Spectrum of Holographic Wilson Loops

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Work in Progress, A. Farragi, LPZ and A. Tirziu

Introduction

- Wilson loops (WL) are an important class of gauge invariant non-local operators in QCD (**B**FKL).
- Wilson loops and amplitudes in $\mathcal{N}=4$ SYM

  .
- In $\mathcal{N}=4$ SYM the half BPS circular WL is captured by a Gaussian Matrix Model (Drukker, Gross; Erickson, Semenoff, Zarembo; Pestun).
- Summing ladders and other diagrammatic intuition.

$$W_R(C) = \mathrm{Tr}_R P \exp\left(i \int_C ds\left(A_\mu \dot{x}^\mu + \phi_I \dot{y}^I\right)\right)$$

Localization: Pestun

- (Topological String)/(String Theory)= ($\mathcal{N}=4$ SYM)/QCD
- Want $\int \exp(S)$:

$$\begin{aligned} \int \exp(S) \quad &\longrightarrow \quad \int \exp\left(S + tQV\right) \\ \frac{d}{dt}\int \exp\left(S+tQV\right) \quad &= \quad \int \{Q,V\}\exp\left(S+tQV\right) \\ \{Q, V\exp\left(S+tQV\right)\} \quad &= \quad 0 \end{aligned} \tag{1}$$

- Independence of t, take $t \to \infty$ $\quad \longrightarrow$ Classical plus one-loop.
- Localizes on a Gaussian action.

$$S = \frac{4\pi^2}{g_{YM}^2}\, r^2 a^2 \tag{2}$$

- a is a constant matrix coming from ϕ^I.
- r is the radius of S^4.

- Significant progress in the holographic description of WL (Maldacena; Rey, Yee; Drukker, Gross, Semenoff Ooguri; Drukker, Fiol, Gomis, Passerini; Yamaguchi; Trancanelli; Kruczenski):
- The branes sum the interaction among strings.

Configuration	Representation of $SU(N)$
F1	Fundamental
D3	Symmetric
D5	Antisymmetric

- We did a semi-classical analysis for the D3-brane.
  - Exact results on the gauge theory side; precision tests of AdS/CFT.
  - Unifying picture in terms of SUSY for all cases.
  - Field theory is ahead (localization): Precision test

Holographic Description of Wilson Loops

- For the D3-brane (Drukker, Fiol)

$$ds^2 = L^2 \left(\cosh^2(u_k) ds^2_{AdS_2} + \sinh^2(u_k) d\Omega_2^2\right),$$

$$F = iL^2 \cosh(u_k) e^{\underline{0}} \wedge e^{\underline{1}}.$$

$AdS_2 \times S^2$ worldvolume with electric flux.

$$\sinh(u_k) = \frac{k\sqrt{\lambda}}{4N} \equiv \kappa.$$

- The bosonic part of the D3 brane action in the is given by

$$S_B = T_{D3} \int d^4\sigma \, \sqrt{\det\left(g + 2\pi\alpha' F\right)} - T_{D3} \int P[C_4].$$

- The D-brane provides a $1/N$ expansion! $T_{D3} = \frac{N}{2\pi^2 L^4}$

$$S_{circle} = -2N \left[\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa \right]$$

How to organize the answer?

- Has the same bosonic symmetries as the field theory operator $SL(2,\mathbb{R}) \times SO(3) \times SO(5)$.
  - $(P_\mu, J_{\mu\nu}, D, K_\mu)$ of $SO(4,2)$
  - The subgroup preserved by the Wilson loop is generated by (P_0, J_{ij}, D, K_0).
  - J_{ij} span the $SU(2) \simeq SO(3)$
  - (P_0, K_0, D) form $SL(2,\mathbb{R})$
  - n^I breaks the $SO(6)$ R-symmetry down to $SO(5) \simeq USp(4)$
- Preserves half of the $AdS_5 \times S^5$ supersymmetries $OSp(4^*|4) \subset SU(2,2|4)$.
  - $OSp\,(4^*|4)$ has $SL(2,\mathbb{R}) \times SO(3) \times SO(5)$ as its even subgroup and 16 fermionic generators

- We studied bosonic and fermionic fluctuations explicitly.
  - Bosonic action:

$$S_{\phi}^{(2)} = \frac{T_{D3}\coth(u_k)}{2}\int d^4\sigma\sqrt{\hat{g}}\hat{g}^{\alpha\beta}\left(\partial_\alpha\phi^{\hat{\underline{4}}}\partial_\beta\phi_{\hat{\underline{4}}} + \partial_\alpha\phi^{\hat{\underline{i}}}\partial_\beta\phi_{\hat{\underline{i}}}\right),$$

$$S_{a}^{(2)} = \frac{T_{D3}\coth(u_k)}{4}\int d^4\sigma\sqrt{\hat{g}}\hat{g}^{\alpha\beta}\hat{g}^{\gamma\delta}f_{\alpha\gamma}f_{\beta\delta}.$$

Six massless scalars and a massless gauge field in $AdS_2\times S^2$.

  - Fermionic action:

$$S_{\Theta}^{(2)} = \frac{T_{D3}\coth(u_k)}{2}\int d^4\sigma\sqrt{\hat{g}}\,\overline{\Theta}\hat{\nabla\!\!\!/}\Theta.$$

Four massless Weyl fermions in $AdS_2\times S^2$.

