Critical Vortices in $2+1$ dimensions
and
Stochastic Background of Gravitational Waves from
Cosmic Strings

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Dedication

To Rukiye and my parents.
Abstract

This thesis consists of two parts. In the first part we consider vortices which are topological defects that emerge upon spontaneously broken local symmetries in 2 + 1 dimensions. We discuss the renormalization of the central charge and the mass of the $\mathcal{N} = 2$ supersymmetric Abelian vortices. At the classical level the mass of the vortex is equal to its central charge, which is referred to as Bogomol’nyi - Prasad - Sommerfield (BPS) saturation. At the quantum level both the mass and the central charge get corrections. We show that the mass and the central charge of the vortex get the same nonvanishing quantum corrections, which preserves BPS saturation at the quantum level.

In the second part, we study the stochastic background of gravitational waves (SBGW) generated by kinks and cusps on cosmic string loops. Cosmic strings, which can be considered as 2+1 dimensional vortices extended along the additional dimension, are one dimensional topological defects predicted by a large class of unified theories as remnants of spontaneously broken local or global symmetries. Grand unified theories have gauge symmetries which are eventually spontaneously broken down to the symmetry of the Standard Model, and certain class of these phase transitions are expected to produce cosmic strings. The interactions of cosmic strings result in cusps or kinks on them which decay by radiating gravitational waves. In this study, we find that kinks contribute at the same order as cusps to the SBGW, and discuss the accessibility of the total background due to kinks as well as cusps to current and planned gravitational wave detectors, as well as to the big bang nucleosynthesis (BBN), the cosmic microwave background (CMB), and pulsar timing constraints. Furthermore we consider anisotropies in the SBGW arising from random fluctuations in the number of sources. Such anisotropies are analogous to the anisotropies observed in the CMB radiation and would carry additional information about the gravitational-wave sources that generated them.
# Contents

Acknowledgements

Dedication

Abstract

List of Figures

1 Introduction
   1.1 Formation of Topological Defects
      1.1.1 Spontaneously Broken Global Symmetries
      1.1.2 Spontaneously Broken Local Symmetries
   1.2 Vortices and Strings

2 Critical Vortices in Three-Dimensional SQED
   2.1 $\mathcal{N} = 2$ supersymmetry
   2.2 Description of the model and classical results
   2.3 Quantum Corrections
      2.3.1 Fayet–Iliopoulos parameter at one loop
      2.3.2 Central Charge
      2.3.3 Renormalization of the vortex mass
      2.3.4 One-loop contribution from the untilded sector
      2.3.5 The tilded sector (regulator) contribution in $M_v$
      2.3.6 Higher orders
   2.4 Calculation of the Noether charge $q
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>Conclusion</td>
<td>27</td>
</tr>
<tr>
<td>3</td>
<td>Cosmic Strings and Gravitational Radiation</td>
<td>29</td>
</tr>
<tr>
<td>3.1</td>
<td>Classical String Theory</td>
<td>32</td>
</tr>
<tr>
<td>3.2</td>
<td>Nambu Goto Action</td>
<td>33</td>
</tr>
<tr>
<td>3.3</td>
<td>Gravitational Radiation</td>
<td>37</td>
</tr>
<tr>
<td>3.3.1</td>
<td>The Weak Field Approximation</td>
<td>37</td>
</tr>
<tr>
<td>3.4</td>
<td>Cusps on Cosmic Strings</td>
<td>40</td>
</tr>
<tr>
<td>3.5</td>
<td>Kinks on Cosmic Strings</td>
<td>43</td>
</tr>
<tr>
<td>4</td>
<td>Stochastic Background</td>
<td>45</td>
</tr>
<tr>
<td>4.1</td>
<td>Small Loops</td>
<td>52</td>
</tr>
<tr>
<td>4.2</td>
<td>Large Loops</td>
<td>53</td>
</tr>
<tr>
<td>4.3</td>
<td>Removing Rare Events</td>
<td>54</td>
</tr>
<tr>
<td>4.4</td>
<td>Numerical Results</td>
<td>55</td>
</tr>
<tr>
<td>4.5</td>
<td>Analytical Approximation for the Stochastic Background</td>
<td>56</td>
</tr>
<tr>
<td>4.5.1</td>
<td>Analytical Approximation for small loop case</td>
<td>57</td>
</tr>
<tr>
<td>4.5.2</td>
<td>Analytical Approximation for large loop case</td>
<td>58</td>
</tr>
<tr>
<td>4.6</td>
<td>Effect of number of kinks</td>
<td>59</td>
</tr>
<tr>
<td>4.7</td>
<td>Parameter Space Constraints and Results</td>
<td>60</td>
</tr>
<tr>
<td>5</td>
<td>Anisotropies in the Stochastic Background of Gravitational Waves</td>
<td>65</td>
</tr>
<tr>
<td>5.1</td>
<td>Anisotropies in the SBGW</td>
<td>66</td>
</tr>
<tr>
<td>5.1.1</td>
<td>Cosmic Strings Case</td>
<td>68</td>
</tr>
<tr>
<td>5.2</td>
<td>Conclusion and Discussion</td>
<td>72</td>
</tr>
<tr>
<td>References</td>
<td></td>
<td>74</td>
</tr>
<tr>
<td>6</td>
<td>Appendix I</td>
<td>79</td>
</tr>
<tr>
<td>6.1</td>
<td>$\mathcal{N} = 2$ supersymmetric Lagrangian</td>
<td>79</td>
</tr>
<tr>
<td>6.2</td>
<td>$\mathcal{N} = 2$ supersymmetry current</td>
<td>80</td>
</tr>
<tr>
<td>6.3</td>
<td>Fermion mode decomposition</td>
<td>82</td>
</tr>
</tbody>
</table>
7 Appendix II

7.1 Energy and Momentum of gravitational waves . . . . . . . . . . . . . . . . . . . . . . . . . . 85
7.2 Correlations in Amplitude . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 86
7.3 Spherical Harmonics Expansion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 91
## List of Figures

1.1 The quartic scalar potential (Mexican hat) ........................................... 3  
2.1 Tadpole diagrams determining one-loop correction to $\xi$. ....................... 16  
2.2 Vertex for the Noether charge calculation .............................................. 27  
3.1 Path of a relativistic particle in the $x-t$ plane ..................................... 32  
3.2 The surfaced traced by the string ........................................................... 34  
3.3 The contour of integration ..................................................................... 39  
4.1 Coordinate system with two sources located at distances $r$ and $r'$ ........ 46  
4.2 The cone of radiation ............................................................................. 50  
4.3 The strip traced by the cone of radiation .............................................. 51  
4.4 Numerical results for kink and cusp spectrum for small loops. .............. 55  
4.5 Numerical results for kink and cusp spectrum for large loops. ............... 56  
4.6 Accessible regions in the $\varepsilon - G\mu$ plane for $p = 10^{-3}$ (small loops). 60  
4.7 Accessible regions in the $\varepsilon - G\mu$ plane for $p = 10^{-2}$ (small loops). 61  
4.8 Accessible regions in the $\varepsilon - G\mu$ plane for $p = 10^{-1}$ (small loops). 62  
4.9 Accessible regions in the $p - G\mu$ plane (large loops). ......................... 62  
5.1 Normalized correlation for $f = 1$ Hz and $f = 10$ Hz. .......................... 70  
5.2 Normalized correlation for $f = 100$ Hz. ............................................... 70  
5.3 Normalized correlation vs. frequency for cusp and kink ....................... 71  
5.4 $\mathcal{N}C$ for cusps and kinks at $f = 10$ Hz. ........................................ 71  
5.5 $\mathcal{N}C$ for cusps and kinks at $f = 10^{-3}$ Hz. .................................... 71  
5.6 $\mathcal{N}C$ for cusps and kinks at $f = 10^{-8}$ Hz. .................................... 72
Chapter 1

Introduction

Topological defects are remnants of spontaneously broken local or global symmetries. They appear in many fields of physics ranging from high energy physics to solid state physics. One of the most well known topological defects appears in magnetic materials. Let us consider a material which is composed of clusters with magnetic moments. The dynamics of the system can be described by a Heisenberg type Hamiltonian, which is invariant under rotations, i.e. there are no preferred directions for the system. However, the physical realization of the ground state of the system is not rotationally invariant. The direction of the magnetic moments are chosen randomly at different locations. Nearby moments align with each other and create a domain structure. The magnetization smoothly interpolates between different domains, and the width of the transition range is the thickness of the domain wall. The domain wall is the topological defect that emerges upon breaking of the rotational symmetry of the system by randomly chosen magnetization. This is an example of spontaneously broken global symmetry. The domain walls are physical objects: they carry (magnetic) energy, and they can be moved or rotated by external currents or magnetic fields.

An example of spontaneously broken local symmetry occurs in superconducting materials. If a superconducting material is placed in a strong magnetic field, the magnetic field penetrates into the material at certain locations at which the superconductivity is lost. The magnetic field forms flux tubes which are one dimensional topological defects known as Abrikosov-Nielsen-Olesen flux tubes [1]. Abrikosov-Nielsen-Olesen flux tubes are topological defects associated with spontaneously broken $U(1)$ gauge symmetry,
which will be discussed in further detail in Sec. 1.2. For the case of high energy physics, vortices or strings may form as a result of spontaneously broken unified theories. In the following sections we first outline the field theoretical background of formation of topological defects. In the first part of the thesis, we focus on vortices. We consider normalization of the mass and central charge of vortices in $\mathcal{N} = 2$ supersymmetric field theory. In the second part of the thesis, we consider strings which can be constructed as vortices extended along an additional dimension. We then discuss the SBGW due cusps and kinks on cosmic strings.

1.1 Formation of Topological Defects

Topological defects are relics of spontaneously broken symmetries. The exact nature of the defect depends on the group of the symmetry broken. Below we consider two important cases.

1.1.1 Spontaneously Broken Global Symmetries

Let us consider the Lagrangian for a complex scalar field:

$$L = \partial_\mu \varphi \partial^\mu \varphi^* - V(\varphi, \varphi^*).$$

(1.1)

The potential can be chosen to be of the form

$$V(\varphi, \varphi^*) = \frac{\lambda^2}{2} \left( |\varphi|^2 - \frac{n^2}{2} \right)^2,$$

(1.2)

which is shown in Fig. 1.1.

The Lagrangian in Eq. (1.1) has a global $U(1)$ symmetry, i.e. it remains invariant under the phase rotations:

$$\varphi \rightarrow e^{i\theta} \varphi,$$

(1.3)

where $\theta$ is a constant real number. Although the field theory defined by the Lagrangian in Eq. (1.1) is invariant under the phase rotations, the vacuum state of the field is not. The vacuum state solution is given by the field configuration that minimizes the potential, which is

$$\varphi_V = \frac{n}{\sqrt{2}} e^{i\varphi_V}.$$

(1.4)
$\theta_V$ is the phase of field at the vacuum state, which has no physical significance since it can be removed by a $U(1)$ rotation. The solution $\varphi_V$ is clearly not invariant under $U(1)$ rotation, hence the $U(1)$ symmetry is spontaneously broken. The results of the broken symmetry can be seen by expanding the field around the vacuum solution. It is convenient to separate out the radial and angular components of the field by using the following expansion \[ (1.5) \]

$$\varphi = \frac{\eta + \xi}{\sqrt{2}} e^{i\alpha},$$

where $\theta_V$ is set to zero. Plugging this expansion to Eq. (1.1) we get

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \xi \partial^{\mu} \xi - \frac{\lambda \eta^2}{2} \xi^2 + \frac{\eta^2}{2} \partial_{\mu} \alpha \partial^{\mu} \alpha + \text{interaction terms}.
\]

The first two terms in the effective Lagrangian in Eq. (1.6) represent a neutral particle $\xi$ which has mass $\lambda \eta$. It is important to note that $\xi$ corresponds to the radial excitation in the potential well, therefore the particle sees the curvature of the potential. On the other hand, the field $\alpha$ has no mass term in Eq. (1.6). It corresponds to the angular excitation in the Mexican hat-shaped potential. This massless mode is referred to as the Goldstone Boson. Whenever a global symmetry is spontaneously broken Goldstone Bosons which correspond to the excitation of the fields along the flat directions of the potential are generated. On the other hand, if the broken symmetry is a local symmetry,
the degrees of freedom of the excitations along the flat directions are absorbed into the longitudinal component of gauge bosons which acquire mass upon spontaneously breaking the symmetry. This mechanism is crucial for the formation of vortices and flux tubes and hence it is discussed in detail below.

1.1.2 Spontaneously Broken Local Symmetries

In order to make the global $U(1)$ symmetry defined in Eq. (1.3) local, one introduces a gauge field $A_\mu$ with the following transformation

$$A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \theta(x), \quad (1.7)$$

where $e$ is the coupling constant. The partial derivatives are replaced with the gauge covariant derivatives

$$D_\mu = \partial_\mu + ieA_\mu. \quad (1.8)$$

With these definitions, the local $U(1)$ invariant Lagrangian can be written as

$$\mathcal{L} = D_\mu \varphi D^\mu \varphi^* - V(\varphi, \varphi^*) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (1.9)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (1.10)$$

is the field strength of the gauge field. The vacuum solutions are still as given in Eq. (1.4) and we can use the expansion in Eq. (1.5). With this expansion the kinetic term in Eq. (1.9) can be written as

$$\mathcal{L}_K = \frac{1}{2} \partial_\mu \xi \partial^\mu \xi - \lambda \xi^2 - \frac{1}{2} A_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \text{interaction terms}, \quad (1.12)$$

We note that $\partial_\mu \alpha$ term can be absorbed into $A_\mu$ by gauging as described in Eq. (1.7), which shows that the would-be Goldstone boson is absorbed into the longitudinal component of the vector field. The full Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \xi \partial^\mu \xi - \frac{\lambda^2 \eta^2}{2} \xi^2 + \frac{1}{2} \eta^2 A_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \text{interaction terms}, \quad (1.12)$$
which describes an interacting theory with a massive scalar and massive vector field. The number of degrees of freedom before and after the symmetry breaking is the same: one degree of freedom from the complex field is transferred to the vector field, which becomes massive, and hence it can have longitudinal polarization.

1.2 Vortices and Strings

In this section we reproduce the vortex solutions for a spontaneously broken local Abelian symmetry in $2 + 1$ dimensions. The Lagrangian for a complex scalar field coupled to the gauge field is given by

$$
L = D_\mu \varphi^* D^\mu \varphi - \frac{e^2}{2} (|\varphi|^2 - \eta^2)^2 - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}.
$$

(1.13)

We would like to consider the static solutions in $2 + 1$ dimensions. Eliminating the terms with time derivatives we can express the energy density as

$$
E = -L = D_k \varphi D^k \varphi^* + \frac{e^2}{2} (|\varphi|^2 - \eta^2)^2 + \frac{1}{2} F_{kl} F^{kl},
$$

(1.14)

where $k = 1, 2$ denotes the space indices. It is important to note that the potential chosen here is a special case of Eq. (1.2), where $\lambda$ is set to $e$, which is the coupling constant (and also note that $\eta$ is re-scaled by a factor $\sqrt{2}$). In this special case the equations of motion, which are a priori second order differential equations, can be reduced to first order differential equations by Bogomol’nyi completion [3]. The energy density given in Eq. (1.14) can be written in the following form

$$
E = \frac{1}{2} |(D_k + i \epsilon_{kl} D_l) \varphi|^2 + \frac{1}{2} \left( F_{12} + e(\varphi^2 - \eta^2) \right)^2, \\
+\epsilon_{kl} \partial_l (\epsilon \eta^2 A_l - i \varphi^* D_l \varphi),
$$

(1.15)

where $\epsilon_{kl}$ is the two dimensional Levi-Civita tensor with the convention $\epsilon_{12} = 1$. The first two terms in this equation are positive definite and the last one is a boundary term. Therefore the energy can be minimized if the following equations are satisfied:

$$
(D_1 \pm i D_2) \varphi = 0, \\
F_{12} + e(\varphi^2 - \eta^2) = 0,
$$

(1.16)
which are first order Bogomol’nyi equations. The last term in Eq. (1.15) is a surface term. If one integrates the last term over space coordinates, the result reads

\[ Z \equiv \int d^2 x \epsilon_{kl} \partial_l (\eta^2 A_l - i \varphi^* \mathcal{D}_l \varphi) \]  

\[ = e \eta^2 \int_{r \to \infty} rd\theta A_\theta, \]  

which is proportional to the winding number of the gauge field (The second term in Eq. (1.17) vanishes exponentially.) \( Z \) is referred to as the central charge, since it commutes with the generators of the supersymmetric extension of the model (To be more precise, for the case of vortex, \( Z \) commutes with a portion of the supersymmetry generators, see Chap. 2). The asymptotic solutions of the Bogomol’nyi equations are

\[ \varphi = \eta e^{i \theta} \]  

\[ A_k = -n \epsilon_{kl} x^l, \]  

where \( n \) is an integer that represents the winding number. The magnetic field corresponding to the vector potential is \( F_{12} \) and it is confined to a region with radius of scale \( 1/\eta \). This is again in agreement with the conclusion from Sec. 1.1.2, where the gauge boson acquires a mass of \( \eta \), and therefore the interaction strength decays exponentially with the distance.

The mass of vortex configuration reads

\[ M \equiv \int d^2 x E = Z = 2\pi \eta^2 |n|. \]  

The equality of the mass and central charge is called the Bogomol’nyi Prasad Sommerfield (BPS) saturation. The BPS saturation is far from coincidence: it holds even under quantum corrections in supersymmetric extensions, which will be discussed in Chap. 2.

If the vortex configuration is extended along the \( z \)-axis, the result is a flux tube. The mass per unit length of the tube can be described as the tension, and Eq. (1.20) shows that the tension is proportional to the square of \( \eta \), which is the energy scale of the phase transition. Therefore the tension of the string critically depends on the energy scale of the symmetry breaking. For the case of cosmic strings, it is convenient to describe the tension with a dimensionless quantity \( G \mu \), where \( G \) is the Newton’s
gravitational constant and \( \mu \) is the tension of the string. For strings formed in phase transitions of grand unified theories, the tension of the string would correspond to \( G\mu \simeq 10^{-7} - 10^{-6} \). To put this number into a perspective, the mass of the string of length of a solar radius is comparable to the solar mass, therefore they can be detected by their gravitational effects on the nearby matter as well as lensing effect on background light sources. Another observation is that the thickness of the strings, which scales as \( 1/\eta \), is much smaller than any length scale in cosmology, therefore the string can be treated as a one dimensional object with zero thickness.
Chapter 2

Critical Vortices in
Three-Dimensional SQED

In this chapter we consider renormalization of the central charge and the mass of the \( \mathcal{N} = 2 \) supersymmetric Abelian vortices in 2 + 1 dimensions [4]. We obtain \( \mathcal{N} = 2 \) supersymmetric theory in 2 + 1 dimensions by dimensionally reducing the \( \mathcal{N} = 1 \) Supersymmetric Quantum Electrodynamics (SQED) in 3 + 1 dimensions with two chiral fields carrying opposite charges. Then we introduce a mass for one of the matter multiplets without breaking \( \mathcal{N} = 2 \) supersymmetry. This massive multiplet is viewed as a regulator in the large mass limit. We show that the mass and the central charge of the vortex get the same nonvanishing quantum corrections, which preserves BPS saturation at the quantum level. Comparison with the operator form of the central extension exhibits fractionalization of a global U(1) charge; it becomes \( \pm 1/2 \) for the minimal vortex. The very fact of the mass and charge renormalization is due to a “reflection” of an unbalanced number of the fermion and boson zero modes on the vortex in the regulator sector.
2.1 $\mathcal{N} = 2$ supersymmetry

$\mathcal{N} = 2$ supersymmetric QED with the Fayet–Iliopoulos term in 2+1 dimensions supports Abrikosov–Nielsen–Olesen (ANO) vortices [5, 6]. These classical solutions are 1/2-BPS saturated (two out of four supercharges are conserved). Quantum corrections to the vortex mass and central charge were discussed in the literature more than once. It is firmly established [7] that there are two fermion zero modes on the vortex implying that the supermultiplet to which the vortex belongs is two-dimensional. This is a short supermultiplet. Hence, the classical BPS saturation cannot be lost in loops.

Particular implementation of the vortex BPS saturation turned out to be a contentious issue, almost to the same extent as it had happened with two-dimensional kinks in $\mathcal{N} = 1$ models (for reviews see [8], Sec. 3.1 in [9], and [10]). The authors of [5] and [11] obtained a vanishing quantum correction to the vortex mass using the following eigenvalue densities:

$$n_B(w) - n_F(w) \propto \delta(w),$$

(2.1)

where $n_{B(F)}$ is the bosonic (fermionic) density of states. The mass correction vanishes since

$$\Delta M_v \propto \int dw \left( n_B(w) - n_F(w) \right) w = 0.$$  

(2.2)

Since the vortex mass $M_v$ is proportional to the Fayet–Iliopoulos (FI) parameter $\xi$, and $\xi$ is renormalized in one loop, the above result caused a problem.

Later new calculations of the vortex mass were undertaken and a nonvanishing one-loop correction to the vortex mass was reported in [12, 13]. It was shown [7] that the central charge also gets a correction, so that the BPS saturation of the vortex persists at the one-loop level. However, the (dimensional) regularization that was used in the most detailed paper [7], expressly written to discuss three-dimensional supersymmetric vortices, does not allow one to treat in a straightforward manner the Chern–Simons (CS) term, whose role in the problem at hand is important. In this study we use another regularization method in which the CS term naturally appears in the limit of large regulator mass. This mass is also crucial in the operator form of the centrally extended algebra which we derive at one loop. Our operator expression for the central extension includes the Noether charge Sec. 2.21.

In this study we revisit the issue using a physically motivated regularization which
is absolutely transparent. We recalculate the renormalization of the vortex mass at one loop

\[ M_{v,R} = 2\pi \left( \xi_R - \frac{m}{4\pi} \right) \]  

and the one-loop effect in the central charge. (Here \( \xi_R \) is the renormalized value of the FI parameter, \( m \) is the matter field mass,

\[ m = e\sqrt{2\xi_R}, \]  

and the subscript \( R \) stands for renormalized. The above result is in agreement with the previous calculations [7, 12]. Needless to say, our direct calculation confirms BPS saturation, \( M_{v,R} = |Z_R| \). Moreover, it demonstrates that, in the limit of the large regulator mass, regulator’s role is taken over by the Chern–Simons term. A new finding obtained by comparing the central charge calculation with the operator form of the central extension is a U(1) global charge fractionalization. The operator expression for the central extension which we derive in our regularization is presented in Eqs. (2.20) and (2.21). Then we discuss the central charge/vortex mass renormalization to all orders in perturbation theory, see Eq. (2.56).

\( \mathcal{N} = 2 \) SQED Lagrangian in 2 + 1 dimensions (four supercharges) can be obtained by dimensional reduction of \( \mathcal{N} = 1 \) supersymmetric Lagrangian in 3 + 1 dimensions. In order to have a well defined anomaly-free SQED in four dimensions, one has to have two matter superfields, say \( \Phi \) and \( \tilde{\Phi} \), with the opposite charges. Since there is no chirality in three dimensions, in three-dimensional SQED, in principle, it is sufficient to keep a single superfield (say, \( \Phi \)), while \( \tilde{\Phi} \) can be eliminated. This is a minimal setup which is routinely considered. The four-dimensional anomaly is reflected in three dimensions in the form of a “parity anomaly” [14, 15] and the emergence of the Chern–Simons term, as will be explained momentarily.

