for any $\tau > 0$, there is a compact subset $K(\tau)$ of $V_\epsilon^2$ such that

$$S_\epsilon(P_\epsilon f, t)\mathcal{B}_\epsilon^1 \subset K(\tau), \quad t \geq \tau.$$ 

The results below are more interesting than Theorem 13. We recall that $\pi_\epsilon(u_0, P_\epsilon f, \tau) = (S_\epsilon(P_\epsilon f, \tau)u_0, (P_\epsilon f)_\tau)$ denote the skew-product semiflow generated by the strong solutions of (2.5).

**Theorem 14.** Assume that the hypotheses of Theorem 13 hold and that $f \in W(Q_3)$ is chosen so that $P_\epsilon f$ belongs to $W^{1, \infty}((0, \infty), H_\epsilon)$ and $H^+(f)$ is compact. Let $\epsilon_0 > 0$ be given by Theorem 13. Then, for $0 < \epsilon \leq \epsilon_0$, the skew-product semiflow $\pi_\epsilon(\cdot, P_\epsilon f, \tau)$ has a unique, maximal, compact (local) attractor $\mathcal{A}_\epsilon$ included in $\mathcal{B}_\epsilon^1 \times \omega(P_\epsilon f)$ which attracts $\mathcal{B}_\epsilon^1 \times H^+(P_\epsilon f)$ in the space $V_\epsilon^1 \times P_\epsilon W(Q_3)$. Furthermore,

$$\mathcal{A}_\epsilon \subset \{u \in V_\epsilon^1; \|A_\epsilon^{\frac{1}{2}}u\|^2 \leq 2C_{11}^2 \tilde{C}_2 \nu^{-2} \epsilon^{2+r}\} \times \omega(P_\epsilon f).$$

Moreover, $\mathcal{A}_\epsilon$ is bounded and compact in $V_\epsilon^2 \times \omega(P_\epsilon f)$ and attracts the bounded set $(\mathcal{B}_\epsilon^1 \cap V_\epsilon^2) \times H^+(P_\epsilon f)$ in the space $V_\epsilon^2 \times P_\epsilon W(Q_3)$. Finally, the attractor $\mathcal{A}_\epsilon$ is the global attractor for the Leray solutions of (2.5).

**Proof.** The first part of this theorem is proved in the same way as Theorem 4. We will only give the argument that $\mathcal{A}_\epsilon$ is the global attractor for the Leray solutions of (2.5), i.e., the weak solutions of (2.5) that satisfy the energy estimate:

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|A_\epsilon^{\frac{1}{2}} u\|^2 \leq \|u\|\|P_\epsilon f\|, \quad t > 0. \quad (7.10)$$

From (7.10) and (7.2), we infer that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|A_\epsilon^{\frac{1}{2}} u\|^2 \leq \frac{\nu}{2} \|A_\epsilon^{\frac{1}{2}} u\|^2 + \frac{C_{11}^2 \epsilon^2}{2\nu} \|P_\epsilon f\|^2$$

which implies that

$$\frac{d}{dt} \|u\|^2 + \nu \|A_\epsilon^{\frac{1}{2}} u\|^2 \leq \frac{C_{11}^2 \epsilon^2}{\nu^2} \|P_\epsilon f\|^2, \quad t > 0, \quad (7.11)$$

67
and also
\[
\frac{1}{t} \int_0^t \|A_\epsilon^\frac{1}{2} u\|^2 ds \leq \frac{\nu^{-1}}{t} \|u_0\|^2 + C_{11}^2 \tilde{C}_2 \nu^{-2} \epsilon^{2+r}, \quad t > 0.
\]
Therefore, for \( T = C_{11}^{-2} \tilde{C}_2^{-1} \nu \epsilon^{-(2+r)} \|u_0\|^2 \), there is a time \( t_0, 0 \leq t_0 \leq T \), such that
\[
\|A_\epsilon^\frac{1}{2} u(t_0)\|^2 \leq 2C_{11}^2 \tilde{C}_2 \nu^{-2} \epsilon^{2+r},
\]
that is, \( u(t_0) \) belongs to \( \mathcal{B}_t^1 \) and, according to the proof of Theorem 13, \( u(t) \) is regular for all \( t \geq t_0 \). This implies that \( \mathfrak{A}_\epsilon \) is the global attractor for the Leray solutions of (2.5). □

