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Asymptotic models in magnetostriction with application to design of sensors

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Abstract

Magnetostrictive wires of diameter in the nanometer scale have been proposed for application as acoustic sensors [Downey et al., 2008], [Yang et al., 2006]. The sensing mechanism is expected to operate in the bending regime. In the first part of this work, we derive a variational theory for the bending of magnetostrictive nanowires starting from a full 3-dimensional continuum theory of magnetostriction. We recover a theory which looks like a typical Euler-Bernoulli bending model but includes an extra term contributed by the magnetic part of the energy. The solution of this variational theory for an important, newly developed magnetostricitve alloy called Galfenol (cf. [Clark et al., 2000]) is compared with the result of experiments on actual nanowires (cf. [Downey, 2008]) which shows agreement.

In the next part of this thesis, Multilayered wires of diameter in the nanometer scale with periodic layering of non-magnetic copper and ferromagnetic galfenol segments are studied. The numerical computation of the physics of magnetization for such geometries is very costly computationally. We use the theory of periodic homogenization to understand the overall behavior of such structures. We first determine a "homogenized theory" after which this "homogenized model" is used to study the nucleation and stability of staturated states. Thus we get a broad generalization of what is known in the magnetic literature as the "fanning model" first introduced in [Jacobs and Bean, 1955] for a chain of spheres geometry. Some further numerical work on computing M vs H curves for such geometries is also presented.

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Chapter I

Introduction

I.1 Magnetostriction

Magnetostriction as a phenomenon can be understood as a coupling between two different physics, that of elasticity and magnetism. As a result, any change in the elastic state of a body induces changes in the magnetic properties of the body and vice versa changes in the magnetization causes the elastic state to change. Due to this property, magnetostrictive materials are well suited and commonly used for the purpose of designing actuators and sensors for a whole range of different applications. Magnetostriction and similar physics like piezoelectricity, shape-memory alloys etc. have been widely studied in the last few years. The purpose of this thesis is to further the understanding of the application of the physics of magnetostriction to design of sensors and transducers at one extreme of the length scale, namely in the nanometer scale.

I.1.1 Origin of Magnetostriction

Magnetism is an inherent property at the atomic scale. Electrons in an atom have both orbital motion and spin which resembles a current loop [Aharoni, 2000]. From a classical viewpoint, charged particles like electrons that are spinning physically would act as a current loop producing a magnetic moment. In most materials, these spins cancel at room temperature. Thus, the only source of externally observable magnetization comes when an external field is applied. Diamagnetic materials are one such class of materials, where spins naturally cancel and on application of an external field, the response of the electron orbitals produces a magnetization in opposition to the external field. Further, the magnitude of the electronic response is proportional to the external field. Diamagnetism as an effect is common to all atoms of all materials, though it is usually a very weak effect. Thus diamagnetic materials are those in which the "weak" diamagnetic effect is not dominated by some stronger effect.

Some elements possess a natural spin imbalance due to the non-symmetric filling order of electrons in orbitals as prescribed by *Hund's Rule* [Chikazumi and Charap, 1978]. Thus, these elements exhibit a higher population of electrons of a particular spin over its opposite. However at temperatures greater than 0^oK, Fermi smearing of this disbalanced spin state due to thermal excitation may be large enough so that there is no net observable imbalance over observable times. In such a case, when applied field is zero, the net magnetic moment of the electron cloud is zero just as in the diamagnetic case. At zero applied field, the temperature at which thermal agitation is strong enough to eliminate the intrinsic magnetization due to spin imbalance is called the Curie temperature, T_c.

Materials at temperatures above T_c are called as paramagnetic. In paramagnetic materials, an external magnetic field will cause the local spins to align parallel to the field in an averaged sense. The net moment thus observed, is proportional to the field magnitude and in the same direction. When the field is removed, the net moment again randomizes to zero. The net moment is often linearly proportional to the field and the proportionality constant is called as the magnetic susceptibility, χ_m. The Curie-Weiss law describes the temperature dependence of χ_m in the paramagnetic regime [Chikazumi and Charap, 1978].

At temperatures below T_c, the spin imbalance persists in-spite of thermal agitation, and a net moment at a lattice site even in the absence of an applied field. Furthermore, the moments at each lattice site, also align favorably in parallel directions. This effect was initially explained by a phenomenological model called the *Weiss mean field*. Later, the mean field theory was clarified in the light of a quantum mechanical effect called "exchange interaction" [Aharoni, 2000]. Exchange interaction is a quantum mechanical effect without a classical analogue. Thus at a scale much larger than an individual lattice site, there persists a spontaneous net magnetic moment. This effect is known as ferromagnetism. The net moment may locally point in any direction. The direction in which the net moment points, depends on many mechanisms, which include the sample shape, the internal crystalline structure, the local state of strain, and even the impact of spin polarized charge carriers from a magnet of dissimilar magnetization orientation.

The origin of magnetostriction lies in understanding of how the net local moment at a lattice point, interacts with the crystalline structure of the material. The exact origin of this effect is believed to be the interaction between the moment of an atomic dipole and the electrostatic charge of the nearby ions. This interaction forms the basis of the crystalline anisotropy and magnetostriction. As a result of this interaction, the local moment aligns favorably along certain crystalline directions and also locally deforms the bond length between adjacent neighbors. Thus the lattice is distorted away from the original lattice.

Magnetostriction as an effect occurs in almost all ferromagnetic materials. However, it is a small effect in the range of 20-200 ppm for commonly occurring ferromagnetic materials like Fe, Co and Ni and their alloys.

I.2 Galfenol : What is it?

In the 1970's giant magnetostrictive alloys like $Tb_{0.3}Dy_{0.7}Fe_2$ were developed. This alloy called Terfenol has high magnetostriction of the order ~ 2000 ppm, but is very brittle and has low tensile strength of the order ~ 100 MPa. For this reason, in most sensor/actuator applications, it is used under compressive strain. Recent research in [Clark et al., 2000] has led to the development of a new alloy called "Galfenol", $Fe_{100-x}Ga_x$ where x ranges from $10\% - 30\%$. This class of alloys, have high magnetostriction ~ 400 ppm and high tensile strengths ~ 400 MPa. While the exact physics of how gallium affects the magnetostriction is still being investigated, a lot of work has been done to study the $Fe_{100-x}Ga_x$ phase diagram and measure the magnetic properties of individual phases.

Figure I.1 taken from [Ikeda et al., 2002], shows the phase diagram for $Fe_{100-x}Ga_x$ alloys. The important phases related to magnetostriction property are the disordered A_2 or α' phase, the ordered $D0_3$ phase, and the ordered B_2 or α'' phase which has a Cs-Cl structure. Magnetostriction is typically measured in cubic materials in terms of a quantity known as $\frac{3}{2}\lambda_{100}$ [Chikazumi and Charap, 1978], which has two peaks for Galfenol. They are at 19% and 28% gallium, with approximate values of 400ppm and 450ppm respectively [Clark et al., 2003]. The Table I.1 shows a comparison of various physical properties of galfenol vs. terfenol which illustrates the basic

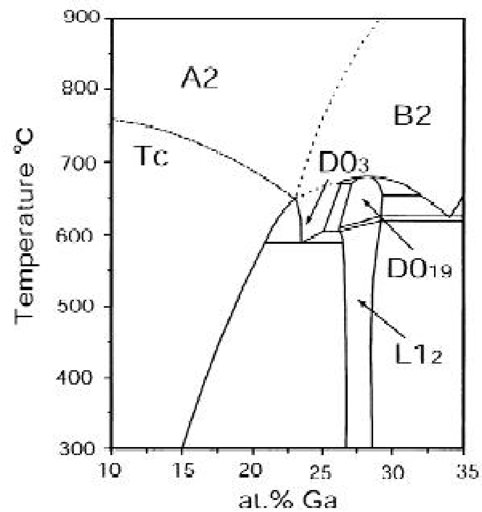

Figure I.1: Fe-Ga phase-diagram from [Ikeda et al., 2002].

problem with terfenol and the reason for development of galfenol. Data presented in this table are compiled from various references as [Rafique et al., 2004], [Petculescu et al., 2005], [Clark et al., 2003].

I.3 Biologically inspired transduction

In nature we observe that organisms across the spectrum have developed organs and organelles for the purpose of sensing various stimuli in their immediate environment. These have been the basis of inspiration for scientists and engineers for new ideas into designing artificial sensors and transducers. One of the most common sensing agents found in many organisms are called cilia.

Cilia refers to organelles which typically look like hair and are found in nearly all biological species. They are typically useful as sensory devices and in smaller single and multicellular organisms as motion sources. In the form of sensors they are seen in several organisms and useful for sensing various stimuli. As an example, on the legs of insects they occur primarily to sense touch [Albert et al., 2001], [Barth, 2004]. They also occur along the lateral line of fish for imaging under water [Coombs, 2001] and in the cochlea of reptiles, birds, and mammals [Manley, 1990], [Pickles, 1988] to sense sound. Figure I.2 shows a schematic idea behind how the cilia works

Table I.1: Comparison of Magnetostrictive Data for Terfenol and Galfenol.

Headings	Terfenol	Galfenol -18	Galfenol -28
Easy Axis	[111]	[100]	[100]
Elastic Modula			
c_{11} (GPa)	141	200	155
c_{12} (GPa)	65	220	210
c_{44} (GPa)	49	120	134
Anisotropy Constant $K_1(kJ/m^3)$	-60	30	-1
Magnetostriction $\frac{3}{2}\lambda$	1800	400	400
Staturation Magnetization M_s (Tesla)	1	1.6	1.15

for sensing sound. Any mechanical deflection of the cilium causes an ion channel to open and the induced chemical potential stimulates the neurons that are attached at the base. The dimensions of the cilia vary over some range with diameters ranging from hundreds of nanometers to tens of microns, with lengths of up to a millimeter. The cilia in the human ear are located in the cochlea. The cochlea is shown in Fig I.3. When acoustic waves hit these cilia, their bending deformation causes the nuerons at the base to send a signal to the brain. This is the basis of “hearing” in human beings. It is the purpose of this thesis, to understand whether and how, magnetostrictive

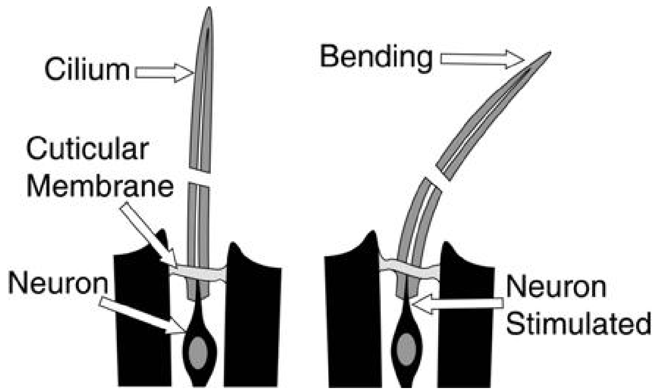

Figure I.2: Schematic of acoustic transduction in cilia of the cochlea is shown, taken from [Chen et al., 2006]

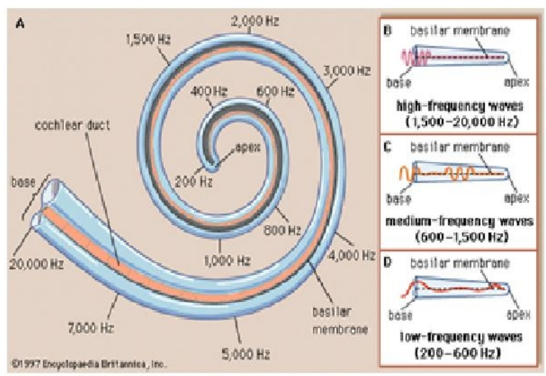

Figure I.3: Schematic figure of Inner ear.

nanowires can be designed to replicate the action of the cilia in the human ear.

I.4 Galfenol nanowire Sensors

I.4.1 Galfenol nanowire fabrication

In recent years a lot of new experimental techniques have been developed to manufacture ferromagnetic wires of nanometer diameter e.g. electron-beam lithography, step growth and template-assisted electrodeposition. One of these, namely template-assisted electrodeposition has become very popular because it is simple and cost-effective especially as compared to the lithography technique [Maeda et al., 1994]. The template technique for Galfenol normally consists of starting with alumina matrix which contains pores and then electrodepositing the iron-gallium alloy within the pores. After the deposition is over, the alumina matrix is etched out using some chemical. Due to the fact that the pores can be produced into the alumina matrix with excellent regularity and great control over the diameter and depth, this technique produces wires with excellent control over geometry. In addition deposition time, voltage and bath composition can be controlled to alter composition of the wire, surface roughness of the wire etc. In this section here onwards, we describe an “imprint-assisted” nanowire electrodeposition process as was developed by Professor Bethanie Stadler at the University of Minnesota along with her student Patrick McGary.

The first step in the process is getting a controlled porous alumina matrix. The normal commercial process produces anodized alumina which has only short range order and relatively high variation in diameters. However with respect to designing cilia inspired sensors, a more uniform and regular porous alumina matrix is required. With this in mind the alumina was first indented with a nanoimprint nitride stamp made via lithography. This initial indentation with the nanoimprint, when further subjected to pore formation gave rise to very nice and regular pore structure with long range order. Figure I.4 shows a schematic of the nitride imprint on the left and an AFM image of the regular pore structure thus produced on the right. Details of this process and the figure are cited from [McGary, 2008].

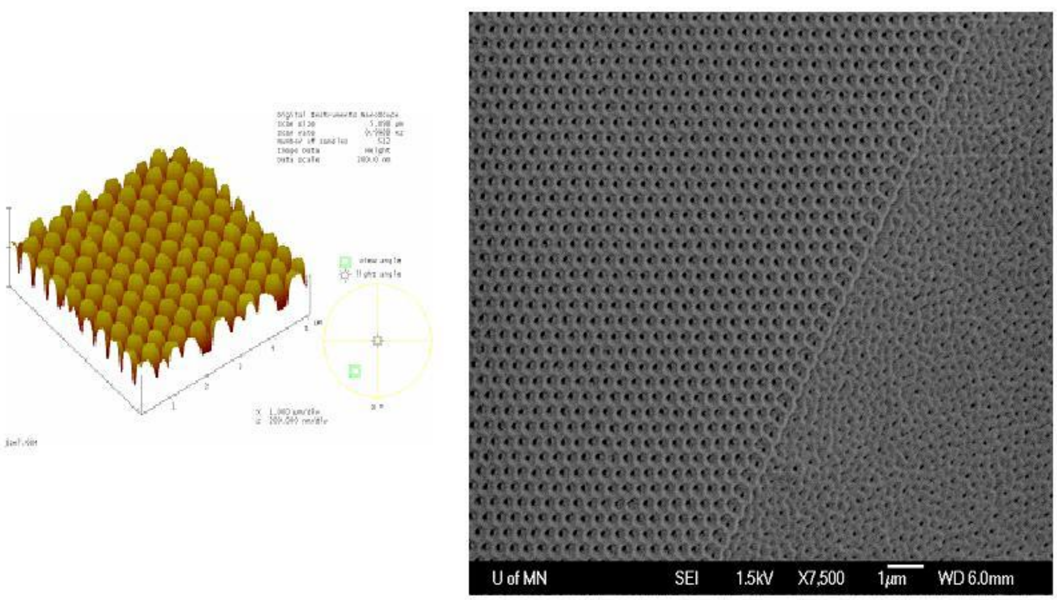

Figure I.4: Left: Nitride imprint , Right: AFM image of regular pore structure after imprinting [McGary, 2008].

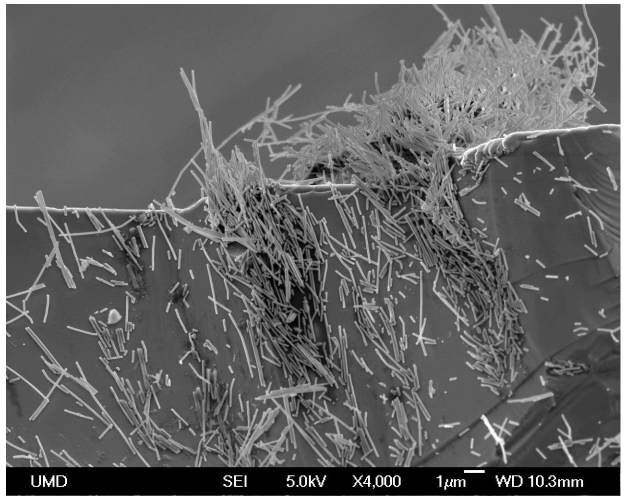

Figure I.5: Several nanowires sticking together in a clump, courtesy [McGary, 2008].

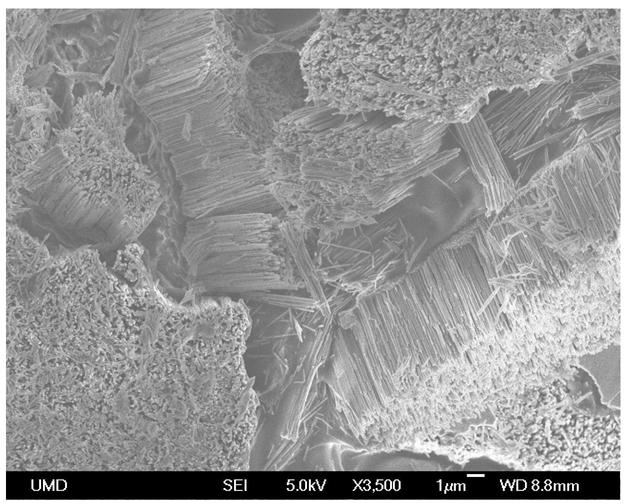

Figure I.6: Matrices of nanowire arrays pointing in different directions, courtesy [McGary, 2008].

Subsequently, the regular porous anodized alumina was electrodeposited with iron-gallium alloy by employing a Hull cell. The matrix of electrodeposited Galfenol wires, were then freed from the alumina matrix by etching using chromic acid. The figures I.5, I.6 and I.7 show some pictures of the nanowires thus produced [McGary, 2008].

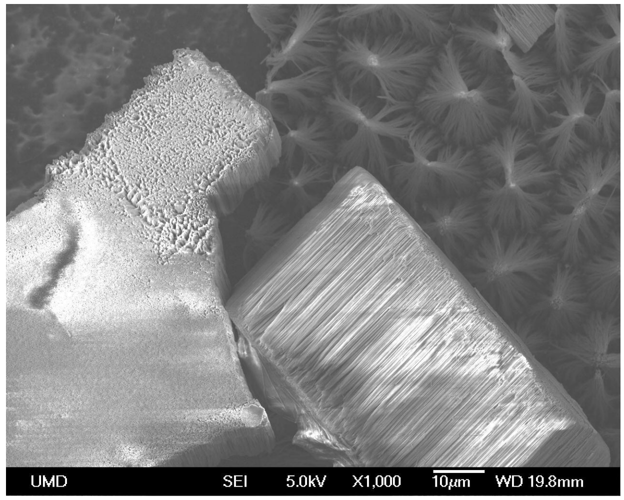

Figure I.7: Two nanowire arrays pointing in orthogonal directions, courtesy [McGary, 2008].

I.4.2 Galfenol nanowire sensors: Proposed designs

The production of Galfenol wires with controllable lengths and diameters, makes a large number of designs feasible. The primary idea of course is to emulate the cilia in the human ear using an array of these nanowires. Figure I.8 is a basic schematic of the sensor that is studied in this thesis. Here the sensor consists of Galfenol wires of equal length which are cantilevered at the base. To treat each wire in the matrix as a sensor will require independent data acquisition from each individual wire. This will make the design very complex as the array matrix of wires usually contains as many as 25×10^6 nanowires/mm^2. As a result the design involves attaching the wire base to a single large sensing element like a giant magneto-resistance (GMR) sensor. While this may make the design incapable of reading the signal from each individual wire, a typical stimuli that this sensor is expected to read, varies in space at a scale much larger than the size of individual wire. Thus a reading using the GMR sensor technique will involve signal sensing over a length scale which is more comparable to the length scale of the stimuli.

More complicated designs were also possible. One way of increasing the complication of the design is by noting that the length of the wire is a very important factor in the sensing characteristic of the wires. As a result, the first step towards complication could be by growing wires of different lengths. Figure I.9 shows a schematic of such a design. It is the expected that this design

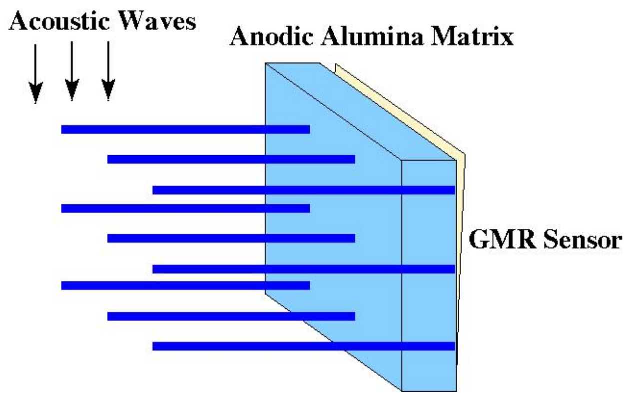

Figure I.8: Proposed Sensor : Sensor using wires of equal length.

will have a broadband response.

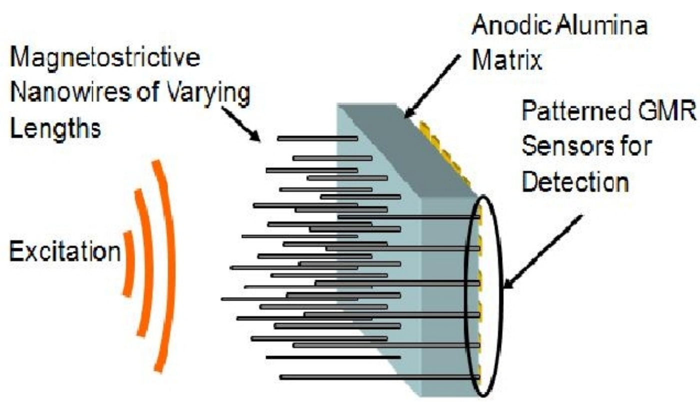

Figure I.9: Proposed Sensor : Sensor using wires of different lengths.

The last design that we will mention here is a wire matrix which uses multi-layered nanowires. The electrodeposition into anodized alumina templates is a flexible enough process so that one can electrodeposit wires into the pores which consist of alternate layers of magnetic Galfenol and non-magnetic Copper. Figure I.10 shows a schematic of the corresponding design.

I.5 Research Objectives

In this thesis we primarily look at the two designs which are portrayed in figures I.8 and I.10. The main methods used to understand both the sensor designs, are tools from the calculus

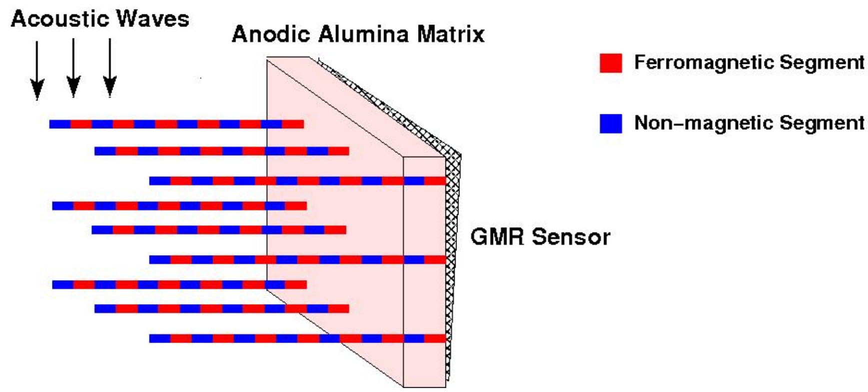

Figure I.10: Proposed Sensor : Sensor using multilayered wires with magnetic and non-magnetic segments.

of variations, notably Γ-convergence and mathematical theory of homogenization. These methods and their results are further complimented by implementing numerical models.

In Chapter II, we investigate the design in figure I.8. As part of this process we first try and understand how individual wires in the size scale of the as grown nanowires behave. Typical size scale of the wires that we propose to study have diameters in the range of 30-100 nm and lengths of upto 1-2 μm. The theory of Γ-convergence provides a very broad set of tools to understand how variational problems behave under limiting conditions. As a rough idea, let ε be some parameter and $\mathscr{P}^{\varepsilon}$ be a minimization problem depending on ε. Loosely speaking Γ-convergence enables us to investigate what happens to the problem $\mathscr{P}^{\varepsilon}$ as the parameter converges to some critical value, typically 0 or ∞. In our case we have wires which have very small diameter. As a result one way to investigate the problem of the behavior of galfenol wires with such small diameters would be to set the parameter ε to be the diameter of the wire and then investigate how the energy which describes the physics of magnetostriction behaves as $\varepsilon \to 0$.

In the literature, similar ideas based out of the theory of Γ-convergence were used to study problems in elasticity. Consequently it was established that common one-dimensional models like the Euler-Bernoulli beam bending theory are the correct approximation to three dimensional elasticity under limiting conditions. The first section in Chapter II give more details and references to the various problems in elasticity where good progress has been made in understanding through the method of Γ-convergence. The remaining part of the chapter then presents a derivation of asymptotic models starting from a linear theory of magnetostriction.

Next we study the design in figure I.10. The geometries we are studying, have a scale of the periodic layering similar to that of the planar density of wires in the matrix. Typical values for these might be as follows: magnetic & non-magnetic segment lengths being 20-50 nm each, wire diameters being 30-80 nm and center to center distance between adjacent wires being 100-150 nm. The mathematical theory of periodic homogenization is a standard tool which is very useful under these circumstances. This is a powerful tool which is very useful to understand phenomenon which express themselves at two or more scales.

In our case, we express the problem of analyzing the design in figure I.10, as a question of understanding the macroscale M vs H behavior of the sensor. Because the M vs H curve for any structure represents the macro-scale behavior of that structure, while the physics of magnetization occurs at the scale of the individual segments in the nanometer scale, the problem is a multi-scale problem. In Chapter III, we use the theory of homogenization to derive a "homogenized" model. This model is then further treated in terms of a second-variation stability problem, to study the stability and nucleation problem of a uniformly staturated structure.

This second variation stability investigation is a common theme in the micromagnetics literature. This was first used to calculate the critical field at which a staturated magnetic state looses stability, i.e. the apex of the M vs. H curve. As a result several modes of instability were discovered and named, for e.g. uniform rotation, curling, and buckling. In our case however, we start the second variation stability investigation, not from the full micromagnetic problem. But rather, we start with the homogenized model and derive a new mode of instability which might be thought of as a three-dimensional generalization of what is known in micromagnetics literature as "symmetric fanning" mode as applicable to chain-of-spheres geometry.

Finally in Chapter IV, we use some of the results derived in chapters II and III, and give them a numerical treatment. As a result, we look at the implications of the results of the earlier chapters towards the design of sensors.

Chapter II

One dimensional model for a magnetostrictive wire

Abstract

Magnetostrictive wires of diameter in the nanometer scale have been proposed for application as acoustic sensors [Downey et al., 2008], [Yang et al., 2006]. The sensing mechanism is expected to operate in the bending regime. In this work we derive a variational theory for the bending of magnetostrictive nanowires starting from a full 3-dimensional continuum theory of magnetostriction. We recover a theory which looks like a typical Euler-Bernoulli bending model but includes an extra term contributed by the magnetic part of the energy. The solution of this variational theory for an important, newly developed magnetostricitve alloy called Galfenol (cf. [Clark et al., 2000]) is compared with the result of experiments on actual nanowires (cf. [Downey, 2008]) which shows agreement.

II.1 Introduction

Magnetostrictive solids are those in which reversible elastic deformations are caused by changes in the magnetization. These materials have a coupling of ferromagnetic energies with elastic energies. In recent years a lot of new experimental techniques have been developed to manufacture ferromagnetic wires of nanometer diameter such as electron-beam lithography, step growth electro-deposition, and template-assisted electro-deposition. A possible application of these nanosize wires is in making acoustic sensors. The inspiration for this application comes from the structure of the human ear. The inner ear has fine cilia like hair whose response to impinging acoustic waves is transmitted by the nervous system to the brain. Such biologically inspired devices have been proposed to detect acoustic, fluid flow and tactile inputs (cf. [Yang et al., 2006]). One possible arrangement of galfenol nanowires is in the form of an array depicted in Fig II.1 . Here impinging acoustic waves are expected to change the magnetization of the wire array by inducing bending deformation.

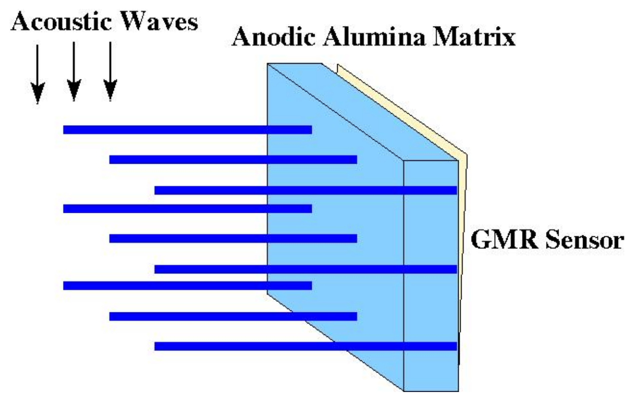

Figure II.1: Proposed model device using nanowires of Galfenol

The models of a vibrating string and the bending of a beam are important models in elasticity which are known to approximate the full 3-D behavior of a deformable body in the linear strain regime. Starting in the 80's rigorous mathematical methods based on the theory of Γ-convergence were used to justify these 1-D models as the correct approximation of 3-D elasticity, loosely speaking under asymptotic conditions as the diameter of the 3-D body approaches zero. The basic references for these results are [Acerbi et al., 1991] and [Anzellotti et al., 1994], while reference for Γ -convergence can be found in [Braides, 2002].

Meanwhile in the micromagnetics literature there has been extensive use of Γ-convergence based methods to derive reduced dimension models for ferromagnetic thin films. The earliest results in this direction are [Gioia and James, 1997] and [Carbou, 2001]. Since our nanowires are expected to be used for the proposed sensor application in the bending deformation regime, the main goal of this paper is to combine the ideas of the references cited above from the elasticity and micromagnetics literature to derive similar asymptotic models for magnetostrictive nanowires in bending. The nanowires we are modeling have diameters in the 10-100nm range with lengths in the range 2-5μm. We will show that the bending behavior of a magnetostrictive nanowire resembles the classical Euler-Bernoulli bending model with an extra term which comes from the magnetic part of the energy.

§ II.2 gives a brief review of the continuum theory of magnetostriction and defines the classical energy $\mathscr{E}(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{u}})$ as a function of the magnetization-deformation pair $(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{u}})$. The section § II.3 gives a simple heuristic argument to show the various scales of elastic and magnetic energy relevant to the final result. In § II.4 we start with the energy $\mathscr{E}(\widetilde{\boldsymbol{m}},\widetilde{\boldsymbol{u}})$ defined on a wire of diameter ε and on rescaling the wire to have unit diameter, recover a new energy $\mathscr{I}^{\varepsilon}(\boldsymbol{m},\boldsymbol{u})$ which equals the energy $\mathscr{E}(\widetilde{\boldsymbol{m}},\widetilde{\boldsymbol{u}})$ per unit wire cross-sectional area, and depends on a rescaled magnetization-deformation pair $(\boldsymbol{m},\boldsymbol{u})$ now defined on the wire with unit diameter. Starting with minimizers $(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon})$ of the energy $\mathscr{I}^{\varepsilon}(\boldsymbol{m},\boldsymbol{u})$ in § II.5 we derive the first variational limit problem which physically represents the magnetoelastic equivalent to the elastic theory of an extensible string. § II.6 gives the next order correction to the first variational problem which only involves magnetic terms. § II.7 gives the following order variational problem which is the main result of this paper and describes the bending behavior of the magnetostrictive nanowires. Here we show that we can extract a deformation $\boldsymbol{w}^{\varepsilon}$ (cf. (II.7.11)) from the energy minimizing pair $(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon})$ which itself minimizes an energy $\mathscr{I}_2^{o}$ (cf. (II.7.19)) where $\mathscr{I}_2^{o}$ is an energy which resembles the classical Euler-Bernoulli bending energy with some correction terms depending on the magnetization. The method of proof involves the idea of convergence of minimizers, and we do not use the more abstract Γ-convergence method. The Appendix II.A treats the magnetostatic energy separately.

Basic notation: $\alpha,\beta,\gamma,\cdots$ are scalars; $\boldsymbol{a},\boldsymbol{u},\boldsymbol{m},\cdots$ denote vectors in $\mathbb{R}^3$; $\boldsymbol{A},\boldsymbol{B},\boldsymbol{E},\cdots$ are tensors in $\mathbb{R}^{3\times3}$ and $S^2\subset\mathbb{R}^3$ represents the surface of the unit ball in $\mathbb{R}^3$. Components of any vector $\boldsymbol{m}$ are denoted by either m_1,m_2,m_3 or m_x,m_y,m_y. For any matrix $\boldsymbol{A}$, $\boldsymbol{A}^T$ denotes the transpose of the matrix. We use standard function space notation of $L^2(\Omega,\mathbb{R}^3)$, $H^1(\mathbb{R}^3,\mathbb{R}^3)$, $H_0^1(\Omega,\mathbb{R}^3)$; for details refer [Adams and Fournier, 2009]. By Young's inequality we mean $2ab\le\delta^{-1}a^2+\delta b^2$ for $\mathbb{R}\ni\delta>0$, a variation of the classical Young's inequality.

II.2 Micromagnetics

The initial model for ferromagnetic solids was proposed in [Landau and Lifshitz, 1935] where they also derived a model for magnetization dynamics. The continuum theory of ferromagnetic materials was developed in the work of Brown [Brown, 1963] which was subsequently expanded to a theory for magnetostriction in [Brown, 1966], where a variational model for magnetostriction with small strain is developed. We give a brief presentation of Brown's work relevant to magnetostriction in this section.

Let Ω_ε be a smooth bounded reference configuration in $\mathbb{R}^3$ depending on a parameter ε. In the following sections we fill specify this dependence. Let $\widetilde{\boldsymbol{m}}(\boldsymbol{y})$ be the magnetization vector at a point $\boldsymbol{y}\in\Omega_\varepsilon$. Below the Curie temperature, the magnetization is constrained to have constant euclidean norm i.e.,

$$|\widetilde{\boldsymbol{m}}(\boldsymbol{y})| = m_s \quad a.e. \qquad \boldsymbol{y} \in \Omega_\varepsilon.$$

For a bounded domain, this constraint implies $\widetilde{\boldsymbol{m}} \in L^p(\Omega_\varepsilon, m_s S^2),\ \forall\ 1 \leq p \leq \infty$. We extend $\widetilde{\boldsymbol{m}}$ by 0 outside Ω_ε whenever necessary and denote it by $\widetilde{\boldsymbol{m}}\chi_{\Omega_\varepsilon} = \widetilde{\boldsymbol{m}}(\boldsymbol{y})\chi_{\Omega_\varepsilon}(\boldsymbol{y})$ which as a result gives $\widetilde{\boldsymbol{m}}\chi_{\Omega_\varepsilon} \in L^p(\mathbb{R}^3, \mathbb{R}^3),\ \forall\ 1 \leq p \leq \infty$. We denote by $\widetilde{\boldsymbol{u}} \in H^1(\Omega_\varepsilon, \mathbb{R}^3)$ the displacement map. The infinitesimal strain corresponding to $\widetilde{\boldsymbol{u}}(\boldsymbol{y})$ is, $\left(\nabla^y \text{ is gradient w.r.t. } \boldsymbol{y}\right)$

$$\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}](\boldsymbol{y}) = \frac{1}{2}\left(\nabla^y \widetilde{\boldsymbol{u}}(\boldsymbol{y}) + \nabla^y \widetilde{\boldsymbol{u}}(\boldsymbol{y})^T\right). \tag{II.2.1}$$

Interaction of the magnetization with the crystalline structure of a magnetic solid generates an interaction energy modeled by a function, $\varphi : m_s S^2 \to [0, \infty)$. This energy has a finite number of wells (say N) along a set of constant magnetization vectors $\left\{\widetilde{\boldsymbol{m}}^{(k)}\right\} \in m_s S^2$ where the index $k \in \left\{1, 2, \cdots N\right\}$ and on which without loss of generality we can set $\varphi\left(\widetilde{\boldsymbol{m}}^{(k)}\right) = 0$. The anisotropy energy thus becomes,

$$E_{anis} = \int_{\Omega_\varepsilon} \varphi\left(\widetilde{\boldsymbol{m}}(\boldsymbol{y})\right) \boldsymbol{dy}.$$

For cubic materials $\varphi(\widetilde{\boldsymbol{m}}) = \frac{\Pi_1}{m_s^4}(\widetilde{m}_1^2\widetilde{m}_2^2 + \widetilde{m}_1^2\widetilde{m}_3^2 + \widetilde{m}_2^2\widetilde{m}_3^2) + \frac{\Pi_2}{m_s^6}(\widetilde{m}_1^2\widetilde{m}_2^2\widetilde{m}_3^2)$, which along with the constraint $|\widetilde{\boldsymbol{m}}| = m_s$ gives that $0 \leq \varphi(\widetilde{\boldsymbol{m}}) \leq K_1$. Thus

$$0 \leq E_{anis} = \int_{\Omega} \varphi\left(\widetilde{\boldsymbol{m}}(\boldsymbol{y})\right) \boldsymbol{dy} \leq K_1 |\Omega_\varepsilon|. \tag{II.2.2}$$

The exchange energy penalizes variations in the magnetization in a body and thus tends to prefer constant magnetizations. It is modeled as follows,

$$E_{exc} = d \int_{\Omega_\varepsilon} \left|\nabla^y \widetilde{\boldsymbol{m}}\right|^2 \boldsymbol{dy}.$$

Here d is called the exchange constant. Magnetized bodies generate a magnetic self field in all of $\mathbb{R}^3$. This field $\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y})$ is given by the following equation,

$$\begin{aligned}
&\nabla^y \cdot \left(-\nabla^y \widetilde{\phi}^\varepsilon(\boldsymbol{y}) + 4\pi \widetilde{\boldsymbol{m}}(\boldsymbol{y})\right) = 0 \qquad \forall \boldsymbol{y} \in \mathbb{R}^3,\\
&\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y}) = -\nabla^y \widetilde{\phi}^\varepsilon(\boldsymbol{y}),\\
&\left[\left|\nabla^y \widetilde{\phi}^\varepsilon \cdot \widetilde{\boldsymbol{n}}\right|\right] = \left[\left|-\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}} \cdot \widetilde{\boldsymbol{n}}\right|\right] = 4\pi\ \widetilde{\boldsymbol{m}} \cdot \widetilde{\boldsymbol{n}} \qquad \text{on } \partial\Omega_\varepsilon.
\end{aligned}$$

$[|\cdot|]$ represents the jump of a quantity across any oriented surface with unit normal $\tilde{\boldsymbol{n}}$. The demagnetization energy is generated by the interaction of the magnetization $\widetilde{\boldsymbol{m}}$ with $\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}$ and equals

$$E_{demag}(\widetilde{\boldsymbol{m}}) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \left|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y})\right|^2 \boldsymbol{dy} = -\frac{1}{2}\int_{\Omega_\varepsilon} \widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y})\cdot\widetilde{\boldsymbol{m}}(\boldsymbol{y})\,\boldsymbol{dy}. \tag{II.2.3}$$

A standard upper and lower bound for E_{demag} is given by

$$0 \le \; E_{demag}(\widetilde{\boldsymbol{m}}) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \left|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y})\right|^2 \boldsymbol{dy} \;\; \le \frac{1}{2}\int_{\Omega_\varepsilon} |\widetilde{\boldsymbol{m}}(\boldsymbol{y})|^2\,\boldsymbol{dy} = \frac{1}{2}|\Omega_\varepsilon|\, m_s^2, \tag{II.2.4}$$

since $|\widetilde{\boldsymbol{m}}| = m_s$. The energy of interaction between an external applied field $\widetilde{\boldsymbol{h}_a} \in L^2(\Omega,\mathbb{R}^3)$ and the magnetization over the body is modeled by the following,

$$E_{app}(\widetilde{\boldsymbol{m}}) = -\int_{\Omega_\varepsilon} \widetilde{\boldsymbol{h}_a}(\boldsymbol{y})\cdot\widetilde{\boldsymbol{m}}(\boldsymbol{y})\,\boldsymbol{dy}.$$

which along with Hölder's inequality gives

$$-K_2 \le E_{app}(\widetilde{\boldsymbol{m}}) \le K_2, \qquad\qquad K_2 = \|\widetilde{\boldsymbol{h}_a}\|_{L^2(\Omega_\varepsilon)} \|\widetilde{\boldsymbol{m}}\|_{L^2(\Omega_\varepsilon)}. \tag{II.2.5}$$

The elastic energy for the magnetoelastic solid for small strains is given by,

$$E_{el} = \int_{\Omega_\varepsilon} \frac{1}{2}\left(\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})\right) : \mathbb{C}\left[\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})\right] \boldsymbol{dy}.$$

In this paper by an abuse of notation, we write the above integrand as

$$\left(\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})\right) : \mathbb{C}\left[\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})\right] = \mathbb{C}\left[\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})\right]^2.$$

$\mathbb{C}$ is a positive definite fourth order tensor. Here $\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})$ is the spontaneous strain due to magnetization

$$\widetilde{\boldsymbol{m}} \mapsto \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}}) \in \boldsymbol{M}^{3\times3}_{sym},$$

where $\boldsymbol{M}^{3\times3}_{sym}$ denotes the set of symmetric matrices of 3×3 dimension. For cubic materials it's form is

$$\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}}) = \frac{3}{2m_s^2}\begin{bmatrix} \lambda_{100}\,\widetilde{m_1}^2 & \lambda_{111}\,\widetilde{m_1}\widetilde{m_2} & \lambda_{111}\,\widetilde{m_1}\widetilde{m_3} \\ \lambda_{111}\,\widetilde{m_1}\widetilde{m_2} & \lambda_{100}\,\widetilde{m_2}^2 & \lambda_{111}\,\widetilde{m_2}\widetilde{m_3} \\ \lambda_{111}\,\widetilde{m_1}\widetilde{m_3} & \lambda_{111}\,\widetilde{m_2}\widetilde{m_3} & \lambda_{100}\,\widetilde{m_3}^2 \end{bmatrix} - \frac{\lambda_{100}}{2} I$$

where I is the Identity matrix in $\mathbb{R}^{3\times 3}$. The form for $\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})$ and $|\widetilde{\boldsymbol{m}}| = m_s$ gives

$$\left|\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})\right| \le K_3, \qquad \text{where } \left|\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})\right|^2 = tr\left(\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})^T \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})\right). \tag{II.2.6}$$

The norm defined used above is the standard Frobenius norm, i.e. for any matrix $\boldsymbol{M}$, $\left|\boldsymbol{M}\right|^2$ is defined as the trace of $\boldsymbol{M}^T\boldsymbol{M}$. Also $\mathbb{C}$ being symmetric positive definite 4^{th} order tensor gives for some $\gamma, \Gamma > 0$,

$$\Gamma\left|\boldsymbol{M}^2\right| \ge \mathbb{C}\left[\boldsymbol{M}\right]^2 \ge \gamma\left|\boldsymbol{M}^2\right|, \qquad \forall \boldsymbol{M} \in \boldsymbol{M}^{3\times 3}_{sym}. \tag{II.2.7}$$

In addition to these, energy due to external force acting on the body in the form of body force or surface traction is included in the general energy. However since these terms are lower order in deformation $\widetilde{\boldsymbol{u}}$, they do not affect the final form of the limit problem. For our investigation in this paper, we neglect this term to reduce the length of the computation. Thus the full energy functional for magnetostriction is,

$$\begin{aligned}\mathscr{E}^{\varepsilon}(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{u}}) &= E_{exc} + E_{anis} + E_{app} + E_{el} + E_{demag} \\ &= \int_{\Omega_\varepsilon} \left\{ d\left|\nabla^y \widetilde{\boldsymbol{m}}\right|^2 + \varphi(\widetilde{\boldsymbol{m}}) - \widetilde{\boldsymbol{h}_a} \cdot \widetilde{\boldsymbol{m}} + \frac{\mathbb{C}}{2}\left[\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})\right]^2 \right\} \boldsymbol{dy} + \frac{1}{8\pi}\int_{\mathbb{R}^3} \left|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}\right|^2 \boldsymbol{dy}.\end{aligned} \tag{II.2.8}$$

II.3 Heuristic Scaling of energy

In § II.3.1 and § II.3.2, we start with a cylindrical domain with radius ε and length 1. We then show how both an isotropic linear elastic energy and the magnetostatic energy defined in equation (II.2.3) scale with respect to ε. The scaling of the linear elastic energy has been know for long in the engineering literature, but a rigorous derivation starting from a three-dimensional linear elastic theory is relatively recent.

II.3.1 Linear Elastic Energy

Let $\Theta = \left\{(y_1, y_2) \in B_\varepsilon(0)\ ,\ y_3 \in (0,1)\right\}$ be a cylindrical domain of radius ε centered at the origin and length 1 with axis aligned along the y_3 axis. Let Y be the Young's Modulus, $A = \pi\varepsilon^2$ is the cross-sectional area, and $I = \frac{\pi}{4}\varepsilon^4$ be the second moment of area of the cross section. Let $(\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3)$ be the displacements in (y_1, y_2, y_3) directions. From the engineering literature we know that the extensional energy of a rod along its axis is given as

$$\int_0^1 Y A \left|\partial_3 \widetilde{u}_3\right|^2 dy_3 = Y \pi \varepsilon^2 \int_0^1 \left|\partial_3 \widetilde{u}_3\right|^2 dy_3 \approx O(\varepsilon^2), \tag{II.3.1}$$

where $\widetilde{u}_3$ is the extension of the rod along its axis. From the Euler-Bernoulli model for a beam bending in the direction of the x_1 axis, the bending energy is

$$\int_0^1 Y I \left|\partial_{33} \widetilde{u}_1\right|^2 dy_3 = Y \frac{\pi \varepsilon^4}{4} \int_0^1 \left|\partial_{33} \widetilde{u}_1\right|^2 dy_3 \approx O(\varepsilon^4). \tag{II.3.2}$$

The different scaling of the two energies with respect to ε suggests to us that a linear elastic isotropic energy of the form

$$\mathscr{W}^{\varepsilon}(\widetilde{\boldsymbol{u}}) = \int_{\Theta} \left\{ \mu \left|\boldsymbol{E}(\widetilde{\boldsymbol{u}})\right|^2 + \frac{\lambda}{2} \left| \operatorname{tr}\left(\boldsymbol{E}(\widetilde{\boldsymbol{u}})\right) \right|^2 \right\} \boldsymbol{dy} \tag{II.3.3}$$

should factor into terms which are of different orders in powers of ε. Using Γ-convergence this factorization into orders of powers of ε has been proven in [Anzellotti et al., 1994]. They have shown that,

$$\mathscr{W}^{\varepsilon}(\widetilde{\boldsymbol{u}}) = \varepsilon^2 \mathscr{W}_1(\widehat{u}_3) + \varepsilon^4 \mathscr{W}_2(\widehat{u}_1, \widehat{u}_2, \widehat{u}_4) + \text{ higher order terms} \tag{II.3.4}$$

where $\widehat{\boldsymbol{u}}(y_3) \equiv (\widehat{u}_1, \widehat{u}_2, \widehat{u}_3)(y_3)$ and $\widehat{\boldsymbol{u}}(y_3) = \frac{1}{|B_{\varepsilon}(0)|} \int_{B_{\varepsilon}(0)} \widetilde{\boldsymbol{u}}(\boldsymbol{y})\, dy_1 dy_2$ is the averaged cross-sectional displacement and $\widehat{u}_4(y_3) = \frac{2}{\varepsilon^2 |B_{\varepsilon}(0)|} \int_{B_{\varepsilon}(0)} \left(y_2 \widetilde{u}_1 - y_1 \widetilde{u}_2\right) dy_1 dy_2$ gives the torsional component.

II.3.2 Magnetostatic energy

For an ellipsoidal body it is well known cf. [Maxwell, 1873] that the demagnetization field $\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}$ and the corresponding demagnetization E_{demag} for a constant magnetization $\widetilde{\boldsymbol{m}}$ are,

$$\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}} = -4\pi \boldsymbol{D} \widetilde{\boldsymbol{m}}, \qquad E_{demag} = 2\pi \left(\text{Volume of body} \right) \times \boldsymbol{D} \widetilde{\boldsymbol{m}} \cdot \widetilde{\boldsymbol{m}} \tag{II.3.5}$$

where $\boldsymbol{D} \in \mathbb{R}^{3\times 3}$ is called demagnetization tensor. $\boldsymbol{D}$ is independent of position $\boldsymbol{y}$, and has trace 1. For non-ellipsoidal bodies supporting a constant magnetization $\widetilde{\boldsymbol{m}}$, it still is true that $\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}} = -4\pi \boldsymbol{D} \widetilde{\boldsymbol{m}}$. However the demagnetization tensor $\boldsymbol{D}$ (with trace still 1) now depends on position $\boldsymbol{y}$. The magnetostatic energy is now given by $E_{demag} = 2\pi\left(\text{Volume of body}\right) \times \widehat{\boldsymbol{D}} \widetilde{\boldsymbol{m}} \cdot \widetilde{\boldsymbol{m}}$, where $\widehat{\boldsymbol{D}}$ is the volumetric average of $\boldsymbol{D}$. For our cylindrical domain $\Theta = B_{\varepsilon}(0) \times (0,1)$, $\widehat{\boldsymbol{D}}$ is a diagonal matrix with entries

$$\widehat{D}_{33} = \frac{8\varepsilon}{3\pi} - \frac{\varepsilon^2}{2} + O(\varepsilon^4), \qquad \widehat{D}_{11} = \widehat{D}_{22} = \frac{1}{2} - \frac{4\varepsilon}{3\pi} + \frac{\varepsilon^2}{4} + O(\varepsilon^4).$$

See [Joseph, 1966] for a simple derivation of this result. The demagnetization energy for a constant magnetization $\widetilde{\boldsymbol{m}} = (\widetilde{m}_1, \widetilde{m}_2, \widetilde{m}_3)$ is given by

$$\begin{aligned} E_{demag} &= 2\pi^2\Big[\varepsilon^2\frac{\widetilde{m}_1^2+\widetilde{m}_2^2}{2} - \varepsilon^3\frac{4}{3\pi}\Big(\big(\widetilde{m}_1^2+\widetilde{m}_2^2\big) - 2\widetilde{m}_3^2\Big) + \frac{\varepsilon^4}{2}\Big(\frac{\widetilde{m}_1^2+\widetilde{m}_2^2}{2} - \widetilde{m}_3^2\Big)\Big] \\ &= \pi^2\big(\widetilde{m}_1^2+\widetilde{m}_2^2\big)\,\varepsilon^2 + \varepsilon^3 Q_1(\widetilde{\boldsymbol{m}}) + \varepsilon^4 Q_2(\widetilde{\boldsymbol{m}}), \end{aligned} \tag{II.3.6}$$

where $Q_1(\widetilde{\boldsymbol{m}}) := \frac{8\pi}{3}\Big(\big(\widetilde{m}_1^2+\widetilde{m}_2^2\big) - 2\widetilde{m}_3^2\Big)$ and $Q_2(\widetilde{\boldsymbol{m}}) := \frac{\pi^2}{2}\Big(\big(\widetilde{m}_1^2+\widetilde{m}_2^2\big) - 2\widetilde{m}_3^2\Big)$. Thus for a cylindrical domain Θ with constant magnetization we can already see the presence of various orders of scales in the magnetostatic and elastic energy.

II.4 Rescaling

In this section we rescale the domain Ω_ε depending on a parameter ε to a fixed domain Ω. The space variable in the original domain Ω_ε is either denoted by $\boldsymbol{y}$ or $\boldsymbol{z}$ and in the rescaled domain by $\boldsymbol{x}$. The gradient operators w.r.t. $\boldsymbol{y}$ and $\boldsymbol{z}$ are denoted by ∇^y and ∇^z respectively and gradient w.r.t. $\boldsymbol{x}$ is denoted as just ∇. All variables in the original domain Ω_ε come with the tilde notation, for e.g. $\widetilde{\boldsymbol{m}}$ while variables in the rescaled domain are plain e.g. $\boldsymbol{m}$. For any vector $\boldsymbol{v} \in \mathbb{R}^3$, we will write $\boldsymbol{v} = (v_1, v_2, v_3) = (\boldsymbol{v}_p, v_3)$ where $p = 1,2$ and $\boldsymbol{v}_p$ denotes the planar component of $\boldsymbol{v}$. Analogously the gradient operator may be denoted by $\nabla = (\nabla_p, \partial_3)$.

Let $\Omega_\varepsilon := \big[\,\boldsymbol{y}_p \in \omega_\varepsilon,\, y_3 \in (0,1)\big]$ be a domain with cross-section $\omega_\varepsilon \subset \mathbb{R}^2$ where ω_ε is any Lipschitz domain in 2-dimensions. While the results of all the subsequent sections in this paper hold for any arbitrary cross-section ω_ε, however for the sake of simplicity we set

$$\omega_\varepsilon = B_\varepsilon(0) \subset \mathbb{R}^2 \tag{II.4.1}$$

a ball of radius ε in 2-D. We rescale the domain Ω_ε to Ω by the following one-to-one map

$$x_1 = \frac{y_1}{\varepsilon} \qquad x_2 = \frac{y_2}{\varepsilon} \qquad x_3 = y_3. \tag{II.4.2}$$

By the rescaling $\Omega = \big[\,\boldsymbol{x}_p \in \omega,\, x_3 \in (0,1)\big]$ where ω is now a ball with unit radius in 2 dimensions.

We rescale the fields $\widetilde{\boldsymbol{m}}(\boldsymbol{y})$, $\widetilde{\boldsymbol{u}}(\boldsymbol{y})$, $\widetilde{\boldsymbol{h}_a}(\boldsymbol{y})$, and $\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y})$ using the one-to-one maps

$$\boldsymbol{m}(\boldsymbol{x}) = \widetilde{\boldsymbol{m}}(\boldsymbol{y}), \qquad \boldsymbol{u}(\boldsymbol{x}) = \widetilde{\boldsymbol{u}}(\boldsymbol{y}), \qquad \boldsymbol{h}_a(\boldsymbol{x}) = \widetilde{\boldsymbol{h}_a}(\boldsymbol{y}), \qquad \boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}(\boldsymbol{x}) = \widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y}). \tag{II.4.3}$$

The map $\boldsymbol{m}(\boldsymbol{x}) = \widetilde{\boldsymbol{m}}(\boldsymbol{y})$ being one-to-one means that we can invert the rescaled magnetization $\boldsymbol{m}(\boldsymbol{x})$ back to the unscaled magnetization $\widetilde{\boldsymbol{m}}(\boldsymbol{y})$. Also while the pair $\left(\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}, \widetilde{\boldsymbol{m}}\right)$ satisfies Maxwell's equation on Ω_ε, the rescaled pair $\left(\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}, \boldsymbol{m}\right)$ does not satisfy Maxwell's equation on Ω. However unscaling the pair $\left(\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}, \boldsymbol{m}\right)$ to $\left(\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}, \widetilde{\boldsymbol{m}}\right)$, solves Maxwell's equation on Ω_ε. Hence the ε superscript on $\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}$.

The gradient operator $\nabla^y = (\nabla^y_p, \partial^y_3)$ operating on $\widetilde{\boldsymbol{m}}(\boldsymbol{y})$ or $\widetilde{\boldsymbol{u}}(\boldsymbol{y})$ correspondingly scales as,

$$\nabla^y_p \, \widetilde{\boldsymbol{m}}(\boldsymbol{y}) = \frac{1}{\varepsilon} \nabla_p \, \boldsymbol{m}(\boldsymbol{x}), \qquad \partial^y_3 \, \widetilde{\boldsymbol{m}}(\boldsymbol{y}) = \partial_3 \, \boldsymbol{m}(\boldsymbol{x}).$$

Using the scaling of gradients, we rescale the strain $\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}](\boldsymbol{y})$ to get a new field $\boldsymbol{\kappa}^{\varepsilon}(\boldsymbol{x})$ as

$$\begin{aligned}
\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}](\boldsymbol{y}) &= \frac{1}{2} \left[\; \nabla^y \widetilde{\boldsymbol{u}}(\boldsymbol{y}) + \nabla^y \widetilde{\boldsymbol{u}}(\boldsymbol{y})^T \; \right] \\
&= \begin{bmatrix} \frac{1}{\varepsilon}\partial_1 u_1(\boldsymbol{x}) & \frac{1}{2\varepsilon}\big(\partial_1 u_2 + \partial_2 u_1\big)(\boldsymbol{x}) & \frac{1}{2}\big(\frac{1}{\varepsilon}\partial_1 u_3 + \partial_3 u_1\big)(\boldsymbol{x}) \\ \frac{1}{2\varepsilon}\big(\partial_1 u_2 + \partial_2 u_1\big)(\boldsymbol{x}) & \frac{1}{\varepsilon}\partial_2 \, u_2(\boldsymbol{x}) & \frac{1}{2}\big(\frac{1}{\varepsilon}\partial_2 u_3 + \partial_3 u_2\big)(\boldsymbol{x}) \\ \frac{1}{2}\big(\frac{1}{\varepsilon}\partial_1 u_3 + \partial_3 u_1\big)(\boldsymbol{x}) & \frac{1}{2}\big(\frac{1}{\varepsilon}\partial_2 u_3 + \partial_3 u_2\big)(\boldsymbol{x}) & \partial_3 u_3(\boldsymbol{x}) \end{bmatrix} \\
&=: \boldsymbol{\kappa}^{\varepsilon}[\boldsymbol{u}](\boldsymbol{x}).
\end{aligned} \tag{II.4.4}$$

Substituting the above transformations into equation (II.2.8) we get

$$\begin{aligned}
\mathscr{E}^{\varepsilon}(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{u}}) = \varepsilon^2 \int_\Omega \Big[\; \frac{d}{\varepsilon^2} \left|\nabla_p \boldsymbol{m}\right|^2 + d \left|\partial_3 \boldsymbol{m}\right|^2 + \varphi(\boldsymbol{m}) - \boldsymbol{h}_a \cdot \boldsymbol{m} + \frac{\mathbb{C}}{2} \left[\boldsymbol{\kappa}^{\varepsilon}[\boldsymbol{u}] - \boldsymbol{E}_s(\boldsymbol{m})\right]^2 \Big] \boldsymbol{dx} \\
+ \frac{\varepsilon^2}{8\pi} \int_{\mathbb{R}^3} \left|\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}(\boldsymbol{x})\right|^2 \boldsymbol{dx}.
\end{aligned}$$

Dividing above by ε^2 and defining a new energy $\mathscr{I}^{\varepsilon}(\boldsymbol{m}, \boldsymbol{u}) := \varepsilon^{-2} \mathscr{E}(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{u}})$ we get,

$$\begin{aligned}
\mathscr{I}^{\varepsilon}(\boldsymbol{m}, \boldsymbol{u}) = \int_\Omega \Big\{ \frac{d}{\varepsilon^2} \left|\nabla_p \boldsymbol{m}\right|^2 + d \; \left|\partial_3 \boldsymbol{m}\right|^2 + \varphi(\boldsymbol{m}) - \boldsymbol{h}_a \cdot \boldsymbol{m} + \frac{\mathbb{C}}{2} \left[\boldsymbol{\kappa}^{\varepsilon}[\boldsymbol{u}] - \boldsymbol{E}_s(\boldsymbol{m})\right]^2 \Big\} \boldsymbol{dx} \\
+ \mathscr{E}^{\varepsilon}_d(\boldsymbol{m}),
\end{aligned}$$

where $\mathscr{E}^{\varepsilon}_d(\boldsymbol{m})$ is defined and bounded by rescaling the standard demag bound in equation (II.2.4)

$$0 \;\leq\; \mathscr{E}^{\varepsilon}_d(\boldsymbol{m}) := \frac{1}{8\pi} \int_{\mathbb{R}^3} \left|\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}(\boldsymbol{x})\right|^2 \boldsymbol{dx} \;\leq\; \frac{1}{2} \int_\Omega |\boldsymbol{m}(\boldsymbol{x})|^2 \boldsymbol{dx} = \frac{1}{2} \left|\Omega\right| m_s^2. \tag{II.4.5}$$

We investigate the asymptotic nature of the problem

$$(\mathscr{P}^\varepsilon)\ \inf_{\mathscr{A}_\varepsilon} \mathscr{I}^\varepsilon(\boldsymbol{m},\boldsymbol{u}), \quad \mathscr{A}_\varepsilon = \left\{ (\boldsymbol{m},\boldsymbol{u}) \in H^1(\Omega, m_s S^2) \times H^1_\sharp(\Omega,\mathbb{R}^3) \right\} \tag{II.4.6}$$

where $H^1_\sharp(\Omega,\mathbb{R}^3) = \left\{\boldsymbol{u}(\boldsymbol{x}) \in H^1(\Omega,\mathbb{R}^3) \mid \boldsymbol{u}(x_1,x_2,0) = \boldsymbol{0}, \quad \forall (x_1,x_2) \in \omega\right\}$ enforces Dirichlet boundary conditions at the base $x_3 = 0$. For the subsequent sections we also use the notation $H^1_\sharp\big((0,1),\mathbb{R}\big)$ defined as

$$H^1_\sharp\big((0,1),\mathbb{R}\big) = \left\{ w(x_3) \in H^1((0,1),\mathbb{R}^3) \mid w(x_3 = 0) = 0 \right\}. \tag{II.4.7}$$

In the next section, we will start with a sequence of minimizers $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ of $\mathscr{I}^\varepsilon(\boldsymbol{m},\boldsymbol{u})$ and show that we can extract a subsequence whose limit relates to the minimizers of a simpler lower dimensional problem $\mathscr{I}^o$.