When we speak of eliminating \( \tilde{\Phi} \) we should be careful. Eliminating does not mean discarding. A perfectly safe method of getting rid of \( \tilde{\Phi} \) is to make the tilded fields heavy [9]. Then the corresponding supermultiplet decouples and does not appear in the low-energy theory. It leaves a trace, however, in the form of the Chern–Simons term [14, 15], as shown in Sec. 2.4.

There is a well-known method of making the tilded fields heavy without altering the masses of the untilded fields. It works in three dimensions. One can introduce a
“real” mass $\tilde{m}$ [16] (a three-dimensional analog of the twisted mass in two dimensions [17, 18, 19]) without breaking $\mathcal{N} = 2$ supersymmetry of three-dimensional SQED. The real mass corresponds to a constant background vector field along the reduced direction.

When the masses of the tilded and untilded fields are equal, the renormalization of the FI term vanishes [26, 27], and so do quantum corrections to the vortex mass. When we make the tilded fields heavy, $\tilde{m} \gg e\sqrt{\xi}$, effectively they become physical regulators. As long as we keep their mass $\tilde{m}$ large but finite it acts as an ultraviolet cut-off in loop integrals. All one-loop corrections, including the linearly divergent part, become well-defined and perfectly transparent. We have a smooth transition as we eventually send $\tilde{m}$ to infinity.

Our analysis is organized as follows. In Sec. 2.2 we describe our basic model obtained from four-dimensional SQED by reducing one of the spatial dimensions. We introduce the real mass $\tilde{m}$, to be treated as a free parameter, for the “second” chiral superfield. Sec. 2.3 is devoted to quantum corrections to the central charge and vortex mass. The operator form of the central extension is discussed in detail in this section. In Sec. 2.4 we consider a global charge fractionalization and a related question of Chern–Simons.

### 2.2 Description of the model and classical results

Our starting point is $\mathcal{N} = 1$ SQED in $3+1$ dimensions with two chiral matter superfields $\Phi$ and $\tilde{\Phi}$ and the Fayet–Iliopoulos term. It has four conserved supercharges. The corresponding Lagrangian is

$$
\mathcal{L} = \left\{ \frac{1}{4e^2} \int d^2 \theta W_\alpha W^\alpha + \text{H.c.} \right\} + \int d^4 \theta \Phi^* e^V \Phi
+ \int d^4 \theta \Phi^* e^{-V} \Phi - \xi \int d^2 \theta d^2 \bar{\theta} \bar{V}(x, \theta, \bar{\theta}),
$$

(2.5)

where $W_\alpha$ is the gauge field multiplet,

$$
W_\alpha = \frac{1}{8} \bar{D}^2 D_\alpha V = \lambda_\alpha - \theta_\alpha D - i\theta^\beta F_{\alpha\beta} + i\theta^2 \partial_{\alpha\bar{\alpha}} \lambda^{\dagger\bar{\alpha}}.
$$

(2.6)

In order to get $\mathcal{N} = 2$ supersymmetry in $2+1$ dimensions we compactify one of the dimensions, say the third axis, keeping the zero Kaluza–Klein modes and discarding nonzero ones. To introduce the tilded field mass we introduce a constant background
gauge field along the compactified axis, $V_{bg}$, where the subscript bg means background. In terms of the components we have

$$V_{bg} = \bar{\theta}^\dagger \gamma^0 \gamma^\mu \theta V^{bg}_\mu , \quad (2.7)$$

$\gamma$-matrices are defined in Eq. (2.32) below. The background vector field is chosen to be a constant field along the compactified axis, i.e. $V^{bg}_\mu = 2\tilde{m} \delta^\mu_3$. It is important to note that this is a new auxiliary field, rather than the expectation value of the original photon field. This background is coupled to $\tilde{\Phi}$ only, with the charge $-1$. Then the Lagrangian takes the form

$$\mathcal{L} = \left\{ \frac{1}{4e^2} \int d^2 \theta W_\alpha W^\alpha + \text{H.c.} \right\} + \int d^4 \theta \Phi^* e^V \Phi$$

$$+ \int d^4 \theta \tilde{\Phi}^* e^{-V-V_{bg}} \tilde{\Phi} - \xi \int d^2 \theta d^2 \theta^\dagger V(x, \theta, \theta^\dagger) , \quad (2.8)$$

Upon introduction of the constant background field, $\tilde{\Phi}$ multiplet becomes massive whereas $\Phi$ multiplet is not affected, since it is chosen to be neutral with respect to the background field. It is clear that the kinetic term for the gauge multiplet is not affected, and similarly, the Fayet-Iliopoulos term remains the same since the superspace integral $\int d^4 \theta V$ is non-vanishing only for the last component of the superfield $V$.

After compactification of the third axis and imposing the Wess–Zumino gauge, we get the following bosonic and fermionic Lagrangians in terms of the component fields (see Sec. 6.1 for details) :

$$\mathcal{L}_B = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + D^\mu \bar{\phi}^* D_\mu \phi + D^\mu \phi^* D_\mu \phi + \frac{1}{2e^2} (\partial_\mu N)^2$$

$$+ \frac{1}{2e^2} D^2 - \xi \bar{D}D(\phi^* \bar{\phi} - \bar{\phi}^* \phi) - N^2 \phi^* \phi - (\bar{m} + N)^2 \bar{\phi}^* \bar{\phi} ,$$

$$\mathcal{L}_F = \frac{1}{e^2} \bar{\lambda} i \theta \lambda + \bar{\psi} i \bar{D} \psi + \bar{\psi} i \bar{D} \bar{\psi} + N \tilde{\psi} \psi - (\bar{m} + N) \bar{\tilde{\psi}} \tilde{\psi}$$

$$+ i \sqrt{2} \left[ (\bar{\lambda} \psi^* - \bar{\psi} \lambda) \phi \right] - i \sqrt{2} \left[ (\bar{\lambda} \bar{\psi}) \bar{\phi}^* - (\bar{\psi} \lambda) \phi \right] , \quad (2.9)$$

where $N = - A_3$ is a real pseudoscalar field, and

$$i D_\mu \phi = (i \partial_\mu + A_\mu) \phi , \quad i D_\mu \bar{\phi} = (i \partial_\mu - A_\mu) \bar{\phi} .$$
Moreover, $D$ is an auxiliary field, which can be eliminated via its equation of motion. The Lagrangian (2.9) is invariant under the following supersymmetry transformations,

\[
\begin{align*}
\delta \phi &= \sqrt{2} \bar{\epsilon} \psi, \\
\delta \tilde{\phi} &= \sqrt{2} \bar{\epsilon} \tilde{\psi}, \\
\delta A_\mu &= i (\bar{\epsilon} \gamma_\mu \lambda - \bar{\lambda} \gamma_\mu \epsilon), \\
\delta \lambda &= -\gamma^\mu (\partial_\mu N - f_\mu) + i e \frac{D}{e},
\end{align*}
\]

(2.10)

where

\[
f_\mu = -\frac{i}{2} \epsilon_{\mu \alpha \beta} F^{\alpha \beta}, \quad D = e^2 \left( |\phi|^2 - |\tilde{\phi}|^2 - \xi \right),
\]

and $\epsilon = (\epsilon_1, \epsilon_2)$ is a complex spinor. The corresponding supersymmetry current is (see Sec. 6.2 for details)

\[
\begin{align*}
J^\mu &= \sqrt{2} (\mathcal{D} \phi^* + ie N \phi^*) \gamma^\mu \psi + \sqrt{2} (\mathcal{D} \tilde{\phi}^* - ie (N + \tilde{m}) \tilde{\phi}^*) \gamma^\mu \tilde{\psi} \\
&\quad + (i \not\partial N - i f + D) \gamma^\mu \lambda.
\end{align*}
\]

(2.11)

The centrally extended algebra of the supercharges is discussed below, in Sec. 2.3.2, see Eq. (2.20). After elimination of the auxiliary $D$ field via equation of motion, we get the following scalar potential:

\[
V = \frac{e^2}{2} \left[ \xi - (\phi^* \phi - \tilde{\phi}^* \tilde{\phi}) \right]^2 + N^2 \phi^* \phi + (\tilde{m} + N)^2 \tilde{\phi}^* \tilde{\phi}.
\]

(2.12)

If $\xi$ is positive (and we will assume it is) the theory is in the Higgs regime and supports the BPS-saturated vortices. We will assume $\tilde{m}$ to be positive too. If $\tilde{m} \neq 0$, the vacuum configuration is as follows:

\[
\tilde{\phi} = 0, \quad N = 0, \quad \phi^* \phi = \xi.
\]

(2.13)

Vortices with nonvanishing winding number correspond to windings of the $\phi$ field [81]. The fermionic fields are set to zero in the classical approximation.

We are interested in static solutions; the relevant part of the Lagrangian, upon the Bogomol’nyi completion [3], takes the form

\[
\mathcal{L}_{\text{BPS}} = -\frac{1}{2 e^2} B^2 - |D^i \phi|^2 - \frac{e^2}{2} \left[ \xi - \phi^* \phi \right]^2 \\
= -|D_+ \phi|^2 - \frac{1}{2 e^2} \left[ B - e^2 (|\phi|^2 - \xi) \right]^2 \\
- \xi B - i \partial_k (\epsilon_{kl} \phi^* D_l \phi),
\]

(2.14)
where $B = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field and $D_+ \equiv D_1 + iD_2$

Since the solution is static we have $H = -\mathcal{L}_{\text{BPS}}$. We will label the fields minimizing $H$ by the subscript (or superscript) $v$. They satisfy the following first-order BPS equations:

$$B_v - e^2 (|\phi_v|^2 - \xi) = 0, \quad D_+^v \phi_v = 0.$$  \tag{2.15}

The boundary conditions are self-evident. Solutions to these BPS equations in different homotopy classes are labeled by the winding number $n$. Needless to say, they are well known. A vortex with the winding number $n$ has the mass

$$M_v = 2\pi n \xi,$$  \tag{2.16}

where, at the classical level, the parameter $\xi$ on the right-hand side is that entering the Lagrangian (2.9). At this level the central charge

$$|Z_v| = \xi \int d^2xB = 2\pi n \xi.$$  \tag{2.17}

The vortex solution breaks $1/2$ of supersymmetry. More precisely, the vortex solution is invariant under the supersymmetry transformations Eq. (2.10) restricted to $\epsilon = (0, \epsilon_2)$. In Sec. 2.3 we will show that this residual symmetry between bosons and fermions is strong enough to preserve the BPS saturation at the quantum level.

## 2.3 Quantum Corrections

In this section we will calculate quantum corrections to the Fayet–Iliopoulos parameter, the vortex mass and the central charge, using the regularization outlined in Sec. 2.1. We will keep $\tilde{m}$ large but finite, taking the limit $\tilde{m} \to \infty$ at the very end. In order to calculate one-loop corrections to the classical results we will expand the fields around the background solutions

$$\phi = \phi_v + \eta, \quad A_\mu = A_\mu^v + a_\mu$$  \tag{2.18}

keeping the terms quadratic in $\eta, a_\mu$. The fields $\eta$ and $a_\mu$ have the mass $m = e\sqrt{2\xi}$; while $\tilde{\phi}$ and $\tilde{\psi}$ have the mass $\tilde{m}$. The superpartners $\tilde{\psi}$ and $\lambda$ do not have definite masses; the mass matrix for these fields can be diagonalized providing us with two diagonal combinations, $\psi' = \frac{\psi + i\lambda}{\sqrt{2}}$ and $\lambda' = \frac{\psi - i\lambda}{\sqrt{2}}$. The latter have masses $e\sqrt{2\xi}$. Note that both parameters, $e$ and $\sqrt{\xi}$, have dimensions $[m]^{1/2}$. 

2.3.1 Fayet–Iliopoulos parameter at one loop

As was mentioned, the Fayet–Iliopoulos parameter receives no corrections if \( \tilde{m} = m \). If \( \tilde{m} \neq m \), there is a one-loop quantum correction. The simplest way to compute the renormalization of \( \xi \) is to consider the Lagrangian before eliminating the auxiliary field \( D \), i.e the bosonic part in Eq. (2.9). In this exercise we treat \( D \) as a constant background field. Fig. 2.1 shows the tadpole diagrams arising from the couplings \( D(\phi^* \phi - \tilde{\phi}^* \tilde{\phi}) \), which renormalize \( \xi \),

\[
\xi_R \equiv \xi + \delta \xi = \xi + \int \frac{d^3k}{(2\pi)^3} \left( \frac{i}{k^2 - \tilde{m}^2} - \frac{i}{k^2 - m^2} \right)
\]

(2.19)

We see that \( \tilde{m} \) plays the role of the ultraviolet cut-off, as was expected. Needless to say, the finite part of the correction, \( m/4\pi \), depends on the definition of the renormalized FI parameter. In fact, it has an infrared origin (otherwise, odd powers of \( m \) could not have entered). The renormalized FI parameter is defined as the coefficient in front of the \( D \) term in \( \Gamma_{\text{one-loop}} \). Here we note a couple of differences between the result in Eq. (2.19) and the results in [7, 12]. The first difference is that \( \tilde{m} \), which represents the linear divergence of \( \xi \), is absent in the previous results since the authors used dimensional and zeta-function regularization, respectively. Another difference is the sign of the \( m/4\pi \) term. The calculation of the vortex mass renormalization in [7, 12] was phrased as a counter term calculation; therefore, the result [7, 12] \( \delta \xi = -\frac{m}{4\pi} \) which superficially has the sign opposite to that in Eq. (2.19) is in full accord with our result and with the central charge renormalization.

2.3.2 Central Charge

The nonvanishing (and linearly divergent) correction to \( \xi \) implies that the classical central charge in Eq. (2.17) must be corrected too, in accordance with Eq. (2.19), so that \( \xi \) is converted to \( \xi_R \) in the central charge. Now we will explain where this correction comes from.
Figure 2.1: Tadpole diagrams determining one-loop correction to $\xi$.

The centrally extended superalgebra is

$$\{Q, (Q^\dagger) \gamma^0\} = 2 \left( P_0 \gamma^0 + P_1 \gamma^1 + P_2 \gamma^2 \right) - 2 \left( P_3 + \xi \int d^2x B \right), \quad (2.20)$$

where our conventions for the gamma matrices are summarized in Eq. (2.32) and $P_3$ is the "momentum" along the reduced direction,

$$P_3 = -\tilde{m} \int d^2x \left( i\tilde{\phi}^* \partial_t \tilde{\phi} + \bar{\tilde{\psi}} \gamma_0 \tilde{\psi} \right) \equiv \tilde{m} q. \quad (2.21)$$

Here $q$ is the Noether charge of the vortex,

$$\tilde{J}_\mu = -\left( i\tilde{\phi}^* \partial_\mu \tilde{\phi} + \bar{\tilde{\psi}} \gamma_\mu \tilde{\psi} \right), \quad \tilde{J}_0 = \int d^2x \tilde{J}_0. \quad (2.22)$$

The current $\tilde{J}_\mu$ defines a global $U(1)$ symmetry acting in the regulator sector. Below we will show that the corresponding charge fractionalizes. (In the low-energy sector it is related to the occurrence of the Chern–Simons term after the tilded fermion is integrated out.)

It is rather obvious that the $P_3$ term is in one-to-one correspondence with the fact that integrating out massive fermions in $2 + 1$ dimensions generates the Chern–Simons
term in the Lagrangian \([14, 15]\), which, in turn, makes the vortex electrically charged \([20]\). Since our theory is fully regularized, the superalgebra Eq. (2.20) presents the exact operator equality in an explicit representation (which is sometimes elusive in other regularizations.) The second line in Eq. (2.20) is \(-2Z_v\). Although the coefficient of the Chern–Simons term in the Lagrangian is dimensionless, integrating out heavy fermions in the central charge produces a term which has mass dimension 1. In fact, in Sec. 2.4 (see also Sec. 6.3) we will calculate the value of the Noether charge \(q\) (at one loop) and will show that \(q = -\frac{n}{2}\). Note that for odd \(n\) the charge is fractional, a well known phenomenon of charge fractionalization \([21]\).

Assembling two terms in the central charge and using the fact that \(q = -\frac{n}{2}\) we get

\[
|Z_{n,v}| = 2\pi n \xi + \hat{m} q = 2\pi n \xi - \frac{\hat{m} n}{2}
\]

\[
= n \left(2\pi \xi_R - \frac{m}{2}\right), \tag{2.23}
\]

where we used Eq. (2.19) to convert \(\xi\) into \(\xi_R\). The contribution due to \(P_3\) comes precisely in the combination ensuring that the bare parameter \(\xi\) is converted into the renormalized \(\xi_R\). Eq. (2.23) demonstrates the emergence of the quantum correction \(-\frac{m n}{2}\).

### 2.3.3 Renormalization of the vortex mass

To calculate the one-loop contribution to the vortex mass, we expand the Lagrangian in Eq. (2.9) around the background field, in the quadratic order, using the definitions in Eq. (2.18). It is convenient to introduce the following gauge-fixing term:

\[
\mathcal{L}_{gf} = -\frac{1}{2} \left( \frac{1}{e} \partial_\mu a^\mu + ie(\phi_v \eta^* - \phi_v^* \eta) \right)^2. \tag{2.24}
\]

The gauge-fixing term is chosen to cancel the terms \((\eta^* \phi_v)^2\) and \((\eta \phi_v^*)^2\) originating from the scalar potential in Eq. (2.12) as well as the term \(\partial_\mu a^\mu (\eta^* \phi_v - \eta \phi_v^*)\) arising from the term \(\mathcal{D}_\mu \phi^* \mathcal{D}_\mu \phi\) in Eq. (2.9). Note that under this gauge choice, \(a_0\) becomes a dynamical field, and one has to take its loop contribution into account. The corresponding ghost Lagrangian is

\[
\mathcal{L}_{gh} = \bar{c} \left[ -\frac{1}{e^2} \partial_\mu \partial^\mu - (2|\phi_v|^2 + \phi_v \eta^* + \phi_v^* \eta) \right] c, \tag{2.25}
\]
where $\bar{c}$ and $c$ are spin-zero complex fields with fermion statistics. We will drop the last two terms in Eq. (2.25) since they show up only in higher-order corrections. Assembling all the bosonic contributions, we get the following bosonic Lagrangian (at the quadratic order)

$$L^{(2)}_B = L^{(2)}_{g f} + L^{(2)}_B + L^{(2)}_{gh}$$

$$= |D^{\mu}_{\nu}\eta|^2 - e^2 (3|\phi_v|^2 - \xi)|\eta|^2$$

$$+ [D^{\mu\nu}\phi_v|^2 - 2ia_m (\eta^* D^{\mu}_{\nu}\phi_v - \eta D^{\nu}_{\mu}\phi^*_v)]$$

$$+ |D^{\mu\nu}\tilde{\phi}|^2 + \frac{1}{2e^2}(\partial_{\mu}a_m)^2 - |\phi_v|^2a_m^2 + \frac{1}{2e^2}(\partial_{\mu}N)^2 - N^2|\phi_v|^2$$

$$+ \bar{c} \left( -\frac{1}{e^2} \partial_{\mu}\partial^\mu - 2|\phi_v|^2 \right)c,$$

(2.26)

where $\mu = 0, 1, 2$ and $m = 1, 2$ (the fields $\eta$ and $a$ are defined in Eq. (2.18)). The last two lines in Eq. (2.26) include one complex scalar field with the fermion statistics and two real scalar fields with the boson statistics, satisfying the same equations of motion. If we impose the same boundary conditions on the fields $a_0, N, \bar{c}$ and $c$, (and we do), they produce the same determinants, and their contributions to the vortex mass cancel each other [12]. With this observation in mind, we will drop this line in what follows.

The transverse components of the gauge field, $a_1$ and $a_2$, can be combined into complex fields by defining

$$a^\pm = \frac{a^1 \pm ia^2}{\sqrt{2e}}.$$  

(2.27)

By the same token, we define $D^{\mu\nu}_\pm = D^{\mu\nu}_1 \pm iD^{\mu\nu}_2$. With these definitions Eq. (2.26) can be rewritten as follows:

$$L^{(2)}_B = |D^{\mu\nu}_\eta|^2 - e^2 (3|\phi_v|^2 - \xi)|\eta|^2$$

$$+ \partial_{\mu}a^+ \partial^\mu a^- - 2e^2|\phi_v|^2a^+a^- - \sqrt{2}ie (\eta^* a^+ D^{\nu}_{\mu}\phi_v - \eta a^- D^{\nu}_{\mu}\phi^*_v)$$

$$+ |D^{\mu\nu}_{\tilde{\phi}}|^2 + (e^2 (|\phi_v|^2 - \xi^2) - \tilde{m}^2) |\tilde{\phi}|^2.$$  

(2.28)

Note that, at the quadratic order, the tilded bosonic sector is decoupled from the fluctuations of the nontilded one, i.e. $\tilde{\phi}$ is coupled to the background fields only. (We will
soon see that the same decoupling occurs for the fermionic sector.) This allows us to consider the contributions of tilded and untilded fields separately.