**Theorem 15.** Assume that the hypotheses of Theorem 14 hold. Then we have

\[
(7.12) \quad \sup_{(u, h) \in \mathfrak{A}_\epsilon} \|u\| \to 0 \text{ as } \epsilon \to 0.
\]

If in addition, \( r > -2 \), then the first components of the attractors \( \mathfrak{A}_\epsilon \) converge to 0 in \( D(A_\epsilon^\frac{1}{2}) \), i.e.,

\[
(7.13) \quad \sup_{(u, h) \in \mathfrak{A}_\epsilon} \|A_\epsilon^\frac{1}{2} u\| \to 0 \text{ as } \epsilon \to 0.
\]

**Proof.** Property (7.13) is an obvious consequence of Theorems 13 and 14, and property (7.12) is a direct consequence of Theorem 14 and (7.2) and (7.11). Indeed, from (7.2) and (7.11), we infer that

\[
\frac{d}{dt} \|u\|^2 + \frac{\nu \epsilon^{-2}}{C_{11}^2} \|u\|^2 \leq \frac{C_{11}^2 \epsilon^2}{\nu^2} \|P f\|^2, \quad t > 0,
\]

which by Gronwall inequality implies that

\[
(7.14) \quad \|u\|^2 \leq (\exp(-\nu C_{11}^{-2} \epsilon^{-2} t)) \|u_0\|^2 + C_{11}^4 \tilde{C}_2 \nu^{-3} \epsilon^{4+r}, \quad t \geq 0.
\]

Now (7.12) follows from (7.14), the fact that \( r > -3 \), and the invariance of \( \mathfrak{A}_\epsilon \). □

**8. Appendix: Proofs of Auxiliary Estimates**

In this section we give the proof of the estimates (3.3) in the case of periodic boundary conditions and of the corresponding estimates in the case of other boundary conditions. We shall use \( c_1, c_2, \ldots \) to denote constants which do not depend on \( \epsilon \) for \( 0 < \epsilon \leq 1 \).

**8.1 Periodic Boundary Conditions.** We will keep the notation of Sections 2 and 3. Let us begin with the following lemma.
LEMMA 8.1. For any $q$, $2 \leq q \leq 6$, there exist two positive constants $c_1$ and $c_2$, such that for any $w$ satisfying $Mw = 0$, one has

(8.1) \[ \|w\|_{L^q(Q_3)} \leq c_1 \epsilon^{2q-1} \left( \|w\|_{H^1(Q_3)} + \epsilon^{-1} \|D_3 w\|_{L^2(Q_3)} \right) \]

and

(8.2) \[ \|w\|_{L^q(Q_3)} \leq c_2 \epsilon^{2q-1} \|A_\epsilon^\frac{1}{2} w\| \]

for $0 < \epsilon \leq 1$.

Proof. Inequality (8.2) is a direct consequence of (8.1) and (2.17). In order to prove (8.1) we will use two inequalities from Hale and Raugel, (1989b, Lemma 4.1 and Proposition 4.2), which can be written (in the notation of Section 2) as

(8.3) \[ \|w\|_{L^q(Q_3)} \leq c_3 \epsilon \left( \|w\|_{H^1(Q_3)} + \epsilon^{-1} \|D_3 w\|_{L^2(Q_3)} \right) \]

and

(8.4) \[ \|w\|_{L^q(Q_3)} \leq c_4 \epsilon^{\frac{3}{4}} \left( \|w\|_{H^1(Q_3)} + \epsilon^{-1} \|D_3 w\|_{L^2(Q_3)} \right) \]

wherever $Mw = 0$. Inequality (8.1) is then obtained by interpolation between (8.3) and (8.4). (Note that inequality (8.1) could also be derived by replacing $q = 6$ in the proof of Hale and Raugel, (1989b, Proposition 4.2) by any $q$, $2 \leq q \leq 6$.) \[ \]

The next step is to prove the following result.