II.5 First variational limiting problem

Let $(\widehat{\cdot})$ denote the cross-sectional average of any scalar/vector, i.e. for any field $\boldsymbol{a}(\boldsymbol{x})$ set

$$\widehat{\boldsymbol{a}}(x_3) = \fint_\omega \boldsymbol{a}(\boldsymbol{x}_p, x_3) \boldsymbol{d}\boldsymbol{x}_p. \tag{II.5.1}$$

For ε fixed, let $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ be a minimizer of $\mathscr{I}^\varepsilon(\boldsymbol{m},\boldsymbol{u})$. We look at the behavior of $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ as $\varepsilon \to 0$. For that we will first show that $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ is bounded above and below independent of ε. Then we will show that from the sequence $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$, we can extract a subsequence(unrelabeled) such that $(\boldsymbol{m}^\varepsilon, \widehat{u_3}^\varepsilon)$ on the subsequence converges weakly to some $(\boldsymbol{m}^o, v^o)$ in an appropriate space. This convergence will be improved to strong and the limit $(\boldsymbol{m}^o, v^o)$ will be shown to minimize a new functional $\mathscr{I}^o$ in Theorem II.5.1 .

II.5.1 Boundedness of $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$

For an upper bound on $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ we compare its energy with a test function $(\boldsymbol{m}, \boldsymbol{0})$ with $\boldsymbol{m}$ any constant vector on $m_s S^2$ and $\boldsymbol{u} = 0$ to get,

$$\begin{aligned}\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \le \mathscr{I}^\varepsilon(\boldsymbol{m}, \boldsymbol{0}) &= \int_\Omega \left[\phi(\boldsymbol{m}) + \frac{1}{2}\mathbb{C}\big[\boldsymbol{E}_s(\boldsymbol{m})\big]^2 - \boldsymbol{h}_a \cdot \boldsymbol{m}\right] \boldsymbol{d}\boldsymbol{x} + \mathscr{E}^\varepsilon_d(\boldsymbol{m}) \\ &\le K_4 + \frac{m_s^2}{2}|\Omega|,\end{aligned} \tag{II.5.2}$$

where the anisotropy, elastic, Zeeman and magnetostatic terms are bounded using equations (II.2.2), (II.2.6), (II.2.7), (II.2.5) and (II.4.5). The positivity of all terms in $\mathscr{I}^{\varepsilon}(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon})$ except possibly of the Zeeman energy along with equation (II.2.5) gives the lower bound,

$$\mathscr{I}^{\varepsilon}(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon}) \geq -\int_{\Omega} \boldsymbol{h}_a \cdot \boldsymbol{m}\, \boldsymbol{dx} \geq -K_2.$$

II.5.2 Weak compactness of minimizers $(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon})$ as $\varepsilon \to 0$

The upper and lower bound on $\mathscr{I}^{\varepsilon}(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon})$ gives,

$$K_5 \;>\; \int_{\Omega}\left\{ \frac{d}{\varepsilon^2}\left|\nabla_p \boldsymbol{m}^{\varepsilon}\right|^2 + d\;\left|\;\partial_3 \boldsymbol{m}^{\varepsilon}\right|^2 \right\}\boldsymbol{dx} \;\geq\; d\int_{\Omega} |\nabla \boldsymbol{m}^{\varepsilon}|^2 \boldsymbol{dx}. \tag{II.5.3}$$

Then for some unrelabeled subsequence

$$\left\|\nabla_p \boldsymbol{m}^{\varepsilon}(\boldsymbol{x})\right\|^2_{L^2(\Omega)} \leq \frac{K_5}{d}\varepsilon^2, \qquad \boldsymbol{m}^{\varepsilon} \to \boldsymbol{m}^{o} \text{ in } L^2(\Omega), \qquad \nabla \boldsymbol{m}^{\varepsilon} \rightharpoonup \nabla \boldsymbol{m}^{o} \text{ in } L^2(\Omega). \tag{II.5.4}$$

By the weak convergence of $\nabla \boldsymbol{m}^{\varepsilon}(\boldsymbol{x})$ to $\nabla \boldsymbol{m}^{o}(\boldsymbol{x})$ and the lower semi-continuity of norm operator $\left\|(\,\cdot\,)\right\|_{L^2(\Omega)}$ w.r.t. weak convergence we have using equation (II.5.4)

$$\left\|\nabla_p \boldsymbol{m}^{o}(\boldsymbol{x})\right\|_{L^2(\Omega)} \leq \liminf_{\varepsilon\to 0} \left\|\nabla_p \boldsymbol{m}^{\varepsilon}(\boldsymbol{x})\right\|_{L^2(\Omega)} \leq \liminf_{\varepsilon\to 0} \sqrt{\frac{K_5}{d}}\varepsilon = 0,$$

which implies

$$\lim_{\varepsilon\to 0} \boldsymbol{m}^{\varepsilon}(\boldsymbol{x}) = \boldsymbol{m}^{o}(\boldsymbol{x}) = \boldsymbol{m}^{o}(x_3) \qquad \text{in } L^2(\Omega). \tag{II.5.5}$$

Strong convergence of $\boldsymbol{m}^{\varepsilon}$ to $\boldsymbol{m}^{o}$ in $L^2(\Omega)$ gives convergence pointwise $a.e.$ for a (unrelabelled) subsequence. The cross-sectional average of this subsequence $\widehat{\boldsymbol{m}}^{\varepsilon}(x_3) = \fint_{\omega} \boldsymbol{m}^{\varepsilon}(\boldsymbol{x})\boldsymbol{dx}_p$ then converges pointwise $a.e.$ to $\fint_{\omega} \boldsymbol{m}^{o}(x_3)\boldsymbol{dx}_p = \boldsymbol{m}^{o}(x_3)$. Since from Jensen's inequality we have $|\widehat{\boldsymbol{m}}^{\varepsilon}(x_3)| \leq |\boldsymbol{m}^{\varepsilon}| = m_s$, the pointwise a.e. convergence of the unrelabeled subsequence $\widehat{\boldsymbol{m}}^{\varepsilon}(x_3)$ to $\boldsymbol{m}^{o}(x_3)$ gives on using L^p Dominated convergence theorem

$$\widehat{\boldsymbol{m}}^{\varepsilon}(x_3) \to \boldsymbol{m}^{o}(x_3) \qquad \text{in } L^p(0,1) \text{ as } \varepsilon \to 0, \quad \forall 1 \leq p \leq \infty. \tag{II.5.6}$$

Also the strong convergence of $\boldsymbol{m}^{\varepsilon}$ to $\boldsymbol{m}^{o}$ in $L^2(\Omega)$ gives convergence of $|m_i^{\varepsilon}|^2$ to $|m_i^{o}|^2$ in $L^2(\Omega)$ because of the fact that $\left|\,|m_i^{\varepsilon}|^2 - |m_i^{o}|^2\,\right|^2 = |m_i^{\varepsilon} - m_i^{o}|^2\; |m_i^{\varepsilon} + m_i^{o}|^2$ and domination of $|m_i^{\varepsilon}|^2$ and $|m_i^{o}|^2$ by m_s^2 and $i \in \{1,2,3\}$ denoting any of the 3 components of $\boldsymbol{m}^{\varepsilon}$. Then using the same argument as above we can derive an unrelabeled subsequence such that

$$\widehat{|m_i^\varepsilon|^2}(x_3) \to \widehat{|m_i^o|^2}(x_3) = |m_i^o|^2(x_3) \qquad \text{in } L^2(0,1) \text{ as } \varepsilon \to 0. \tag{II.5.7}$$

We will need equation (II.5.7) for showing convergence of the elastic energy in Theorem II.5.1. Next we prove a proposition which we need for extracting weak compactness of the elastic terms.

Proposition II.5.1. *Given* $\widehat{\boldsymbol{m}}^\varepsilon \in H^1(\Omega)$ *and* $\widehat{\boldsymbol{u}}^\varepsilon \in H^1_\sharp(\Omega)$ *using eqn.* (II.5.1)*, we have the following,*

$$\begin{aligned}
&\|\widehat{\boldsymbol{m}}^\varepsilon\|^2_{L^2(\Omega)} = |\omega| \ \|\widehat{\boldsymbol{m}}^\varepsilon\|^2_{L^2(0,1)} \le \|\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)},\\
&\|\partial_3\widehat{\boldsymbol{m}}^\varepsilon\|^2_{L^2(\Omega)} = |\omega| \ \|\partial_3\widehat{\boldsymbol{m}}^\varepsilon\|^2_{L^2(0,1)} \le \|\partial_3\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)},\\
&\|\partial_3\widehat{\boldsymbol{u}}^\varepsilon\|^2_{L^2(\Omega)} = |\omega| \ \|\partial_3\widehat{\boldsymbol{u}}^\varepsilon\|^2_{L^2(0,1)} \le \|\partial_3\boldsymbol{u}^\varepsilon\|^2_{L^2(\Omega)},
\end{aligned}$$

and for $i = \{1,2,3\}$

$$\|\widehat{u_i}^\varepsilon\|^2_{L^2(\Omega)} \le K_6 \|\partial_3\widehat{u_i}^\varepsilon\|^2_{L^2(\Omega)} \le K_6 \|\partial_3 u_i^\varepsilon\|^2_{L^2(\Omega)}.$$

Proof. The first result is easily seen using Jensen's inequality

$$\int_0^1 |\widehat{\boldsymbol{m}}^\varepsilon|^2 dx_3 = \int_0^1 \left|\frac{1}{|\omega|}\int_\omega \boldsymbol{m}^\varepsilon \boldsymbol{dx}_p\right|^2 dx_3 \le \int_0^1 \frac{1}{|\omega|}\int_\omega |\boldsymbol{m}^\varepsilon|^2 \boldsymbol{dx}_p dx_3 = \frac{1}{|\omega|}\int_\Omega |\boldsymbol{m}^\varepsilon|^2 \boldsymbol{dx}.$$

To see the second result, note for $i \in \{1,2,3\}$ using Jensen's inequality

$$\begin{aligned}
\int_0^1 |\partial_3\widehat{m_i}^\varepsilon|^2 dx_3 &= \int_0^1 \left| \partial_3 \left\{\frac{1}{|\omega|}\int_\omega m_i^\varepsilon \boldsymbol{dx}_p\right\} \right|^2 dx_3 = \int_0^1 \frac{1}{|\omega|^2} \left|\int_\omega \partial_3 m_i^\varepsilon \boldsymbol{dx}_p\right|^2 dx_3\\
&\le \int_0^1 \frac{1}{|\omega|}\int_\omega \left|\partial_3 m_i^\varepsilon\right|^2 \boldsymbol{dx}_p \ dx_3 = \frac{1}{|\omega|}\int_\Omega \left|\partial_3 m_i^\varepsilon\right|^2 \boldsymbol{dx}.
\end{aligned}$$

Integrating over ω and summing over i gives us the first result. Similar calculation with $\boldsymbol{u}^\varepsilon$ replacing $\boldsymbol{m}^\varepsilon$ gives the third result. Noting the Dirichlet Boundary conditions on $\widehat{\boldsymbol{u}}^\varepsilon(x_3)$ at $x_3 = 0$ we get using 1-D Poincaré inequality over $(0,1)$,

$$\int_0^1 |\widehat{u_i}^\varepsilon|^2 dx_3 \ \le \ K_6 \int_0^1 |\partial_3\widehat{u_i}^\varepsilon|^2 dx_3 \ \le \ K_6 \int_\Omega \frac{1}{|\omega|} \left|\partial_3 u_i^\varepsilon\right|^2 \boldsymbol{dx}$$

where K_6 is the Poincaré constant on $(0,1)$. Integrating over ω gives the result. □

Using positive definiteness of $\mathbb{C}$ in (II.2.7), (II.2.6) and bounds on $\mathscr{I}^{\varepsilon}(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon})$, we get

$$\int_{\Omega}\left|\boldsymbol{\kappa}^{\varepsilon}[\boldsymbol{u}^{\varepsilon}]\right|^{2}=\int_{\Omega}\left|\boldsymbol{\kappa}^{\varepsilon}[\boldsymbol{u}^{\varepsilon}]-\boldsymbol{E}_{s}(\boldsymbol{m}^{\varepsilon})+\boldsymbol{E}_{s}(\boldsymbol{m}^{\varepsilon})\right|^{2}\leq 2\int_{\Omega}\left\{\left|\boldsymbol{\kappa}^{\varepsilon}[\boldsymbol{u}^{\varepsilon}]-\boldsymbol{E}_{s}(\boldsymbol{m}^{\varepsilon})\right|^{2}+\left|\boldsymbol{E}_{s}(\boldsymbol{m}^{\varepsilon})\right|^{2}\right\}$$
$$\leq\frac{2}{\gamma}\int_{\Omega}\mathbb{C}\left[\boldsymbol{\kappa}^{\varepsilon}[\boldsymbol{u}^{\varepsilon}]-\boldsymbol{E}_{s}(\boldsymbol{m}^{\varepsilon})\right]^{2}+2\int_{\Omega}\left|\boldsymbol{E}_{s}(\boldsymbol{m}^{\varepsilon})\right|^{2}\ \leq\ K_{7}.$$

Combining this with the fourth result in Proposition II.5.1 we have

$$\int_{\Omega}\left|\widehat{u_3}^{\varepsilon}\right|^{2}dx_3\leq K_6\int_{\Omega}\left|\partial_3\widehat{u_3}^{\varepsilon}\right|^{2}dx_3\leq K_6\int_{\Omega}\left|\partial_3 u_3^{\varepsilon}\right|^{2}\boldsymbol{dx}\leq K_6\int_{\Omega}\left|\boldsymbol{\kappa}^{\varepsilon}[\boldsymbol{u}^{\varepsilon}]\right|^{2}\boldsymbol{dx}<K_8.$$

Thus $\|\widehat{u_3}^{\varepsilon}\|_{H^1(0,1)}\leq\infty$ and due to Dirichlet conditions on $\boldsymbol{u}^{\varepsilon}$ we get $\widehat{u_3}^{\varepsilon}\in H^1_{\sharp}(0,1)$. For an unrelabeled subsequence we have,

$$\widehat{u_3}^{\varepsilon}(x_3)\to v^{o}(x_3)\text{ in }L^2(0,1),\qquad\qquad \partial_3\widehat{u_3}^{\varepsilon}(x_3)\rightharpoonup\partial_3 v^{o}(x_3)\text{ in }L^2(0,1)\tag{II.5.8}$$

Already from the fact that $\boldsymbol{m}^{o}$ and v^{o} depends only on the x_3 space variable the 1-D nature of the limit problem becomes evident. The magnetostatic estimate in equation (II.A.20) gives

$$\mathscr{E}_d^{\varepsilon}(\boldsymbol{m}^{\varepsilon})-\pi|\omega|\int_0^1\left|\widehat{\boldsymbol{m}}_p^{\varepsilon}(y_3)\right|^2dy_3=O(\varepsilon)+O(\varepsilon^{3/4})$$

which implies on using strong convergence of $\widehat{\boldsymbol{m}}^{\varepsilon}$ to $\boldsymbol{m}^{o}$ in $L^2(0,1)$ in equation (II.5.6),

$$\lim_{\varepsilon\to 0}\ \mathscr{E}_d^{\varepsilon}(\boldsymbol{m}^{\varepsilon})=\lim_{\varepsilon\to 0}\ \pi|\omega|\int_0^1\left|\widehat{\boldsymbol{m}}_p^{\varepsilon}(x_3)\right|^2dx_3=\pi|\omega|\int_0^1\left|\boldsymbol{m}_p^{o}(x_3)\right|^2dx_3.\tag{II.5.9}$$

The magnetostatic estimate in equation (II.A.20) and Remark II.A.3 also gives

$$\mathscr{E}_d^{\varepsilon}(\boldsymbol{m}^{o})=\pi|\omega|\int_0^1\left|\boldsymbol{m}_p^{o}(x_3)\right|^2dx_3+O(\varepsilon)+O(\varepsilon^{3/4})=\pi|\omega|\int_0^1\left|\boldsymbol{m}_p^{o}(x_3)\right|^2dx_3\tag{II.5.10}$$

and thus,

$$\lim_{\varepsilon\to 0}\ \mathscr{E}_d^{\varepsilon}(\boldsymbol{m}^{\varepsilon})=\lim_{\varepsilon\to 0}\ \mathscr{E}_d^{\varepsilon}(\boldsymbol{m}^{o})=\pi|\omega|\int_0^1\left|\boldsymbol{m}_p^{o}(x_3)\right|^2dx_3.\tag{II.5.11}$$

II.5.3 Strong compactness of $(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon})$ and variational problem

$$\text{Set}\quad f_0(s)\ :=\ \min\left[\mathbb{C}[\boldsymbol{E}]:\boldsymbol{E}\ ;\ \boldsymbol{E}\in\boldsymbol{M}^{3\times 3}_{sym}\text{ and }E_{33}=s\right].\tag{II.5.12}$$

Note that f_0 defined above in (II.5.12) can be evaluated as

$$f_0(s) = c_{11}|s|^2 - 2\sigma c_{12}|s|^2 := Y|s|^2, \qquad \int_0^1 f_0(s(x_3))dx_3 = Y\left\|s(x_3)\right\|^2_{L^2(0,1)} \tag{II.5.13}$$

where $\sigma = \left(\dfrac{c_{12}}{c_{11}+c_{12}}\right)$ is the Poisson's ratio and $Y = \left(c_{11} - 2\sigma c_{12}\right)$ is the Young's modulus. We now state the main result of this section.

Theorem II.5.1. *There exists a subsequence* $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ *not relabeled such that* $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}^o$ *strongly in* $H^1(\Omega,\mathbb{R}^3)$*,* $\widehat{u_3}^\varepsilon \to v^o$ *strongly in* $H^1_\sharp((0,1),\mathbb{R})$ *and* $(\boldsymbol{m}^o, v^o)$ *minimizes* $\mathscr{I}^o(\boldsymbol{m},v)$ *in* $\mathscr{A}_o = \Big\{(\boldsymbol{m}(x_3),$ $v(x_3)) \in H^1((0,1),m_sS^2)\times H^1_\sharp((0,1),\mathbb{R})\Big\}$ *where* $\mathscr{I}^o(\boldsymbol{m},v)$ *is defined as*

$$\mathscr{I}^o(\boldsymbol{m},v) = \int_0^1 d\left|\partial_3\boldsymbol{m}\right|^2 + \varphi(\boldsymbol{m}) + \pi|\boldsymbol{m}_p|^2 + \frac{1}{2}f_0\left(\partial_3 v - E_{s_{33}}(\boldsymbol{m})\right) - \boldsymbol{h}_a\cdot\boldsymbol{m}. \tag{II.5.14}$$

Proof. Comparing energy of $\mathscr{I}^\varepsilon$ at its minimizer $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ with the test function $(\boldsymbol{m}^o, \boldsymbol{u}^\varepsilon)$ we get $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon) \le \mathscr{I}^\varepsilon(\boldsymbol{m}^o,\boldsymbol{u}^\varepsilon)$ which expands out as

$$\begin{aligned}
&\int_\Omega \Big[\frac{d}{\varepsilon^2}|\nabla_p\boldsymbol{m}^\varepsilon|^2 + d\left|\partial_3\boldsymbol{m}^\varepsilon\right|^2 + \varphi(\boldsymbol{m}^\varepsilon) - \boldsymbol{h}_a\cdot\boldsymbol{m}^\varepsilon + \frac{\mathbb{C}}{2}\left[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\right]^2\Big]\boldsymbol{dx} + \mathscr{E}^\varepsilon_d(\boldsymbol{m}^\varepsilon)\\
&\le \int_\Omega \Big[d\left|\partial_3\boldsymbol{m}^o\right|^2 + \varphi(\boldsymbol{m}^o) - \boldsymbol{h}_a\cdot\boldsymbol{m}^o + \frac{\mathbb{C}}{2}\left[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^o)\right]^2\Big] + \mathscr{E}^\varepsilon_d(\boldsymbol{m}^o).
\end{aligned}$$

Equation (II.5.11) gives that $\lim_{\varepsilon\to0}\ \mathscr{E}^\varepsilon_d(\boldsymbol{m}^\varepsilon) = \lim_{\varepsilon\to0}\ \mathscr{E}^\varepsilon_d(\boldsymbol{m}^o)$. Then taking lim-sup of both sides w.r.t. ε, canceling common terms, and noting that $\boldsymbol{m}^\varepsilon(\boldsymbol{x}) \to \boldsymbol{m}^o(x_3)$ strongly in $L^2(\Omega)$, we can simplify the above equation to get

$$\limsup_{\varepsilon\to0}\int_\Omega \Big[\frac{d}{\varepsilon^2}|\nabla_p\boldsymbol{m}^\varepsilon|^2 + d\left|\partial_3\boldsymbol{m}^\varepsilon\right|^2\Big]\boldsymbol{dx} \ \le\ \int_\Omega d\left|\partial_3\boldsymbol{m}^o\right|^2\boldsymbol{dx}.$$

But weak convergence of $\nabla\boldsymbol{m}^\varepsilon$ in equation (II.5.4) implies $\liminf_{\varepsilon\to0}\int_\Omega\left|\partial_3\boldsymbol{m}^o\right|^2 \le \int_\Omega\left|\partial_3\boldsymbol{m}^\varepsilon\right|^2$ which combined with the lim sup condition above gives the strong convergence,

$$\partial_3\boldsymbol{m}^\varepsilon \to \partial_3\boldsymbol{m}^o \text{ in } L^2(\Omega), \qquad \frac{1}{\varepsilon}\nabla_p\boldsymbol{m}^\varepsilon \to \boldsymbol{0} \text{ in } L^2(\Omega). \tag{II.5.15}$$

Now we show strong convergence of the elastic terms. Set $s^\varepsilon(\boldsymbol{x})$ and $\widehat{s}^{\,\varepsilon}(x_3)$ as

$$s^\varepsilon(\boldsymbol{x}) := \partial_3 u_3^\varepsilon(\boldsymbol{x}) - E_{s_{33}}(\boldsymbol{m}^\varepsilon), \qquad \widehat{s}^{\,\varepsilon}(x_3) = [\partial_3\widehat{u_3^\varepsilon - E_{s_{33}}}(\boldsymbol{m}^\varepsilon)](x_3) = \widehat{\partial_3 u_3^\varepsilon}(x_3) - \widehat{E_{s_{33}}(\boldsymbol{m}^\varepsilon)}$$

where $\widehat{s}^{\,\varepsilon}(x_3)$ is defined using equation (II.5.1). Noting $f_0(s) = Y|s|^2$, using Jensen's inequality

$$\begin{aligned}\int_\Omega f_0(\partial_3\widehat{u_3}^{\varepsilon} - \widehat{E_{s_{33}}(\boldsymbol{m}^\varepsilon)})\boldsymbol{dx} = \int_0^1\int_\omega Y|\widehat{s}^{\,\varepsilon}|^2\boldsymbol{dx} \le \int_0^1 Y\Big[\int_\omega |s^\varepsilon|^2\boldsymbol{dx}_p\Big]dx_3 = \int_\Omega f_0(s^\varepsilon(\boldsymbol{x}))\boldsymbol{dx}\\ \le \int_\Omega \mathbb{C}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\big]^2\boldsymbol{dx}\end{aligned} \tag{II.5.16}$$

where in the last step we have used the definition of f_0 from equation (II.5.12).
By definition $E_{s_{33}}(\boldsymbol{m}^\varepsilon) = \frac{3}{2m_s^2}\big(|m_3^\varepsilon|^2 - \frac{1}{3}\big)$ which using $\widehat{|m_3^\varepsilon|^2} \to |m_3^o|^2$ in $L^2(0,1)$ from equation (II.5.7) gives

$$\widehat{E_{s_{33}}(\boldsymbol{m}^\varepsilon)} \to \boldsymbol{E}_{s_{33}}(\boldsymbol{m}^o) \quad \text{in } L^2(0,1).$$

The above combined with weak convergence $\partial_3\widehat{u_3}^{\varepsilon} \rightharpoonup \partial_3 v^o$ in $L^2(0,1)$ in eqn. (II.5.8) gives

$$\partial_3\widehat{u_3}^{\varepsilon} - \widehat{E_{s_{33}}(\boldsymbol{m}^\varepsilon)} \rightharpoonup \partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^o) \text{ in } L^2(0,1)\,.$$

Then noting from eqn. (II.5.13) that $\int_\Omega f_0(s(x_3))\boldsymbol{dx} = Y|\omega|\;\|s(x_3)\|^2_{L^2(0,1)}$ and weak lower semi-continuity of norm in $L^2(0,1)$ gives

$$\begin{aligned}\int_\Omega f_0\big(\partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^o)\big)\boldsymbol{dx} &\le \liminf_{\varepsilon\to 0}\int_\Omega f_0(\partial_3\widehat{u_3}^{\varepsilon} - \widehat{E_{s_{33}}(\boldsymbol{m}^\varepsilon)})\boldsymbol{dx}\\ &\le \liminf_{\varepsilon\to 0}\int_\Omega \mathbb{C}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\big]^2\boldsymbol{dx}\end{aligned} \tag{II.5.17}$$

where in the last step we use eqn. (II.5.16).

To get strong convergence we will show the converse inequality of equation (II.5.17). For that we need to compare the energy $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ with some test function based on $\boldsymbol{m}^o$ and v^o. But the lack of regularity of $v^o \in H^1(0,1)$ requires a mollification procedure. Let $v^h(x_3) \in \mathscr{D}(0,1)$ and $v^h(x_3) \to v^o(x_3)$ in $H^1(0,1)$ as $h \to 0$. Set $s^h(x_3) := \big(\partial_3 v^h(x_3) - E_{s_{33}}(\boldsymbol{m}^o)\big)$. Note $\lim_{h\to 0} s^h = \big(\partial_3 v^o(x_3) - E_{s_{33}}(\boldsymbol{m}^o)\big)$. Define

$$\boldsymbol{v}_h^\varepsilon(\boldsymbol{x}) := \begin{bmatrix} \varepsilon E_{s_{11}}(\boldsymbol{m}^o)x_1 + \varepsilon E_{s_{12}}(\boldsymbol{m}^o)x_2 - \varepsilon\sigma\; s^h(x_3)x_1\\ \varepsilon E_{s_{22}}(\boldsymbol{m}^o)x_2 + \varepsilon E_{s_{12}}(\boldsymbol{m}^o)x_1 - \varepsilon\sigma\; s^h(x_3)x_2\\ v^h(x_3) + 2\varepsilon\big(E_{s_{13}}(\boldsymbol{m})x_1 + 2E_{s_{23}}(\boldsymbol{m})x_2\big)\end{bmatrix}. \tag{II.5.18}$$

For $\boldsymbol{v}_h^\varepsilon$ defined above, $\boldsymbol{\kappa}^\varepsilon[\boldsymbol{v}_h^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^o)$ is given by

$$\boldsymbol{\kappa}^\varepsilon[\boldsymbol{v}_h^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^o) = \begin{bmatrix} -\sigma s^h & 0 & -\frac{\varepsilon}{2}\left(\sigma x_1\ \partial_3 s^h + x_1\ \partial_3 E_{s_{11}}(\boldsymbol{m}^o) + x_2\ \partial_3 E_{s_{12}}(\boldsymbol{m}^o)\right) \\ \cdot & -\sigma s^h & -\frac{\varepsilon}{2}\left(\sigma x_2\ \partial_3 s^h + x_2\ \partial_3 E_{s_{22}}(\boldsymbol{m}^o) + x_1\ \partial_3 E_{s_{12}}(\boldsymbol{m}^o)\right) \\ \cdot & \cdot & s^h + 2\varepsilon\left(x_1\ \partial_3 E_{s_{13}}(\boldsymbol{m}^o) + x_2\ \partial_3 E_{s_{23}}(\boldsymbol{m}^o)\right) \end{bmatrix},$$

where we have left out terms below the diagonal due to symmetry. A straight forward computation gives, (Recall $Y = c_{11} - \sigma c_{12}$ from eqn. (II.5.13))

$$\int_\Omega \mathbb{C}\left[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{v}_h^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^o)\right]^2 \boldsymbol{dx} = \int_\Omega f_0(s^h)\ \boldsymbol{dx} + O(\varepsilon^2). \tag{II.5.19}$$

Then comparing energy of the test function $(\boldsymbol{m}^o, \boldsymbol{v}_h^\varepsilon)$ with $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ gives

$$\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)\ \le\ \mathscr{I}^\varepsilon(\boldsymbol{m}^o, \boldsymbol{v}_h^\varepsilon).$$

Fixing h and taking lim-sup of both sides w.r.t. ε, using strong convergence in (II.5.15), and equation (II.5.19), the above simplifies to

$$\begin{aligned} \limsup_{\varepsilon\to 0} \int_\Omega \frac{\mathbb{C}}{2}\left[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\right]^2 \boldsymbol{dx}\ &\le\ \limsup_{\varepsilon\to 0} \int_\Omega \frac{\mathbb{C}}{2}\left[\boldsymbol{\kappa}^\varepsilon(\boldsymbol{v}_h^\varepsilon) - \boldsymbol{E}_s(\boldsymbol{m}^o)\right]^2 \boldsymbol{dx} \\ &= \int_\Omega \frac{1}{2} f_0(s^h) \boldsymbol{dx}. \end{aligned}$$

Now taking $\lim_{h\to 0}$ of L.H.S. and noting that $\lim_{h\to 0} s^h = (\partial_3 v^o(x_3) - E_{s_{33}}(\boldsymbol{m}^o))$ gives

$$\limsup_{\varepsilon\to 0} \int_\Omega \frac{\mathbb{C}}{2}\left[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\right]^2 \boldsymbol{dx}\ \le\ \int_\Omega \frac{1}{2} f_0\left(\partial_3 v^o(x_3) - E_{s_{33}}(\boldsymbol{m}^o)\right) \boldsymbol{dx}. \tag{II.5.20}$$

Then (II.5.17) and (II.5.20) together give along with eqn. (II.5.12), the strong convergence

$$\lim_{\varepsilon\to 0}\ \partial_3 \widehat{u_3}^\varepsilon \to \partial_3 v^o \text{ in } L^2(0,1), \tag{II.5.21a}$$

$$\lim_{\varepsilon\to 0} \int_\Omega \frac{\mathbb{C}}{2}\left[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\right]^2 \boldsymbol{dx} \to \int_\Omega \frac{1}{2} f_0\left(\partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^o)\right) \boldsymbol{dx}. \tag{II.5.21b}$$

Finally its easy to see that the strong convergence from (II.5.15) and (II.5.21) together gives,

$$\lim_{\varepsilon\to 0} \mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) = |\omega| \mathscr{I}^o(\boldsymbol{m}^o, v^o).$$

Now we show that $\left(\boldsymbol{m}^o, v^o\right)$ minimizes $\mathscr{I}^o(\boldsymbol{m}, v)$ in $\mathscr{A}_o$. For any $\left(\boldsymbol{m}, v\right)$ with $\boldsymbol{m} \in H^1((0,1), m_s S^2)$

and $v \in C^{\infty}(0,1) \cap \{v(0)=0\}$, let us define as in eqn. (II.5.18)

$$\boldsymbol{V}^{\varepsilon}(\boldsymbol{x}) := \begin{bmatrix} \varepsilon E_{s_{11}}(\boldsymbol{m})x_1 + \varepsilon E_{s_{12}}(\boldsymbol{m})x_2 - \varepsilon\sigma\,\big(\partial_3 v - E_{s_{33}}(\boldsymbol{m})\big)x_1 \\ \varepsilon E_{s_{22}}(\boldsymbol{m})x_2 + \varepsilon E_{s_{12}}(\boldsymbol{m})x_1 - \varepsilon\sigma\,\big(\partial_3 v - E_{s_{33}}(\boldsymbol{m})\big)x_2 \\ v(x_3) + 2\varepsilon\big(E_{s_{13}}(\boldsymbol{m})x_1 + 2E_{s_{23}}(\boldsymbol{m})x_2\big) \end{bmatrix}.$$

Then comparing energy of $\mathscr{I}^{\varepsilon}(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon})$ with $\mathscr{I}^{\varepsilon}(\boldsymbol{m},\boldsymbol{V}^{\varepsilon})$ we get $\mathscr{I}^{\varepsilon}(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon}) \le \mathscr{I}^{\varepsilon}(\boldsymbol{m},\boldsymbol{V}^{\varepsilon})$. Taking $\lim_{\varepsilon}$ of the inequality we note that L.H.S converges from above $\lim_{\varepsilon\to 0} \mathscr{I}^{\varepsilon}(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon}) = |\omega|\mathscr{I}^{o}(\boldsymbol{m}^{o},v^{o})$. Using equations (II.5.10) and (II.5.19) with $\boldsymbol{m}$ and $\boldsymbol{V}^{\varepsilon}$ replacing $\boldsymbol{m}^{o}$ and $\boldsymbol{v}_h^{\varepsilon}$ in the respective equations it is easy to check that the R.H.S converges as $\lim_{\varepsilon\to 0} \mathscr{I}^{\varepsilon}(\boldsymbol{m},\boldsymbol{V}^{\varepsilon}) = |\omega|\mathscr{I}^{o}(\boldsymbol{m},v)$. We thus get our minimizing principle on noticing that $v \in C^{\infty}(0,1) \cap \{v(0)=0\}$ is dense in $H^1_{\sharp}(0,1)$ □

II.5.4 Minimization of limit problem

The minimization of $\mathscr{I}^{o}(\boldsymbol{m},v)$ is a substantially simpler problem than the original one. One can see that if the applied field is a constant over the domain, the terms $\varphi(\boldsymbol{m})+\pi|\boldsymbol{m}_p|^2 - \boldsymbol{h}_a\cdot\boldsymbol{m}$ behaves like an "effective anisotropy". If this is minimized over constant vector $\boldsymbol{m} \in m_s S^2$ to give $\boldsymbol{m}^{o}$, then its easy to see that $(\boldsymbol{m}^{o}, E_{s_{33}}(\boldsymbol{m}^{o})x_3)$ minimizes $\mathscr{I}^{o}(\boldsymbol{m},v)$.

For a large class of ferromagnetic materials, the largest energy in the "effective anisotropy" for typical applied fields is the demagnetization term $\pi|\boldsymbol{m}_p|^2$ which finds its minimum if $\boldsymbol{m}^{o}$ is an axial magnetization $(0,0,m_s)$. In particular for our nanowires of Galfenol this is true. Experimentally produced nanowires of Galfenol of 30-100 nanometer diameter show strong alignment of magnetization along the axis in the absence of applied fields and need large applied fields in transverse direction to alter this state. Experimental verification of these results for Galfenol wires can be seen from Magnetic Force Microscopy (MFM) scans in Figures II.2 taken from [Downey, 2008].

These scans are done for wires with 100 nanometer diameter and <110> crystallographic orientation with no applied field. For cubic anisotropy, <110> is a local minimum of the anisotropy energy and gives zero magnetostatic energy contribution making it a global minimum of the "effective anisotropy". The uniformity of the scan along the wire length depicts a uniform state of magnetization and the bright and dark spots at the two ends are interpreted to be the field lines due to an axial magnetization producing net positive and negative poles at the ends.

With these observations in mind, for the following sections we will assume that the field $\boldsymbol{h}_a$ is constant. This assumption simplifies the calculation in the following sections without effecting

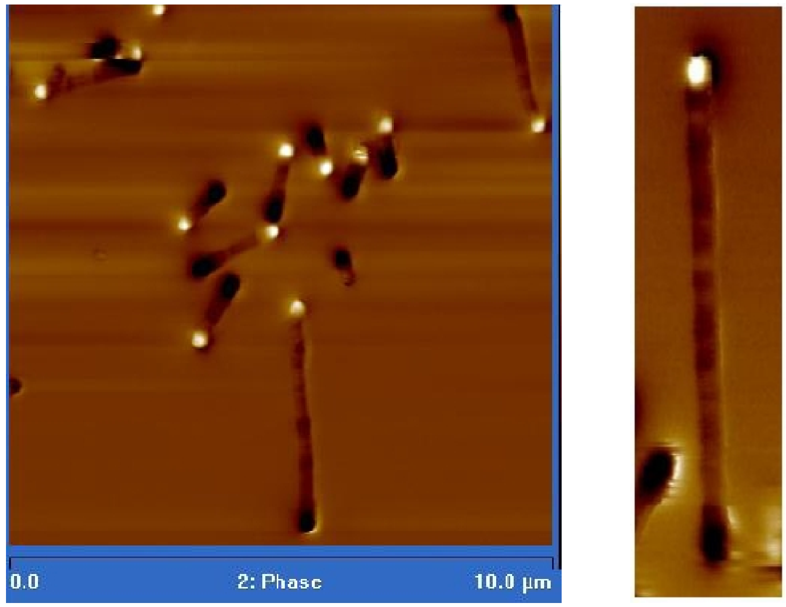

Figure II.2: Left: MFM scan for several Galfenol nanowires, Right: Magnified scan of single nanowire, scale of the wires shown in bottom of left figure (Scans courtesy of Downey [Downey, 2008]).

the main presentation of the asymptotic limiting problem. Let us then set

$$Q_0 = \int_\Omega \varphi(\boldsymbol{m}^o) + \pi|\boldsymbol{m}_p^o|^2 - \boldsymbol{h}_a \cdot \boldsymbol{m}^o \tag{II.5.22}$$

where $\boldsymbol{m}^o$ minimizes $\varphi(\boldsymbol{m}) + \pi|\boldsymbol{m}_p|^2 - \boldsymbol{h}_a \cdot \boldsymbol{m}$ in $m_s S^2$. Then $(\boldsymbol{m}^o, v^o)$ minimizes $\mathscr{I}^o$ where $v^o := E_{s_{33}}(\boldsymbol{m}^o)x_3$. Set

$$\boldsymbol{u}^o(\boldsymbol{x}) := \begin{bmatrix} \varepsilon E_{s_{11}}(\boldsymbol{m}^o)x_1 + \varepsilon E_{s_{12}}(\boldsymbol{m}^o)x_2 \\ \varepsilon E_{s_{22}}(\boldsymbol{m}^o)x_2 + \varepsilon E_{s_{12}}(\boldsymbol{m}^o)x_1 \\ E_{s_{33}}(\boldsymbol{m}^o)x_3 + +2\varepsilon\big(x_1 E_{s_{13}}(\boldsymbol{m}^o) + x_2 E_{s_{23}}(\boldsymbol{m}^o)\big) \end{bmatrix}$$

where we have abused notation a little as $\boldsymbol{u}^o(\boldsymbol{x})$ depends on ε but does not reflect that. Then it is easy to check that $\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o] = \boldsymbol{E}_s(\boldsymbol{m}^o)$ since $\boldsymbol{m}^o$ is constant. Using eqn. (II.3.6) ,

$$\int_\Omega \frac{\mathbb{C}}{2}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o] - \boldsymbol{E}_s(\boldsymbol{m}^o)\big]^2 \boldsymbol{dx} = \int_\Omega \frac{1}{2} f_0\big(\partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^o)\big)\, \boldsymbol{dx} = 0 \tag{II.5.23a}$$

$$\mathscr{I}^\varepsilon(\boldsymbol{m}^o, \boldsymbol{u}^o) = Q_0 + \mathscr{E}_d^\varepsilon(\boldsymbol{m}^o) - \int_\Omega \pi|\boldsymbol{m}_p^o|^2 = Q_0 + \varepsilon Q_1(\boldsymbol{m}^o) + \varepsilon^2 Q_2(\boldsymbol{m}^o) \tag{II.5.23b}$$

$$|\omega| \inf_{\mathscr{A}_o} \mathscr{I}^o(\boldsymbol{m}, v^o) = Q_0. \tag{II.5.23c}$$

II.6 Second order variational limit problem

§ II.5 gives a rigorous derivation of the first order variational approximation $\mathscr{I}^o(\boldsymbol{m},v)$ in the sense that for a sequence of minimizers $(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$ of $\mathscr{I}^\varepsilon(\boldsymbol{m},\boldsymbol{u})$, $\lim_{\varepsilon\to 0}\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon) = |\omega|\mathscr{I}^o(\boldsymbol{m}^o,v^o) + o(\varepsilon)$ with $(\boldsymbol{m}^o,v^o)$ minimizing $\mathscr{I}^o(\boldsymbol{m},v)$ in an appropriate space. Correctors to this approximation come up as higher order theories which involve an expansion of the $o(\varepsilon)$ term. These higher terms can be understood as an asymptotic Γ- series of variational problems in the sense of [Anzellotti and Baldo, 1993].

With this in mind we define $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon) := \varepsilon^{-1}(\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon) - Q_0)$. We look at the limit minimization problem corresponding to $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$. For this we first show that $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$ is bounded above and below independently of ε so that its limit $\varepsilon\to 0$ makes sense. We then show that a limit exists as $\varepsilon\to 0$ for the quantity $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$. Note that

$$\begin{aligned}\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon) - Q_0 &= \left[\int_\Omega \frac{d}{\varepsilon^2}|\nabla_p \boldsymbol{m}^\varepsilon|^2 \boldsymbol{dx}\right] + \left[\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \int_\Omega \pi|\boldsymbol{m}_p^\varepsilon|^2 \boldsymbol{dx}\right] + \left[\int_\Omega \Big\{ d\,|\partial_3 \boldsymbol{m}^\varepsilon|^2 \right.\\ &\qquad \left. + \varphi(\boldsymbol{m}^\varepsilon) - \boldsymbol{h}_a\cdot\boldsymbol{m}^\varepsilon + \pi|\boldsymbol{m}_p^\varepsilon|^2 + \frac{\mathbb{C}}{2}\left[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\right]^2 \Big\}\boldsymbol{dx} - Q_0\right]\\ &= \mathfrak{A}^\varepsilon + \mathfrak{B}^\varepsilon + \mathfrak{C}^\varepsilon \end{aligned} \tag{II.6.1}$$

where $\mathfrak{A}^\varepsilon, \mathfrak{B}^\varepsilon$ and $\mathfrak{C}^\varepsilon$ are the terms in the big square brackets.

II.6.1 Bounds for $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$

Since $(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$ minimizes $\mathscr{I}^\varepsilon(\boldsymbol{m},\boldsymbol{u})$, we have $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon) \le \mathscr{I}^\varepsilon(\boldsymbol{m}^o,\boldsymbol{u}^o)$ which on using equation (II.5.23b) along with the definition of $\mathscr{I}_1^\varepsilon(\boldsymbol{m},\boldsymbol{u})$ gives us the inequality

$$\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon) \le \mathscr{I}_1^\varepsilon(\boldsymbol{m}^o,\boldsymbol{u}^o) = \frac{1}{\varepsilon}\left(\varepsilon Q_1(\boldsymbol{m}^o) + \varepsilon^2 Q_2(\boldsymbol{m}^o)\right) \le K_9. \tag{II.6.2}$$

The lower bound for $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$ requires the following technical condition. See Result 8.2 in [Bhattacharya and James, 1999] and Definition 5.2 from [Le Dret and Meunier, 2005] to see other contexts where such a condition is necessary to get lower bound estimates.

Definition II.6.1. *We say that a minimizer* $(\boldsymbol{m}^o,v^o)$ *of* $\mathscr{I}^o(\boldsymbol{m},v)$ $\big($*cf.Eqn.* (II.5.14)$\big)$*, satisfies the strong second variation condition if for any* $(\boldsymbol{m}(x_3),v(x_3)) \in \mathscr{A}_o$ *there exists* $\Lambda > 0$ *such that,*

$$\begin{aligned}\mathscr{I}^o(\boldsymbol{m},v) - \mathscr{I}^o(\boldsymbol{m}^o,v^o) = \mathscr{I}^o(\boldsymbol{m},v) - Q_0 \ge \Lambda\int_0^1 \Big\{ &\left|\partial_3\boldsymbol{m}(x_3) - \partial_3\boldsymbol{m}^o(x_3)\right|^2 + \left|\boldsymbol{m} - \boldsymbol{m}^o\right|^2 \\ &+ \left|\partial_3 v(x_3) - \partial_3 v^o(x_3)\right|^2\Big\} dx_3. \end{aligned} \tag{II.6.3}$$

provided $\|\boldsymbol{m}-\boldsymbol{m}^o\|_{H^1(0,1)} < K_{10}\varepsilon$ *and* $\|v-v^o\|_{H^1(0,1)} < K_{11}\varepsilon$ *for some* $\varepsilon > 0$ *sufficiently small and* K_{10}, K_{11} *arbitrary constants independent of* ε.

$$\text{Set} \qquad \boldsymbol{M}^\varepsilon := \boldsymbol{m}^\varepsilon - \boldsymbol{m}^o. \tag{II.6.4}$$

Using the hypothesis that $\mathscr{I}^o(\boldsymbol{m}, v^o)$ satisfies strong second variation condition let us show the following Lemma,

Lemma II.6.1. *For* $\mathfrak{C}^\varepsilon$ *defined as in equation* (II.6.1)*,*

$$\mathfrak{C}^\varepsilon \;\geq\; \int_\omega \mathscr{I}^o(\boldsymbol{m}^\varepsilon, u_3^\varepsilon)\boldsymbol{dx} - Q_0 \;\geq\; \Lambda\Big(\|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} \Big).$$

Proof. For fixed $\boldsymbol{x}_p = (x_1, x_2) \in \omega$ we define $\mathfrak{M}^\varepsilon(x_3) := \boldsymbol{m}^\varepsilon(\boldsymbol{x}_p, x_3)$, $\mathfrak{V}^\varepsilon(x_3) := \boldsymbol{u}^\varepsilon(\boldsymbol{x}_p, x_3)$. Then using the strong second variation condition on $\mathscr{I}^o$ we get

$$\mathscr{I}^o(\mathfrak{M}^\varepsilon(x_3), \mathfrak{V}_3^\varepsilon(x_3)) - \mathscr{I}^o(\boldsymbol{m}^o, v^o) \geq \Lambda \int_0^1 \Big\{ \big|\partial_3\big(\mathfrak{M}^\varepsilon - \boldsymbol{m}^o\big)\big|^2 + \big|\mathfrak{M}^\varepsilon - \boldsymbol{m}^o\big|^2 \Big\} dx_3.$$

For fixed $\boldsymbol{x}_p \in \omega$, $\boldsymbol{M}^\varepsilon(\boldsymbol{x}_p, x_3) = \mathfrak{M}^\varepsilon(x_3) - \boldsymbol{m}^o(x_3)$. Integrating above result over $\boldsymbol{x}_p \in \omega$ gives

$$\int_\omega \mathscr{I}^o(\mathfrak{M}^\varepsilon, \mathfrak{V}_3^\varepsilon)\boldsymbol{dx}_p - Q_0 = \int_\omega \mathscr{I}^o(\boldsymbol{m}^\varepsilon, u_3^\varepsilon)\boldsymbol{dx} - Q_0 = \Lambda\big(\|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\big)$$

noting $Q_0 = \int_\omega \mathscr{I}^o(\boldsymbol{m}^o, v^o)\boldsymbol{dx}_p$. From eqn. (II.5.12), $\mathbb{C}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\big]^2 \geq f_0\big(\partial_3 u_3^\varepsilon - \boldsymbol{E}_{s_{33}}(\boldsymbol{m}^\varepsilon)\big)$ which gives $\mathfrak{C}^\varepsilon \geq \int_\omega \mathscr{I}^o(\boldsymbol{m}^\varepsilon, u_3^\varepsilon)\boldsymbol{dx} - Q_0$ and our final result. □

We use the result of Lemma II.6.1 above and Proposition II.A.6 to get

$$\begin{aligned}
\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) - Q_0 &= \mathfrak{A}^\varepsilon + \mathfrak{B}^\varepsilon + \mathfrak{C}^\varepsilon \\
&\geq \frac{d}{\varepsilon^2}\|\nabla_p \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \Big[\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \int_\Omega \pi\big|\boldsymbol{m}_p^\varepsilon\big|^2\boldsymbol{dx}\Big] + \Lambda\Big(\|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} \Big) \\
&\geq \frac{d}{2\varepsilon^2}\|\nabla_p \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \frac{\Lambda}{2}\Big(\|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} \Big) + \varepsilon Q_1(\boldsymbol{m}^o) + \varepsilon^2 Q_2(\boldsymbol{m}^o) - D_{18}\varepsilon^2 \\
&\geq -K_{12}\varepsilon
\end{aligned} \tag{II.6.5}$$

where $Q_1(\boldsymbol{m}^o) = \frac{8\pi}{3}\Big(|\boldsymbol{m}_p^o|^2 - 2|m_3^o|^2\Big)$ and $Q_2(\boldsymbol{m}^o) = \pi^2\Big(\frac{|\boldsymbol{m}_p^o|^2}{2} - |m_3^o|^2\Big)$ as in equation (II.3.6).

II.6.2 Convergence of $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$

Theorem II.6.1. *We have the following convergence,*

$$\lim_{\varepsilon\to 0} \mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) = Q_1 = \frac{16\pi}{3}|m_3^o|^2 - \frac{8\pi}{3}|\boldsymbol{m}_p^o|^2.$$

Proof. Dividing equation (II.6.5) by ε gives

$$\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \geq \frac{1}{\varepsilon}\big(\varepsilon Q_1 + \varepsilon^2 Q_2\big) = \frac{8\pi}{3}\Big(|\boldsymbol{m}_p^o|^2 - 2|m_3^o|^2\Big) + \varepsilon\pi^2\Big(\frac{|\boldsymbol{m}_p^o|^2}{2} - |m_3^o|^2\Big)$$

Taking $\liminf_{\varepsilon\to 0}$ above to get,

$$\liminf_{\varepsilon\to 0} \mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \geq \frac{8\pi}{3}\Big(|\boldsymbol{m}_p^o|^2 - 2|m_3^o|^2\Big).$$

To get the reverse inequality we take divide eqn. (II.6.2) by ε and then take $\limsup$ to get,

$$\limsup_{\varepsilon\to 0} \mathscr{I}_1(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \leq \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big(\varepsilon Q_1 + \varepsilon^2 Q_2\Big) = Q_1 = \frac{8\pi}{3}\Big(|\boldsymbol{m}_p^o|^2 - 2|m_3^o|^2\Big).$$

The $\limsup$ and $\liminf$ inequality together gives our result. □

Remark II.6.2. *In the limit we get that $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ converges to a fixed quantity $Q_1(\boldsymbol{m}^o)$ depending only on $\boldsymbol{m}^o$. $Q_1(\boldsymbol{m}^o)$ consists of the mutual interaction of the poles generated by $\boldsymbol{m}^o$ on one end $\omega(0)$ of the wire domain Ω with the other end $\omega(1)$ giving the term $\frac{16}{3}\pi|m_3^o|^2$, and the self-interaction of the poles created by $\boldsymbol{m}^o$ on the curved surface $\partial\omega\times(0,1)$ giving the term $-\frac{8}{3}\pi|\boldsymbol{m}_p^o|^2$.*

II.7 Third variational limit problem

As in the previous section we first define $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) := \varepsilon^{-2}\big(\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) - Q_0 - \varepsilon Q_1(\boldsymbol{m}^o)\big)$. We will show that $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ is bounded above and below independent of ε. Then we define $\boldsymbol{w}^\varepsilon$ in (II.7.11) and prove a weak compactness result for it. The convergence is improved to strong in Theorem II.7.1 where we also define a new variational problem $\mathscr{I}_2^o$ and show its relation with $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$.

Recalling $\mathfrak{A}^\varepsilon, \mathfrak{B}^\varepsilon$ and $\mathfrak{C}^\varepsilon$ from equation (II.6.1) in § II.6 , we note

$$\begin{aligned}
&\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) - Q_0 - \varepsilon Q_1(\boldsymbol{m}^o) \\
&= \Big[\int_\Omega \frac{d}{\varepsilon^2}|\nabla_p \boldsymbol{m}^\varepsilon|^2 \boldsymbol{dx}\Big] + \Big[\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \pi\int_\Omega |\boldsymbol{m}_p^\varepsilon|^2 \boldsymbol{dx} - \varepsilon Q_1\Big] + \Big[\int_\Omega \Big\{d\ |\partial_3 \boldsymbol{m}^\varepsilon|^2 + \varphi(\boldsymbol{m}^\varepsilon) \\
&\quad - \boldsymbol{h}_a\cdot \boldsymbol{m}^\varepsilon + 2|\boldsymbol{m}_p^\varepsilon|^2 + \frac{1}{2}\mathbb{C}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2\Big\}\boldsymbol{dx} - Q_0\Big] \\
&= \mathfrak{A}^\varepsilon + \big(\mathfrak{B}^\varepsilon - \varepsilon Q_1(\boldsymbol{m}^o)\big) + \mathfrak{C}^\varepsilon.
\end{aligned}$$

II.7.1 Boundedness of $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$

To get an upper bound on $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ we use the upper bound on $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ from eqn. (II.6.2) and subtract $Q_1(\boldsymbol{m}^o)$ from both sides to get

$$\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) = \frac{1}{\varepsilon}\big[\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) - \varepsilon Q_1\big] \le \frac{1}{\varepsilon}\big[\mathscr{I}_1^\varepsilon(\boldsymbol{m}^o, \boldsymbol{u}^o) - \varepsilon Q_1\big] = Q_2(\boldsymbol{m}^o) \le K_{12}. \tag{II.7.1}$$

To get a lower bound on $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$, we subtract $\varepsilon Q_1(\boldsymbol{m}^o)$ from the lower bound in the previous section in equation (II.6.5) to get

$$\begin{aligned}
&\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) - Q_0 - \varepsilon Q_1(\boldsymbol{m}^o) = \mathfrak{A}^\varepsilon + \big(\mathfrak{B}^\varepsilon - \varepsilon Q_1(\boldsymbol{m}^o)\big) + \mathfrak{C}^\varepsilon \\
&\ge \frac{d}{2\varepsilon^2}\big\|\nabla_p \boldsymbol{M}^\varepsilon\big\|^2_{L^2(\Omega)} + \frac{\Lambda}{2}\Big(\ \|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\ \Big) + \varepsilon^2 Q_2(\boldsymbol{m}^o) - D_{18}\varepsilon^2 \qquad \text{(II.7.2)} \\
&\ge -K_{13}\varepsilon^2.
\end{aligned}$$

The upper bound II.7.1 and lower bound in II.7.2 together give with Sobolev inequality on Ω

$$\begin{aligned}
K_{14}\varepsilon^2 &\ge \frac{d}{2\varepsilon^2}\big\|\nabla_p \boldsymbol{M}^\varepsilon\big\|^2_{L^2(\Omega)} + \frac{\Lambda}{2}\Big(\ \|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\ \Big) \\
&\ge \frac{\Lambda}{2}\big\|\boldsymbol{M}^\varepsilon\big\|^2_{H^1(\Omega)} \ge C_q\big\|\boldsymbol{M}^\varepsilon\big\|^2_{L^q(\Omega)}, \qquad \forall 1 \le q \le 6 \qquad \text{(II.7.3)}
\end{aligned}$$

with C_q being the appropriate Sobolev constant.

II.7.2 Weak convergence of $\boldsymbol{w}^\varepsilon$

In this subsection we will extract some energy terms from the elastic energy and define a new variable $\boldsymbol{w}^\varepsilon$ from the extracted terms. For this we need an improvement on Lemma II.6.1 . For that first note that using a truncated Taylor Expansion we write $\boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)$ as

$$\boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)-\boldsymbol{E}_s(\boldsymbol{m}^o)=\boldsymbol{E}_s'(\boldsymbol{m}^o)\cdot\boldsymbol{M}^\varepsilon+\frac{1}{2}\boldsymbol{E}_s''(\boldsymbol{m}^o)\boldsymbol{M}^\varepsilon\cdot\boldsymbol{M}^\varepsilon+o(|\boldsymbol{M}^\varepsilon|^2):=\Delta(\boldsymbol{M}^\varepsilon) \tag{II.7.4}$$

where we recall $\boldsymbol{m}^\varepsilon=\boldsymbol{m}^o+\boldsymbol{M}^\varepsilon$ and $\boldsymbol{E}_s'(\boldsymbol{m})$ and $\boldsymbol{E}_s''(\boldsymbol{m})$ are the 1$^{\text{st}}$ and 2$^{\text{nd}}$ derivatives of $\boldsymbol{E}_s(\boldsymbol{m})$ w.r.t. $\boldsymbol{m}$. Since $\boldsymbol{E}_s(\boldsymbol{m})$ is a polynomial function of $\boldsymbol{m}$, both $\boldsymbol{E}_s'$ and $\boldsymbol{E}_s''$ are bounded in L^∞ for $|\boldsymbol{m}|=m_s$. Then using (II.7.3) we get

$$\Delta(\boldsymbol{M}^\varepsilon)\le K_{15}|\boldsymbol{M}^\varepsilon|+K_{16}|\boldsymbol{M}^\varepsilon|^2 \qquad\text{and}\qquad \left\|\Delta(\boldsymbol{M}^\varepsilon)\right\|^2_{L^2(\Omega)}\le K_{17}\varepsilon^2. \tag{II.7.5}$$

Set $\boldsymbol{u}^\varepsilon=\boldsymbol{u}^o+\boldsymbol{U}^\varepsilon$. Note $\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon]=\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o]+\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]$. Eqn. (II.2.7) and Young's inequality gives

$$\left|\mathbb{C}\big[\Delta(\boldsymbol{M}^\varepsilon)\big]:\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right| \;\le\; \Gamma\left|\Delta(\boldsymbol{M}^\varepsilon)\right|\,\left|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right| \;\le\; \frac{4\Gamma^2}{\gamma}\left|\Delta(\boldsymbol{M}^\varepsilon)\right|^2+\frac{\gamma}{4}\left|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right|^2. \tag{II.7.6}$$

From (II.5.23) note that $\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o]-\boldsymbol{E}_s(\boldsymbol{m}^o)=\boldsymbol{0}$. Then equations (II.2.7), (II.5.12) and (II.7.6) gives

$$\begin{aligned}
&\frac{\mathbb{C}}{2}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon]-\boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\big]^2=\frac{\mathbb{C}}{2}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o]+\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]-\boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\big]^2\\
&=\frac{\mathbb{C}}{2}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o]-\boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\big]^2+\frac{\mathbb{C}}{2}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\big]^2+\mathbb{C}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o]-\boldsymbol{E}_s(\boldsymbol{m}^o)\big]:\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]+\mathbb{C}\big[\Delta(\boldsymbol{M}^\varepsilon)\big]:\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\\
&\ge\frac{1}{2}f_0\big(\partial_3 v^o-E_{s_{33}}(\boldsymbol{m}^\varepsilon)\big)+\frac{\gamma}{2}\left|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right|^2-\frac{4\Gamma^2}{\gamma}\left|\Delta(\boldsymbol{M}^\varepsilon)\right|^2-\frac{\gamma}{4}\left|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right|^2\\
&=\frac{1}{2}f_0\big(\partial_3 v^o-E_{s_{33}}(\boldsymbol{m}^\varepsilon)\big)+\frac{\gamma}{4}\left|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right|^2-\frac{4\Gamma^2}{\gamma}\left|\Delta(\boldsymbol{M}^\varepsilon)\right|^2.
\end{aligned} \tag{II.7.7}$$

Then using equation (II.7.5), and above result (II.7.7), we improve Lemma II.6.1 to get

$$\begin{aligned}
\mathfrak{E}^\varepsilon&=\int_\Omega\Big\{d\;|\partial_3\boldsymbol{m}^\varepsilon|^2+\varphi(\boldsymbol{m}^\varepsilon)-\boldsymbol{h}_a\cdot\boldsymbol{m}^\varepsilon+\pi|\boldsymbol{m}^\varepsilon_p|^2+\frac{1}{2}\mathbb{C}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon]-\boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\big]^2\Big\}\boldsymbol{dx}-Q_0\\
&\ge\int_\Omega\Big\{d\;|\partial_3\boldsymbol{m}^\varepsilon|^2+\varphi(\boldsymbol{m}^\varepsilon)-\boldsymbol{h}_a\cdot\boldsymbol{m}^\varepsilon+\pi|\boldsymbol{m}^\varepsilon_p|^2+\frac{1}{2}f_0\big(\partial_3 v^o-\boldsymbol{E}_{s_{33}}(\boldsymbol{m}^\varepsilon)\big)\Big\}\boldsymbol{dx}\\
&\qquad\qquad+\int_\Omega\Big\{\frac{\gamma}{4}\left|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right|^2-\frac{4\Gamma^2}{\gamma}|\Delta(\boldsymbol{M}^\varepsilon)|^2\Big\}\boldsymbol{dx}-Q_0\\
&\ge\Big[\int_\omega\mathscr{I}^o(\boldsymbol{m}^\varepsilon,v^o)\boldsymbol{dx}_p-Q_0\Big]+\frac{\gamma}{4}\left\|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right\|^2_{L^2(\Omega)}-4\Gamma^2\gamma^{-1}\left\|\Delta(\boldsymbol{M}^\varepsilon)\right\|^2_{L^2(\Omega)}\\
&\ge\Lambda\Big(\left\|\partial_3\boldsymbol{M}^\varepsilon\right\|^2_{L^2(\Omega)}+\left\|\boldsymbol{M}^\varepsilon\right\|^2_{L^2(\Omega)}\Big)+\frac{\gamma}{4}\left\|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right\|^2_{L^2(\Omega)}-K_{18}\varepsilon^2
\end{aligned} \tag{II.7.8}$$

where we have used the strong second variation condition on $\mathscr{I}^o$ in the last step. Using this we revisit the lower bound equation (II.7.2) using Proposition II.A.6 to give

$$\begin{aligned}\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)-Q_0-\varepsilon Q_1(\boldsymbol{m}^o)&=\mathfrak{A}^\varepsilon+\left(\mathfrak{B}^\varepsilon-\varepsilon Q_1(\boldsymbol{m}^o)\right)+\mathfrak{C}^\varepsilon\\ &\geq\int_\Omega\left\{\frac{d}{\varepsilon^2}|\nabla_p\boldsymbol{M}^\varepsilon|^2+\Lambda|\partial_3\boldsymbol{M}^\varepsilon|^2+\Lambda|\boldsymbol{M}^\varepsilon|^2+\frac{\gamma}{4}\left|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right|^2\right\}-K_{18}\varepsilon^2+\left(\mathfrak{B}^\varepsilon-\varepsilon Q_1\right)\\ &\geq\frac{\gamma}{4}\left\|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\right\|^2_{L^2(\Omega)}+\varepsilon^2Q_2(\boldsymbol{m}^o)-D_{18}\varepsilon^2-K_{18}\varepsilon^2.\end{aligned}\tag{II.7.9}$$

The upper bound (II.7.1) and the above equation then gives

$$K_{19}\varepsilon^2\geq\int_\Omega\left|\boldsymbol{\kappa}^\varepsilon(\boldsymbol{U}^\varepsilon)\right|^2.\tag{II.7.10}$$

Set $\boldsymbol{w}^\varepsilon=(\boldsymbol{u}_p^\varepsilon-\boldsymbol{u}_p^o,\varepsilon^{-1}(u_3^\varepsilon-u_3^o))=(\boldsymbol{U}_p^\varepsilon,\varepsilon^{-1}U_3^\varepsilon)$ and note

$$\begin{aligned}\boldsymbol{\kappa}^\varepsilon(\boldsymbol{U}^\varepsilon)&=\begin{bmatrix}\frac{1}{\varepsilon}\partial_1 w_1^\varepsilon & \frac{1}{2\varepsilon}(\partial_1 w_2^\varepsilon+\partial_2 w_1^\varepsilon) & \frac{1}{2}(\partial_1 w_3^\varepsilon+\partial_3 w_1^\varepsilon)\\ \frac{1}{2\varepsilon}(\partial_1 w_2^\varepsilon+\partial_2 w_1^\varepsilon) & \frac{1}{\varepsilon}\partial_2 w_2^\varepsilon & \frac{1}{2}(\partial_2 w_3^\varepsilon+\partial_3 w_2^\varepsilon)\\ \frac{1}{2}(\partial_1 w_3^\varepsilon+\partial_3 w_1^\varepsilon) & \frac{1}{2}(\partial_2 w_3^\varepsilon+\partial_3 w_2^\varepsilon) & \varepsilon\partial_3 w_3^\varepsilon\end{bmatrix}\\ &=:\boldsymbol{\chi}(\boldsymbol{w}^\varepsilon).\end{aligned}\tag{II.7.11}$$

Note $\left|\frac{\boldsymbol{\chi}(\boldsymbol{w}^\varepsilon)}{\varepsilon}\right|\geq\left|\boldsymbol{E}(\boldsymbol{w}^\varepsilon)\right|$ where $\boldsymbol{E}(\boldsymbol{w}^\varepsilon)$ is the elastic strain of field $\boldsymbol{w}^\varepsilon$. Korn's inequality in (II.7.10) gives,

$$K_{19}\geq\int_\Omega\left|\frac{\boldsymbol{\kappa}^\varepsilon(\boldsymbol{U}^\varepsilon)}{\varepsilon}\right|^2\boldsymbol{dx}=\int_\Omega\left|\frac{\boldsymbol{\chi}(\boldsymbol{w}^\varepsilon)}{\varepsilon}\right|^2\boldsymbol{dx}\geq\int_\Omega\left|\boldsymbol{E}(\boldsymbol{w}^\varepsilon)\right|^2\boldsymbol{dx}\geq\alpha\int_\Omega\left(\left|\nabla\boldsymbol{w}^\varepsilon\right|^2+|\boldsymbol{w}^\varepsilon|^2\right)\boldsymbol{dx}$$

where $\alpha(\Omega)>0$ is the Korn's constant. These results together imply for some unrelabeled subsequence

$$\boldsymbol{w}^\varepsilon\rightharpoonup\boldsymbol{w}^o\text{ in }H^1(\Omega;\mathbb{R}^3)\qquad\qquad\boldsymbol{w}^\varepsilon\to\boldsymbol{w}^o\text{ in }L^2(\Omega;\mathbb{R}^3)\tag{II.7.12a}$$

$$\boldsymbol{E}(\boldsymbol{w}^\varepsilon)\rightharpoonup\boldsymbol{E}(\boldsymbol{w}^o)\text{ in }L^2(\Omega;\mathbb{R}^3)\qquad\qquad\frac{\boldsymbol{\chi}^\varepsilon}{\varepsilon}\rightharpoonup\boldsymbol{\nu}^0\text{ in }L^2(\Omega;\mathbb{R}^3).\tag{II.7.12b}$$

Note from (II.7.11),

$$\frac{\chi_{ij}^\varepsilon}{\varepsilon}=\frac{1}{\varepsilon^2}E_{ij}(\boldsymbol{w}^\varepsilon)\text{ for }(i,j)\in\{1,2\},\qquad\qquad\frac{\chi_{i3}^\varepsilon}{\varepsilon}=\frac{1}{\varepsilon}E_{i3}(\boldsymbol{w}^\varepsilon)\text{ for }i\in\{1,2\}$$

which together imply after using lower semi-continuity of norm w.r.t weak convergence

$$\left\|E_{ij}(\boldsymbol{w}^o)\right\|_{L^2(\Omega)} \leq \liminf_{\varepsilon\to 0} \left\|E_{ij}(\boldsymbol{w}^\varepsilon)\right\|_{L^2(\Omega)} = 0 \tag{II.7.13}$$

when $i \in \{1,2\}$ and $j \in \{1,2,3\}$.