### 2.3.4 One-loop contribution from the untilded sector

In the first part of this subsection we will compute the classical Hamiltonian (density) of the fluctuations. In the second part we will quantize the Hamiltonian by imposing canonical (anti)commutation relations. Finally we will compute the sum of the energies, which turns out to be vanishing. We first start with the bosonic Hamiltonian corresponding to the untilded part of the Lagrangian in Eq. (2.28), which can be written in the matrix form,

\[
\mathcal{H}^{(2)}_B = \left( \hat{\eta}, \ i\dot{a}_+ \right)^* \left( \begin{array}{c} \dot{\eta} \\ i\dot{a}_+ \end{array} \right) + \left( \begin{array}{c} \eta, \ ia_+ \end{array} \right)^* \mathcal{D}^2_B \left( \begin{array}{c} \eta \\ ia_+ \end{array} \right),
\]  

(2.29)

where we defined the quadratic bosonic operator

\[
\mathcal{D}^2_B = \left( \begin{array}{cc} -(\mathcal{D}_k^v)^2 + e^2(3|\phi_v|^2 - \xi) & \sqrt{2}e\mathcal{D}_v\phi_v \\ \sqrt{2}e(\mathcal{D}_v\phi_v)^* & -\partial_k^2 + 2e^2|\phi_v|^2 \end{array} \right).
\]  

(2.30)

Eq. (2.29) gives the classical Hamiltonian for the bosonic fields. The fermionic Lagrangian in Eq. (2.9) is already quadratic in the fermionic fields. Setting the bosonic fields to their background values gives the following quadratic Lagrangian for the untilded fermionic fields:

\[
\mathcal{L}^{(2)}_F = \frac{1}{e^2} \bar{\lambda}i\hat{\phi}\lambda + \bar{\psi}i\mathcal{D}\psi + i\sqrt{2}\left[\bar{\lambda}\psi\phi^* - \bar{\psi}\lambda\phi\right].
\]  

(2.31)

We choose the following set of \(\gamma\) matrices:

\[
\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2, \quad \gamma^2 = i\sigma_1.
\]  

(2.32)
With the chosen representation of $\gamma$ matrices the Hamiltonian corresponding to the Lagrangian in Eq. (2.31) reads

$$\mathcal{H}_F^{(2)} = -i \begin{pmatrix} \psi_1 & \psi_2 & \lambda_1/e & \lambda_2/e \end{pmatrix} \begin{pmatrix} 0 & D^u_+ & -\sqrt{2}e\phi_v & 0 \\ D^u_- & 0 & 0 & \sqrt{2}e\phi_v \\ \sqrt{2}e\phi_v^* & 0 & 0 & \partial_- \\ 0 & -\sqrt{2}e\phi_v^* & \partial_+ & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \lambda_1/e \\ \lambda_2/e \end{pmatrix}$$

$$= \begin{pmatrix} U \\ V \end{pmatrix} \begin{pmatrix} 0 & -iD_F \\ iD_F^\dagger & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix},$$

(2.33)

where we regrouped the components of $\lambda$ and $\psi$,

$$\{ U; \ V \} = \{ (\psi_1, \lambda_2/e); (\psi_2, \lambda_1/e) \},$$

(2.34)

and defined the fermionic operator,

$$D_F \equiv \begin{pmatrix} D^u_+ & -\sqrt{2}e\phi_v \\ -\sqrt{2}e\phi_v^* & \partial_- \end{pmatrix},$$

$$D_F^\dagger = \begin{pmatrix} -D^u_- & -\sqrt{2}e\phi_v \\ -\sqrt{2}e\phi_v^* & -\partial_+ \end{pmatrix}.$$ 

(2.35)

Supersymmetry of the Lagrangian reveals itself when we calculate the following quadratic fermionic operator:

$$D_F^\dagger D_F = \begin{pmatrix} - (D^u_k)^2 + e^2(3|\phi_v|^2 - \xi) & \sqrt{2}eD^u_- \phi_v \\ \sqrt{2}e(D^u_- \phi_v)^* & -\partial_k^2 + 2e^2|\phi_v|^2 \end{pmatrix},$$

(2.36)

which coincides with $D_B^2$ defined in Eq. (2.30),

$$D_B^2 = D_F^\dagger D_F.$$ 

(2.37)

By virtue of this identification we rewrite the full Hamiltonian for untilded fields in terms of the operators $D_F$ and $D_F^\dagger$,

$$\mathcal{H}^{(2)} = \left( \dot{\eta}, i\dot{a}_+ \right)^* \begin{pmatrix} \dot{\eta} \\ i\dot{a}_+ \end{pmatrix} + \left( \eta, i\dot{a}_+ \right)^* D_F^\dagger D_F \begin{pmatrix} \eta \\ i\dot{a}_+ \end{pmatrix}$$

$$- iU^\dagger D_F V + iV^\dagger D_F^\dagger U.$$ 

(2.38)
To quantize the Hamiltonian in Eq. (2.38) we will follow methods worked out long ago (e.g. [22]). First, we impose boundary conditions which are compatible with the residual supersymmetry. We place the system into a spherical two-dimensional “box” of radius $R$, with the assumption that $R$ is much larger than any length scale in the model at hand. To ensure that the energy associated with the boundary vanishes, we require all the fields to vanish at $r = R$. This condition does not break the residual supersymmetry since it is compatible with the transformations defined in Eq. (2.10).

Then we expand the fields in Eq. (2.38) in eigenmodes of the operators $D_B^2$ and the associated operator $D_B^{2'}$ defined as follows:

$$D_B^{2'} = D_FD_F^\dagger.$$  

(2.39)

The eigenvalue equations for these operators are

$$D_B^2 \xi_{n,\sigma} \equiv w_n^2 \xi_{n,\sigma}, \quad D_B^{2'} \xi'_{n,\sigma} \equiv w_n^2 \xi'_{n,\sigma}.$$  

(2.40)

The eigenvalues for both operators are the same: the eigenfunctions can be related to each other by

$$\xi'_{n,\sigma} = \frac{1}{w_n} D_F \xi_{n,\sigma}, \quad \xi_{n,\sigma} = \frac{1}{w_n} D_F^\dagger \xi'_{n,\sigma}.$$  

(2.41)

For each $w_n^2$ there are two independent solutions, which are labeled by subscript $\sigma$. The above statement excludes the zero modes, $w_n = 0$, which occur only in one of these operators, namely $D_B^2$, reflecting the translational invariance in the problem at hand. Usually, they are referred to as translational. Their fermion counterparts, the zero modes of $D_F$, are supertranslational modes. $D_F^\dagger$ has no zero modes.

The eigenfunctions $\xi_{n,\sigma}$ form an orthonormal and complete basis, in which we expand
the fields in Eq. (2.38)

\[
\begin{pmatrix}
\eta(t, x) \\
i a_+(t, x)
\end{pmatrix} = \sum_{n \neq 0} a_{n,\sigma}(t) \xi_{n,\sigma}(x),
\]

\[
\sigma = 1, 2
\]

\[
V(t, x) = \sum_{n \neq 0} v_{n,\sigma}(t) \xi_{n,\sigma}(x),
\]

\[
\sigma = 1, 2
\]

\[
U(t, x) = \sum_{n \neq 0} u_{n,\sigma}(t) \xi'_{n,\sigma}(x).
\]

(2.42)

Note that the zero modes do not enter in the expansion in Eq. (2.42), nor do they appear in the Hamiltonian in Eq. (2.38). For nonzero modes the ratio of the bosonic to fermionic modes is 1:2, i.e., we have two complex expansion coefficients \(a_{n,\sigma}(t)\) for bosons and four complex expansion coefficients \(v_{n,\sigma}(t)\) and \(u_{n,\sigma}(t)\) for fermions, for each value of \(w_n^2\). As we will see below, this is precisely what is needed for cancelation. Let us note in passing that for zero modes the ratio is 1:1. We have one complex bosonic modulus and one fermionic.

Using the above mode decompositions in Eq. (2.38), we arrive at an infinite set of oscillators,

\[
\mathcal{H}^{(2)} = \sum_{n, n' \neq 0, \sigma, \sigma'} \left( \hat{a}_{n,\sigma}^* \hat{a}_{n',\sigma'} \xi_{n,\sigma}^\dagger \xi_{n',\sigma'}^\dagger + a_{n,\sigma}^* a_{n',\sigma'} \xi_{n,\sigma}^\dagger D_F^\dagger D_F \xi_{n',\sigma'}^\dagger 
- \frac{i}{w_n} v_{n,\sigma}^* u_{n',\sigma'} \xi_{n,\sigma}^\dagger D_F^\dagger D_F \xi_{n',\sigma'}^\dagger + i w_n u_{n,\sigma}^* v_{n',\sigma'} \xi_{n,\sigma}^\dagger \xi_{n',\sigma'}^\dagger \right). \tag{2.43}
\]

To be accurate, we should note that here we use integration by parts in the last term, which means that Eq. (2.43) is valid up to a full spatial derivative. Now, for each oscillator, the coefficients \(a, \dot{a}, v, u\) and their complex conjugated must be represented as linear combinations of the corresponding creation and annihilation operator subject to the standard (anti)commutation relations. This procedure parallels that discussed in
detail in Ref. [8]. The only difference is that in [8] for each mode one has an oscillator for one real degree of freedom, while in the case at hand we deal with a complex degree of freedom which is equivalent to two real degrees of freedom. We will not dwell on details referring the reader to Ref. [8]. Imposing the appropriate (anti)commutation relations on the creation and annihilation operators, we get for expectation values of bilinears in the vortex ground state

\[ \langle a_{n,\sigma}^* a_{n',\sigma'} \rangle_{\text{vor}} = \frac{1}{2w_n} \delta_{nn'} \delta_{\sigma\sigma'}, \quad \langle \hat{a}_{n,\sigma}^* \hat{a}_{n',\sigma'} \rangle_{\text{vor}} = \frac{w_n}{2} \delta_{nn'} \delta_{\sigma\sigma'}, \]

\[ \langle u_{n,\sigma}^* v_{n',\sigma'} \rangle_{\text{vor}} = \frac{i}{2} \delta_{nn'} \delta_{\sigma\sigma'}, \quad \langle v_{n,\sigma}^* u_{n',\sigma'} \rangle_{\text{vor}} = -\frac{i}{2} \delta_{nn'} \delta_{\sigma\sigma'}, \] (2.44)

where the angular brackets mark the vortex expectation value. Expectation values of all other bilinears vanish. If we substitute these results in Eq. (2.43) we immediately see that the one-loop correction in the untilded sector vanishes locally, i.e. in the Hamiltonian density. Needless to say, it vanishes in the integral \( \int d^2x \mathcal{H}^{(2)} \) too.

Thus, we demonstrated the cancelation of the bosonic and fermionic contributions mode by mode, for each given \( n \). This vanishing result shows that the vortex mass receives no correction from the untilded sector. If we did not have the \( \tilde{\Phi} \) multiplet, this would be the final answer. However, the theory per se is ill-defined without the tilded sector.

From Eq. (2.19) we see that in the absence of \( \tilde{\phi} \), the FI parameter \( \xi \) would be linearly divergent at one loop. With \( \tilde{\phi} \) included, the theory is regularized; cancelation of loops in Fig. 2.1 takes place. The linear divergence is replaced by the linear dependence of \( \xi \) on \( \tilde{m} \). The latter parameter is kept large, but finite till the very end. It is only natural that the linear dependence of \( M_{v,R} \) on \( \tilde{m} \) will be provided by the tilded sector contribution (Sec. 2.3.5).

### 2.3.5 The tilded sector (regulator) contribution in \( M_v \)

The Lagrangian for the tilded sector is

\[ \tilde{\mathcal{L}}^{(2)} = |D_\mu \tilde{\phi}|^2 + \left( e^2 (|\phi_v|^2 - \xi^2) - \tilde{m}^2 \right) |\tilde{\phi}|^2 + \bar{\tilde{\psi}} i \tilde{\gamma} \tilde{\psi} - \tilde{m} \tilde{\psi} \tilde{\psi}. \] (2.45)
The corresponding Hamiltonian density then takes the form
\[ \mathcal{H}^{(2)} = |\tilde{\phi}|^2 + \tilde{\phi}^* (-\mathcal{D}_+^v \mathcal{D}_-^v + \tilde{m}^2) \tilde{\phi} \]
\[ + \left( \begin{array}{c} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{array} \right)^* \left( \begin{array}{cc} \tilde{m} & -i \mathcal{D}_+^v \\ -i \mathcal{D}_-^v & -\tilde{m} \end{array} \right) \left( \begin{array}{c} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{array} \right), \] (2.46)

where \( \mathcal{D}_+^v = \mathcal{D}_1^v \pm i \mathcal{D}_2^v \) and we used Eq. (2.15). For what follows it is important to know that the operator \( \mathcal{D}_+^v \mathcal{D}_-^v \) has no zero modes.

If we denote the eigenvalues of the bosonic operator
\[ -\mathcal{D}_+^v \mathcal{D}_-^v + \tilde{m}^2 \] (2.47)
by \( \Delta \) (\( \Delta \) is strictly larger than \( \tilde{m}^2 \)), for each given \( \Delta \) we have two eigenmodes of the associated fermion equation
\[ \left( \begin{array}{cc} \tilde{m} & -i \mathcal{D}_+^v \\ -i \mathcal{D}_-^v & -\tilde{m} \end{array} \right) \left( \begin{array}{c} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{array} \right) = \pm \sqrt{\Delta} \left( \begin{array}{c} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{array} \right). \] (2.48)

The eigenfunctions have the following structure. If \( \tilde{\psi}_1 \) is the normalized eigenfunction of the operator \( -\mathcal{D}_+^v \mathcal{D}_-^v \), then \( \tilde{\psi}_2 \) is the corresponding eigenfunction of the conjugated operator \( -\mathcal{D}_+^v \mathcal{D}_-^v \) times
\[ \sqrt{\frac{\pm \sqrt{\Delta} - \tilde{m}}{\pm \sqrt{\Delta} + \tilde{m}}} \]
depending on the sign in the eigenvalue Eq. (2.48). Thus, for each complex boson mode with the eigenvalue \( \Delta \) we have two complex fermion modes with the eigenvalues \( \pm \sqrt{\Delta} \). This balance of modes guarantees that the corresponding quantum corrections to \( M_v \) vanish.

This is not the end of the story, however. There is one additional (complex) fermion mode with \( \Delta \) exactly equal to \( \tilde{m}^2 \). (The above statement refers to the elementary vortex with the unit winding number. Generalization to higher winding numbers is straightforward.) Let us focus on this unbalanced mode which will be solely responsible for the contribution of the tilded sector in \( M_v \).

From Eqs. (2.48) and (2.15) it is clear that this fermion mode has the form
\[ (2.49) \]
where the eigenvalue on the right-hand side of Eq. (2.48) is $-\tilde{m}$. This gives rise to the following contribution in the energy density:

$$E^{(0)} = -\tilde{m} (\tilde{\psi}_2^2)^* \tilde{\psi}_2^2. \tag{2.50}$$

We proceed to quantization in the standard manner. To this end we represent

$$\tilde{\psi}_2^2 = \alpha^\dagger(t) \varphi(x), \tag{2.51}$$

where $\varphi(x)$ is the normalized $c$-numerical part of the zero mode while $\alpha^\dagger$ is the operator part with the appropriate anticommutation relation implying

$$\langle \alpha \alpha^\dagger \rangle = \frac{1}{2}. \tag{2.52}$$

Now, the contribution of the tilded sector to $M_v$ obviously reduces to

$$\delta M \equiv \int d^2 x \langle E^{(0)} \rangle = -\tilde{m} \langle \alpha \alpha^\dagger \rangle = -\frac{\tilde{m}}{2}. \tag{2.53}$$

Eq. (2.53) gives the only nonvanishing quantum correction,

$$M_R \equiv M + \delta M = 2\pi \xi - \frac{\tilde{m}}{2} = 2\pi \xi_R - \frac{m}{2}, \tag{2.54}$$

where we again used Eq. (2.19) to convert $\xi$ to $\xi_R$. Comparing this result with the renormalization of the central charge in Eq. (2.23), we conclude that the BPS saturation does indeed hold at the quantum level.

### 2.3.6 Higher orders

Let us discuss now what changes as we pass to higher orders of perturbation theory. Returning to Sec. 2.3.2 and, in particular, to Eq. (2.21), it is not difficult to understand that the relation $Z = 2\pi \xi - \frac{1}{2} \tilde{m}$ (for the elementary vortex) remains exact to all orders. Indeed, $q$ is half-integer and the relation $q = -\frac{1}{2}$ for the elementary vortex cannot receive corrections in $e/\sqrt{\xi}$ (A simple dimensional analysis shows that perturbative corrections run in powers of $e/\sqrt{\xi}$). If we define $\tilde{\xi}$ as

$$\tilde{\xi} = \xi - \frac{\tilde{m}}{4\pi}, \tag{2.55}$$
where $\xi$ and $\tilde{m}$ are bare parameters, then the statement that

$$M_v = Z = 2\pi \tilde{\xi}$$

(2.56)

is valid to all orders. The term $\tilde{m}$ comes from the ultraviolet, and, therefore, it is natural to refer to $\tilde{\xi}$ as to an “effective ultraviolet parameter.” Eq. (2.56) is akin to the NSVZ theorem for the gauge coupling renormalization in four dimensions \cite{23, 24, 25}: being expressed of terms of the ultraviolet (bare) parameters the gauge coupling renormalization is limited to one loop (see also the second paper in \cite{26}).

Corrections in powers of $e/\sqrt{\xi}$ arise if we decide to express the result in terms of $\xi_R$, a parameter defined in the infrared; the expression of $\xi_R$ in terms of $\xi$ does contain an infrared contribution (otherwise, odd powers of $e$ could not have entered, see Sec. 2.3.1). Generalizing the arguments of \cite{26} we can write, instead of Eq. (2.19)

$$\tilde{\xi} = \xi_R \left( 1 - \frac{1}{2\sqrt{2\pi}} \frac{e}{\sqrt{\xi_R}} \right).$$

(2.57)

Eqs. (2.56) and (2.57) assembled together present a perturbatively exact result for $M_v = Z$.

### 2.4 Calculation of the Noether charge $q$

In Sec. 2.3.2 we used the fact that the Noether U(1) charge of the elementary vortex is $-1/2$. The Noether charge is saturated by the fermion term in Eq. (2.21),

$$q = - \int d^2 x \bar{\psi} \gamma^0 \psi.$$  

(2.58)

Here we will explore this issue in more detail. The vortex Noether charge can be calculated in a number of ways. The most straightforward calculation is that of the Feynman diagram depicted in Fig. 2.2, using the background field expansion. This expansion is justified because the background photon field is small compared to the value of $\tilde{m}$ (in the very end we want to tend $\tilde{m}$ to infinity).

We trace the term linear in the momentum $k$ of the produced photon (assuming $k$ to be small). Terms of the zeroth order vanish because of the gauge invariance. Quadratic and higher order terms in $k$ are irrelevant since they are suppressed by powers
Figure 2.2: Vertex for the Noether charge calculation

of $1/\tilde{m}$. Therefore for our purposes it is sufficient to limit ourselves to the leading term (proportional to $F_{\alpha\beta}$). Using $\gamma^\mu$ in the upper vertex in Fig. 2.2 (denoted by the closed circle) we get the Noether current in the background field in the form

$$\bar{\tilde{\psi}} \gamma^\mu \tilde{\psi} \rightarrow \tilde{m} F_{\alpha\beta} \epsilon^{\mu\alpha\beta} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(p^2 + \tilde{m}^2)^2} = \frac{1}{8\pi} F_{\alpha\beta} \epsilon^{\mu\alpha\beta}. \quad (2.59)$$

The current in Eq. (2.59) couples to the gauge field $A_{\mu}$, giving a term of the form $A_{\mu} F_{\alpha\beta} \epsilon^{\mu\alpha\beta}$, which is nothing but the Chern–Simons term. Now, if we set $\mu = 0$ in Eq. (2.59) and invoke the standard value of the magnetic flux,

$$\int d^2 x B = 2\pi,$$

we immediately get

$$\langle q \rangle = -1/2. \quad (2.60)$$

**2.5 Conclusion**

In this study we showed that the mass and the central charge of the $\mathcal{N} = 2$ vortices in $2 + 1$ dimensions, being expressed in terms of $\xi_R$, get a quantum correction $-m n/2$ where $m$ is the mass of the charged bosons (fermions) and $n$ is the winding number of the vortex. The equality of the corrections to the vortex mass/central charge shows that the BPS saturation persists at the quantum level. Our result is in agreement with the previous ones [12, 10].

New elements of our work (compared to [12] and [10]) are as follows: we use a more straightforward and physically transparent regularization scheme which captures linearly divergent terms invisible in the regularization methods used in the previous
papers. In our scheme we have a massive regulator multiplet acting in loops as an ultraviolet cutoff. In the limit of infinitely large regulator mass, regulator’s role is taken over by the Chern–Simons term. We establish a contact between one-loop calculations and the general operator expression for the central charge (obtained within the same regularization scheme). Analyzing both, in a single package, we are able to reveal a simple physical interpretation behind the occurrence of the $-mn/2$ shift, and obtain all-order results (Sec. 2.3.6). This concludes our discussion of the normalization of the central charge and the mass of the vortex, and we move to the discussion of strings which can be treated as extension of 2+1 dimensional vortices into the usual 3+1 dimensional space-time.
Chapter 3

Cosmic Strings and Gravitational Radiation

Although topological defects can form as a result of spontaneously broken global symmetries, most of the attention in the literature has been focused on defects originating from broken gauge symmetries. This is because grand unified theories have gauge symmetries which are eventually spontaneously broken down to the symmetry of the Standard Model. Various types of topological defects are predicted to have formed during the phase transitions of unified theories of electromagnetic, strong and weak forces. The nature of the topological defect formed depends on the symmetry broken and the energy scale at which the transition occurs. It is well known that inflation dilutes monopoles along with any other topological defects, therefore there may be observable defects left if they were formed after the inflation or in the latest stages of it.

Cosmic strings are one dimensional topological defects predicted by a large class of unified theories [2, 28, 29]. They were first considered as the seeds of structure formation [30, 31], however, later, it was discovered that cosmic strings would result in a cosmic microwave background (CMB) angular power spectrum which would be different than the observed one. Therefore although cosmic strings can still contribute to structure formation, they cannot be the dominant source. They are still candidates for the generation of other observable astrophysical phenomena such as high energy cosmic rays, gamma ray bursts and gravitational waves [2, 32, 33, 34]. Furthermore, recently it has been
shown that in string-theory-inspired cosmological scenarios cosmic strings may also be generated [35, 36, 37, 38]. They are referred to as cosmic superstrings. This realization has revitalized interest in cosmic strings and their potential observational signatures. There are some important differences between cosmic strings and cosmic superstrings. The reconnection probability, $p$, which is the probability of colliding strings to inter-commute, is unity for cosmic strings [2, 39]. Cosmic superstrings, on the other hand, have reconnection probability less than unity. This is a result of the probabilistic nature of their interaction and also the fact that it is less probable for strings to meet since they live in higher dimensional space [40]. The value of $p$ ranges from $10^{-3}$ to 1 in different theories [41]. The stability of cosmic strings depends on the topological properties of the vacuum manifold, therefore they could also be unstable, decaying long before the present time. In this case, however, they may also leave behind a detectable gravitational wave signature [42].

In the early universe, a network of cosmic strings evolves toward an attractor solution called the “scaling regime”. The scaling regime can be understood as follows: if the density of strings increases, the strings meet more frequently, creating small loops which eventually decay by emitting gravitational radiation. This reduces the density of strings in the network, reducing the probability of further collisions. These two opposing effects balance in the scaling regime. In the scaling regime the statistical properties of the network, such as the average distance between strings and the size of loops at formation, scale with the cosmic time. In addition, the energy density of the network remains a small fraction of the energy density of the universe. For cosmic superstrings in the scaling regime, the density of the network $\rho$ is inversely proportional to the reconnection probability $p$, that is $\rho \propto p^{-\beta}$. The value of $\beta$ is still under debate [43, 44, 45], and as a placeholder in our analysis we assume that $\beta = 1$.

The gravitational interaction of strings is characterized by their tension $\mu$, or more conveniently by the dimensionless parameter $G\mu$. The current CMB bound on the tension is $G\mu < 6.1 \times 10^{-7}$ [46, 47]. It was first believed that gravitational radiation from cosmic strings with $G\mu \ll 10^{-7}$ would be too weak to observe. However it was later shown that gravitational radiation produced at cusps, which have large Lorentz boosts, could lead to a detectable signal [48, 49, 50]. Gravitational radiation bursts from (super)strings could be observable by current and planned gravitational wave detectors.
for values of $G\mu$ as low as $10^{-13}$, which may provide a test for a certain class of string theories [51]. Indeed, searches for burst signals using ground-based detectors are already underway [52].

A gravitational background produced by the incoherent superposition of cusp bursts from a network of cosmic strings and superstrings was considered in [53]. In this study the computation is extended to include kinks, long-lived sharp edges on strings that result from intercommutations. It is found that kinks contribute at almost the same level as cusps. The detectability of the total background produced by cusps and kinks on cosmic string loops by a wide range of current and planned experiments is investigated.