LEMMA 8.2. There exist positive constants $c_5$, $c_6$ and $c_7$ such that for all $u^1 \in D(A_\epsilon^\frac{1}{2})$, $u^2 \in D(A_\epsilon)$ and $u^3 \in H_\epsilon$ the following hold:

1. If $Mu^1 = 0$, then

(8.5) \[ |b_\epsilon(u^1, u^2, u^3)| \leq c_5 \epsilon^{\frac{3}{4}} \|A_\epsilon^\frac{1}{2} u^1\| \|A_\epsilon^\frac{1}{2} u^2\| \|A_\epsilon^\frac{1}{2} u^2\| \|u^3\|. \]

2. If $Mu^2 = 0$, then

(8.6) \[ |b_\epsilon(u^1, u^2, u^3)| \leq c_6 \epsilon^{\frac{3}{4}} \|A_\epsilon^\frac{1}{2} u^1\| \|A_\epsilon^\frac{1}{2} u^2\| \|A_\epsilon^\frac{1}{2} u^2\| \|u^3\|. \]

3. If $Mu^1 = Mu^2 = 0$, then

(8.7) \[ |b_\epsilon(u^1, u^2, u^3)| \leq c_7 \epsilon^{\frac{3}{4}} \|A_\epsilon^\frac{1}{2} u^1\| \|A_\epsilon^\frac{1}{2} u^2\| \|A_\epsilon^\frac{1}{2} u^2\| \|u^3\|. \]
Proof. Let us recall that

$$b_\epsilon(u^1, u^2, u^3) = \sum_{i,j=1}^{3} \int_{Q_3} e^{-(i)} u^1_i D_i u^2_j u^3_j dx$$

where \(\{1\} = \{2\} = 0\) and \(\{3\} = 1\). Using the Hölder inequality several times, we obtain

(8.8) \[ |b_\epsilon(u^1, u^2, u^3)| \leq \sum_{i,j=1}^{3} \|u^1_i\|_{L^6(Q_3)} \|e^{-(i)} D_i u^2_j\|_{L^2(Q_3)} \|u^3_j\|_{L^2(Q_3)} \]

Assume now that \(Mu^1 = 0\). Then we deduce from (2.17), (8.2) and (8.8) that

(8.9) \[ |b_\epsilon(u^1, u^2, u^3)| \leq c_8 \epsilon^{\frac{1}{2}} \|A_{\epsilon}^{\frac{1}{2}} u^1\|_1 \|A_{\epsilon}^{\frac{1}{2}} u^2\|_2 \|u^3\|_2 \left( \sum_{i=1}^{3} \|e^{-(i)} D_i u^2\|_{H^1(Q_3)} \right) \]

and (8.5) is now a direct consequence of (8.9) and (2.18).

Assume next that \(Mu^2 = 0\). Then obviously, \(MD_i u^2 = 0\) for \(i = 1, 2\). Since \(u^2\) is periodic with respect to the third variable, we also have \(MD_3 u^2 = 0\). Therefore we can apply inequality (8.1) to \(w = e^{-(i)} D_i u^2\) to obtain

(8.10) \[ \|e^{-(i)} D_i u^2\|_{L^6(Q_3)} \leq c_9 \epsilon^{\frac{1}{2}} \left( \|e^{-(i)} D_i u^2\|_{H^1(Q_3)} + \epsilon^{-1} \|e^{-(i)} D_3 D_i u^2\|_{L^2(Q_3)} \right)^{\frac{1}{2}} \]

The estimate (8.6) is a direct consequence of (8.8), (8.10) and (2.18).

The case \(Mu^1 = Mu^2 = 0\) is a combination of the above, and (8.7) is a straightforward consequence of the inequalities (8.2), (8.8), (8.10), (2.17) and (2.18). \(\square\)

Note that (8.7) establishes the first inequality in (3.3). In order to prove the other two inequalities in (3.3), we need the following results.