Lemma II.7.1. *Let the strain corresponding to a displacement field $\boldsymbol{w}^o \in H^1_\sharp(\Omega)$, $\boldsymbol{E}(\boldsymbol{w}^o)$ be such that $E_{ij}(\boldsymbol{w}^o) = 0$ for $i \in \{1,2\}$ and $j \in \{1,2,3\}$. Then $\exists \gamma^o(x_3) \in H^1_\sharp(0,1)$ so that $\boldsymbol{w}^o$ is given by,*

$$\begin{aligned} &w_1^o(\boldsymbol{x}) = w_1^o(x_3) \in H^2_{\sharp\sharp}(0,1)\,, \qquad\qquad w_2^o(\boldsymbol{x}) = w_2^o(x_3) \in H\sharp\sharp^2(0,1)\,,\\ &w_3^o(\boldsymbol{x}) = -x_1\ \partial_3 w_1^o(x_3) - x_2\ \partial_3 w_2^o(x_3) + \gamma^o(x_3) \end{aligned} \tag{II.7.14}$$

and $H^2_{\sharp\sharp}(0,1) = \left\{w \in H^2(0,1) : w(x_3 = 0) = \partial_3 w(x_3 = 0) = 0\,\right\}$.

Proof. Given $E_{11}(\boldsymbol{w}^o) = \partial_1 w_1^o(\boldsymbol{x}) = 0$ and $E_{22}(\boldsymbol{w}^o) = \partial_2 w_2^o(\boldsymbol{x}) = 0$ together gives $w_1^o(\boldsymbol{x}) = \alpha_1(x_2,x_3)$ and $w_2^o(\boldsymbol{x}) = \alpha_2(x_1,x_3)$. $E_{12}(\boldsymbol{w}^o) = 0$ gives us,

$$\partial_2 w_1^o(\boldsymbol{x}) + \partial_1 w_2^o(\boldsymbol{x}) = \partial_2\alpha_1(x_2,x_3) + \partial_1\alpha_2(x_1,x_3) = 0.$$

This implies $\partial_2\alpha_1(x_2,x_3) = -\partial_1\alpha_2(x_1,x_3) = \beta(x_3)$. Thus, $w_1^o(\boldsymbol{x}) = \gamma_1(x_3) + x_2\beta(x_3)$ and $w_2^o(\boldsymbol{x}) = \gamma_2(x_3) - x_1\beta(x_3)$. Also given $E_{13}(\boldsymbol{w}^o) = E_{23}(\boldsymbol{w}^o) = 0$, we have

$$\partial_2 E_{13}(\boldsymbol{w}^o) - \partial_1 E_{23}(\boldsymbol{w}^o) = \frac{1}{2}(\partial_{32} w_1^o + \partial_{12} w_3^o - \partial_{31} w_2^o - \partial_{12} w_3^o) = \partial_3\beta(x_3) = 0.$$

This gives us $\beta(x_3) = K_{20}$ is constant. Using the Dirichlet boundary conditions at the base $x_3 = 0$, we have $w_1^o(x_2,0) = \gamma_1(0) + x_2 K_{20} = 0$ which gives us $K_{20} = 0$. So $w_1^o(\boldsymbol{x}) = \gamma_1(x_3)$ and $w_2^o(\boldsymbol{x}) = \gamma_2(x_3)$. We finally have,

$$\begin{aligned} &E_{13}(\boldsymbol{w}^o) = 0 \Rightarrow \partial_1 w_3^o(\boldsymbol{x}) = -\partial_3 w_1^o(\boldsymbol{x}) = -\partial_3\gamma_1(x_3),\\ &E_{2,3}(\boldsymbol{w}^o) = 0 \Rightarrow \partial_2 w_3^o(\boldsymbol{x}) = -\partial_3 w_2^o(\boldsymbol{x}) = -\partial_3\gamma_2(x_3). \end{aligned}$$

which gives us for w_3^o on integrating above equations

$$\begin{aligned} &w_3^o(\boldsymbol{x}) = -x_1\,\partial_3\gamma_1(x_3) - x_2\,\partial_3\gamma_2(x_3) + \gamma^o(x_3) = -x_1\ \partial_3 w_1^o(x_3) - x_2\ \partial_3 w_2^o(x_3) + \gamma^o(x_3),\\ &\partial_3 w_3^o(\boldsymbol{x}) = -x_1\ \partial_{33} w_1^o(x_3) - x_2\ \partial_{33} w_2^o(x_3) + \partial_3\gamma^o(x_3). \end{aligned}$$

Note also that $\boldsymbol{w}^o \in H^1(\Omega)$ gives $\partial_3 w_3^o \in L^2(\Omega)$. Equation (II.7.14) gives then that $\partial_{33} w_i^o(x_3) \in L^2(\Omega)$

for $i = 1,2$ and $\partial_3\gamma_3 \in L^2(\Omega)$ and thus $w_i^o(x_3) \in H^2(\Omega)$ and $\gamma \in H^1(\Omega)$. Note also that the Dirichlet boundary conditions at the base gives $w_3^o(\boldsymbol{x}_p, 0) = -x_1\, \partial_3 w_1^o(0) - x_2\, \partial_3 w_2^o(0) + \gamma^o(0) \equiv 0$ which means $\partial_3 w_1^o(0) = \partial_3 w_2^o(0) = \gamma^o(0) = 0$. □

The displacement solution in Lemma II.7.1 are well known in literature as the Bernoulli-Navier displacements. See Theorem 4.3 in [Trabucho and Viano, 1996] for more details.

II.7.3 Strong convergence of $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$

For the third limit variational problem we assume that $\boldsymbol{m}^o = (0,0,m_s)$. This assumption greatly simplifies our final limit problem while essentially describing the underlying physics. Refer to Remark II.7.2 to see more regarding this assumption and the more general case. From (II.5.14), $\boldsymbol{M}^\varepsilon = \boldsymbol{m}^\varepsilon - \boldsymbol{m}^o$ which gives $|\boldsymbol{m}^\varepsilon|^2 - |\boldsymbol{m}^o|^2 = |\boldsymbol{M}^\varepsilon|^2 + 2\boldsymbol{m}^o \cdot \boldsymbol{M}^\varepsilon = |\boldsymbol{M}^\varepsilon|^2 + 2m_3^o M_3^\varepsilon \equiv 0$. Also note $\kappa_{33}^\varepsilon(\boldsymbol{u}^\varepsilon) = \partial_3 u_3^o + \chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon) = \partial_3 v^o + \chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon) = E_{s_{33}}(\boldsymbol{m}^o) + \chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)$. Then

$$
\begin{aligned}
|\boldsymbol{M}^\varepsilon|^2 &= -2\boldsymbol{m}^o \cdot \boldsymbol{M}^\varepsilon = -2m_s M_3^\varepsilon, && \text{(II.7.15)}\\
E_{s_{33}}(\boldsymbol{m}^\varepsilon) &= \frac{3\lambda_{100}}{2m_s^2}\Big(|m_3^\varepsilon|^2 - \frac{m_s^2}{3}\Big) = E_{s_{33}}(\boldsymbol{m}^o) - \frac{3\lambda_{100}}{2m_s^2}\, |\boldsymbol{M}_p^\varepsilon|^2, && \text{(II.7.16)}\\
\kappa_{33}^\varepsilon(\boldsymbol{u}^o) - E_{s_{33}}(\boldsymbol{m}^\varepsilon) &= \partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^o) + \frac{3\lambda_{100}}{2m_s^2}\, |\boldsymbol{M}_p^\varepsilon|^2 = \frac{3\lambda_{100}}{2m_s^2}\, |\boldsymbol{M}_p^\varepsilon|^2,
\end{aligned}
$$

and using Hölder's inequality and (II.7.3)

$$
\begin{aligned}
\int_\Omega f_0\big(\kappa_{33}^\varepsilon(\boldsymbol{u}^\varepsilon) - E_{s_{33}}(\boldsymbol{m}^\varepsilon)\big) &= \int_\Omega f_0\big(\partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^\varepsilon)\big) + f_0\big(\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\big) + \frac{3Y\lambda_{100}}{2m_s^2}\, |\boldsymbol{M}_p^\varepsilon|^2\, \chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\\
&\ge \int_\Omega f_0\big(\partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^\varepsilon)\big) + \int_\Omega f_0\big(\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\big) - Y\frac{3\lambda_{100}}{2m_s^2}\, \|\boldsymbol{M}^\varepsilon\|_{L^4(\Omega)}^2 \|\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\|_{L^2(\Omega)}\\
&\ge \int_\Omega f_0\big(\partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^\varepsilon)\big) + \int_\Omega f_0\big(\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\big) - K_{21}\varepsilon^2 \|\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\|_{L^2(\Omega)}. && \text{(II.7.17)}
\end{aligned}
$$

Using (II.5.12) $\mathbb{C}\big[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon]-\boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)\big] \geq f_0\big(\kappa^\varepsilon_{33}(\boldsymbol{u}^\varepsilon)-E_{s_{33}}(\boldsymbol{m}^\varepsilon)\big)$ which gives

$$
\begin{aligned}
\mathfrak{C}^\varepsilon &\geq \int_\Omega d\ |\partial_3\boldsymbol{m}^\varepsilon|^2+\varphi(\boldsymbol{m}^\varepsilon)-\boldsymbol{h}_a\cdot\boldsymbol{m}^\varepsilon+2|\boldsymbol{m}^\varepsilon_p|^2+\frac{1}{2}f_0\big(\kappa^\varepsilon_{33}(\boldsymbol{u}^\varepsilon)-E_{s_{33}}(\boldsymbol{m}^\varepsilon)\big)-\boldsymbol{Q}_0\\
&\geq \int_\Omega d\ |\partial_3\boldsymbol{m}^\varepsilon|^2+\varphi(\boldsymbol{m}^\varepsilon)-\boldsymbol{h}_a\cdot\boldsymbol{m}^\varepsilon+2|\boldsymbol{m}^\varepsilon_p|^2+\frac{1}{2}f_0\big(\partial_3 v^o-E_{s_{33}}(\boldsymbol{m}^\varepsilon)\big)-\boldsymbol{Q}_0\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\int_\Omega f_0(\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon))-K_{21}\varepsilon^2\big\|\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon)\big\|_{L^2(\Omega)}\\
&=\Big[\int_\omega \mathscr{I}^o(\boldsymbol{m}^\varepsilon,v^o)\boldsymbol{dx}_p-\boldsymbol{Q}_0\Big]+\int_\Omega f_0(\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon))-K_{21}\varepsilon^2\big\|\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon)\big\|_{L^2(\Omega)}\\
&=\Lambda\Big(\big\|\partial_3\boldsymbol{M}^\varepsilon\big\|^2_{L^2(\Omega)}+\big\|\boldsymbol{M}^\varepsilon\big\|^2_{L^2(\Omega)}\Big)+\int_\Omega f_0(\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon))-K_{21}\varepsilon^2\big\|\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon)\big\|_{L^2(\Omega)}
\end{aligned}
\tag{II.7.18}
$$

where we have used strong second variation condition on $\mathscr{I}^o(\boldsymbol{m}^\varepsilon,v^o)$ in the last step.

Define $\mathscr{I}^o_2(w_1(x_3),w_2(x_3),\nu(x_3))$ in function space $\mathscr{A}_2$ as

$$
\mathscr{I}^o_2(w_1,w_2,\gamma)=\frac{1}{2}\int_\Omega\Big[f_0\big(x_1\partial_{33}w_1(x_3)\big)+f_0\big(x_2\partial_{33}w_2(x_3)\big)+f_0\big(\partial_3\gamma(x_3)\big)\Big]\boldsymbol{dx}+\boldsymbol{Q}_2(\boldsymbol{m}^o)
\tag{II.7.19}
$$

and $\mathscr{A}_2:=\big\{(w_1(x_3),w_2(x_3),\gamma(x_3))\in H^2_{\sharp\sharp}(0,1)\times H^2_{\sharp\sharp}(0,1)\times H^1_\sharp(0,1)\big\}$.

Theorem II.7.1. *There exists a subsequence $\boldsymbol{w}^\varepsilon$ not relabeled such that $\boldsymbol{w}^\varepsilon\to\boldsymbol{w}^o$ strongly in $H^1(\Omega,\mathbb{R}^3)$. $\boldsymbol{w}^o$ is given as in Lemma II.7.1 and $(w^o_1(x_3),w^o_2(x_3),\gamma^o(x_3))$ minimizes $\mathscr{I}^0_2$ in $\mathscr{A}_2$ and $\lim_{\varepsilon\to 0}\mathscr{I}^\varepsilon_2(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)=\mathscr{I}^o_2(w^o_1,w^o_2,\gamma^o)$ where (w^o_1,w^o_2,γ^o) minimizes $\mathscr{I}^o_2(w_1(x_3),w_2(x_3),\nu(x_3))$ in $\mathscr{A}_2$.*

Proof. Because $\boldsymbol{m}^o=(0,0,m_s)$ we use the relevant magnetostatic estimate from equation (II.A.23) in the remark II.A.4 following Proposition II.A.6 and equation (II.7.18) to give

$$
\begin{aligned}
\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)-\boldsymbol{Q}_0-\varepsilon\boldsymbol{Q}_1(\boldsymbol{m}^o)&=\mathfrak{A}^\varepsilon+\big(\mathfrak{B}^\varepsilon-\varepsilon\boldsymbol{Q}_1(\boldsymbol{m}^o)\big)+\mathfrak{C}^\varepsilon\\
&\geq\int_\Omega\Big\{\frac{d}{\varepsilon^2}|\nabla_p\boldsymbol{M}^\varepsilon|^2+\Lambda|\partial_3\boldsymbol{M}^\varepsilon|^2+\Lambda|\boldsymbol{M}^\varepsilon|^2+\frac{1}{2}f_0\big(\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon)\big)\Big\}-K_{21}\varepsilon^3\|\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon)\|_{L^2(\Omega)}\\
&\quad+\Big(\mathscr{E}^\varepsilon_d(\boldsymbol{m}^\varepsilon)-\pi\int_\Omega\big|\boldsymbol{m}^\varepsilon_p\big|^2-\varepsilon\boldsymbol{Q}_1(\boldsymbol{m}^o)\Big)\\
&\geq\int_\Omega\frac{1}{2}f_0\big(\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon)\big)-K_{21}\varepsilon^3\Big\|\frac{\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon)}{\varepsilon}\Big\|_{L^2(\Omega)}+\varepsilon^2\boldsymbol{Q}_2(\boldsymbol{m}^o).
\end{aligned}
\tag{II.7.20}
$$

Dividing by ε^2, noting $\varepsilon^{-1}\chi^\varepsilon_{33}(\boldsymbol{w}^\varepsilon)=\partial_3 w^\varepsilon_3$ and $\partial_3 w^\varepsilon_3\rightharpoonup\partial_3 w^o_3$ in $L^2(\Omega)$, we get on taking $\liminf_{\varepsilon\to 0}$,

$$
\liminf_{\varepsilon\to 0}\mathscr{I}^\varepsilon_2(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)\geq\int_\Omega\frac{1}{2}f_0\big(\partial_3 w^o_3\big)+\boldsymbol{Q}_2(\boldsymbol{m}^o).
\tag{II.7.21}
$$

The first term in the R.H.S comes from the fact that (II.5.12) gives, $\int_\Omega f_0(\partial_3 w^\varepsilon_3)=Y\big\|\partial_3 w^\varepsilon_3\big\|^2_{L^2(\Omega)}$ and $\partial_3 w^\varepsilon_3\rightharpoonup\partial_3 w^o_3$ in $L^2(\Omega)$ which implies $\big\|\partial_3 w^o_3\big\|_{L^2(\Omega)}\leq\liminf\big\|\partial_3 w^\varepsilon_3\big\|_{L^2(\Omega)}$.

To get the lim sup inequality, we compare energy of $\mathscr{I}_2^\varepsilon$ at its minimizer $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ against a test function $(\boldsymbol{m}^o, \boldsymbol{U} = \boldsymbol{u}^o + (\boldsymbol{W}_p, \varepsilon W_3))$ where

$$\begin{aligned}
W_1 &= w_1^o(x_3) - \varepsilon^2 \sigma x_1 \partial_3 \gamma(x_3) + \varepsilon^2 \frac{\sigma}{2}\left(x_1^2 \partial_{33} w_1^o - x_2^2 \partial_{33} w_1^o + 2x_1 x_2 \partial_{33} w_2^o\right) \\
W_2 &= w_2^o(x_3) - \varepsilon^2 \sigma x_2 \partial_3 \gamma(x_3) + \varepsilon^2 \frac{\sigma}{2}\left(x_2^2 \partial_{33} w_2^o - x_1^2 \partial_{33} w_2^o + 2x_1 x_2 \partial_{33} w_1^o\right) \\
W_3 &= w_3^o(\boldsymbol{x}) = \gamma^o(x_3) - x_1 \partial_3 w_1^o - x_2 \partial_3 w_2^o
\end{aligned} \tag{II.7.22}$$

Then $\boldsymbol{\kappa}^\varepsilon(\boldsymbol{U}) - \boldsymbol{E}_s(\boldsymbol{m}^o) = \boldsymbol{\kappa}^\varepsilon(\boldsymbol{u}^o) - \boldsymbol{E}_s(\boldsymbol{m}^o) + \boldsymbol{\chi}^\varepsilon(\boldsymbol{W}) = \boldsymbol{\chi}^\varepsilon(\boldsymbol{W})$ where $\varepsilon^{-1}\boldsymbol{\chi}^\varepsilon(\boldsymbol{W})$ converges as:

$$\begin{aligned}
&\varepsilon^{-1}\boldsymbol{\chi}_{11}^\varepsilon(\boldsymbol{W}) = \varepsilon^{-1}\boldsymbol{\chi}_{22}^\varepsilon(\boldsymbol{W}) = -\sigma\left(\partial_3\gamma^o(x_3) - x_1\partial_{33}w_1^o - x_2\partial_{33}w_2^o\right), \qquad \varepsilon^{-1}\boldsymbol{\chi}_{12}^\varepsilon(\boldsymbol{W}) = 0 \\
&\varepsilon^{-1}\boldsymbol{\chi}_{12}^\varepsilon(\boldsymbol{W}) = \frac{\varepsilon}{2}\left\{-\sigma x_1 \partial_{33}\gamma^o(x_3) + \frac{\sigma}{2}\left(x_1^2\partial_{333}w_1^o - x_2^2\partial_{333}w_1^o + 2x_1x_2\partial_{333}w_2^o\right)\right\} \approx O(\varepsilon) \\
&\varepsilon^{-1}\boldsymbol{\chi}_{13}^\varepsilon(\boldsymbol{W}) = \frac{\varepsilon}{2}\left\{-\sigma x_2 \partial_{33}\gamma^o(x_3) + \frac{\sigma}{2}\left(x_2^2\partial_{333}w_2^o - x_1^2\partial_{333}w_2^o + 2x_1x_2\partial_{333}w_1^o\right)\right\} \approx O(\varepsilon) \\
&\varepsilon^{-1}\boldsymbol{\chi}_{33}^\varepsilon(\boldsymbol{W}) = \partial_3 W_3 = \partial_3\gamma^o(x_3) - x_1\partial_{33}w_1^o - x_2\partial_{33}w_2^o.
\end{aligned}$$

Its easy to check that $\mathbb{C}\left[\boldsymbol{\kappa}^\varepsilon(\boldsymbol{U}) - \boldsymbol{E}_s(\boldsymbol{m}^o)\right]^2 = f_0\left(\varepsilon\partial_3 W_3\right) + O(\varepsilon^3)$ gives

$$\begin{aligned}
\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \le \mathscr{I}_2^\varepsilon(\boldsymbol{m}^o, \boldsymbol{U}) &= \frac{\mathscr{I}^\varepsilon(\boldsymbol{m}^o, \boldsymbol{U}) - \varepsilon Q_1(\boldsymbol{m}^o) - Q_0}{\varepsilon^2} \\
&= Q_2(\boldsymbol{m}^o) + \int_\Omega f_0\left(\partial_3\gamma^o(x_3) - x_1\partial_{33}w_1^o(x_3) - x_2\partial_{33}w_2^o(x_3)\right) + O(\varepsilon).
\end{aligned}$$

Taking lim sup as $\varepsilon \to 0$ we get our result. We finally need to show that (w_1^o, w_2^o, γ^o) minimizes $\mathscr{I}^o(w_1, w_2, \gamma)$ in $\mathscr{A}_2$. Again as in Theorem II.5.1 we start of with smooth (w_1, w_2, γ) satisfying the boundary conditions. We set up a displacement $\boldsymbol{W}$ exactly as in eqn. (II.7.22) with (w_1, w_2, γ) replacing (w_1^o, w_2^o, γ^o). Comparing energy of $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ with the test function $\left(\boldsymbol{m}^o, \boldsymbol{u}^o + (\boldsymbol{W}_p^\varepsilon, W_3^\varepsilon)\right)$ we get $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \le \mathscr{I}_2^\varepsilon\left(\boldsymbol{m}^o, \boldsymbol{u}^o + (\boldsymbol{W}_p^\varepsilon, W_3^\varepsilon)\right)$. We get our result on taking limit as $\varepsilon \to 0$ and noting that smooth functions (w_1, w_2, γ) satisfying the appropriate boundary conditions are dense in $\mathscr{A}_2$.□

Remark II.7.2. *The assumption $\boldsymbol{m}^o = (0,0,m_s)$ is not necessary, but it greatly simplifies the form of the third variational limit problem $\mathscr{I}_2^o$. If $\boldsymbol{m}^o$ is a more general constant magnetization, then the magnetization $\boldsymbol{m}^o$ will have a non-trivial corrector. In fact the third variational limit problem then will be a more complicated problem involving both elastic and magnetic corrector terms. The elastic part of the problem will however still retain the Euler-Bernoulli type terms and the problem however will simplify to the limit problem of Theorem II.7.1 if $\boldsymbol{m}^o = (0,0,m_s)$.*

We however do not present that result here, as we are more interested in nanowires made of Galfenol. For these wires made of Galfenol, as expressed in § II.5.4, the demagnetization term

$\pi|\boldsymbol{m}_p|^2$ is the largest term in the "effective anisotropy" $\varphi(\boldsymbol{m})+\pi|\boldsymbol{m}_p|^2-\boldsymbol{h}_a\cdot\boldsymbol{m}$ by an order of magnitude for typical applied fields. Thus the minimizer $\boldsymbol{m}^o$ of this "effective anisotropy" is expected to be at or very close to $(0,0,m_s)$.

II.8 Summary and Discussion

Figure II.3: Bent wires of Galfenol

We have presented in this paper the derivation of simple models for nanometer diameter wires to be used in sensors/devices using the physics of magnetostriction. Though the starting point for these problems is an infinite dimensional variational problem with a non-convex non-linear constraint and variational energy contains terms which are non-local, using the method of variational convergence we have derived much simpler 1-dimensional models which is expected to approximate the actual physics of the starting model. The Theorems II.5.1 and II.7.1 clearly set up these simpler models $\mathscr{I}^o(\boldsymbol{m},v)$ and $\mathscr{I}_2^o(w_1,w_2,v)$ respectively.

The bending behavior of the nanowires is described by $\mathscr{I}_2^o(w_1,w_2,v)$ if $\boldsymbol{m}^o=(0,0,m_s)$ solves the first variational limit problem $\mathscr{I}^o(\boldsymbol{m},v)$ as is expected for Galfenol wires. The form of second variational limit and the third variational limit suggest that the magnetization remains strongly stabilized at $\boldsymbol{m}^o$ and higher order theories do not add correctors to $\boldsymbol{m}^o$ within the framework of geometrically linear theory of magnetostriction. The displacement solution $\boldsymbol{u}^o$ corresponding to the first variational problem is however corrected due to the appearance of the bending energy terms in the third variational limit $\mathscr{I}_2^o(w_1,w_2,v)$.

Although we have not included any external applied force in our analysis, it can be included with very minor changes to our presentation. The galfenol wires in bending behave like purely elastic beams with additional magnetic term which comes thorough the interaction of the positive and negative poles created at the two ends of the wire by the magnetization $\boldsymbol{m}^o=(0,0,m_s)$. This contribution is a fixed energy at the order at which bending elastic terms appear.

The strong stabilization of the magnetization is borne out by experiments where nanowires have been bend using an AFM tip. The Figure II.3 shows the MFM scan for a galfenol wire in bent shape. The details of the experiment are available from [Downey, 2008]. The MFM scan shows the same bright and dark spots at the two ends of the wire characteristic of axially magnetized wires as seen in Figure II.2. The bright spot in the middle was detected to be a topological defect. It is clear that even the large bending is unable to alter the axial magnetization, which can be interpreted as being equal to $\boldsymbol{m}^o$.

The bending behavior of the nanowires will be more complicated if $\boldsymbol{m}^o \neq (0,0,\pm m_s)$ solves the first variational limit problem $\mathscr{I}^o(\boldsymbol{m}, v)$ as mentioned in Remark II.7.2. This case is however not very important for Galfenol nanowires with the geometry that we are interested in.

The highly nonlinear deformation of the nanowires in Figure II.3 also suggests to start of with a geometrically nonlinear theory for magnetostriction. For geometrically nonlinear deformations however, the problem is significantly harder as the magnetic energies in the starting energy (II.2.8) will be defined on the deformed configuration, while typically in nonlinear elasticity, the free energy is defined over the reference configuration.
Recall the energy $\mathscr{I}_2^o(w_1(x_3), w_2(x_3), v(x_3))$ was defined in the previous section as

$$\mathscr{I}_2^o(w_1, w_2, v) = \int_\Omega \frac{1}{2}\Big\{ f_0(x_1\partial_{33}w_1) + f_0(x_2\partial_{33}w_2) + f_0(\partial_3\gamma)\Big\} \boldsymbol{dx} + Q_2(\boldsymbol{m}^o).$$

Note that the first and second term are exactly the bending energy that appears in classical Euler-Bernoulli theory. To see this note that from the definition of f_o in (II.5.13) we get,

$$\begin{aligned}\int_\Omega f_0(x_1\partial_{33}w_1(x_3))\boldsymbol{dx} = \int_0^1 \int_\omega Y x_1^2 \left|\partial_{33}w_1(x_3)\right|^2 = \int_0^1 Y \left\{ \int_\omega x_1^2 \boldsymbol{dx}_p \right\} \left|\partial_{33}w_1(x_3)\right|^2 dx_3 \\ = \int_0^1 Y I_{22} \left|\partial_{33}w_1(x_3)\right|^2 dx_3\end{aligned}$$

where I_{22} is the polar moment of inertia.

From the point of view of using Galfenol as a potential material for sensor application, the strong stabilization of magnetization $\boldsymbol{m}^o = (0,0,m_s)$ is not encouraging, as a designer would hope that the magnetization would change drastically from $\boldsymbol{m}^o$ on imposing any bending deformation. Newer proposals for sensor design using Galfenol have been made which replace the wire array of Galfenol with an array where each wire is multi-layered with fine layers of magnetic Galfenol and non-magnetic Copper (cf. [Park et al., 2010]).

II.A Magnetostatic calculations

II.A.1 Introduction

Recall in (II.4.5) we defined $\mathscr{E}_d^\varepsilon(\boldsymbol{m}) = \frac{1}{8\pi}\int_{\mathbb{R}^3} |\boldsymbol{h}_{\boldsymbol{m}}^\varepsilon(\boldsymbol{x})|^2 \boldsymbol{dx} = \frac{1}{\varepsilon^2}\frac{1}{8\pi}\int_{\mathbb{R}^3} |\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}}^\varepsilon(\boldsymbol{y})|^2 \boldsymbol{dy}$. In this section we will work in the unrescaled magnetization $\widetilde{\boldsymbol{m}}$ and demag field $\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}}^\varepsilon$. We define

$$\widehat{\boldsymbol{m}}^\varepsilon(y_3) := \fint_{\omega_\varepsilon} \widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{y}_p, y_3)\, \boldsymbol{dy}_p. \tag{II.A.1}$$

Note on rescaling $\widetilde{\boldsymbol{m}}^\varepsilon$ as in (II.4.3) $\widehat{\boldsymbol{m}}^\varepsilon$ also corresponds to the cross-sectional average defined in (II.5.1) i.e., $\widehat{\boldsymbol{m}}^\varepsilon = \fint_\omega \boldsymbol{m}^\varepsilon(\boldsymbol{x}_p, x_3)\, \boldsymbol{dx}_p$. Thus proposition II.5.1 on $\widehat{\boldsymbol{m}}^\varepsilon$ gives

$$\begin{aligned}
&\|\widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(0,1)}^2 \le |\omega|^{-1}\|\boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}^2, \qquad\qquad \|\partial_3^y \widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(0,1)}^2 \le |\omega|^{-1}\|\partial_3 \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}^2,\\
&\|\widehat{\boldsymbol{m}}^\varepsilon\|_{H^1(0,1)}^2 = \|\widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(0,1)}^2 + \|\partial_3^y \widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(0,1)}^2 \le \frac{1}{|\omega|}\big(\, \|\boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}^2 + \|\partial_3 \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}^2\big).
\end{aligned} \tag{II.A.2}$$

Let $\widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon$ solve Maxwell equation for $\widehat{\boldsymbol{m}}^\varepsilon$ on Ω_ε. We first prove the following Lemma to estimate the difference in magnetostatic energy between $\widetilde{\boldsymbol{m}}^\varepsilon$ and $\widehat{\boldsymbol{m}}^\varepsilon$.

Lemma II.A.1. *The following inequality holds:*

$$\frac{1}{8\pi\varepsilon^2}\Big|\, \|\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^\varepsilon}^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 - \|\widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon\|_{L^2(\mathbb{R}^3)}^2\Big| \le D_2 \|\boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}\, \|\nabla_p \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}.$$

Proof. First we recall the basic demagnetization energy bound in equation (II.2.4),

$$\frac{1}{8\pi}\|\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}}^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \le \frac{1}{2}\|\widetilde{\boldsymbol{m}}\|_{L^2(\Omega_\varepsilon)}^2, \qquad \forall\ \ \widetilde{\boldsymbol{m}} \in L^2(\Omega_\varepsilon, m_s S^2). \tag{II.A.4}$$

We have $\|\widetilde{\boldsymbol{m}}^\varepsilon - \widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(\omega^\varepsilon)}^2 \le D_1\varepsilon^2\|\nabla_p^y \widetilde{\boldsymbol{m}}^\varepsilon\|_{L^2(\omega^\varepsilon)}^2$ using Poincaré inequality on a cross-section plane $\omega_\varepsilon(y_3)$, which on integrating on $y_3 \in (0,1)$ gives

$$\|\widetilde{\boldsymbol{m}}^\varepsilon - \widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \ < \ D_1\,\varepsilon^2\,\|\nabla_p^y \widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{y})\|_{L^2(\Omega_\varepsilon)}^2.$$

Since $\widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon$ satisfies Maxwell equation for $\widehat{\boldsymbol{m}}^\varepsilon$, by linearity $(\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^\varepsilon}^\varepsilon - \widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon)$ satisfies Maxwell equation for $(\,\widetilde{\boldsymbol{m}}^\varepsilon - \widehat{\boldsymbol{m}}^\varepsilon\,)$. Then using basic bound eqn. (II.A.4) for Maxwell equation we have,

$$\frac{1}{8\pi}\|\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^\varepsilon}^\varepsilon - \widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \le \frac{1}{2}\|\widetilde{\boldsymbol{m}}^\varepsilon - \widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 < \frac{D_1}{2}\,\varepsilon^2\,\|\nabla_p^y \widetilde{\boldsymbol{m}}^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2.$$

Using triangle inequality we also have,

$$\Big|\, \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}^{\varepsilon}}\|_{L^2(\mathbb{R}^3)} - \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|_{L^2(\mathbb{R}^3)} \,\Big| \le \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}^{\varepsilon}} - \widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|_{L^2(\mathbb{R}^3)} < D_2\,\varepsilon\,\|\nabla^{\boldsymbol{y}}_p \widetilde{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)}.$$

Jensen's inequality gives $|\widehat{\boldsymbol{m}}^{\varepsilon}| \le |\widetilde{\boldsymbol{m}}^{\varepsilon}|$ and using basic demag bound in eqn. (II.A.4) again for $\widetilde{\boldsymbol{m}}^{\varepsilon}$ and $\widehat{\boldsymbol{m}}^{\varepsilon}$ we have,

$$\frac{1}{8\pi}\Big|\, \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}^{\varepsilon}}\|_{L^2(\mathbb{R}^3)} + \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|_{L^2(\mathbb{R}^3)} \,\Big| \le \frac{1}{2}\Big(\|\widetilde{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)} + \|\widehat{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)}\Big) \le \|\widetilde{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)}.$$

Then on combining the two and rescaling $\widetilde{\boldsymbol{m}}^{\varepsilon}$ to $\boldsymbol{m}^{\varepsilon}$ we have

$$\begin{aligned}\Big|\, \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)} - \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)} \Big| &= \Big|\, \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}^{\varepsilon}}\|_{L^2} - \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|_{L^2} \Big| \cdot \Big|\, \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}^{\varepsilon}}\|_{L^2} + \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|_{L^2} \Big| \\ &\le D_2\,\varepsilon\,\|\nabla^{\boldsymbol{y}}_p \widetilde{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)} \cdot\ 8\pi\ \|\widetilde{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)} \\ &= 8\pi\ D_2\,\varepsilon^2\ \|\boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)}\ \|\nabla_p \boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)},\end{aligned}$$

by noting that $\|\widetilde{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)} = \varepsilon\|\boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)}$ and $\|\nabla^{\boldsymbol{y}}_p \widetilde{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)} = \|\nabla_p \boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)}$. □

Remark II.A.2. *From* (II.5.3) *we know that the exchange energy of magnetization* $\boldsymbol{m}^{\varepsilon}$ *is bounded as* $K_5 \ge \frac{d}{\varepsilon^2}\|\nabla_p \boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)}$. *Since* $\mathscr{E}^{\varepsilon}_d(\boldsymbol{m}) = \frac{1}{8\pi}\|\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}\|^2_{L^2(\mathbb{R}^3)} = \frac{1}{8\pi\varepsilon^2}\|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}\|_{L^2(\mathbb{R}^3)}$, *we then get from the Lemma*

$$\begin{aligned}\big|\mathscr{E}^{\varepsilon}_d(\boldsymbol{m}^{\varepsilon}) - \mathscr{E}^{\varepsilon}_d(\widehat{\boldsymbol{m}}^{\varepsilon})\big| = \frac{1}{8\pi\varepsilon^2}\Big|\, \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)} - \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)} \Big| &\le D_2\|\boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)}\|\nabla_p \boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)} \\ &\le D_2\, m_s |\Omega|^{1/2}\sqrt{\frac{K_5}{d}}\,\varepsilon = O(\varepsilon).\end{aligned}$$

Thus the difference in magnetostatic energy between $\mathscr{E}^{\varepsilon}_d(\boldsymbol{m}^{\varepsilon})$ *and* $\mathscr{E}^{\varepsilon}_d(\widehat{\boldsymbol{m}}^{\varepsilon})$ *is of order* $O(\varepsilon)$*. We will see that for the convergence arguments it is enough to estimate* $\mathscr{E}^{\varepsilon}_d(\widehat{\boldsymbol{m}}^{\varepsilon}) = \frac{1}{8\pi\varepsilon^2}\|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)}$.

For $\widehat{\boldsymbol{m}}(y_3)$ note $\nabla^{\boldsymbol{y}} \cdot \widehat{\boldsymbol{m}}(\boldsymbol{y}) = \partial^{\boldsymbol{y}}_3 \widehat{m}_3(y_3)$. It is well know that for magnetization $\widehat{\boldsymbol{m}}^{\varepsilon}$, the energy $\frac{1}{8\pi\varepsilon^2}\|\widehat{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)}$ can be written as a convolution of fundamental solutions with $\widehat{\boldsymbol{m}}^{\varepsilon}$,

$$
\begin{aligned}
\frac{\varepsilon^{-2}}{8\pi}\|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2} &= \frac{1}{2}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon}\frac{\nabla^y\cdot\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{y})\;\nabla^y\cdot\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{z})}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} + \frac{1}{2}\int_{\partial\Omega_\varepsilon}\int_{\partial\Omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{y})\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y})\;\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{z})\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z})}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\
&\quad -\int_{\Omega_\varepsilon}\int_{\partial\Omega_\varepsilon}\frac{\nabla^y\cdot\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{y})\;\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{z})\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z})}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\
&= \frac{1}{2}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon}\frac{\partial^y_3\widehat{m}^{\varepsilon}_3(y_3)\;\partial^y_3\widehat{m}^{\varepsilon}_3(z_3)}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} + \frac{1}{2}\int_{\partial\Omega_\varepsilon}\int_{\partial\Omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y})\;\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z})}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\
&\quad -\int_{\Omega_\varepsilon}\int_{\partial\Omega_\varepsilon}\frac{\partial^y_3\widehat{m}^{\varepsilon}_3(y_3)\;\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z})}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\
&= \frac{1}{2}J^{\varepsilon}_1(\widehat{\boldsymbol{m}}^{\varepsilon}) + \frac{1}{2}J^{\varepsilon}_2(\widehat{\boldsymbol{m}}^{\varepsilon}) + J^{\varepsilon}_3(\widehat{\boldsymbol{m}}^{\varepsilon}).
\end{aligned} \tag{II.A.6}
$$

Note that $J^\varepsilon_1, J^\varepsilon_2$ and J^ε_3 respectively represent that "Bulk-Bulk", the "Boundary-Boundary" and the "Bulk-Boundary" terms of the magnetostatic energy. The body $\Omega_\varepsilon = \omega_\varepsilon \times (0,1)$ and the boundary $\partial\Omega_\varepsilon$ can be decomposed as as $\partial\Omega_\varepsilon = \big\{\partial\omega_\varepsilon \times (0,1)\big\}\bigcup\omega_\varepsilon(y_3=0)\bigcup\omega_\varepsilon(y_3=1)$.

II.A.2 Estimates of $J^\varepsilon_1(\widehat{\boldsymbol{m}}^{\varepsilon}), J^\varepsilon_2(\widehat{\boldsymbol{m}}^{\varepsilon})$, and $J^\varepsilon_3(\widehat{\boldsymbol{m}}^{\varepsilon})$

The magnetostatic estimates in this section are inspired by similiar estimates in other works like [Kohn and Slastikov, 2005] and [Carbou, 2001]. We use the following integral inequality in this section: for arbitrary $a \neq b \in \mathbb{R}$ and $q, L \in \mathbb{R}$ using the fact that $q(q^2+L^2)^{-1/2} \leq 1$ we have

$$
\int_a^b \frac{dq}{\big\{L^2+q^2\big\}^{3/2}} = \frac{1}{L^2}\frac{q}{(L^2+q^2)^{1/2}}\Big|_a^b = \frac{1}{L^2}\Big(\frac{b}{(L^2+b^2)^{1/2}} - \frac{a}{(L^2+a^2)^{1/2}}\Big) \leq \frac{2}{L^2}. \tag{II.A.7}
$$

We also need an estimate of the following term, where we use the change of variable $\boldsymbol{w}_p = \boldsymbol{y}_p - \boldsymbol{z}_p$, $\boldsymbol{dw}_p = \boldsymbol{dy}_p$ to get, $\Big($Recall ω_ε is a ball of radius ε in 2-d$\Big)$

$$
\begin{aligned}
\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{\boldsymbol{dy}_p\;\boldsymbol{dz}_p}{|\boldsymbol{y}_p-\boldsymbol{z}_p|} = \int_{\omega_\varepsilon}\boldsymbol{dz}_p\int_{\omega_\varepsilon-\boldsymbol{z}_p}\frac{\boldsymbol{dw}_p}{|\boldsymbol{w}_p|} &\leq \int_{\omega_\varepsilon}\boldsymbol{dz}_p\int_{\omega_{3\varepsilon}}\frac{\boldsymbol{dw}_p}{|\boldsymbol{w}_p|} \\
&= \int_{\omega_\varepsilon}\boldsymbol{dz}_p\int_0^{2\pi}\int_0^{3\varepsilon}\frac{|\boldsymbol{w}_p|\;\boldsymbol{d}(|\boldsymbol{w}_p|)\;d\theta}{|\boldsymbol{w}_p|} \\
&= (\pi\varepsilon^2)\,(2\pi)\,(3\varepsilon) = 6\pi^2\varepsilon^3,
\end{aligned} \tag{II.A.8}
$$

where we have used the fact that $(\omega_\varepsilon - \boldsymbol{z}_p) \subset \omega_{3\varepsilon}$ for $\boldsymbol{z}_p \in \omega_\varepsilon$. Henceforth we drop the $\boldsymbol{y}$ superscript on the derivative operator.

Also note that if $\widehat{\boldsymbol{m}}^\varepsilon \in H^1(0,1)$, Sobolev embedding gives along with (II.A.2)

$$\sup_{y_3} \left|\widehat{\boldsymbol{m}}^\varepsilon(y_3)\right| \leq D_3 \|\widehat{\boldsymbol{m}}^\varepsilon\|_{H^1(0,1)} \leq \frac{D_3}{|\omega|}\big(\|\boldsymbol{m}^\varepsilon\|_{L^2(\Omega)} + \left\|\partial_3 \boldsymbol{m}^\varepsilon\right\|_{L^2(\Omega)}\big). \tag{II.A.9}$$

Proposition II.A.1.

$$\left|J_1^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)\right| \leq D_4 \varepsilon\big(\|\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}\big).$$

Proof. Recalling definition of J_1^ε from Equation (II.A.6) and noting $|\boldsymbol{y}_p - \boldsymbol{z}_p| \leq |\boldsymbol{y}-\boldsymbol{z}|$ we have

$$\begin{aligned}\left|\varepsilon^2 J_1^\varepsilon(\widehat{\boldsymbol{m}})\right| &\leq \int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} \frac{|\partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(y_3)\, \partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(z_3)|}{|\boldsymbol{y}-\boldsymbol{z}|} \boldsymbol{dy dz} \leq \int_0^1\int_0^1\int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{|\partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(y_3)\, \partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(z_3)|}{|\boldsymbol{y}_p-\boldsymbol{z}_p|} \\ &= \int_0^1\int_0^1 \left|\partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(y_3)\, \partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(z_3)\right| \int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{\boldsymbol{dy}_p \boldsymbol{dz}_p}{|\boldsymbol{y}_p-\boldsymbol{z}_p|} \leq D_4 \varepsilon^3 \left\|\partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon\right\|^2_{L^2(0,1)}\end{aligned}$$

where we have used Hölder's inequality on the term $\int_0^1\int_0^1 \left|\partial_3^{\boldsymbol{y}} \widehat{m}_3(y_3)\, \partial_3^{\boldsymbol{y}} \widehat{m}_3(z_3)\right| dy_3 dz_3$ and equation (II.A.8) in the last step. Using equation (II.A.2) we get our result. □

Proposition II.A.2.

$$\left|J_3^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)\right| \leq D_5 \varepsilon\big(\|\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}\big).$$

Proof. Recalling definition of J_1^ε from Equation (II.A.6), we split of $J_3^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)$ into 2 parts,

$$\begin{aligned}-\varepsilon^2 J_3^\varepsilon(\widehat{\boldsymbol{m}}) = \int_{\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^\varepsilon(z_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{|\boldsymbol{y}-\boldsymbol{z}|} \partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(y_3) + \int_{\omega_\varepsilon}\int_0^1 \partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(y_3) \times \\ \Big[\int_{\omega_\varepsilon(0)} \frac{\widehat{\boldsymbol{m}}^\varepsilon(z_3=0)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z})}{|\boldsymbol{y}-\boldsymbol{z}|} + \int_{\omega_\varepsilon(1)} \frac{\widehat{\boldsymbol{m}}^\varepsilon(z_3=1)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z})}{|\boldsymbol{y}-\boldsymbol{z}|}\Big] =: \varepsilon^2 J_{31}^\varepsilon + \varepsilon^2 J_{32}^\varepsilon\end{aligned}$$

with $\varepsilon^2 J_{31}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)$ being first term and $\varepsilon^2 J_{32}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)$ is the remaining term of the R.H.S. For J_{31}^ε using divergence theorem on $\partial\omega_\varepsilon(z_3)$ gives,

$$\begin{aligned}\varepsilon^2 J_{31}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) &= \int_{\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^\varepsilon(z_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{|\boldsymbol{y}-\boldsymbol{z}|} \partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(y_3) \boldsymbol{dy}_p \boldsymbol{d\sigma}(\boldsymbol{z}_p) dy_3 dz_3 \\ &= \int_{\omega_\varepsilon}\int_0^1\int_0^1 \partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(y_3)\, \boldsymbol{dy}_p dy_3 dz_3 \int_{\omega_\varepsilon} \nabla_p^{\boldsymbol{z}} \cdot \Big(\frac{\widehat{\boldsymbol{m}}^\varepsilon(z_3)}{|\boldsymbol{y}-\boldsymbol{z}|}\Big)\, \boldsymbol{dz}_p \\ &= \int_{\omega_\varepsilon}\int_0^1\int_0^1\int_{\omega_\varepsilon} \partial_3^{\boldsymbol{y}} \widehat{m}_3^\varepsilon(y_3) \frac{\widehat{\boldsymbol{m}}^\varepsilon(z_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{\big\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2 + (y_3-z_3)^2\big\}^{3/2}} \boldsymbol{dy dz}.\end{aligned} \tag{II.A.10}$$

Setting $q = (z_3 - y_3)$ and $dz_3 = dq$ gives,

$$\left|\varepsilon^2 J^{\varepsilon}_{31}(\widehat{\boldsymbol{m}}^{\varepsilon})\right| \le \sup_{z_3} |\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)| \int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\int_0^1 \left|\partial_3^{\boldsymbol{y}} \widehat{m}^{\varepsilon}_3(y_3)\right| \int_{-y_3}^{1-y_3} \frac{|\,\boldsymbol{y}_p - \boldsymbol{z}_p\,|}{\left\{|\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + q^2\right\}^{3/2}} dq.$$

Using equation (II.A.7) on the inner integral gives

$$\begin{aligned}
\left|\varepsilon^2 J^{\varepsilon}_{31}(\widehat{\boldsymbol{m}}^{\varepsilon})\right| &\le 2 \sup_{z_3} \left|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\right| \int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\int_0^1 \frac{|\partial_3^{\boldsymbol{y}} \widehat{m}^{\varepsilon}_3(y_3)|}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} \\
&= 2 \sup_{z_3} \left|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\right| \left\{ \int_0^1 |\partial_3^{\boldsymbol{y}} \widehat{m}^{\varepsilon}_3(y_3)|\, dy_3 \right\} \left\{ \int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{\boldsymbol{dy}_p \boldsymbol{dz}_p}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} \right\} \\
&\le D_6 \varepsilon^3 \sup_{z_3} \left|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\right| \left\|\partial_3^{\boldsymbol{y}} \widehat{\boldsymbol{m}}^{\varepsilon}\right\|_{L^2(0,1)} \le D_7 \varepsilon^3 \left(\left\|\boldsymbol{m}^{\varepsilon}\right\|^2_{L^2(\Omega)} + \left\|\partial_3 \boldsymbol{m}^{\varepsilon}\right\|^2_{L^2(\Omega)}\right),
\end{aligned}$$

using equations (II.A.8), (II.A.9) and (II.A.2). Also we estimate $J^{\varepsilon}_{32}(\widehat{\boldsymbol{m}}^{\varepsilon})$ as

$$\begin{aligned}
\varepsilon^2 \left|J^{\varepsilon}_{32}\right| &\le \left(|\widehat{m}^{\varepsilon}_3(0)| + |\widehat{m}^{\varepsilon}_3(1)|\right) \int_{\omega_\varepsilon}\int_0^1\int_{\omega_\varepsilon} \left[\frac{|\partial_3^{\boldsymbol{y}} \widehat{m}^{\varepsilon}_3(y_3)|}{\sqrt{|\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + y_3^2}} + \frac{|\partial_3^{\boldsymbol{y}} \widehat{m}^{\varepsilon}_3(y_3)|}{\sqrt{|\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + (1-y_3)^2}} \right] \\
&\le 4 \sup_{z_3} \left|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\right| \left\{ \int_0^1 |\partial_3^{\boldsymbol{y}} \widehat{m}^{\varepsilon}_3(y_3)| dy_3 \right\} \left\{ \int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{\boldsymbol{dy}_p \boldsymbol{dz}_p}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} \right\} \\
&\le D_8 \varepsilon^3 \sup_{z_3} \left|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\right| \left\|\partial_3^{\boldsymbol{y}} \widehat{\boldsymbol{m}}^{\varepsilon}\right\|_{L^2(0,1)} \le D_9 \varepsilon^3 \left(\left\|\boldsymbol{m}^{\varepsilon}\right\|^2_{L^2(\Omega)} + \left\|\partial_3 \boldsymbol{m}^{\varepsilon}\right\|^2_{L^2(\Omega)}\right)
\end{aligned}$$

again using equations (II.A.8), (II.A.9) and (II.A.2). Combining estimates for $J^{\varepsilon}_{31}(\widehat{\boldsymbol{m}}^{\varepsilon})$ and $J^{\varepsilon}_{32}(\widehat{\boldsymbol{m}}^{\varepsilon})$ we get our result. □

Recalling $J^{\varepsilon}_2(\widehat{\boldsymbol{m}}^{\varepsilon})$ from eqn. (II.A.6) we write $J^{\varepsilon}_2 = J^{\varepsilon}_{21} + J^{\varepsilon}_{22} + J^{\varepsilon}_{23} + J^{\varepsilon}_{24}$ where,

$$\begin{aligned}
\varepsilon^2 J^{\varepsilon}_{21} &= \int_{\partial\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y})\, \widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z})}{\left\{\,|\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + (y_3 - z_3)^2\,\right\}^{1/2}}, \\
\varepsilon^2 J^{\varepsilon}_{22} &= \int_{\omega_\varepsilon(0)}\int_{\omega_\varepsilon(0)} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(0)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\, \widehat{\boldsymbol{m}}^{\varepsilon}(0)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} + \int_{\omega_\varepsilon(1)}\int_{\omega_\varepsilon(1)} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(1)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\, \widehat{\boldsymbol{m}}^{\varepsilon}(1)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{|\boldsymbol{y}_p - \boldsymbol{z}_p|}, \\
\varepsilon^2 J^{\varepsilon}_{23} &= 2\int_{\omega_\varepsilon(0)}\int_{\omega_\varepsilon(1)} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(0)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\, \widehat{\boldsymbol{m}}^{\varepsilon}(1)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\left\{\,|\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + 1\,\right\}^{1/2}} \quad \text{and,} \\
\frac{\varepsilon^2 J^{\varepsilon}_{24}}{2} &= \int_{\partial\omega_\varepsilon}\int_0^1 \left[\int_{\omega_\varepsilon(0)} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y})\, \widehat{\boldsymbol{m}}^{\varepsilon}(0)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\left\{\,|\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + y_3^2\,\right\}^{1/2}} + \int_{\omega_\varepsilon(1)} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y})\, \widehat{\boldsymbol{m}}^{\varepsilon}(1)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\left\{|\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + (1-y_3)^2\right\}^{1/2}} \right].
\end{aligned}$$

Noting that $|\widehat{\boldsymbol{m}}^{\varepsilon}(t)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z})| = |\widehat{m}^{\varepsilon}_3(t)| \le \sup_{z_3} |\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)|$ for $t = 0$ and $t = 1$ we have using equations

(II.A.8) and (II.A.9),

$$\varepsilon^2 J^\varepsilon_{22}(\widehat{\boldsymbol{m}}^\varepsilon) = \sup_{z_3} \left|\widehat{\boldsymbol{m}}^\varepsilon(z_3)\right|^2 \int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{\boldsymbol{dy}_p\ \boldsymbol{dz}_p}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} \le D_{10}\varepsilon^3\left(\|\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}\right). \tag{II.A.11}$$

$$\varepsilon^2 J^\varepsilon_{24}(\widehat{\boldsymbol{m}}^\varepsilon) = 2\int_{\partial\omega_\varepsilon}\int_0^1\int_{\omega_\varepsilon} \frac{-\widehat{m}^\varepsilon_3(0)\ \widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y})}{\left\{\ |\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + y_3^2\ \right\}^{1/2}} + \frac{\widehat{m}^\varepsilon_3(1)\ \widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y})}{\left\{|\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + (1-y_3)^2\right\}^{1/2}}. \tag{II.A.12}$$

Note $\left(\ |\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + 1\ \right)^{-\frac{1}{2}} \le 1$. Then eqn. (II.A.9) gives

$$\begin{aligned} \left|\varepsilon^2 \frac{J^\varepsilon_{23}}{2}(\widehat{\boldsymbol{m}}^\varepsilon)\right| &\le \left|\widehat{m}^\varepsilon_3(0)\widehat{m}^\varepsilon_3(1)\right| \left\{\int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \boldsymbol{dy}_p \boldsymbol{dz}_p\right\} = \pi^2\varepsilon^4 \sup_{z_3}\left|\widehat{\boldsymbol{m}}^\varepsilon(z_3)\right|^2 \\ &\le D_{11}\varepsilon^4\left(\ \|\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}\right). \end{aligned} \tag{II.A.13}$$

Proposition II.A.3.

$$\left|J^\varepsilon_{24}(\widehat{\boldsymbol{m}}^\varepsilon)\right| \le D_{12}\varepsilon\left(\ \|\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}\right).$$

Proof. As for term the J^ε_{31} in Proposition II.A.2, first using divergence theorem in J^ε_{24} from (II.A.12) on $\partial\omega_\varepsilon(y_3)$ we get

$$\varepsilon^2 J^\varepsilon_{24}(\widehat{\boldsymbol{m}}^\varepsilon) = 2\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\int_0^1 \left\{ \frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p - \boldsymbol{z}_p)\ \widehat{m}^\varepsilon_3(0)}{\left\{\ |\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + y_3^2\right\}^{3/2}} - \frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p - \boldsymbol{z}_p)\ \widehat{m}^\varepsilon_3(1)}{\left\{\ |\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + (1-y_3)^2\right\}^{3/2}} \right\}.$$

Then using $\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p - \boldsymbol{z}_p) \le |\boldsymbol{y}_p - \boldsymbol{z}_p|\ \sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|$ and (II.A.7) we get,

$$\begin{aligned} \left|\varepsilon^2 J^\varepsilon_{24}\right| &\le 2\sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2 \int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\left|\int_0^1 \frac{|\boldsymbol{y}_p - \boldsymbol{z}_p|\ dy_3}{\left\{|\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + y_3^2\right\}^{\frac{3}{2}}} - \int_0^1 \frac{|\boldsymbol{y}_p - \boldsymbol{z}_p|\ d(1-y_3)}{\left\{|\boldsymbol{y}_p - \boldsymbol{z}_p|^2 + (1-y_3)^2\right\}^{\frac{3}{2}}}\right| \\ &\le 8\sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2 \int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{\boldsymbol{dy}_p \boldsymbol{dz}_p}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} = D_{12}\varepsilon^3\left(\|\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}\right). \end{aligned}$$

and eqns. (II.A.9) and (II.A.8) in the last step. □

We will now show that $J^\varepsilon_{21}(\widehat{\boldsymbol{m}}^\varepsilon)$ is the largest term in the magnetostatic terms. It contributes energy of $O(1)$ which appears in the first limit problem $\mathscr{I}_0$. We split $J^{\varepsilon}_{21}(\widehat{\boldsymbol{m}}^{\varepsilon})$ as follows:

$$\begin{aligned}
J^{\varepsilon}_{21}(\widehat{\boldsymbol{m}}^{\varepsilon}) &= \int_{\partial\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\,\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\
&= \int_{\partial\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\
&\qquad - \int_{\partial\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\,(\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(z_3))\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\
&= J^{\varepsilon}_{211}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J^{\varepsilon}_{212}(\widehat{\boldsymbol{m}}^{\varepsilon}).
\end{aligned}$$

Next we show the following proposition.