We organize the discussion of the stochastic background of gravitational waves (SBGW) as follows:

We have introduced the field theoretical description of strings in Sec. 1.2, and have shown that the dynamics of the formed strings can be described in the thin wire approximation. In Sec. 3.1 the equations of motion for strings are derived in the thin wire approximation, and the energy momentum tensor for the string configuration is calculated. The energy momentum tensor of cusps and kinks is used to calculate gravitational waves in the weak field limit. We also discuss the energy carried in the gravitational waves (see Sec. 7.1) which is important for detection of the waves in experiments. This section is mainly intended for developing the necessary background and introducing the notation used in the second part of the thesis. The notation used in this study follows the conventions of [48, 49], and more details can be found in these references.

In Chap. 4 we derive the expression for the stochastic background, which is a double integral over redshift and loop length, evaluated numerically and analytically (with certain approximations). In Sec. 4.7 we discuss the observability by various experiments. In Chap. 5, spatial anisotropies in SBGW arising from random fluctuations in the number of sources are discussed. Such anisotropies are analogous to the anisotropies observed in the CMB radiation and would carry additional information about the gravitational-wave sources that generated them.
3.1 Classical String Theory

As discussed in Sec. 1.2, for string or vortex solutions, the gauge field decays exponentially with the distance. The characteristic length scale of the field range is inversely proportional to the gauge boson mass, which is acquired after the gauge symmetry is broken spontaneously. Therefore, the flux of gauge field forming the string is confined to a length scale which is much smaller than any astrophysical scale, which means that the string can be treated as a zero thickness wire, i.e. one dimensional object. The action for a one dimensional string can be obtained by covariantly extending the relativistic action for a point particle. The action for a point particle can be written as

\[ S = -m \int ds = -m \int \frac{ds}{dt} dt, \]  

(3.1)

where \( m \) is the mass of the particle and \( ds \) is the proper differential length. The integral limits are the initial and final times. Therefore the action is essentially the proper distance between the initial and final space-time coordinates, as illustrated in Fig. 3.1.

![Figure 3.1: Path of a relativistic particle in the x - t plane](image-url)
A string can be treated as a collection of point particles of mass $\mu ds$, which requires an additional integration over the parameter that parameterizes the string.

### 3.2 Nambu Goto Action

The action for a string can be written as the world sheet area of the surface traced by the string. Let us first consider an Euclidean space with the metric $g^{\mu\nu}$. A two dimensional surface can be parameterized by two parameters, say $\xi^1$ and $\xi^2$. In this space the differential length can be written as

$$
 ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} d\xi^a d\xi^b \\
 \equiv \gamma_{ab} d\xi^a d\xi^b,
$$

where $\gamma_{ab}$ is the induced metric which is defined as

$$
 \gamma_{ab} = g^{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b}.
$$

(3.2)

The induced metric can be used to write the differential lengths or areas on a given surface, $X^\mu(\xi^1, \xi^2)$. The differential area can be written as

$$
 dA = \sqrt{\gamma_{00}} \sqrt{\gamma_{11}} d\xi^0 d\xi^1 \sin \theta = \sqrt{\gamma_{00}} \sqrt{\gamma_{11}} d\xi^1 d\xi^2 \sqrt{1 - \cos^2 \theta} \\
 = \sqrt{\gamma_{00} \gamma_{11} - \gamma_{01}^2} d\xi^1 d\xi^2 \\
 = \sqrt{\gamma} d^2 \xi,
$$

(3.4)

where $\theta$ is the angle between the infinitesimal line segments and $\gamma = \det \gamma_{ab}$. We also used the equality $\cos^2 \theta = \frac{\gamma_{01}^2}{\gamma_{00} \gamma_{11}}$. Extending this definition to world-sheet coordinates, the action for a string can be written as proportional to the world-sheet area traced by the string as illustrated in Fig. 3.2. Therefore the action can be written as

$$
 S = -\mu \int d\tau d\sigma \sqrt{-\gamma},
$$

(3.5)

where the proportionality constant $\mu$ is the tension of the string. We also choose the world-sheet coordinates $\xi^a$ as $\sigma$ and $\tau$. The sign of the determinant is a reflection of the transition from the Euclidean space to Minkowskian space. This action is the Nambu-Goto action, which can be used to describe cosmic strings in the thin wire approximation [2, 32].
Let us consider the strings in the Minkowskian space. In this case the induced metric can be written as

$$\gamma_{ab} = \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \quad (3.6)$$

where $a$ and $b$ denote world sheet coordinates, and the corresponding Lagrangian becomes

$$\mathcal{L} = -\mu \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}, \quad (3.7)$$

where dot and prime denote differentiation with respect to $\tau$ and $\sigma$, respectively. The equation of motion can be obtained by varying the Lagrangian with respect to the coordinates. To this end it is useful to use the following formula for the determinant:

$$\gamma = e^{tr \ln \gamma_{ab}}, \quad (3.8)$$

and the variation of the determinant with respect to the matrix elements reads

$$\delta \gamma = e^{tr \ln (\gamma_{ab} + \delta \gamma_{ab})} - \gamma \simeq \gamma^{a} b \delta \gamma_{ab}, \quad (3.9)$$
where $\gamma^{ab}$ is the inverse of the induced metric. The variation of the action can be written as

$$\delta L = \frac{\partial L}{\partial \gamma} \frac{\partial \gamma}{\partial \gamma^{ab}} \left( \frac{\partial \gamma^{ab}}{\partial X^\mu} \delta X^\mu + \frac{\partial \gamma^{ab}}{\partial X^\mu, c} \delta X^\mu, c \right)$$

$$= -\mu \sqrt{-\gamma} \gamma^{ab} X^\mu, a \delta X^\mu, b ,$$

(3.10)

where $X^\mu, a \equiv \frac{\partial X^\mu}{\partial \sigma^a}$. The equation of motion following from Eq. (3.10) is

$$\partial_\mu L - \partial_a \frac{\partial L}{\partial \partial X^\mu, a} = \mu \partial_a (\sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu) = 0 .$$

(3.11)

A properly chosen parametrization yields

$$\sqrt{-\gamma} \gamma^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(3.12)

This simplifies the equation of motion to the well-known wave equation,

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0$$

(3.13)

The parametrization also entails the Virasoro conditions given by

$$\gamma_{00} + \gamma_{11} = \ddot{X} \cdot \dot{X} + X' \cdot X' = 0 \text{ and } \gamma_{01} = \dddot{X} \cdot \dot{X} = 0,$$

(3.14)

where dot and prime denote derivatives with respect to $\tau$ and $\sigma$ respectively. If we define $\sigma_\pm = \tau \pm \sigma$, the equation of motion becomes

$$\partial_+ \partial_- X^\mu = 0,$$

(3.15)

which is solved by left and right moving waves,

$$X^\mu = \frac{1}{2} \left( X^\mu_+ (\sigma_+) + X^\mu_- (\sigma_-) \right) .$$

(3.16)

Furthermore Virasoro conditions in Eq. (3.14) simplify to

$$\dot{X}^2_\pm = 1,$$

(3.17)

where dot now represents the derivative with respect to the (unique) argument of the functions $X^\mu_\pm$. We require that $X^\mu(\sigma, \tau)$ is periodic in $\sigma$ with period $l$, which is the
length of the loop. This implies that the functions $X^\mu_{\pm}$ are periodic functions with the same period. The period in $t$ is $l/2$ since

$$X^\mu(\sigma + l/2, \tau + l/2) = X^\mu(\sigma, \tau). \quad (3.18)$$

The energy momentum tensor corresponding to the Nambu-Goto action can be calculated by varying Eq. (3.10) with respect to the metric, which yields

$$T_{\mu\nu}(x) = -2 \frac{\delta S}{\delta \eta_{\mu\nu}} = \mu \int d\tau d\sigma (\dot{X}^\mu \dot{X}^\nu - X^\nu X'^\mu) \delta^{(4)}(x-X)$$

$$= \frac{\mu}{2} \int d\sigma_- d\sigma_+ (\dot{X}_+^\mu \dot{X}_-^\nu + \dot{X}_-^\nu \dot{X}_+^\mu) \delta^{(4)}(x-X). \quad (3.19)$$

It is convenient to transform the energy momentum tensor into the momentum space by Fourier transformation

$$T_{\mu\nu}(k) = \frac{1}{T} \int_0^T dt \int d^3 \vec{x}' e^{i(wt - \vec{k} \cdot \vec{x}')} T_{\mu\nu}(\vec{x}', t), \quad (3.20)$$

where $T$ is the fundamental period of the source, and $k = (w, \vec{k})$ Transforming Eq. (3.19) gives us the energy momentum tensor in momentum space

$$T_{\mu\nu}(k) = \frac{\mu}{T_l} \int d\sigma_- d\sigma_+ (\dot{X}_+^\mu \dot{X}_-^\nu + \dot{X}_-^\nu \dot{X}_+^\mu) e^{-\frac{i}{2}(k \cdot X_+ + k \cdot X_-)}, \quad (3.21)$$

where we define

$$\dot{X}_+^\mu \dot{X}_-^\nu = \frac{1}{2}(\dot{X}_+^\mu \dot{X}_-^\nu + \dot{X}_-^\nu \dot{X}_+^\mu). \quad (3.22)$$

The nice property of Eq. (3.21) is that integration variables decouple, hence two integrals can be calculated independently,

$$I_{\pm}^\mu(k) = \int_0^l d\sigma_+ \dot{X}_+^\mu e^{-\frac{i}{2}k \cdot X_\pm}. \quad (3.23)$$

The energy momentum tensor can be expressed in terms of $I_{\pm}^\mu$ as follows;

$$T_{\mu\nu}(k) = \frac{\mu}{T} I_{\pm}^\mu I_{\pm}^\nu, \quad (3.24)$$

where we used $T_l = \frac{l}{2}$. This equation expresses the energy momentum tensor of the string in terms of its geometrical shape. In Sec. 3.4, we will calculate the energy momentum tensor separately for kinks and cusps since they have different geometrical shapes. The energy momentum tensor is the source of gravitational waves, which are small perturbations in the metric. The generation of gravitational waves for arbitrary sources is discussed in the following chapter, which will be applied to the case of cosmic strings in the subsequent sections.
3.3 Gravitational Radiation

In this section we consider gravitational waves created by cusps and kinks. For completeness we follow closely the analysis in [48, 49], and reproduce a number of their results. We begin with a derivation for the metric perturbation in terms of the Fourier transform of the stress energy tensor of the source. We then write the stress energy tensor for a relativistic string and compute its Fourier transform. Using these results we then compute the gravitational waveforms produced by cusps and kinks on cosmic strings.

3.3.1 The Weak Field Approximation

Gravitational waves from a source can be calculated using the weak field approximation [54],

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (3.25) \]

where \( \eta_{\mu\nu} \) is the Minkowski metric with positive signature and \( h_{\mu\nu} \) is the metric perturbation. At the linear order the connection can be written as

\[ \Gamma^{\lambda}_{\mu\kappa} = \frac{1}{2} g^{\lambda\beta} (\partial_\kappa g_{\beta\mu} + \partial_\mu g_{\kappa\beta} - \partial_\beta g_{\mu\kappa}) \]

\[ \simeq \frac{1}{2} \eta^{\lambda\beta} (\partial_\kappa h_{\beta\mu} + \partial_\mu h_{\kappa\beta} - \partial_\beta h_{\mu\kappa}), \quad (3.26) \]

and the Riemann tensor simplifies to

\[ R^{\lambda}_{\mu\nu\kappa} = \partial_\kappa \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\lambda}_{\kappa\delta} \Gamma^{\delta}_{\mu\nu} - \Gamma^{\lambda}_{\mu\delta} \Gamma^{\delta}_{\kappa\nu} \simeq \partial_\kappa \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\kappa}. \quad (3.27) \]

It is convenient to work in the Harmonic gauge which is given by

\[ g^{\mu\nu} \Gamma^{\lambda}_{\mu\nu} = 0. \quad (3.28) \]

In this gauge, the linearized Ricci tensor is

\[ R_{\mu\kappa} \simeq \frac{1}{2} \partial_\lambda \partial^{\lambda} h_{\mu\kappa}. \quad (3.29) \]

Substituting into Einstein’s equations yields

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \simeq \frac{1}{2} (\partial_\lambda \partial^{\lambda} h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \partial_\lambda \partial^{\lambda} h) = -8\pi G T_{\mu\nu}, \quad (3.30) \]
where \( R \) is the Ricci scalar, \( T_{\mu\nu} \) is the energy momentum tensor of matter and \( h = \eta_{\mu\nu}h^{\mu\nu} \). Defining \( \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu}h \) further simplifies Eq. (3.30),

\[
\partial_{\lambda} \partial^{\lambda} \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu},
\]

which is a wave equation with a source term. We can rewrite this equation in the frequency domain as,

\[
(w^2 + \nabla^2)\bar{h}_{\mu\nu}(\vec{x}, w) = -16\pi G T_{\mu\nu}(x, w),
\]

where

\[
\bar{h}_{\mu\nu}(\vec{x}, w) = \int dt e^{iwt} \bar{h}_{\mu\nu}(\vec{x}, t).
\]

The nonhomogeneous wave equation in Eq. (3.32) has the solution of the form

\[
\bar{h}_{\mu\nu}(\vec{x}, w) = -16\pi G \int d^3 \vec{x}' G(\vec{x} - \vec{x}', w) T_{\mu\nu}(x', w),
\]

where \( G(\vec{x} - \vec{x}', w) \) is the Green’s function for the operator \( w^2 + \nabla^2 \) defined by

\[
(w^2 + \nabla^2)G(\vec{x} - \vec{x}', w) = -4\pi \delta^{(3)}(\vec{x} - \vec{x}').
\]

The partial differential in Eq. (3.35) can be easily solved in the momentum space:

\[
G(k, w) = \frac{1}{2\pi^2} \frac{1}{k^2 - w^2},
\]

where

\[
G(\vec{k}, w) = \int d^3 \vec{r} e^{i\vec{k} \cdot \vec{r}} G(\vec{r}, w).
\]

The Green’s function can be Fourier transformed back to the \( \vec{x} \)-space by integrating on the contour shown in Fig. 3.3, which yields

\[
G(\vec{x} - \vec{x}', w) = \int d^3 k e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} G(\vec{k}, w)
\]

\[
= \frac{1}{i\pi r} \int_0^{\infty} dk \frac{1}{(k + w)(k - w)} \left( e^{-ikr} - e^{ikr} \right)
\]

\[
= \frac{e^{iwr}}{r},
\]

(3.38)
where $r = |\vec{x} - \vec{x}'|$. Inserting the Green’s function into Eq. (3.34), the metric perturbations are calculated as

$$\tilde{h}_{\mu\nu}(\vec{x}, w) = -16\pi G e^{iw|\vec{x}|} \frac{e^{iw|\vec{x}|}}{|\vec{x}|} \mathcal{T}_{\mu\nu}(\vec{k}, w),$$

(3.39)

where $\vec{k} = w\hat{x}$ and $\mathcal{T}_{\mu\nu}(\vec{k}, w)$ is the energy momentum tensor in the momentum space, as defined in Eq. (3.20). Eq. (3.39) relates gravitational waves to energy momentum tensor of the source. The gravitational wave detectors measure the energy associated with the gravitational waves. Therefore we will also need to calculate the energy carried away by the gravitational waves, which is considered in Chap. 7. We now proceed to calculate the metric perturbations generated by cusps and kinks.
3.4 Cusps on Cosmic Strings

Let us start with the geometrical interpretation of, $X^2_{\pm} = 1$, Eq. (3.17). It tells us that $X_{\pm}$ trace a unit sphere centered at the origin, which is called Kibble-Turok sphere [55]. Integrating $\dot{X}_{\pm}$ and using the periodicity, we get

$$\int_0^l \dot{X}_{\pm}(\sigma_{\pm}) d\sigma_{\pm} = 0, \quad (3.40)$$

which implies that $\dot{X}_+$ and $\dot{X}_-$ cannot lie completely in a single hemisphere and therefore they may intersect at some point(s). We choose our parametrization and the coordinate system such that the intersection occurs at the parameters $\sigma_{\pm} = 0$ at the origin, that is $X^\mu_\pm(0) = 0$. $X_{\pm}(\sigma_{\pm})$ and $\dot{X}_{\pm}(\sigma_{\pm})$ can be expanded around $\sigma_{\pm} = 0$

$$X^\mu_\pm(\sigma_{\pm}) = l^\mu_\pm \sigma_{\pm} + \frac{1}{2} \ddot{X}^\mu_{\pm} \sigma_{\pm}^2 + \frac{1}{6} X^{(3)\mu}_{\pm} \sigma_{\pm}^3, \quad (3.41)$$

$$\dot{X}^\mu_\pm(\sigma_{\pm}) = l^\mu_\pm + \ddot{X}^\mu_{\pm} \sigma_{\pm} + \frac{1}{2} X^{(3)\mu}_{\pm} \sigma_{\pm}^2, \quad (3.42)$$

where $l^\mu_\pm = \dot{X}^\mu_\pm(0)$. We can calculate $I^\mu_\pm$ for cusps using the expansion in Eq. (3.41).

First of all, we note that the first term Eq. (3.42) is pure gauge, it can be removed by a coordinate transformation [48]. Furthermore imposing Virasoro condition in Eq. (3.17) gives

$$l_\pm \cdot \ddot{X}_\pm = 0, \quad \text{and} \quad l_\pm \cdot X^{(3)}_\pm = -\ddot{X}_\pm^2. \quad (3.43)$$

When the line of sight $k$ is in the direction of $l$ we have $l_\pm = k/w$, which gives

$$-i k \cdot X_\pm = \frac{i}{6} w \ddot{X}_\pm^2 \sigma_{\pm}^3. \quad (3.44)$$

If we plug in the expansion in Eq. (3.41) into Eq. (3.23) we get,

$$I^\mu_\pm(k) = \ddot{X}^\mu_\pm \int_0^l d\sigma \sigma e^{i \frac{w}{12} \ddot{X}^2_\pm \sigma^3} \simeq \frac{\ddot{X}^\mu_\pm}{\left(\frac{1}{12} w \ddot{X}_\pm^2\right)^{2/3}} \int_0^\infty du u e^{i u^3} \quad (3.44)$$

$$= \frac{2\pi i \ddot{X}^\mu_\pm}{3 \Gamma(1/3) \left(\frac{1}{12} w \ddot{X}_\pm^2\right)^{2/3}},$$

where we extended the upper limit of the integration to infinity and defined

$$u = \left(\frac{1}{12} w \ddot{X}_\pm^2\right)^{1/3} \sigma. \quad (3.45)$$
Replacing $w$ with $2\pi f$ gives

$$I_{\pm}^\mu(k) = C_{\pm}^\mu f^{-\frac{2}{3}},$$  \hspace{2cm} (3.46)
$$T_{\mu\nu}(k) = \frac{\mu}{f} |f|^{-\frac{4}{3}} C_{\mu}^{(\mu) \nu},$$  \hspace{2cm} (3.47)

where

$$C_{\pm}^\mu = i \frac{(32\pi/3)^{1/3}}{\Gamma(1/3)} \frac{\dot{X}_{\pm}^\mu}{|\dot{X}_{\pm}|^{\frac{2}{3}}}. \hspace{2cm} (3.48)$$

It is important to note that in the derivation of Eq. (3.44) we assumed that the line of sight $k$ is in the direction of the motion of the cusp, $l_\pm$. We can estimate the maximum angle between $k$ and $l_\pm$ such that Eq. (3.44) can still be used [49]. Consider the case where $\dot{k} = N(\hat{l} + \sin \theta \hat{\delta})$ where $\theta$ is a small angle and $N$ is a normalization factor such that $\dot{k}$ is a unit vector. $\hat{\delta}$ is orthogonal to $\hat{l}$. Therefore the phase of Eq. (3.44) reads

$$-\frac{1}{2} k \cdot X_\pm \simeq \frac{w}{12} \ddot{X}_{\pm}^2 \sigma_\pm^3 + \frac{w}{4} \theta |\dot{X}_{\pm}| \sigma_\pm^2 + \frac{w}{4} \theta^2 \sigma_\pm$$  \hspace{2cm} (3.49)

where the first term is the one we used in the zeroth order approximation to calculate Eq. (3.44). If the second term and the third term are small relative to the first one, we can still approximate the actual result with the approximation in Eq. (3.44). Using the same definition in Eq. (3.45), we see that the first term can be used to approximate the phase if $\theta$ is smaller than a critical value given by

$$\theta_m \simeq \left( \frac{w}{|\ddot{X}_\pm|} \right)^{-\frac{1}{3}}, \hspace{2cm} (3.50)$$

see Fig. 4.2 for the illustration. If $\theta$ is larger than $\theta_m$, the phase defined in Eq. (3.49) will have no saddle points and the integral will vanish due to the oscillatory nature of the integrand. Therefore the conclusion is that the result in Eq. (3.44) is valid for $\theta < \theta_m$, and the integral vanishes otherwise.

Let us now describe the shape of the string, $X_{\pm}^\mu$, at $\tau = 0 \ (\sigma_\pm = \pm \sigma)$ for the case of $l_- = l_+ = k/w$. Using the expansion Eq. (3.41) at $\tau = 0$, we get

$$X^\mu(\sigma, \tau = 0) = \frac{1}{2} \left( X_+^\mu(\sigma) + X_-^\mu(-\sigma) \right)$$
$$= \frac{1}{4} (\ddot{X}_+^\mu + \ddot{X}_-^\mu) \sigma^2 + \frac{1}{12} (X_+^{(3)} + X_-^{(3)}) \sigma^3. \hspace{2cm} (3.51)$$
In order to visualize the shape of the string around the origin, we can choose the coordinate system such that \( (\dddot{X}_+ + \dddot{X}_-)^2 \) lies on the \( x \)-axis, and define
\[
x = \frac{1}{4} (\dddot{X}_+ + \dddot{X}_-)^2 \sigma^2.
\] (3.52)

Let us also denote the direction of \( \dot{X}^{(3)}_+ + \dot{X}^{(3)}_- \) by \( \hat{y} \), which is not necessarily orthogonal to \( \hat{x} \). If we define \( y = \frac{1}{12} (X^{(3)}_+ + X^{(3)}_-)^3 \), we see that
\[
y \propto x^{3/2},
\] (3.53)

which has a sharp turn at \( x = 0 \), which is referred to as cusp.

Finally we need to estimate \( |\dddot{X}_\pm| = |\dot{X}_\pm| \). Since \( X_\pm \) are periodic with period \( l \), \( \dot{X} \) expanded as
\[
\dot{X}(\pm) = \sum_n c_n e^{i \frac{2\pi n}{l} \pm} \eta,
\] (3.54)

where the expansion coefficients \( c_n \) are constrained by \( |\dot{X}_\pm| = 1 \). If the string is not too wiggly, \( c_n \) is nonvanishing for only small \( n \), therefore we can estimate \( |\dddot{X}_\pm| \sim \frac{2\pi}{l} \).

Combining Eqs. (3.39), (3.47) and (3.48) the strain of the gravitational waves can be estimated as
\[
h^{(c)}(f) = \frac{G\mu l^2}{r} |f|^{-\frac{4}{3}}.
\] (3.55)

It is convenient to express \( r \) as a function of \( z \)
\[
r = \frac{1}{H_0} \int_0^z \frac{dz'}{H(z')} = \frac{1}{H_0} \varphi_r(z),
\] (3.56)

where \( H_0 \) is the Hubble constant today and \( H(z) \) is the Hubble function given by
\[
H(z) = (\Omega_M(1+z)^3 + \Omega_R(1+z)^4 + \Omega_\Lambda)^{1/2}.
\] (3.57)

The numerical values for the constants in this equation are \( \Omega_M = 0.25 \), \( \Omega_R = 4.6 \times 10^{-5} \), \( \Omega_\Lambda = 1 - \Omega_R - \Omega_M \) and \( H_0 = 73 \text{km/s/Mpc} \) [56].