**Lemma 8.3.** The following statements are valid:

1. For any real numbers \(r\) and \(\theta\), satisfying \(2 \leq r \leq 6, \frac{1}{2} \leq \theta \leq 1\) and \(r \theta - 6(1 - \theta) > 0\), there exists a positive constant \(c_{10} = c_{10}(r, \theta)\) such that, for any \(w \in D(A_{\epsilon})\) with \(Mw = 0\), and any \(u^2 \in D(A_{\epsilon}^{\frac{1}{2}})\) and any \(u^3\) in \(H_{\epsilon}\), one has

(8.11) \[ |b_\epsilon(w, u^2, u^3)| \leq c_{10} \epsilon^{\frac{2(1-\theta)}{r}} \|A_{\epsilon} w\|^\theta \|A_{\epsilon}^{\frac{1}{2}} u^2\|^{1-\theta} \|A_{\epsilon}^{\frac{1}{2}} u^3\|. \]

2. For any real number \(q\), \(2 < q \leq 6\), there exists a positive constant \(c_{11}\) such that, for any \(w \in D(A_{\epsilon})\) with \(Mw = 0\), and any \(v \in \mathcal{R}(M) \cap D(A_{\epsilon}^{\frac{1}{2}})\) and any \(u \in H_{\epsilon}\), we have

(8.12) \[ |b_\epsilon(v, w, u)| \leq c_{11} \epsilon^{\frac{1}{2}} \|A_{\epsilon}^{\frac{1}{2}} v\| \|A_{\epsilon}^{\frac{1}{2}} w\|^{\frac{1}{2}} \|A_{\epsilon} w\|^{\frac{1}{2}} \|u\|. \]
Proof. Using the inequalities (2.17) and the Cauchy-Schwarz inequality, we obtain

\[ |b_\varepsilon(w, u^2, u^3)| \leq \sum_{i,j=1}^3 \|w_i\|_{L^\infty(Q_3)} \|\varepsilon^{-(i)} D_i u_j\|_{L^2(Q_3)} \|u_j\|_{L^2(Q_3)}, \]

or

(8.13) \[ |b_\varepsilon(w, u^2, u^3)| \leq c_{12} \|w\|_{L^\infty(Q_3)} \|A_\varepsilon^{\frac{1}{2}} u^2\| \|u^3\|. \]

It is well known that, for any \( p > 3 \), there exists a positive constant \( c_{12} \) such that

(8.14) \[ \|w\|_{L^\infty(Q_3)} \leq c_{12} \|w\|_{W^{1,p}(Q_3)}. \]

Now, using a Gagliardo-Nirenberg inequality (see Friedman (1964, Theorem 10.1)) for instance, we obtain, for \( \frac{1}{2} \leq \theta \leq 1 \), and \( r(\theta - 2 + \frac{6}{p}) = 6(1 - \theta) \), that

(8.15) \[ \|w\|_{W^{1,p}(Q_3)} \leq c_{13} \|w\|_{H^2(Q_3)} \|w\|_{L^\infty(Q_3)}^{1-\theta} \]

where \( c_{13} \) is a positive constant depending only on \( r, \theta, p \). Combining the inequalities (8.14) and (8.15), we see that actually, for any real numbers \( r, \theta \), satisfying \( 2 \leq r \leq 6 \), \( \frac{1}{2} \leq \theta \leq 1 \) and \( r\theta - 6(1 - \theta) > 0 \), we have

(8.16) \[ \|w\|_{L^\infty(Q_3)} \leq c_{14} \|w\|_{H^2(Q_3)} \|w\|_{L^\infty(Q_3)}^{1-\theta} \]

where \( c_{14} \) is a positive constant depending only on \( r, \theta \). Now the estimate (8.11) is a direct consequence of the inequalities (8.13), (8.16), (8.2) and (2.18).