Proposition II.A.4.

$$\left|J^{\varepsilon}_{212}(\widehat{\boldsymbol{m}}^{\varepsilon})\right| \le D_{13}\varepsilon^{3/4}\left(\|\boldsymbol{m}^{\varepsilon}\|^2_{L^2(\Omega)} + \|\partial_3\boldsymbol{m}^{\varepsilon}\|^2_{L^2(\Omega)}\right).$$

Proof. Using Divergence theorem in $\boldsymbol{y}_p$ variable as in (II.A.10) and Fubini's theorem we get,

$$\begin{aligned}
\varepsilon^2 J^{\varepsilon}_{212} &= \int_{\partial\omega_\varepsilon}\int_0^1\int_0^1\int_{\partial\omega_\varepsilon} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)}{\sqrt{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2}}\,(\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3))\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)\boldsymbol{d\sigma}(\boldsymbol{y}_p) \\
&= \int_{\partial\omega_\varepsilon}\int_0^1\int_0^1\int_{\omega_\varepsilon} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot(\boldsymbol{z}_p-\boldsymbol{y}_p)}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}^{3/2}}\,(\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3))\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)\boldsymbol{d}\boldsymbol{y}_p \\
&= \int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)\cdot\int_0^1\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot(\boldsymbol{z}_p-\boldsymbol{y}_p)\int_0^1\frac{\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}^{3/2}}\,dz_3.
\end{aligned}$$

Now note that $\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}^{1/2}} \le 1$ and $|\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)| = 1$ which gives

$$\left|\varepsilon^2 J^{\varepsilon}_{212}(\widehat{\boldsymbol{m}}^{\varepsilon}(y_3))\right| \le \sup_{y_3}\left|\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\right|\ \left\{\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\int_0^1\int_0^1\frac{|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)|}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}}\,dz_3\right\}. \tag{II.A.14}$$

Note that,

$$\frac{1}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}} \le \frac{1}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^{1/4}}\frac{1}{|y_3-z_3|^{7/4}} \tag{II.A.15}$$

Then

$$\int_0^1\int_0^1 \frac{|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)|dz_3dy_3}{\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\}} \leq \frac{1}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^{1/4}}\int_0^1\int_0^1 \frac{|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)|}{|y_3-z_3|^{7/4}}dz_3dy_3$$
$$\leq \frac{1}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^{1/4}}\left\|\partial_3^{\boldsymbol{y}}\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\right\|_{L^1(0,1)} \tag{II.A.16}$$

because of the fact that $\int_0^1\int_0^1 \frac{|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)|}{|y_3-z_3|^{7/4}}dz_3dy_3$ denotes the seminorm in the fractional Sobolev space $W^{\frac{3}{4},1}(0,1)$ and by the continuous embedding of $W^{1,1}(0,1)\subset W^{\frac{3}{4},1}(0,1)$. The integral above cannot be bounded by norm in $W^{1,1}$ alone unless $\widehat{\boldsymbol{m}}^{\varepsilon}$ is a constant, which is shown by the surprising result Proposition 1 in [Brézis, 2002]. Also note using Hölder inequality

$$\left\|\partial_3^{\boldsymbol{y}}\widehat{\boldsymbol{m}}^{\varepsilon}\right\|_{L^1(0,1)} = \int_0^1\left|\partial_3^{\boldsymbol{y}}\widehat{\boldsymbol{m}}^{\varepsilon}\right|\chi_{(0,1)}dy_3 \leq \left\|\chi_{(0,1)}\right\|_{L^2(0,1)}\left\|\partial_3^{\boldsymbol{y}}\widehat{\boldsymbol{m}}^{\varepsilon}\right\|_{L^2(0,1)} = \left\|\partial_3^{\boldsymbol{y}}\widehat{\boldsymbol{m}}^{\varepsilon}\right\|_{L^2(0,1)}.$$

Then using (II.A.16) in eqn. (II.A.14) along with the above result we get,

$$\left|\varepsilon^2 J_{212}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)) \leq \sup_{y_3}\left|\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\right| \left\{\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\int_0^1\int_0^1 \frac{|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)|}{\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\}}dz_3\right\}\right.$$
$$\leq \sup_{y_3}\left|\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\right| . \left\|\partial_3^{\boldsymbol{y}}\widehat{\boldsymbol{m}}^{\varepsilon}\right\|_{L^2(0,1)} \left\{\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{\boldsymbol{dy}_p\boldsymbol{d\sigma}(\boldsymbol{z}_p)}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^{1/4}}\right\}$$
$$= D_{14}\ \varepsilon^2\ \varepsilon^{3/4}\ \left(\left\|\boldsymbol{m}^{\varepsilon}\right\|^2_{L^2(\Omega)}+\left\|\partial_3\boldsymbol{m}^{\varepsilon}\right\|^2_{L^2(\Omega)}\right)$$

using calculation like in eqn. (II.A.8) to get the $\varepsilon^2\varepsilon^{3/4}$ term and eqns. (II.A.9) and (II.A.2). □

In 2-dimensional micromagnetics on a domain $\Psi\in\mathbb{R}^2$ for a constant magnetization $\boldsymbol{m}\in H^1(\Psi,m_sS^2)$, the demagnetization field is given by,

$$\boldsymbol{h_m}(\boldsymbol{x}) = \int_{\partial\Psi}\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^2}\boldsymbol{m}\cdot\boldsymbol{n}(\boldsymbol{y})\boldsymbol{dy} \tag{II.A.17}$$

and magnetostatic energy is given by,

$$\mathscr{E}_{2d} = \int_\Psi\int_{\partial\Psi}\boldsymbol{m}\cdot\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^2}\ \ \boldsymbol{m}\cdot\boldsymbol{n}(\boldsymbol{y})\boldsymbol{dy}. \tag{II.A.18}$$

Proposition II.A.5.

$$\left|\ J_{211}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})-2\pi|\omega_\varepsilon|\int_0^1\left|\widehat{\boldsymbol{m}}_p^{\varepsilon}(y_3)\right|^2dy_3\ \right| \leq D_{15}\ \varepsilon\left(\left\|\boldsymbol{m}^{\varepsilon}\right\|^2_{L^2(\Omega)}+\left\|\partial_3\boldsymbol{m}^{\varepsilon}\right\|^2_{L^2(\Omega)}\right).$$

Proof. Using the Divergence theorem on $\boldsymbol{z}_p$ as in (II.A.10) and a subsequent change of variables $q(z_3) = z_3 - y_3$, followed by (II.A.7) (as in Proposition II.A.2) we get

$$
\begin{aligned}
\int_{\partial\omega_\varepsilon}\int_0^1\int_0^1\int_{\partial\omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\sqrt{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2}} &= \int_{\partial\omega_\varepsilon}\int_0^1\int_0^1\int_{\omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\,\}^{3/2}} \\
&= \int_{\partial\omega_\varepsilon}\int_0^1\int_{\omega_\varepsilon}\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\int_{-y_3}^{1-y_3}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+q^2\,\}^{3/2}}dq \\
&= \int_{\partial\omega_\varepsilon}\int_0^1\int_{\omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|^2}\left\{\frac{y_3\,\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}+\frac{(1-y_3)\,\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)}{\sqrt{(1-y_3)^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}\right\} \\
&=: \varepsilon^2\big(\,J^\varepsilon_{2111}(\widehat{\boldsymbol{m}}^\varepsilon)+J^\varepsilon_{2112}(\widehat{\boldsymbol{m}}^\varepsilon)\,\big)
\end{aligned}
$$

where $\varepsilon^2 J^\varepsilon_{2111}(\widehat{\boldsymbol{m}}^\varepsilon) = \int_{\partial\omega_\varepsilon}\int_0^1\int_{\omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|^2}\;\frac{y_3\,\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}$ and $\varepsilon^2 J^\varepsilon_{2111}(\widehat{\boldsymbol{m}}^\varepsilon)$ the remaining term. Set $J^\varepsilon_0(\widehat{\boldsymbol{m}}^\varepsilon)$ as

$$
J^\varepsilon_0(\widehat{\boldsymbol{m}}^\varepsilon) := \int_0^1\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}\boldsymbol{d\sigma}(\boldsymbol{y}_p)\boldsymbol{dz}_p\,dy_3,
$$

Let $R := \max\ 2\varepsilon^{-1}|\boldsymbol{x}_p-\boldsymbol{y}_p|,\ \big(\boldsymbol{z}_p\in\omega_\varepsilon,\boldsymbol{y}_p\in\partial\omega_\varepsilon\big)$, and note

$$
1-\frac{y_3}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}} \;\le\; \begin{cases}\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}{2y_3^2}, & \text{for } y_3\ge R\varepsilon\ ,\\ 1 & \text{for } y_3\le R\varepsilon\ .\end{cases} \tag{II.A.19}
$$

Noting that $|\tilde{\boldsymbol{n}}| = 1$, $|\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)| \le |\widehat{\boldsymbol{m}}^\varepsilon(y_3)|$ and $|\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)| \le |\widehat{\boldsymbol{m}}^\varepsilon(y_3)|\,|\boldsymbol{y}_p-\boldsymbol{z}_p|$,

$$
\begin{aligned}
&\int_0^1\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|^2}\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\Big(1-\frac{y_3}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}\Big)dy_3 \\
&\quad\le \int_0^{R\varepsilon}\frac{|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|}\Big|1-\frac{y_3}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}\Big|+\int_{R\varepsilon}^1\frac{|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|}\Big|1-\frac{y_3}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}\Big| \\
&\quad\le \int_0^{R\varepsilon}\frac{|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|}dy_3+\int_{R\varepsilon}^1\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2\left(\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{2y_3^2}\right)dy_3 \\
&\quad\le \sup_{y_3\in(0,1)}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2\ \left\{\int_0^{R\varepsilon}\frac{dy_3}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|}+\int_{R\varepsilon}^1\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{2y_3^2}dy_3\right\}.
\end{aligned}
$$

Using above result and noting that $\partial_3\big(y_3^{-1}\big) = -y_3^{-2}$ we get

$$\begin{aligned}
\varepsilon^2 |J_0^\varepsilon - J_{2111}^\varepsilon| &\leq \left| \int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|^2}\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\Big(1-\frac{y_3}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}\Big)\right| \\
&\leq \sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\left\{\int_0^{R\varepsilon}\frac{dy_3}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|}+\int_{R\varepsilon}^1\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{2y_3^2}dy_3\right\} \\
&\leq \sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\left\{\frac{R\varepsilon}{|\boldsymbol{y}_p-\boldsymbol{z}_p|}-\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{2}\,\frac{1}{y_3}\Big|_{R\varepsilon}^1\right\} \\
&\leq \sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\left\{\frac{R\varepsilon}{|\boldsymbol{y}_p-\boldsymbol{z}_p|}-\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{2}+\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{2R\varepsilon}\right\}\boldsymbol{d\sigma(y_p)dz_p}
\end{aligned}$$

Note from equation (II.A.8), the term $\int_{\omega_\varepsilon}\frac{1}{|\boldsymbol{y}_p-\boldsymbol{z}_p|}\,\boldsymbol{dz}_p = D_{15}\varepsilon$. So the first integral above is $R\varepsilon\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{1}{|\boldsymbol{y}_p-\boldsymbol{z}_p|}\boldsymbol{dz}_p \approx$ $D_{16}\varepsilon^3$. The second integrand is $O(\varepsilon)$ and so its integral is of $O(\varepsilon^4)$. The third integrand is bounded by 1, since by definition $R\varepsilon \geq |\boldsymbol{y}_p - \boldsymbol{z}_p|$. So the third integral $\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{R\varepsilon}\boldsymbol{dz}_p \approx D_{17}\varepsilon^3$. So $|J_0^\varepsilon - J_{2111}^\varepsilon| \leq D_{18}\varepsilon\ \sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2$. J_{2112}^ε can be treated the same way to give the result on using eqn (II.A.2)

$$\Big|\ J_{211}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) - 2J_0^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)\ \Big| \leq D_{18}\varepsilon\ \sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2 \leq D_{19}\varepsilon\big(\|\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}\big).$$

We get our result noting that $J_0^\varepsilon(\widehat{\boldsymbol{m}})$ is exactly the 2-D magnetostatic energy $\mathscr{E}_{2d}$ defined in (II.A.18) and for a circular cross-section ω_ε it is well know that

$$\mathscr{E}_{2d}(\widehat{\boldsymbol{m}}^\varepsilon) = J_0^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) = \pi|\omega_\varepsilon|\int_0^1\left|\widehat{\boldsymbol{m}}_p^\varepsilon(y_3)\right|^2 dy_3 = \varepsilon^2\pi|\omega|\int_0^1\left|\widehat{\boldsymbol{m}}_p^\varepsilon(x_3)\right|^2 dx_3. \qquad \square$$

II.A.3 Final Estimate for $\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon)$

The excange energy of $\boldsymbol{m}^\varepsilon$ is bounded by equation (II.5.3), $\frac{K_5}{d} > \varepsilon^{-2}\|\nabla_p\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}$. Using Remark II.A.2 we get first,

$$\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \mathscr{E}_d^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) = O(\varepsilon).$$

Combining Propositions II.A.1 , II.A.2 , II.A.3 , II.A.4 , II.A.5 and and equations (II.A.11) and (II.A.13) we get,

$$\mathscr{E}_d^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) - \pi|\omega|\int_0^1\left|\widehat{\boldsymbol{m}}_p^\varepsilon(y_3)\right|^2 dy_3 = \big(O(\varepsilon) + O(\varepsilon^{3/4})\big)\Big(\|\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}\Big).$$

Combining the two we get

$$\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \pi|\omega|\int_0^1 \left|\widehat{\boldsymbol{m}}_p^\varepsilon(y_3)\right|^2 dy_3 = O(\varepsilon) + O(\varepsilon^{3/4}). \tag{II.A.20}$$

Remark II.A.3. *The above result* (II.A.20) *is true for any magnetization* $\boldsymbol{m}$ *as long as the magnetization satisfies the exchange bound* $K_5 \geq \varepsilon^{-2}\|\nabla_p \boldsymbol{m}\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{m}\|^2_{L^2(\Omega)}$.

Let $\widetilde{\boldsymbol{m}}^o$ be a constant vector in $m_s S^2$. If $\boldsymbol{m}^o$ is the rescaled version of $\widetilde{\boldsymbol{m}}^o$, recall the result in equation (II.3.6) gives,

$$\begin{aligned}
\mathscr{E}_d^\varepsilon(\boldsymbol{m}^o) = \varepsilon^{-2} E_{demag} &= \pi^2|\boldsymbol{m}_p^o|^2 - \varepsilon\frac{8\pi}{3}\Big(\,|\boldsymbol{m}_p^o|^2 - 2|m_3^o|^2\Big) + \pi^2\varepsilon^2\Big(\frac{|\boldsymbol{m}_p^o|^2}{2} - |m_3^o|^2\Big)\\
&= \pi^2|\boldsymbol{m}_p^o|^2 + \varepsilon Q_1 + \varepsilon^2 Q_2,
\end{aligned}$$

where we define Q_1 and Q_2 as in equation (II.3.6).

Proposition II.A.6. *Let* $\widetilde{\boldsymbol{m}}^o$ *be a constant on* $m_s S^2$ *and* $H^1(\Omega_\varepsilon; m_s S^2) \ni \widetilde{\boldsymbol{m}}^\varepsilon = \widetilde{\boldsymbol{m}}^o + \widetilde{\boldsymbol{M}}^\varepsilon$. *Then the following holds in terms of the rescaled magnetizations* $(\boldsymbol{m}^\varepsilon, \boldsymbol{m}^o, \boldsymbol{M}^\varepsilon)$,

$$\begin{aligned}
&\frac{d}{\varepsilon^2}\|\nabla_p \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)} + \Lambda\big(\|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\big) + \mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \pi\int_\Omega |\boldsymbol{m}_p^\varepsilon|^2\\
&\quad\geq \frac{d}{2\varepsilon^2}\|\nabla_p \boldsymbol{M}^\varepsilon\|_{L^2(\Omega)} + \frac{\Lambda}{2}\big(\|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\big) + \varepsilon Q_1 + \varepsilon^2 Q_2 - D_{18}\varepsilon^2.
\end{aligned}$$

Proof. First note $\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}^o}$ as given in (II.3.6) on rescaling to $\boldsymbol{h}^\varepsilon_{\boldsymbol{m}^o}$ gives

$$\boldsymbol{h}^\varepsilon_{\boldsymbol{m}^o} = -2\pi\begin{bmatrix}\boldsymbol{m}_p^o\\ 0\end{bmatrix} + \frac{16\varepsilon}{3}\begin{bmatrix}\boldsymbol{m}_p^o\\ -2m_3^o\end{bmatrix} + \pi\varepsilon^2\begin{bmatrix}\boldsymbol{m}_p^o\\ -2m_3^o\end{bmatrix}.$$

Note $\nabla_p \boldsymbol{m}^\varepsilon = \nabla_p(\boldsymbol{m}^o + \boldsymbol{M}^\varepsilon) = \nabla_p \boldsymbol{M}^\varepsilon$ as $\boldsymbol{m}^o$ is constant. Lemma II.A.1 gives along with Young's inequality gives

$$\begin{aligned}
\frac{d}{\varepsilon^2}\|\nabla_p \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \mathscr{E}_d^\varepsilon(\boldsymbol{M}^\varepsilon) - \mathscr{E}_d^\varepsilon(\widehat{\boldsymbol{M}}^\varepsilon) &\geq \frac{d}{\varepsilon^2}\|\nabla_p \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} - D_0\|\boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}\,\|\nabla_p \boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}\\
&\geq \frac{d}{2\varepsilon^2}\|\nabla_p \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} - \frac{D_0\varepsilon^2}{2d}\|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}.
\end{aligned}$$

Using propositions II.A.1 , II.A.2 , II.A.3 , II.A.4 , II.A.5 and equations (II.A.11) and (II.A.13) we get

$$\mathscr{E}_d^\varepsilon(\widehat{\boldsymbol{M}}^\varepsilon) - \pi\int_\Omega \left|\boldsymbol{M}_p^\varepsilon\right|^2 \geq \mathscr{E}_d^\varepsilon(\widehat{\boldsymbol{M}}^\varepsilon) - \pi\int_\Omega \left|\widehat{\boldsymbol{M}}_p^\varepsilon\right|^2 \geq -\left(D_{19}\varepsilon + D_{20}\varepsilon^{3/4}\right)\left(\|\boldsymbol{M}^\varepsilon\|_{L^2}^2 + \|\partial_3 \boldsymbol{M}^\varepsilon\|_{L^2}^2\right)$$
$$\geq -D_{20}\varepsilon^{3/4}\left(\|\boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}^2 + \|\partial_3 \boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}^2\right).$$

Adding the two together we get,

$$\frac{d}{\varepsilon^2}\|\nabla_p \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}^2 + \mathscr{E}_d^\varepsilon(\boldsymbol{M}^\varepsilon) - \pi\int_\Omega \left|\boldsymbol{M}_p^\varepsilon\right|^2$$
$$\geq \frac{d}{2\varepsilon^2}\|\nabla_p \boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}^2 - D_{20}\varepsilon^{3/4}\left(\|\boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}^2 + \|\partial_3 \boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}^2\right). \tag{II.A.21}$$

Note by the linearity of Maxwell's equation

$$\frac{1}{8\pi}\int_{\mathbb{R}^3} |\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^\varepsilon}^\varepsilon|^2 \boldsymbol{dy} = \frac{1}{8\pi}\int_{\mathbb{R}^3} |\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^o}^\varepsilon|^2 \boldsymbol{dy} + \frac{1}{8\pi}\int_{\mathbb{R}^3} |\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{M}}^\varepsilon}^\varepsilon|^2 \boldsymbol{dy} - \int_{\Omega_\varepsilon} \widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^o}^\varepsilon \cdot \widetilde{\boldsymbol{M}}^\varepsilon \boldsymbol{dy}.$$

Dividing by ε^2, noting that $\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) = \frac{1}{\varepsilon^2}\frac{1}{8\pi}\int_{\mathbb{R}^3}\left|\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^\varepsilon}^\varepsilon\right|^2 \boldsymbol{dy}$, and rescaling $\widetilde{\boldsymbol{m}}^\varepsilon$ & $\widetilde{\boldsymbol{M}}^\varepsilon$,

$$\begin{aligned}
\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) &= \mathscr{E}_d^\varepsilon(\boldsymbol{m}^o) + \mathscr{E}_d^\varepsilon(\boldsymbol{M}^\varepsilon) - \int_\Omega \boldsymbol{h}_{\boldsymbol{m}^o}^\varepsilon \cdot \boldsymbol{M}^\varepsilon \boldsymbol{dx} \\
&= \mathscr{E}_d^\varepsilon(\boldsymbol{m}^o) + \mathscr{E}_d^\varepsilon(\boldsymbol{M}^\varepsilon) + 2\pi\int_\Omega \boldsymbol{m}_p^o \cdot \boldsymbol{M}_p^\varepsilon \boldsymbol{dx} + \frac{8\varepsilon}{3}\int_\Omega \left(\boldsymbol{m}_p^o \cdot \boldsymbol{M}_p^\varepsilon - 2m_3^o M_3^\varepsilon\right)\boldsymbol{dx} \\
&\qquad - \varepsilon^2\pi\int_\Omega \left(\boldsymbol{m}_p^o \cdot \boldsymbol{M}_p^\varepsilon - 2m_3^o M_3^\varepsilon\right)\boldsymbol{dx} \qquad\qquad \text{(II.A.22)} \\
&\geq \mathscr{E}_d^\varepsilon(\boldsymbol{m}^o) + \mathscr{E}_d^\varepsilon(\boldsymbol{M}^\varepsilon) + 2\pi\int_\Omega \boldsymbol{m}_p^o \cdot \boldsymbol{M}_p^\varepsilon \boldsymbol{dx} - D_{21}\varepsilon\|\boldsymbol{m}_p^o \cdot \boldsymbol{M}_p^\varepsilon\|_{L^1} - D_{22}\varepsilon\|m_3^o M_3^\varepsilon\|_{L^1} \\
&\geq \mathscr{E}_d^\varepsilon(\boldsymbol{m}^o) + \mathscr{E}_d^\varepsilon(\boldsymbol{M}^\varepsilon) + 2\pi\int_\Omega \boldsymbol{m}_p^o \cdot \boldsymbol{M}_p^\varepsilon \boldsymbol{dx} - \frac{\Lambda}{4}\|\boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}^2 - D_{23}\varepsilon^2\|\boldsymbol{m}^o\|_{L^2(\Omega)}^2,
\end{aligned}$$

where we have used Young's inequality to bound the last two terms in the final step i.e. $D_{21}\varepsilon\left|\boldsymbol{m}_p^o \cdot \boldsymbol{M}_p^\varepsilon\right| \leq \left(D_{21}\varepsilon\frac{1}{\sqrt{\Lambda}}\left|\boldsymbol{m}_p^o\right|\right).\left(\sqrt{\Lambda}\ \left|\boldsymbol{M}_p^\varepsilon\right|\right) \leq \frac{D_{21}^2\varepsilon^2}{\Lambda}\left|\boldsymbol{m}_p^o\right|^2 + \frac{\Lambda}{2}\left|\boldsymbol{M}_p^\varepsilon\right|^2$ and similarly for the $m_3^o M_3^\varepsilon$ term. Also $\pi\int_\Omega\left|\boldsymbol{m}_p^\varepsilon\right|^2 = \pi\int_\Omega\left|\boldsymbol{m}_p^o\right|^2 + \pi\int_\Omega\left|\boldsymbol{M}_p^\varepsilon\right|^2 + 2\pi\int_\Omega \boldsymbol{m}_p^o \cdot \boldsymbol{M}_p^\varepsilon$. Subtracting the two and noting $\|\boldsymbol{m}^o\|_{L^2(\Omega)}^2 = m_s^2|\Omega|$ since $|\boldsymbol{m}^o| = m_s$, we get

$$\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \pi\int_\Omega \left|\boldsymbol{m}_p^\varepsilon\right|^2 \geq \varepsilon Q_1 + \varepsilon^2 Q_2 + \mathscr{E}_d^\varepsilon(\boldsymbol{M}^\varepsilon) - \pi\int_\Omega \left|\boldsymbol{M}_p^\varepsilon\right|^2 - \frac{\Lambda}{4}\|\boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}^2 - D_{23}\varepsilon^2 m_s^2|\Omega|.$$

Using this and eqn. (II.A.21) we get (note $\nabla_p \boldsymbol{m}^o = \nabla_p \boldsymbol{M}^\varepsilon$)

$$\begin{aligned}
&\frac{d}{\varepsilon^2}\|\nabla_p \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)} + \Lambda\big(\|\boldsymbol{M}^\varepsilon\|_{L^2(\Omega)} + \|\partial_3 \boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}\big) + \mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \pi\int_\Omega |\boldsymbol{m}_p^\varepsilon|^2 \\
&\quad \geq \frac{d}{2\varepsilon^2}\|\nabla_p \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \Lambda\big(\|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\big) + \varepsilon \boldsymbol{Q}_1 + \varepsilon^2 \boldsymbol{Q}_2 \\
&\qquad - D_{20}\varepsilon^{3/4}\Big(\|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\Big) - \frac{\Lambda}{4}\|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} - D_{23}\varepsilon^2 m_s^2|\Omega| \\
&\quad \geq \varepsilon \boldsymbol{Q}_1 + \varepsilon^2 \boldsymbol{Q}_2 + \frac{d}{2\varepsilon^2}\|\nabla_p \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \frac{\Lambda}{2}\big(\|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\big) - D_{18}\varepsilon^2
\end{aligned}$$

for ε small enough. □

Remark II.A.4. *If $\boldsymbol{m}^o = (0,0,m_s)$, we get a simpler estimate than in above Proposition II.A.6 . Note $\boldsymbol{m}^\varepsilon = \boldsymbol{m}^o + \boldsymbol{M}^\varepsilon$ gives $|\boldsymbol{m}^\varepsilon|^2 = m_s^2 = |\boldsymbol{m}^o|^2 + |\boldsymbol{M}^\varepsilon|^2 + 2\boldsymbol{m}^o\cdot\boldsymbol{M}^\varepsilon = m_s^2 + |\boldsymbol{M}^\varepsilon|^2 + 2m_3^\varepsilon M_3^\varepsilon$ which means $-2m_3^o M_3^\varepsilon = |\boldsymbol{M}^\varepsilon|^2$ and $\boldsymbol{m}_p^\varepsilon\cdot\boldsymbol{M}_p^\varepsilon = \boldsymbol{0}$. Substituting these in* (II.A.22) *we get*

$$\begin{aligned}
\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) &= \mathscr{E}_d^\varepsilon(\boldsymbol{m}^o) + \mathscr{E}_d^\varepsilon(\boldsymbol{M}^\varepsilon) + 2\pi\int_\Omega \boldsymbol{m}_p^o\cdot\boldsymbol{M}_p^\varepsilon\,\boldsymbol{dx} + \frac{8\varepsilon}{3}\int_\Omega\Big(\boldsymbol{m}_p^o\cdot\boldsymbol{M}_p^\varepsilon - 2m_3^o M_3^\varepsilon\Big)\boldsymbol{dx} \\
&\qquad - \varepsilon^2\pi\int_\Omega\big(\boldsymbol{m}_p^o\cdot\boldsymbol{M}_p^\varepsilon - 2m_3^o M_3^\varepsilon\big)\boldsymbol{dx} \\
&= \mathscr{E}_d^\varepsilon(\boldsymbol{m}^o) + \mathscr{E}_d^\varepsilon(\boldsymbol{M}^\varepsilon) + 2\pi\int_\Omega \boldsymbol{m}_p^o\cdot\boldsymbol{M}_p^\varepsilon\,\boldsymbol{dx} + \big(\frac{8\varepsilon}{3} - \varepsilon^2\pi\big)\int_\Omega |\boldsymbol{M}^\varepsilon|^2\,\boldsymbol{dx}.
\end{aligned}$$

Using the above and proceeding with the remaining part of the estimate in Proposition II.A.6 we get the following result,

$$\begin{aligned}
&\frac{d}{\varepsilon^2}\|\nabla_p \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)} + \Lambda\big(\|\boldsymbol{M}^\varepsilon\|_{L^2(\Omega)} + \|\partial_3 \boldsymbol{M}^\varepsilon\|_{L^2(\Omega)}\big) + \mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \pi\int_\Omega |\boldsymbol{m}_p^\varepsilon|^2 \\
&\quad \geq \varepsilon \boldsymbol{Q}_1 + \varepsilon^2 \boldsymbol{Q}_2 + \frac{d}{2\varepsilon^2}\|\nabla_p \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \frac{\Lambda}{2}\big(\|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\big)
\end{aligned} \tag{II.A.23}$$

for ε small enough.

Chapter III

Homogenized model for multi-layered ferromagnetic wires

Abstract

Multilayered wires of diameter in the nanometer scale with periodic layering of non-magnetic copper and ferromagnetic galfenol segments are studied in this chapter. The numerical computation of the physics of magnetization for such geometries is very costly computationally. We use the theory of periodic homogenization to understand the overall behavior of such structures. We first determine a "homogenized theory" after which this "homogenized model" is used to study the nucleation and stability of staturated states. Thus we get a broad generalization of what is known in the magnetic literature as the "fanning model" first introduced in [Jacobs and Bean, 1955] for a chain of spheres geometry.

III.1 Introduction

A lot of new experimental techniques have been developed to manufacture ferromagnetic wires of nanometer diameter e.g. electron-beam lithography, step growth and template-assisted electrodeposition. These techniques have also been adapted to growing multilayer wires which consist of layers of ferromagnetic material separated by nonmagnetic material.

A possible application of these multilayered nanosize wires is in making acoustic sensors. The inspiration for this application comes from the structure of the human ear. The inner ear has fine cilia like hair whose response to impinging acoustic waves is transmitted through the nervous system to the brain. One arrangement of multilayered nanowires is in the form of an array depicted in Fig III.1. Here impinging acoustic waves are expected to induce a detectable

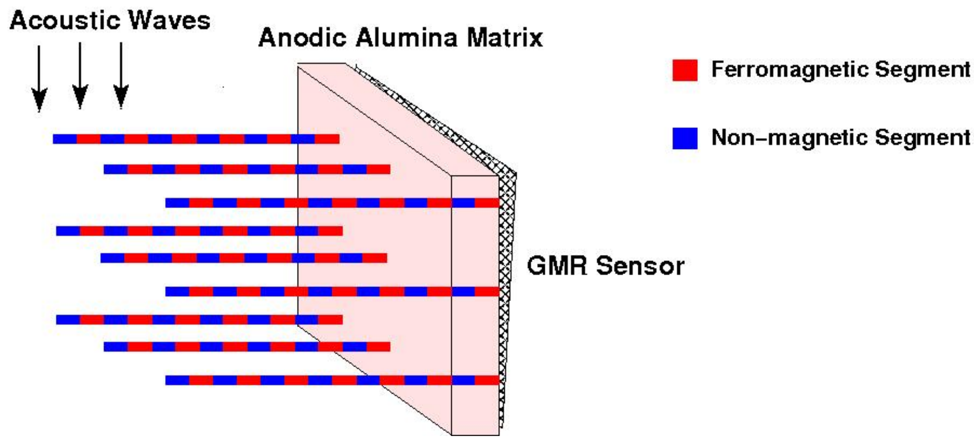

Figure III.1: Model Device using nanowires of Galfenol

change in the magnetization of the array.

In [Park et al., 2010] an investigation is conducted using Magnetic Force Microscopy (MFM) on multilayered wires with Galfenol as the ferromagnetic material. The multilayered nanowires we are trying to model have diameters in the 10-40nm range and length of the individual magnetic and nonmagnetic segments in the range of 10-30nm. The general problem of micromagnetics for such geometries becomes prohibitively expensive for numerical simulations as the segment length of these wires becomes small. A possible approach to deal with this problem is to use homogenization theory to simplify this problem. General references for homogenization theory can be found in [Bensoussan et al., 1978]. Newer methods of homogenization using two-scale convergence and Γ-convergence can be referenced in [Cioranescu and Donato, 1999] and [Braides and Defranceschi, 1998] respectively.

Homogenization methods have been previously used on ferromagnetic problems. Composites with magnetostrictive inclusions have been investigated in [Liu et al., 2006] using the constrained theory of magnetostriction of [DeSimone and James, 2002]. Multilayered ferromagnetic structures have also been investigated by [Hamdache, 2002] using Rado-Weertman surface energy and exchange coupling between the magnetic layers. Homogenization limits for the Landau-Lifschitz equation have also been investigated in [Santugini-Repiquet, 2007]. In this paper we look at a different limit from the aforementioned references where the surface energies or exchange coupling are not present and as a result the homogenized limit completely looses the exchange energy term.

Basic notation: $\alpha,\beta,\gamma,\cdots$ are scalars; $\boldsymbol{a},\boldsymbol{u},\boldsymbol{m},\cdots$ denotes vectors in $\mathbb{R}^3$; $\boldsymbol{A},\boldsymbol{B},\boldsymbol{E},\cdots$ are matrices in $\mathbb{R}^{3\times 3}$. Components of vector $\boldsymbol{m}$ are denoted by either m_1,m_2,m_3 or m_x,m_y,m_y. For any matrix $\boldsymbol{A}$, $\boldsymbol{A}^T$ denotes the transpose of the matrix. We use function space notation of $L^2(\Omega,\mathbb{R}^3)$, $H^1(\mathbb{R}^3,\mathbb{R}^3)$,

$H_0^1(\Omega,\mathbb{R}^3)$; for details refer [Adams and Fournier, 2009]. Section § II.2 gives a brief review of the micromagnetic theory of ferromagnetism. Section § III.3 gives the basic notion of two-scale convergence with important references and the behavior of the magnetostatic energy with respect to two-scale convergence. Section § III.4 describes the main results of this paper in Theorem III.4.1.

III.2 Micromagnetics

Let Ω be a smooth bounded reference configuration in $\mathbb{R}^3$. Let Y denote the unit cell in $\mathbb{R}^3$. Let $Y^* \Subset Y$ represent the magnetic segment, Y_1 the nonmagnetic segment and Y_2, the gap between the wires within Y. Let Ω_ε denote the intersection with Ω of the periodic extensions of εY^*. Figure III.2 shows the details of the geometry with periodic scale ε and the upscaled reference cell Y with the corresponding segments Y^*, Y_1 and Y_2.

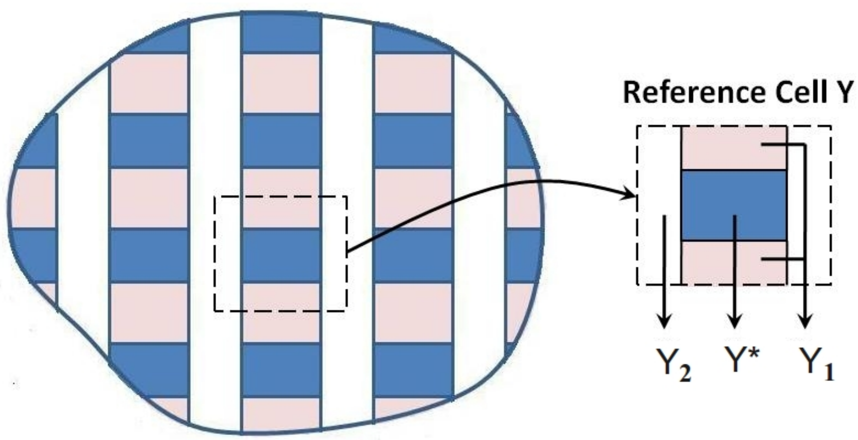

Figure III.2: Domain

Let $\boldsymbol{m}(\boldsymbol{x})$ be the magnetization vector at a point $\boldsymbol{x} \in \Omega_\varepsilon$. Below the Curie temperature, the magnetization is constrained to have constant euclidean norm i.e,

$$|\boldsymbol{m}(\boldsymbol{x})| = m_s \quad a.e. \quad \boldsymbol{x} \in \Omega_\varepsilon.$$

Interaction of magnetization with crystalline structure of magnetic solid gives rise to an anisotropy energy modeled by a function $\varphi : m_s S^2 \to [0,\infty)$. This energy has wells along a finite set of magnetization vectors $\{\boldsymbol{m}^{(k)}\}$ with $k \in \{1,2,\cdots N\}$ on which without loss of generality we can set $\Phi(\boldsymbol{m}^{(k)}) = 0$. For cubic materials the form of $\Phi(\boldsymbol{m})$ is given below along with a bound due to the constraint

$|\boldsymbol{m}| = m_s$,

$$\Phi(\boldsymbol{m}) = \Pi_1\big(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2\big) + \Pi_2\big(m_1^2 m_2^2 m_3^2\big) \tag{III.2.1}$$

$$0 \le E_{anis} = \int_{\Omega_\varepsilon} \Phi\big(\boldsymbol{m}(\boldsymbol{x})\big)\, \boldsymbol{dx} \;\le\; K_1 |\Omega_\varepsilon|. \tag{III.2.2}$$

The exchange energy penalizes variations in the magnetization in a body and thus tends to prefer constant magnetizations. It is modeled as follows,

$$E_{exc} = \int_{\Omega_\varepsilon} d\big|\nabla \boldsymbol{m}(\boldsymbol{x})\big|^2 \, \boldsymbol{dx}$$

where d is called the exchange constant. The energy of interaction of the magnetization $\boldsymbol{m}$ with an external applied field $\boldsymbol{h}_a \in L^2(\Omega, \mathbb{R}^3)$ is modeled and bounded using Hölder's inequality as follows,

$$-K_2 \;\le\; E_{app} = \int_{\Omega_\varepsilon} \boldsymbol{h}_a \cdot \boldsymbol{m} \;\le K_2, \qquad\qquad K_2 = \|\widetilde{\boldsymbol{h}_a}\|_{L^2(\Omega)} \;\|\widetilde{\boldsymbol{m}}\|_{L^2(\Omega)}. \tag{III.2.3}$$

Magnetized bodies generate a magnetic self field in all of $\mathbb{R}^3$. This energy term is given by the L^2 norm of the field $\boldsymbol{h_m}$ which is defined by the following equation,

$$\nabla \cdot \big(-\nabla\psi + 4\pi\ \boldsymbol{m}\ \chi_{\Omega_\varepsilon}\big)(\boldsymbol{x}) = 0 \qquad \forall \boldsymbol{x} \in \mathbb{R}^3, \qquad\qquad \boldsymbol{h_m}(\boldsymbol{x}) = -\nabla\psi(\boldsymbol{x}),$$
$$[|\nabla\psi \cdot \boldsymbol{n}|] = [|-\boldsymbol{h_m} \cdot \boldsymbol{n}|] = 4\pi\ \boldsymbol{m} \cdot \boldsymbol{n} \qquad \text{on } \partial\Omega_\varepsilon.$$

$[|\cdot|]$ represents the jump of a quantity across any oriented surface with unit normal $\boldsymbol{n}$. This term known as the demagnetization energy (we call it $\mathscr{E}_d^\varepsilon(\boldsymbol{m})$) enjoys standard upper and lower bounds,

$$0 \;\le\; \mathscr{E}_d^\varepsilon(\boldsymbol{m}) := \frac{1}{8\pi}\int_{\mathbb{R}^3} |\boldsymbol{h_m}|^2 \boldsymbol{dx} = -\frac{1}{2}\int_{\mathbb{R}^3} \boldsymbol{h_m} \cdot \boldsymbol{m}\boldsymbol{dx} \;\le\; \frac{1}{2}\int_{\Omega_\varepsilon} |\boldsymbol{m}(\boldsymbol{x})|^2\, \boldsymbol{dx} = \frac{1}{2}|\Omega_\varepsilon|\, m_s^2. \tag{III.2.4}$$

It is useful to have a positive total energy. Noting that the only negative term is the Zeeman energy, we add a constant energy term $\frac{1}{2}\int_\Omega \big(\ |\boldsymbol{h}_a|^2 + |\boldsymbol{m}|^2\ \big)\ \boldsymbol{dx} = \frac{1}{2}\|\boldsymbol{h}_a\|_{L^2(\Omega)}^2 + |\Omega| m_s^2 = C$ to the energy. The total energy is then given by,

$$\mathscr{E}^\varepsilon(\boldsymbol{m}) = \int_{\Omega_\varepsilon} \Big\{ d|\nabla \boldsymbol{m}|^2 + \Phi(\boldsymbol{m}) - \boldsymbol{h}_a \cdot \boldsymbol{m} \Big\} \boldsymbol{dx} + \mathscr{E}_d^\varepsilon(\boldsymbol{m}) + C. \tag{III.2.5}$$

We then try and minimize the problem $\inf_{\mathscr{A}^\varepsilon}\ \mathscr{E}^\varepsilon(\boldsymbol{m})$ in $\mathscr{A}^\varepsilon := \big\{\boldsymbol{m} \in H^1\big(\Omega_\varepsilon, m_s S^2\big)\ \big\}$.

In the subsequent sections we will investigate the convergence of the sequence of problems $\inf_{\mathscr{A}^\varepsilon}\ \mathscr{E}^\varepsilon(\boldsymbol{m})$ as ε goes to 0. Starting with a sequence of minimizers $\boldsymbol{m}^\varepsilon$ we will show in the next section that as $\varepsilon \to 0$, $\boldsymbol{m}^\varepsilon$ converges to a piecewise constant vector on $m_s S^2$.

III.3 Basics: Two scale convergence

In this section we will define the notion of two-scale convergence and a basic compactness theorem for any bounded sequence of functions in L^p space, as described in Chapter. 9 of [Cioranescu and Donato, 1999]. Then we will show how the demagnetization energy converges if the corresponding magnetizations converge two-scale. Let $\{\varepsilon\}$ be a sequence of reals converging to 0.

Definition III.3.1. *A sequence of functions $\boldsymbol{m}^\varepsilon(\boldsymbol{x})$ in $L^2(\Omega;\mathbb{R}^3)$ (respectively $L^2(\mathbb{R}^3;\mathbb{R}^3)$) is said to weakly two-scale converge to $\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})$ in $L^2(\Omega\times Y)$ (respectively $L^2(\mathbb{R}^3\times Y)$) if for all $\eta(\boldsymbol{x},\boldsymbol{y})\in L^2\big(\Omega, C^\infty_\sharp(Y)\big)$ $\Big($ respectively $L^2\big(\mathbb{R}^3, C^\infty_\sharp(Y)\big)\Big)$*

$$\lim_{\varepsilon\to 0}\int_\Omega m_i^\varepsilon \eta(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\boldsymbol{dx} = \int_{\Omega\times Y} m_i(\boldsymbol{x},\boldsymbol{y})\ \eta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dxdy}$$

for each component m_i^ε with $i\in\{1,2,3\}$ and we write $\boldsymbol{m}^\varepsilon(\boldsymbol{x}) \xrightharpoonup{2} \boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})$ in $L^2(\Omega\times Y)$.

Theorem III.3.1. *If $\boldsymbol{m}^\varepsilon(\boldsymbol{x})\in L^2(\Omega)$ is a sequence such that $\boldsymbol{m}^\varepsilon(\boldsymbol{x}) \xrightharpoonup{2} \boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})$ in $L^2(\Omega\times Y)$ then*

$$\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m}^o(\boldsymbol{x}) := \fint_Y \boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} \qquad \text{in } L^2(\Omega)$$
$$\|\boldsymbol{m}^o(\boldsymbol{x})\|_{L^2(\Omega)} \le \|\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\|_{L^2(\Omega\times Y)} \le \liminf_\varepsilon \|\boldsymbol{m}^\varepsilon(\boldsymbol{x},\boldsymbol{y})\|_{L^2(\Omega)}.$$

The inequality in the above theorem shows that two-scale convergence captures more information than typical weak convergence for a sequence of functions in L^p spaces. If $\|\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\|_{L^2(\Omega\times Y)} = \lim_\varepsilon \|\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\|_{L^2(\Omega)}$, then we say $\boldsymbol{m}^\varepsilon(\boldsymbol{x}) \xrightarrow{2} \boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})$.

Theorem III.3.2. *Let $\boldsymbol{m}^\varepsilon(\boldsymbol{x})$ be a uniformly bounded sequence of functions in $L^2(\Omega)$ (resp. $L^2(\mathbb{R}^3)$). Then there exists a unrealabeled subsequence $\boldsymbol{m}^\varepsilon(\boldsymbol{x})$ and a function $\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\in L^2(\Omega\times Y)$ (resp. $L^2(\mathbb{R}^3\times Y)$) such that $\boldsymbol{m}^\varepsilon(\boldsymbol{x})$ two-scale converges to $\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})$. Moreover*

$$\boldsymbol{m}^\varepsilon(\boldsymbol{x}) \rightharpoonup \fint_Y \boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy}. \quad \text{in } L^2(\Omega)\ \big(\text{resp. } L^2(\mathbb{R}^3)\big).$$

Theorem III.3.3. *Let $u^\varepsilon(\boldsymbol{x})$ be a sequence of functions in $W^{1,2}(\Omega)$ $\Big($ resp. $W_0^{1,2}(\mathbb{R}^3;\mathbb{R}^3)$ $\Big)$ and converges weakly to $u^o(\boldsymbol{x})$ in the same space. Then $u^\varepsilon(\boldsymbol{x})$ two-scale converges to $u^o(\boldsymbol{x})$ and there exists an unrelabeled subsequence $u^\varepsilon(\boldsymbol{x})$ and a function $u^1(\boldsymbol{x},\boldsymbol{y})\in L^2\big(\Omega, \mathscr{W}^{1,2}_\sharp(Y)\big)$ $\Big($ resp. $L^2\big(\mathbb{R}^3, \mathscr{W}^{1,2}(Y)\big)\Big)$ such that $\nabla u^\varepsilon(\boldsymbol{x})$ two-scale converges to $\nabla u^o(\boldsymbol{x}) + \nabla^y u^1(\boldsymbol{x},\boldsymbol{y})$.*

We now qoute Theroem 4 from [Lukkassen et al., 2002] which we use in this work.

Theorem III.3.4. *Let $B_p(\Omega;Y)$, $1 \le p < \infty$ denote any of the spaces $L_p\big(\Omega;C_\sharp(Y)\big), L^p_\sharp\big(Y;C(\overline{\Omega})\big)$. Then for $f \in B_p(\Omega;Y)$ we have*

$$\lim_{\varepsilon\to\infty}\int_\Omega \left|f\big(\boldsymbol{x},\varepsilon^{-1}\boldsymbol{x}\big)\right|\boldsymbol{dx} = \int_\Omega\int_Y \left|f(\boldsymbol{x},\boldsymbol{y})\right|^p \boldsymbol{dxdy}.$$

III.3.1 Two-scale limit of Demagnetization energy

Let $\boldsymbol{m}^\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} \boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})$ in $L^2(\Omega)$. Let us also denote the weak limit of $\boldsymbol{m}^\varepsilon$ by $\boldsymbol{m}^o$. Recall

$$\boldsymbol{m}^\varepsilon(\boldsymbol{x}) \rightharpoonup \boldsymbol{m}^o(\boldsymbol{x}) = \fint_Y \boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy}.$$

Set $\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{m}^o(\boldsymbol{x}) + \boldsymbol{m}^1(\boldsymbol{x},\boldsymbol{y})$ and note $\fint_Y \boldsymbol{m}^1(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = 0$.

We investigate the corresponding convergence of $\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}$ and $\mathscr{E}^\varepsilon_d(\boldsymbol{m}^\varepsilon)$. Multiplying Maxwell's with a test function $\varphi(\boldsymbol{x},\varepsilon^{-1}\boldsymbol{x})$ where $\varphi(\boldsymbol{x},\boldsymbol{y}) \in \mathscr{D}(\mathbb{R}^3, C^\infty_\sharp(Y))$ we get,

$$\begin{aligned}
&\int_{\mathbb{R}^3}\big(-\nabla\psi^\varepsilon(\boldsymbol{x}) + 4\pi\boldsymbol{m}^\varepsilon\big)\cdot\nabla\varphi\big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\big)\boldsymbol{dx} \\
&\qquad = \int_{\mathbb{R}^3}\big(-\nabla\psi^\varepsilon(\boldsymbol{x}) + 4\pi\boldsymbol{m}^\varepsilon\big)\cdot\big(\nabla^{\boldsymbol{x}}\varphi\big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\big) + \frac{1}{\varepsilon}\nabla^{\boldsymbol{y}}\varphi\big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\big)\big) = 0
\end{aligned} \tag{III.3.1}$$

Proposition III.3.1. *Let $\boldsymbol{m}^\varepsilon\chi_\varepsilon$ be a bounded sequence in $L^2(\mathbb{R}^3)$ such that $\boldsymbol{m}^\varepsilon\chi_\varepsilon \overset{2}{\rightharpoonup} \boldsymbol{m}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{m}^o(\boldsymbol{x}) + \boldsymbol{m}^1(\boldsymbol{x},\boldsymbol{y})$. Then $\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}(\boldsymbol{x}) \overset{2}{\rightharpoonup} \boldsymbol{h}_{\boldsymbol{m}^o}(\boldsymbol{x}) + \nabla^{\boldsymbol{y}}\psi^1(\boldsymbol{x},\boldsymbol{y})$ in $L^2(\mathbb{R}^3)$ where $\boldsymbol{h}_{\boldsymbol{m}^o}$ solves Maxwell's equation for $\boldsymbol{m}^o$ and $\psi(\boldsymbol{x},\boldsymbol{y})$ solves for all $\varphi(\boldsymbol{y}) \in C^\infty_\sharp(Y)$*

$$\int_Y \big(-\nabla^{\boldsymbol{y}}\psi^1(\boldsymbol{x},\boldsymbol{y}) + 4\pi\boldsymbol{m}^1(\boldsymbol{x},\boldsymbol{y})\big)\cdot\nabla^{\boldsymbol{y}}\varphi(\boldsymbol{y}) = 0 \qquad \forall\boldsymbol{x} \tag{III.3.2}$$

which we call the Periodic Maxwell's problem. Also if $\boldsymbol{m}^\varepsilon(\boldsymbol{x}) \overset{2}{\to} \boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})$ then $\boldsymbol{h}^\varepsilon_{\boldsymbol{m}} \overset{2}{\to} \boldsymbol{h}_{\boldsymbol{m}^o}(\boldsymbol{x}) + \nabla^{\boldsymbol{y}}\psi^1(\boldsymbol{x},\boldsymbol{y})$.

Proof. It is standard to show that $\psi^\varepsilon(\boldsymbol{x})$ which solves Maxwell's equation for a uniformly bounded sequence $\boldsymbol{m}^\varepsilon$ is uniformly bounded in $W^{1,2}_0(\mathbb{R}^3)$. Let an unrelabeled subsequence converge weakly in $W^{1,2}(\mathbb{R}^3)$ to some $\psi^o(\boldsymbol{x})$. Also using Theorem III.3.3 we also have for the same subsequence, $\nabla\psi^\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} \nabla\psi^o(\boldsymbol{x}) + \nabla^{\boldsymbol{y}}\psi^1(\boldsymbol{x},\boldsymbol{y})$ with in $\psi^1(\boldsymbol{x},\boldsymbol{y}) \in L^2\big(\mathbb{R}^3, \mathscr{W}^{1,2}_\sharp(Y)\big)$.

In eqn. (III.3.1) choosing $\varphi(\boldsymbol{x},\boldsymbol{y}) = \varphi(\boldsymbol{x})$ independent of $\boldsymbol{y}$ and taking $\lim_{\varepsilon\to 0}$ in (III.3.1) we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\big(-\nabla\psi^{\varepsilon}(\boldsymbol{x}) + 4\pi\boldsymbol{m}^{\varepsilon}\chi_{\varepsilon}(\boldsymbol{x})\big)\cdot\nabla_{\boldsymbol{x}}\varphi(\boldsymbol{x})\boldsymbol{dx}$$
$$= \int_{\mathbb{R}^3}\big(-\nabla\psi^{o}(\boldsymbol{x}) + 4\pi\boldsymbol{m}^{o}\chi_{\Omega}(\boldsymbol{x})\big)\cdot\nabla_{\boldsymbol{x}}\varphi(\boldsymbol{x})\boldsymbol{dx} = 0. \tag{III.3.3}$$

Now multiplying (III.3.1) by ε and taking $\lim_{\varepsilon\to 0}$ we get

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\big(-\nabla\psi^{\varepsilon}(\boldsymbol{x}) + 4\pi\boldsymbol{m}^{\varepsilon}\chi_{\varepsilon}(\boldsymbol{x})\big)\cdot\big(\varepsilon\nabla_{\boldsymbol{x}}\varphi(\boldsymbol{x},\varepsilon^{-1}\boldsymbol{x}) + \nabla^{\boldsymbol{y}}\varphi(\boldsymbol{x},\varepsilon^{-1}\boldsymbol{x})\big)\boldsymbol{dx}$$
$$= \int_{\mathbb{R}^3}\int_{Y}\big(-\nabla\psi^{o}(\boldsymbol{x}) - \nabla^{\boldsymbol{y}}\psi^{1}(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x},\boldsymbol{y}) + 4\pi(\boldsymbol{m}^{o}(\boldsymbol{x}) + \boldsymbol{m}^{1}(\boldsymbol{x},\boldsymbol{y}))\big)\cdot\nabla^{\boldsymbol{y}}\varphi(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{dx}\,\boldsymbol{dy} = 0.$$

The periodicity of $\varphi(\boldsymbol{x},\boldsymbol{y})$ in $\boldsymbol{y}$ gives $\int_Y \nabla_y\varphi(\boldsymbol{x},\boldsymbol{y}) = 0$. The first term in the L.H.S. then becomes $\int_{\mathbb{R}^3}\int_Y -\nabla\psi^{o}(\boldsymbol{x})\cdot\nabla^{\boldsymbol{y}}\varphi(\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbb{R}^3} -\nabla\psi^{o}(\boldsymbol{x})\,\big)\cdot\int_Y\nabla^{\boldsymbol{y}}\varphi(\boldsymbol{x},\boldsymbol{y}) = 0$ and similarly $\int_{\mathbb{R}^3}\int_Y \boldsymbol{m}^{o}(\boldsymbol{x})\cdot\nabla^{\boldsymbol{y}}\varphi(\boldsymbol{x},\boldsymbol{y}) = 0$ which gives

$$\int_{\mathbb{R}^3}\int_{Y}\big(-\nabla^{\boldsymbol{y}}\psi^{1}(\boldsymbol{x},\boldsymbol{y}) + 4\pi\boldsymbol{m}^{1}(\boldsymbol{x},\boldsymbol{y})\big)\cdot\nabla^{\boldsymbol{y}}\varphi(\boldsymbol{x},\boldsymbol{y}) = 0 \tag{III.3.4}$$

and we get the periodic problem by choosing $\varphi(\boldsymbol{x},\boldsymbol{y}) = \varphi_1(\boldsymbol{x})\varphi_2(\boldsymbol{y})$ with $\varphi_1(\boldsymbol{x}) \in C^{\infty}(\mathbb{R}^3)$ and $\varphi_2(\boldsymbol{y}) \in C^{\infty}_{\sharp}(Y)$ so that $\int_Y\big(-\nabla^{\boldsymbol{y}}\psi^{1}(\boldsymbol{x},\boldsymbol{y}) + 4\pi\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\big)\cdot\nabla^{\boldsymbol{y}}\varphi_2(\boldsymbol{y}) = 0$ for all $\boldsymbol{x}$. Also in equation (III.3.4) by choosing a sequence test functions $\varphi_j(\boldsymbol{x},\boldsymbol{y}) \to \psi^{1}(\boldsymbol{x},\boldsymbol{y})$ in $L^2\big(\mathbb{R}^3,\mathscr{W}^{1,2}_{\sharp}\big)$ we get

$$\int_{\mathbb{R}^3}\int_{Y}\left|\nabla^{\boldsymbol{y}}\psi^{1}(\boldsymbol{x},\boldsymbol{y})\right|^2 = 4\pi\int_{\mathbb{R}^3}\int_{Y}\boldsymbol{m}^{1}(\boldsymbol{x},\boldsymbol{y})\cdot\nabla^{\boldsymbol{y}}\psi^{1}(\boldsymbol{x},\boldsymbol{y}) \tag{III.3.5}$$

Now we show the strong two-scale convergence of the demag field. Since $\boldsymbol{m}^{\varepsilon} \xrightarrow{2} \boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}} \xrightarrow{2} \boldsymbol{h}_{\boldsymbol{m}^{o}}(\boldsymbol{x}) + \nabla^{\boldsymbol{y}}\psi^{1}(\boldsymbol{x},\boldsymbol{y})$, then $\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}\cdot\boldsymbol{m}^{\varepsilon} \rightharpoonup \fint_Y\big(\boldsymbol{h}_{\boldsymbol{m}^{o}}(\boldsymbol{x}) + \nabla^{\boldsymbol{y}}\psi^{1}(\boldsymbol{x},\boldsymbol{y})\big)\cdot\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{dy}$ in $\mathscr{D}'(\mathbb{R}^3)$.

Let $\chi^{\delta}_{\Omega}(\boldsymbol{x})$ be a mollification of $\chi_{\Omega}(\boldsymbol{x})$ so that $\chi^{\delta}_{\Omega}(\boldsymbol{x}) = 1$ for all $\boldsymbol{x}\in\Omega$ and $\chi^{\delta}_{\Omega}(\boldsymbol{x}) \to \chi_{\Omega}(\boldsymbol{x})$. Then

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\big(\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}\cdot\boldsymbol{m}^{\varepsilon}\big)\,\chi^{\delta}_{\Omega}(\boldsymbol{x})\,\boldsymbol{dx} = \lim_{\varepsilon\to 0}\int_{\Omega}\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}(\boldsymbol{x})\cdot\boldsymbol{m}^{\varepsilon}(\boldsymbol{x})\,\boldsymbol{dx}$$
$$= \int_{\mathbb{R}^3}\fint_{Y}\big(\boldsymbol{h}_{\boldsymbol{m}^{o}}(\boldsymbol{x}) + \nabla^{\boldsymbol{y}}\psi^{1}(\boldsymbol{x},\boldsymbol{y})\big)\cdot\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\,\chi^{\delta}_{\Omega}(\boldsymbol{x})\,\boldsymbol{dy}\boldsymbol{dx}.$$

Taking further $\lim_{\delta\to 0}$ we get $\int_{\Omega}\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}\cdot\boldsymbol{m}^{\varepsilon}\boldsymbol{dx} \to \int_{\Omega}\fint_{Y}\big(\boldsymbol{h}_{\boldsymbol{m}^{o}}(\boldsymbol{x}) + \nabla^{\boldsymbol{y}}\psi^{1}(\boldsymbol{x},\boldsymbol{y})\big)\cdot\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{dy}\boldsymbol{dx}$. Then using eqns. (III.2.4) and (III.3.5) we have

$$-4\pi\|\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}\|^2_{L^2(\mathbb{R}^3)} = \int_\Omega \boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}\cdot\boldsymbol{m}^{\varepsilon}\boldsymbol{dx} \to \int_\Omega \boldsymbol{h}_{\boldsymbol{m}^o}\cdot\boldsymbol{m}^o\boldsymbol{dx} + \int_\Omega\fint_Y \nabla^y\psi^1(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{m}^1(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dxdy}$$
$$= -4\pi\|\boldsymbol{h}_{\boldsymbol{m}^o}\|^2_{L^2(\mathbb{R}^3)} - 4\pi\,|Y|^{-1}\|\nabla^y\psi^1\|^2_{L^2(\mathbb{R}^3\times Y)}. \qquad \square$$

Analogous to eqn. (III.2.4) if $\boldsymbol{m}$ is a magnetization with support in Ω we define

$$\mathscr{E}_d(\boldsymbol{m}) := \frac{1}{8\pi}\int_\Omega |\boldsymbol{h}_{\boldsymbol{m}}|^2\boldsymbol{dx} \;\le\; \frac{m_s^2}{2}|\Omega| \tag{III.3.6}$$

where $\boldsymbol{h}_{\boldsymbol{m}}$ solves Maxwell's equation for $\boldsymbol{m}$. And if $\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y})$ is a magnetization with support on $L^2(\Omega\times Y)$ we define for every $\boldsymbol{x}\in\Omega$, the term $\mathscr{E}_{per}(\boldsymbol{m})$ as

$$\mathscr{E}_{per}(\boldsymbol{m}) := \frac{1}{8\pi}\fint_Y |\nabla^y\psi^1(\boldsymbol{x},\boldsymbol{y})|^2\boldsymbol{dy} \tag{III.3.7}$$

where $\nabla^y\psi^1$ solves equation (III.3.2) for $\boldsymbol{m}^1(\boldsymbol{x},\boldsymbol{y})$.

III.4 Convergence of variational problem

Let $\boldsymbol{m}^\varepsilon$ be a minimizer of the problem $\mathscr{E}^\varepsilon(\boldsymbol{m})$ in $\mathscr{A}^\varepsilon$. We will first show that the energy $\mathscr{E}^\varepsilon(\boldsymbol{m}^\varepsilon)$ is bounded above and below independant of ε.