Note that \( f \) in Eq. (3.55) is the frequency of the radiation in the frame of emission. In order to convert it to the frequency we observe today, the effect of the cosmological redshift must be included. The frequency in the frame of emission, \( f \), is related to the frequency we observe now, \( f_{\text{now}} \), by the relation \( f = (1 + z)f_{\text{now}} \). The redshift effect on
Eq. (3.55) needs to be implemented carefully. One should note that replacing $f$ in Eq. (3.55) with $(1 + z)f_{\text{now}}$ is not correct, since this replacement will scale the argument and the amplitude of $h^{(c)}(f)$ by a factor of $\frac{1}{1+z}$, which is the reflection of the fact that the measure of Fourier integral is not dimensionless. Since redshifting should change the argument but not the amplitude, one needs to multiply the result by $1 + z$ so that the amplitude remains the same. Equivalently, one can define Logarithmic Fourier Transform, as discussed in Ref. [48], such that the measure of the transform becomes dimensionless. After redshifting properly, we get

$$h^{(C)}(f, z, l) = \frac{G_H H_0 l^2}{(1 + z)^{\frac{3}{2}} \varphi_r(z)} |f|^{-\frac{2}{3}}, \quad (3.58)$$

where we dropped the subscript “now”. The radiation decays exponentially as the angle between the line of sight and the direction of the radiation. Therefore Eq. (3.58) is valid for angles smaller than $\theta_m$. With the approximation $|\dot{X}_\pm| \sim \frac{2\pi}{f}$, and after the redshifting effect, we can rewrite the maximum angle as

$$\theta_m \simeq (f l(1 + z))^{-\frac{1}{3}}, \quad (3.59)$$

where we replaced $w$ with $2\pi f$.

### 3.5 Kinks on Cosmic Strings

Calculation of kink radiation is similar to the cusp case. The form of $I^\mu_+ \mu$ is the same as the cusp result. $I^\mu_+ \mu$ has a discontinuity at the cusp point and needs a different treatment. Let us describe the kink (at $\sigma_- = 0$ and $X_\pm = 0$) as a jump of the tangent vector from $l_1^\mu$ to $l_2^\mu$. At the first order one can replace approximate $\dot{X}^\mu_\pm$ by $l_1^\mu$ for $\sigma_- < 0$ and $l_2^\mu$ for $\sigma_+ > 0$. At this approximation, one gets

$$I^\mu_+ (k) = \int_{-1/2}^{1/2} d\sigma_- \dot{X}^\mu_\pm e^{-\frac{i}{2} k \cdot X_\pm} \simeq \frac{2i}{w} \left( \frac{l_1^\mu}{l_1 \cdot k} - \frac{l_2^\mu}{l_2 \cdot k} \right), \quad (3.60)$$

where we dropped two oscillatory terms. The exact value of Eq. (3.60) depends on the sharpness of the kink, $l_1 \cdot l_2$ [57], however we will assume that the average value of this
quantity is of order one. Combining this result with $I^l_1$ we get the frequency
distribution of the radiation from a kink as

$$h^{(K)}(f, z, l) = \frac{G\mu l^{\frac{1}{2}}H_0}{(1 + z)^{\frac{2}{3}}\phi_r(z)} f^{-5/3}. \quad (3.61)$$

This concludes the calculation of the strain of radiation due cusps and kinks. In the
next chapter we discuss the stochastic background associated with this radiation.
Chapter 4

Stochastic Background

A stochastic background of gravitational-wave (SBGW) radiation is produced by a large number of weak, independent and unresolved gravitational wave sources. The energy density associated with the gravitational waves can be written as

\[ \rho_{gw}(t, \vec{x}_o) = \frac{1}{32\pi G} \langle \dot{h}_{ab}(t, \vec{x}_o) \dot{h}_{ab}(t, \vec{x}_o) \rangle, \]  

(4.1)

where \( \langle \rangle \) denotes averaging over several wavelengths of the wave (see Chap. 7 for a detailed discussion on the averaging). The measurable quantity is the energy density at a small band of frequency, which is usually defined in the logarithmic scale. This quantity is the stochastic background which is defined as

\[ \Omega_{gw}(f, t, \vec{x}_o) \equiv \frac{f}{\rho_c} \frac{\Delta \rho_{gw}(t, \vec{x}_o)}{\Delta f}, \]  

(4.2)

where \( \rho_c \equiv 3H_0^2/8\pi G \) is the critical energy density. In order to derive the explicit expression for Eq. (4.2), we will will start from a gravitational wave of frequency \( f \), created at point \( \vec{y} \) which can be written as

\[ h_{ab}(t, \vec{x}_o, f, \vec{y}, \zeta) = e^{2\pi i f(t-|\vec{x}_o-\vec{y}|)} e^{A}_{ab}(\hat{\Omega}) A_{A}(\vec{y}, f, \zeta), \]  

(4.3)

where \( t \) and \( \vec{x}_o \) are the observation point and time and \( \hat{\Omega} \) is the direction of propagation, which is in the direction of \( \vec{x}_o - \vec{y} \), as shown in Fig. 4.1. We also use the natural units, where the speed of light \( c \) is set to unity.

\( e^{A}_{ab}(\hat{\Omega}) \) is the polarization tensor, and \( A_{A}(\vec{y}, f, \zeta) \) is the (complex) amplitude of the wave. In order to express the polarization tensors explicitly, one can use the unit
vectors of spherical coordinates:

\[
\hat{\Omega} = \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z},
\hat{l} = \hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z},
\hat{m} = \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}.
\] (4.4)

In this coordinate system, the polarization tensors can be written as

\[
e^+_{ab} (\hat{\Omega}) = \hat{l}_a \hat{l}_b - \hat{m}_a \hat{m}_b,
\] 
\[
e^x_{ab} (\hat{\Omega}) = \hat{l}_a \hat{m}_b + \hat{m}_a \hat{l}_b,
\] (4.5)

which satisfy

\[
e^A_{ab} (\hat{\Omega}) e^B_{ab} (\hat{\Omega}) = 2 \delta^{AB}.
\] (4.6)

\(\zeta\) in Eq. (4.3) is the set of parameters specifying the source of the gravitational wave. For example, in the case of gravitational waves generated by kinks and cusps on cosmic strings, the \(\zeta\) parameters are the loop length and the redshift (assuming fixed
string tension). The amplitudes in Eq. (4.3) are random fields whose mean values can be assumed to be zero:

$$\langle A_A(\vec{x}, f, \zeta) \rangle = 0.$$  \hspace{1cm} (4.7)

Eq. (4.3) represents a wave generated by the fixed source parameters. In order to include all the contributions to the gravitational waves observed at point $\vec{x}_o$ at time $t$, we need to integrate over the volume, the frequency and the parameters of the source, i.e.

$$h_{ab}(t, \vec{x}_o) = \int df d\zeta d^3\vec{y} h_{ab}(t, \vec{x}_o, f, \vec{y}, \zeta) = \int df d\zeta d^3\vec{y} e^{2\pi i f(t-|\vec{x}_o-\vec{y}|)} e^{A_A(\hat{\Omega})} A_A(\vec{y}, f, \zeta), \hspace{1cm} (4.8)$$

The average energy density associated with the wave reads

$$\rho_{gw}(t, \vec{x}_o) \equiv \frac{\pi}{8G} \int df df' df'' df''' \int d\zeta d\zeta' d^3x d^3x' d^3\vec{y} d^3\vec{y}' \left| A_A(\vec{x}+\vec{x}_o, f, \zeta) A_B^*(\vec{y}'+\vec{x}_o, f', \zeta') \right|^2 \times e^{2\pi i f(t-r) - f'(t'-r')} \hspace{1cm} (4.9)$$

In order to simplify the exponent we shift our coordinates and define $\vec{y} = \vec{x} + \vec{x}_o$ and $\vec{y}' = \vec{x}' + \vec{x}_o$, which also gives $\hat{\Omega} = -\hat{x}$ and $\hat{\Omega}' = -\hat{x}'$, see Fig. 4.1. In this coordinate system we can rewrite Eq. (4.9) as

$$\rho_{gw}(t, \vec{x}_o) = \frac{1}{32\pi G} \langle \hat{h}_{ab}^* (t, \vec{x}_o) \hat{h}_{ab} (t, \vec{x}_o) \rangle = \frac{\pi}{8G} \int df df' df'' df''' \int d\zeta d\zeta' d^3x d^3x' d^3\vec{y} d^3\vec{y}' \left| A_A(\vec{x}+\vec{x}_o, f, \zeta) A_B^*(\vec{y}'+\vec{x}_o, f', \zeta') \right|^2 \times e^{2\pi i f(t-r) - f'(t'-r')} A_A(\vec{x}+\vec{x}_o, f, \zeta) A_B^*(\vec{y}'+\vec{x}_o, f', \zeta') \hspace{1cm} (4.10)$$

where $r = |\vec{x}|$ and $r' = |\vec{x}'|$. The stochastic background corresponding to $\rho_{gw}$ reads

$$\Omega_{gw}(f, t, \vec{x}_o) \equiv \frac{f}{\rho_c} \frac{d \rho_{gw}(t, \vec{x}_o)}{df} = \frac{\pi^2 f^2}{3H_0^2} \int df' df'' df''' \int d\zeta d\zeta' d^3x d^3x' d^3\vec{y} d^3\vec{y}' \left| A_A(\vec{x}+\vec{x}_o, f, \zeta) A_B^*(\vec{y}'+\vec{x}_o, f', \zeta') \right|^2 \times e^{2\pi i f(t-r) - f'(t'-r')} A_A(\vec{x}+\vec{x}_o, f, \zeta) A_B^*(\vec{y}'+\vec{x}_o, f', \zeta') \hspace{1cm} (4.11)$$

If we assume that the amplitude of the sources at two different locations and with different polarizations are uncorrelated, we can impose the following bilinear
expectation:

\[
C_2 \equiv \left\langle e^{2\pi i (f'(t-r')-f(t-r))} A_A(\vec{x} + \vec{x}_o, f, \zeta) A_B^*(\vec{x}' + \vec{x}_o, f', \zeta') \rightangle
= \frac{1}{4\pi} |A(\vec{x}, f, \zeta)|^2 \delta(f-f') \delta(\zeta-\zeta') \delta(\vec{x}-\vec{x}')
\] (4.12)

which is the most general expression for an unpolarized, Gaussian and stationary background. Note that the dependence on the location of the observation \(\vec{x}_o\) drops. However, if the sources have correlation over a distance, the correlator need not be vanishing for \(\vec{x} \neq \vec{x}'\). This case is discussed separately in Sec. 7.2. Note that \(\delta^3(\vec{x} - \vec{x}')\) requires \(\hat{\Omega} = \hat{\Omega}'\) (and also \(r = r'\)), for which \(e_{ab}^A(\hat{\Omega})e_{ab}^B(\hat{\Omega})\delta_{AB} = 4\).

Substituting Eq. (4.12) into Eq. (4.11), we get,

\[
\Omega_{gw}(f) = \frac{\pi f^3}{3H_0^2} \int d\zeta d^3\vec{x} \left| A(\vec{x}, f, \zeta) \right|^2,
\] (4.13)

where we note that the \(\Omega_{gw}\) depends on \(f\) only, i.e. \(t\) dependence drops because of \(\delta(f-f')\). At this point it is convenient to separate the angular dependence and the radial dependence and parameterize the radial distance and the differential volume with redshift \(z\), i.e.

\[
d^3\vec{x} = r^2(z) dr(z) d\hat{\Omega} = r^2(z) \frac{dr(z)}{dz} d\Omega dz \equiv \Phi(z) dz d\hat{\Omega}.
\] (4.14)

With this parametrization of \(r\) and \(d^3\vec{x}\) we rewrite Eq. (4.13) as

\[
\Omega_{gw}(f) = \frac{\pi f^3}{3H_0^2} \int d\zeta d\Omega dz \mathcal{R}(\zeta, z, \hat{\Omega}) h^2(\zeta, f, z, \hat{\Omega}),
\] (4.15)

where we defined

\[
\mathcal{R}(\zeta, z, \hat{\Omega}) h^2(\zeta, f, z, \hat{\Omega}) = \Phi(z) \left| A(\vec{x}, f, \zeta) \right|^2.
\] (4.16)

In Eq. (4.15), \(\mathcal{R}(\zeta, z, \hat{\Omega})\) is the rate of emission of gravitational waves per redshift and per parameter space volume. It represents the observable part of the gravitational radiation from the source, i.e. it incorporates the propagation of the wave in the expanding universe as well as possible beaming effects, see [48, 49]. Note also that in In Eq. (4.15) it is clear that we are adding up the powers, \(h^2\), from individual sources weighed by their rates, \(\mathcal{R}\), as opposed to adding amplitudes and squaring the total amplitude to get the total power. Obviously this is the reflection of the fact that the
sources are incoherent. \( h(\zeta, f, z, \hat{\Omega}) \) is the strain of gravitational wave emitted by a single source at \( \hat{\Omega} \), \( r \), \( f \) and with parameters \( \zeta \). In the isotropic case, i.e.

\[
\mathcal{R}(\zeta, z, \hat{\Omega}) h(\zeta, f, z, \hat{\Omega}) = \mathcal{R}(\zeta, z) h(\zeta, f, z),
\]

we get

\[
\Omega_{gw}(f) = \frac{4\pi^2 f^3}{3\mathcal{H}_0^2} \int d\zeta dz \mathcal{R}(\zeta, z) h^2(\zeta, f, z). \tag{4.17}
\]

This is the most general expression for SBGW with sources specified by the redshift and the set of parameters \( \zeta \).

Now we would like to apply this to the case of cosmic strings, for which the only parameter \( \zeta \) is the loop length \( l \) (Note that redshift \( z \) is also such a parameter, however, starting from Eq. (4.15), we are treating it separately rather than embedding it in the set of parameters \( \zeta \)). In this case, the stochastic gravitational background can be written as

\[
\Omega_{gw}(f) = \frac{4\pi^2 f^3}{3\mathcal{H}_0^2} \int dz \int dl h^2(f, z, l) \frac{d^2 R(z, l)}{dz dl}, \tag{4.18}
\]

where \( h(f, z, l) \) is given in Eqs. (3.58) and (3.61) for cusps and kinks, respectively. We also defined \( \mathcal{R}(\zeta, z) \equiv \frac{d^2 R(z, l)}{dz dl} \) to emphasize that it is the observable burst rate per length per redshift, which will be defined below. We take the number of cusps (kinks) to be one per loop (We will discuss how the number of cusps or kinks affects SBGW in Sec. 4.5.) If we define the density (per volume) of the loops of length \( l \) at time \( t \) as \( n(l, t) \), the rate of burst (per loop length per volume) can be expressed as \( \frac{n(l, t)}{l/2} \), where \( l/2 \) factor is the fundamental period of the string. However, this is not the observable burst rate since we can observe only the fraction of bursts that is beamed toward us. Including this fraction we obtain

\[
\frac{d^2 R}{dl dz} = H_0^{-3} \varphi_V(z)(1 + z)^{-1} \frac{2n(l, t)}{l} \Delta(z, f, l), \tag{4.19}
\]

where \( (1 + z)^{-1} \) comes from converting emission rate to observed rate, and \( H_0^{-3} \varphi_V(z) \) follows from converting differential volume element to the corresponding function of redshift \( z \),

\[
dV = 4\pi a^3(t)r^2 dr = \frac{4\pi H_0^{-3} \varphi_V^2(z)}{(1 + z)^3 \mathcal{H}(z)} dz \equiv H_0^{-3} \varphi_V(z) dz, \tag{4.20}
\]

where \( a(t) \) is the cosmological scale factor. \( \Delta(z, f, l) \) is the fraction of the bursts we can observe. Geometrically the radiation will be in a conic region, as shown in Fig.
4.2, with half opening angle $\theta_m$ (Eq. (3.59)) and outside the cone it will decay exponentially, see Eq. (3.49) and the discussion following it.

![Figure 4.2: The cone of radiation](image)

Figure 4.2: The cone of radiation

To simplify the calculation we assume that the radiation amplitude vanishes outside this conic region, which will be implemented by a $\Theta$-function. It is important to note that cusps are instantaneous events, and it is possible to observe their radiation only if the line of sight happens to be inside the cone of radiation. In order to estimate the effect of this beaming, we can first calculate the solid angle spanned by the cone by using the following relation

$$\Omega_m = 2\pi (1 - \cos \theta_m) \approx \pi \theta_m^2,$$

(4.21)

where $\theta_m$ is defined in Eq. (3.59). Thus the probability that the line of sight is within this solid angle is

$$\frac{\Omega_m}{4\pi} \approx \frac{\theta_m^2}{4},$$

(4.22)

which is referred to as the beaming fraction of the cusp. The beaming fraction, Eq. (4.22), which is proportional to $\theta_m^2$, is the fraction of the time the line of sight is inside the cone of radiation.

In contrast, kinks radiate continuously—as kinks travel around a string loop they radiate in a fan-like pattern, as shown in Fig. 4.3. Therefore radiation cone of a kink
will sweep a strip of width $2\theta_m$ and an average length $\pi$ on the surface of the unit sphere as it travels around the cosmic string loop. That is, the probability of observing radiation from a kink is

$$\frac{\Omega_m}{4\pi} \sim \frac{2\theta_m \pi}{4\pi} = \frac{\theta_m}{2}. \quad (4.23)$$

![Figure 4.3: The strip traced by the cone of radiation](image)

We combine the cutoff for large angles and the beaming effect into

$$\Delta^{(C)}(z, f, l) \approx \frac{\theta_m^2(z, f, l)}{4} \Theta(1 - \theta_m(z, f, l)). \quad (4.24)$$

For kinks we have

$$\Delta^{(K)}(z, f, l) \approx \frac{\theta_m(z, f, l)}{2} \Theta(1 - \theta_m(z, f, l)). \quad (4.25)$$

Inserting this result into Eq. (4.18) gives the background radiation $\Omega_{gw}(f)$ as a double integral over $l$ and $z$, which needs to be evaluated numerically.

Finally we need to discuss the form of the loop density, $n(l, t)$, in Eq. (4.19). To do this, it is convenient to first convert the cosmic time $t$ to a suitable function of redshift $z$ using the following relation

$$\frac{dz}{dt} = -(1 + z)H_0 \mathcal{H}(z), \quad (4.26)$$
which can be integrated to give
\[ t = H_0^{-1} \int_z^\infty \frac{dz'}{(1 + z')H(z')} = H_0^{-1} \varphi_t(z). \] (4.27)

Below we discuss the two main contending scenarios for the size of cosmic string loops.

### 4.1 Small Loops

Early simulations suggested that the size of loops was dictated by gravitational back reaction. In this case the size of the loops is fixed by the cosmic time \( t \), and all the loops present at a cosmic time \( t \), are of the same size \( \alpha t \). The value of \( \alpha \) is set by the gravitational back reaction, that is \( \alpha \propto \Gamma G\mu \). We parameterize \( \alpha \) by \( \alpha = \epsilon \Gamma G\mu \) where \( \epsilon \) is a parameter we scan over. The constant \( \Gamma \) is the ratio of the power radiated into gravitational waves by loops to \( G\mu^2 \). Numerical simulation results suggest that \( \Gamma \approx 50 \). Therefore the density is of the form

\[ n(l, t) \propto (p \Gamma G\mu)^{-1} t^{-3} \delta(l - \alpha t), \] (4.28)

where \( p \) is the reconnection probability. The overall coefficient is estimated by simulations (for a review see [2]) which show that the density in the radiation domination era is about 10 times larger the one in the matter domination era. This behavior of the density can be implemented by a function, \( c(z) \), which converges to 10 for \( z \gg z_{eq} \) and to 1 for \( z \ll z_{eq} \). Therefore the density can be written as

\[ n(l, t) = c(z) (p \Gamma G\mu)^{-1} t^{-3} \delta(l - \alpha t), \] (4.29)

where [48]

\[ c(z) = 1 + \frac{9z}{z + z_{eq}}. \] (4.30)

Such a distribution simplifies the calculation of SBGW since the \( l \)-integral in Eq. (4.18) can be evaluated trivially to yield

\[ \Omega_{gw,R}^{(C)}(f) = \frac{4\pi^2 f^3}{3H_0^2} \int dz \int dl h^2(f, z, l) \frac{d^2 R(z, l)}{dzdl} \]

\[ = \frac{2cG\mu}{3p\alpha^{1/3}\Gamma f^{1/3}} \int dz \frac{c(z)\varphi_V \Theta (1 - [f(1 + z)\alpha\varphi_t]^{-1/3})}{(1 + z)^{7/3} \varphi_t \varphi_t^{10/3}}. \] (4.31)
For kinks, we have a similar integral,
\[
\Omega_{gw,R}^{(K)}(f) = \frac{4 c G \mu \pi^2 H_0^{1/3}}{3 p \alpha^{2/3} \Gamma f^{2/3}} \int dz \frac{c(z) \varphi \theta (1 - [f(1 + z)\alpha \varphi]^{-1/3})}{(1 + z)^{5/3} \varphi_t^{11/3}}. \tag{4.32}
\]
We note the different frequency dependence of SBGW for cusps and kinks. Eqs. (4.31) and (4.32) have overall \( f^{-1/3} \) and \( f^{-2/3} \) factors, however frequency also enters into the integrand in a nontrivial way. In Sec. 4.5 we analytically evaluate the integrals in Eqs. (4.31) and (4.32) with certain approximations, and in Sec. 4.7 perform numerical integration.

4.2 Large Loops

Recent simulations \([58, 59, 60]\) suggest that the size of the loops is set by the large scale dynamics of the network, and that the gravitational back-reaction scale is irrelevant. It is important to emphasize that the sizes of large loops, which are set by the value of \( \alpha \), are still under debate. Refs. \([58, 59]\) suggest lower values of \( \alpha \), whereas in Ref. \([60]\) it is found that the loop production functions have peaks around \( \alpha \approx 0.1 \). The dependence of SBGW on \( \alpha \) can be found in Sec. 4.5. In order to contrast the large loop and the small loop cases, we adopt \( \alpha = 0.1 \) for numerical computations. For long-lived loops, the distribution can be calculated if a scaling process is assumed (see [2]). In the radiation era it is
\[
n(l, t) = \chi_R t^{-\frac{3}{2}}(l + \Gamma G \mu t)^{-\frac{5}{2}}, \quad \alpha t < t < t_{eq}, \tag{4.33}
\]
where \( \chi_R \approx 0.4 \zeta_l \alpha^{1/2} \), and \( \zeta_l \) is a parameter related to the correlation length of the network \([50]\). The numerical value of \( \zeta_l \) is found in numerical simulations of radiation era evolution to be about 15 (see Table 10.1 in [2]). The upper bound on the length arises because no loops are formed with sizes larger than \( \alpha t \). For \( t > t_{eq} \) (the matter era) the distribution has two components, loops formed in the matter era and survivors from the radiation era. Loops formed in the matter era have lengths distributed according to,
\[
n_1(l, t) = \chi_M t^{-2}(l + \Gamma G \mu t)^{-2}, \quad \alpha t_{eq} - \Gamma G \mu (t - t_{eq}) < l < \alpha t, \quad t > t_{eq}, \tag{4.34}
\]
with $\chi_M \approx 0.12 \zeta_1$, with $\zeta_1 \approx 4$ (see Table 10.1 in [2]). The lower bound on the length is due to the fact that the smallest loops present in the matter era started with a length $\alpha t_{eq}$ when they were formed and their lengths have since decreased due to gravitational wave emission. Additionally there are loops formed in the radiation era that survive into the matter era. Their lengths are distributed according to,

$$n_2(l, t) = \chi R t_{eq}^{1/2} t^{-2} (l + \Gamma G \mu)^{-\frac{5}{2}},$$

$$l < \alpha t_{eq} - \Gamma G \mu (t - t_{eq}), \quad t > t_{eq},$$

(4.35)

where the upper bound on the length comes from the fact that the largest loops formed in the radiation era had a size $\alpha t_{eq}$ but have since shrunk due to gravitational wave emission.