Let us now prove the estimate (8.12). Using a Hölder inequality, we obtain, for any \( 1 < \tilde{q} \leq 3 \), that

(8.17) \[ |b_\varepsilon(v, w, u)| \leq \sum_{i,j=1}^3 \|v\|_{L^4(Q_3)} \|\varepsilon^{-(i)} D_i u_j\|_{L^{2\tilde{q}}(Q_3)} \|\varepsilon^{-(i)} D_i u\|_{L^{2\tilde{q}}(Q_3)} \|u_j\|_{L^2(Q_3)}, \]

where \( p = \frac{\tilde{q}}{\tilde{q}-1} \). Let us point out that the inequality (8.17) has a meaning since the vector \( v \) depends only on the variables \( x_1, x_2 \) and therefore belongs to any space \( L^4 p(Q_3) \), \( \frac{1}{4} \leq p < +\infty \), as soon as it belongs to \( H^1(Q_3) \). As in the proof of Proposition 8.2, we remark that \( MD_i w = 0, i = 1, 2, 3 \); whence we may apply the inequality (8.1) to \( w = \varepsilon^{-(i)} D_i w \).

Using the estimate (2.18) in addition we obtain

(8.18) \[ |b_\varepsilon(v, w, u)| \leq c_{15} \varepsilon^{\frac{1}{2}} \|A_\varepsilon^{\frac{1}{2}} v\| \|A_\varepsilon w\| \|A_\varepsilon^{\frac{1}{2}} w\| \|u\| \]

where \( c_{15} \) is a positive constant depending only on \( \tilde{q} \). By replacing \( 2\tilde{q} \) with \( q \) we see that (8.12) follows from (8.18).

The second estimate in (3.3) is simply the estimate (8.11) in the particular case where \( r = 6, \theta = \frac{17}{32} \). Likewise the third estimate in (3.3) is derived from (8.12) by choosing \( q = 4 \).
8.2 Other Boundary Conditions. In the proofs of Section 8.1, we never used the fact that the boundary conditions on $\partial Q_2 \times (0,1)$ were periodic ones. In particular, the estimate (8.1) is independent of the boundary conditions. Therefore, by using (2.17) and (2.18), one easily checks that the Lemma 8.1 and the Propositions 8.2 and 8.3 still hold if we replace the periodic boundary conditions on $\partial Q_3$ by homogeneous Dirichlet boundary conditions on $\partial Q_2 \times (0,1)$ and periodic boundary conditions on $(Q_2 \times \{0\}) \cup (Q_2 \times \{1\})$. Hence the estimates (3.3) are still true in this case.

Finally, let us consider the case where we have homogeneous Dirichlet boundary conditions on $\partial Q_3$. Arguing as in Hale and Raugel (1989b, Lemma 6.1) and in Lemma 8.1, one can prove the following result:

**Lemma 8.4.** For any $q$, $2 \leq q \leq 6$, there exists a positive constant $c_{16}$ such that, for any $u \in H^1(Q_3)$ with $u = 0$ on $(Q_2 \times \{0\}) \cup (Q_2 \times \{1\})$, one has

$$\|u\|_{L^q(Q_3)} \leq c_{16} \epsilon^2 \left( \|u\|_{H^1(Q_3)} + \epsilon^{-1} \|D_3 u\|_{L^2(Q_3)} \right). \tag{8.19}$$

This lemma enables us to prove the following result.

**Lemma 8.5.** There exists a positive constant $c_{17}$ such that, for $u^1 \in D(A_\epsilon^{\frac{1}{2}})$, $u^2 \in D(A_\epsilon)$ and $u^3 \in H_\epsilon$, one has

$$|b_\epsilon(u^1, u^2, u^3)| \leq c_{17} \epsilon^2 \|A^{\frac{3}{2}}_\epsilon u^1\| \|A^{\frac{1}{2}}_\epsilon u^2\| \|u^3\|$$

$$\times \left( \sum_{i=1}^{3} \|e^{-(i)} D_i u^2\| + \epsilon^{-1} \|e^{-(i)} D_3 D_i u^2\| \right)^\frac{1}{2}. \tag{8.20}$$