III.4.1 Bounds on energy $\mathscr{E}^\varepsilon(\boldsymbol{m}^\varepsilon)$

Comparing energy of $\boldsymbol{m}^\varepsilon$ w.r.t an arbitrary constant vector $\boldsymbol{m}^o$ on m_sS^2 gives

$$\begin{aligned}\mathscr{E}^\varepsilon(\boldsymbol{m}^\varepsilon) \;&\le\; \mathscr{E}^\varepsilon(\boldsymbol{m}^o) = \int_{\Omega_\varepsilon}\Big\{d|\nabla\boldsymbol{m}^o|^2 + \Phi(\boldsymbol{m}^o) - \boldsymbol{h}_a\cdot\boldsymbol{m}^o\Big\}\boldsymbol{dx} + \mathscr{E}^\varepsilon_d(\boldsymbol{m}^o) + C\\ &\le\; K_1|\Omega_\varepsilon| + K_2 + \frac{1}{2}|\Omega_\varepsilon| + Cm_s^2 \;\le\; K_3\end{aligned} \tag{III.4.1}$$

where we have used the standard bounds on each term from (III.2.2), (III.2.3) and (III.2.4). A straightforward lower bound comes from the fact that we added the constant term C to make $\mathscr{E}^\varepsilon(\boldsymbol{m}^\varepsilon) \;\ge\; 0$. The upper from (III.4.1) and the positivity together give independantly of ε

$$K_4 \;\ge\; \|\nabla\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}. \tag{III.4.2}$$

Also since $|\boldsymbol{m}^{\varepsilon}| = m_s$ on Ω_ε, extending $\boldsymbol{m}^{\varepsilon}$ by $\mathbf{0}$ on $\Omega/\Omega_\varepsilon$ we get for some unrelabeled subsequence

$$\boldsymbol{m}^{\varepsilon} \rightharpoonup \boldsymbol{m}^{o} \qquad \text{in} \quad L^2(\Omega). \tag{III.4.3}$$

III.4.2 Improved bound on exchange energy and convergence of $\boldsymbol{m}^{\varepsilon}$

Here we will show that the exchange energy $\boldsymbol{m}^{\varepsilon}$ is in fact $O(\varepsilon)$ and thus the minimizing magnetizations $\boldsymbol{m}^{\varepsilon}$ are very close to constants on each segment constituting Ω_ε. Let the individual magnetic segments in Ω_ε be indexed as W_k with k running from $k=1$ to some $N(\varepsilon)$ so as to cover all the segments. Using Poincaré inequality on W_k for any $k \in \{1, 2, \cdots N(\varepsilon)\}$, we have

$$\|\boldsymbol{m}^{\varepsilon} - \widehat{\boldsymbol{m}}_k^{\varepsilon}\|^2_{L^2(W_k)} = |W_k| \left(m_s^2 - |\widehat{\boldsymbol{m}}_k^{\varepsilon}|^2\right) \leq C_o \varepsilon^2 \|\nabla \boldsymbol{m}^{\varepsilon}\|^2_{L^2(W_k)} \tag{III.4.4}$$

where $\widehat{\boldsymbol{m}}_k^{\varepsilon} := \fint_{W_k} \boldsymbol{m}^{\varepsilon}\, \boldsymbol{dx}$ is the local average within each cell W_k and C_o is the Poincaré constant for the scaled cell Y^*. Let us define

$$\widetilde{\Omega}_\varepsilon := \left\{ \boldsymbol{x} \in \Omega_\varepsilon \ : \ |\nabla \boldsymbol{m}^{\varepsilon}|^2 \geq \frac{m_s^2}{2C_o\varepsilon^2} \right\} \tag{III.4.5}$$

and using Chebyshev's inequality note that

$$|\widetilde{\Omega}_\varepsilon| = \left| \left\{ \boldsymbol{x} \in \Omega_\varepsilon : |\nabla \boldsymbol{m}^{\varepsilon}|^2 \geq \frac{m_s^2}{2C_o\varepsilon^2} \right\} \right| \leq \frac{2C_o\varepsilon^2}{m_s^2} \int_{\Omega_\varepsilon} |\nabla \boldsymbol{m}^{\varepsilon}|^2 \boldsymbol{dx} \ \leq \ \frac{2K_4 C_o \varepsilon^2}{m_s^2}. \tag{III.4.6}$$

Let us define a test function as follows:

$$\boldsymbol{M}^{\varepsilon}(\boldsymbol{x}) = \begin{cases} m_s \left| \widehat{\boldsymbol{m}}_k^{\varepsilon}(\boldsymbol{x}) \right|^{-1} \widehat{\boldsymbol{m}}_k^{\varepsilon}(\boldsymbol{x}) & \text{for } \ \boldsymbol{x} \in W_k \ \text{ if } \ \left|W_k \cap \left(\Omega_\varepsilon/\widetilde{\Omega}_\varepsilon\right)\right| > 0 \\ \text{any arbitrary vector on } m_s S^2 \text{ otherwise.} \end{cases} \tag{III.4.7}$$

To make sure the above definition makes sense, we need to check that $|\widehat{\boldsymbol{m}}_k^{\varepsilon}| > 0$ on all W_k if $|W_k \cap (\Omega_\varepsilon/\widetilde{\Omega}_\varepsilon)| > 0$. To see that note if $\left|W_k \cap \left(\Omega_\varepsilon/\widetilde{\Omega}_\varepsilon\right)\right| > 0$, from (III.4.5) we have $\int_{W_k} |\nabla \boldsymbol{m}^{\varepsilon}|^2 \boldsymbol{dx} \leq \frac{m_s^2}{2C_o\varepsilon^2} |W_k|$. This along with the Poincaré inequality in (III.4.4) gives,

$$\|\boldsymbol{m}^{\varepsilon}\|^2_{L^2(W_k)} - \|\widehat{\boldsymbol{m}}_k^{\varepsilon}\|^2_{L^2(W_k)} = |W_k| \left(m_s^2 - |\widehat{\boldsymbol{m}}_k^{\varepsilon}|^2 \right) \ \leq \ C_o\varepsilon^2 \|\nabla \boldsymbol{m}^{\varepsilon}\|^2_{L^2(W_k)} = \frac{m_s^2}{2} |W_k|$$

which on rearranging implies $|\widehat{\boldsymbol{m}}_k^{\varepsilon}|^2 \geq \frac{m_s^2}{2}$. Next we show the $\boldsymbol{m}^{\varepsilon}$ and $\boldsymbol{M}^{\varepsilon}$ are close.

Proposition III.4.1. *The following holds,* $\|\boldsymbol{m}^{\varepsilon} - \boldsymbol{M}^{\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} \ \leq \ K_6\varepsilon$.

Proof. Let $\mathbb{K}$ be an index such that $\Omega_\varepsilon = \bigcup_{k\in\mathbb{K}} W_k$. We can split the index $\mathbb{K}$ into two parts $\mathbb{K} = \mathbb{K}_1 \cup \mathbb{K}_2$ such that

$$k \in \mathbb{K}_1 \text{ if } \left|W_k \bigcap \left(\Omega_\varepsilon/\widetilde{\Omega}_\varepsilon\right)\right| > 0 \qquad \text{and} \qquad k \in \mathbb{K}_2 \text{ if } W_k \subseteq \widetilde{\Omega}_\varepsilon.$$

For $k \in \mathbb{K}_1$,

$$\int_{W_k} \left|\boldsymbol{m}^\varepsilon - \boldsymbol{M}^\varepsilon\right|^2 \boldsymbol{dx} = 2m_s^2\left|W_k\right| - 2m_s \frac{\widehat{\boldsymbol{m}}_k^\varepsilon}{|\widehat{\boldsymbol{m}}_k^\varepsilon|} \cdot \int_{W_k} \boldsymbol{m}^\varepsilon = 2m_s\left|W_k\right| \left(m_s - \left|\widehat{\boldsymbol{m}}_k^\varepsilon\right|\right).$$

Since $m_s \le |\widehat{\boldsymbol{m}}_k^\varepsilon| + m_s$, the above gives along with Poincaré inequality in eqn. (III.4.4)

$$\int_{W_k} \left|\boldsymbol{m}^\varepsilon - \boldsymbol{M}^\varepsilon\right|^2 \le 2\left|W_k\right| \left(m_s^2 - \left|\widehat{\boldsymbol{m}}_k^\varepsilon\right|^2\right) \le 2C_o\varepsilon^2 \left\|\nabla \boldsymbol{m}^\varepsilon\right\|^2_{L^2(W_k)}.$$

Summing up for all $k \in \mathbb{K}_1$ we get

$$\sum_{k\in\mathbb{K}_1} \left\|\boldsymbol{m}^\varepsilon - \boldsymbol{M}^\varepsilon\right\|^2_{L^2(W_k)} \le 2C_o\varepsilon^2 \sum_{k\in\mathbb{K}_1} \left\|\nabla \boldsymbol{m}^\varepsilon\right\|^2_{L^2(W_k)} \le 2C_o\varepsilon^2 \left\|\nabla \boldsymbol{m}^\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)} \le 2C_oK_4\varepsilon^2.$$

For $k \in \mathbb{K}_2$ we have

$$\begin{aligned}\sum_{k\in\mathbb{K}_2} \left\|\boldsymbol{m}^\varepsilon - \boldsymbol{M}^\varepsilon\right\|^2_{L^2(W_k)} &\le 2\sum_{k\in\mathbb{K}_2} \left[\left\|\boldsymbol{m}^\varepsilon\right\|^2_{L^2(W_k)} + \left\|\boldsymbol{M}^\varepsilon\right\|^2_{L^2(W_k)}\right] = 2m_s^2 \sum_{k\in\mathbb{K}_2} \left|W_k\right| \\ &= 2m_s^2 \left|\widetilde{\Omega}_\varepsilon\right| \le 4K_4C_o\varepsilon^2.\end{aligned}$$

Adding the two estimates over $k \in \mathbb{K}$ gives our result. □

Now we evaluate the difference in magnetostatic energy between $\boldsymbol{m}^\varepsilon$ and $\boldsymbol{M}^\varepsilon$. Lemma III.A.1 gives us

$$\begin{aligned}\frac{1}{8\pi}\left| \left\|\boldsymbol{h}_{\boldsymbol{m}^\varepsilon}\right\|^2_{L^2(\mathbb{R}^3)} - \left\|\boldsymbol{h}_{\boldsymbol{M}^\varepsilon}\right\|^2_{L^2(\mathbb{R}^3)} \right| &\le \frac{1}{2}\left\|\boldsymbol{m}^\varepsilon - \boldsymbol{M}^\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \left(\left\|\boldsymbol{m}^\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + \left\|\boldsymbol{M}^\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\right) \\ &\le K_7\varepsilon \cdot 2m_s\left|\Omega_\varepsilon\right|^{1/2} = K_8\varepsilon \end{aligned} \tag{III.4.8}$$

where we have used Proposition III.4.1 to estimate $\left\|\boldsymbol{m}^\varepsilon - \boldsymbol{M}^\varepsilon\right\|_{L^2(\Omega)}$ and equation (III.2.4) to estimate $\left\|\boldsymbol{m}^\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + \left\|\boldsymbol{M}^\varepsilon\right\|_{L^2(\Omega_\varepsilon)}$.
Taylor expansion of the polynomial function $\Phi(\boldsymbol{m}^\varepsilon)$ gives, $\left(\Phi'(\boldsymbol{m})\text{ is derivative of }\Phi\text{ w.r.t. }\boldsymbol{m}\right)$

$$\int_{\Omega_\varepsilon} \Phi(\boldsymbol{m}^\varepsilon)\boldsymbol{dx} = \int_{\Omega_\varepsilon} \Phi(\boldsymbol{M}^\varepsilon)\boldsymbol{dx} + \int_{\Omega_\varepsilon} \Phi'(\boldsymbol{M}^\varepsilon)\cdot(\boldsymbol{m}^\varepsilon - \boldsymbol{M}^\varepsilon)\boldsymbol{dx} + O\left(\left\|\boldsymbol{m}^\varepsilon - \boldsymbol{M}^\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)}\right).$$

Comparing energy of the minimizer $\boldsymbol{m}^\varepsilon$ with the test function $\boldsymbol{M}^\varepsilon$ we have,

$$\int_{\Omega_\varepsilon}\left(c\,|\nabla \boldsymbol{m}^\varepsilon|^2+\Phi(\boldsymbol{m}^\varepsilon)-\boldsymbol{h}_a\cdot\boldsymbol{m}^\varepsilon\right)\boldsymbol{dx}+\frac{1}{8\pi}\|\boldsymbol{h}_{\boldsymbol{m}^\varepsilon}\|^2_{L^2(\mathbb{R}^3)}\le\int_{\Omega_\varepsilon}\left(\Phi(\boldsymbol{M}^\varepsilon)-\boldsymbol{h}_a\cdot\boldsymbol{M}^\varepsilon\right)\boldsymbol{dx}$$
$$+\frac{1}{8\pi}\|\boldsymbol{h}_{\boldsymbol{M}^\varepsilon}\|^2_{L^2(\mathbb{R}^3)}$$

which gives using the truncated Taylor expansion, Cauchy-Schwarz inequality and eqn. (III.4.8)

$$\begin{aligned}\int_{\Omega_\varepsilon}c|\nabla\boldsymbol{m}^\varepsilon|^2\boldsymbol{dx}\;&\le\;\int_{\Omega_\varepsilon}\left(\Phi'(\boldsymbol{M}^\varepsilon)+\boldsymbol{h}_a\right)\cdot\left(\boldsymbol{m}^\varepsilon-\boldsymbol{M}^\varepsilon\right)\,\boldsymbol{dx}+K_8\varepsilon+O\left(\|\boldsymbol{m}^\varepsilon-\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\right)\\&\le\;\left(\|\Phi'(\boldsymbol{M}^\varepsilon)\|_{L^2(\Omega_\varepsilon)}+\|\boldsymbol{h}_a\|_{L^2(\Omega_\varepsilon)}\right)\|\boldsymbol{m}^\varepsilon-\boldsymbol{M}^\varepsilon\|_{L^2(\Omega_\varepsilon)}+K_8\varepsilon\;=\;K_9\varepsilon\end{aligned}$$

noting that $\Phi'(\boldsymbol{M}^\varepsilon)\in L^\infty(\Omega_\varepsilon)$ as Φ is a polynomial function of $\boldsymbol{M}^\varepsilon\in m_sS^2$. Its also easy to then show that

$$\mathscr{E}^\varepsilon(\boldsymbol{m}^\varepsilon)\;\ge\;\mathscr{E}^\varepsilon(\boldsymbol{M}^\varepsilon)-K_{10}\varepsilon. \tag{III.4.9}$$

III.4.3 The limit variational problem

We first setup notation which simplifies the computations for following sections. Let kY be a k-"sided" supercell consiting of k^3 copies of our basic Y cell. Let $kY^*\subset kY$ be the k^3 copies of Y^* subsets within each cell Y. Individual members of kY^* or kY will be indexed as Y_i^* or Y_i for $i\in\{1,2,\cdots k^3\}$.

We now state the main result of this section.

Theorem III.4.1. *From sequence of minimizers $\boldsymbol{m}^\varepsilon$ of $\inf_{\mathscr{A}^\varepsilon}\ \mathscr{E}^\varepsilon(\boldsymbol{m})$ in $\mathscr{A}^\varepsilon$, we can extract a subsequence (unrelabeled) which converges weakly in $L^2(\Omega)$ to $\boldsymbol{m}^o$, where $\boldsymbol{m}^o$ minimizes a limit energy $\mathscr{E}^o(\boldsymbol{m})$ $\left(\text{defined for all } |\boldsymbol{m}(\boldsymbol{x})|\le\theta m_s\right)$ defined as*

$$\mathscr{E}^o(\boldsymbol{m})=\int_\Omega\left[-\boldsymbol{h}_a(\boldsymbol{x})\cdot\boldsymbol{m}(\boldsymbol{x})+\liminf_{k\to\infty}\inf_{\boldsymbol{w}\in\mathscr{A}_o^k(\boldsymbol{m}^o)}\left\{\fint_{kY}\left(\Phi(\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y}))+\frac{1}{8\pi}|\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}}(\boldsymbol{x},\boldsymbol{y})|^2\right)\boldsymbol{dy}\right\}\right]\boldsymbol{dx}$$
$$+\int_{\mathbb{R}^3}\frac{1}{8\pi}|\boldsymbol{h}_{\boldsymbol{m}}(\boldsymbol{x})|^2\,\boldsymbol{dx}, \tag{III.4.10}$$

$$\mathscr{A}_o^k(\boldsymbol{m}):=\Big\{\,\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\in(L^2(\Omega);L^2_\sharp(kY))\ ;\ \boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\text{ is a constant vector in } m_sS^2 \text{ on each}$$
$$Y_i^*\in kY \text{ and } \boldsymbol{0} \text{ on each } Y_i/Y_i^*\in kY;\ \fint_{kY}\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{dy}=\boldsymbol{m}(\boldsymbol{x})\,\Big\},$$

and $\boldsymbol{h}_{\boldsymbol{m}}(\boldsymbol{x})$ solves the Maxwell's problem for $\boldsymbol{m}(\boldsymbol{x})$ and $\psi^1_{\boldsymbol{w}}(\boldsymbol{x},\boldsymbol{y})$ solves the Periodic Maxwell's prob-

lem in eqn. (III.3.2) *from Proposition III.3.1 for all* $\boldsymbol{x}$.

We already have shown the weak convergence $\boldsymbol{m}^{\varepsilon} \rightharpoonup \boldsymbol{m}^{o}$ in $L^2(\Omega)$ from eqn. (III.4.3). The proof can be decomposed into a set of lemmas which set up a lim sup upper bound and a lim inf lower bound on $\mathscr{E}^{\varepsilon}(\boldsymbol{m}^{\varepsilon})$ w.r.t. $\mathscr{E}^{o}(\boldsymbol{m}^{o})$. We follow the line of argument presented in [Cioranescu et al., 2006] and [Fonseca and Krömer, 2010]. We first establish a lim sup upper bound which resembles Lemma 2.6 and Proposition 3.7 of the aforementioned references respectively.

Lemma III.4.1.

$$\limsup_{\varepsilon\to 0} \mathscr{E}^{\varepsilon}(\boldsymbol{m}^{\varepsilon}) \le \mathscr{E}^{o}(\boldsymbol{m}^{o}).$$

Proof. Let $k \in \mathbb{N}$ be fixed. Let $V = kY$ where Y is the unit cube. Let $\Omega^{k,\varepsilon}$ be the intersection of Ω with a tiling of $\mathbb{R}^3$ by εV. As we have done in Appeendix Section III.B, let $\boldsymbol{z} \in \mathbb{Z}^3$ and let $V_{\varepsilon,\boldsymbol{z}}$ and $Z_{k,\varepsilon}$ be defined as

$$V_{\varepsilon,\boldsymbol{z}} := \varepsilon k \boldsymbol{z} + \varepsilon V, \qquad Z_{k,\varepsilon} = \left\{ \boldsymbol{z} \in \mathbb{Z}^3 \mid V_{\varepsilon,\boldsymbol{z}} \bigcap \Omega \neq \emptyset \right\}. \tag{III.4.11}$$

If $\boldsymbol{x} \in V_{\varepsilon,\boldsymbol{z}}$, then we can write $\boldsymbol{x} = \varepsilon k \boldsymbol{z} + \varepsilon \boldsymbol{y}$ for $\boldsymbol{y} \in V = kY$. Conversely for any $\boldsymbol{x} \in \Omega$ we can get $z_i = \left\lfloor \frac{x_i}{\varepsilon k} \right\rfloor$ where the operator $\lfloor \cdot \rfloor$ is the largest integer less than equal to its operand, and $y_i := \left\{ \frac{x_i}{k\varepsilon} \right\} = \boldsymbol{x} - \varepsilon k \left\lfloor \frac{\boldsymbol{x}}{\varepsilon k} \right\rfloor$ is the fractional part of x_i modulo εk with $i \in \{1,2,3\}$. Also as in Section III.B , set $\widehat{\Omega}^{k\varepsilon} = \cup_{\boldsymbol{z}\in Z_{k,\varepsilon}} V_{\varepsilon,\boldsymbol{z}}$ if $V_{\varepsilon,\boldsymbol{z}} \Subset \Omega$ and $\Lambda_{k\varepsilon} = \Omega / \widehat{\Omega}^{k\varepsilon}$ as the boundary layer.

Let $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y}) \in \mathscr{A}_o^k$. Given the definition of $\mathscr{A}_o^k$ it is clear that for $a.e.\boldsymbol{x} \in \Omega$ fixed, $\boldsymbol{w}$ is a constant in $m_s S^2$ on each $Y_i^* \in kY$. Effectively $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})$ comprises k^3 functions $\boldsymbol{u}_i(\boldsymbol{x}) \in L^2(\Omega; m_s S^2)$ and $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_i(\boldsymbol{x})$ when $\boldsymbol{y} \in Y_i^*$. Using Lemma III.B.1, we can then construct a sequence of k^3 functions $\boldsymbol{u}_i^{\delta} \in C\left(\overline{\Omega}; m_s S^2\right)$ converging to $\boldsymbol{u}_i$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Let $\boldsymbol{w}^{\delta}(\boldsymbol{x},\boldsymbol{y})$ be a sequence in $L^2_\sharp(kY; C(\overline{\Omega}))$ which takes on value $\boldsymbol{u}_i^{\delta}$ for $\boldsymbol{y} \in Y_i^*$ and 0 elsewhere. Then $\boldsymbol{w}^{\delta}(\boldsymbol{x},\boldsymbol{y}) \to \boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})$ in the $L^2\left(\Omega; L^2_\sharp(kY)\right)$ norm, $\left|\boldsymbol{w}^{\delta}(\boldsymbol{x},\boldsymbol{y})\right| = m_s$ on kY^* and for any $\boldsymbol{x} \in \Omega$, it is constant on each $Y_i^* \in kY$. We then define a function $\boldsymbol{w}^{\delta,k,\varepsilon}(\boldsymbol{x})$ as

$$\boldsymbol{w}^{\delta,k,\varepsilon}(\boldsymbol{x}) := \begin{cases} ⨍_{V_{\varepsilon,\boldsymbol{z}}} \boldsymbol{w}^{\delta}\left(\boldsymbol{p}, \frac{\boldsymbol{x}}{\varepsilon}\right) \boldsymbol{dp} = ⨍_V \boldsymbol{w}^{\delta}\left(\varepsilon k \boldsymbol{z} + \varepsilon \boldsymbol{q}, \left\{\frac{\boldsymbol{x}}{k\varepsilon}\right\}\right) \boldsymbol{dq} & \text{on } \widehat{\Omega}^{k\varepsilon} \\ \text{arbitrary fixed vector in } m_s S^2 & \text{on } \Lambda^{k\varepsilon} \end{cases}$$

where we have changed variables in the last step to $\varepsilon \boldsymbol{q} = \boldsymbol{p} - \varepsilon k \boldsymbol{z}$ and used the periodicity of $\boldsymbol{w}^{\delta}(\boldsymbol{x},\boldsymbol{y})$ in it's second variable. Note that $\boldsymbol{w}^{\delta,k,\varepsilon} = U^{k\varepsilon}(\boldsymbol{w}^{\delta})$, where $U^{k\varepsilon}$ is the averaging operator with Definition III.B.2 in Appendix III.B but with scale kY. Equation (III.B.2) then gives us that

$$\boldsymbol{w}^{\delta}\big(\boldsymbol{x},\varepsilon^{-1}\boldsymbol{x}\big)-\boldsymbol{w}^{\delta,k,\varepsilon}(\boldsymbol{x})=\boldsymbol{w}^{\delta}\big(\boldsymbol{x},\varepsilon^{-1}\boldsymbol{x}\big)-U^{\varepsilon}(\boldsymbol{w}^{\delta})(\boldsymbol{x})\xrightarrow{\varepsilon\to 0}0 \qquad \text{in} \quad L^2(\Omega) \tag{III.4.12}$$

Using $\boldsymbol{w}^{\delta,k,\varepsilon}$ as a test function for $\mathscr{E}^{\varepsilon}(\boldsymbol{m})$, we get

$$\mathscr{E}^{\varepsilon}(\boldsymbol{m}^{\varepsilon})\leq\int_{\Omega_\varepsilon}\Big(\Phi\big(\boldsymbol{w}^{\delta,k,\varepsilon}(\boldsymbol{x})\big)-\boldsymbol{h}_a(\boldsymbol{x})\cdot\boldsymbol{w}^{\delta,k,\varepsilon}(\boldsymbol{x})\Big)\,\boldsymbol{dx}+\mathscr{E}^{\varepsilon}_d(\boldsymbol{w}^{\delta,k,\varepsilon})+C$$

Using equation (III.4.12) and noting the continuity of the R.H.S. of the above equation, we get

$$\mathscr{E}^{\varepsilon}(\boldsymbol{m}^{\varepsilon})\leq\int_{\Omega_\varepsilon}\Big(\Phi\big(\boldsymbol{w}^{\delta}(\boldsymbol{x},\tfrac{\boldsymbol{x}}{\varepsilon})\big)-\boldsymbol{h}_a(\boldsymbol{x})\cdot\boldsymbol{w}^{\delta}(\boldsymbol{x},\tfrac{\boldsymbol{x}}{\varepsilon})\Big)\,\boldsymbol{dx}+\mathscr{E}^{\varepsilon}_d\big(\boldsymbol{w}^{\delta}(\boldsymbol{x},\tfrac{\boldsymbol{x}}{\varepsilon})\big)+C+\sigma^{\varepsilon} \tag{III.4.13}$$

where $\lim_{\varepsilon\to 0}\sigma^{\varepsilon}\to 0$ as $\boldsymbol{w}^{\delta}\big(\boldsymbol{x},\varepsilon^{-1}\boldsymbol{x}\big)-\boldsymbol{w}^{\delta,k,\varepsilon}(\boldsymbol{x})\to 0$.

We deal with the RHS of the above equation to determine how it behaves as $\varepsilon\to 0$. To see that we first deal with the first two terms in the integrand, i.e. the anisotropy and the Zeeman energies.

Let $f(\boldsymbol{x},\boldsymbol{y}):=\Big(\Phi\big(\boldsymbol{w}^{\delta}(\boldsymbol{x},\boldsymbol{y})\big)-\boldsymbol{h}_a(\boldsymbol{x})\cdot\boldsymbol{w}^{\delta}(\boldsymbol{x},\boldsymbol{y})\Big)$ and note then that $f(\boldsymbol{x},\boldsymbol{y})\in L^1\big(kY,C(\overline{\Omega})\big)$ since both terms constituting $f(\boldsymbol{x},\boldsymbol{y})$ are polynomial functions in $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\in L^2\big(kY,C(\overline{\Omega})\big)$ which are pointwise uniformly bounded i.e. $\big|\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\big|\leq m_s\ a.e.$ Then from Lemma III.3.4 we have that if $f^{\varepsilon}(\boldsymbol{x}):=f\big(\boldsymbol{x},\varepsilon^{-1}\boldsymbol{x}\big)$, we get

$$\lim_{\varepsilon\to 0}\int_{\Omega}f^{\varepsilon}(\boldsymbol{x})\boldsymbol{dx}=\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\Big(\Phi\big(\boldsymbol{w}^{\delta}(\boldsymbol{x},\tfrac{\boldsymbol{x}}{\varepsilon})\big)-\boldsymbol{h}_a(\boldsymbol{x})\cdot\boldsymbol{w}^{\delta}(\boldsymbol{x},\tfrac{\boldsymbol{x}}{\varepsilon})\Big)\,\boldsymbol{dx}=\int_{\Omega}\fint_{kY}f(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dxdy}. \tag{III.4.14}$$

Now we deal with the demag energy in the R.H.S of equation (III.4.13). From equation (III.B.1) we get $\boldsymbol{w}^{\delta}(\boldsymbol{x},\varepsilon^{-1}\boldsymbol{x})\xrightarrow{2}\boldsymbol{w}^{\delta}(\boldsymbol{x},\boldsymbol{y})$ in $L^2\big(\Omega,L^2_\sharp(kY)\big)$, and using Proposition III.3.1 gives $\boldsymbol{h}_{\boldsymbol{w}^{\delta}(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon})}(\boldsymbol{x})\xrightarrow{2}\boldsymbol{h}_{\left(\fint_{kY}\boldsymbol{w}^{\delta}\boldsymbol{dy}\right)}(\boldsymbol{x})+\nabla^{\boldsymbol{y}}\psi^{1}_{\boldsymbol{w}^{\delta}}(\boldsymbol{x},\boldsymbol{y})$. Thus taking $\lim_{\varepsilon}$ of R.H.S of (III.4.13) gives

$$\lim_{\varepsilon\to 0}R.H.S.=C+\int_{\Omega}\fint_{kY}\Big(\Phi\big(\boldsymbol{w}^{\delta}(\boldsymbol{x},\boldsymbol{y})\big)-\boldsymbol{h}_a\cdot\boldsymbol{w}^{\delta}(\boldsymbol{x},\boldsymbol{y})\Big)+\frac{1}{8\pi}\int_{\mathbb{R}^3}\big|\boldsymbol{h}_{\left(\fint\boldsymbol{w}^{\delta}\boldsymbol{dy}\right)}\big|^2\boldsymbol{dx}+\frac{1}{8\pi}\int_{\mathbb{R}^3}\fint_{kY}\big|\nabla^{\boldsymbol{y}}\psi^{1}_{\boldsymbol{w}^{\delta}}\big|^2$$

Further taking $\lim_{\delta}$ of above and noting continuity of all the terms in $\boldsymbol{w}^{\delta}$ gives

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}R.H.S.=C+\int_{\Omega}\fint_{kY}\Big(\Phi\big(\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\big)-\boldsymbol{h}_a\cdot\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\Big)+\frac{1}{8\pi}\int_{\mathbb{R}^3}\big|\boldsymbol{h}_{\boldsymbol{m}^{o}}(\boldsymbol{x})\big|^2\boldsymbol{dx}+\frac{1}{8\pi}\int_{\mathbb{R}^3}\fint_{kY}\big|\nabla^{\boldsymbol{y}}\psi^{1}_{\boldsymbol{w}}\big|^2$$

Thus using Lemma III.B.2, there exists a cross-sequence $\boldsymbol{w}^{\delta(\varepsilon),k,\varepsilon}$ such that taking $\limsup_{\varepsilon\to 0}$ of eqn. (III.4.13) we get

$$\limsup_{\varepsilon\to 0}\mathscr{E}^{\varepsilon}(\boldsymbol{m}^{\varepsilon}) \le \int_{\Omega}⨍_{kY}\Big(\Phi\big(\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\big)-\boldsymbol{h}_a(\boldsymbol{x})\cdot\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\Big)\boldsymbol{dydx}+\frac{1}{8\pi}\int_{\mathbb{R}^3}\left|\boldsymbol{h}_{\boldsymbol{m}^o}(\boldsymbol{x})\right|^2\boldsymbol{dx}$$
$$+\frac{1}{8\pi}\int_{\mathbb{R}^3}⨍_{kY}\left|\nabla^{y}\psi^{1}_{\boldsymbol{w}}(\boldsymbol{x},\boldsymbol{y})\right|^2\boldsymbol{dydx}+C.$$

Taking inf of left hand side w.r.t. $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\in\mathscr{A}_o^k$ and further $\sup_k$ for $k\in\mathbb{N}$, we get our result. □

We will now show a lim inf lower bound with the help of the next two lemmata. This uses the same line of argument as Lemma 2.8 in [Cioranescu et al., 2006] and Propositiom 3.8 in [Fonseca and Krömer, 2010]. For the next Lemma we will write the sequence ε as ε_h where $h\in\mathbb{N}$.

Lemma III.4.2. *Let $\delta=(\nu h)^{-1}$ where $\nu,h\in\mathbb{N}$. Then*

$$\liminf_{\varepsilon_h\to 0}\mathscr{E}^{\varepsilon_h}(\boldsymbol{m}^{\varepsilon_h})\ge\sup_{\nu\in\mathbb{N}}\liminf_{h\to 0}\int_{\Omega_\delta}\Big(\Phi(\boldsymbol{m}^{\nu,h}(\boldsymbol{x}))-\boldsymbol{h}_a(\boldsymbol{x})\cdot\boldsymbol{m}^{\nu,h}(\boldsymbol{x})\Big)\boldsymbol{dx}+\mathscr{E}_d^{\delta}(\boldsymbol{m}^{\nu,h})+C$$

for a sequence of functions $\boldsymbol{m}^{\nu,h}$ with $\left|\boldsymbol{m}^{\nu,h}\right|=m_s$ on Ω_δ and $\boldsymbol{m}^{\nu,h}$ is constant on each segment consituting Ω_δ. Also $\boldsymbol{m}^{\nu,h}\rightharpoonup\boldsymbol{m}^o$ in $L^2(\Omega)$ as $h\to\infty$.

Proof. We choose a subsequence $\{\varepsilon_n\}\subset\{\varepsilon_h\}$ so that $\liminf_{\varepsilon_h\to 0}\mathscr{E}^{\varepsilon_h}(\boldsymbol{m}^{\varepsilon_h})=\lim_{\varepsilon_n\to 0}\mathscr{E}^{\varepsilon_n}(\boldsymbol{m}^{\varepsilon_n})$. Recall equation (III.4.9) gives us

$$\mathscr{E}^{\varepsilon_n}(\boldsymbol{m}^{\varepsilon_n})\ge\mathscr{E}^{\varepsilon_n}(\boldsymbol{M}^{\varepsilon_n})-K_{10}\varepsilon_n \qquad\Longrightarrow\qquad \lim_{\varepsilon_n\to 0}\mathscr{E}^{\varepsilon_n}(\boldsymbol{m}^{\varepsilon_n})\ge\lim_{\varepsilon_n\to 0}\mathscr{E}^{\varepsilon_n}(\boldsymbol{M}^{\varepsilon_n}) \tag{III.4.15}$$

In the proof from here on at each step we may choose subsequences of the limit attaining sequence ε_n which will be unrelabelled and hence still called ε_n.

Fix $\nu\in\mathbb{N}$. Let us choose a subsequence ε_n such that $k_{\nu,n}=\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor$ is an increasing sequence of numbers where $\lfloor x\rfloor$ is the largest integer smaller than x. Let $\theta_n=\nu\varepsilon_n\left\lfloor\frac{1}{\nu\varepsilon_n}\right\rfloor$ and note that $\theta_n\to 1^-$ as $n\to\infty$. Let n be large enough so that $\Omega'\Subset\Omega$ and $\theta_n\Omega'\Subset\Omega$. Since $\mathscr{E}^{\varepsilon}\ge 0$ we have on change of variables $\boldsymbol{z}=\theta_n^{-1}\boldsymbol{x}$

$$\mathscr{E}^{\varepsilon_n}(\boldsymbol{M}^{\varepsilon_n})\ge\int_{\theta_n\Omega'}\Big(\Phi\big(\boldsymbol{M}^{\varepsilon_n}(\boldsymbol{x})\big)-\boldsymbol{h}_a(\boldsymbol{x})\cdot\boldsymbol{M}^{\varepsilon_n}(\boldsymbol{x})\Big)\,\boldsymbol{\chi}_{Y}^{}_{*}(\varepsilon_n^{-1}\boldsymbol{x})\boldsymbol{dx}+\mathscr{E}_d^{\varepsilon_n}\big(\boldsymbol{M}^{\varepsilon_n}\boldsymbol{\chi}_{\theta_n\Omega'}(\boldsymbol{x})\big)+C$$
$$\ge|\theta_n|^3\int_{\Omega'}\Big(\Phi\big(\boldsymbol{M}^{\varepsilon_n}(\theta_n\boldsymbol{z})\big)-\boldsymbol{h}_a(\theta_n\boldsymbol{z})\cdot\boldsymbol{M}^{\varepsilon_n}(\theta_n\boldsymbol{z})\Big)\,\boldsymbol{\chi}_{Y^*}(\varepsilon_n^{-1}\theta_n\boldsymbol{z})\,\boldsymbol{dz}$$
$$+|\theta_n|^3\,\mathscr{E}_d^{\varepsilon_n}\big(\boldsymbol{M}^{\varepsilon_n}(\theta_n\boldsymbol{z})\boldsymbol{\chi}_{\Omega'}(\boldsymbol{z})\big)+C.$$

Note that $\varepsilon_n^{-1}\theta_n = \nu\lfloor\frac{1}{\nu\varepsilon_n}\rfloor = \nu k_{\nu,n}$ and as $n\to\infty$ we have $\theta_n\boldsymbol{z}\to\boldsymbol{z}$ and $|\theta_n|^3\to 1$. So

$$\lim_{n\to\infty}\mathscr{E}^{\varepsilon_n}(\boldsymbol{M}^{\varepsilon_n}) \ \geq\ \liminf_{n\to\infty}\Big\{\int_{\Omega'}\Big(\Phi\big(\boldsymbol{M}^{\varepsilon_n}(\theta_n\boldsymbol{z})\big)-\boldsymbol{h}_a(\boldsymbol{z})\cdot\boldsymbol{M}^{\varepsilon_n}(\theta_n\boldsymbol{z})\Big)\,\chi_{Y^*}\big(\nu k_{\nu,n}\boldsymbol{z}\big)\boldsymbol{dz} + \mathscr{E}_d^{\varepsilon_n}\big(\boldsymbol{M}^{\varepsilon_n}(\theta_n\boldsymbol{z})\chi_{\Omega'}(\boldsymbol{z})\big)+C\Big\}.$$

Let us set $\boldsymbol{w}^{\nu,k_{\nu,n}}(\boldsymbol{z}) = \boldsymbol{M}^{\varepsilon_n}(\theta_n\boldsymbol{z})$ and note that $\boldsymbol{w}^{\nu,k_{\nu,n}}(\boldsymbol{z})$ is supported on Ω_δ where $\delta = 1/(\nu k_{\nu,n})$. Also note since $\boldsymbol{M}^{\varepsilon_n}\rightharpoonup\boldsymbol{m}^o$, we also have $\boldsymbol{w}^{\nu,k_{\nu,n}}\rightharpoonup\boldsymbol{m}^o$ in $L^2(\Omega')$ as $k_{\nu,n}\to\infty$. Letting $\big|\Omega/\Omega'\big|\to 0$ gives

$$\lim_{n\to\infty}\mathscr{E}^{\varepsilon_n}(\boldsymbol{M}^{\varepsilon_n}) \ \geq\ \liminf_{n\to\infty}\Big\{\int_{\Omega}\Big(\Phi\big(\boldsymbol{w}^{\nu,k_{\nu,n}}(\boldsymbol{z})\big)-\boldsymbol{h}_a(\boldsymbol{z})\cdot\boldsymbol{w}^{\nu,k_{\nu,n}}(\boldsymbol{z})\Big)\,\chi_{Y^*}\big(\nu k_{\nu,n}\boldsymbol{z}\big)\boldsymbol{dz} + \mathscr{E}_d^{\delta}\big(\boldsymbol{w}^{\nu,k_{\nu,n}}(\boldsymbol{z})\chi_{\Omega}(\boldsymbol{z})\big)+C\Big\}.$$

Then recalling we started with a sequence ε_h with $h\in\mathbb{N}$ we get

$$\lim_{h\to\infty}\mathscr{E}^{\varepsilon_h}(\boldsymbol{m}^{\varepsilon_h}) \ \geq\ \liminf_{n\to\infty}\Big\{\int_{\Omega}\Big(\Phi\big(\boldsymbol{w}^{\nu,k_{\nu,n}}(\boldsymbol{z})\big)-\boldsymbol{h}_a(\boldsymbol{z})\cdot\boldsymbol{w}^{\nu,k_{\nu,n}}(\boldsymbol{z})\Big)\,\chi_{Y^*}\big(\nu k_{\nu,n}\boldsymbol{z}\big)\boldsymbol{dz} + \mathscr{E}_d^{\delta}\big(\boldsymbol{w}^{\nu,k_{\nu,n}}(\boldsymbol{z})\chi_{\Omega}(\boldsymbol{z})\big)+C\Big\}.$$

Recalling that $k_{\nu,n}$ is a strictly increasing sequence for $\nu\in\mathbb{N}$, we set

$$\boldsymbol{m}^{\nu,h} = \begin{cases} \boldsymbol{w}^{\nu,k_{\nu,n}} & \text{if } h = k_{\nu,n} \\ \text{any function in } L^2(\Omega_\delta; m_s S^2) \text{ constant on each segment of } \Omega_\delta \text{ otherwise.} \end{cases}$$

Using this we get

$$\lim_{h\to\infty}\mathscr{E}^{\varepsilon_h}(\boldsymbol{m}^{\varepsilon_h}) \ \geq\ \liminf_{h\to\infty}\Big\{\int_{\Omega}\Big(\Phi\big(\boldsymbol{m}^{\nu,h}(\boldsymbol{z})\big)-\boldsymbol{h}_a(\boldsymbol{z})\cdot\boldsymbol{m}^{\nu,h}(\boldsymbol{z})\Big)\,\chi_{Y^*}\big(\nu h\boldsymbol{z}\big)\boldsymbol{dz} + \mathscr{E}_d^{\delta}\big(\boldsymbol{m}^{\nu,h}(\boldsymbol{z})\chi_{\Omega}(\boldsymbol{z})\big)+C\Big\}.$$

and our result is proved by taking sup w.r.t. ν. □

Next we state a technical result which we need for getting our final upper bound in Lemma III.4.4. The proof of this result is presented in the Appendix.

Lemma III.4.3. *Let $\boldsymbol{v}^h(\boldsymbol{y})$ be a sequence in $L^2(Y)$ with support in hY^* and $\boldsymbol{v}^h\in m_sS^2$ is constant vector on each $W^i\in hY^*$ with $\boldsymbol{v}^h\rightharpoonup\boldsymbol{\alpha}^o$ in $L^2(Y)$. Then exists a sequence $\boldsymbol{w}^h$ in $L^2_\sharp(hY^*; m_sS^2)$ with $\boldsymbol{w}^h$ constant on each W^i and $\int_Y\boldsymbol{w}^h(\boldsymbol{y})\,\boldsymbol{dy} = \int_Y\boldsymbol{\alpha}^o(\boldsymbol{y})\,\boldsymbol{dy}$ and $\boldsymbol{v}^h-\boldsymbol{w}^h\to 0$ as $h\to\infty$.*

Now we state and prove our final upper bound Lemma.

Lemma III.4.4.

$$\liminf_{\varepsilon\to 0} \mathscr{E}(\boldsymbol{m}^{\varepsilon}) \geq \int_{\Omega} -\boldsymbol{h}_a(\boldsymbol{x})\cdot\boldsymbol{m}^o(\boldsymbol{x})\,\boldsymbol{dx} + \int_{\mathbb{R}^3}\frac{1}{8\pi}|\boldsymbol{h}_{\boldsymbol{m}^o}(\boldsymbol{x})|^2\,\boldsymbol{dx}$$
$$+\int_{\Omega}\liminf_{h\to\infty}\inf_{\boldsymbol{w}\in\mathscr{A}_o^k}\left\{ \fint_{hY}\left(\Phi(\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y}))+\frac{1}{8\pi}|\nabla^{\boldsymbol{y}}\psi^{1}_{\boldsymbol{w}}(\boldsymbol{x},\boldsymbol{y})|^2\right)\boldsymbol{dy}\right\}\boldsymbol{dx}$$

Proof. We will show our result by showing that the R.H.S of Lemma III.4.2 is greater than R.H.S of our result. Let $\Omega' \Subset \Omega$ and let $\nu\in\mathbb{N}$. Recalling that Y is a unit cube, let $\forall\,\boldsymbol{z}\in\mathbb{Z}^3$

$$Y_{\nu,\boldsymbol{z}} := \frac{1}{\nu}\boldsymbol{z}+\frac{1}{\nu}Y, \qquad \boldsymbol{z}=\lfloor\nu\boldsymbol{x}\rfloor,\ \boldsymbol{y}=\nu\boldsymbol{x}-\lfloor\nu\boldsymbol{x}\rfloor, \qquad \& \qquad Z_\nu=\left\{\boldsymbol{z}\in\mathbb{Z}^3 \mid Y_{\nu,z}\bigcap\Omega'\neq\emptyset\right\}$$

Note unlike the appendix Section III.B, we are now working with domain Ω' and not Ω. Thus for ν large enough we can expect $Y_{\nu,z}\Subset\Omega$ for all $\boldsymbol{z}\in Z_\nu$ and let us set $\bigcup_{\boldsymbol{z}\in Z_\nu} Y_{\nu,\boldsymbol{z}} =: \Omega_\nu \Subset \Omega$. For $\boldsymbol{x}\in Y_{\nu,\boldsymbol{z}}$ and $\boldsymbol{y}\in Y$ note $T_{1/\nu}(\boldsymbol{m}^{\nu,h})(\boldsymbol{x},\boldsymbol{y}) = T_{1/\nu}(\boldsymbol{m}^{\nu,h})(\frac{\boldsymbol{z}}{\nu},\boldsymbol{y}) = \boldsymbol{m}^{\nu,h}(\frac{\boldsymbol{z}+\boldsymbol{y}}{\nu}) := \boldsymbol{v}^{\nu,h,\boldsymbol{z}}(\boldsymbol{y})$.

Since $\boldsymbol{m}^{\nu,h}(\boldsymbol{x}) \rightharpoonup \boldsymbol{m}^o(\boldsymbol{x})$ as $h\to\infty$ for fixed ν, we get for fixed $\boldsymbol{z}\in Z_\nu$, $\boldsymbol{x}\in Y_{\nu,\boldsymbol{z}}$, by change of variables $(\,\boldsymbol{x}\longmapsto\boldsymbol{y},\quad \boldsymbol{dx}=\nu^{-3}\boldsymbol{dy},\quad |Y_{\nu,\boldsymbol{z}}|=\nu^{-3}|Y|\,)$

$$\fint_{Y_{\nu,\boldsymbol{z}}}\boldsymbol{m}^{\nu,h}(\boldsymbol{x})\boldsymbol{dx} = \fint_Y \boldsymbol{m}^{\nu,h}(\frac{\boldsymbol{z}+\boldsymbol{y}}{\nu})\boldsymbol{dy} = \fint_Y T_{\frac{1}{\nu}}(\boldsymbol{m}^{\nu,h})(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = \fint_Y \boldsymbol{v}^{\nu,h,\boldsymbol{z}}(\boldsymbol{y})\boldsymbol{dy} \xrightarrow{h\to\infty} \fint_{Y_{\nu,\boldsymbol{z}}}\boldsymbol{m}^o(\boldsymbol{x})\boldsymbol{dx}.$$

For fixed ν we use Lemma III.4.3 to get a function $\boldsymbol{u}^{\nu,h,\boldsymbol{z}}(\boldsymbol{y})\in L^2_\sharp(Y)$ such that $\boldsymbol{u}^{\nu,h,\boldsymbol{z}}-\boldsymbol{v}^{\nu,h,\boldsymbol{z}}\to\boldsymbol{0}$ in $L^2(Y)$ as $h\to\infty$ and $\fint_Y \boldsymbol{u}^{\nu,h,\boldsymbol{z}}(\boldsymbol{y})\boldsymbol{dy} = \fint_{Y_{\nu,\boldsymbol{z}}}\boldsymbol{m}^o(\boldsymbol{x})\boldsymbol{dx}$. Let us define

$$\boldsymbol{u}^{\nu,h}(\boldsymbol{x}) := \sum_{\boldsymbol{z}\in Z_\nu}\chi_{\Omega_\nu\bigcap Y_{\nu,\boldsymbol{z}}}(\frac{\lfloor\nu\boldsymbol{x}\rfloor}{\nu})\boldsymbol{u}^{\nu,h,\boldsymbol{z}}(\{\nu\boldsymbol{x}\}), \qquad \& \qquad \boldsymbol{s}^{\nu,h}(\boldsymbol{x},\boldsymbol{y}) := \sum_{\boldsymbol{z}\in Z_\nu}\chi_{\Omega_\nu\cap Y_{\nu,\boldsymbol{z}}}(\boldsymbol{x})\,\boldsymbol{u}^{\nu,h,\boldsymbol{z}}(\boldsymbol{y}).$$

Then we get on changing of variables,

$$\begin{aligned}\int_{\Omega_\nu}|\boldsymbol{m}^{\nu,h}-\boldsymbol{u}^{\nu,h}|^2\boldsymbol{dx} &= \sum_{\boldsymbol{z}\in Z_\nu}\frac{1}{\nu^3}\int_{Y_{\nu,\boldsymbol{z}}}|\boldsymbol{m}^{\nu,h}(\frac{\boldsymbol{z}+\boldsymbol{y}}{\nu})-\boldsymbol{u}^{\nu,h}(\frac{\boldsymbol{z}+\boldsymbol{y}}{\nu})|^2\boldsymbol{dy}\\ &= \sum_{\boldsymbol{z}\in Z_\nu}\int_{Y_{\nu,\boldsymbol{z}}}\frac{1}{|Y|}\boldsymbol{dx}\int_Y\left|T_{1/\nu}(\boldsymbol{m}^{\nu,h}-\boldsymbol{u}^{\nu,h})(\boldsymbol{x},\boldsymbol{y})\right|^2\boldsymbol{dy}\\ &= \frac{1}{|Y|}\sum_{\boldsymbol{z}\in Z_\nu}\int_{Y_{\nu,\boldsymbol{z}}}\fint_Y\left|\boldsymbol{v}^{\nu,h,\boldsymbol{z}}(\boldsymbol{y})-\boldsymbol{u}^{\nu,h,\boldsymbol{z}}(\boldsymbol{y})\right|^2\boldsymbol{dydx} \xrightarrow{h\to\infty} 0,\end{aligned} \tag{III.4.16}$$

by recalling that $\boldsymbol{u}^{\nu,h,\boldsymbol{z}}-\boldsymbol{v}^{\nu,h,\boldsymbol{z}}\to\boldsymbol{0}$ in $L^2(Y)$. Thus $\boldsymbol{u}^{\nu,h}\rightharpoonup\boldsymbol{m}^o\chi_{\Omega_\nu}$ on $L^2(\Omega_\nu)$. We first show an inequality for the demag energy of $\boldsymbol{u}^{\nu,h}$ i.e. $\mathscr{E}^\delta_d(\boldsymbol{u}^{\nu,h})$. Let $\eta^{\nu,h,\boldsymbol{z}}_\sharp$ be the solution to the minimization problem under periodic boundary conditions

$$\inf_{\mathcal{W}_\sharp^{1,2}(Y_{\nu,\boldsymbol{z}})} \int_{Y_{\nu,\boldsymbol{z}}} \Big(\frac{1}{2}|\nabla\eta|^2 - 4\pi\big(\boldsymbol{u}^{\nu,h} - \boldsymbol{m}^o\big)\cdot\nabla\eta \Big)\boldsymbol{dx}, \qquad \forall \boldsymbol{z} \in Z_\nu.$$

Using $\psi_{\boldsymbol{m}^o}\chi_{\Omega_\nu} + \sum_{\boldsymbol{z}\in Z_\nu} \chi_{Y_{\nu,\boldsymbol{z}}}(\boldsymbol{x})\, \eta_\sharp^{\nu,h,\boldsymbol{z}}(\boldsymbol{x})$ as a test function with Maxwell's equation for magnetization $\boldsymbol{u}^{\nu,h}$ and recalling Lemma III.A.3 gives

$$\liminf_{h\to\infty}\mathcal{E}_d^\delta\big(\boldsymbol{u}^{\nu,h}\big) \;\geq\; \mathcal{E}_d\big(\boldsymbol{m}^o\,\chi_{\Omega_\nu}\big) + \liminf_{h\to\infty}\sum_{\boldsymbol{x}\in Z_\nu}\int_{Y_{\nu,\boldsymbol{z}}}\frac{1}{8\pi}|\eta_\sharp^{\nu,h,\boldsymbol{z}}|^2\boldsymbol{dx}. \tag{III.4.17}$$

Since the anisotropy and demag energies are positive, and $\Omega_\nu \subset \Omega$ we get

$$\begin{aligned}
I_{\nu,h} &:= \int_\Omega \Phi\big(\boldsymbol{m}^{\nu,h}(\boldsymbol{x})\big)\chi_{Y^*}\big(\nu h\boldsymbol{x}\big)\boldsymbol{dx} + \mathcal{E}_d^\delta\big(\boldsymbol{m}^{\nu,h}\big) \geq \int_{\Omega_\nu}\Phi\big(\boldsymbol{m}^{\nu,h}\big)\,\chi_{Y^*}\big(\nu h\boldsymbol{x}\big)\boldsymbol{dx} + \mathcal{E}_d^\delta\big(\boldsymbol{m}^{\nu,h}\,\chi_{\Omega_\nu}\big)\\
&\geq \int_{\Omega_\nu}\Phi\big(\boldsymbol{u}^{\nu,h}\big)\,\chi_{Y^*}\big(\nu h\boldsymbol{x}\big)\boldsymbol{dx} + \mathcal{E}_d^\delta\big(\boldsymbol{u}^{\nu,h}\big) + \tau_h\\
&\geq \int_{\Omega_\nu}\Phi\big(\boldsymbol{u}^{\nu,h}\big)\,\chi_{Y^*}\big(\nu h\boldsymbol{x}\big)\boldsymbol{dx} + \mathcal{E}_d\big(\boldsymbol{m}^o\,\chi_{\Omega_\nu}\big) + \sum_{\boldsymbol{x}\in Z_\nu}\int_{Y_{\nu,\boldsymbol{z}}}\frac{1}{8\pi}|\eta_\sharp^{\nu,h,\boldsymbol{z}}|^2\boldsymbol{dx} + \tau_h
\end{aligned} \tag{III.4.18}$$

where $\lim_h \tau_h \to 0$ due to eqn. III.4.16 and using estimate eqn. (III.4.17) in the last step. We look at the anisotropy and demag energy terms in the R.H.S. of above equation. Using the change of variables again and recalling $\boldsymbol{s}^{\nu,h}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{z}\in Z_\nu}\chi_{\Omega_\nu\cap Y_{\nu,\boldsymbol{z}}}(\boldsymbol{x})\,\boldsymbol{u}^{\nu,h,\boldsymbol{z}}(\boldsymbol{y})$ we get

$$\begin{aligned}
&\int_{\Omega_\nu}\Phi\big(\boldsymbol{u}^{\nu,h}(\boldsymbol{x})\big)\,\chi_{Y^*}\big(\nu h\boldsymbol{x}\big)\boldsymbol{dx} = \sum_{\boldsymbol{z}\in Z_\nu}\frac{1}{\nu^3}\int_Y\Phi\big(\boldsymbol{u}^{\nu,h}\big(\frac{\boldsymbol{z}+\boldsymbol{y}}{\nu}\big)\big)\,\chi_{Y^*}\big(h(\boldsymbol{z}+\boldsymbol{y})\big)\boldsymbol{dy}\\
&= \sum_{\boldsymbol{z}\in Z_\nu}\int_{Y_{\nu,\boldsymbol{z}}}\frac{1}{|Y|}\boldsymbol{dx}\int_Y\Phi\big(T_{1/\nu}(\boldsymbol{u}^{\nu,h})(\nu^{-1}\boldsymbol{z},\boldsymbol{y})\big)\,\chi_{Y^*}\big(h\boldsymbol{y}\big)\boldsymbol{dy}\\
&= \sum_{\boldsymbol{z}\in Z_\nu}\int_{Y_{\nu,\boldsymbol{z}}}\Big\{\fint_Y\Phi\big(\boldsymbol{u}^{\nu,h,\boldsymbol{z}}(\boldsymbol{y})\big)\,\chi_{Y^*}\big(h\boldsymbol{y}\big)\boldsymbol{dy}\Big\}\boldsymbol{dx} = \int_{\Omega_\nu}\fint_Y\Phi\big(\boldsymbol{s}^{\nu,h}(\boldsymbol{x},\boldsymbol{y})\big)\,\chi_{Y^*}\big(h\boldsymbol{y}\big)\boldsymbol{dxdy}
\end{aligned}$$

using $\int_{Y_{\nu,\boldsymbol{z}}}\boldsymbol{dx} = \nu^{-3}|Y|$, χ_{Y^*} is $\boldsymbol{z}$ periodic, and the periodic unfolding operator giving $T_{\frac{1}{\nu}}(\boldsymbol{u}^{\nu,h})(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}^{\nu,h}\big(\frac{\lfloor\nu\boldsymbol{x}\rfloor}{\nu} + \frac{\boldsymbol{y}}{\nu}\big) = T_{\frac{1}{\nu}}(\boldsymbol{u}^{\nu,h})\big(\frac{\lfloor\nu\boldsymbol{x}\rfloor}{\nu},\boldsymbol{y}\big) = T_{\frac{1}{\nu}}(\boldsymbol{u}^{\nu,h})\big(\frac{\boldsymbol{z}}{\nu},\boldsymbol{y}\big)$. Similarly the magnetostatic terms resolve as

$$\sum_{\boldsymbol{z}\in Z_\nu}\frac{1}{8\pi}\int_{Y_{\nu,\boldsymbol{z}}}|\nabla\eta_\sharp^{\nu,h,\boldsymbol{z}}|^2\boldsymbol{dx} = \frac{1}{8\pi}\int_{\Omega_\nu}\fint_Y\big|T_{1/\nu}\big(\nabla\eta_\sharp^{\nu,h,\boldsymbol{z}}\big)(\boldsymbol{x},\boldsymbol{y})\big|^2\boldsymbol{dxdy}.$$

Note for a.e. $\boldsymbol{x} \in Y_{\nu,\boldsymbol{z}}$ we have $L^2(Y) \ni T_{1/\nu}\big(\nabla \eta_\sharp^{\nu,h,\boldsymbol{z}}\big)(\boldsymbol{x},\cdot) = \nabla^{\boldsymbol{y}}\big(T_{1/\nu}\big(\nu\eta_\sharp^{\nu,h,\boldsymbol{z}}\big)\big)$. Also for a.e.

$$\boldsymbol{x} \in Y_{\nu,\boldsymbol{z}} \; ⨍_Y T_{1/\nu}\big(\nu\eta_\sharp^{\nu,h,\boldsymbol{z}}\big)(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = \nu ⨍_{Y_{\nu,\boldsymbol{z}}} \eta_\sharp^{\nu,h,\boldsymbol{z}}(\boldsymbol{p})\boldsymbol{dp} = 0 \qquad \text{as} \quad \eta_\sharp^{\nu,h,\boldsymbol{z}} \in \mathscr{W}_\sharp^{1,2}\big(Y_{\nu,\boldsymbol{z}}\big)$$

Hence we get $T_{1/\nu}\big(\nu\eta_\sharp^{\nu,h,\boldsymbol{z}}\big) \in L^2\big(Y_{\nu,\boldsymbol{z}}, \mathscr{W}_\sharp^{1,2}(Y)\big)$. Also note for a.e. $\boldsymbol{x}$, $T_{1/\nu}\big(\nu\eta_\sharp^{\nu,h,\boldsymbol{z}}\big)$ solves Maxwell's periodic problem for magnetization $T_{1/\nu}\big(\boldsymbol{s}^{\nu,h} - \boldsymbol{m}^o\big)$ because

$$\begin{aligned}
&\int_{Y_{\nu,\boldsymbol{z}}} \big(\nabla\eta_\sharp^{\nu,h,\boldsymbol{z}} - 4\pi(\boldsymbol{u}^{\nu,h} - \boldsymbol{m}^o)\big)\cdot\nabla\phi = \int_{Y_{\nu,\boldsymbol{z}}} ⨍_Y T_{1/\nu}\big(\nabla\eta_\sharp^{\nu,h,\boldsymbol{z}} - 4\pi(\boldsymbol{u}^{\nu,h} - \boldsymbol{m}^o)\big)\cdot T_{1/\nu}(\nabla\phi) \\
&= \int_{Y_{\nu,\boldsymbol{z}}} ⨍_Y \Big(\nabla^{\boldsymbol{y}} T_{\frac{1}{\nu}}\big(\nu\eta_\sharp^{\nu,h,\boldsymbol{z}}\big) - 4\pi\big[\boldsymbol{s}^{\nu,h} - T_{\frac{1}{\nu}}\big(\boldsymbol{m}^o\big)\big]\Big)(\boldsymbol{x},\boldsymbol{y})\cdot\nabla^{\boldsymbol{y}} T_{\frac{1}{\nu}}(\nu\phi)(\boldsymbol{x},\boldsymbol{y}) = 0.
\end{aligned} \tag{III.4.19}$$

Combining all the above results and substituting in eqn. (III.4.18) we get

$$\begin{aligned}
I_{\nu,h} &\geq \int_{\Omega_\nu} \Phi\big(\boldsymbol{u}^{\nu,h}\big)\,\chi_{Y^*}(\nu h\boldsymbol{x})\boldsymbol{dx} + \mathscr{E}_d\big(\boldsymbol{m}^o\,\chi_{\Omega_\nu}\big) + \sum_{\boldsymbol{x}\in Z_\nu}\int_{Y_{\nu,\boldsymbol{z}}} \frac{1}{8\pi}\big|\nabla\eta_\sharp^{\nu,h,\boldsymbol{z}}\big|^2\boldsymbol{dx} + \tau_h \\
&\geq \sum_{\boldsymbol{z}\in Z_\nu}\int_{Y_{\nu,\boldsymbol{z}}} ⨍_Y \Big\{\Phi\big(\boldsymbol{s}^{\nu,h}\big)\,\chi_{Y^*}(h\boldsymbol{y}) + \frac{1}{8\pi}\big|\nabla^{\boldsymbol{y}} T_{1/\nu}\big(\nu\eta_\sharp^{\nu,h,\boldsymbol{z}}\big)\big|^2\Big\}\boldsymbol{dxdy} + \mathscr{E}_d\big(\boldsymbol{m}^o\,\chi_{\Omega_\nu}\big) + \tau_h \\
&\geq \int_{\Omega_\nu} ⨍_Y \Big\{\Phi\big(\boldsymbol{s}^{\nu,h}\big)\,\chi_{Y^*}(h\boldsymbol{y}) + \frac{1}{8\pi}\big|\nabla^{\boldsymbol{y}} T_{1/\nu}\big(\nu\eta_\sharp^{\nu,h,\boldsymbol{z}}\big)\big|^2\Big\}\boldsymbol{dxdy} + \mathscr{E}_d\big(\boldsymbol{m}^o\,\chi_{\Omega_\nu}\big) + \tau_h
\end{aligned} \tag{III.4.20}$$

with $⨍_Y \boldsymbol{s}^{\nu,h}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = ⨍_Y \boldsymbol{u}^{\nu,h,\boldsymbol{z}}(\boldsymbol{y})\boldsymbol{dy} = ⨍_{Y_{\nu,\boldsymbol{z}}} \boldsymbol{m}^o(\boldsymbol{p})\boldsymbol{dp} \neq \boldsymbol{m}^o(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in Y_{\nu,\boldsymbol{z}}$ and $T_{1/\nu}\big(\nu\eta_\sharp^{\nu,h,\boldsymbol{z}}\big)$ solves eqn. (III.4.19).

We note that the magnetization $\boldsymbol{s}^{\nu,h}$ is piecewise constant in $\boldsymbol{x}$ and does not satisfy the need for the magnetization $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})$ to have $⨍_Y \boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = \boldsymbol{m}^o(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Omega_\nu$. To get such $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})$ we use the same technique as the proof of Lemma III.4.3 in deforming $\boldsymbol{s}^{\nu,h}(\boldsymbol{x},\boldsymbol{y})$ to satisfy this condition.