### 4.3 Removing Rare Events

Before we start calculating the SBGW, we should mention a crucial observation due Damour and Vilenkin [48]. SBGW generated by a network of cosmic strings includes bursts which occur infrequently, and the computation of $\Omega_{gw}(f)$ should not be biased by including these large rare events (i.e. events with low rate). If the loop density is taken of the form given in Eq. (4.28) (small loop), the rate is specified by the redshift only. Therefore the condition on the rate can be implemented by a cutoff on redshifts such that large events for which the rate is smaller than the relevant time-scale of the experiment are excluded (see Eq. (6.17) of [48]). However, when loops are large the situation is more complicated because at any given redshift there are loops of many different sizes given in Eqs. (4.33) and (4.34). This case has been dealt with in Ref. [53] as follows: instead of integrating over the variables $l$ and $z$ in Eq. (4.18) one integrates over $h$ and $z$ (where $h$ is defined in Eqs. (3.58) and (3.61)) and imposes the cutoff limit on the $h$ integral. The cutoff is defined as

$$\int_{h^*}^{\infty} dh \int \frac{d^2 R}{dz dh} = f,$$

(4.36)

where $\frac{d^2 R}{dz dh} = \frac{d^2 R}{dz dh}$. Eq. (4.36) is solved for $h^*$ and used to exclude rare event using the following integral (instead of Eq. (4.18))

$$\Omega_{gw}(f) = \frac{4\pi^2}{3H_0^2} f^3 \int_0^{h^*} dh \frac{h^2}{h^*} \int \frac{d^2 R}{dz dh}.$$

(4.37)
This procedure removes large amplitude events (those with strain $h > h^*$) that occur at a rate smaller than $f$.

### 4.4 Numerical Results

We numerically evaluate Eq. (4.18) for small loops and large loops cases for kinks and cusps separately (The removal of rare events described in Sec. 4.3 is also implemented). Fig. 4.4 shows the SBGW spectrum for kinks and cusps for small loops. For the top curves (red and green) we have, $G\mu = 2 \times 10^{-6}$, $p = 10^{-3}$ and $\epsilon = 10^{-4}$, whereas for the bottom two curves (blue and pink) $G\mu = 10^{-7}$, $p = 5 \times 10^{-3}$ and $\epsilon = 1$ ($\epsilon \equiv \frac{\alpha}{G\mu}$).

Figure 4.4: Numerical results for kink and cusp spectrum for small loops.

Fig. 4.5 shows the spectrum for large loops. For the top curves (blue and pink), which are almost identical, we have, $G\mu = 10^{-7}$ and $p = 5 \times 10^{-3}$, whereas for the bottom two curves (red and green) $G\mu = 10^{-9}$ and $p = 5 \times 10^{-2}$.

Here we note that for $f \gg \frac{H_2}{G\mu}$, the spectrum is flat for both cusps and kinks, which
Numerical results depicted in Figs. 4.4 and 4.5 show that the spectrum is constant for "large" values of $f$ (we will quantify "large" momentarily). In this section we show that the spectrum is flat for $f \gg H_0 \sqrt{z_{eq}}/\alpha$ for small loops and for $f \gg H_0 \sqrt{z_{eq}}/G\mu^4$ for large loops. This is rather unexpected since $\Omega_{gw}(f)$ has an explicit $f^{-1/3}$ and $f^{-2/3}$ dependence for cusps and kinks, respectively. The only other $f$ dependence comes from the $\Theta$-functions. In the following section we show analytically that the $f$ dependence coming from the $\Theta$-function is of the form $f^{1/2}$ and $f^{3/2}$ for cusps and kinks respectively so that the spectrum is indeed flat for large values of the frequency $f$.

Furthermore we discuss the dependence of the spectrum on the parameters: $G\mu$, $\epsilon \equiv \frac{\alpha}{G\mu}$ and $p$ for small loops and $G\mu$ and $p$ for large loops.
We limit our discussion to large values of $f$, for which the spectrum gets the dominant contribution from the loops in the radiation era. Matter era loops contribute to lower frequency part of the spectrum. It is relatively easier to verify this in the case of small loops. If one limits the redshift integration in Eqs. (4.31) and (4.32) to matter domination and uses the corresponding approximate cosmological functions, it is found that $\Omega(f)$ depends on the negative powers of $f$, which are negligible for large $f$. The same argument also applies to the large loop case. Since we want to get an estimate of the spectrum we will neglect the complications arising from removing rare burst. In the radiation domination, $z > z_{eq} = \sqrt{\Omega_R} \simeq 5440$, the Hubble function in Eq. (3.57), can be approximated as
\[
H(z) \simeq \sqrt{\Omega_R} z^2 = \frac{z^2}{2\sqrt{z_{eq}}}. \quad (4.38)
\]
The cosmological functions that appear in the SBGW formula can be approximated as
\[
\varphi_l(z) = \int_z^\infty \frac{dz'}{(1 + z')H(z')} \simeq \int_z^\infty \frac{dz'}{z'H(z')} \simeq \sqrt{z_{eq}} z^{-2}. \quad (4.39)
\]
\[
\varphi_r(z) = \int_0^z \frac{dz'}{H(z')} = \int_0^{z_{eq}} \frac{dz'}{H(z')} + \int_{z_{eq}}^z \frac{dz'}{H(z')} \simeq 3.6. \quad (4.40)
\]
\[
\varphi_V(z) = \frac{4\pi\varphi_r^2}{(1 + z)^3H(z)} \simeq 325 \sqrt{z_{eq}} z^{-5}. \quad (4.41)
\]

### 4.5.1 Analytical Approximation for small loop case

We first consider the small loop case, for which the expression for SBGW reduces to an integral over redshift given in Eqs. (4.31) and (4.32). Inserting the result in Eqs. (4.39-4.41) into Eq. (4.31) we get
\[
\Omega_{gw,R}(f) \propto \frac{G\mu}{p\alpha^{1/3}f^{1/3}} \int_{z_{eq}}^{z_{max}} \frac{dz}{z^{2/3}} \Theta \left(1 - \left[\frac{fz_{eq}^{1/2}}{H_0 z}\right]^{-1/3}\right) \propto \frac{G\mu}{p}, \quad (4.42)
\]
where we dropped a term with $1/f$ dependence since it is small in large $f$ limit, and the subscript $R$ reminds us that this is the contribution from radiation era loops. The upper limit of the integration, $z_{max}$, is the redshift at the time of the creation of the strings, which depends on the energy scale of the phase transition. The result in Eq.
(4.42) is valid for \( \frac{z_{1/2}}{\alpha} \ll \frac{f}{H_0} < \frac{\tilde{z}_{\text{eq}}}{\alpha z_{\text{eq}}} \), for which the upper limit of the integral is set by the Θ-function. If \( \frac{f}{H_0} > \frac{\tilde{z}_{\text{eq}}}{\alpha z_{\text{eq}}} \), the integral does not depend on \( f \) and the frequency dependence of \( \Omega_{gw,R}(f) \) is given by the prefactor, which has \( f^{-1/3} \) behavior (Note that this case is outside the range of the frequencies plotted in the Figs. 4.5 and 4.4).

For kinks we get

\[
\Omega_{gw,R}^{(K)}(f) \propto \frac{G\mu}{p^{2/3} f^{2/3}} \int_{z_{\text{eq}}}^{z_{\text{max}}} \frac{dz}{z^{1/3}} \Theta \left( 1 - \left[ \frac{f z_{\text{eq}}^{1/2} \alpha}{H_0 z} \right]^{-1/3} \right)
\]

\[
\propto \frac{G\mu}{p} .
\]

Eqs. (4.42) and (4.43) show that for \( \frac{z_{1/2}}{\alpha} \ll \frac{f}{H_0} \) the spectrum is constant and it scales with \( G\mu/p \). The amplitude does not depend on the parameter \( \alpha \), however the spectrum shifts to the right linearly in \( \alpha \).

This result is in perfect agreement with Fig. 4.4. For the bottom curves \( \frac{G\mu}{p} = 2 \times 10^{-5} \) where as \( \frac{G\mu}{p} = 2 \times 10^{-3} \) for the top curves, which have two orders of magnitude larger amplitude, exactly agreeing with the figure. Furthermore, the top curves (\( \epsilon = 10^{-4} \)) are shifted to the right compared to the bottom curves (\( \epsilon = 1 \)) by about 4-orders in \( f \) as predicted by our results above. One can easily understand the parametrical dependence of SBGW on \( G\mu \) and \( p \): The strain, \( h \), is proportional to \( G\mu \) and number density, \( n \), is proportional to \( (G\mu p)^{-1} \). Therefore \( \Omega_{gw,R} \propto nh^2 \propto G\mu/p \).

### 4.5.2 Analytical Approximation for large loop case

Now we consider large loops in the radiation domination, for which the density \( n(l,t) \) is given in Eq. (4.33), where \( t \) is to be replaced with \( \varphi_l(z)/H_0 \). Substituting the results in Eqs. (4.39-4.41) into Eq. (4.18) we get

\[
\Omega_{gw,R}^{(C)}(f) = A(f) \int dz \int dl \frac{z(lz)^{-\frac{4}{3}}}{(lz^2 + \delta^2)^2} \Theta(1 - \frac{1}{f z l}) \Theta(\beta z^2 - l)
\]

\[
= A(f) \int dz \int_z^{z_{\text{eq}}} \frac{du}{\frac{1}{4}(uz + \delta^2)^{\frac{2}{3}}},
\]

where we define

\[
A(f) = \frac{165 \alpha^2 \delta^2 \chi_R}{p z_{\text{eq}}^{1/4} H_0^3 \Gamma^2 f^{1/3}} .
\]
with $\delta = \frac{G\mu\Gamma}{\alpha}$ and $\beta = \frac{\alpha\sqrt{z_{eq}}}{H_0}$ ($\alpha \approx 0.1$ for large loop case) and the dummy integration variable $u = lz$. The upper limit of the $z$ integral, $z^*$ will be set by requiring $\beta/z > 1/f$, that is, $z < f \beta$. If $f < z_{max}/\beta$ we have,

$$
\Omega_{gw,R}^{(C)}(f) = A(f) \int_{z_{eq}}^{\beta/f} dz \int_{\frac{1}{f}}^{\beta} du \frac{u^{-\frac{1}{3}}}{(uz + \beta\delta)^{\frac{2}{3}}}
= A(f) \int_{\frac{1}{f}}^{\beta} du \int_{z_{eq}}^{\frac{\beta}{uz^*}} dz \frac{u^{-\frac{1}{3}}}{(uz + \beta\delta)^{\frac{2}{3}}}
= -\frac{2}{3} A(f) \int_{\frac{1}{f}}^{\frac{\beta}{uz_{eq}}} du \frac{1}{u^3} \left( \frac{1}{(\beta + \beta\delta)^{\frac{2}{3}}} - \frac{1}{(uz_{eq} + \beta\delta)^{\frac{2}{3}}} \right). 
$$

If $\frac{1}{f} < \frac{\delta\beta}{z_{eq}} = \frac{G\mu\Gamma}{H_0\sqrt{z_{eq}}}$, we can split the integration range $[1/f, \beta/z_{eq}]$ in the second integral into $[1/f, \delta\beta/z_{eq}]$ and $[\delta\beta/z_{eq}, \beta/z_{eq}]$ and neglect $uz_{eq}$ and $\beta\delta$ respectively in these two integrals. Combining all terms and keeping the lowest order in $\delta$ we get,

$$
\Omega_{gw,R}^{(C)}(f) = A(f) \left( \frac{2f^{\frac{1}{2}}}{(\delta\beta)^{\frac{1}{3}}} - \frac{18z_{eq}^{\frac{1}{2}}}{11(\delta\beta)^{4/3}} \right) = \frac{330\alpha^2 \delta^{\frac{1}{2}} \chi_R}{p} \int_{z_{eq}}^{\frac{1}{4}H_0^{\frac{1}{2}}\Gamma^2\beta^{\frac{3}{2}}} dz
\simeq 3.2 \times 10^{-4} \sqrt{\frac{G\mu}{p}}, 
$$

which is valid for $f > \frac{3.6 \times 10^{-18}}{G\mu}$ Hz.

The calculation for the case of kinks is very similar to cusp case, following the same steps we get

$$
\Omega_{gw,R}^{(K)}(f) \simeq 3.2 \times 10^{-4} \sqrt{\frac{G\mu}{p}}, 
$$

for $f > \frac{3.6 \times 10^{-18}}{G\mu}$ Hz. This result is identical to the cusp result. Eqs. (4.47) and (4.48) show that the spectrum is flat for $f > \frac{3.6 \times 10^{-18}}{G\mu} Hz$ and its amplitude scales with $\sqrt{G\mu/p}$, which is in excellent agreement with Fig. 4.5. The flat value of the spectrum for the top curves ($G\mu = 10^{-7}$ and $p = 5 \times 10^{-3}$) is $2.1 \times 10^{-5}$ and for the bottom curve ($G\mu = 10^{-9}$ and $p = 5 \times 10^{-2}$) is $2.1 \times 10^{-7}$. These results are to be compared with the analytical results $2.0 \times 10^{-5}$ and $2.0 \times 10^{-7}$ predicted by Eqs. (4.47) and (4.48).

### 4.6 Effect of number of kinks

It is important to note that, in this study we assume that the number of kinks, $N$, is order of one. This assumption enters in the estimation $|\dot{X}_\pm| \sim \frac{2\pi}{\tau}$, and if there are $N$
kinks on strings, it needs to be replaced by $|\tilde{X}_\pm| \sim \frac{2\pi}{l/N}$. The replacement of $l$ with $l/N$ should also be done in the opening angle of the cone of the radiation, Eq. (3.59), which will result in a nontrivial dependence on $N$. However we can simply convert the resultant expression to the one we calculated in Eq. (4.32) by defining $\alpha = \alpha'N$.

Since we have shown that $\alpha$ has the effect of moving the spectrum horizontally, one effect of having $N$ kinks will be shifted spectrum compared to one kink spectrum. The other effect will be an overall scaling of the spectrum by $1/N$.

### 4.7 Parameter Space Constraints and Results

In this section we discuss certain experimental bounds on SBGW due to cosmic string cusps and kinks. For the case of large loops the free model parameters are $G\mu$ and $p$, and for small loops the parameters are $G\mu$, $\epsilon$ and $p$. It is important to note that the nontrivial dependence on $p$ follows from excluding rare bursts as described in Eqs. (4.36) and (4.37) (if rare events were included $\Omega_{\text{gw}}(f)$ would simply scale with $1/p$).

We add up the contributions to SBGW from the kinks and cups.

![Figure 4.6: Accessible regions in the $\epsilon - G\mu$ plane for $p = 10^{-3}$ (small loops).](image)
Figure 4.7: Accessible regions in the $\varepsilon - G\mu$ plane for $p = 10^{-2}$ (small loops).

Accessible regions corresponding to different experiments and bounds are shown in Figs. 4.6-4.8 for small loops (loop sizes are determined by gravitational back-reaction). Fig. 4.9 shows the accessible regions in the $p - G\mu$ plane for the large long-lived loop models. The accessible regions are to the right of the corresponding curves. The entire parameter space considered is within reach of LISA [61] and Advanced LIGO [62], and most are within the projected pulsar bound. The shaded regions, from darkest to lightest, are: LIGO S4 [63] limit, LIGO S5 [64], LIGO H1H2 projected sensitivity (cross-correlating the data from the two LIGO interferometers at Hanford, WA (H1 and H2)), and AdvLIGO H1H2 projected sensitivity. All projections assume 1 year of exposure and either LIGO design sensitivity or Advanced LIGO sensitivity tuned for binary neutron star inspiral search. The solid black curve corresponds to the BBN [65] bound, the dot-dashed curve to the pulsar bound[66], the +s to the projected pulsar sensitivity, the circles to the bound based on the CMB and matter spectra [67], the $\times$s to the projected sensitivity of the LIGO burst [50] search, and the $\diamond$-curve to the LISA projected sensitivity [68]. These burst searches are based on individual cusp burst,
Figure 4.8: Accessible regions in the $\varepsilon - G\mu$ plane for $p = 10^{-1}$ (small loops).

Figure 4.9: Accessible regions in the $p - G\mu$ plane (large loops).
rather than the SBGW. The BBN and CMB bounds are integral bounds, i.e. they are upper limits for the integral of $\Omega(f)$ over $\ln f$, therefore a model is excluded if it predicts an integral larger than the limit. It is also important to note that redshifts applicable to BBN and CMB bounds are $z > z_{\text{BBN}}$ and $z > z_{\text{CMB}}$, respectively. On the other hand, the pulsar and LIGO bounds apply in specific frequency bands, thus a model is excluded if it has $\Omega(f)$ larger than the limit (or projected sensitivity) for any $f$ in the range of the pulsar experiments. For the case of LIGO bound, the spectrum is integrated over the LIGO frequency band, i.e. it is an integral bound. The range of the redshift integral in Eq. (4.18) must chosen properly for a given experiment. For BBN bound, the integration is performed for $z > 5.5 \times 10^9$. Similarly, for the bound based on the CMB and matter spectra, the integration is performed for $z > 1100$.

First, we note that smaller values of $p$ are more accessible, which follows from the fact that the loop density is inversely proportional to $p$. This makes cosmic superstrings more accessible than field theoretical strings. Second, we note that LIGO stochastic search constrains large $G\mu$, small $\epsilon$ part of the parameter space, whereas pulsar limit constrains large $G\mu$ and large $\epsilon$ part of the parameter space. Similarly, the LIGO burst bound applies to large $G\mu$ and intermediate $\epsilon$ part of the parameter space. Therefore large $G\mu$ part of the parameter space is covered by these three experiments. Furthermore since they also overlap for large $G\mu$ and intermediate $\epsilon$, in the case of detection, the two LIGO searches could potentially confirm each other. We also see that the BBN [75] and CMB [67] bounds are not very sensitive to $\epsilon$: the corresponding curves are rather vertical in $\epsilon - G\mu$ plane. This result is in perfect agreement our results (Eqs. (4.42) and (4.43)) that show $\Omega_{\text{gw,R}}(f) \propto G\mu/p$, which does not depend on $\epsilon$ (Also note that $\Omega_{\text{gw,R}}(f)$ is a good approximation since the dominant contribution comes from the radiation era in the given range of the frequency).

For the case of large loops, GW background is significantly larger than the small loop one, see Figs. 4.4 and 4.5. Therefore more of the parameter space is accessible to the current and proposed experiments, as depicted in Fig. 4.9. The strongest constraint is the pulsar bound, which rules out cosmic (super)string models with $G\mu > 10^{-12}$ and $p < 8 \times 10^{-3}$. This bound also rules out field theoretical strings ($p = 1$) with $G\mu > 2 \times 10^{-9}$. One can compare these results with the case where only cusps are included [53]. In that case cosmic (super)string models with $G\mu > 10^{-12}$ and
$p < 3 \times 10^{-3}$ and field theoretical strings with $G_\mu > 10^{-9}$ are ruled out. This result illustrates that kinks contribute to SBGW at the same order as cusps.
Chapter 5

Anisotropies in the Stochastic Background of Gravitational Waves

The sources of the SBGW can be isotropic or anisotropic. For the case of sources of cosmological origin [69, 70, 71] the distribution of the gravitational wave sources is expected to be isotropic, while astrophysical sources such as rotating neutron stars [72] or magnetars [73] may have an anisotropic distribution. Even in the case of an a priori isotropic source distribution, random fluctuations in the number of sources will (in general) give rise to anisotropies. Such anisotropies are analogous to the anisotropies observed in the cosmic microwave background radiation and would carry additional information about the gravitational-wave sources that generated them.

In this study, we develop a general formalism to treat SBGW anisotropies. In particular, we consider two-point correlations in SBGW between two different directions in the sky, which arise from random fluctuations in the number of gravitational-wave sources. While this formalism is applicable to a variety of cosmological and astrophysical SBGW models (see [74, 75] and the references therein), we illustrate it for the specific case of cosmic (super)string cusps and kinks.
5.1 Anisotropies in the SBGW

We start from formalism developed in Chap. 4 and extend it to treat angular dependence. We extend Eq. (4.18) such that we can define the energy density of SBGW at frequency $f$ corresponding to sources in the direction $\hat{\Omega}$ as follows.

$$\Omega_{gw}(f, \hat{\Omega}) \equiv f \frac{d\rho_{gw}(\hat{\Omega})}{df}. \quad (5.1)$$

Let us assume that sources are characterized by a set of parameters $\zeta$ - in the case of cosmic strings, redshift $z$ is one such parameter. Therefore $\Omega_{gw}$ is an integral over the parameter space

$$\Omega_{gw}(f, \hat{\Omega}) = \int d\zeta n(\zeta, \hat{\Omega}) w(f, \zeta, \hat{\Omega}), \quad (5.2)$$

which is equivalent to Eq. (4.18) up to integration over $\hat{\Omega}$. We also single out the number density of sources (i.e. number per parameter space volume), $n(\zeta, \hat{\Omega})$, and define the rest of the integrand as $w(f, \zeta, \hat{\Omega})$, which is the contribution to $\Omega_{gw}$ of one source at frequency $f$, in the direction $\hat{\Omega}$, and with the parameter set $\zeta$. Comparing Eq. (5.2) with Eq. (4.15) we can easily write the contribution of one source

$$w(f, \zeta, \hat{\Omega}) = \frac{\pi f^3}{3H^2_0} h^2(f, \zeta, \hat{\Omega}) \tilde{R}(f, \zeta, \hat{\Omega}), \quad (5.3)$$

where $\tilde{R}(f, \zeta, \hat{\Omega}) \equiv \frac{R(f, \zeta, \hat{\Omega})}{n(\zeta, \hat{\Omega})}$, and $R(f, \zeta, \hat{\Omega})$ is as defined in Eq. (4.16). The division by the loop density, $n(\zeta, \hat{\Omega})$, is simply the reflection of the fact that we explicitly factor out $n(\zeta, \hat{\Omega})$ in the integral in Eq. (5.2). We then propose to discretize the integral in Eq. (5.2) as follows:

$$\Omega_{gw}(f, \hat{\Omega}) = \int d\zeta n(\zeta, \hat{\Omega}) w(f, \zeta, \hat{\Omega})$$

$$\approx \sum_i \Delta(\zeta_i) n(\zeta_i, \hat{\Omega}) w(f, \zeta_i, \hat{\Omega})$$

$$\equiv \sum_i N(\zeta_i, \hat{\Omega}) w(f, \zeta_i, \hat{\Omega}). \quad (5.4)$$

Here we assume that the parameter space can be divided into disjoint volumes $\Delta(\zeta_i)$, centered at $\zeta_i$, whose size is large compared to the correlation length of the number of sources. In other words, a statistical fluctuation in the number of sources in one
volume would have no implications on the number of sources in any other volume. In the direction \( \hat{\Omega} \) with the parameter set \( \zeta_i \), and \( w(f, \zeta_i, \hat{\Omega}) \) as the contribution to \( \Omega_{gw} \) of one source at frequency \( f \), in the direction \( \hat{\Omega} \), and with the parameter set \( \zeta_i \). We also define

\[
N(\zeta_i, \hat{\Omega}) \equiv n(\zeta_i, \hat{\Omega})\Delta(\zeta_i),
\]

which is the total number of sources with the parameters in the range from \( \zeta_i \) to \( \zeta_i + \Delta(\zeta_i) \) and in the direction \( \hat{\Omega} \).