**Proof.** From (8.8), (8.19) and (2.17), we deduce that

$$|b_\epsilon(u^1, u^2, u^3)| \leq c_{18} \epsilon^2 \|A^{\frac{1}{2}}_\epsilon u^1\| \|u^2\| \|A^{\frac{1}{2}}_\epsilon u^2\| \frac{1}{2} \left( \sum_{i,j=1}^{3} \|e^{-(i)} D_i u^2\| \|A^{\frac{3}{2}}_\epsilon u^3\| \right). \tag{8.21}$$

It remains to estimate $\|e^{-(i)} D_i u_j\|_{L^2(Q_3)}$ for $1 \leq i, j \leq 3$. Since $D_i u_j$ is equal to zero on $(Q_2 \times \{0\}) \cup (Q_2 \times \{1\})$ if $i = 1, 2$ and since $MD_3 u_j = 0$, for $1 \leq j \leq 3$, one can apply Lemmas 8.4 and 8.1 to $D_i u_j$, $i = 1, 2$, and $D_3 u_j$, respectively, for $1 \leq j \leq 3$. From (8.1), (8.18) and (8.21), we at once infer the estimate (8.20). $\Box$
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<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>582</td>
<td>E.G. Kalnins and W. Miller, Jr.,</td>
<td>Separation of Variables Methods for Systems of Differential Equations in Mathematical Physics</td>
</tr>
<tr>
<td>583</td>
<td>Mitchell Luskin and George R. Sell</td>
<td>The Construction of Inertial Manifolds for Reaction-Diffusion Equations by Elliptic Regularization</td>
</tr>
<tr>
<td>584</td>
<td>Konstantin Mishaikow</td>
<td>Dynamic Phase Transitions: A Connection Matrix Approach</td>
</tr>
<tr>
<td>585</td>
<td>Philippe Le Floc’h and Li Tatsien</td>
<td>A Global Asymptotic Expansion for the Solution to the Generalized Riemann Problem</td>
</tr>
<tr>
<td>586</td>
<td>Matthew Witten, Ph.D.</td>
<td>Computational Biology: An Overview</td>
</tr>
<tr>
<td>587</td>
<td>Matthew Witten, Ph.D.</td>
<td>Peering Inside Living Systems: Physiology in a Supercomputer</td>
</tr>
<tr>
<td>588</td>
<td>Michael Renardy</td>
<td>An existence theorem for model equations resulting from kinetic theories of polymer solutions</td>
</tr>
<tr>
<td>590</td>
<td>Luigi Preziosi</td>
<td>An Invariance Property for the Propagation of Heat and Shear Waves</td>
</tr>
<tr>
<td>591</td>
<td>Gregory M. Constantine and John Bryant</td>
<td>Sequencing of Experiments for Linear and Quadratic Time Effects</td>
</tr>
<tr>
<td>592</td>
<td>Prabir Daripa</td>
<td>On the Computation of the Beltrami Equation in the Complex Plane</td>
</tr>
<tr>
<td>593</td>
<td>Philippe Le Floc’h</td>
<td>Shock Waves for Nonlinear Hyperbolic Systems in Nonconservative Form</td>
</tr>
<tr>
<td>595</td>
<td>Mark J. Friedman and Eusebius J. Doedel</td>
<td>Numerical computation and continuation of invariant manifolds connecting fixed points</td>
</tr>
<tr>
<td>596</td>
<td>Scott J. Spector</td>
<td>Linear Deformations as Global Minimizers in Nonlinear Elasticity</td>
</tr>
<tr>
<td>597</td>
<td>Denis Serre</td>
<td>Richness and the classification of quasilinear hyperbolic systems</td>
</tr>
<tr>
<td>598</td>
<td>L. Preziosi and F. Rosso</td>
<td>On the stability of the shearing flow between pipes</td>
</tr>
<tr>
<td>599</td>
<td>Avner Friedman and Wenxiong Liu</td>
<td>A system of partial differential equations arising in electrophotography</td>
</tr>
<tr>
<td>600</td>
<td>Jonathan Bell, Avner Friedman, and Andrew A. Lacey</td>
<td>On solutions to a quasilinear diffusion problem from the study of soft tissue</td>
</tr>
<tr>
<td>601</td>