From Lebesgue Differentiation Theorem we have that for a.e. $\boldsymbol{x} \in \Omega' \cap Y_{\nu,\boldsymbol{z}}$

$$\Big\{ ⨍_Y \boldsymbol{s}^{\nu,h}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} - \boldsymbol{m}^o(\boldsymbol{x}) \Big\} = \Big\{ ⨍_{Y_{\nu,\boldsymbol{z}}} \boldsymbol{m}^o(\boldsymbol{p})\,\boldsymbol{dp} - \boldsymbol{m}^o(\boldsymbol{x}) \Big\} = \sigma_\nu \quad \text{where } \sigma_\nu \xrightarrow{\nu\to\infty} 0.$$

Since $\sigma_\nu \to 0$, for a.e. fixed $\boldsymbol{x}_o \in \Omega_\nu$ we deform $\boldsymbol{s}^{\nu,h}(\boldsymbol{x}_o,\boldsymbol{y})$ using Lemma III.4.3 to get $\boldsymbol{w}^{\nu,h}(\boldsymbol{x}_o,\boldsymbol{y})$ such that $\boldsymbol{w}^{\nu,h}(\boldsymbol{x}_o,\boldsymbol{y}) - \boldsymbol{s}^{\nu,h}(\boldsymbol{x}_o,\boldsymbol{y}) \xrightarrow{\nu\to\infty} \boldsymbol{0}$ in $L_\sharp^2(Y)$ and $⨍_Y \boldsymbol{w}^{\nu,h}(\boldsymbol{x}_o,\boldsymbol{y})\boldsymbol{dy} = \boldsymbol{m}^o(\boldsymbol{x})$. It is also clear then that $\boldsymbol{w}^{\nu,h}(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{s}^{\nu,h}(\boldsymbol{x},\boldsymbol{y}) \to \boldsymbol{0}$ in $L^2\big(\Omega, L_\sharp^2(Y)\big)$. Thus substituting this into quation (III.4.20), using the strong continuity of the anisotropy and demag energues w.r.t. $\boldsymbol{s}^{\nu,h}$ and the fact that

$T_{1/\nu}\boldsymbol{m}^o \xrightarrow{\nu\to\infty} \boldsymbol{m}^o$ gives

$$\begin{aligned} I_{\nu,h} &\geq \int_{\Omega_\nu} \fint_Y \left\{ \Phi\left(\boldsymbol{s}^{\nu,h}\right) \chi_{Y^*}(h\boldsymbol{y}) + \frac{1}{8\pi} \left|\nabla^y T_{1/\nu}\left(\nu\eta_\sharp^{\nu,h,\boldsymbol{z}}\right)\right|^2 \right\} \boldsymbol{dxdy} + \mathscr{E}_d\left(\boldsymbol{m}^o \chi_{\Omega_\nu}\right) + \tau_h \\ &\geq \int_{\Omega_\nu} \fint_Y \left\{ \Phi\left(\boldsymbol{w}^{\nu,h}\right) \chi_{Y^*}(h\boldsymbol{y}) + \frac{1}{8\pi} \left|\nabla^y \psi_{\boldsymbol{w}^{\nu,h}}\right|^2 \right\} \boldsymbol{dxdy} + \mathscr{E}_d\left(\boldsymbol{m}^o \chi_{\Omega_\nu}\right) + \tau_h + \sigma_\nu \end{aligned}$$

where $\psi_{\boldsymbol{w}^{\nu,h}}$ solved periodic Maxwell's equation for $\boldsymbol{w}^{\nu,h}(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{m}^o(\boldsymbol{x})$. Hence our result follows when we let $\Omega' \to \Omega$ and use Lemma III.4.2. □

Proof of Theorem III.4.1. : The proof is now clear. By combining the upper bound in Lemma III.4.1 and the lower bounds in Lemma III.4.2 and III.4.4, we get our result. □

III.5 Critical Field & Stability

In the previous section we showed that in the limit as ε goes to zero, the full micromagnetics problem of minimizing $\mathscr{E}^\varepsilon(\boldsymbol{m})$ on $H^1\left(\Omega_\varepsilon, m_s S^2\right)$ can instead be replaced by the limit problem of minimizing $\mathscr{E}^o(\boldsymbol{m}(\boldsymbol{x}))$ on $L^2(\Omega)$.

In this section we follow a semi-rigourous approach of using the limit problem for studying nucleation and hysteresis of multilayered nanowires. The major problem in the minimization of $\mathscr{E}^o(\boldsymbol{m}(\boldsymbol{x}))$ is the $\liminf_{k\to\infty}$ in it's definition. Numerical or theoretical investigations have to deal with this $\liminf$ using some kind of approximation. Thus some kind of semi-rigourous approach has to be used. Since we are interested in the hysteresis properties of the multilayered wire geomtries, we will first investigate the classical problem of nucleation of eigenmodes. We will generalize what is know in magnetics literature as the fanning model which was first proposed for elongated magnetic particles in [Jacobs and Bean, 1955] and has subsequently been used to explain hysteresis and coercivity bahavior in a variety of situations including magnetic nanowires, nanochains, magnetic nanodots etc. To begin the analysis let us fix $k \in \mathbb{N}$.

III.5.1 Nucleation problem

Fix arbitrary $k \in \mathbb{N}$. To tackle with the $\liminf_k$ in Theorem III.4.1 we will replace the problem of minimizing $\mathscr{E}^o(\boldsymbol{m})$ by a simpler energy $\mathscr{I}^o_{(k)}(\boldsymbol{m})$ defined as follows for all $\boldsymbol{m} \in L^2(\Omega)$ & $|\boldsymbol{m}| \leq \theta m_s$,

$$\begin{aligned} \mathscr{I}^o_{(k)}(\boldsymbol{m}) = &\int_\Omega \left[-\boldsymbol{h}_a(\boldsymbol{x}) \cdot \boldsymbol{m}(\boldsymbol{x}) + \inf_{\boldsymbol{w}\in\mathscr{A}^k_o} \left\{ \fint_{kY} \left(\Phi(\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})) + \frac{1}{8\pi} |\nabla^y \psi^1_{\boldsymbol{w}}(\boldsymbol{x},\boldsymbol{y})|^2 \right) \boldsymbol{dy} \right\} \right] \boldsymbol{dx} \\ &+ \int_{\mathbb{R}^3} \frac{1}{8\pi} |\boldsymbol{h}_{\boldsymbol{m}}(\boldsymbol{x})|^2 \, \boldsymbol{dx} \end{aligned} \tag{III.5.1}$$

with $\mathscr{A}_o^k$ defined as in Theorem III.4.1 .

As is typical for all nucleation calculations, we assume that Ω is an ellipsoid. Refer to Chapter 9 in [Aharoni, 2000] to get a background with respect to this assumption. The chief simplification which this assumption provides is that for a uniformly magnetized ellipsoid, the demag field generated is also uniform inside the ellipsoid. In principle it is not possible to completely magnetize a body even with very large fields unless the body is an allipsoid or one of its limiting shapes like a cylinder or an infinite plate or a half space.

Assuming the principal axis of the ellipsoid are aligned along the coordinate axis, we will show that $\mathscr{I}^o_{(k)}(\boldsymbol{m})$ affords a uniform stable staionary critical point $\boldsymbol{m}^*(\boldsymbol{x})$ along any coordinate axis provided the easy axis of the anisotropy $\Phi(\boldsymbol{m})$ is also aligned with the same axis and the applied field h_a is a constant and parallel to the same.

Recall the notation that kY is a supercell of k^3 cells and Y_i^* indexes members of kY^* with $i = 1,2,\cdots k^3$. Let $\boldsymbol{Q} \in SO(3)$ represent generic rotations in $\mathbb{R}^3$ and let $\boldsymbol{W} \in M_{skw}(\mathbb{R}^3)$ where $M_{skw}(\mathbb{R}^3)$ denotes generic skew-symmetric matrices in $\mathbb{R}^3$.

Let the z-axis be a principal axis of the ellipsoid, with the easy axis of the anisotropy energy and constant applied field $\boldsymbol{h}_a$ alinged along it. Let $\boldsymbol{m}^*(\boldsymbol{x}) = \{0,0,\theta m_s\}$ and $\boldsymbol{w}^*(\boldsymbol{x},\boldsymbol{y}) = \{0,0,m_s\}\ \chi_{kY^*}(\boldsymbol{y})$. A small perturbation of $\boldsymbol{m}^*$ is of the form,

$$\boldsymbol{m}^\delta(\boldsymbol{x}) = \fint_{kY} \boldsymbol{w}^\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{d}\boldsymbol{y}, \qquad \boldsymbol{w}^\delta(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{R}^\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{w}^*, \quad \boldsymbol{R}^\delta(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{Q}_i^\delta(\boldsymbol{x}) \in SO(3) \text{ for } \boldsymbol{y} \in Y_i^*$$

where $\delta << 1$ and $\boldsymbol{Q}_i^\delta \to \boldsymbol{I}$ as $\delta \to 1$ for all i. Any such rotation $\boldsymbol{Q}_i^\delta(\boldsymbol{x})$ can be expanded as a Taylor Series around the identity $\boldsymbol{I}$ and gives a corresponding expansion on $\boldsymbol{w}^\delta(\boldsymbol{x},\boldsymbol{y})$ as

$$\boldsymbol{Q}_i^\delta(\boldsymbol{x}) = \boldsymbol{I} + \delta \boldsymbol{U}_i(\boldsymbol{x}) + \frac{1}{2}\delta^2 \boldsymbol{U}_i^2(\boldsymbol{x}) + O(\delta^3), \quad \boldsymbol{U}_i(\boldsymbol{x}) \in M_{skw}(\mathbb{R}^3), \quad \text{on } Y_i^*, \quad i \in \{1,2,\cdots k^3\}.$$

The Figure III.3 shows a typical variation $\boldsymbol{w}^\delta$ for $k = 2$ in 2-dimensions. Defining $\boldsymbol{W}(\boldsymbol{x},\boldsymbol{y})$ we get,

$$\boldsymbol{W}(\boldsymbol{x},\boldsymbol{y}) := \sum_{i=1}^{k^3} \boldsymbol{U}_i(\boldsymbol{x})\chi_{Y_i^*}(\boldsymbol{y}), \qquad \text{where } \boldsymbol{U}_i(\boldsymbol{x}) \in M_{skw}(\mathbb{R}^3) \tag{III.5.2}$$

$$\boldsymbol{w}^\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{R}^\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{w}^* = \boldsymbol{w}^*(\boldsymbol{x},\boldsymbol{y}) + \delta\ \boldsymbol{W}(\boldsymbol{x},\boldsymbol{y})\ \boldsymbol{w}^*(\boldsymbol{x},\boldsymbol{y}) + \frac{\delta^2}{2}\ \boldsymbol{W}^2(\boldsymbol{x},\boldsymbol{y})\ \boldsymbol{w}^* + O(\delta^3). \tag{III.5.3}$$

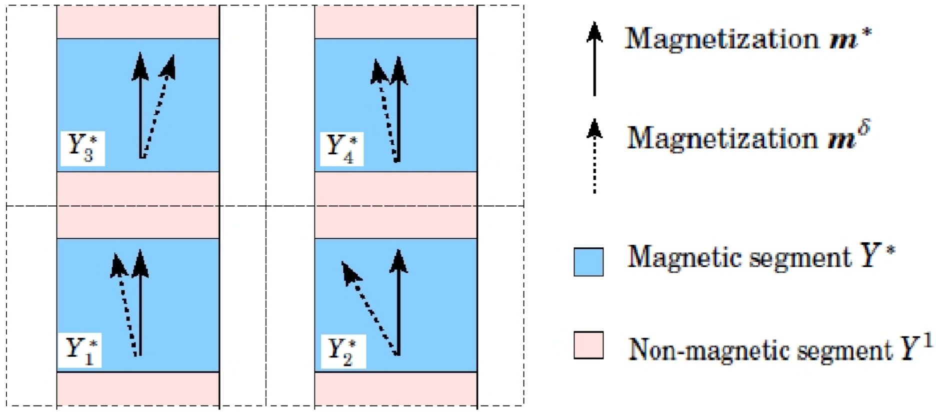

Figure III.3: Typical variation $\boldsymbol{w}^\delta$ with $k = 2$ shown in 2-dimensions. The 4 magnetic segments are indexed as Y_i^* with $i \in \{1,2,3,4\}$.

Breakup of W into average and oscillatory parts

Let us define the operator $(\overline{\cdot})$ and $(\widetilde{\cdot})$ as the average and oscillatory part of any quantity defined on kY^*. Recalling that $\boldsymbol{w}^*(\boldsymbol{x},\boldsymbol{y}) = \{0,0,m_s\}\chi_{kY^*}(\boldsymbol{y})$, we set

$$\overline{\boldsymbol{W}}(\boldsymbol{x}) = \frac{1}{|kY^*|}\int_{kY^*}\boldsymbol{W}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = \frac{1}{k^3}\sum_{i}^{k^3}\boldsymbol{U}_i(\boldsymbol{x}), \qquad \fint_{kY}\boldsymbol{W}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = \theta\overline{\boldsymbol{W}}(\boldsymbol{x})$$

$$\widetilde{\boldsymbol{W}}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{W}(\boldsymbol{x},\boldsymbol{y}) - \overline{\boldsymbol{W}}(\boldsymbol{x})\chi_{kY^*}(\boldsymbol{y}) = \sum_{i=1}^{k^3}\boldsymbol{U}_i(\boldsymbol{x})\chi_{Y_i^*}(\boldsymbol{y}) - \overline{\boldsymbol{W}}(\boldsymbol{x})\chi_{kY^*}(\boldsymbol{y}), \qquad \fint_{kY}\widetilde{\boldsymbol{W}}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = \boldsymbol{0}. \tag{III.5.4}$$

Recalling $\boldsymbol{w}^*(\boldsymbol{x},\boldsymbol{y}) = \{0,0,m_s\}\ \chi_{kY^*}(\boldsymbol{y})$ and $\boldsymbol{m}^*(\boldsymbol{x}) = \{0,0,\theta m_s\}$, we have

$$\fint_{kY}\boldsymbol{W}\boldsymbol{w}^*(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = \overline{\boldsymbol{W}}(\boldsymbol{x})\ \boldsymbol{m}^*(\boldsymbol{x}), \qquad \fint_{kY}\boldsymbol{W}^2\boldsymbol{w}^*(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = \overline{\boldsymbol{W}^2}(\boldsymbol{x})\ \boldsymbol{m}^*(\boldsymbol{x}),$$

$$\boldsymbol{w}^\delta(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{w}^*(\boldsymbol{x},\boldsymbol{y}) = \delta\boldsymbol{W}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{w}^*(\boldsymbol{x},\boldsymbol{y}) + \frac{\delta^2}{2}\boldsymbol{W}^2(\boldsymbol{x},\boldsymbol{y})\boldsymbol{w}^*(\boldsymbol{x},\boldsymbol{y}) + O(\delta^3) \approx O(\delta), \tag{III.5.5}$$

$$\boldsymbol{m}^\delta(\boldsymbol{x}) - \boldsymbol{m}^*(\boldsymbol{x}) = \fint_{kY}\left(\boldsymbol{w}^\delta - \boldsymbol{w}^*\right)\boldsymbol{dy} = \delta\overline{\boldsymbol{W}}(\boldsymbol{x})\ \boldsymbol{m}^*(\boldsymbol{x}) + \frac{\delta^2}{2}\overline{\boldsymbol{W}^2}(\boldsymbol{x})\ \boldsymbol{m}^*(\boldsymbol{x}) + O(\delta^3). \tag{III.5.6}$$

Similiarly we define $\overline{\boldsymbol{W}^2}(\boldsymbol{x})$. Using the Taylor expansion (III.5.3), we expand out the energy $\mathscr{E}^o(\boldsymbol{m})$.

Taylor expansion of Zeeman Energy

Using Taylor expansions in equations (III.5.5) and (III.5.6), the Zeeman energy becomes

$$\int_\Omega \boldsymbol{h}_a(\boldsymbol{x})\cdot(\boldsymbol{m}^\delta-\boldsymbol{m}^*)\boldsymbol{dydx} = \int_\Omega \boldsymbol{h}_a\cdot\left(\,\delta\,\overline{\boldsymbol{W}}(\boldsymbol{x})+\frac{\delta^2}{2}\,\overline{\boldsymbol{W}^2}(\boldsymbol{x})\,\right)\boldsymbol{m}^*(\boldsymbol{x})\boldsymbol{dx}+O(\delta^3).$$

Taylor expansion of Anisotropy Energy

The Taylor expansion of the Anisotropy energy gives using equations (III.5.6) and (III.5.5) and noting $\left|\boldsymbol{w}^\delta-\boldsymbol{w}^*\right|\approx O(\delta)$,

$$\begin{aligned}\int_\Omega ⨍_{kY}\left(\Phi(\boldsymbol{w}^\delta)-\Phi(\boldsymbol{w}^*)\right) &= \int_\Omega ⨍_{kY}\Phi'(\boldsymbol{w}^*)\cdot(\boldsymbol{w}^\delta-\boldsymbol{w}^*)+\frac{\Phi''(\boldsymbol{w}^*)}{2}(\boldsymbol{w}^\delta-\boldsymbol{w}^*)\cdot(\boldsymbol{w}^\delta-\boldsymbol{w}^*)+O\left(\left|\boldsymbol{w}^\delta-\boldsymbol{w}^*\right|^3\right)\\ &= \int_\Omega ⨍_{kY}\left\{\Phi'(\boldsymbol{w}^*)\cdot\left(\delta\boldsymbol{W}\boldsymbol{w}^*+\frac{\delta^2}{2}\boldsymbol{W}^2\boldsymbol{w}^*\right)+\delta^2\frac{\Phi''(\boldsymbol{w}^*)}{2}\boldsymbol{W}\boldsymbol{w}^*\cdot\boldsymbol{W}\boldsymbol{w}^*\right\}\boldsymbol{dy}\,\boldsymbol{dx}+O(\delta^3)\end{aligned}$$

Taylor expansion of $\mathscr{E}_d(\boldsymbol{m})$

Let $\boldsymbol{h}_{\boldsymbol{m}^\delta}$ and $\boldsymbol{h}_{\boldsymbol{m}^*}$ be solution to Maxwell for $\boldsymbol{m}^\delta$ and $\boldsymbol{m}^*$ respectively. Then Lemma III.A.2 gives

$$\begin{aligned}\int_{\mathbb{R}^3}\left(\left|\boldsymbol{h}_{\boldsymbol{m}^\delta}\right|^2-\left|\boldsymbol{h}_{\boldsymbol{m}^*}\right|^2\right)\boldsymbol{dx} &= \int_{\mathbb{R}^3}\left(\boldsymbol{h}_{\boldsymbol{m}^\delta}-\boldsymbol{h}_{\boldsymbol{m}^*}\right)\cdot\left(\boldsymbol{h}_{\boldsymbol{m}^\delta}-\boldsymbol{h}_{\boldsymbol{m}^*}+2\boldsymbol{h}_{\boldsymbol{m}^*}\right)\boldsymbol{dx}\\ &= \int_{\mathbb{R}^3}\left|\boldsymbol{h}_{\boldsymbol{m}^\delta}-\boldsymbol{h}_{\boldsymbol{m}^*}\right|^2\boldsymbol{dx}+2\int_{\mathbb{R}^3}\left(\boldsymbol{h}_{\boldsymbol{m}^*}\right)\cdot\left(\boldsymbol{h}_{\boldsymbol{m}^\delta}-\boldsymbol{h}_{\boldsymbol{m}^*}\right)\,\boldsymbol{dx}\\ &= \int_{\mathbb{R}^3}\left|\boldsymbol{h}_{\boldsymbol{m}^\delta}-\boldsymbol{h}_{\boldsymbol{m}^*}\right|^2\boldsymbol{dx}-8\pi\int_{\mathbb{R}^3}\left(\boldsymbol{h}_{\boldsymbol{m}^*}\right)\cdot\left(\boldsymbol{m}^\delta-\boldsymbol{m}^*\right)\,\boldsymbol{dx}\\ &= \int_{\mathbb{R}^3}\left|\boldsymbol{h}_{\boldsymbol{m}^\delta}-\boldsymbol{h}_{\boldsymbol{m}^*}\right|^2\boldsymbol{dx}-8\pi\int_\Omega\left(\boldsymbol{h}_{\boldsymbol{m}^*}\right)\cdot\left(\delta\,\overline{\boldsymbol{W}}\boldsymbol{m}^*+\frac{\delta^2}{2}\,\overline{\boldsymbol{W}^2}\boldsymbol{m}^*\right)+O(\delta^3).\end{aligned} \tag{III.5.7}$$

By linearity of Maxwell's equation $\left(\boldsymbol{h}_{\boldsymbol{m}^\delta}-\boldsymbol{h}_{\boldsymbol{m}^*}\right)$ is the demag field corresponding to magnetization $\left(\boldsymbol{m}^\delta-\boldsymbol{m}^*\right)$ which gives on recalling eqn. (III.5.6),

$$\frac{1}{8\pi}\int_{\mathbb{R}^3}\left|\boldsymbol{h}_{\boldsymbol{m}^\delta}-\boldsymbol{h}_{\boldsymbol{m}^*}\right|^2\boldsymbol{dx}=\mathscr{E}_d\left(\delta\overline{\boldsymbol{W}}\boldsymbol{m}^*+\frac{\delta^2}{2}\,\overline{\boldsymbol{W}^2}\boldsymbol{m}^*+O(\delta^3)\right)=\delta^2\mathscr{E}_d(\overline{\boldsymbol{W}}\boldsymbol{m}^*)+O(\delta^3)$$

Then dividing eqn. (III.5.7) by 8π and using above we get,

$$\mathscr{E}_d(\boldsymbol{m}^\delta)-\mathscr{E}_d(\boldsymbol{m}^*)=-\delta\left\{\int_\Omega\boldsymbol{h}^*_{\boldsymbol{m}}\cdot\overline{\boldsymbol{W}}\boldsymbol{m}^*\right\}+\frac{\delta^2}{2}\left\{2\mathscr{E}_d(\overline{\boldsymbol{W}}\boldsymbol{m}^*)-\int_\Omega\left(\boldsymbol{h}_{\boldsymbol{m}^*}\right)\cdot\overline{\boldsymbol{W}^2}\boldsymbol{m}^*\right\}+O(\delta^3).$$

Taylor expansion of Periodic part of magnetostatice energy $\mathscr{E}_{per}(\boldsymbol{m})$

Let $\left(\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^\delta}\right.$ and $\left.\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\right)$ be solutions to the periodic Maxwell's equation for $\boldsymbol{w}^\delta$ and $\boldsymbol{w}^*$ respectively. Then by linearity $\left(\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^\delta} - \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\right)$ solves the periodic Maxwell problem for magnetization $\left(\boldsymbol{w}^\delta - \boldsymbol{w}^*\right)$ and we have

$$\frac{1}{8\pi}\fint_{kY} |\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^\delta} - \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}|^2 \boldsymbol{dy} = \mathscr{E}_{per}\left(\delta \boldsymbol{W}\boldsymbol{w}^* + \frac{\delta^2}{2}\boldsymbol{W}^2\boldsymbol{w}^* + O(\delta^3)\right) = \delta^2\mathscr{E}_{per}(\boldsymbol{W}\boldsymbol{w}^*) + O(\delta^3). \quad \text{(III.5.8)}$$

Then for any $\boldsymbol{x} \in \Omega$, using Lemma III.A.2 and eqn. (III.5.8) we get

$$\begin{aligned}
\fint_{kY} \left(|\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^\delta}|^2 - |\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}|^2\right)\boldsymbol{dy} &= \fint_{kY} \left(\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^\delta} - \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\right)\cdot\left(\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^\delta} - \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*} + 2\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\right)\boldsymbol{dy} \\
&= \fint_{kY} |\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^\delta} - \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}|^2\boldsymbol{dy} + 2\fint_{kY} \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\cdot\left(\nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^\delta} - \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\right)\boldsymbol{dy} \\
&= 8\pi\delta^2\mathscr{E}_{per}(\boldsymbol{W}\boldsymbol{w}^*) - 8\pi\fint_{kY} \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\cdot\left(\delta\boldsymbol{W}\boldsymbol{w}^* + \frac{\delta^2}{2}\boldsymbol{W}^2\boldsymbol{w}^* + O(\delta^3)\right) + O(\delta^3),
\end{aligned}$$

which gives on dividing above by 8π, $\forall \boldsymbol{x} \in \Omega$,

$$\begin{aligned}
\mathscr{E}_{per}(\boldsymbol{w}^\delta) - \mathscr{E}_{per}(\boldsymbol{w}^*) = -\delta\left\{\fint_{kY} \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\cdot\boldsymbol{W}\boldsymbol{w}^*\boldsymbol{dy}\right\} + \frac{\delta^2}{2}\Big\{ &-\fint_{kY} \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\cdot\boldsymbol{W}^2\boldsymbol{w}^*\boldsymbol{dy} \\
&+ 2\mathscr{E}_{per}(\boldsymbol{W}\boldsymbol{w}^*)\Big\} + O(\delta^3).
\end{aligned}$$

First and Second variation conditions

Combining the individual expansions of the different energy terms, we have

$$\begin{aligned}
&\mathscr{I}^o_{(k)}(\boldsymbol{m}^\delta) - \mathscr{I}^o_{(k)}(\boldsymbol{m}^*) = \delta\, I_1(\boldsymbol{w}^\delta, \boldsymbol{w}^*) + \delta^2\, I_2(\boldsymbol{w}^\delta, \boldsymbol{w}^*) + O(\delta^3), \qquad \text{where} \\
&I_1(\boldsymbol{w}^\delta, \boldsymbol{w}^*) = \int_\Omega\fint_{kY} \left(\Phi'(\boldsymbol{w}^*) - \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\right)\cdot\boldsymbol{W}\boldsymbol{w}^*\boldsymbol{dy}\boldsymbol{dx} - \int_\Omega \left(\boldsymbol{h}_a(\boldsymbol{x}) + \boldsymbol{h}_{\boldsymbol{m}^*}(\boldsymbol{x})\right)\cdot\overline{\boldsymbol{W}}\boldsymbol{m}^*\boldsymbol{dx}, \quad \text{(III.5.9)} \\
&I_2(\boldsymbol{w}^\delta, \boldsymbol{w}^*) = \frac{1}{2}\int_\Omega\fint_{kY} \left\{\left(\Phi'(\boldsymbol{w}^*) - \nabla^{\boldsymbol{y}}\psi^1_{\boldsymbol{w}^*}\right)\cdot\boldsymbol{W}^2\boldsymbol{w}^* + \Phi''(\boldsymbol{w}^*)\boldsymbol{W}\boldsymbol{w}^*\cdot\boldsymbol{W}\boldsymbol{w}^*\right\}\boldsymbol{dy}\boldsymbol{dx} \\
&\qquad + \mathscr{E}_d(\overline{\boldsymbol{W}}\boldsymbol{m}^*) + \frac{1}{2}\int_\Omega -\left(\boldsymbol{h}_a + \boldsymbol{h}_{\boldsymbol{m}^*}\right)\cdot\overline{\boldsymbol{W}^2}\boldsymbol{m}^*\boldsymbol{dx} + \int_\Omega \mathscr{E}_{per}(\boldsymbol{W}\boldsymbol{w}^*)\boldsymbol{dx}. \qquad \text{(III.5.10)}
\end{aligned}$$

Now we can check that $\boldsymbol{m}^*$ is a stationary critical point of $\mathscr{I}^o_{(k)}(\boldsymbol{m})$ if it satisfies the vanishing first variation condition. From a standard representation for any $\boldsymbol{U}_i(\boldsymbol{x}) \in M_{skw}(\mathbb{R}^3)$ we get

$$\boldsymbol{W}(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{k^3} \boldsymbol{U}_i(\boldsymbol{x})\ \chi_{Y_i^*}(\boldsymbol{y}) = \sum_{i=1}^{k^3} \begin{bmatrix} 0 & \varkappa_i(\boldsymbol{x}) & \xi_i(\boldsymbol{x}) \\ -\varkappa_i(\boldsymbol{x}) & 0 & \eta_i(\boldsymbol{x}) \\ -\xi_i(\boldsymbol{x}) & -\eta_i(\boldsymbol{x}) & 0 \end{bmatrix} \chi_{Y_i^*}(\boldsymbol{y}),$$

$$\boldsymbol{W}\boldsymbol{w}^* = \sum_{i=1}^{k^3} m_s\{\xi_i,\eta_i,0\}^T(\boldsymbol{x})\ \chi_{Y_i^*}(\boldsymbol{y}), \qquad \boldsymbol{W}^2\boldsymbol{w}^* = \sum_{i=1}^{k^3} m_s\{\varkappa_i\eta_i, -\varkappa_i\xi_i, -\xi_i^2-\eta_i^2\}^T(\boldsymbol{x})\ \chi_{Y_i^*}(\boldsymbol{y}). \tag{III.5.11}$$

Setting

$$\beta(\boldsymbol{x},\boldsymbol{y}) := \sum_{i=1}^{k^3} \xi_i(\boldsymbol{x})\ \chi_{Y_i^*}(\boldsymbol{y}), \quad \gamma(\boldsymbol{x},\boldsymbol{y}) := \sum_{i=1}^{k^3} \eta_i(\boldsymbol{x})\ \chi_{Y_i^*}(\boldsymbol{y}), \quad \text{gives} \quad \boldsymbol{W}\boldsymbol{w}^* = m_s\{\beta,\gamma,0\}^T. \tag{III.5.12}$$

We then recall equation (III.5.4) to split $\beta(\boldsymbol{x},\boldsymbol{y})$ as

$$\overline{\beta}(\boldsymbol{x}) = \fint_{kY^*} \beta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = \frac{1}{k^3}\sum_{i=1}^{k^3}\xi_i(\boldsymbol{x}), \quad \widetilde{\beta}(\boldsymbol{x},\boldsymbol{y}) = \beta(\boldsymbol{x},\boldsymbol{y}) - \overline{\beta}\chi_{kY^*} \qquad \text{giving} \ \fint_{kY^*} \widetilde{\beta}\boldsymbol{dy} = 0 \tag{III.5.13}$$

and $\gamma(\boldsymbol{x},\boldsymbol{y}) = \overline{\gamma}(\boldsymbol{x})\ \chi_{kY^*}(\boldsymbol{y}) + \widetilde{\gamma}(\boldsymbol{x},\boldsymbol{y})$ similarly. Since $\boldsymbol{m}^* = \{0,0,\theta m_s\}^T$, using eqn. (III.5.11),

$$\overline{\boldsymbol{W}}\boldsymbol{m}^*(\boldsymbol{x}) = \fint_{kY^*} \boldsymbol{W}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{m}^*(\boldsymbol{x})\boldsymbol{dy} = \frac{\theta m_s}{k^3}\sum_{i=1}^{k^3}\{\xi_i,\eta_i,0\}^T(\boldsymbol{x}) = \theta m_s\{\overline{\beta},\overline{\gamma},0\}^T(\boldsymbol{x}).$$

The applied field $\boldsymbol{h}_a(\boldsymbol{x}) = \{0,0,H_a\}^T$ is along z-axis. For $\boldsymbol{m}^* = \{0,0,\theta m_s\}$, the demag field $\boldsymbol{h}_{\boldsymbol{m}^*}(\boldsymbol{x}) = \{0,0,-\theta N_z m_s\}^T$. Then

$$\big(\ \boldsymbol{h}_a(\boldsymbol{x}) + \boldsymbol{h}_{\boldsymbol{m}^*}(\boldsymbol{x})\ \big)\cdot\overline{\boldsymbol{W}}\boldsymbol{m}^* \equiv 0.$$

For $\Phi(\boldsymbol{m})$ given as in eqn. (III.2.1), the derivative of anisotropy $\Phi'(\boldsymbol{w}^*) \equiv \boldsymbol{0}$ at $\boldsymbol{w}^* = \{0,0,m_s\}^T$. So we just need to check the term in the first variation which comes from the periodic part of the magnetostatic energy. Since $\boldsymbol{w}^* = (0,0,m_s)\chi_{kY^*}(\boldsymbol{y})$ is symmetric about the z-axis, we can check that $\int_{Y_i}\nabla^y\psi^*_{\boldsymbol{m}}\boldsymbol{dy} = \boldsymbol{0}$ for each $Y_i^* \in k^3Y$. Recalling $\boldsymbol{W}\boldsymbol{w}^* = m_s\{\xi_i,\eta_i,0\}^T(\boldsymbol{x})\chi_{Y_i^*}(\boldsymbol{y})$ from eqn. (III.5.11), and equation (III.5.25) from Lemma III.5.1 we get,

$$\fint_{kY} \nabla^y \psi^{\mathit{1}}_{\boldsymbol{w}^*}(\boldsymbol{x},\boldsymbol{y})\cdot \boldsymbol{W}\boldsymbol{w}^* \boldsymbol{dy} = \frac{m_s}{k^3}\sum_{i}^{k^3}\{\xi_i,\eta_i,0\}^T(\boldsymbol{x})\cdot \fint_{Y_i^*} \nabla^y \psi^{\mathit{1}}_{\boldsymbol{w}^*}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy}$$

$$= \frac{m_s}{k^3}\left\{0,0,\fint_{kY^*}\partial^{y_3}\psi^{\mathit{1}}_{\boldsymbol{w}^*}(\boldsymbol{y})\boldsymbol{dy}\right\}\cdot \sum_{i}^{k^3}\{\xi_i,\eta_i,0\}^T(\boldsymbol{x}) = 0. \qquad \text{(III.5.14)}$$

Combining these and substituting into eqn. (III.5.9) we have $I_1(\boldsymbol{w}^\delta,\boldsymbol{w}^*) = 0$ so that $\boldsymbol{m}^*$ is a stationary critical point.

Stability of critical point $\boldsymbol{m}^*$

Stability of $\boldsymbol{m}^*$ can be analyzed in terms of the positivity of the second variation $I_2(\boldsymbol{m}^\delta,\boldsymbol{m}^*)$ to determine a critical value of the field $\boldsymbol{h}_a$ so that $\boldsymbol{m}^*$ is second-variation stable. Thus the critical applied field $\boldsymbol{h}_a = (0,0,H_a)$ can be thought of as the smallest field H_a such that $I_2(\boldsymbol{w}^\delta,\boldsymbol{w}^*) \geq 0$. To simplify the second varition analysis forst note that since $\Phi'(\boldsymbol{w}^*) \equiv 0$, the first term in the integrand constituting $I_2(\boldsymbol{w}^\delta,\boldsymbol{w}^*)$ is 0. The form of $\Phi(\boldsymbol{m})$ in eqn. (III.2.1) gives

$$\Phi''(\boldsymbol{w}^*) = \begin{bmatrix} 2\Pi_1 & 0 & 0 \\ 0 & 2\Pi_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{at} \quad \boldsymbol{w}^* = \{0,0,m_s\}^T.$$

Then using eqn. (III.5.12), we get

$$\frac{1}{2}\int_\Omega \fint_{kY} \Phi''(\boldsymbol{w}^*)\boldsymbol{W}\boldsymbol{w}^*\cdot \boldsymbol{W}\boldsymbol{w}^* = \Pi_1 m_s^2 \int_\Omega \fint_{kY}(\beta^2+\gamma^2)\boldsymbol{dx} = \Pi_1 m_s^2\,\theta\int_\Omega(\overline{\beta^2}+\overline{\gamma^2})\boldsymbol{dx}$$

$$-(\,\boldsymbol{h}_a + \boldsymbol{h}_{\boldsymbol{m}^*}\,)\cdot \overline{\boldsymbol{W}^2}\boldsymbol{m}^* = \theta m_s(H_a - \theta N_z m_s)(\overline{\beta^2}+\overline{\gamma^2}).$$

Recall $\beta(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{k^3}\xi_i(\boldsymbol{x})\chi_{Y_i^*}(\boldsymbol{y})$. Then $|\beta(\boldsymbol{x},\boldsymbol{y})|^2 = \sum_{i=1}^{k^3}|\xi_i(\boldsymbol{x})|^2\chi_{Y_i^*}(\boldsymbol{y})$ which gives

$$\overline{\beta^2}(\boldsymbol{x}) = \fint_{kY^*}|\beta(\boldsymbol{x},\boldsymbol{y})|^2\boldsymbol{dy} = \frac{1}{k^3}\sum_{i=1}^{k^3}|\xi_i(\boldsymbol{x})|^2,$$

and similar for $\overline{\gamma^2}(\boldsymbol{x})$ Similar to equation (III.5.14), using equation (III.5.25) from Lemma III.5.1 we get

$$\begin{aligned}\fint_{kY}\nabla\psi^1_{\boldsymbol{w}^*}\cdot\boldsymbol{W}^2\boldsymbol{w}^*\boldsymbol{dy} &= \frac{m_s}{k^3}\Big\{0,0,\fint_{kY^*}\partial^{y_3}\psi^1_{\boldsymbol{w}^*}(\boldsymbol{y})\boldsymbol{dy}\Big\}\cdot\sum_i^{k^3}\{\varkappa_i\eta_i,-\varkappa_i\xi_i,-\xi_i^2-\eta_i^2\}^T(\boldsymbol{x})\\ &= -m_s\Big\{\fint_{kY^*}\partial^{y_3}\psi^1_{\boldsymbol{w}^*}(\boldsymbol{y})\boldsymbol{dy}\Big\}\cdot\ (\overline{\beta^2}+\overline{\gamma^2})(\boldsymbol{x})\end{aligned}$$

Using the above results $I_2(\boldsymbol{w}^\delta,\boldsymbol{w}^*)$ can be rewritten as just a function of (β,γ) as follows

$$\begin{aligned}I_2(\boldsymbol{w}^\delta,\boldsymbol{w}^*) = \int_\Omega\Big\{\frac{m_s}{2}\Big(\fint_{kY^*}\partial^{y_3}\psi^1_{\boldsymbol{w}^*}(\boldsymbol{y})\boldsymbol{dy}\Big)+\theta\Pi_1m_s^2+\frac{\theta m_s}{2}(H_a-\theta N_z m_s)\Big\}\big(\overline{\beta^2}+\overline{\gamma^2}\big)\boldsymbol{dx}\\ +\theta^2m_s^2\mathscr{E}_d\big(\{\overline{\beta},\overline{\gamma},0\}\big)+m_s^2\int_\Omega\mathscr{E}_{per}\big(\{\beta,\gamma,0\}\big)\boldsymbol{dx} =: I_2(\beta,\gamma;H_a).\end{aligned}\tag{III.5.15}$$

Let

$$\mathscr{B}^k = \big\{\beta\in L^2\big(\Omega;L^2_\sharp(kY)\big)\,\big|\,\forall\boldsymbol{x}\in\Omega,\ supp\big(\beta(\boldsymbol{x},\cdot)\big)=kY^*\ \&\ \beta \text{ is constant on } Y_i^*\in kY^*\big\}.\tag{III.5.16}$$

The critical field H_a is the largest field such that $\inf_{(\beta,\gamma)\in(\mathscr{B}^k)^2} I_2(\beta,\gamma;H_a)$ where $(\mathscr{B}^k)^2=\mathscr{B}^k\times\mathscr{B}^k$. This can be used to alternately characterize the critical field as a Rayleigh quotient

$$\frac{H_a^{cr}}{m_s} = \theta N_z-2\Pi_1-\Big(\fint_{k^3Y^*}\partial^{y_3}\psi^1_{\boldsymbol{w}^*}(\boldsymbol{y})\boldsymbol{dy}\Big)-2\inf_{(\beta,\gamma)\in(\mathscr{B}^k)^2}\frac{\theta^2\mathscr{E}_d\big(\{\overline{\beta},\overline{\gamma},0\}\big)+\int_\Omega\mathscr{E}_{per}\big(\{\beta,\gamma,0\}\big)\boldsymbol{dx}}{\int_\Omega\theta\ \big(\ \overline{\beta^2}+\overline{\gamma^2}\ \big)\boldsymbol{dx}}.\tag{III.5.17}$$

To draw conclusions from the above, we go back to the decomposition $\beta(\boldsymbol{x},\boldsymbol{y})=\overline{\beta}(\boldsymbol{x})\chi_{kY^*}(\boldsymbol{y})+\widetilde{\beta}(\boldsymbol{x},\boldsymbol{y})$ from equation (III.5.13) and similarly for $\gamma(\boldsymbol{x},\boldsymbol{y})$. Recall also that $\fint_{kY^*}\widetilde{\beta}\boldsymbol{dy}=\fint_{kY^*}\widetilde{\gamma}\boldsymbol{dy}=0$. This decomposition involves a similar decomposition of the space $\mathscr{B}^k$ as follows:

$$\begin{aligned}\overline{\mathscr{B}}^k &= \big\{\beta(\boldsymbol{x},\boldsymbol{y})\in\mathscr{B}^k\,\big|\,\forall\boldsymbol{x}\in\Omega,\ \beta(\boldsymbol{x},\cdot)\text{ is constant on } kY^*\big\}\\ \widetilde{\mathscr{B}}^k &= \big\{\beta(\boldsymbol{x},\boldsymbol{y})\in\mathscr{B}^k\,\big|\,\forall\boldsymbol{x}\in\Omega,\quad \fint_{kY}\beta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy}=0\big\},\end{aligned}\tag{III.5.18}$$

while noting $\overline{\beta}\boldsymbol{\chi}_{k^3Y^*} \in \overline{\mathscr{B}}^k$ and $\widetilde{\beta} \in \widetilde{\mathscr{B}}^k$. Let $\overline{\psi}^1$, $\widetilde{\psi}^1$ solve the periodic Maxwell problem for $\{\overline{\beta},\overline{\gamma},0\}^T\boldsymbol{\chi}_{kY^*}$ and $\{\widetilde{\beta},\widetilde{\gamma},0\}^T$ respectively. For $\boldsymbol{x} \in \Omega$, using Lemma III.A.2

$$\begin{aligned}\fint_{kY} |\nabla^{\boldsymbol{y}}\overline{\psi}^1 + \nabla^{\boldsymbol{y}}\widetilde{\psi}^1|^2 \boldsymbol{dy} &= \fint_{kY} \left(|\nabla^{\boldsymbol{y}}\overline{\psi}^1|^2 + |\nabla^{\boldsymbol{y}}\widetilde{\psi}^1|^2 \right)\boldsymbol{dy} + 2\fint_{kY} \nabla^{\boldsymbol{y}}\overline{\psi}^1 \cdot \nabla^{\boldsymbol{y}}\widetilde{\psi}^1 \boldsymbol{dy} \\ &= \fint_{kY} \left(|\nabla^{\boldsymbol{y}}\overline{\psi}^1|^2 + |\nabla^{\boldsymbol{y}}\widetilde{\psi}^1|^2 \right)\boldsymbol{dy} - 8\pi \fint_{kY} \nabla^{\boldsymbol{y}}\overline{\psi}^1 \cdot \{\widetilde{\beta},\widetilde{\gamma},0\}^T \boldsymbol{dy}.\end{aligned}$$

Note that $\{\widetilde{\beta},\widetilde{\gamma},0\}^T = \{\xi_i(\boldsymbol{x},\boldsymbol{y}) - \overline{\beta}(\boldsymbol{x})\boldsymbol{\chi}_{Y_i^*}(\boldsymbol{y}), \eta_i(\boldsymbol{x},\boldsymbol{y}) - \overline{\gamma}(\boldsymbol{x})\boldsymbol{\chi}_{Y_i^*}(\boldsymbol{y}), 0\}^T$. Using $(c.)$ from Lemma III.5.1

$$\begin{aligned}\fint_{kY} \nabla^{\boldsymbol{y}}\overline{\psi}^1 \cdot (\widetilde{\beta}, \widetilde{\gamma}, 0)^T \boldsymbol{dy} &= \theta \sum_i^{k^3} \frac{\{\xi_i - \overline{\beta}, \eta_i - \overline{\gamma}, 0\}^T}{k^3} \cdot \fint_{Y_i^*} \nabla^{\boldsymbol{y}}\overline{\psi}^1 \boldsymbol{dy} \\ &= \left(\fint_{kY^*} \nabla^{\boldsymbol{y}}\overline{\psi}^1 \boldsymbol{dy}\right) \cdot \sum_i^{k^3} \{\xi_i - \overline{\beta}, \eta_i - \overline{\gamma}, 0\}^T = 0,\end{aligned}$$

using the fact that $\overline{\beta}(\boldsymbol{x}) = \frac{1}{k^3}\sum_{i=1}^{k^3} \xi_i(\boldsymbol{x})$ from equation (III.5.13). Thus we have

$$\mathscr{E}_{per}(\{\beta,\gamma,0\}) = \mathscr{E}_{per}(\{\overline{\beta},\overline{\gamma},0\}\boldsymbol{\chi}_{kY^*}) + \mathscr{E}_{per}(\{\widetilde{\beta},\widetilde{\gamma},0\}). \qquad \text{(III.5.19)}$$

Also $\beta^2(\boldsymbol{x},\boldsymbol{y}) = |\overline{\beta}(\boldsymbol{x})|^2 \boldsymbol{\chi}_{kY^*}(\boldsymbol{y}) + |\widetilde{\beta}(\boldsymbol{x},\boldsymbol{y})|^2 + 2\overline{\beta}(\boldsymbol{x})\widetilde{\beta}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\chi}_{kY^*}(\boldsymbol{y})$ giving

$$\overline{\beta^2}(\boldsymbol{x}) = \fint_{kY^*} |\overline{\beta}(\boldsymbol{x})|^2 \boldsymbol{dy} + \fint_{kY^*} |\widetilde{\beta}(\boldsymbol{x},\boldsymbol{y})|^2 \boldsymbol{dy} + 2\,\overline{\beta}(\boldsymbol{x}) \fint_{kY^*} \widetilde{\beta}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = |\overline{\beta}(\boldsymbol{x})|^2 + \overline{|\widetilde{\beta}|^2}(\boldsymbol{x}) \qquad \text{(III.5.20)}$$

and a similiar expansion of $\overline{|\widetilde{\gamma}|^2}$. Then combining eqns. (III.5.19) and (III.5.20)

$$\begin{aligned}&\inf_{(\beta,\gamma)} \frac{\theta^2 \mathscr{E}_d(\{\overline{\beta},\overline{\gamma},0\}) + \int_\Omega \mathscr{E}_{per}(\{\beta,\gamma,0\})\boldsymbol{dx}}{\int_\Omega \theta \left(\overline{\beta^2} + \overline{\gamma^2} \right)\boldsymbol{dx}} \\ &\quad = \frac{1}{\theta} \inf_{(\beta,\gamma)} \frac{\left[\theta^2 \mathscr{E}_d(\{\overline{\beta},\overline{\gamma},0\}) + \int_\Omega \mathscr{E}_{per}(\{\overline{\beta},\overline{\gamma},0\}\boldsymbol{\chi}_{kY^*})\boldsymbol{dx}\right] + \left[\int_\Omega \mathscr{E}_{per}(\{\widetilde{\beta},\widetilde{\gamma},0\})\boldsymbol{dx}\right]}{\int_\Omega (|\overline{\beta}|^2 + |\overline{\gamma}|^2)\boldsymbol{dx} + \int_\Omega (\overline{|\widetilde{\beta}|^2} + \overline{|\widetilde{\gamma}|^2})\boldsymbol{dx}}.\end{aligned}$$

Finally we will now show that for any $\overline{\beta},\overline{\gamma}$, there exists $\widetilde{\beta},\widetilde{\gamma}$ such that

$$\begin{aligned}&\int_\Omega (|\overline{\beta}|^2 + |\overline{\gamma}|^2)\boldsymbol{dx} = \int_\Omega (\overline{|\widetilde{\beta}|^2} + \overline{|\widetilde{\gamma}|^2})\boldsymbol{dx} \\ &\int_\Omega \mathscr{E}_{per}(\{\overline{\beta},\overline{\gamma},0\}\boldsymbol{\chi}_{kY^*})\boldsymbol{dx} \geq \int_\Omega \mathscr{E}_{per}(\{\widetilde{\beta},\widetilde{\gamma},0\})\boldsymbol{dx}\end{aligned} \qquad \text{(III.5.21)}$$

and as a result the critical field H_a^{cr} in eqn. (III.5.17) is easily seen to minimize only over $\widetilde{\beta},\widetilde{\gamma}$.

Thus

$$\frac{H_a^{cr}}{m_s} = \theta N_z - 2\Pi_1 - \left(\fint_{kY^*} \partial^{y_3} \psi^1_{\boldsymbol{w}^*}(\boldsymbol{y}) \boldsymbol{dy} \right) - 2 \inf_{(\widetilde{\beta},\widetilde{\gamma})\in(\mathscr{B}^k)^2} \frac{\int_\Omega \mathscr{E}_{per}\left(\{\widetilde{\beta},\widetilde{\gamma},0\}\right)\boldsymbol{dx}}{\int_\Omega \theta\left(\overline{|\widetilde{\beta}|^2} + \overline{|\widetilde{\gamma}|^2}\right)\boldsymbol{dx}}. \tag{III.5.22}$$

To show equation (III.5.21) we will first restrict the result to only even k, i.e. $k = 2,4,6\cdots$ and assume that Y^* is a rectangular domain. The results however hold for any star shaped domain Y^*. Recall Y is just the unit cube and let $Y^* = \prod_{i=1}^3 \left((1+l_i)/2,(1+l_i)/2\right)$.

Let first $k = 2$. We put the origin of the periodic k^3Y sized block at it's center so that it looks like Figure III.4 with Y_i^* shown where $i = \{1,2,3,4\}$ and the Y_i^* with $i = \{5,6,7,8\}$ lying below $\{1,2,3,4\}$ and having the same magnetizations. Let the left image on Figure III.4 represent

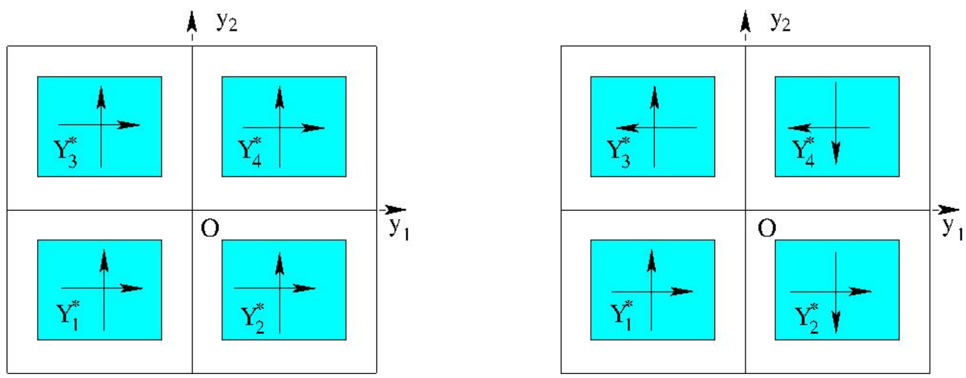

Figure III.4: Right: Example of a magnetization $\{\overline{\beta}(\boldsymbol{x}),\overline{\gamma}(\boldsymbol{x}),0\}^T \boldsymbol{\chi}_{kY^*}(\boldsymbol{y})$, Left: A proposed magnetization $\{\widetilde{\beta},\widetilde{\gamma},0\}^T(\boldsymbol{x},\boldsymbol{y})$

a given magnetization $\{\overline{\beta},\overline{\gamma},0\}^T(\boldsymbol{x})\boldsymbol{\chi}_{kY^*}(\boldsymbol{y})$. Let the right side image in Figure III.4 represent a magnetization $\{\widetilde{\beta},\widetilde{\gamma},0\}^T(\boldsymbol{x},\boldsymbol{y})$ which we will use to show the result eqn. (III.5.21) where

$$\widetilde{\beta}(\boldsymbol{x},\boldsymbol{y}) = \overline{\beta}(\boldsymbol{x})\{\boldsymbol{\chi}_{Y_1^*\cup Y_3^*\cup Y_5^*\cup Y_7^*}(\boldsymbol{y}) - \boldsymbol{\chi}_{Y_2^*\cup Y_4^*\cup Y_6^*\cup Y_8^*}(\boldsymbol{y})\}$$

and same for $\widetilde{\gamma}(\boldsymbol{x},\boldsymbol{y})$. We first show the following simple Lemma.

Lemma III.5.1. *For magnetizations* $\{\overline{\beta},\overline{\gamma},0\}^T(\boldsymbol{x})\boldsymbol{\chi}_{kY^*}(\boldsymbol{y})$ *and* $\{\widetilde{\beta},\widetilde{\gamma},0\}^T(\boldsymbol{x},\boldsymbol{y})$ *as defined in Figure III.4,* *we have*

$$
\begin{aligned}
&a.)\ \mathscr{E}_{per}\left(\{\overline{\beta},\overline{\gamma},0\}\chi_{kY^*}\right)=\mathscr{E}_{per}\left(\{0,\overline{\gamma},0\}\chi_{kY^*}\right)+\mathscr{E}_{per}\left(\{\overline{\beta},0,0\}\chi_{kY^*}\right)\\
&b.)\ \mathscr{E}_{per}\left(\{\widetilde{\beta},\widetilde{\gamma},0\}\right)=\mathscr{E}_{per}\left(\{\widetilde{\beta},0,0\}\right)+\mathscr{E}_{per}\left(\{0,\widetilde{\gamma},0\}\right)\\
&c.)\ \int_{Y_i^*}\nabla^{\boldsymbol{y}}\overline{\psi}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy}=\frac{1}{k^3}\int_{kY^*}\left\{\partial^{y_1}\overline{\psi}(\boldsymbol{x},\boldsymbol{y}),\partial^{y_2}\overline{\psi}(\boldsymbol{x},\boldsymbol{y}),0\right\}^T\boldsymbol{dy},\qquad \textit{for any } i\in\{1,2,\cdots k^3\}.
\end{aligned}
$$

where $\overline{\psi}(\boldsymbol{x},\boldsymbol{y})$ *solves periodic Maxwell's equation for magnetization* $\{\overline{\beta},\overline{\gamma},0\}^T\chi_{kY^*}$*. Of these* $(a.)$ *and* $(c.)$ *are valied for any* k *and corresponding magnetization* $\{\overline{\beta},\overline{\gamma},0\}^T\chi_{kY^*}$*.*

Proof. Let us write $(\overline{\beta}\chi_{kY^*},\overline{\gamma}\chi_{kY^*})$ as a fourier series. Since both are even function in $\boldsymbol{y}$

$$
\begin{aligned}
\overline{\beta}(\boldsymbol{x})\ \chi_{kY^*}(\boldsymbol{y})&=\overline{\beta}_o(\boldsymbol{x})+\sum_{l,m,n}A_{lmn}(\boldsymbol{x})\cos\left(l\pi y_1\right)\ \cos\left(m\pi y_2\right)\ \cos\left(n\pi y_3\right),\\
\overline{\gamma}(\boldsymbol{x})\ \chi_{kY^*}(\boldsymbol{y})&=\overline{\gamma}_o(\boldsymbol{x})+\sum_{l,m,n}C_{lmn}(\boldsymbol{x})\cos\left(l\pi y_1\right)\ \cos\left(m\pi y_2\right)\ \cos\left(n\pi y_3\right),
\end{aligned}
$$

with $\overline{\beta}_o(\boldsymbol{x})=\fint_{kY}\overline{\beta}(\boldsymbol{x})\ \chi_{kY^*}(\boldsymbol{y})\boldsymbol{dy}=\theta\overline{\beta}(\boldsymbol{x})$ and similarly for $\overline{\gamma}_o(\boldsymbol{x})=\theta\overline{\gamma}(\boldsymbol{x})$.

Noting $Y_4^*=\prod_{i=1}^3\left(1/2-l_i/2,1/2+l_i/2\right)$, using fourier cosine formula we get

$$
\begin{aligned}
\left\{A_{l,m,n},C_{l,m,n}\right\}(\boldsymbol{x})&=\frac{512}{\pi^3 lmn}\left\{\overline{\beta},\overline{\gamma}\right\}(\boldsymbol{x})\ \cos\left(\frac{l\pi}{2}\right)\cos\left(\frac{m\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right)\sin\left(\frac{l\pi l_1}{2}\right)\sin\left(\frac{m\pi l_2}{2}\right)\sin\left(\frac{n\pi l_3}{2}\right)\\
&=\frac{512}{\pi^3 lmn}\left\{\overline{\beta},\overline{\gamma}\right\}(\boldsymbol{x})\ (-1)^{(l+m+n)/2}\sin\left(\frac{l\pi l_1}{2}\right)\sin\left(\frac{m\pi l_2}{2}\right)\sin\left(\frac{n\pi l_3}{2}\right)
\end{aligned}
\tag{III.5.23}
$$

if l,m,n even. Thus $\left\{A_{l,m,n},C_{l,m,n}\right\}$ takes non-zero value only for l,m,n even. Let $\overline{\psi}(\boldsymbol{x},\boldsymbol{y})$ be solution to the Maxwell periodic problem with magnetization $\{\overline{\beta},\overline{\gamma},0\}^T\chi_{kY^*}$ i.e.

$$
\int_{kY}\left(-\nabla^{\boldsymbol{y}}\overline{\psi}(\boldsymbol{x},\boldsymbol{y})+4\pi\left(\{\overline{\beta},\overline{\gamma},0\}^T-\{\overline{\beta}_o,\overline{\gamma}_o,0\}^T\right)\ \chi_{kY^*}(\boldsymbol{y})\right)\cdot\nabla^{\boldsymbol{y}}\phi(\boldsymbol{y})=0,\qquad \phi(\boldsymbol{y})\in C_\sharp^\infty(kY).
$$

Noting the even structure of $\{\overline{\beta},\overline{\gamma},0\}^T\chi_{kY^*}$, we look for solutions $\psi_{\overline{\boldsymbol{m}}}$ such that

$$
\overline{\psi}=\overline{\psi}_o+\sum_{l,m,n}B_{lmn}(\boldsymbol{x})\sin\left(l\pi y_1\right)\cos\left(m\pi y_2\right)\cos\left(n\pi y_3\right)+D_{lmn}(\boldsymbol{x})\cos\left(l\pi y_1\right)\sin\left(m\pi y_2\right)\cos\left(n\pi y_3\right)
$$

where $\overline{\psi}_o=\fint_{kY}\overline{\psi}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy}$.