The angular dependence of \( N \) can originate from anisotropic source distribution. Moreover, even in the case of an a priori isotropic SBGW, random fluctuations in the number of sources will (in general) give rise to anisotropies. Note that \( N(\zeta_i, \hat{\Omega}) \) are dimensionless numbers, which are by construction uncorrelated for different values of the index \( i \). Assuming Poisson distribution, the statistical fluctuations of \( N(\zeta_i, \hat{\Omega}) \) are of order \( \sqrt{N(\zeta_i, \hat{\Omega})} \). The corresponding fluctuation in \( \Omega_{gw} \) is

\[
\delta\Omega_{gw}(f, \hat{\Omega}) = \sum_i \delta N(\zeta_i, \hat{\Omega})w(f, \zeta_i, \hat{\Omega}),
\]

The two-point correlation of \( \delta\Omega_{gw}(f, \hat{\Omega}) \) at two different directions reads

\[
C \equiv \left\langle \delta\Omega_{gw}(f, \hat{\Omega})\delta\Omega_{gw}(f, \hat{\Omega}') \right\rangle \equiv \sum_{i,j} w(f, \zeta_i, \hat{\Omega})w(f, \zeta_j, \hat{\Omega}') \left\langle \delta N(\zeta_i, \hat{\Omega})\delta N(\zeta_j, \hat{\Omega}') \right\rangle.
\]

Since the fluctuations in the number of gravitational-wave sources are Poissonian, we propose the following bilinear expectation:

\[
\left\langle \delta N(\zeta_i, \hat{\Omega})\delta N(\zeta_j, \hat{\Omega}') \right\rangle \sim N(\zeta_i, \hat{\Omega})F(\gamma, \zeta_i)\delta_{ij},
\]

where \( \gamma \) is the angle between \( \hat{\Omega} \) and \( \hat{\Omega}' \), as shown in Fig. 4.1. \( F \) is a function that incorporates the correlation properties of the gravitational wave sources. Although the precise form of \( F \) will depend on the problem at hand, we can discuss several properties of this function. Firstly, we expect to see the maximum correlation if the two sources are close to each other in the physical space as well as the parameter space. Therefore \( F \) must assume its maximum value at \( \gamma = 0 \), and it should decrease for larger values of \( \gamma \). Since \( F \) constrains \( \gamma \), the angle between \( \hat{\Omega} \) and \( \hat{\Omega}' \), to small
values, we keep only $\hat{\Omega}$ at the right hand side of Eq. (5.8). This is a good approximation as long as $N$ changes slowly with $\hat{\Omega}$. Below we will consider an explicit example and discuss form of $F$ in more detail. Inserting this into Eq. (5.7) gives

$$C = \sum_i \Delta(\zeta_i) n(\zeta_i, \hat{\Omega}) w^2(f, \zeta_i, \hat{\Omega}) F(\gamma, \zeta_i)$$

$$\rightarrow \int d\zeta n(\zeta, \hat{\Omega}) w^2(f, \zeta, \hat{\Omega}) F(\gamma, \zeta),$$

(5.9)

where we take the integral limit of the sum. This is a general expression applicable to both cosmological and astrophysical problems in which the correlation properties of the sources are specified by the function $F$.

5.1.1 Cosmic Strings Case

We now apply this formalism to the case of cosmic strings, including gravitational-wave bursts from cusps and kinks, in which the distribution of sources is specified by the redshift $z$. Using the dimensionless cosmological functions defined in Eqs. (3.56), (4.27) and (4.20), we explicitly construct the integral in Eq. (5.9). Firstly, the parameter space volume $d\zeta$ in this case is simply the co-moving volume. It can be written as $H_0^{-3} \varphi_V(z) dz$, where $H_0$ is the present value of the Hubble constant. This converts the co-moving differential volume $r^2 dr$ to the corresponding differential volume as a function of the redshift. The next quantity in Eq. (5.9) is the number density of the loops. For the remaining part of the study, we consider small loops, for which the loop density is given Eq. (4.28). Upon integration over loop length, the only remaining parameter is the redshift, and the density of small loops as a function of redshift reads

$$n(z) = \frac{c(z)}{p \Gamma G \mu^3(z)},$$

(5.10)

In order to define the $F$-function in Eq. (5.9), we assume that the number density of cosmic string cusps and kinks at a given $z$ is correlated over the length scale $R(z)$ given by the Hubble size, $R(z) \approx t(z)$. The angular size spanned by this length scale at the distance $r(z)$ can be calculated using the standard angular diameter-redshift relation as

$$\gamma_z = 2 \arctan \left[ \frac{(1 + z) R(z)}{r(z)} \right],$$

(5.11)
where \( r(z) \) can be written as \( r(z) = \varphi_r(z)/H_0 \). Therefore, for the given redshift \( z \), two directions on the sky are correlated if their angular separation, \( \gamma \), is less than \( \gamma_z \). This condition can be imposed by the \( \mathcal{F} \) function,

\[
\mathcal{F}(\gamma, z) \equiv \Theta \left[ 1 - \frac{\gamma}{\gamma_z} \right],
\]

which vanishes if \( \gamma \) is larger than \( \gamma_z \), the angle subtended by the length scale \( R(z) \). We emphasize that the correlations considered here are large scale, and arise from the fluctuations in the number of cosmic string loops in an evolving cosmic string network. This is different from the correlations associated with the correlation length of a single cosmic string loop, which are important in determining the cosmic string signatures in the CMB [76]. For cusps and kinks on cosmic string loops with sizes given by the gravitational back-reaction scale (small loops) we have [77]

\[
w^{(C)}(f, z) = \frac{2\pi^2(G\mu)^2}{3(1+z)^{7/3}c^2\varphi_t^{1/3}} \Theta \left[ 1 - \left( \frac{2f}{H_0}(1+z)\varphi_t \right)^{-1} \right] \Theta \left[ 1 - \left( \frac{\alpha f}{H_0}(1+z)\varphi_t \right)^{-1} \right],
\]

\[
w^{(K)}(f, z) = \frac{4\pi^2(G\mu)^2}{3(1+z)^{8/3}c^2\varphi_t^{2/3}} \Theta \left[ 1 - \left( \frac{2f}{H_0}(1+z)\varphi_t \right)^{-1} \right] \Theta \left[ 1 - \left( \frac{\alpha f}{H_0}(1+z)\varphi_t \right)^{-1} \right],
\]

where \( \alpha \equiv \epsilon \Gamma G\mu \) is the parameter that sets the length of the loops. Since the function \( w \) has no angle dependence for kinks and cusps, Eq. (5.9) simplifies to

\[
\mathcal{C} = \mathcal{C}(f, \gamma) = \int dz H_0^{-3} \varphi_V(z)n(z)w^2(f, z)\mathcal{F}(\gamma, z) = \int dz \varphi_V(z) \frac{c(z)(p\Gamma G\mu)^{-1}}{\varphi_t^3(z)} w^2(f, z) \Theta \left[ 1 - \frac{\gamma}{\gamma_z} \right]
\]

which is a function of the opening angle, \( \gamma \), and the frequency only. The integrand of Eq. (5.14) quickly vanishes with increasing redshift, implying that the dominant contribution comes from low redshifts. The small values of redshift correspond to closer sources, which have larger angular size in the sky. Therefore the angular dependence of correlations will be rather flat for small angles, and it will rapidly vanish for large angles, for which small values of redshift are excluded from the integral by \( \mathcal{F}(\gamma, z) \). In order to understand the relative strength of the fluctuations at a given
frequency $f$ compared to $\Omega_{gw}(f)$ (integrated over all sky) we define the following quantity: \( \mathcal{NC}(f, \gamma) \equiv \frac{\sqrt{C(f, \gamma)}}{\Omega_{gw}(f)} \), which we refer to as the normalized correlation. We numerically evaluate the integrals in Eq. (5.14) for kinks and cusps and calculate the normalized correlations. Figs. 5.1 and 5.2 show the correlations as a function of angle at various frequencies. Fig. 5.3 shows the correlations as a function of frequency at various angles. It is important to note that large rare events which occur at rates smaller than the relevant time-scale of the experiment are excluded from $\Omega_{gw}(f)$ (see Sec. 4.3) We also do a parameter scan in $\epsilon - G\mu$ space. Figs. 5.4-5.6 show the density plot for the strength of the background $\Omega_{gw}$ and $\mathcal{NC}$ at various values of $f$ for cusps and kinks. The frequencies are applicable to ground-based detectors (10 Hz) [52], satellite-based detectors (1 mHz) [78], and pulsar-based observations ($10^{-8}$ Hz) [79]. The (base 10 logarithm of) numerical values of $\mathcal{NC}$ are denoted in the color bar for each plot.

![Figure 5.1: Normalized correlation for $f = 1$ Hz and $f = 10$ Hz.](image1)

![Figure 5.2: Normalized correlation for $f = 100$ Hz.](image2)

In the plots the darkest regions represents the strongest normalized anisotropy, which
Figure 5.3: Normalized correlation vs. frequency for cusp and kink

Figure 5.4: $NC$ for cusps and kinks at $f = 10$ Hz.

Figure 5.5: $NC$ for cusps and kinks at $f = 10^{-3}$ Hz.
Figure 5.6: $\mathcal{N}C$ for cusps and kinks at $f = 10^{-8}$ Hz.

become as high as $10^{-3}$. The accessibility of the regions are determined by the value of $\Omega_{gw}(f, \hat{\Omega})$ (as discussed in Sec. 4.7) at the given parameters and the sensitivity of the measurement $\Omega_{gw}(f, \hat{\Omega})$. Therefore in order to be able to disentangle the fluctuations calculated here from the stochastic background, the precision must be better than one part in a thousand. This precision may be in the reach of the planned third-generation detectors.

5.2 Conclusion and Discussion

In this study we have developed the formalism for calculating the spatial anisotropies in the stochastic background of gravitational waves associated with the random fluctuations in the number of sources. The formalism is applicable to a variety of cosmological and astrophysical models. We applied it to the case of SBGW due to cosmic (super)string cusps and kinks, and observed that the relative strength of the anisotropies, $\sqrt{C}/\Omega_{gw}$, can be estimated by $1/\sqrt{N} = \sqrt{\Gamma G \mu}$, which can be as high as $10^{-3}$. While observation of these spatial anisotropies is unlikely for the second-generation detectors that are currently being built (Advanced LIGO and Advanced Virgo), the planned third-generation detectors (such as Einstein Telescope) should be sufficiently sensitive to measure them over a large part of the parameter space. We emphasize that the general formalism developed here can be used to distinguish between different SBGW models - that is, between models that predict
similar frequency spectra and different spatial anisotropies. This technique will be crucial for the identification of the source of SBGW which is expected to be observed by the future generations of the gravitational-wave detectors.
References


Chapter 6

Appendix I

6.1 $\mathcal{N} = 2$ supersymmetric Lagrangian

In this section we show the calculations starting from Eq. (2.8), which is gives the Lagrangian in superspace formalism, to Eq. (2.9), which is the Lagrangian in component form. The $\mathcal{N} = 2$ supersymmetric Lagrangian in superspace is written as

$$\mathcal{L} = \int d^4\theta \Phi^* e^V \Phi + \left\{ \frac{1}{4e^2} \int d^2\theta W_{\alpha} W^{\alpha} + \text{H.c.} \right\}$$

$$+ \int d^4\theta \tilde{\Phi}^* e^{-V-V_{\text{bg}}} \tilde{\Phi} - \xi \int d^2\theta d^2\theta^\dagger V(x, \theta, \theta^\dagger),$$

(A.1)

where

$$\Phi = \phi + i\theta \sigma^\mu \bar{\theta} \partial_\mu \phi - \frac{1}{4} \partial_\mu \partial^\mu \phi \theta^2 \bar{\theta}^2 + \sqrt{2}\theta \psi + \frac{i}{\sqrt{2}} \theta^2 \bar{\theta} \bar{\sigma}^\mu \partial_\mu \psi + \theta^2 \Gamma,$$

(A.2)

and

$$V = 2\theta \sigma^\mu \bar{\theta} V_\mu + 2i\theta^2 \bar{\theta} \bar{\lambda} - 2i\bar{\theta}^2 \theta \lambda + \theta^2 \bar{\theta}^2 D.$$

(A.3)

The first integrand in Eq. (A.1) can be written as

$$\Phi^* e^V \Phi = \left( \phi + i\theta \sigma^\mu \bar{\theta} \partial_\mu \phi - \frac{1}{4} \partial_\mu \partial^\mu \phi \theta^2 \bar{\theta}^2 + \sqrt{2}\theta \psi + \frac{i}{\sqrt{2}} \theta^2 \bar{\theta} \bar{\sigma}^\mu \partial_\mu \psi \right)^*$$

$$\times \left( 1 + 2\theta \sigma^\mu \bar{\theta} V_\mu + 2i\theta^2 \bar{\theta} \bar{\lambda} - 2i\bar{\theta}^2 \theta \lambda + \theta^2 \bar{\theta}^2 (D + V_\mu V^\mu) \right)$$

$$\times \left( \phi + i\theta \sigma^\mu \bar{\theta} \partial_\mu \phi - \frac{1}{4} \partial_\mu \partial^\mu \phi \theta^2 \bar{\theta}^2 + \sqrt{2}\theta \psi + \frac{i}{\sqrt{2}} \theta^2 \bar{\theta} \bar{\sigma}^\mu \partial_\mu \psi \right),$$

(A.4)
where we dropped the $F$ term in Eq. (A.2). We need to collect only the terms with $\theta^2 \bar{\theta}^2$ since all the other terms will vanish upon integration over the super-space.

Integration over the superspace gives

$$
\int d^4 \bar{\theta} \Phi^* e^V \Phi = D^\mu \phi^* D_\mu \phi + D \phi^* \phi - N^2 \phi^* \phi
+ \bar{\psi} i \bar{\gamma} \psi + N \bar{\psi} \psi + i \sqrt{2} (\bar{\lambda} \psi \phi^* - \bar{\psi} \lambda \phi),
$$  \hspace{1cm} (A.5)

where $N = - A_3$ is a real pseudoscalar field, and

$$
i D^\mu \phi = (i \partial^\mu + A_\mu) \phi.
$$

With vector multiple given in Eq. (A.3) the kinetic term for the gauge field can be written using the following construction:

$$
W_\alpha = \frac{1}{8} \bar{D}^2 D_\alpha V = \lambda_\alpha - \theta_\alpha D - i \theta^\beta F_{\alpha \beta} + i \theta^2 \partial_{a\bar{a}} \alpha^{\dagger} \phi.
$$  \hspace{1cm} (A.6)

We need to collect $\theta^2$ terms in $W_\alpha W^\alpha$. Upon integration we get

$$
\left\{ \frac{1}{4e^2} \int d^2 \theta W_\alpha W^\alpha + \text{H.c.} \right\} = \frac{D^2}{2e^2} - \frac{1}{4e^2} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2e^2} (\partial_\mu N)^2 + \frac{1}{e^2} \bar{\lambda} i \partial^\lambda \lambda \lambda \lambda
$$  \hspace{1cm} (A.7)

The derivation of the Lagrangian for the tilde fields is very similar. The last term in Eq. (A.1) is very easy to calculate. The superspace integration simply picks up the last term:

$$
\int d^2 \theta d^2 \bar{\theta} V(x, \theta, \bar{\theta}^\dagger) = D.
$$  \hspace{1cm} (A.8)

Combining all the terms we get the Lagrangian given in Eq. (2.9).

### 6.2 $\mathcal{N} = 2$ supersymmetry current

In this section we calculate the supersymmetry current corresponding to the supersymmetry transformations. Let us concentrate on the non-tilde section of the Lagrangian:

$$
\mathcal{L} = - \frac{1}{4e^2} F_{\mu \nu} F^{\mu \nu} + D^\mu \phi^* D_\mu \phi + \frac{1}{2e^2} (\partial_\mu N)^2 - N^2 \phi^* \phi + \frac{1}{e^2} \bar{\lambda} i \partial^\lambda \lambda
+ \bar{\psi} i \bar{\gamma} \psi + N \bar{\psi} \psi + i \sqrt{2} (\bar{\lambda} \psi \phi^* - \bar{\psi} \lambda \phi) + \frac{1}{2e^2} D^2 - \xi D + D \phi^* \phi,
$$  \hspace{1cm} (A.9)
which is invariant under the following supersymmetry transformations:

\[
\begin{align*}
\delta \phi &= \sqrt{2} \bar{\epsilon} \psi, \\
\delta \psi &= \sqrt{2} \left( i \partial \phi - e N \phi \right) \epsilon, \\
\delta A_\mu &= i (\bar{\epsilon} \gamma_\mu \lambda - \bar{\lambda} \gamma_\mu \epsilon), \\
\delta \lambda &= -\gamma^\mu \epsilon (\partial_\mu N - f_\mu) + i \epsilon \frac{D}{e}.
\end{align*}
\] (A.10)

where

\[
f_\mu = -\frac{i}{2} \epsilon_{\alpha \beta} F^{\alpha \beta},
\] (A.11)

and \( \epsilon = (\epsilon_1, \epsilon_2) \) is a complex spinor. The corresponding equations of motion are

\[
\begin{align*}
\partial_\mu F^{\mu \nu} - ie \phi^* \overleftrightarrow{D}^\nu \phi - e \bar{\psi} \gamma^\nu \psi &= 0, \\
i \partial \psi - i \sqrt{2} \lambda \phi + N \psi &= 0, \\
i \frac{e}{\sqrt{2}} \partial \lambda + i \sqrt{2} \psi \phi^* &= 0, \\
D_\mu D^{\mu} \phi + N^2 \phi + i \sqrt{2} \lambda \psi + D \phi &= 0, \\
\frac{1}{e^2} \partial_\mu \partial^\mu N - 2N \phi^* \phi - \bar{\psi} \psi &= 0.
\end{align*}
\] (A.12)

The Lagrangian is invariant under the supersymmetry transformations up to a total derivative. It requires a little care to construct the conserved current. Let us consider the variation of the Lagrangian in a generic approach:

\[
\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\partial \varphi_i} \delta \varphi_i + \frac{\delta \mathcal{L}}{\partial \partial_\mu \varphi_i} \delta \partial_\mu \varphi_i
= \left( \frac{\delta \mathcal{L}}{\partial \varphi_i} - \partial_\mu \left( \frac{\delta \mathcal{L}}{\partial \partial_\mu \varphi_i} \delta \varphi_i \right) \right) \delta \varphi_i + \partial_\mu \left( \frac{\delta \mathcal{L}}{\partial \partial_\mu \varphi_i} \delta \varphi_i \right),
\] (A.13)

where \( \varphi_i \) represents a generic field. The first parenthesis vanishes due to the equations of motion. The conserved supersymmetry current can be constructed by moving the full derivative on the right to the left as follows:

\[
\frac{\delta \mathcal{L}}{\delta \epsilon} - \partial_\mu \left( \frac{\delta \mathcal{L}}{\partial \partial_\mu \varphi_i} \delta \varphi_i \right) \equiv \partial_\mu j^\mu.
\] (A.14)

The bosonic part of the Lagrangian does not contribute to the current due to the cancelations. The fermionic part does contribute to the supersymmetry current which can be calculated as
\[ j^\mu = \frac{\delta \bar{\psi}}{\delta \epsilon} \gamma^\mu \psi + i \frac{\delta \bar{\psi}}{\delta \epsilon} \gamma^\mu \bar{\psi} + i \frac{\delta \bar{\lambda}}{\delta \epsilon} \gamma^\mu \lambda \]

\[ = \sqrt{2} (\bar{\psi} \phi^* + ieN\phi^*) \gamma^\mu \psi + \sqrt{2} \left( \bar{\phi} \phi^* - ie(N + \tilde{m})\phi^* \right) \gamma^\mu \bar{\psi} \]

\[ + (i\partial N - i\mathcal{F} + D) \gamma^\mu \lambda, \quad (A.15) \]

where we added the contribution of the fermions from the tilde sector which has the following transformation

\[ \delta \bar{\psi} = \sqrt{2} \left( i\bar{\psi} \phi + e(N + \tilde{m})\phi \right) \epsilon. \quad (A.16) \]

### 6.3 Fermion mode decomposition

It is instructive to illustrate calculations of the charge \( q \) by inspecting the fermion mode decomposition discussed in Sec. 2.3.5. It is important to note that the mode decomposition in Sec. 2.3.5 is not the canonical expansion. A similar charge calculation by virtue of the canonical expansion was first performed in [80]. We will discuss both methods.

First, we expand the fields \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) in terms of the eigenfunctions of the operators \( -D_v^u D_v^u \) and \( -D_v^u D_v^u \), namely, in \( \eta_{n,\sigma} \) and \( \eta'_{n,\sigma} \), respectively:

\[ \tilde{\psi}_1 = \sum_{n \neq 0} \sum_{\sigma = 1, 2} v_{n,\sigma}(t) \eta_{n,\sigma}(x), \]

\[ \tilde{\psi}_2 = \tilde{\psi}_2^{(0)} + \sum_{n \neq 0} \sum_{\sigma = 1, 2} u_{n,\sigma}(t) \eta'_{n,\sigma}(x), \quad (A.17) \]

where \( \sigma \) labels two independent solutions corresponding to the same eigenvalue, and \( \tilde{\psi}_2^{(0)} \) is the zero mode defined in Eq. (2.51). The nonvanishing bilinears constructed from \( u_{n,\sigma}(t) \) and \( v_{n,\sigma}(t) \) are given in Eq. (2.44). With the expansion in Eq. (A.17) in hands, it is easy to see that the only nonvanishing contribution to \( q \) comes from the
zero mode of the operator $D^{\mu}_{v}$. This statement is a consequence of the following expansion of $q$:

$$
\langle q \rangle = -\int d^2x \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle = -\langle \alpha \alpha^\dagger \rangle - \sum_{n \neq 0} \langle u^*_{n,\sigma} u_{n,\sigma} + v^*_{n,\sigma} v_{n,\sigma} \rangle . \quad (A.18)
$$

Using Eqs. (2.44) and (2.52) we get

$$
\langle q \rangle = -1/2 , \quad (A.19)
$$

in perfect agreement with the previous result in Eq. (2.60).