<td>David G. Schaeffer and Michael Shearer</td>
<td>Loss of hyperbolicity in yield vertex plasticity models under nonproportional loading</td>
</tr>
<tr>
<td>602</td>
<td>Herbert C. Kranzer and Barbara Lee Keyfitz</td>
<td>A strictly hyperbolic system of conservation laws admitting singular shocks</td>
</tr>
<tr>
<td>603</td>
<td>S. Laederich and M. Levi</td>
<td>Qualitative dynamics of planar chains</td>
</tr>
<tr>
<td>604</td>
<td>Milan Miklavčič</td>
<td>A sharp condition for existence of an inertial manifold</td>
</tr>
<tr>
<td>605</td>
<td>Charles Collins, David Kinderlehrer, and Mitchell Luskin</td>
<td>Numerical approximation of the solution of a variational problem with a double well potential</td>
</tr>
<tr>
<td>606</td>
<td>Todd Arbogast</td>
<td>Two-phase incompressible flow in a porous medium with various nonhomogeneous boundary conditions</td>
</tr>
<tr>
<td>607</td>
<td>Peter Poláčik</td>
<td>Complicated dynamics in scalar semilinear parabolic equations in higher space dimension</td>
</tr>
<tr>
<td>608</td>
<td>Bei Hu</td>
<td>Diffusion of penetrant in a polymer: a free boundary problem</td>
</tr>
<tr>
<td>609</td>
<td>Mohamed Sami ElBialy</td>
<td>On the smoothness of the linearization of vector fields near resonant hyperbolic rest points</td>
</tr>
<tr>
<td>610</td>
<td>Max Jodeit, Jr. and Peter J. Olver</td>
<td>On the equation ( \nabla f = M \nabla g )</td>
</tr>
<tr>
<td>611</td>
<td>Shui-Nee Chow, Kening Lu, and Yun-Qiu Shen</td>
<td>Normal form and linearization for quasiperiodic systems</td>
</tr>
<tr>
<td>612</td>
<td>Prabir Daripa</td>
<td>Theory of one dimensional adaptive grid generation</td>
</tr>
<tr>
<td>613</td>
<td>Michael C. Mackey and John G. Milton</td>
<td>Feedback, delays and the origin of blood cell dynamics</td>
</tr>
<tr>
<td>614</td>
<td>D.G. Aronson and S. Kamin</td>
<td>Disappearance of phase in the Stefan problem: one space dimension</td>
</tr>
<tr>
<td>615</td>
<td>Martin Krupa</td>
<td>Bifurcations of relative equilibria</td>
</tr>
<tr>
<td>616</td>
<td>D.D. Joseph, P. Singh, and K. Chen</td>
<td>Couette flows, rollers, emulsions, tall Taylor cells, phase separation and inversion, and a chaotic bubble in Taylor-Couette flow of two immiscible liquids</td>
</tr>
<tr>
<td>617</td>
<td>Artemio González-López, Niky Kamran, and Peter J. Olver</td>
<td>Lie algebras of differential operators in two complex variables</td>
</tr>
<tr>
<td>618</td>
<td>L.E. Fraenkel</td>
<td>On a linear, partly hyperbolic model of viscoelastic flow past a plate</td>
</tr>
<tr>
<td>619</td>
<td>Stephen Schecter and Michael Shearer</td>
<td>Undercompressive shocks for nonstrictly hyperbolic conservation laws</td>
</tr>
<tr>
<td>620</td>
<td>Xinfu Chen</td>
<td>Axially symmetric jets of compressible fluid</td>
</tr>
</tbody>
</table>
J. David Logan, Wave propagation in a qualitative model of combustion under equilibrium conditions

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Allan P. Fordy, Isospectral flows: their Hamiltonian structures, Miura maps and master symmetries

Daniel D. Joseph, John Nelson, Michael Renardy, and Yuriko Renardy, Two-Dimensional cusped interfaces

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