Using for test function $\phi(\boldsymbol{y})=\sin\left(l\pi y_1\right)\ \cos\left(m\pi y_2\right)\ \cos\left(n\pi y_3\right)$ for $(l,m,n)\in\mathbb{N}^3$ and using the orthogonality of sines and cosines we get

$$\pi^2\Big\{l^2+m^2+n^2\Big\}B_{lmn}(\boldsymbol{x}) = 4l\pi^2 A_{lmn}(\boldsymbol{x}),$$

and using test function $\phi(\boldsymbol{y}) = \cos\big(l\pi y_1\big)\ \sin\big(m\pi y_2\big)\ \cos\big(n\pi y_3\big)$ for $(l,m,n)\in\mathbb{N}^3$ gives D_{lmn} as

$$\pi^2\Big\{l^2+m^2+n^2\Big\}D_{lmn}(\boldsymbol{x}) = 4m\pi^2 C_{lmn}(\boldsymbol{x}).$$

Since A_{lmn} and C_{lmn} are 0 if any of l,m,n are odd, the above relations give us that B_{lmn} and D_{lmn} are also 0 if any of l,m,n are odd. The energy $\mathscr{E}_{per}\big(\{\overline{\beta},\overline{\gamma},0\}\boldsymbol{\chi}_{_{kY^*}}\big)$ becomes using Parseval's theorem

$$\begin{aligned}\mathscr{E}_{per}\big(\{\overline{\beta},\overline{\gamma},0\}\boldsymbol{\chi}_{_{kY^*}}\big) &= \frac{1}{8\pi}\int_{kY}\big|\nabla^y\psi_{\overline{\boldsymbol{m}}}(\boldsymbol{x},\boldsymbol{y})\big|^2\boldsymbol{dy} = \frac{3\pi^2}{64\pi}\sum_{l,m,n}\Big(\frac{l^2}{L_1^2}+\frac{m^2}{L_2^2}+\frac{n^2}{L_3^2}\Big)\Big\{\big|B_{lmn}\big|^2+\big|D_{lmn}\big|^2\Big\}\\ &= \mathscr{E}_{per}\big(\{\overline{\beta},0,0\}\boldsymbol{\chi}_{_{kY^*}}\big)+\mathscr{E}_{per}\big(\{0,\overline{\gamma},0\}\boldsymbol{\chi}_{_{kY^*}}\big)\end{aligned}$$

because of the absence of cross terms involving B_{lmn} and D_{lmn}. So $(a.)$ is proved. By an abuse of notation, if we set $s_1 = \sin\big(l\pi y_1\big)$, $c_1 = \cos\big(l\pi y_1\big)$ and s_2,c_2,s_3,c_3 similarly, we note

$$\nabla^y\overline{\psi} = \sum_{l,m,n}^{even}\Big\{B_{lmn}c_1c_2c_3 - D_{lmn}s_1s_2c_3, -B_{lmn}s_1s_2c_3 + D_{lmn}c_1c_2c_3, -B_{lmn}s_1c_2s_3 - D_{lmn}c_1s_2c_3\Big\}^T.$$

Recalling that $Y_4^* = \prod_{i=1}^3\big(1/2-l_i/2, 1/2+l_i/2\big)$

$$\begin{aligned}&\int_{(1-l_1)/2}^{(1+l_1)/2}\sin\big(l\pi y_1\big)dy_1 = \frac{-1}{l\pi}\cos\big(l\pi y_1\big)\Big|_{(1-l_1)/2}^{(1+l_1)/2} = \frac{2}{l\pi}\sin\big(\frac{l\pi}{2}\big)\sin\big(\frac{l\pi l_1}{2}\big) = 0 \quad \text{for } l \text{ even},\\ &\int_{(1-l_1)/2}^{(1+l_1)/2}\cos\big(l\pi y_1\big)dy_1 = \frac{1}{l\pi}\sin\big(l\pi y_1\big)\Big|_{(1-l_1)/2}^{(1+l_1)/2} = \frac{2}{l\pi}\cos\big(\frac{l\pi}{2}\big)\sin\big(\frac{l\pi l_1}{2}\big) \neq 0 \quad \text{for } l \text{ even}.\end{aligned}$$

Similarly for integrals of s_2,c_2,s_3,c_3 on Y_4^*. Using these and that B_{lmn},D_{lmn} are 0 for l,m,n odd

$$\begin{aligned}\int_{Y_4^*}\nabla^y\overline{\psi}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} &= \sum_{l,m,n}^{even}\int_{Y_4^*}\Big\{B_{lmn}c_1c_2c_3, D_{lmn}c_1c_2c_3, 0\Big\}^T\boldsymbol{dy}\\ &= \sum_{l,m,n}^{even}\Big\{B_{lmn}, D_{lmn}, 0\Big\}^T(-1)^{(l+m+n)/2}\frac{8}{lmn\pi^3}\sin\big(\frac{l\pi l_1}{2}\big)\sin\big(\frac{m\pi l_2}{2}\big)\sin\big(\frac{n\pi l_3}{2}\big)\end{aligned} \tag{III.5.24}$$

with other terms becoming zero for l,m,n even. We can note also that the above result is independant of $Y_i^* \in kY^*$, i.e. for any $i \in \{1, 2, \cdots k^3\}$

$$\int_{Y_i^*} \nabla^y \overline{\psi}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y} \equiv \int_{Y_4^*} \nabla^y \overline{\psi}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y} = \frac{1}{k^3} \int_{kY^*} \nabla^y \overline{\psi}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y} = \frac{1}{k^3} \int_{kY^*} \left\{ \partial^{y_1} \overline{\psi}, \partial^{y_2} \overline{\psi}, 0 \right\}^T d\boldsymbol{y}$$

which proves $(b.)$. Almost the same calculation for magnetization $\boldsymbol{w}^* = \{0, 0, m_s\}^T$ gives us that

$$\int_{Y_i^*} \nabla^y \psi^1_{\boldsymbol{w}^*}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y} = \frac{1}{k^3} \int_{kY^*} \left\{ 0, 0, \partial^{y_3} \psi^1_{\boldsymbol{w}^*}(\boldsymbol{y}) \right\}^T d\boldsymbol{y} \tag{III.5.25}$$

where $\psi^1_{\boldsymbol{w}^*}(\boldsymbol{y})$ solves periodic Maxwell for magnetization $\boldsymbol{w}^* = \{0, 0, m_s\}^T$ and is independant of $\boldsymbol{x}$ becasuse $\boldsymbol{w}^*$ is independant of $\boldsymbol{x}$. From Figure III.4 we can write $\widetilde{\beta}(\boldsymbol{x},\boldsymbol{y}) = \overline{\beta}(\boldsymbol{x})(\boldsymbol{\chi}_{Y_1^* \cup Y_5^*} + \boldsymbol{\chi}_{Y_2^* \cup Y_6^*} - \boldsymbol{\chi}_{Y_3^* \cup Y_7^*} - \boldsymbol{\chi}_{Y_4^* \cup Y_8^*})(\boldsymbol{y})$ and similar for $\widetilde{\gamma}(\boldsymbol{x},\boldsymbol{y})$

$$\widetilde{\beta}(\boldsymbol{x},\boldsymbol{y}) = \sum_{l,m,n} P_{lmn}(\boldsymbol{x}) \cos(l\pi y_1)\ \sin(m\pi y_2)\ \cos(n\pi y_3),$$
$$\widetilde{\gamma}(\boldsymbol{x},\boldsymbol{y}) = \sum_{l,m,n} R_{lmn}(\boldsymbol{x}) \sin(l\pi y_1)\ \cos(m\pi y_2)\ \cos(n\pi y_3).$$

Its easy to check using the formula for fourier coeffecients

$$P_{lmn}(\boldsymbol{x}) = \frac{512}{\pi^3 lmn} \overline{\beta}(\boldsymbol{x})\ (-1)^{(l+m+n+3)/2} \sin\Big(\frac{l\pi l_1}{2}\Big) \sin\Big(\frac{m\pi l_2}{2}\Big) \sin\Big(\frac{n\pi l_3}{2}\Big),$$

if (l, n) even and m odd and 0 otherwise. Correspndingly $R_{l,m,n}(\boldsymbol{x})$ is given by

$$R_{lmn}(\boldsymbol{x}) = -\frac{512}{\pi^3 lmn} \overline{\gamma}(\boldsymbol{x})\ (-1)^{(l+m+n+3)/2} \sin\Big(\frac{l\pi l_1}{2}\Big) \sin\Big(\frac{m\pi l_2}{2}\Big) \sin\Big(\frac{n\pi l_3}{2}\Big),$$

if (m, n) even and l odd and 0 otherwise. We look for solution $\widetilde{\psi}(\boldsymbol{x},\boldsymbol{y})$ to periodic Maxwell with magnetization $\{\widetilde{\beta}, \widetilde{\gamma}, 0\}^T(\boldsymbol{x},\boldsymbol{y})$ in the form

$$\widetilde{\psi}(\boldsymbol{x},\boldsymbol{y}) = \sum_{l,m,n} Q_{lmn}(\boldsymbol{x}) \sin(l\pi y_1) \sin(m\pi y_2) \cos(n\pi y_3)$$

and using test functions of the type $\phi(\boldsymbol{y}) = \sin(l\pi y_1)\ \sin(m\pi y_2)\ \cos(n\pi y_3)$ for $(l, m, n) \in \mathbb{N}^3$ and using the orthogonality of sines and cosines we get

$$\pi^2 \left\{ l^2 + m^2 + n^2 \right\} Q_{lmn}(\boldsymbol{x}) = \begin{cases} 4l\pi^2 P_{lmn}(\boldsymbol{x}) & \text{if } (l,n) \text{ is even and } m \text{ odd} \\ 4m\pi^2 R_{lmn}(\boldsymbol{x}) & \text{if } (m,n) \text{ is even and } l \text{ odd,} \end{cases}$$

the energy $\mathscr{E}_{per}(\{\widetilde{\beta},\widetilde{\gamma},0\})$ becomes using Parseval's theorem

$$\begin{aligned}\mathscr{E}_{per}(\{\widetilde{\beta},\widetilde{\gamma},0\}) &= \frac{1}{8\pi}\int_{kY}\left|\nabla^{y}\psi_{\widetilde{\boldsymbol{m}}}(\boldsymbol{x},\boldsymbol{y})\right|^{2}\boldsymbol{dy} = \frac{3\pi^{2}}{64\pi}\sum_{l,m,n}\left(l^{2}+m^{2}+n^{2}\right)\left|Q_{lmn}(\boldsymbol{x})\right|^{2}\\ &= \frac{48\pi^{4}}{64\pi}\sum_{\{(l,n)\ even\}}^{\{m\ odd\}}\frac{l^{2}\left|P_{lmn}(\boldsymbol{x})\right|^{2}}{l^{2}+m^{2}+n^{2}}+\frac{48\pi^{4}}{64\pi}\sum_{\{(m,n)\ even\}}^{\{l\ odd\}}\frac{m^{2}\left|R_{lmn}(\boldsymbol{x})\right|^{2}}{l^{2}+m^{2}+n^{2}}.\\ &= \mathscr{E}_{per}(\{\widetilde{\beta},0,0\})+\mathscr{E}_{per}(\{0,\widetilde{\gamma},0\})\end{aligned}$$

again due to non cross terms between P_{lmn} and R_{lmn} which proves $(c.)$. □

Lemma III.5.2. *For any k even and any $(\overline{\beta}\boldsymbol{\chi}_{k^3Y^*},\overline{\gamma}\boldsymbol{\chi}_{k^3Y^*})\in\overline{\mathscr{B}}^{k}\times\overline{\mathscr{B}}^{k}$, there exists $(\widetilde{\beta},\widetilde{\gamma})\in\widetilde{\mathscr{B}}^{k}\times\widetilde{\mathscr{B}}^{k}$ such that*

$$\int_{\Omega}(|\overline{\beta}|^{2}+|\overline{\gamma}|^{2})\boldsymbol{dx}=\int_{\Omega}(\overline{|\widetilde{\beta}|^{2}}+\overline{|\widetilde{\gamma}|^{2}})\boldsymbol{dx}$$

$$\int_{\Omega}\mathscr{E}_{per}(\{\overline{\beta},\overline{\gamma},0\}\boldsymbol{\chi}_{k^3Y^*})\boldsymbol{dx}\geq\int_{\Omega}\mathscr{E}_{per}(\{\widetilde{\beta},\widetilde{\gamma},0\})\boldsymbol{dx}$$

Proof. Let $k=2$. We will show the result for $k=2$ and for any even $k>2$, use the same construction. We also translate origin of the periodic k^3Y sized block so that it looks like Figure III.5. As before we index the members Y_i with $i\in\{1,2,..12\}$ where with them arranged as in Figure III.5 where the cells numbered $\{7,8,\cdots 12\}$ sit below the ones marked $\{1,2,\cdots 6\}$ and in the same order and the same magnetizations as the ones under which they sit. Note here that Y_i runs with index i from $1,\cdots 12$ and not upto $k^3=8$ because, by the translation of the origin we included 8 half sections of Y. To compare the energy of $\overline{\beta}$ to the proposed $\widetilde{\beta}$ we first use the breakup shown in

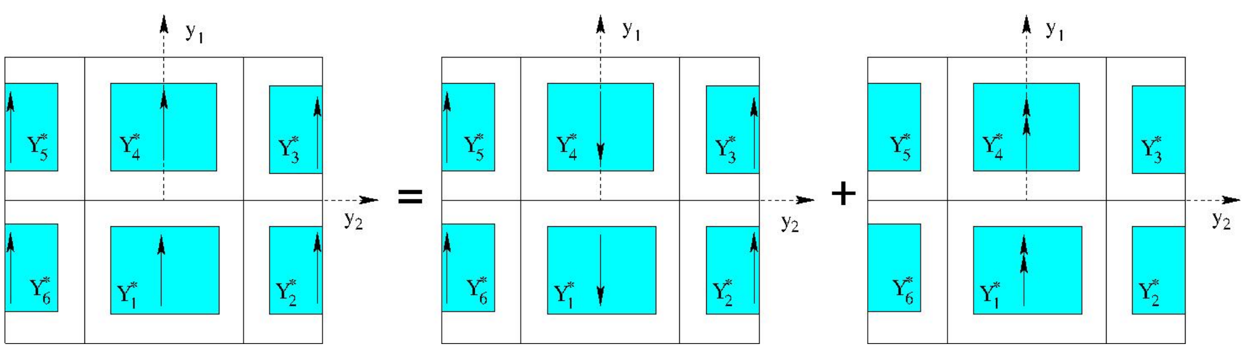

Figure III.5: Decomposition of $\overline{\beta}(\boldsymbol{x})\boldsymbol{\chi}_{kY^*}(\boldsymbol{y})$ as $\overline{\beta}(\boldsymbol{x})\boldsymbol{\chi}_{kY^*}(\boldsymbol{y})=\widetilde{\beta}(\boldsymbol{x},\boldsymbol{y})+\widehat{\beta}(\boldsymbol{x},\boldsymbol{y})$.

Figure III.5 as follows: $\overline{\beta}(\boldsymbol{x})\boldsymbol{\chi}_{kY^*}(\boldsymbol{y})=\widetilde{\beta}(\boldsymbol{x},\boldsymbol{y})+\widehat{\beta}(\boldsymbol{x},\boldsymbol{y})$ where $\widehat{\beta}(\boldsymbol{x},\boldsymbol{y})=\{2\overline{\beta}(\boldsymbol{x})\ \boldsymbol{\chi}_{Y_1^*\cup Y_4^*\cup Y_7^*\cup Y_{10}^*}(\boldsymbol{y}),0,0\}$.

Let $\overline{\psi}$, $\widetilde{\psi}$ and $\widehat{\psi}$ be solutions to the periodic Maxwell problem for magnetizations $\{\overline{\beta}\boldsymbol{\chi}_{kY^*}(\boldsymbol{y}),0,0\}^T$, $\{\widetilde{\beta},0,0\}^T$ and $\{\widehat{\beta},0,0\}^T$ respectively. Then using Lemma III.A.2

$$\begin{aligned}\frac{1}{8\pi}\int_{kY}\left\{\left|\nabla^{\boldsymbol{y}}\overline{\psi}\right|^2-\left|\nabla^{\boldsymbol{y}}\widetilde{\psi}\right|^2\right\}\boldsymbol{dy} &= \frac{1}{8\pi}\int_{kY}\left|\nabla^{\boldsymbol{y}}\widehat{\psi}\right|^2\boldsymbol{dy}-\int_{kY^*}\nabla^{\boldsymbol{y}}\widehat{\psi}\cdot\widetilde{\beta}\boldsymbol{dy}\\ &=-\frac{1}{2}\int_{kY^*}\nabla^{\boldsymbol{y}}\widehat{\psi}\cdot\widehat{\beta}\boldsymbol{dy}-\int_{kY^*}\nabla^{\boldsymbol{y}}\widehat{\psi}\cdot\widetilde{\beta}\boldsymbol{dy}\\ &=-\overline{\beta}(\boldsymbol{x})\int_{Y_1^*\cup Y_4^*\cup Y_7^*\cup Y_{10}^*}\partial^{y_1}\widehat{\psi}\,\boldsymbol{dy}+\overline{\beta}(\boldsymbol{x})\int_{Y_1^*\cup Y_4^*\cup Y_7^*\cup Y_{10}^*}\partial^{y_1}\widehat{\psi}\,\boldsymbol{dy}-\overline{\beta}(\boldsymbol{x})\int_{Y_2^*\cup Y_3^*\cup Y_5^*\cup Y_6^*}\partial^{y_1}\widehat{\psi}\,\boldsymbol{dy}-\overline{\beta}(\boldsymbol{x})\int_{Y_8^*\cup Y_9^*\cup Y_{11}^*\cup Y_{12}^*}\partial^{y_1}\widehat{\psi}\,\boldsymbol{dy}\\ &=-\overline{\beta}(\boldsymbol{x})\int_{Y_2^*\cup Y_3^*\cup Y_5^*\cup Y_6^*}\partial^{y_1}\widehat{\psi}\,\boldsymbol{dy}-\overline{\beta}(\boldsymbol{x})\int_{Y_8^*\cup Y_9^*\cup Y_{11}^*\cup Y_{12}^*}\partial^{y_1}\widehat{\psi}\,\boldsymbol{dy}.\end{aligned}$$

Dividing through by $|kY|$ we get that, for all $\boldsymbol{x}\in\Omega$,

$$\mathscr{E}_{per}\left(\{\overline{\beta},\overline{\gamma},0\}\boldsymbol{\chi}_{kY^*}\right)-\mathscr{E}_{per}\left(\{\widetilde{\beta},\widetilde{\gamma},0\}\right)=-\overline{\beta}(\boldsymbol{x})\left\{\int_{Y_2^*\cup Y_3^*\cup Y_5^*\cup Y_6^*}+\int_{Y_8^*\cup Y_9^*\cup Y_{11}^*\cup Y_{12}^*}\right\}\partial^{y_1}\widehat{\psi}\,\boldsymbol{dy} \tag{III.5.26}$$

We try to show that the L.H.S above is negative which will prove our result. Using the even structure of $\widehat{\beta}(\boldsymbol{x},\boldsymbol{y})$ we write it as a fourier series:

$$\widehat{\beta}(\boldsymbol{x},\boldsymbol{y})=\left\{⨍_{kY}\widehat{\beta}\,\boldsymbol{dy}\right\}(\boldsymbol{x})+\sum_{l,m,n}A_{lmn}(\boldsymbol{x})\cos\left(l\pi y_1\right)\,\cos\left(m\pi y_2\right)\,\cos\left(n\pi y_3\right).$$

Noting the above, we look for solutions $\psi_{\widehat{\beta}}$ such that

$$\widehat{\psi}(\boldsymbol{x},\boldsymbol{y})=\sum_{l,m,n}B_{lmn}(\boldsymbol{x})\sin\left(l\pi y_1\right)\cos\left(m\pi y_2\right)\cos\left(n\pi y_3\right)$$

recalling that the periodic Maxwell problem is solved for $\widehat{\psi}(\boldsymbol{x},\boldsymbol{y})$ such that $⨍_{kY}\widehat{\psi}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy}$. Setting $s_1=\sin\left(l\pi y_1\right)$, $c_1=\cos\left(l\pi y_1\right)$ etc, we get

$$-\nabla^{\boldsymbol{y}}\widehat{\psi}=-\sum_{l,m,n}B_{lmn}(\boldsymbol{x})\left\{l\pi c_1c_2c_3,-m\pi s_1s_2c_3,-n\pi s_1c_2s_3\right\}^T$$

Test function $\phi(\boldsymbol{y})=\sin\left(l\pi y_1\right)\,\cos\left(m\pi y_2\right)\,\cos\left(n\pi y_3\right)$ for $(l,m,n)\in\mathbb{N}^3$ we get

$$\int_{kY}\left(-\nabla^{\boldsymbol{y}}\psi_{\widehat{\beta}}+4\pi(\widehat{\beta}-\widehat{\beta}_o)\right)\cdot\nabla^{\boldsymbol{y}}\phi(\boldsymbol{y})\boldsymbol{dy}=\sum_{l,m,n}\left\{-B_{lmn}(\boldsymbol{x})\pi^2\left(l^2+m^2+n^2\right)+4l\pi^2A_{lmn}(\boldsymbol{x})\right\}=0$$

which gives $B_{lmn}(\boldsymbol{x})=\frac{4l\pi^2A_{lmn}(\boldsymbol{x})}{\pi^2(l^2+m^2+n^2)}$. To determine $A_{lmn}(\boldsymbol{x})$ we use multidimensional fourier cosine forumla to get, $\left\{\text{Note } Y_4^*\cap Y=\left(\frac{1+l_1}{2},\frac{1-l_1}{2}\right)\times\left(0,\frac{l_2}{2}\right)\times\left(\frac{1+l_3}{2},\frac{1-l_3}{2}\right)\right\}$

$$
\begin{aligned}
A_{lmn} &= 8\int_0^1\int_0^1\int_0^1 \widehat{\beta}(\boldsymbol{x})\ \chi_{Y_4^*}(\boldsymbol{y})\ \cos\big(l\pi y_1\big)\ \cos\big(m\pi y_2\big)\ \cos\big(n\pi y_3\big)\boldsymbol{dy} \\
&= \frac{16\,\overline{\beta}(\boldsymbol{x})}{lmn\pi^3}\int_{\frac{1-l_1}{2}}^{\frac{1+l_1}{2}}\int_0^{\frac{l_2}{2}}\int_{\frac{1-l_3}{2}}^{\frac{1+l_3}{2}} d\Big(\sin\big(l\pi y_1\big)\Big)d\Big(\sin\big(m\pi y_2\big)\Big)d\Big(\sin\big(n\pi y_3\big)\Big)\boldsymbol{dy} \\
&= \frac{64\overline{\beta}(\boldsymbol{x})}{\pi^3 lmn}\cos\big(\frac{l\pi}{2}\big)\cos\big(\frac{n\pi}{2}\big)\sin\big(\frac{l\pi l_1}{2}\big)\sin\big(\frac{m\pi l_2}{2}\big)\sin\big(\frac{n\pi l_3}{2}\big) \\
&= \frac{64\overline{\beta}(\boldsymbol{x})}{\pi^3 lmn}(-1)^{\frac{l+n}{2}}\sin\big(\frac{l\pi l_1}{2}\big)\sin\big(\frac{m\pi l_2}{2}\big)\sin\big(\frac{n\pi l_3}{2}\big) \qquad \text{if } (l,n) \text{ even}
\end{aligned}
$$

and 0 otherwise. Then, $\Big\{\text{Noting } Y_3^* = \big(\frac{1+l_1}{2},\frac{1-l_1}{2}\big)\times\big(1-\frac{l_2}{2},1\big)\times\big(\frac{1+l_3}{2},\frac{1-l_3}{2}\big)\Big\}$

$$
\begin{aligned}
\int_{Y_3^*}\partial^{y_1}\widehat{\psi}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} &= \sum_{l\ge1}\sum_{(m,n)even}\int_{Y_3^*} B_{lmn}(\boldsymbol{x})\ l\pi\cos\big(l\pi y_1\big)\ \cos\big(m\pi y_2\big)\ \cos\big(n\pi y_3\big)\boldsymbol{dy} \\
&= \sum_{l\ge1}\sum_{(m,n)even}\frac{l\pi B_{lmn}(\boldsymbol{x})}{lmn\pi^3}\int_{\frac{1-l_1}{2}}^{\frac{1+l_1}{2}}\int_{1-\frac{l_2}{2}}^{1}\int_{\frac{1-l_3}{2}}^{\frac{1+l_3}{2}} d\Big(\sin\big(l\pi y_1\big)\Big)d\Big(\sin\big(m\pi y_2\big)\Big)d\Big(\sin\big(n\pi y_3\big)\Big) \\
&= \sum_{l\ge1}\sum_{(m,n)even}\frac{4l\pi B_{lmn}(\boldsymbol{x})}{lmn\pi^3}\cos\big(\frac{l\pi}{2}\big)\cos(m\pi)\cos\big(\frac{n\pi}{2}\big)\sin\big(\frac{l\pi l_1}{2}\big)\sin\big(\frac{m\pi l_2}{2}\big)\sin\big(\frac{n\pi l_3}{2}\big) \\
&= \sum_{m\ge1}\sum_{(l,n)even}(-1)^m\frac{1024\,\overline{\beta}(\boldsymbol{x})}{\pi^5\big(l^2+m^2+n^2\big)}\frac{1}{m^2n^2}\sin^2\big(\frac{l\pi l_1}{2}\big)\sin^2\big(\frac{m\pi l_2}{2}\big)\sin^2\big(\frac{n\pi l_3}{2}\big)
\end{aligned}
$$

where in the last step we substitute for B_{lmn}. It is also clear that

$$
\int_{Y_i^*}\partial^{y_1}\widehat{\psi}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} = \int_{Y_3^*}\partial^{y_1}\widehat{\psi}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{dy} \qquad \text{for } i\in\{2,5,6,8,9,11,12\}
$$

Then substituting this result in equation (III.5.26) gives

$$
\begin{aligned}
&\mathscr{E}_{per}\big(\{\overline{\beta},\overline{\gamma},0\}\chi_{kY^*}\big)-\mathscr{E}_{per}\big(\{\widetilde{\beta},\widetilde{\gamma},0\}\big) = -8\overline{\beta}(\boldsymbol{x})\Big\{\int_{Y_3^*}\partial^{y_1}\widehat{\psi}\,\boldsymbol{dy}\Big\} \\
&= \sum_{m\ge1}\sum_{(l,n)even}(-1)^{(m+1)}\frac{8192\,\big|\overline{\beta}(\boldsymbol{x})\big|^2}{\pi^5\big(l^2+m^2+n^2\big)}\frac{1}{m^2n^2}\sin^2\big(\frac{l\pi l_1}{2}\big)\sin^2\big(\frac{m\pi l_2}{2}\big)\sin^2\big(\frac{n\pi l_3}{2}\big) \\
&= \sum_{(l,n)even}\frac{8192\,\big|\overline{\beta}(\boldsymbol{x})\big|^2}{n^2\pi^5}\sin^2\big(\frac{l\pi l_1}{2}\big)\sin^2\big(\frac{n\pi l_3}{2}\big)\Big\{\sum_{m\ge1}\frac{(-1)^{(m+1)}\sin^2\big(\frac{m\pi l_2}{2}\big)}{m^2\big(l^2+m^2+n^2\big)}\Big\}
\end{aligned}
$$

and let $G_m(l,n) = \frac{(-1)^{(m+1)}\sin^2\left(\frac{m\pi l_2}{2}\right)}{m^2\left(l^2+m^2+n^2\right)} = \frac{(-1)^{(m+1)}\left(1-\cos(m\pi l_2)\right)}{2m^2\left(l^2+m^2+n^2\right)}$. We make use of two formulas numbered

1.443.4 and 1.445.3 from [Gradshteyn et al., 2000] and get

$$\sum_{m=1}^{\infty}(-1)^{(m+1)}\frac{\cos(mx)}{m^2}=\frac{\pi^2}{12}-\frac{x^2}{4}, \qquad \sum_{m=1}^{\infty}\frac{(-1)^{(m+1)}}{m^2}=\frac{\pi^2}{12},$$
$$\sum_{m=1}^{\infty}(-1)^{(m+1)}\frac{\cos(mx)}{m^2+a^2}=\frac{1}{2a^2}-\frac{\pi}{2a}\frac{\cosh\left(ax\right)}{\sinh\left(a\pi\right)}, \qquad \sum_{m=1}^{\infty}\frac{(-1)^{(m+1)}}{m^2+a^2}=\frac{1}{2a^2}-\frac{\pi}{2a\sinh\left(a\pi\right)}.$$

Then setting $a^2=l^2+n^2$ and noting $\cosh(a\pi l_2)\le\cosh(a\pi)$ we get

$$\begin{aligned}\sum_{m\ge1}G_m(l,n)&=\sum_{m\ge1}\frac{(-1)^{(m+1)}\left(1-\cos(m\pi l_2)\right)}{2\left(l^2+n^2\right)}\left\{\frac{1}{m^2}-\frac{1}{\left(l^2+m^2+n^2\right)}\right\}\\&=\frac{1}{2a^2}\left\{\frac{\pi^2 l_2^2}{4}+\frac{\pi\left(1-\cosh(a\pi l_2)\right)}{2a\sinh(a\pi)}\right\}=\frac{\pi^2 l_2^2}{8a^2\sinh(a\pi)}\left\{\sum_{k=1}^{\infty}\frac{(a\pi)^{2k-1}}{2k-1!}-\frac{2}{\pi a l_2^2}\sum_{k=1}^{\infty}\frac{(a\pi l_2)^{2k}}{2k!}\right\}\\&=\frac{\pi^2 l_2^2}{8a^2\sinh(a\pi)}\left\{\sum_{k=1}^{\infty}\frac{(a\pi)^{2k-1}}{2k-1!}\left(1-\frac{l_2^{2(k-1)}}{k}\right)\right\}\ge0\end{aligned}$$

as $l_2\le1$. Thus we get $\mathscr{E}_{per}\left(\{\overline{\beta},0,0\}\chi_{kY^*}\right)\ge\mathscr{E}_{per}\left(\{\widetilde{\beta},0,0\}\right)$. A similar calculation for $\overline{\gamma}$ gives $\mathscr{E}_{per}\left(\{0,\overline{\gamma},0\}\chi_{kY^*}\right)\ge\mathscr{E}_{per}\left(\{0,\widetilde{\gamma},0\}\right)$. Using the previous lemma III.5.1 we get

$$\begin{aligned}\mathscr{E}_{per}\left(\{\overline{\beta},\overline{\gamma},0\}\chi_{kY^*}\right)&=\mathscr{E}_{per}\left(\{\overline{\beta},0,0\}\chi_{kY^*}\right)+\mathscr{E}_{per}\left(\{0,\overline{\gamma},0\}\chi_{kY^*}\right)\\&\ge\mathscr{E}_{per}\left(\{\widetilde{\beta},0,0\}\right)+\mathscr{E}_{per}\left(\{0,\widetilde{\gamma},0\}\right)=\mathscr{E}_{per}\left(\{\widetilde{\beta},\widetilde{\gamma},0\}\right).\end{aligned}$$

Hence we get our result. □

III.6 Summary

We showed in the preceding section that the critical field for our homogenized model is given in equation (III.5.22) as

$$\frac{H_a^{cr}}{m_s}=\theta N_z-2\Pi_1-\left(\fint_{kY^*}\partial^{y_3}\psi^1_{\boldsymbol{w}^*}(\boldsymbol{y})\,\boldsymbol{dy}\right)-2\inf_{(\widetilde{\beta},\widetilde{\gamma})\in(\mathscr{B}^k)^2}\frac{\int_\Omega\mathscr{E}_{per}\left(\{\widetilde{\beta},\widetilde{\gamma},0\}\right)\boldsymbol{dx}}{\int_\Omega\theta\left(\,\left|\overline{\widetilde{\beta}}\right|^2+\left|\overline{\widetilde{\gamma}}\right|^2\,\right)\boldsymbol{dx}}.$$

This expression for the critical field H_a^{cr} is a generalization of the "Fanning mode" for a "Chain-of-spheres" model. This is a well known model in the magnetics literature for a linear infinite chain of magnetic spheres which was first proposed in [Jacobs and Bean, 1955] to explain the low coercivity of elongated magnetic particles. The model has become very useful in recent years with the development of magnetic structures like nanochains, magnetic nanodots and magnetic nanowires. To recover the fanning model we first try and use the form of the critical field H_a^{cr}. First we ignore the fact that the above result was shown for a ellipsoid with finely distributed periodic magnetic domains and assume it is true for a linear chain of fine magnetic domains. Then if we compute H_a^{cr} for $k=2$, we first note that $\beta=\beta_1$ on Y_1^* and $\beta=\beta_2$ on Y_2^* and analogous for γ. But the condition that

$$\fint_{2Y^*} \beta\, \boldsymbol{dx} = \left|Y^*\right| (\beta_1+\beta_2) = 0 \qquad\qquad \Rightarrow \beta_1 = -\beta_2.$$

Similarly $\gamma_1 = -\gamma_2$. This mode is exactly the form of the symmetric fanning mode for the chain-of-spheres model. M vs H curves in physical experiments for such geometries also typically reveal that this average condition for out of plan magnetization continues to be zero even beyond the critical field when the staturated state looses stability.

III.A Magnetostatic calculations

Lemma III.A.1. *Let $\boldsymbol{m}$ and $\boldsymbol{M}$ be any two vectors in $L^\infty(\Omega)$. Let $\boldsymbol{h_m}$ and $\boldsymbol{h_M}$ be the corresponding demag fields. Then following inequality holds:*

$$\left| \|\boldsymbol{h_m}\|^2_{L^2(\mathbb{R}^3)} - \|\boldsymbol{h_M}\|^2_{L^2(\mathbb{R}^3)} \right| \leq 4\pi \|\boldsymbol{m}-\boldsymbol{M}\|_{L^2(\Omega)} \left(\|\boldsymbol{m}\|_{L^2(\Omega)} + \|\boldsymbol{M}\|_{L^2(\Omega)} \right). \tag{III.A.1}$$

Proof. By linearity of Maxwell's equation we know $(\boldsymbol{h_m} - \boldsymbol{h_M})$ satisfies Maxwell's equation for $(\boldsymbol{m}-\boldsymbol{M})$. Thus using basic bound (III.2.4) for Maxwell's equation we have,

$$\frac{1}{8\pi}\|\boldsymbol{h_m} - \boldsymbol{h_M}\|^2_{L^2(\mathbb{R}^3)} \leq \frac{1}{2}\|\boldsymbol{m}-\boldsymbol{M}\|^2_{L^2(\Omega)}.$$

Using triangle inequality we also have,

$$| \, \|\boldsymbol{h_m}\|_{L^2(\mathbb{R}^3)} - \|\boldsymbol{h_M}\|_{L^2(\mathbb{R}^3)} \, | \leq \|\boldsymbol{h_m} - \boldsymbol{h_M}\|_{L^2(\mathbb{R}^3)} \leq 2\sqrt{\pi}\|\boldsymbol{m}-\boldsymbol{M}\|_{L^2(\Omega)}.$$

Using (III.2.4) for $\boldsymbol{m}$ and $\boldsymbol{M}$ separately we have,

$$| \, \|\boldsymbol{h_M}\|_{L^2(\mathbb{R}^3)} + \|\boldsymbol{h_m}\|_{L^2(\mathbb{R}^3)} \, | \leq 2\sqrt{\pi} \left(\|\boldsymbol{m}\|_{L^2(\Omega)} + \|\boldsymbol{M}\|_{L^2(\Omega)} \right) = 2\sqrt{\pi} m_s |\Omega|^{1/2}.$$

Thus,

$$| \, \|\boldsymbol{h_m}\|^2_{L^2(\mathbb{R}^3)} - \|\boldsymbol{h_M}\|^2_{L^2(\mathbb{R}^3)} \, | \leq 4\pi \|\boldsymbol{m}-\boldsymbol{M}\|_{L^2(\Omega)} \left(\|\boldsymbol{m}\|_{L^2(\Omega)} + \|\boldsymbol{M}\|_{L^2(\Omega)} \right).$$ □

We now show a Lemma, which is popularly known as the Reciprocity Theorem in micromagnetics literature.

Lemma III.A.2.
(1) Let $\boldsymbol{h}_{\boldsymbol{m}_1}(\boldsymbol{x})$ and $\boldsymbol{h}_{\boldsymbol{m}_2}(\boldsymbol{x})$ be the demagnetization field corresponding to magnetizations $\boldsymbol{m}_1(\boldsymbol{x})$ and $\boldsymbol{m}_2(\boldsymbol{x})$ on Ω. Then we have

$$\frac{1}{4\pi}\int_{\mathbb{R}^3} \boldsymbol{h}_{\boldsymbol{m}_1}(\boldsymbol{x}) \cdot \boldsymbol{h}_{\boldsymbol{m}_2}(\boldsymbol{x}) \boldsymbol{dx} = -\int_\Omega \boldsymbol{h}_{\boldsymbol{m}_1}(\boldsymbol{x}) \cdot \boldsymbol{m}_2(\boldsymbol{x}) \boldsymbol{dx} = -\int_\Omega \boldsymbol{m}_1(\boldsymbol{x}) \cdot \boldsymbol{h}_{\boldsymbol{m}_2}(\boldsymbol{x}) \boldsymbol{dx}.$$

(2) Let $\psi_{\boldsymbol{m}_1}(\boldsymbol{x},\boldsymbol{y})$ and $\psi_{\boldsymbol{m}_2}(\boldsymbol{x},\boldsymbol{y})$ be the solution to the periodic Maxwell's equation for $\boldsymbol{m}_1(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{m}_2(\boldsymbol{x},\boldsymbol{y})$ belonging to $L^2(\Omega; L^2_\sharp(k^3Y))$. Then for all $\boldsymbol{x} \in \Omega$,

$$\frac{1}{4\pi}⨍_{k^3Y} \nabla\psi_{\boldsymbol{m}_1} \cdot \nabla\psi_{\boldsymbol{m}_2} \boldsymbol{dy} = -⨍_{k^3Y} \nabla\psi_{\boldsymbol{m}_1}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{m}_2(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{dy} = -⨍_{k^3Y} \boldsymbol{m}_1(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla\psi_{\boldsymbol{m}_2}(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{dy}.$$

Proof. Writing Maxwell's equation for $\boldsymbol{m}_1(\boldsymbol{x})$ in distributional form, we have for all $\zeta \in C_0^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \boldsymbol{h}_{\boldsymbol{m}_1}(\boldsymbol{x}) \cdot \nabla\zeta(\boldsymbol{x})\, \boldsymbol{dx} = -\int_{\mathbb{R}^3} 4\pi\ \boldsymbol{m}_1(\boldsymbol{x}) \cdot \nabla\zeta(\boldsymbol{x})\, \boldsymbol{dx}.$$

Choosing a sequence $\zeta_i(\boldsymbol{x})$ for $i \in \mathbb{N}$ so that $\nabla\zeta_i(\boldsymbol{x}) \to \boldsymbol{h}_{\boldsymbol{m}_2}(\boldsymbol{x})$ we have on taking $\lim_i$

$$\int_{\mathbb{R}^3} \boldsymbol{h}_{\boldsymbol{m}_1}(\boldsymbol{x}) \cdot \boldsymbol{h}_{\boldsymbol{m}_2}(\boldsymbol{x})\, \boldsymbol{dy} = -4\pi \int_{\mathbb{R}^3} \boldsymbol{m}_1(\boldsymbol{x}) \cdot \boldsymbol{h}_{\boldsymbol{m}_2}(\boldsymbol{x})\, \boldsymbol{dy}.$$

The other part of result (1) can be proved the same way. For the periodic Maxwell's equation for $\boldsymbol{m}_1(\boldsymbol{x},\boldsymbol{y})$ in distributional form, we have for all $\varphi(\boldsymbol{y}) \in C_\sharp^\infty(k^3 Y)$

$$⨍_{k^3Y} \nabla^y \psi_{\boldsymbol{m}_1}(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla^y \varphi(\boldsymbol{y})\, \boldsymbol{dy} = -⨍_{k^3Y} 4\pi\ \boldsymbol{m}_1(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla^y \varphi(\boldsymbol{y})\, \boldsymbol{dy}.$$

Choosing a sequence $\varphi_i(\boldsymbol{y})$ for $i \in \mathbb{N}$ so that for fixed $\boldsymbol{x} \in \Omega$, $\lim_i \nabla^y \varphi_i(\boldsymbol{y}) \to \nabla^y \psi_{\boldsymbol{m}_2}(\boldsymbol{x},\boldsymbol{y})$ we have on taking $\lim_i$

$$⨍_{k^3Y} \nabla^y \psi_{\boldsymbol{m}_1}(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla^y \psi_{\boldsymbol{m}_2}(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{dy} = -⨍_{k^3Y} 4\pi\ \boldsymbol{m}_1(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla^y \psi_{\boldsymbol{m}_2}(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{dy}.$$

The remaining part of (2) can be proved analogously. □

Lemma III.A.3. *Let $\boldsymbol{m}^h$ be a magnetization which converges weakly to $\boldsymbol{m}^o$ i.e. $\boldsymbol{m}^h \rightharpoonup \boldsymbol{m}^o$ in $L^2(\Omega, m_s S^2)$. Let $\eta_\sharp^h \in \mathscr{W}_\sharp^{1,2}(Y)$ minimize the energy*

$$\mathscr{E}^h(\eta) = \int_Y \left(\frac{1}{2}|\nabla\eta|^2 - 4\pi(\boldsymbol{m}^h - \boldsymbol{m}^o) \cdot \nabla\eta \right) \boldsymbol{dx}$$

with periodic boundary conditions. Then

$$\liminf_{h\to\infty} \mathscr{E}_d(\boldsymbol{m}^h) \geq \mathscr{E}_d(\boldsymbol{m}^o) + \liminf_{h\to\infty} \mathscr{E}^h(\eta_\sharp^h).$$

Proof. First note that if $\psi_{\boldsymbol{m}^h}$ solve Maxwell's equation for $\boldsymbol{m}^h$ and $\psi_{\boldsymbol{m}^o}$ solve the same for $\boldsymbol{m}^o$. Then we already have $\psi_{\boldsymbol{m}^h} \rightharpoonup \psi_{\boldsymbol{m}^o}$ in $W^{1,2}(\mathbb{R}^3)$. Next taking first variation on $\mathscr{E}^h(\eta)$ we get $\forall \phi \in \mathscr{W}_\sharp^{1,2}(Y)$

$$\int_D \left(\nabla\eta_\sharp^h - 4\pi(\boldsymbol{m}^h - \boldsymbol{m}^o) \right) \cdot\ \nabla\phi\, \boldsymbol{dx} = 0, \qquad \mathscr{E}^h(\eta_\sharp^h) = -\frac{1}{2}\int_Y |\eta_\sharp^h|^2 \boldsymbol{dx}$$

where the second equality above comes by choosing $\phi = \eta_\sharp^h$. Next by choosing $\phi \in \mathscr{W}_\sharp^{1,2}(Y)$ independant of h and taking $\lim_h$ we get $\nabla\eta_\sharp^h \rightharpoonup \boldsymbol{0}$ in $L^2(Y)$ and as a result $\nabla\eta_\sharp^h$ is bounded, i.e.

$\|\nabla\eta_\sharp^h\|_{L^2(Y)} \le C$. Then Poincare inequality and the Rellich compactness theorem together give

$$\|\eta_\sharp^h\|_{L^2(Y)} \le C_1\|\nabla\eta_\sharp^h\|_{L^2(Y)} \le C_2, \qquad \eta_\sharp^h \rightharpoonup 0 \text{ in } H^1(Y), \qquad \eta_\sharp^h \to 0 \text{ in } L^2(Y).$$

All three magnetizations $\boldsymbol{m}^h, \boldsymbol{m}^o, (\boldsymbol{m}^h - \boldsymbol{m}^o)$ being $L^\infty(Y)$ functions, by standard regularity we get the estimate

$$\|\nabla\psi_{\boldsymbol{m}^h}\|_{L^p(Y)} \le C_p\|\boldsymbol{m}^h\|_{L^p(Y)} \le D_p, \qquad \|\nabla\psi_{\boldsymbol{m}^o}\|_{L^p(Y)} \le C_p\|\boldsymbol{m}^o\|_{L^p(Y)} \le D_p,$$
$$\|\nabla\eta_\sharp^h\|_{L^p(Y)} \le C_p\|\boldsymbol{m}^h - \boldsymbol{m}^o\|_{L^p(Y)} \le D_p \quad \forall 1 \le p < \infty.$$

Thus $\nabla\psi_{\boldsymbol{m}^h}, \nabla\psi_{\boldsymbol{m}^o}, \nabla\eta_\sharp^h$ are L^2-equi-integrable.

Let ϕ^δ be a cut-off function such that $\phi^\delta = 1$ when $\text{dist}(x,\partial Y) \ge \delta$, $\phi^\delta = 0$ on ∂Y and $|\nabla\phi^\delta| \le K\delta^{-1}$. Then $\eta_\sharp^h(\phi^\delta - 1) \in H_0^1(Y)$ and $|\nabla(\eta_\sharp^h(\phi^\delta - 1))| \le |(\phi^\delta - 1)\nabla\eta_\sharp^h| + |\nabla\phi^\delta\eta_\sharp^h| \le 2|\nabla\eta_\sharp^h| + K\delta^{-1}|\eta_\sharp^h|$. Then, (Recalling that $\eta_\sharp^h \to 0$ in $L^2(Y)$)

$$\begin{aligned}\liminf_{h\to\infty}\int_Y |\nabla(\eta_\sharp^h(\phi^\delta - 1))|^2 \boldsymbol{dx} &\le \liminf_{h\to\infty}\int_{Y/Y_\delta}\left\{2|\nabla\eta_\sharp^h|^2 + \frac{K^2}{\delta^2}|\eta_\sharp^h|^2\right\}\boldsymbol{\chi}_{Y/Y_\delta}\boldsymbol{dx} \\ &\le 2\liminf_{h\to\infty}\left\{\int_{Y/Y_\delta}|\nabla\eta_\sharp^h|^4\boldsymbol{dx}\right\}^{1/2}\left\{\int_{Y/Y_\delta}|\chi_{Y/Y_\delta}|^2\boldsymbol{dx}\right\}^{1/2} \le D_4^2\,|Y/Y_\delta|. \end{aligned} \tag{III.A.2}$$

If f^h be uniformly bounded in $L^2(Y)$ with $\|f^h\|_{L^2(Y)} \le C$ using Hölder's inequality,

$$\begin{aligned}\liminf_{h\to\infty}\int_Y f^h\nabla(\eta_\sharp^h(\phi^\delta - 1))|^2\boldsymbol{dx} &\le \liminf_{h\to\infty}\left\{\int_Y |f^h|^2\boldsymbol{dx}\right\}^{1/2}\left\{\int_{Y/Y_\delta}|\nabla(\eta_\sharp^h(\phi^\delta - 1))|^2|^4\boldsymbol{dx}\right\}^{1/2} \\ &\le CD_4\,|Y/Y_\delta|^{1/2} = K_1\,|Y/Y_\delta|^{1/2}. \end{aligned} \tag{III.A.3}$$

Using $(\psi_{\boldsymbol{m}^o} + \eta_\sharp^h\phi^\delta) = (\psi_{\boldsymbol{m}^o} + \eta_\sharp^h(\phi^\delta - 1) + \eta_\sharp^h\phi^\delta)$ as a test function for $\boldsymbol{m}^h$ we get,

$$\begin{aligned}-4\pi\mathscr{E}_d(\boldsymbol{m}^h) &= \inf_{\psi\in W^{1,2}(\mathbb{R}^3)}\int_{\mathbb{R}^3}\left\{\frac{1}{2}|\nabla\psi|^2 - 4\pi\,\boldsymbol{m}^h\cdot\nabla\psi)\right\}\boldsymbol{dx} \\ &\le \int_{\mathbb{R}^3}\left\{\frac{1}{2}|\nabla(\psi_{\boldsymbol{m}^o} + \eta_\sharp^h\phi^\delta)|^2 - 4\pi\,\boldsymbol{m}^h\cdot\nabla(\psi_{\boldsymbol{m}^o} + \eta_\sharp^h\phi^\delta)\right\}\boldsymbol{dx}. \end{aligned}$$

Taking negative of above relation, $\liminf_h$ of both sides and using estimates (III.A.2), (III.A.3),

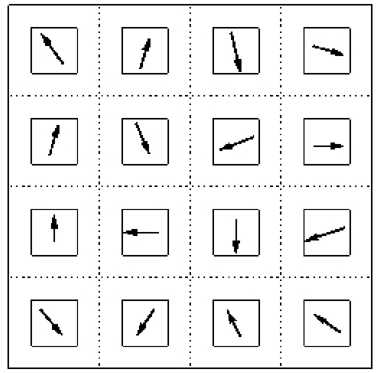

Figure III.6: Example of magnetization for $k = 4$

$\eta_\sharp^h \to 0$ in $L^2(Y)$ and $\nabla\eta_\sharp^h \rightharpoonup 0$ in $L^2(Y)$ gives

$$\liminf_{h\to\infty} 4\pi\mathscr{E}_d(\boldsymbol{m}^h) \geq \liminf_{h\to\infty}\Big[\int_{\mathbb{R}^3}\Big\{4\pi\boldsymbol{m}^o\cdot\nabla\psi_{\boldsymbol{m}^o} - \frac{|\nabla\psi_{\boldsymbol{m}^o}|^2}{2}\Big\} + \Big\{4\pi\big(\boldsymbol{m}^h - \boldsymbol{m}^o\big)\cdot\nabla\eta_\sharp^h - \frac{|\nabla\eta_\sharp^h|^2}{2}\Big\}\Big]$$
$$- K_1\big|Y/Y_\delta\big| - K_2\big|Y/Y_\delta\big|^{1/2}.$$

Taking next a limit as $\delta\to 0$, we get our result. Note because of the equi-integrability of the fields $\nabla\psi_{\boldsymbol{m}^h}, \nabla\psi_{\boldsymbol{m}^o}, \nabla\eta_\sharp^h$, we avoid the more complicated De Giorgi slicing argument to match boundary values. □

Let $\widetilde{h^{-1}Y^*}$ be a periodic extension of $h^{-1}Y^*$ in $\mathbb{R}^3$. Note $\big(Y\bigcap\widetilde{h^{-1}Y^*}\big)$ consists of h^3 segments which we index by W^i with $i\in\{1,2,3,\cdots h^3\}$. We now recall and prove Lemma III.4.3.

Lemma III.4.3 Let $\boldsymbol{v}^h(\boldsymbol{y})$ be a sequence in $L^2(Y)$ with support in $Y\bigcap\widetilde{h^{-1}Y^*}$ and $\boldsymbol{v}^h$ be a constant vector on each W^i in $\big(Y\bigcap\widetilde{h^{-1}Y^*}\big)$, $i\in\{1,2,3,\cdots h^3\}$ taking values in m_sS^2. Let $\boldsymbol{v}^h \rightharpoonup \boldsymbol{\alpha}^o$ in $L^2(Y)$. Then exists a sequence $\boldsymbol{w}^h$ in $L^2_\sharp(Y\bigcap\widetilde{h^{-1}Y^*}; m_sS^2)$ with $\boldsymbol{w}^h$ constant on each W^i in $\big(Y\bigcap\widetilde{h^{-1}Y^*}\big)$, $i\in\{1,2,3,\cdots h^3\}$ and $\int_Y \boldsymbol{w}^h(\boldsymbol{y})\,\boldsymbol{dy} = \int_Y \boldsymbol{\alpha}^o(\boldsymbol{y})\,\boldsymbol{dy}$ and $\boldsymbol{v}^h - \boldsymbol{w}^h \to 0$ as $h\to\infty$.

Figure III.6 is a typical example of $\boldsymbol{v}^h$ for $h = 4$. To prove Lemma III.4.3 we first show this simple result.

Lemma III.A.4. *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be two vectors on m_sS^2 such that $p_1 \geq q_1 > 0$. Then there exist two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in m_sS^2 such that (u_1+v_1) takes up any value between (p_1-q_1, p_1+q_1), with $(u_2+v_2) = (p_2+q_2)$ and $(u_3+v_3) = (p_3+q_3)$.*

Proof. Let us reorient the x_2 and x_3 axis in such a way that we can write $\boldsymbol{p} = \{p_1, p_2, 0\}$ with $p_2 \geq 0$. In these coordinates let $\boldsymbol{q} = \{q_1, q_2, q_3\}$ where again without loss of generality we can assume $q_3 \geq 0$.

For $t \in \mathbb{R}$ with $0 \leq t \leq 2q_1$, set $\delta(t) = \sqrt{q_3^2 + 2q_1 t - t^2} - q_3$. Note $\delta(0) = 0$ and $\delta(2q_1) = \sqrt{q_3^2 + 4q_1^2 - 4q_1^2} - q_3 = 0$ and its easy to check that $\delta(t) \geq 0$ for all $0 \leq t \leq 2q_1$. Define $f(t)$ as the continuous function of t given by

$$f(t) = \left\{\sqrt{p_1^2 - |\delta(t)|^2} + (q_1 - t)\right\}.$$

Then note that

$$f(t=0) = \left\{\sqrt{p_1^2 - |\delta(0)|^2} + (q_1 - t)\right\} = p_1 + q_1,$$
$$f(t=2q_1) = \left\{\sqrt{p_1^2 - |\delta(2q_1)|^2} + (q_1 - t)\right\} = p_1 - q_1.$$

Thus for any value $r \in (p_1 - q_1, p_1 + q_1)$, there exists by continuity t_0 such that $f(t_0) = r$.

Then define two vectors on $m_s S^2$ as $\boldsymbol{u} := \left\{\sqrt{p_1^2 - \delta(t_0)^2}, p_2, -\delta(t_0)\right\}$ and $\boldsymbol{v} := \{q_1 - t_0, q_2, q_3 + \delta(t_0)\}$ and note that

$$(u_2 + v_2) = (p_2 + q_2), \qquad \text{and} \qquad (u_3 + v_3) = \delta(t_0) + q_3 + \delta(t_0) = q_3 = (p_3 + q_3)$$

which completes our proof. If $q_3 < 0$ then we need to define $\delta(t) = \sqrt{q_3^2 + 2q_1 t - t^2} + q_3$, $f(t)$ the same as before and $\boldsymbol{u} := \left\{\sqrt{p_1^2 - \delta(t_0)^2}, p_2, +\delta(t_0)\right\}$ and $\boldsymbol{v} := \{q_1 - t_0, q_2, q_3 - \delta(t_0)\}$ to get our proof.□

Proof of Lemma 4.3. Let $\int_Y \boldsymbol{u}^h(\boldsymbol{y})\, \boldsymbol{dy} = \boldsymbol{\beta}^h$ and note as $\boldsymbol{u}^h \rightharpoonup \boldsymbol{\alpha}^o$ which means

$$\boldsymbol{\beta}^h = \int_Y \boldsymbol{u}^h(\boldsymbol{y})\, \boldsymbol{dy} \longrightarrow \int_Y \boldsymbol{\alpha}^o(\boldsymbol{y})\, \boldsymbol{dy} = \boldsymbol{\beta}^o. \tag{III.A.5}$$

We will explicitly construct the sequence $\boldsymbol{w}^h$.

Step. (1)

If $\boldsymbol{\beta}^o = \boldsymbol{0}$, then move to Step.(2). Else assume without loss of generality $\boldsymbol{\beta}^o = (\beta_1^o, 0, 0)$ is parallel to the X-axis and $\beta_1^o \geq 0$. If $\boldsymbol{\beta}^h$ is also parallel to the X-axis, then move to Step (2).

Let $\cos(\theta^h) = \frac{\boldsymbol{\beta}^h \cdot \boldsymbol{\beta}^o}{|\boldsymbol{\beta}^h||\boldsymbol{\beta}^o|} \neq 1$, i.e. $\boldsymbol{\beta}^h$ is not in the same direction as $\boldsymbol{\beta}^o$. Without loss of generality again we can assume $\boldsymbol{\beta}^h$ and $\boldsymbol{\beta}^o$ are in $X-Y$ plane and for $h \geq H$ for some H large enough $|\boldsymbol{\beta}^h| \geq \frac{1}{2}|\boldsymbol{\beta}^o|$. If $\boldsymbol{R}(\theta_h) \in SO(3)$ is a rotation with Z-axis of angle θ^h, it rotates $\boldsymbol{\beta}^h$ in the direction

$\boldsymbol{\beta}^o$, i.e. X-axis and it is easy to check that

$$\|\boldsymbol{R}(\theta_h)\boldsymbol{u}^h - \boldsymbol{u}^h\|^2_{L^2(Y)} \le 2\big(1-\cos(\theta_h)\big)\|\boldsymbol{v}^h\|^2_{L^2(Y)} \le m_s^2\ |Y|\big(1-\cos(\theta_h)\big).$$

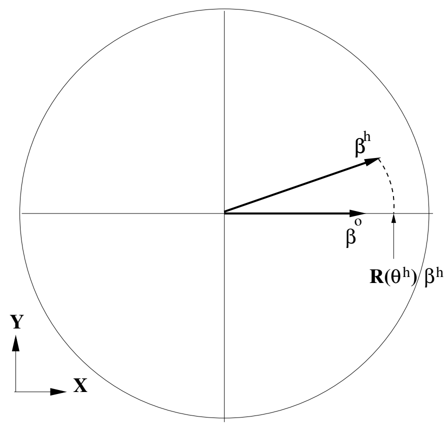

Figure III.7: Rotation of magnetizations by $\boldsymbol{R}(\theta_h)$ causing rotation of average magnetization $\boldsymbol{\beta}^o$

Also note that using the fact that $|\boldsymbol{\beta}^o|^2 + |\boldsymbol{\beta}^h|^2 \le 2|\boldsymbol{\beta}^h|\cdot|\boldsymbol{\beta}^o|$ we get

$$\big(1-\cos(\theta_h)\big) = \Big(1 - \frac{\boldsymbol{\beta}^h\cdot\boldsymbol{\beta}^o}{|\boldsymbol{\beta}^h||\boldsymbol{\beta}^o|}\Big) \le \frac{|\boldsymbol{\beta}^o|^2 + |\boldsymbol{\beta}^h|^2 - 2\boldsymbol{\beta}^h\cdot\boldsymbol{\beta}^o}{2|\boldsymbol{\beta}^h||\boldsymbol{\beta}^o|} = \frac{|\boldsymbol{\beta}^o - \boldsymbol{\beta}^h|^2}{|\boldsymbol{\beta}^h||\boldsymbol{\beta}^o|} \le C|\boldsymbol{\beta}^o - \boldsymbol{\beta}^h|^2$$

where C is independant of h as $|\boldsymbol{\beta}^h| \ge \frac{1}{2}|\boldsymbol{\beta}^o|$. Then setting $\boldsymbol{v}^h := \boldsymbol{R}(\theta_h)\boldsymbol{u}^h$ we get

$$\|\boldsymbol{v}^h - \boldsymbol{u}^h\|^2_{L^2(Y)} = \|\boldsymbol{R}(\theta_h)\boldsymbol{u}^h - \boldsymbol{u}^h\|^2_{L^2(Y)} \le m_s^2\ |Y|\big(1-\cos(\theta_h)\big) = Cm_s^2\ |Y|\ |\boldsymbol{\beta}^o - \boldsymbol{\beta}^h|^2 \xrightarrow{h\to\infty} 0. \quad \text{(III.A.6)}$$

Henceforth we assume that w.l.o.g that $\boldsymbol{\beta}^o = (\beta_1^o, 0, 0)$ and $\boldsymbol{\beta}^h = (\beta_1^h, 0, 0)$ are both along X-axis.
Step. (2) : Let the h^3 values that $\boldsymbol{v}^h$ takes, be enumerated as $\boldsymbol{v}^{h,(i)}$ with support $\mathrm{supp}(\boldsymbol{v}^{h,(i)}) = W^i$ and $i \in \{1,2,3\cdots h^3\}$. As $\boldsymbol{v}^{h,(i)}$ is constant on W^i we have

$$\int_Y v_1^h \boldsymbol{dy} = \sum_{i=1}^{h^3}\int_{W^i} v_1^{h,(i)}\boldsymbol{dy} = \sum_{i=1}^{h^3}|W^i| v_1^{h,(i)} = \Big|\frac{Y^*}{h^3}\Big|\ \sum_{i=1}^{h^3} v_1^{h,(i)}$$

$$\int_Y |v_1^h|^2\boldsymbol{dy} = \Big|\frac{Y^*}{h^3}\Big|\ \sum_{i=1}^{h^3}|v_1^{h,(i)}|^2 \quad \text{and} \quad \Big|\int_Y v_1^h\boldsymbol{dy}\Big|^2 = \Big|\frac{Y^*}{h^3}\Big|^2\ \Big(\sum_{i=1}^{h^3} v_1^{h,(i)}\Big)^2. \quad \text{(III.A.7)}$$

We renumber the index i so that the h^3 values can be split up into sets, $\mathcal{M}(\boldsymbol{v}^h) \cup \mathcal{N}(\boldsymbol{v}^h)$ where

$$i \in \begin{cases} \mathcal{M}(\boldsymbol{v}^h) & \text{if } v_1^{h,(i)} \geq 0, \\ \mathcal{N}(\boldsymbol{v}^h) & \text{if } v_1^{h,(i)} < 0 \end{cases} \qquad \text{and } \boldsymbol{v}^h = \sum_{i \in \mathcal{M}} \boldsymbol{v}^{h,(i)} + \sum_{i \in \mathcal{N}} \boldsymbol{v}^{h,(i)}.$$

Also let the sets $\mathcal{M}(\boldsymbol{v}^h)$ and $\mathcal{N}(\boldsymbol{v}^h)$ be ordered on the basis of the X axis values of its members going from highest to the lowest, i.e. if $\boldsymbol{v}^{h,(j)}, \boldsymbol{v}^{h,(k)} \in \mathcal{M}(\boldsymbol{v}^h)$ and $j > k$, then $|v_1^{h,(j)}| \geq |v_1^{h,(k)}|$.

Recall from Step. (1) we have $\boldsymbol{\beta}^o = (\beta_1^o, 0, 0)$ and $\boldsymbol{\beta}^h = (\beta_1^h, 0, 0)$. We have now 3 possibilities:

Case:	I	II	III
	$\beta_1^o > 0$	$\beta_1^o < 0$	$\beta_1^o = 0$

Table III.1: Three cases based on sign of β_1^o

We will only take up the case $\beta_1^0 > 0$ and argue that the rest of the cases can be proved in a similar fashion. Within this case we have 3 subcases

Case:	A	B	C
	$\beta_1^o > \beta_1^h$	$\beta_1^o < \beta_1^h$	$\beta_1^o = \beta_1^h$

Table III.2: Cases A, B & C

Case C need not be solved as setting $\boldsymbol{w}^h = \boldsymbol{v}^h$, and using (III.A.6) from Step.(1) gives our result.

<u>Case A</u>: $\beta_1^o > \beta_1^h$

Set $\gamma^h := \beta_1^o - \beta_1^h$.

Cases A.1 $\int_Y 2\Big\{\sum_{\mathcal{N}(\boldsymbol{v}^h)} v_1^h\Big\} \boldsymbol{dy} = -\gamma^h$.

$\boldsymbol{v}^{h,(i)} \in \mathcal{N}(\boldsymbol{v}^h)$ implies $v_1^{h,(i)} < 0$. So $\Big|\sum_{\mathcal{N}(\boldsymbol{v}^h)} v_1^h\Big| = \sum_{\mathcal{N}(\boldsymbol{v}^h)} |v_1^h|$, giving (Recall $|v_1^{h,(i)}| \leq m_s$)

$$\int_Y \Big|\sum_{\mathcal{N}(\boldsymbol{v}^h)} v_1^h\Big| \boldsymbol{dy} = \int_Y \Big\{\sum_{\mathcal{N}(\boldsymbol{v}^h)} |v_1^h|\Big\} \boldsymbol{dy} = \gamma^h \qquad \text{and} \qquad \sum_{\mathcal{N}(\boldsymbol{v}^h)} |v_1^h|^2 \leq m_s \sum_{\mathcal{N}(\boldsymbol{v}^h)} |v_1^h|. \tag{III.A.8}$$

Let us thus define $\boldsymbol{w}^h$ through its h^3 values $\boldsymbol{w}^{h,(i)}$ and $i \in \{1, 2, 3, \cdots h^3\}$

$$\boldsymbol{w}^{h,(i)} := \begin{cases} \boldsymbol{v}^{h,(i)} & \text{if } \boldsymbol{v}^{h,(i)} \notin \mathcal{N}(\boldsymbol{v}^{h,(i)}) \\ \{-v_1^{h,(i)}, v_2^{h,(i)}, v_3^{h,(i)}\} & \text{if } \boldsymbol{v}^{h,(i)} \in \mathcal{N}(\boldsymbol{v}^h), \end{cases}$$

where we have defined $\boldsymbol{w}^h$ by flipping elements $\boldsymbol{v}^{h,(i)} \in \mathscr{N}(\boldsymbol{v}^h)$ one-by-one into the set $\mathscr{M}(\boldsymbol{v}^h)$ thorugh the map $\boldsymbol{v}^{h,(i)} = \{v_1^{h,(i)}, v_2^{h,(i)}, v_3^{h,(i)}\} \mapsto \{-v_1^{h,(i)}, v_2^{h,(i)}, v_3^{h,(i)}\}$. Figure III.8 shows an individual element being flipped. Using eqn. (III.A.8) check that

$$\begin{aligned}\|\boldsymbol{v}^h - \boldsymbol{w}^h\|^2_{L^2(Y)} &= \int_Y \Big\{ \sum_{\mathscr{N}(\boldsymbol{v}^h)} |\boldsymbol{v}^h - \boldsymbol{w}^h|^2 \Big\} \boldsymbol{dy} = \int_Y \Big\{ \sum_{\mathscr{N}(\boldsymbol{v}^h)} \Big(|v_1^h - w_1^h|^2 + |\boldsymbol{v}_p^h - \boldsymbol{w}_p^h|^2 \Big) \boldsymbol{dy} \Big\} \\ &= \int_Y \sum_{\mathscr{N}(\boldsymbol{v}^h)} \Big\{ |v_1^h - w_1^h|^2 \Big\} \boldsymbol{dy} = \int_Y \Big\{ \sum_{\mathscr{N}(\boldsymbol{v}^h)} |2v_1^h|^2 \Big\} \boldsymbol{dy} \\ &= 4 \int_Y \Big\{ \sum_{\mathscr{N}(\boldsymbol{v}^h)} |v_1^h|^2 \Big\} \boldsymbol{dy} \le 4m_s \int_Y \Big\{ \sum_{\mathscr{N}(\boldsymbol{v}^h)} |v_1^h| \Big\} \boldsymbol{dy} = 4m_s \gamma^h.\end{aligned}$$

Then if $\boldsymbol{\beta}^h$ is not in the same direction as $\boldsymbol{\beta}^0$, we combine equation (III.A.6) from Step. (1) and the

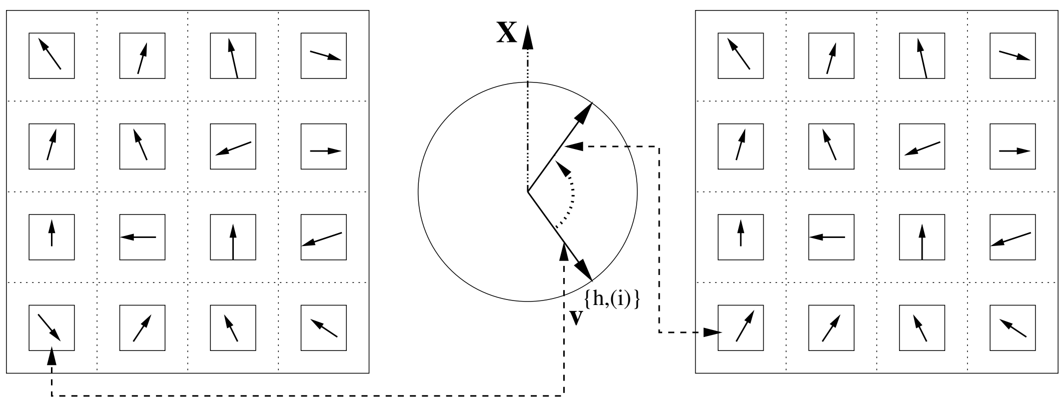

Figure III.8: Flipping an individual $\boldsymbol{v}^{h,(i)} \in \mathscr{N}(\boldsymbol{v}^h)$

above using triangle ineqality to get (Recall $\boldsymbol{v}^h = \boldsymbol{R}(\theta_h)\boldsymbol{u}^h$)

$$\|\boldsymbol{u}^h - \boldsymbol{w}^h\|_{L^2(Y)} \le \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{L^2(Y)} + \|\boldsymbol{v}^h - \boldsymbol{w}^h\|_{L^2(Y)} \le C\Big(|\boldsymbol{\beta}^o - \boldsymbol{\beta}^h| + |\boldsymbol{\beta}^o - \boldsymbol{\beta}^h|^{1/2} \Big) \xrightarrow{h\to\infty} 0$$

and $\boldsymbol{w}^h$ satisfies our other requirements too,

$$\int_Y w_1^h \boldsymbol{dy} = \beta_1^o, \qquad \text{and} \qquad \int_Y \boldsymbol{w}_p^h \boldsymbol{dy} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Cases A.2 $\int_Y 2 \Big\{ \sum_{\mathscr{N}(\boldsymbol{v}^h)} v_1^h \Big\} \boldsymbol{dy} < \gamma^h$.