The fact that $q = -1/2$ on the vortex is in one-to-one correspondence with the fact that integrating out the massive fermion $\tilde{\psi}$ we generate the Chern–Simons term with $\kappa = \frac{e}{4\pi}$ [15]. It is well known that selfdual $n$-vortices with the Chern–Simons term have charge $q = -\frac{2\pi n \kappa}{e} = -\frac{n}{2}$ where $n$ is the winding number [83, 20].

We can carry out a slightly different calculation of the $q$ charge by expanding the tilded fermion field in the canonical basis. However, we should remember that, generally speaking, the U(1) charge of the vacuum is infinite in the absence of proper regularization. This infinity does not show up in Eq. (A.19) because a regularized definition of the $q$ charge is built in in the expansion coefficients. The same “vacuum” infinity then shows up in $q$. In fact, we are interested in the difference between the values of $q$ on the vortex and in the vacuum.

This problem is automatically solved if, instead of the charge $-\int d^2x \tilde{\psi}^\dagger \tilde{\psi}$, one uses the following definition:

$$
q = -\frac{1}{2} \int d^2x \left( \tilde{\psi}^\dagger \tilde{\psi} - \tilde{\psi}_{c}^\dagger \tilde{\psi}_{c} \right) , \quad (A.20)
$$

where $\tilde{\psi}_c = -i(\tilde{\psi}^\dagger \gamma_2)^T$ is the charge-conjugated fermion field. We now expand the fermionic field $\tilde{\psi}$ in the canonical basis,

$$
\tilde{\psi} = a_0^\dagger \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix} + \sum_{n \neq 0} \sum_{\sigma = 1,2} \left( e^{-iw_n t} a^\dagger_{n,\sigma} \frac{\varphi_{n,\sigma}}{\sqrt{2}} + e^{iw_n t} b^\dagger_{n,\sigma} \frac{\varphi^*_{n,\sigma}}{\sqrt{2}} \right) , \quad (A.21)
$$

83
where \( \varphi_{n,\sigma} \) are the energy eigenfunctions of the fermionic Hamiltonian with the eigenvalues \( w_n \). The operators \( a_0, a_{n,\sigma} \) and \( b_{n,\sigma} \) obey the canonical anticommutation relations

\[
\{a_0, a_0^\dagger\} = 1, \quad \{a_{n,\sigma}, a_{n',\sigma'}^\dagger\} = \delta_{n,n'} \delta_{\sigma,\sigma'}, \quad \{b_{n,\sigma}, b_{n',\sigma'}^\dagger\} = \delta_{n,n'} \delta_{\sigma,\sigma'}.
\] (A.22)

Needless to say, all other anticommutators, not indicated in (A.22), vanish. The operators \( a_{n,\sigma} \) and \( b_{n,\sigma}^\dagger \) are the annihilation and creation operators associated with the positive and negative energy solutions. The operators \( a_0 \) and \( a_0^\dagger \) are not necessarily required to be particle annihilation and creation operators, see Ref. [80] for details.

The first term in the expansion (A.21) is the zero mode. Inserting the expansion (A.21) into Eq. (A.20), we get

\[
\langle q \rangle = -\frac{1}{2} \langle a_0 a_0^\dagger - a_0^\dagger a_0 \rangle
\]

\[
- \sum_{n \neq 0} \left( a_{n,\sigma}^\dagger a_{n,\sigma} - b_{n,\sigma}^\dagger b_{n,\sigma} - a_{n,\sigma} a_{n,\sigma}^\dagger + b_{n,\sigma} b_{n,\sigma}^\dagger \right). \tag{A.23}
\]

The condition we impose on \( a \) is \( a |\text{vor}\rangle = 0 \). With this condition we get

\[
\langle q \rangle = -1/2, \tag{A.24}
\]

which again agrees with the previous results.
Chapter 7

Appendix II

7.1 Energy and Momentum of gravitational waves

The calculation of the energy momentum tensor of gravitational waves [54] requires expansion of the metric perturbations including the second order:

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)}, \]  

(A.1)

where \( h_{\mu\nu}^{(1)} \) and \( h_{\mu\nu}^{(2)} \) are the first and second order metric perturbations. The Ricci tensor can also be expanded similarly

\[ R_{\mu\nu} = R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)}, \]  

(A.2)

where \( R_{\mu\nu}^{(1)} \) is a linear function of its argument and \( R_{\mu\nu}^{(2)} \) is a quadratic function of its argument. The Einstein Equation for vacuum requires

\[ R_{\mu\nu} = 0, \]  

(A.3)

which needs to be satisfied at each order of the expansion. The first order results in

\[ R_{\mu\nu}^{(1)}(h^{(1)}) = 0, \]  

(A.4)

and the second order gives

\[ R_{\mu\nu}^{(1)}(h^{(2)}) + R_{\mu\nu}^{(2)}(h^{(1)}) = 0. \]  

(A.5)
With these expansions, the Einstein Equation at the second order can be written as

\[ R_{\mu\nu}(h^{(2)}) \left( h^{(2)} \right) - \frac{1}{2} R_{\mu\nu}(h^{(1)}) = 8\pi G t_{\mu\nu}, \]  

where the tensor on the right hand side is defined as

\[ t_{\mu\nu} \equiv -\frac{1}{8\pi G} \left( R_{\mu\nu}(h^{(1)}) - R_{\mu\nu}(h^{(2)})\eta_{\mu\nu} \right). \]

\( t_{\mu\nu} \) is a conserved quantity and it can be interpreted as the energy momentum tensor of the gravitational wave. However there are two things to be addressed in this interpretation. The first one is that \( t_{\mu\nu} \) is not gauge invariant, and the second is that it is not possible to define the energy and momentum of gravitational waves locally since it is associated with the curvature. However both issues can be resolved if we average the right hand side of Eq. (A.7) over few wavelengths. This operation will capture the curvature and will also make the definition gauge invariant. Imposing the transverse traceless gauge, the energy momentum tensor takes a very simple form

\[ t_{\mu\nu} = \frac{1}{32\pi G} \left( \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \right), \]

where \( \langle \cdots \rangle \) denotes the averaging over a volume that corresponds to few wavelengths of the gravitational wave.

### 7.2 Correlations in Amplitude

In this section we extend the calculation of the SBGW to the case in which the amplitudes have correlations. This case is relevant when sources have correlations over certain distances or over a certain range of other possible parameters that characterize them. We start from

\[ \Omega_{gw}(f,t,\bar{x}_o) \equiv \frac{f}{\rho_c} \frac{d\rho_{gw}(t,\bar{x}_o)}{df} = \frac{\pi^2 f^2}{3H_0^2} \int df' d\zeta d\zeta' d^3\bar{x} d^3\bar{x}' f' e^A_{ab}(\hat{\Omega}) e^B_{ab}(\hat{\Omega}') \times \langle e^{2\pi i [f(t-r)-f'(t'-r')] A_A(\bar{x} + \bar{x}_o, f, \zeta) A^*_B(\bar{x}' + \bar{x}_o, f', \zeta') \rangle \]  

(A.9)

In order to treat possible correlations in the amplitude of the sources, we need to revisit the bilinear expectation

\[ \mathcal{C}_2 \equiv \langle e^{2\pi i [f(t-r)-f'(t'-r')] A_A(\bar{x} + \bar{x}_o, f, \zeta) A^*_B(\bar{x}' + \bar{x}_o, f', \zeta') \rangle. \]

(A.10)
If the sources have a characteristic length scale over which the amplitudes are correlated, the $\vec{x} - \vec{x}' = 0$ requirement imposed by the delta function must be relaxed, i.e. $\delta^3(\vec{x} - \vec{x}')$ must be replaced with a smooth function of $|\vec{x} - \vec{x}'|$. This function must have a peak when the argument is zero, and must vanish rapidly when the argument becomes larger than the correlation length of the sources. It is convenient to parameterize the correlator with redshift $z$ and the angle $\hat{\Omega}$. To this end we use the coordinate system depicted in Fig. 4.1, in which we have $\vec{x} = \vec{x}_o - r(z)\hat{\Omega}$ and $\vec{x}' = \vec{x}_o - r(z')\hat{\Omega}'$. Using these definitions we impose

$$C_2 = \frac{1}{4\pi} \mathcal{R}(\zeta, z) h^2(\zeta, f, z, \hat{\Omega}) \delta(f - f') \delta(\zeta - \zeta') \delta_{AB} \mathcal{F}_\zeta(\gamma, z, z') \delta(\hat{\Omega} - \hat{\Omega}') \delta_{AB} \mathcal{F}(\gamma, z, z'),$$

(A.11)

where $\gamma$ is the angle between $\hat{\Omega}$ and $\hat{\Omega}'$. $\mathcal{F}_\zeta(\gamma, z, z')$ is a function that encodes the correlation properties of the source. Although the exact form of $\mathcal{F}$ depends on the properties of the sources, we can still describe few properties of the function. First of all, $\mathcal{F}$ must have its maximum value at $\gamma = 0$ and $z = z'$, and it must vanish for larger values of $\gamma$ and $|z - z'|$. $\mathcal{F}_\zeta(\gamma, z, z')$ ensures that if $\vec{x}$ and $\vec{x}'$ are separated by distance which is larger than the correlation length of the source, the correlator vanishes. To illustrate Eqs. (4.12) and (A.11), one can use the analogy of light intensity created by incoherent light sources. In this case the intensity is the sum (integral) of the intensities of the sources weighted by their rates. $\delta(\hat{\Omega} - \hat{\Omega}')$ in Eq. (4.12) forces two source points to be the same for a non-vanishing correlator. In terms of the light source analogy, this means that sources are points in the sky with zero (or negligible) size. The correlation of two different sources is zero. On the other hand, Eq. (A.11) implies that the sources have certain sizes, and if two lines of sights differ by a small angle, and the difference of their distances to the observer is small (i.e. smaller than the depth of the source), then the light observed is originating from the same source, for which the correlator is nothing but the intensity of that particular light source.

In order to calculate $\Omega_{gw}$ the corresponding to Eq. (A.11), we substitute it into Eq.
\[ (4.11) \text{ to get} \]
\[
\Omega_{gw}(f, t, \vec{x}_o) = \frac{\pi f^3}{12H_0^2} \int d\xi dV dV' e^A_{ab}(\hat{\Omega}) e^B_{ab}(\hat{\Omega}') \\
\times \mathcal{R}(\zeta, z) h^2(\zeta, f, z, \hat{\Omega}) F(\gamma, z, z') \\
= \frac{\pi f^3}{12H_0^2} \int d\xi dz \frac{dr(z)}{dz} r^2(z) dz' \frac{dr(z')}{dz'} r^2(z') d\hat{\Omega} d\hat{\Omega}' \\
e_{ab}^A(\hat{\Omega}) e_{ab}^B(\hat{\Omega}') \mathcal{R}(\zeta, z) h^2(\zeta, f, z, \hat{\Omega}) F(\gamma, z, z').
\] (A.12)

Note that \( t \) dependence disappears because of \( \delta(f - f') \), but \( \vec{x} \) dependence still remains. One can easily verify that Eq. (A.12) reduces to Eq. (4.13) for \( \mathcal{F} = \delta \): \( \delta \) will force the angles to be the same after \( \hat{\Omega}' \) integral, and the sum over the polarizations can now be calculated to give 4, reproducing the correct overall coefficient. Consider the isotropic case, for which \( h(\zeta, f, z, \hat{\Omega}) = h(\zeta, f, z) \). In this case Eq. (A.12) simplifies to
\[
\Omega_{gw}(f) = \frac{\pi f^3}{12H_0^2} \int d\xi d\hat{\Omega}' e^A_{ab}(\hat{\Omega}) e^A_{ab}(\hat{\Omega}') \int d\zeta \mathcal{R}(\zeta) h^2(\zeta, f, z) F(\gamma, z),
\] (A.13)

where we defined
\[
F(\gamma, z) = \int dz' \frac{dr(z')}{dz'} r^2(z') F(\gamma, z, z').
\] (A.14)

In order to simplify the angle integrations, we decompose the \( \mathcal{F} \) and \( e^A_{ab}(\hat{\Omega}) \) in spherical harmonics. We first expand \( F(\gamma, z) \) in Legendre polynomials:
\[
F(\gamma, z) = \sum_{k=0}^{\infty} \frac{2k + 1}{2} F^k(\zeta) P_k(\cos(\gamma)),
\] (A.15)

where the expansion coefficients are given by
\[
F^k(\zeta) = \int_{-1}^{1} d\cos\gamma F(\gamma, z) P_k(\cos(\gamma)).
\] (A.16)

Then we expand the Legendre polynomials in spherical harmonics
\[
P_k(\cos(\gamma)) = \frac{4\pi}{2k + 1} \sum_{j=-k}^{k} Y_{kj}(\hat{\Omega}') Y_{kj}^*(\hat{\Omega}),
\] (A.17)
which yields
\[ \mathcal{F}_{\xi}(\gamma, z) = 2\pi \sum_{k=0}^{\infty} F_{\xi}^{k}(z) \sum_{j=-k}^{k} Y_{kj}(\hat{\Omega}') Y_{kj}^*(\hat{\Omega}). \]  
(A.18)

For the polarization tensors, we have the following expansion:
\[ e_{ab}^{A}(\hat{\Omega}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathcal{E}_{lm,ab}^{A} Y_{lm}(\hat{\Omega}), \]
\[ e_{ab}^{A}(\hat{\Omega}') = \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \mathcal{E}_{lm',ab}^{A*} Y_{lm'}^*(\hat{\Omega}'), \]  
(A.19)

where we used the fact that \( e_{ab}^{A}(\hat{\Omega}') \) is real. The expansion coefficients are given by
\[ \mathcal{E}_{lm,ab}^{A} = \int d\hat{\Omega} Y_{lm}^*(\hat{\Omega}) e_{ab}^{A}(\hat{\Omega}), \]
\[ \mathcal{E}_{lm',ab}^{A*} = \int d\hat{\Omega}' Y_{lm'}^{*}(\hat{\Omega}') e_{ab}^{A}(\hat{\Omega}'). \]  
(A.20)

The first few non-vanishing \( \mathcal{E}_{lm,ab}^{A} \) are listed in Sec. 7.3. Plugging the expansion in Eqs. (A.18) and (A.19) into Eq. (A.13), we get
\[ \Omega_{gw}(f) = \frac{\pi^2 f^3}{6 H_0^2} \mathcal{E}_{lm,ab}^{A} \mathcal{E}_{lm',ab}^{A*} \int dz \frac{dr(z)}{dz} r^2(z)d\zeta R(\zeta) h^2(\zeta, f, z) F_{\xi}^{k}(z) \]
\[ \times \int d\hat{\Omega} Y_{lm}(\hat{\Omega}) Y_{kj}^*(\hat{\Omega}) \int d\hat{\Omega}' Y_{lm'}^*(\hat{\Omega}') Y_{kj}(\hat{\Omega}') \]
\[ = \frac{\pi^2 f^3}{6 H_0^2} \mathcal{E}_{lm,ab}^{A} \mathcal{E}_{lm',ab}^{A*} \int dz \frac{dr(z)}{dz} r^2(z)d\zeta R(\zeta) h^2(\zeta, f, z) F_{\xi}^{k}(z) \]
\[ \times \delta_{lk} \delta_{mj} \delta_{lk} \delta_{m'j} \]
\[ = \frac{4\pi^2 f^3}{3 H_0^2} \int dz \frac{dr(z)}{dz} r^2(z)d\zeta R(\zeta) h^2(\zeta, f, z) \left[ \frac{1}{8} \mathcal{E}_{km,ab}^{A} \mathcal{E}_{km,ab}^{A*} F_{\xi}^{k}(z) \right] \]  
(A.21)

where sum over all the indices is implied. At this point, let us verify that we can recover the previous result if we replace \( F_{\xi}(\gamma, z) = \delta(\hat{\Omega} - \hat{\Omega}'). \) One can easily find the spherical harmonics decomposition of \( \delta(\hat{\Omega} - \hat{\Omega}') \) by the completeness of the spherical harmonics, i.e.
\[ \delta(\hat{\Omega} - \hat{\Omega}') = \sum_{k=0}^{\infty} \sum_{j=-k}^{k} Y_{kj}(\hat{\Omega}') Y_{kj}^*(\hat{\Omega}). \]  
(A.22)

Comparing this equation with Eq. (A.18), we deduce that
\[ F_{\xi}^{k}(\zeta) = \frac{1}{2\pi}. \]  
(A.23)
Plugging Eq. (A.23) into Eq. (A.21) gives
\[ \Omega_{gw}(f) = \frac{4\pi^2f^3}{3H_0^2} \int dz \frac{dr(z)}{dz} r^2(z) d\zeta R(\zeta) h^2(\zeta, f, z) \]
\[ \times \left[ \frac{1}{16\pi} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \mathcal{E}^A_{km,ab} \mathcal{E}^{A*}_{km,ab} \right]. \] (A.24)

We can further simplify this result using
\[ \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \mathcal{E}^A_{km,ab} \mathcal{E}^{A*}_{km,ab} = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \int \tilde{d}\tilde{\Omega} Y^*_{km}(\tilde{\Omega}) e^A_{ab}(\tilde{\Omega}) \int \tilde{d}\tilde{\Omega}' Y_{km}(\tilde{\Omega}') e^{A*}_{ab}(\tilde{\Omega}') \]
\[ = \int \tilde{d}\tilde{\Omega} \int \tilde{d}\tilde{\Omega}' \left[ \sum_{k=0}^{\infty} \sum_{m=-k}^{k} Y^*_{km}(\tilde{\Omega}) Y_{km}(\tilde{\Omega}') \right] e^A_{ab}(\tilde{\Omega}) e^{A*}_{ab}(\tilde{\Omega}') \]
\[ = \int \tilde{d}\tilde{\Omega} \delta(\tilde{\Omega} - \tilde{\Omega}') e^A_{ab}(\tilde{\Omega}) e^{A*}_{ab}(\tilde{\Omega}') \]
\[ = \int \tilde{d}\tilde{\Omega} e^A_{ab}(\tilde{\Omega}) e^{A*}_{ab}(\tilde{\Omega}) \]
\[ = 4 \times \int \tilde{d}\tilde{\Omega} = 16\pi. \] (A.25)

Plugging this result into Eq. (A.24) recovers the result in Eq. (4.13).

For the case of a general \( F_\zeta(\gamma, z) \), we need to calculate the Legendre coefficients, \( \mathcal{F}_\zeta(z) \), using Eq. (A.16). These coefficients are multiplied by
\[ \mathcal{E}_k \equiv \sum_{m=-k}^{k} \mathcal{E}^A_{km,ab} \mathcal{E}^{A*}_{km,ab}, \] (A.26)

and then summed over \( k \). One can tabulate the numerical values of \( \mathcal{E}_k \), as discussed in Sec. 7.3:
\[ \mathcal{E}_k \simeq \pi \times (2.7, 5.3, 4, 1.9, 0.8, 0.42, 0.25, 0.16, 0.11, 0.08, 0.06, \cdots). \] (A.27)

Finally we need to calculate the integral
\[ \Omega_{gw}(f) = \frac{4\pi^2f^3}{3H_0^2} \int dz \frac{dr(z)}{dz} r^2(z) d\zeta dV R(\zeta) h^2(\zeta, f, z) \left[ \frac{1}{8} \sum_{k=0}^{\infty} \mathcal{E}_k \mathcal{F}_\zeta^k(z) \right]. \] (A.28)
7.3 Spherical Harmonics Expansion

Given the line of sight unit vector \( \hat{\Omega} \), one can construct a right angle coordinate system out of three unit vectors,

\[
\hat{\Omega} = \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z},
\]
\[
\hat{l} = \hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z},
\]
\[
\hat{m} = \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}.
\] (A.29)

In this coordinate system, the polarization tensors can be written as

\[
e^+_{ab}(\hat{\Omega}) = \hat{l}_a \hat{l}_b - \hat{m}_a \hat{m}_b,
\]
\[
e^\times_{ab}(\hat{\Omega}) = \hat{l}_a \hat{m}_b + \hat{m}_a \hat{l}_b,
\] (A.30)

which satisfy

\[
e^A_{ab}(\hat{\Omega})e^B_{ab}(\hat{\Omega}) = 2\delta^{AB}.
\] (A.31)

We expand the polarization tensors in spherical harmonics,

\[
e^A_{ab}(\hat{\Omega}) = \sum_{lm} \mathcal{E}^A_{lm,ab} Y_{lm}(\hat{\Omega}),
\] (A.32)

where the expansion coefficients are given by

\[
\mathcal{E}^A_{lm,ab} = \int d\hat{\Omega} Y^*_{lm}(\hat{\Omega}) e^A_{ab}(\hat{\Omega}).
\] (A.33)

The first few nonzero values of the expansion coefficients are given below:

\[
\mathcal{E}^+_{00,ab} = \frac{2\sqrt{\pi}}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathcal{E}^+_{20,ab} = \frac{2}{3} \sqrt{\frac{\pi}{5}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix},
\] (A.34)

\[
\mathcal{E}^+_{21,ab} = \sqrt{\frac{2\pi}{15}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}, \quad \mathcal{E}^+_{2-1,ab} = -\sqrt{\frac{2\pi}{15}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix},
\] (A.35)

\[
\mathcal{E}^+_{22,ab} = \sqrt{\frac{3\pi}{10}} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{E}^+_{2-2,ab} = \sqrt{\frac{3\pi}{10}} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\] (A.36)
\[ E_{42,ab}^+ = \sqrt{\frac{\pi}{10}} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{4-2,ab}^+ = -\sqrt{\frac{\pi}{10}} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (A.37) \]

\[ E_{11,ab}^\times = \sqrt{\frac{2\pi}{3}} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix}, \quad E_{1-1,ab}^\times = \sqrt{\frac{2\pi}{3}} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & -1 \\ i & -1 & 0 \end{pmatrix}, \quad (A.38) \]

\[ E_{32,ab}^\times = \sqrt{\frac{7\pi}{30}} \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{3-2,ab}^\times = \sqrt{\frac{7\pi}{30}} \begin{pmatrix} -i & 1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (A.39) \]

\[ E_{52,ab}^\times = \sqrt{\frac{11\pi}{210}} \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{5-2,ab}^\times = \sqrt{\frac{11\pi}{210}} \begin{pmatrix} -i & 1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (A.40) \]

For \( k \geq 2 \), \( E_{k\pm2,ab}^+ \) is nonzero for even \( k \) and \( E_{k\pm2,ab}^\times \) is nonzero for odd \( k \). In Eq. (A.21), we have the following sum

\[ \sum_{m=-k}^{k} E_{km,ab}^A E_{km,ab}^{A*} = E_k, \quad (A.41) \]

which can be computed from the coefficients given above.

\[ E_k = \pi \times \left( \frac{8}{3}, \frac{16}{3}, 4, \frac{28}{15}, \frac{44}{5}, \frac{26}{10}, \frac{34}{3}, \frac{38}{495}, \frac{28}{495}, \cdots \right) \]

\[ \simeq \pi \times (2.7, 5.3, 4, 1.9, 0.8, 0.42, 0.25, 0.16, 0.11, 0.08, 0.06, \cdots), \quad (A.42) \]

where we cut the expansion at \( k = 10 \). The analytical result for \( \sum_{k=0}^{\infty} E_k = 16\pi \). If the expansion is cut at \( k = 10 \), we have \( \sum_{k=0}^{10} E_k = 0.983 \times 16\pi \), which is a very good approximation.