As in the previous cases, use the map defined above to flip all the $\boldsymbol{v}^h \in \mathscr{N}(\boldsymbol{v}^h)$. Let us thus define

$\boldsymbol{U}^h$ through its h^3 values $\boldsymbol{U}^{h,(i)}$ and $i \in \{1,2,3,\cdots h^3\}$

$$\boldsymbol{U}^{h,(i)} := \begin{cases} \boldsymbol{v}^{h,(i)} & \text{if } \boldsymbol{v}^{h,(i)} \in \mathscr{M}(\boldsymbol{v}^h) \\ \{-v_1^{h,(i)}, v_2^{h,(i)}, v_3^{h,(i)}\} & \text{if } \boldsymbol{v}^{h,(i)} \in \mathscr{N}(\boldsymbol{v}^h) \end{cases}$$

and we repeat the calculation of previous case to get

$$\|\boldsymbol{v}^h - \boldsymbol{U}^h\|^2_{L^2(Y)} \leq 4 m_s \gamma^h. \tag{III.A.9}$$

By construction the h^3 values $\boldsymbol{U}^{h,(i)}$ of $\boldsymbol{U}^h$ exist only in $\mathscr{M}(\boldsymbol{U}^h)$ and $\beta_1^o - \int_Y U_1^h \boldsymbol{dy} \leq \gamma^h$ and $\int_Y \boldsymbol{U}_p^h = \{\ 0\ ,\ 0\ \}^T$. Define a new map $\Gamma_\lambda : m_s S^2 \mapsto m_s S^2$ acting on $\boldsymbol{U}^h = (U_1^h, \boldsymbol{U}_p^h)$ as,

$$\Gamma_\lambda : (U_1^{h,(i)}, \boldsymbol{U}_p^{h,(i)}) \mapsto (U_1^{h,(i)} + \alpha_\lambda^{h,(i)}, \lambda \boldsymbol{U}_p^{h,(i)}) := \boldsymbol{W}_\lambda^{h,(i)}$$

where $\alpha_\lambda^{h,(i)} = \sqrt{|U_1^{h,(i)}|^2 + (1-\lambda^2)|\boldsymbol{U}_p^{h,(i)}|^2} - U_1^{h,(i)} > 0$ is choosen so as to make $(U_1^{h,(i)} + \alpha_\lambda^{h,(i)}, \lambda \boldsymbol{U}_p^{h,(i)}) \in m_s S^2$. In Figure III.9 we have given a schematic version of the contraction map Γ_λ operating in 2

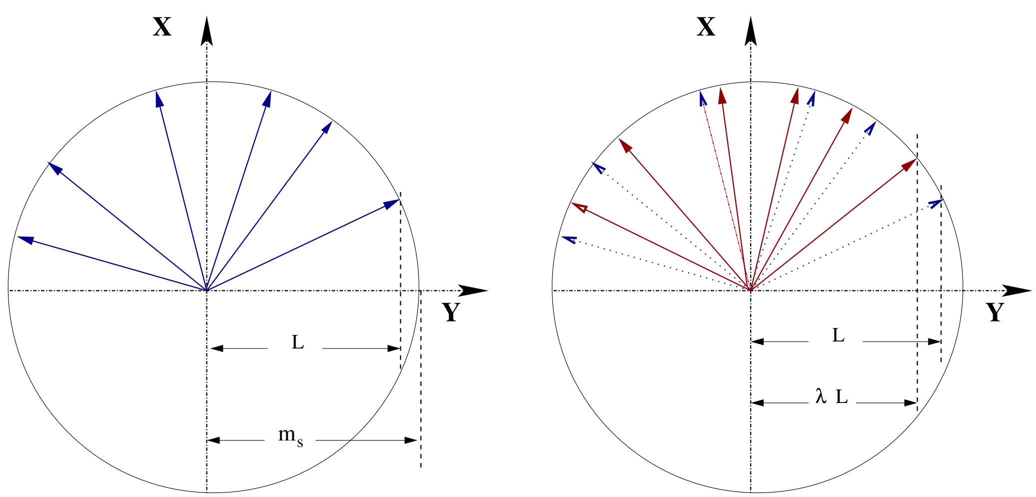

Figure III.9: Contraction by λ of all $\boldsymbol{v}^{h,(i)}$: Note contraction by λ increases the average magnetization along X-axis

dimensions with all the $\boldsymbol{v}^{h,(i)}$ constituting $\boldsymbol{v}^h$ on a 2-dimensional ball of radius m_s. Then we have on rearranging and squaring

$$(1-\lambda^2)|\boldsymbol{U}_p^{h,(i)}|^2 = |\alpha_\lambda^{h,(i)}|^2 + 2\alpha_\lambda^{h,(i)} U_1^{h,(i)} \leq |\alpha_\lambda^{h,(i)}|^2 + 2|\alpha_\lambda^{h,(i)}|\,|U_1^{h,(i)}|. \tag{III.A.10}$$

Also by definition $m_s \geq |\alpha_\lambda^{h,(i)}| \geq 0$ which then gives us

$$\int_Y \Big\{ \sum_{i=0}^{h^3} |\alpha_\lambda^{h,(i)}|^2 \Big\} \boldsymbol{dy} \leq m_s \int_Y \Big\{ \sum_{i=0}^{h^3} |\alpha_\lambda^{h,(i)}| \Big\} \boldsymbol{dy} = m_s \int_Y \Big\{ \sum_{i=0}^{h^3} \alpha_\lambda^{h,(i)} \Big\} \boldsymbol{dy} \tag{III.A.11}$$

and using $m_s \geq |\boldsymbol{U}_p^{h,(i)}|$ with (III.A.10) gives

$$\begin{aligned}(1-\lambda^2) \int_Y \Big\{ \sum_{i=0}^{h^3} |\boldsymbol{U}_p^{h,(i)}|^2 \Big\} \boldsymbol{dy} &\leq \int_Y \Big\{ \sum_{i=0}^{h^3} |\alpha_\lambda^{h,(i)}|^2 \Big\} \boldsymbol{dy} + \int_Y \Big\{ \sum_{i=0}^{h^3} 2|\alpha_\lambda^{h,(i)}|\, |U_1^{h,(i)}| \Big\} \boldsymbol{dy} \\ &\leq m_s \int_Y \Big\{ \sum_{i=0}^{h^3} \alpha_\lambda^{h,(i)} \Big\} \boldsymbol{dy} + 2m_s \int_Y \Big\{ \sum_{i=0}^{h^3} |\alpha_\lambda^{h,(i)}| \Big\} \boldsymbol{dy} \\ &\leq 3m_s \int_Y \Big\{ \sum_{i=0}^{h^3} \alpha_\lambda^{h,(i)} \Big\} \boldsymbol{dy}. \end{aligned} \tag{III.A.12}$$

Next we choose λ such that

$$\int_Y \Big\{ \sum_{i=1}^{h^3} \alpha_\lambda^{h,(i)} \Big\} \boldsymbol{dy} = \int_Y \Big\{ \sum_{i=1}^{h^3} |\alpha_\lambda^{h,(i)}| \Big\} \boldsymbol{dy} = \beta_1^o - \int_Y U_1^h \boldsymbol{dy} \leq \gamma^h. \tag{III.A.13}$$

Set $\boldsymbol{W}_\lambda^h = \Gamma_\lambda(\boldsymbol{U}^h)$. Then

$$|\boldsymbol{W}_\lambda^{h,(i)} - \boldsymbol{U}^{h,(i)}|^2 = |\{\alpha_\lambda^{h,(i)}, (\lambda-1)\boldsymbol{U}_p^{h,(i)}\}|^2 = |\alpha_\lambda^{h,(i)}|^2 + (\lambda-1)^2 |\boldsymbol{U}_p^{h,(i)}|^2$$

and using equations (III.A.11), (III.A.12) and (III.A.13) gives

$$\begin{aligned}\int_Y |\boldsymbol{W}_\lambda^h - \boldsymbol{U}^h|^2 \boldsymbol{dy} &= \int_Y \Big\{ \sum_{i=0}^{h^3} |\alpha_\lambda^{h,(i)}|^2 \Big\} \boldsymbol{dy} + (\lambda-1)^2 \int_Y \Big\{ \sum_{i=0}^{h^3} |\boldsymbol{U}_p^{h,(i)}|^2 \Big\} \boldsymbol{dy} \\ &\leq m_s \int_Y \Big\{ \sum_{i=0}^{h^3} |\alpha_\lambda^{h,(i)}| \Big\} \boldsymbol{dy} + 3m_s \frac{(\lambda-1)^2}{1-\lambda^2} \int_Y \Big\{ \sum_{i=0}^{h^3} |\alpha_\lambda^{h,(i)}| \Big\} \boldsymbol{dy} \\ &= m_s \Big(1 + 3\frac{1-\lambda}{1+\lambda}\Big) \int_Y \Big\{ \sum_{i=0}^{h^3} |\alpha_\lambda^{h,(i)}| \Big\} \boldsymbol{dy} \leq 4m_s \gamma^h \overset{h\to\infty}{\to} 0. \end{aligned} \tag{III.A.14}$$

Thus for this value of λ using eqn. (III.A.13), setting $\boldsymbol{w}^h = \boldsymbol{W}_\lambda^h$, gives our result as $\int_Y w_1^h \boldsymbol{dy} = \beta_1^o$, $\int_Y \boldsymbol{w}_p^h \boldsymbol{dy} = \boldsymbol{0}$ and combining eqns. (III.A.6) from Step. (1), (III.A.9) and (III.A.14) gives using triangle inequality (Recall $\boldsymbol{v}^h = \boldsymbol{R}(\theta_h)\boldsymbol{u}^h$)

$$\begin{aligned}\|\boldsymbol{u}^h - \boldsymbol{w}^h\|_{L^2(Y)} &\le \|\boldsymbol{u}^h - \boldsymbol{v}^h\|_{L^2(Y)} + \|\boldsymbol{v}^h - \boldsymbol{U}^h\|_{L^2(Y)} + \|\boldsymbol{U}^h - \boldsymbol{W}^h_\lambda\|_{L^2(Y)} \\ &\le C\left(\left|\boldsymbol{\beta}^o - \boldsymbol{\beta}^h\right| + \left|\boldsymbol{\beta}^o - \boldsymbol{\beta}^h\right|^{1/2}\right) \xrightarrow{h\to\infty} 0.\end{aligned}$$

Cases A.3 $\int_Y 2\Big\{\sum_{\mathcal{N}(\boldsymbol{v}^h)} v_1^h\Big\}\boldsymbol{dy} > -\gamma^h$

Define for some $l \le |\mathcal{N}(\boldsymbol{v}^h)|$, the vector $\boldsymbol{W}^{h,l}$ throguh it's h^3 values as

$$\boldsymbol{W}^{h,l,(i)} := \begin{cases} \{-v_1^{h,(i)}, v_2^{h,(i)}, v_3^{h,(i)}\} & \text{if} \quad i \in \mathcal{N}(\boldsymbol{v}^h) \text{ and } i \le l, \\ \boldsymbol{v}^{h,(i)} & \quad\text{otherwise .}\end{cases} \tag{III.A.15}$$

Recall that we ordered the set $\mathcal{N}(\boldsymbol{v}^h)$ in the sense that $v_1^{h,(l)} \le -v_1^{h,(m)}$ if $l > m$. Let $(j+1) \in \mathbb{N}$ be the minimum value such that using the above defnition of $\boldsymbol{W}^{h,l}$

$$\begin{aligned}\int_Y W_1^{h,j+1}\boldsymbol{dy} &= \int_Y \Big\{\sum_{\mathcal{M}(\boldsymbol{v}^h)} v_1^{h,(i)} + \sum_{\mathcal{N}(\boldsymbol{v}^h)}^{\{i>(j+1)\}} v_1^{h,(i)} + \sum_{\mathcal{N}(\boldsymbol{v}^h)}^{\{i\le(j+1)\}} \left|v_1^{h,(i)}\right|\Big\}\boldsymbol{dy} \\ &= \int_Y \Big\{\sum_{\mathcal{M}(\boldsymbol{v}^h)} v_1^{h,(i)} + \sum_{\mathcal{N}(\boldsymbol{v}^h)}^{\{i>(j+1)\}} v_1^{h,(i)} + \sum_{\mathcal{N}(\boldsymbol{v}^h)}^{\{i\le(j-1)\}} \left|v_1^{h,(i)}\right|\Big\}\boldsymbol{dy} + \frac{|Y^*|}{h^3}\left(\left|v_1^{h,j}\right| + \left|v_1^{h,j+1}\right|\right) \\ &= \mathcal{L} + h^{-3}\,|Y^*|\,\left(\left|v_1^{h,j}\right| + \left|v_1^{h,j+1}\right|\right) = \beta^+ \ge \beta^o,\end{aligned} \tag{III.A.16}$$

where we have set $\mathcal{L} := \int_Y \Big\{\sum_{\mathcal{M}(\boldsymbol{v}^h)} v_1^{h,(i)} + \sum_{\mathcal{N}(\boldsymbol{v}^h)}^{\{i>(j+1)\}} v_1^{h,(i)} + \sum_{\mathcal{N}(\boldsymbol{v}^h)}^{\{i\le(j-1)\}} \left|v_1^{h,(i)}\right|\Big\}\boldsymbol{dy}$ and

$$\begin{aligned}\int_Y W_1^{h,j}\boldsymbol{dy} &= \int_Y \Big\{\sum_{\mathcal{M}(\boldsymbol{v}^h)} v_1^{h,(i)} + \sum_{\mathcal{N}(\boldsymbol{v}^h)}^{\{i>(j+1)\}} v_1^{h,(i)} + \sum_{\mathcal{N}(\boldsymbol{v}^h)}^{\{i\le(j-1)\}} \left|v_1^{h,(i)}\right|\Big\}\boldsymbol{dy} + \frac{|Y^*|}{h^3}\left(\left|v_1^{h,j}\right| - \left|v_1^{h,j+1}\right|\right) \\ &= \mathcal{L} + h^{-3}\,|Y^*|\,\left(\left|v_1^{h,j}\right| - \left|v_1^{h,j+1}\right|\right) = \beta^- \le \beta^o.\end{aligned} \tag{III.A.17}$$

Set $\boldsymbol{p} := \{-v_1^{h,(j)}, v_2^{h,(j)}, v_3^{h,(j)}\}$ and $\boldsymbol{q} := \{-v_1^{h,(j+1)}, v_2^{h,(j+1)}, v_3^{h,(j+1)}\}$ and note that equations (III.A.16) and (III.A.17) can be written as

$$\mathcal{L} + h^{-3}\,|Y^*|\,(p_1 + q_1) = \beta^+ \ge \beta^o, \qquad \text{and} \qquad \mathcal{L} + h^{-3}\,|Y^*|\,(p_1 - q_1) = \beta^- \le \beta^o.$$

Then using Lemma III.A.4 we get two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that

$$\mathcal{L} + h^{-3}\,|Y^*|\,(u_1 + v_1) = \beta^o,$$

and $h^{-3}|Y^*|(\boldsymbol{u}_p+\boldsymbol{v}_p)=h^{-3}|Y^*|(\boldsymbol{p}_p+\boldsymbol{q}_p)$. Define then $\boldsymbol{w}^h$ through its h^3 values as

$$\boldsymbol{w}^{h,(i)} := \begin{cases} \boldsymbol{u} \ \text{ if } \{i=j \ \& \ i\in\mathcal{N}(\boldsymbol{v}^h)\}, \quad \text{and } \boldsymbol{v} \ \text{ if } \{i=j+1 \ \& \ i\in\mathcal{N}(\boldsymbol{v}^h)\}, \\ \boldsymbol{W}^{h,j+1,(i)} \ \text{ otherwise }. \end{cases}$$

It's easy to check that $\boldsymbol{w}^h$ satisfies all our requirements.

The only remaining case is if $j=0$, i.e. for the first element of $\mathcal{N}(\boldsymbol{v}^h)$ flipped, we already cross β_1^o. First define $\boldsymbol{W}^{h,l=1}$ exactly as in equation (III.A.15) with $l=1$ and the only element being flipped is $\boldsymbol{v}^{h,(l=1)}$. Then note that

$$\mathcal{M}(\boldsymbol{W}^{h,1})=\mathcal{M}(\boldsymbol{v}^h)\bigcup\{\boldsymbol{v}^{h,(1)}\} \qquad \text{and} \qquad \int_Y W_1^{h,1}\boldsymbol{dy}=(\beta_1^h+\gamma^+)>\beta_1^o.$$

Let the element just flipped i.e. $\boldsymbol{v}^{h,(1)}$ be indexed as per the ordering specified to some element $\boldsymbol{W}^{h,1,(k)}$ with $k\in\mathcal{M}(\boldsymbol{W}^{h,1})$. If $k>1$, i.e. the element just flipped is not the element with the largest x_1 component, then choose

$$\boldsymbol{p}=\boldsymbol{W}^{h,1,(i=1)} \ \text{ and } i\in\mathcal{M}(\boldsymbol{W}^h), \qquad \boldsymbol{q}=\boldsymbol{W}^{h,1,(i=k)} \ \text{ and } i\in\mathcal{M}(\boldsymbol{W}^h).$$

Then using Lemma III.A.4 again, generate $\boldsymbol{u}$ and $\boldsymbol{v}$ so that if we define

$$\boldsymbol{w}^{h,(i)} := \begin{cases} \boldsymbol{u} \ \text{ if } \{i=1 \ \& \ i\in\mathcal{N}(\boldsymbol{v}^h)\}, \quad \text{and } \boldsymbol{v} \ \text{ if } \{i=k \ \& \ i\in\mathcal{N}(\boldsymbol{v}^h)\}, \\ \boldsymbol{W}^{h,1,(i)} \ \text{ otherwise }. \end{cases}$$

and again one can check that $\boldsymbol{w}^h$ satisfies our requirements.

If $k=1$, i.e. the flipped element has the largest x_1 component and gets indexed as $\boldsymbol{W}^{h,1,(i=1)}$ and $i\in\mathcal{M}(\boldsymbol{W}^{h,1})$, then from $\mathcal{M}(\boldsymbol{W}^{h,1})$ flip the elements starting from index $i=2$ to some j, i.e. the elements with index $\{2,3,\cdots j\}\in\mathcal{M}(\boldsymbol{W}^{h,1})$ such that if we define the magnetizations $\boldsymbol{U}^{h,j}$ and $\boldsymbol{U}^{h,j+1}$ thorugh it's h^3 values as

$$\boldsymbol{U}^{h,l,(i)} := \begin{cases} \{-W_1^{h,1,(i)},W_2^{h,1,(i)},W_3^{h,1,(i)}\} \ \text{ if } \quad i\in\mathcal{M}(\boldsymbol{W}^{h,1}) \text{ and } 1<i\le l, \\ \boldsymbol{W}^{h,1,(i)} \qquad\qquad\qquad\qquad \text{otherwise }, \end{cases} \tag{III.A.18}$$

then

$$\int_Y U_1^{h,j} \boldsymbol{dy} = \int_Y \Big\{ \sum_{\mathcal{N}(\mathbf{W}^{h,1})} W_1^{h,1,(i)} + \sum_{\mathcal{M}(\mathbf{W}^{h,1})}^{\{i>j\}} W_1^{h,1,(i)} - \sum_{\mathcal{M}(\mathbf{W}^{h,1})}^{\{1<i\leq j\}} W_1^{h,1,(i)} \Big\} \boldsymbol{dy} + \frac{|Y^*|}{h^3}\big(W_1^{h,1,1} + W_1^{h,1,j+1}\big)$$
$$= \mathscr{L} + h^{-3}\,|Y^*|\,\big(W_1^{h,1,1} + W_1^{h,1,j+1}\big) = \beta^+ \geq \beta^o, \tag{III.A.19}$$

with $\mathscr{L} = \int_Y \Big\{ \sum_{\mathcal{N}(\mathbf{W}^{h,1})} W_1^{h,1,(i)} + \sum_{\mathcal{M}(\mathbf{W}^{h,1})}^{\{i>j\}} W_1^{h,1,(i)} - \sum_{\mathcal{M}(\mathbf{W}^{h,1})}^{\{1<i\leq j\}} W_1^{h,1,(i)} \Big\} \boldsymbol{dy}$ and

$$\int_Y U_1^{h,j+1} \boldsymbol{dy} = \int_Y \Big\{ \sum_{\mathcal{N}(\mathbf{W}^{h,1})} W_1^{h,1,(i)} + \sum_{\mathcal{M}(\mathbf{W}^{h,1})}^{\{i>j+1\}} W_1^{h,1,(i)} - \sum_{\mathcal{M}(\mathbf{W}^{h,1})}^{\{1<i\leq j+1\}} W_1^{h,1,(i)} \Big\} \boldsymbol{dy} + \frac{|Y^*|}{h^3}\big(W_1^{h,1,1} + W_1^{h,1,j+2}\big)$$
$$= \mathscr{L} + h^{-3}\,|Y^*|\,\big(W_1^{h,1,1} + W_1^{h,1,j+2}\big) = \beta^- \leq \beta^o, \tag{III.A.20}$$

Then use $\boldsymbol{p} = \boldsymbol{W}^{h,(1)}$ and $\boldsymbol{q} = \boldsymbol{W}^{h,(j+1)}$ and proceed as before.

<u>**Case B**</u>: $\beta_1^h > \beta_1^o$

Starting from $i = 1$ upto some index $i = k$, define $\boldsymbol{U}^h$ as

$$\boldsymbol{U}^{h,(i)} := \begin{cases} \{-v_1^{h,(i)}, v_2^{h,(i)}, v_3^{h,(i)}\} & \text{if} \quad i \in \mathcal{M}(\boldsymbol{v}^h) \text{ and } i \leq k, \\ \boldsymbol{v}^{h,(i)} & \text{otherwise}. \end{cases}$$

where we are now flipping members with positive X-axis contribution, so that

$$\int_Y \boldsymbol{U}_1^h \boldsymbol{dy} < \beta_1^o \qquad \text{and} \qquad \|\boldsymbol{v}^h - \boldsymbol{U}^h\|_{L^2(Y)}$$

Then we have reduced ourselves to Case A, except our function is $\boldsymbol{U}^h$ and not $\boldsymbol{v}^h$. It can be solved using the methods of Case A. □

III.B Periodic Unfolding

In this section we define the periodic unfolding operator and the averaging operator and list some of their properties. The section's contents are based on [Cioranescu et al., 2008].

As usual let Y be the unit cube and Ω be a domain with $\partial\Omega$ bounded. For $\boldsymbol{z} \in \mathbb{Z}^3$, let use define

$$Y_{\varepsilon,\boldsymbol{z}} = \varepsilon\boldsymbol{z} + \varepsilon Y \qquad\qquad Z_\varepsilon = \Big\{\boldsymbol{z} \in \mathbb{Z}^3 \,\big|\, \Omega \bigcap Y_{\varepsilon,\boldsymbol{z}} \neq \emptyset\Big\}$$

It is clear from above that $\bigcup_{\boldsymbol{z}\in Z_\varepsilon} Y_{\varepsilon,\boldsymbol{z}}$ is a covering of Ω by εY sized tiles. Thus any $\boldsymbol{x} \in \Omega$ has to belong to some $Y_{\varepsilon,\boldsymbol{z}_o}$ where $\boldsymbol{z}_o \in Z_\varepsilon$. Such an $\boldsymbol{x}$ can be written as

$$\boldsymbol{x} = \varepsilon \boldsymbol{z}_o + \varepsilon \boldsymbol{y}_o = \varepsilon \left\lfloor \frac{\boldsymbol{x}}{\varepsilon} \right\rfloor + \varepsilon \left\{ \frac{\boldsymbol{x}}{\varepsilon} \right\}$$

where $\boldsymbol{z}_o = \left\lfloor \frac{\boldsymbol{x}}{\varepsilon} \right\rfloor$ is the floor function. The floor function $\lfloor \cdot \rfloor$ is defined as the greatest integer less than or equal to its operand. Also the operator $\{\cdot\}$ above represnts the fractional part of it's operand. Let us also set

$$\widehat{\Omega}^\varepsilon := \bigcup_{\boldsymbol{z}\in Z_\varepsilon} Y_{\varepsilon,\boldsymbol{z}}, \qquad \text{if } Y_{\varepsilon,\boldsymbol{z}} \Subset \Omega \qquad\qquad \Lambda^\varepsilon = \Omega - \widehat{\Omega}^\varepsilon$$

i.e. $\widehat{\Omega}^\varepsilon$ is the union of all $Y_{\varepsilon,\boldsymbol{z}}$ which are proper subsets of Ω and Λ^ε is the remaining $Y_{\varepsilon,\boldsymbol{z}}$ which are so close to the boundary that some part of these elements lies outside Ω. Then we define our unfolding operator as follows:

Definition III.B.1. *For any measurable function ϕ on Ω, we define the unfolding operator T_ε as follows:*

$$T_\varepsilon(\phi)(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \phi\left(\varepsilon \left\lfloor \frac{\boldsymbol{x}}{\varepsilon} \right\rfloor + \varepsilon \boldsymbol{y}\right) & \textit{for} \quad (\boldsymbol{x},\boldsymbol{y}) \in \widehat{\Omega}^\varepsilon \times Y, \\ 0 & \textit{for} \quad (\boldsymbol{x},\boldsymbol{y}) \in \Lambda^\varepsilon \times Y. \end{cases}$$

Similarly let us now define the averaging operator U^ε next.

Definition III.B.2. *For $1 \le p \le \infty$ we define the averaging operator $U_\varepsilon : L^p(\Omega \times Y) \mapsto L^p(\Omega)$ as follows:*

$$U_\varepsilon(\Phi)(\boldsymbol{x}) = \begin{cases} \displaystyle\fint_Y \Phi\left(\varepsilon \left\lfloor \frac{\boldsymbol{x}}{\varepsilon} \right\rfloor + \varepsilon \boldsymbol{p}, \left\{ \frac{\boldsymbol{x}}{\varepsilon} \right\}\right) \boldsymbol{dp} & \textit{for} \quad (\boldsymbol{x},\boldsymbol{y}) \in \widehat{\Omega}^\varepsilon, \\ 0 & \textit{for} \quad (\boldsymbol{x},\boldsymbol{y}) \in \Lambda^\varepsilon. \end{cases}$$

We than qoute the following theorems all of which can be found in [Cioranescu et al., 2008].

Theorem III.B.1. *Let w^ε be a bounded sequence of functions in $L^p(\Omega)$ with $1 < p < \infty$. Then the following assertions are equivalent:*
$(a.) T^\varepsilon(w^\varepsilon)$ converges weakly (respectively strongly) to w in $L^p(\Omega \times Y)$
$(b.)$ w^ε weakly (respectively strongly) two-scale converges to w in $L^p(\Omega \times Y)$.

We state as a theorem Proposition 2.18 from [Cioranescu et al., 2008].

Theorem III.B.2. *Let w^ε be a uniformly bounded sequence of functions in $L^p(\Omega)$ with $1 \le p < \infty$. Then the following assertions are equivalent:*
$(a.) T^\varepsilon(w^\varepsilon) \to w$ *in* $L^p(\Omega \times Y)$
$(b.)$ $w^\varepsilon - U^\varepsilon(w) \to 0$ *in* $L^p(\Omega)$.

Based on the definition of the unfolding operator and Theorem III.B.1, its easy to check that if $w(\boldsymbol{x},\boldsymbol{y}) \in L^2_\sharp\big(Y; C(\overline{\Omega})\big)$ and we set $w^\varepsilon(\boldsymbol{x}) = w(\boldsymbol{x}, \varepsilon^{-1}\boldsymbol{x})$, then

$$w^\varepsilon \xrightarrow{2} w \qquad \text{in } L^2(\Omega \times Y) \qquad \text{or equivalently} \qquad T^\varepsilon(w^\varepsilon) \to w \qquad \text{in } L^2(\Omega\times Y) \tag{III.B.1}$$

This combined with Theorem III.B.2 gives us that if $w^\varepsilon(\boldsymbol{x}) = w(\boldsymbol{x}, \varepsilon^{-1}\boldsymbol{x})$ as above for $w(\boldsymbol{x},\boldsymbol{y}) \in L^2_\sharp\big(Y; C(\overline{\Omega})\big)$, then

$$w^\varepsilon(\boldsymbol{x}) - U^\varepsilon(w)(\boldsymbol{x}) = w(\boldsymbol{x}, \varepsilon^{-1}\boldsymbol{x}) - U^\varepsilon(w)(\boldsymbol{x}) \to 0 \qquad \text{in} \quad L^p(\Omega) \tag{III.B.2}$$

Finally using the tiling decomposition of any set Ω, we first show an approximation result for any function $\boldsymbol{u} \in L^2(\Omega, m_s S^2)$. This result follows Assertion 1 in [De Simone, 1993].

Lemma III.B.1. *Let $\boldsymbol{u} \in L^2(\Omega; m_s S^2)$. Then there exists a sequence $\boldsymbol{u}^\varepsilon \in C(\overline{\Omega}; m_s S^2)$ such that $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ in $L^2(\Omega; m_s S^2)$.*

Proof. Let $\varepsilon = 1/m$ for $m \in \mathbb{N}$. We define $\boldsymbol{u}^m(\boldsymbol{x})$ as

$$\boldsymbol{u}^m(\boldsymbol{x}) := \fint_{V_{1/m,\boldsymbol{z}}} \boldsymbol{u}(\boldsymbol{p})\, d\boldsymbol{p}, \qquad \text{if} \quad \boldsymbol{x} \in V_{1/m,\boldsymbol{z}}$$

with $\boldsymbol{u}$ extended by 0 on cells $V_{1/m,\boldsymbol{z}} \Subset \Omega$. The Lebesgue Differentiation theorem and Dominated Convergence theorem then gives us that $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ in $L^2(\Omega)$. We next define

$$\widetilde{\boldsymbol{u}}^m(\boldsymbol{x}) := \begin{cases} m_s \frac{\boldsymbol{u}^m(\boldsymbol{x})}{|\boldsymbol{u}^m(\boldsymbol{x})|} & \text{if } |\boldsymbol{u}^m(\boldsymbol{x})| \neq 0 \\ m_s \boldsymbol{v} & \text{otherwise} \end{cases}$$

where $\boldsymbol{v}$ is an arbitrary fixed unit vector. Then note

$$\left|\widetilde{\boldsymbol{u}}^m(\boldsymbol{x}) - \boldsymbol{u}^m(\boldsymbol{x})\right| = \left|m_s - |\boldsymbol{u}^m(\boldsymbol{x})|\right| = \left||\boldsymbol{u}(\boldsymbol{x})| - |\boldsymbol{u}^m(\boldsymbol{x})|\right| \to 0 \quad a.e. \quad \boldsymbol{x} \in \Omega.$$

Then triangle inequality in the form $\left|\widetilde{\boldsymbol{u}}^m - \boldsymbol{u}\right| \le \left|\widetilde{\boldsymbol{u}}^\varepsilon - \boldsymbol{u}^m\right| + \left|\boldsymbol{u}^m - \boldsymbol{u}\right|$ and Dominated Convergence theorem gives us that $\widetilde{\boldsymbol{u}}^m \to \boldsymbol{u}$ in $L^2(\Omega; m_s S^2)$. Thus we have approximated $\boldsymbol{u}$ with functions in

$\widetilde{\boldsymbol{u}}^m \in L^2(\Omega; m_s S^2)$ but with finite range. To mollify them we do the following. We define

$$\boldsymbol{g}^m(\boldsymbol{x}) = \begin{cases} \widetilde{\boldsymbol{u}}^m(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega \\ m_s \boldsymbol{v} & \text{if } \boldsymbol{x} \in \mathbb{R}^3/\Omega \end{cases}$$

with $\boldsymbol{v}$ again being arbitrary fixed unit vector and clearly $\boldsymbol{g}^m$ is also a function with finite range in $L^2_{loc}(\mathbb{R}^3; m_s S^2)$. Let $m_s S^2 \ni \boldsymbol{z} \in \text{range}\big(\boldsymbol{g}^m\big)$ be one of the values taken by $\boldsymbol{g}^m$. Let $\pi_{\boldsymbol{z}}$ be the stereographic projection with projection center $\boldsymbol{z}$. Then becuase of the finite range of $\boldsymbol{g}^m$, the stereographic projection $\pi_{\boldsymbol{z}} \circ \boldsymbol{g}^m$ lies in a bounded set in $\mathbb{R}^2$. Next let $\rho_{1/n} \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$ be a standard mollifier. Then define $\big(*$ is the convolution operator $\big)$

$$\widetilde{\boldsymbol{g}}^{m,n}(\boldsymbol{x}) = \pi_{\boldsymbol{z}}^{-1} \circ \big(\rho_{1/n} * \pi_{\boldsymbol{z}} \circ \boldsymbol{g}^m\big) \quad \in C^1\big(\mathbb{R}^3; m_s S^2\big).$$

Note $\big|\widetilde{\boldsymbol{g}}^{m,n}\big| = m_s$ and $\widetilde{\boldsymbol{g}}^{m,n} \to \boldsymbol{g}^m$ in $L^2_{loc}(\mathbb{R}^3; \mathbb{R}^3)$. Restricting $\widetilde{\boldsymbol{g}}^{m,n}$ to Ω and calling the restriction $\overline{\boldsymbol{g}}^{m,n}$, we note $\overline{\boldsymbol{g}}^{m,n} \overset{n\to\infty}{\longrightarrow} \widetilde{\boldsymbol{u}}^m$ in $L^2(\Omega; m_s S^2)$ and $\widetilde{\boldsymbol{u}}^m \overset{m\to\infty}{\longrightarrow} \boldsymbol{u}$ in $L^2(\Omega; m_s S^2)$. Then standard diagonalization arguments gives an increasing mapping $m \mapsto n(m)$ such that $\lim_{m\to\infty} \overline{\boldsymbol{g}}^{m,n(m)} = \boldsymbol{u}$ in $L^2(\Omega; m_s S^2)$. □

We also qoute a very elegant lemma frequently used to justify diagonal sequence construction in the calculus of variations. This result is directly qouted as Lemma 11.1.1 in [Attouch et al., 2006] and proved originally in [Attouch, 1984].

Lemma III.B.2. *Let $a_{m,n}$ be a sequence in a first coutable topological space X with $(m,n) \in \mathbb{N} \times \mathbb{N}$, such that*

$$\lim_{n\to\infty} a_{m,n} = a_m \qquad \textit{and} \qquad \lim_{m\to\infty} a_m = a.$$

Then there exists an non-decreasing map $n \mapsto m(n)$ from $\mathbb{N}$ to $\mathbb{N}$, so that

$$\lim_{n\to\infty} a_{m(n),n} = a.$$

Chapter IV

Discussion and Summary

IV.1 Discussion and numerical results

In this work we have used asymptotic methods and variational analysis to obtain simpler models for studying the physics of magnetostriction and ferromagnetism for two different geometries proposed for use as an acoustic sensor. In the following sections, we will present in more detail, the implications of the results of chapter II and III towards the design considerations for devices based on the corresponding geometries.

IV.1.1 Sensor : homogeneous wires

In chapter II, we studied the behavior of thin homogeneous nanowires made of magnetostrictive materials. Using the inspiration of one-dimensional models like Euler-Bernoulli beam bending theory in 3-D elasticity, we similarly derived one-dimensional models for such wires. The corresponding theories thus derived were then solved for the case of Galfenol and the results were shown to agree very well with physical observations. The bending model for magnetostrictive nanobeams is non-trivial and while it is related to the Euler-Bernoulli model, it contains terms coming from the magnetization.

The basic design that we are trying to understand is given in figure IV.1. As mentioned in chapter II in greater detail, the Galfenol wires as grown have a wire axis which is along the <110> crystallographic axis and the natural minimizing state of magnetization even under large bending is an axial state of magnetization. Figure IV.2 shows individual wires which have been bent

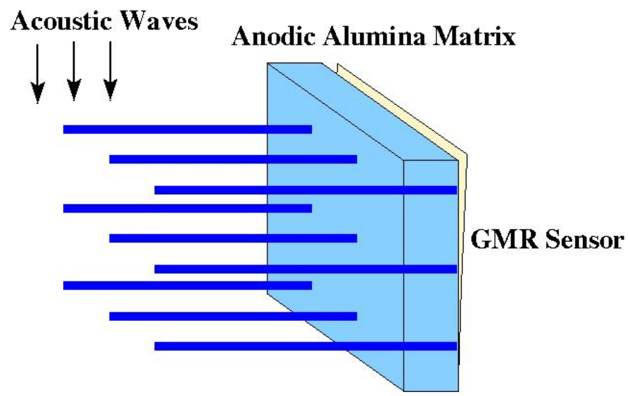

Figure IV.1: Proposed Sensor : Sensor using wires of equal length

significantly by an AFM tip, yet retaining their axial minimizing magnetization. This hardness

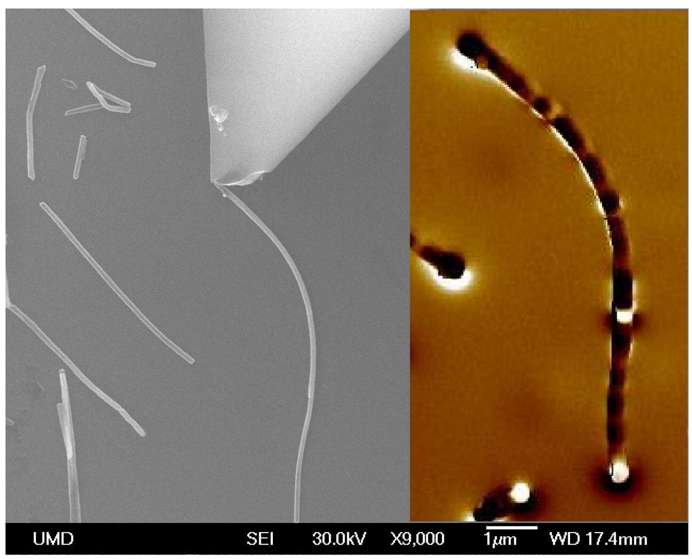

Figure IV.2: Individual wires sustaining large bending using AFM tip, yet retaining axial magnetization [Downey, 2008]

associated with the axial minimizing state was shown to be an outcome of the scale separation of total energy into orders, with the axial magnetization minimizing the first Γ-limit problem, while the bending problem itself appearing as the third Γ-limit problem. Thus right away, it is clear that the phenomenon of magnetostriction plays no role in the working of such a sensor, as the magnetization and the associated spontaneous strain is a fixed quantity, and any bending strain superimposed over that, acts as a purely elastic term.

Next, we look at the field change due to a 2-dimensional array of nanowires as in the design IV.1, when we allow the wire matrix to bend. The bending law that we use is the modified Euler-Bernoulli law of equation (II.7.19) from Theorem II.7.1 in Chapter II. The right side part of Figure IV.3 shows the basic design, with wire diameters being 60nm, center-to-center distance between adjacent wires in a square matrix being 300nm (T in the figure), length of the wires being 1μm (L in the figure), and finally the point at which the field is being measured (point P in the figure), being 300nm below the nanowires(a in the figure). We plot graphs for field readings at this location. Assuming that the wire orientation is along the z-axis, the figure IV.3 shows the field reading in the z-direction, i.e. H_z in Oerstads as a function of the angle that the wires are bending. Next we plot the field generated at the same point P, but in the x-direction, assuming the

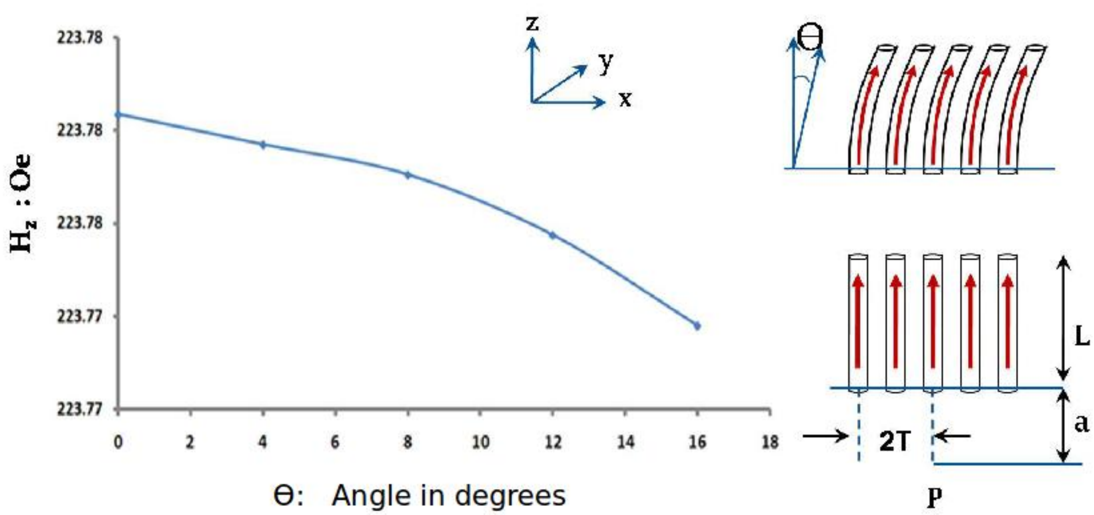

Figure IV.3: H_z field change at P below the wire matrix

wires are bending in the x-direction. Figure IV.4 shows the corresponding graph. It is clear from these two figures, that there is very minor change in the field for fairly large angles of bending. The reason for this is not very obscure though. Simple estimates show that the field reading at any point close to this geometry, is affected mostly by the distribution of free poles created by the magnetization at a free surface. In our case, the magnetizations being always axial, the poles are created at the top and bottom faces of the nanowires. For a point below the bottom face like P, the bending of the wires, moves the poles at an end which is very far away from P. Meanwhile the

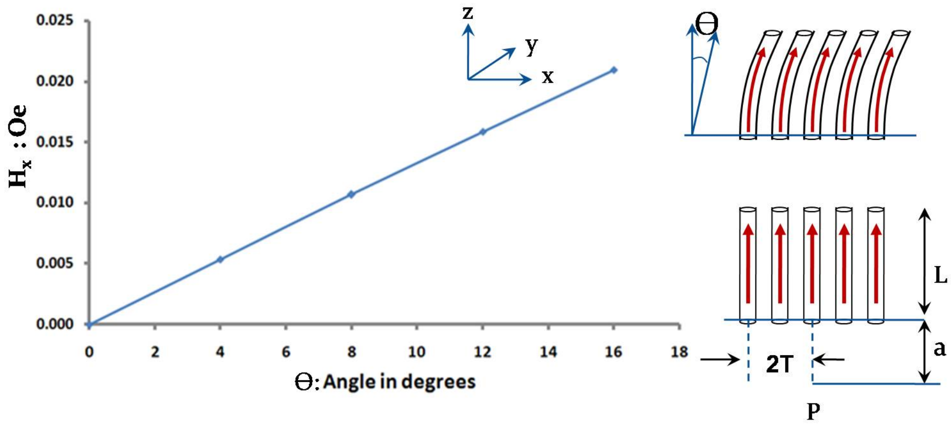

Figure IV.4: H_x field change at P below the wire matrix

bending leaves unaffected the poles at the cantilevered end, which are very close to P. As a result the field changes are very small.

As a result, the first change that was thought off to remedy this situation is, to put a sensing element like a GMR, above the wire array. This would keep the sensing elements much closer to the ends which move. Figures IV.5 and IV.6, show graphs of the corresponding reading of the field at a point P which is now located 300nm above the wire matrix.

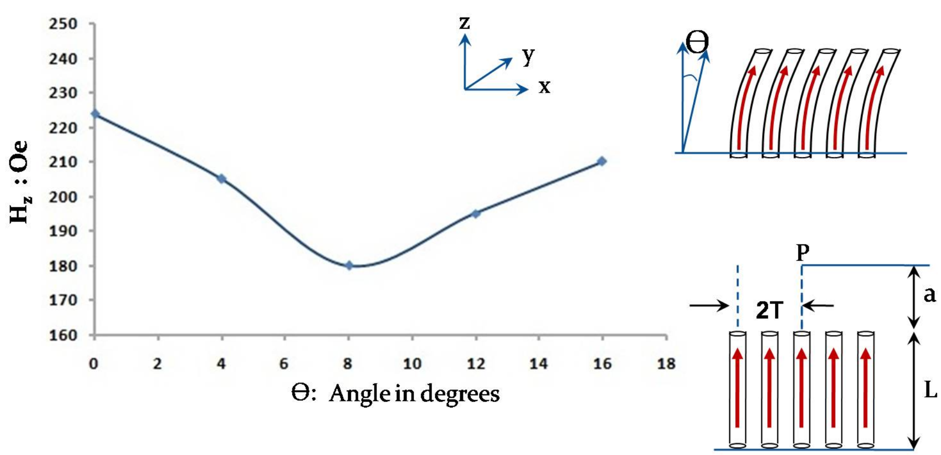

Figure IV.5: H_z field change at P above the wire matrix

It is clear that in this case, the field changes are much more substantial, with the graph in

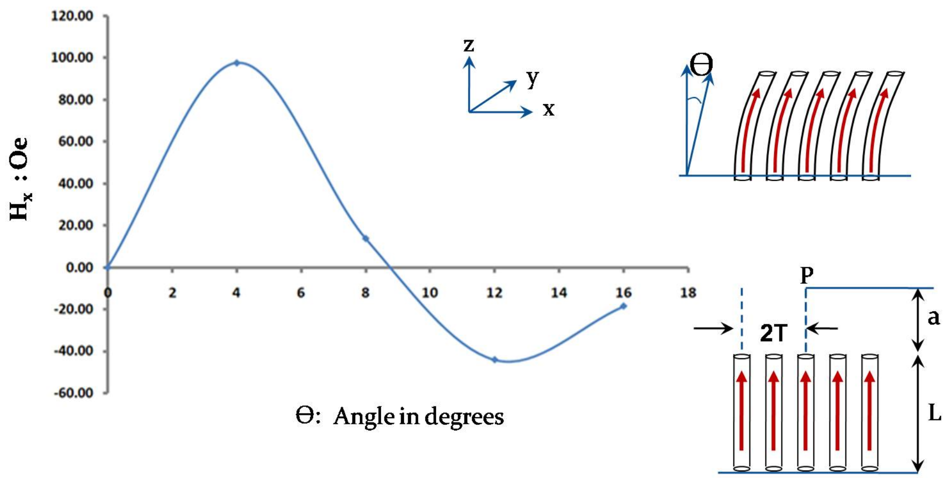

Figure IV.6: H_x field change at P above the wire matrix

figure IV.6 showing a change of 100 Oerstads and a reversal in sign. The change of sign and the fact that it is close to 8^o in angles, is also easy to understand. At close to 8^o bent, the moving ends of the wire are in between the basal faces of adjacent neighbors. As a result the poles at the free end, form a symmetric distribution around P, and produce a net zero x-field.

IV.1.2 Sensor : heterogeneous multilayered wires

This section deals with the analysis of the design of figure IV.7. Typically these matrices contain several million wires arranged in a square matrix with each individual wire further consisting of hundreds of layers of non-magnetic and magnetic segments. The main result is the derivation of a homogenized model for this structure. Similar results will also be valid for composites with small ferromagnetic inclusions. We then studied the stability of the staturared state i.e. the apex of the M vs H curve and investigate the nucleation problem as classically presented by Brown. As a result of this we got a broad generalization to three dimensions, of a result known in literature as the fanning model. This symmetric fanning model was proposed as a possible mechanism for instability of a uniformly magnetized long thin wire if one thought of the wire as a chain-of-spheres. Recall from equation (III.5.22) in chapter III, we know the critical field is given by

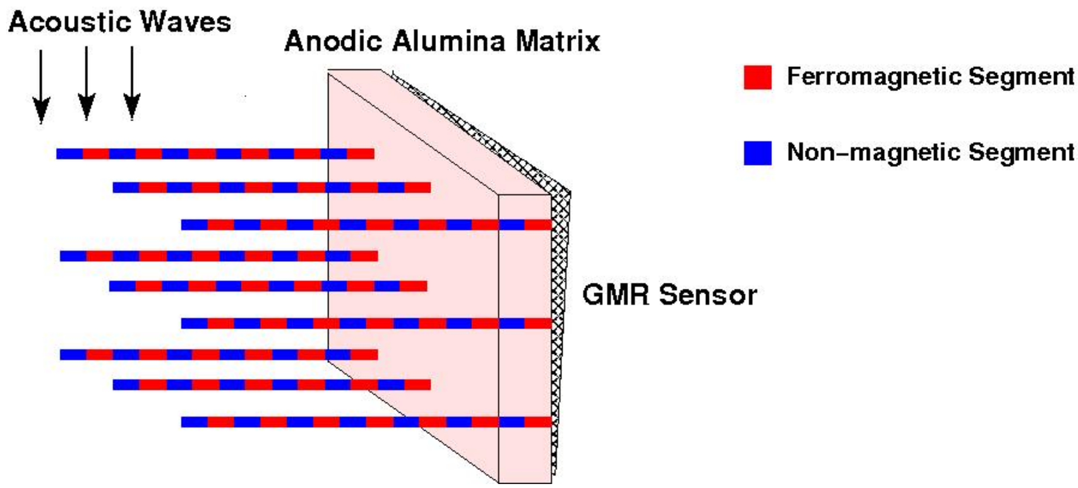

Figure IV.7: Geometry of the multilayered sensor model

$$\frac{H_a^{cr}}{m_s} = \theta N_z - 2\Pi_1 - \left(\fint_{kY^*} \partial^{y_3} \psi^1_{\boldsymbol{w}^*}(\boldsymbol{y}) \boldsymbol{dy} \right) - 2 \inf_{(\widetilde{\beta},\widetilde{\gamma}) \in (\widetilde{\mathscr{B}}^k)^2} \frac{\int_\Omega \mathscr{E}_{per}\left(\{\widetilde{\beta}, \widetilde{\gamma}, 0\}\right) \boldsymbol{dx}}{\int_\Omega \theta \left(\overline{|\widetilde{\beta}|^2} + \overline{|\widetilde{\gamma}|^2} \right) \boldsymbol{dx}},$$

where $\widetilde{\mathscr{B}}^k$ is given from equation (III.5.18) as $\widetilde{\mathscr{B}}^k = \left\{\beta(\boldsymbol{x},\boldsymbol{y}) \in \mathscr{B}^k \,\middle|\, \forall \boldsymbol{x} \in \Omega, \quad \fint_{kY} \beta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{dy} = 0\right\}$. Note in the above equation that the infimum is insensitive to the position $\boldsymbol{x} \in \Omega$. Given a class of functions which are supported on kY^*, have average zero, and are constant on each $Y_i^* \in kY^*$, let there be two functions $b^o(\boldsymbol{y})$ and $c^o(\boldsymbol{y})$ which minimize over this class the following fraction,

$$\frac{\mathscr{E}_{per}\left(\{b^o(\boldsymbol{y}), c^o(\boldsymbol{y}), 0\}\right)}{\theta \left(\overline{|b^o|^2} + \overline{|c^o|^2} \right)} = \inf \frac{\mathscr{E}_{per}\left(\{b(\boldsymbol{y}), c(\boldsymbol{y}), 0\}\right)}{\theta \left(\overline{|b|^2} + \overline{|c|^2} \right)}.$$

Then, it is clear to see that $\widetilde{\beta}(\boldsymbol{x},\boldsymbol{y}) = b^o(\boldsymbol{y})$ and $\widetilde{\gamma}(\boldsymbol{x},\boldsymbol{y}) = c^o(\boldsymbol{y})$, minimize the infimum in equation (III.5.22) and give us our critical field. Thus the critical field calculation collapses to a Maxwell's periodic problem over the domain kY.

Next we come to the question of how we can solve for the energy $\mathscr{E}_{per}$? The typical method would involve either solving the corresponding periodic Maxwell's equations and numerically computing this energy explicitly, or otherwise using some numerical summation method like Ewald summation to evaluate the energy of a periodic pole distribution directly. Talk of solving for such an energy, evokes many deep questions about the typical contradictions that are inherent in summing energies of monopole and dipole periodic distributions and the possibility of some charge distributions having an energy which diverges.

The main contradiction pertaining to magnetic dipoles is as follows: Any periodic cell with it's magnetization, far from the origin looks like some combination of dipoles. Typically dipole fields decay at the rate of $1/r^3$, where r is the distance. However as the volume of dipoles are increasing at the rate r^3 for an infinite periodic distribution, the interaction energy diverges, or at the very best conditionally converges. To resolve this, lets look back at our own case. Note that $\widetilde{\beta}$ and $\widetilde{\gamma}$ have an average zero, which means that for a periodic cell supporting them, they behave as multipoles. The order of the multipole depends on the value of k and the generality of the functions. As a result, the field that they generate has decay faster than $1/r^4$, and they sum quickly even in real space.

Note also that even if a magnetization $\boldsymbol{m}$ supported over a periodic cell did not have a zero average, the periodic Maxwell problem as defined in equation (III.3.2) is given by that potential ψ^1 which solves the periodic problem for $\boldsymbol{m} - \fint_{kY} \boldsymbol{m}\,d\boldsymbol{y}$. Thus ψ^1 in fact solves for any $\varphi(\boldsymbol{y}) \in C^\infty_\sharp(kY)$

$$\int_Y \left(-\nabla^{\boldsymbol{y}} \psi^1(\boldsymbol{x},\boldsymbol{y}) + 4\pi \Big(\boldsymbol{m}(\boldsymbol{x},\boldsymbol{y}) - \fint_{kY} \boldsymbol{m}\,d\boldsymbol{y}\Big) \right) \cdot \nabla^{\boldsymbol{y}} \varphi(\boldsymbol{y}) = 0 \qquad \forall \boldsymbol{x}.$$

Here again it is easy to see that whatever $\boldsymbol{m}$ be, the quantity $\boldsymbol{m} - \fint_{kY} \boldsymbol{m}\,d\boldsymbol{y}$ has an average zero over kY, and in fact it is also a multipole of order greater than a dipole. As a result, for all our numerical computations, the periodic energy $\mathscr{E}_{per}$ will be evaluated by summing the interaction energy over real space upto some accuracy.

With all these observations in mind, we go back to the problem of minimizing energy $\mathscr{I}^o_{(k)}$ over $\mathscr{A}^k_o$ defined as in Theorem III.4.1, where $\mathscr{I}^o_{(k)}$ is given in equation (III.5.1) by,

$$\begin{aligned}\mathscr{I}^o_{(k)}(\boldsymbol{m}) = \int_\Omega \Big[-\boldsymbol{h}_a(\boldsymbol{x}) \cdot \boldsymbol{m}(\boldsymbol{x}) + \inf_{\boldsymbol{w} \in \mathscr{A}^k_o} \Big\{ \fint_{kY} \Big(\Phi(\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})) + \frac{1}{8\pi} |\nabla^{\boldsymbol{y}} \psi^1_{\boldsymbol{w}}(\boldsymbol{x},\boldsymbol{y})|^2 \Big)\, d\boldsymbol{y} \Big\} \Big]\, d\boldsymbol{x} \\ + \int_{\mathbb{R}^3} \frac{1}{8\pi} |\boldsymbol{h}_{\boldsymbol{m}}(\boldsymbol{x})|^2\, d\boldsymbol{x}.\end{aligned}$$

We then use a discretization over Ω, and a conjugate gradient algorithm with numerical continuation using the applied field as a parameter. Such an algorithm has been used in [Kinderlehrer and Ma, 1994] to compute hysteresis curves in regular micromagnetics in the absence of exchange energy. The periodic energies are summed as a real-space sum. The figure IV.8 shows the geometry of our problem, whose details are as follows:

a). $\Omega:$ is a sphere of radius R=20μm,

b). $Y_\varepsilon:$ the scaled periodic cell with $L_1 = 100nm, L_2 = 100nm, L_3 = 80nm$,

c). $Y_\varepsilon^*:$ the magnetic segment $l_1 = 60nm, l_2 = 60nm, l_3 = 40nm$.

The wire axis is along the z-direction. The figure IV.9 shows the M vs. H curve for this geometry.

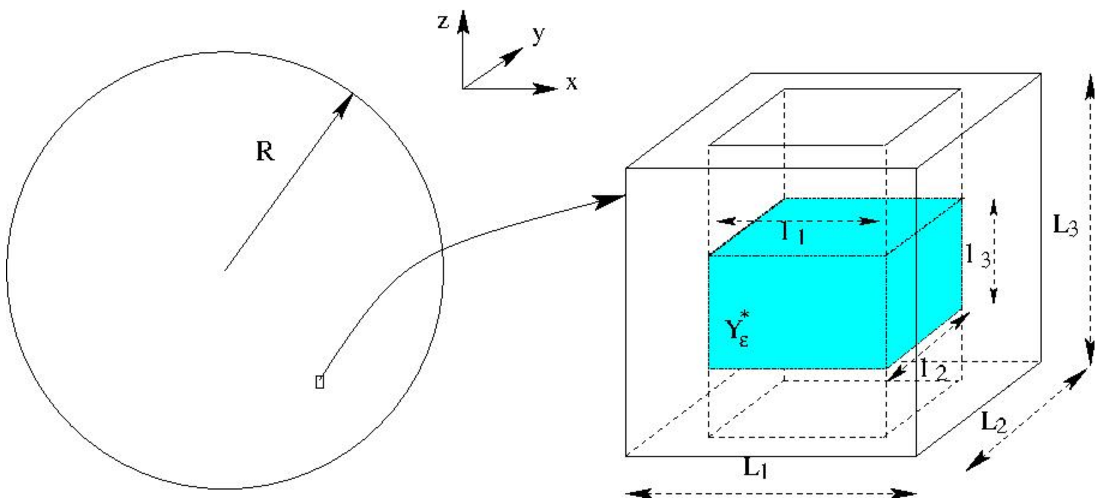

Figure IV.8: Geometry of our model

It is interesting to note that the M vs H curve generated, passes at zero field through a zero magnetization state. As a result, this behaves effectively like a super-paramagnetic material. The best way to explain this is that, as the field reduces, the large number of dis-connected segments allow the magnetization to relax very well. For large enough k, the relaxation can be good enough to allow for zero magnetization, at zero applied fields. Also for k large enough, one can imagine that the hysteresis curve also stabilizes, i.e. further increments in k, do not change the curve in any appreciable way.

More work needs to be done to understand how the sensor geometry can be understood to effect it's design consideration based on these calculations. The generation of the M vs. H curve only signifies the first step towards understanding how a sensor would behave. Further work may also

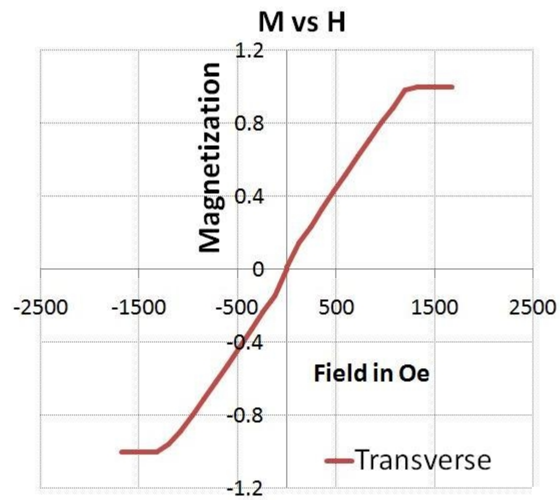

Figure IV.9: M vs. H for a multilayered wire matrix, k=2

be done to understand the dynamics of a vibrating magnetostrictive nanowire. In case of wires which are not magnetized along the wire axis, non-trivial magnetic correctors occurring along with the bending phenomenon, may provide very interesting behavior.

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