- "Deformed" $AdS_2\times S^2$ geometry:

$$d\hat{s}^2 = L^2\sinh^2(u_k)\left(ds^2_{AdS_2} + d\Omega_2^2\right)$$

- We compactify on S^2:

2d field	4d origin	$SL(2,\mathbb{R})$	$SO(3)$	$SO(5)$	
Bosons	embedding in AdS_5	$l+1$	l	$\mathbf{1}$	$l \geq 0$
	embedding in S^5	$l+1$	l	$\mathbf{5}$	$l \geq 0$
	gauge field along AdS_2	$l+1$	l	$\mathbf{1}$	$l \geq 1$
	gauge field along S^2	$l+1$	l	$\mathbf{1}$	$l \geq 1$
Fermions	IIB spinor	$l+1$	l	$\mathbf{4}$	$l \geq \frac{1}{2}$

- Lowest lying modes: 6 massless and 6 massive (two triplets)
- This is quite different from the fundamental string: 8 lowest lying modes (five massless (Sphere) and three massive)

- The KK modes organize into multiplets of $OSp(4^*|4)$.

$$\begin{aligned} \mathbf{j} &= (j+1, j, \mathbf{5}) \oplus (j+\tfrac{3}{2}, j+\tfrac{1}{2}, \mathbf{4}) \oplus (j+2, j+1, \mathbf{1}) \qquad j \geq 1, \\ &\oplus (j+\tfrac{1}{2}, j-\tfrac{1}{2}, \mathbf{4}) \oplus (j+1, j, \mathbf{1}) \\ &\oplus (j, j-1, \mathbf{1}), \\ \mathbf{0} &= (1, 0, \mathbf{5}) \oplus (\tfrac{3}{2}, \tfrac{1}{2}, \mathbf{4}) \oplus (2, 1, \mathbf{1}), \qquad j = 0. \end{aligned}$$

$SL(2,\mathbb{R}) \times SO(3) \times SO(5)$ content of supermultiplets.

Holographic dual of k-antisymmetric WL: D5 on $AdS_5 \times S^5$

- D5 brane with worldvolume $AdS_2 \times S^4$ in $AdS_5 \times S^5$ and flux in its worldvolume.
- D5 brane wraps the S^4 inside the S^5.

$$ds^2 = L^2 \left(\cosh^2(u) ds_H^2 + \sinh^2(u) d\Omega_2^2 + du^2 + d\theta^2 + \sin^2(\theta) d\Omega_4^2\right), \quad (3)$$

- Classical solution: sits at θ_k
- Where k is the fundamental string charge on the brane

$$\theta = \theta_k, \qquad k = \frac{2N}{\pi}\left(\frac{1}{2}\theta_k - \frac{1}{4}\sin 2\theta_k\right), \quad (4)$$
$$2\pi\alpha' F = i L^2 \cos(\theta_k) e^{\underline{0}} \wedge e^{\underline{1}},$$

- Some fluctuations [Camino, Paredes and Ramallo]:

$$\theta = \theta_k + \xi, \qquad F_{0r} = \cos(\theta_k) + f. \tag{5}$$

$$m_l^2 = \begin{cases} (l+3)(l+4) & \text{for } l = 0, 1, \ldots \\ l(l-1) & \text{for } l = 1, 2, \ldots \end{cases} \tag{6}$$

- Here l is related to the eigenvalue $l(l+2)$ of the spherical harmonic on S^4.

- The full spectrum

$$\begin{aligned} h_{triplet} &= f+1 &\longrightarrow\qquad m^2_{triplet} &= f(f+1), \\ h_{singlet1} &= f+2 &\longrightarrow\qquad m^2_{singlet1} &= (f+1)(f+2), \\ h_{singlet2} &= f &\longrightarrow\qquad m^2_{singlet2} &= (f-1)f \\ h_{singlet3} &= f+1 &\longrightarrow\qquad m^2_{singlet3} &= f(f+1). \end{aligned} \tag{7}$$

One-loop Corrections

- Taking into account the $SO(3) \times SO(5)$ quantum numbers, the partition function for one multiplet $j \geq 1$ is

$$Z_B^{(j)} =$$

$$\left[\det\left(-\Box + \frac{j(j+1)}{R^2}\right)\right]^{-5(2j+1)/2} \left[\det\left(-\Box + \frac{(j+1)(j+2)}{R^2}\right)\right]^{-(2j+3)/2}$$

$$\times \left[\det\left(-\Box + \frac{j(j+1)}{R^2}\right)\right]^{-(2j+1)/2} \left[\det\left(-\Box + \frac{(j-1)j}{R^2}\right)\right]^{-(2j-1)/2},$$

$$Z_F^{(j)} =$$

$$\left[\det\left(i\nabla\!\!\!/ + \frac{j+1}{R}\gamma\right)\right]^{4(j+1)} \left[\det\left(i\nabla\!\!\!/ + \frac{j}{R}\gamma\right)\right]^{4j}.$$

(8)

(9)

The Fundamental String

- In the case $j = 0$ we have [Gross, Tseytlin]

$$Z_B^{(0)} = [\det(-\Box)]^{-5/2} \left[\det\left(-\Box + \frac{2}{R^2}\right)\right]^{-3/2}, \tag{12}$$

$$Z_F^{(0)} = \left[\det\left(i\nabla\!\!\!/ + \frac{1}{R}\gamma\right)\right]^4 .$$

The method (Straight String)

$$\det\left(-\Box+\frac{j(j+1)}{R^2}\right)=\prod_{p\in\mathbb{Z}}\det\left(-\sigma^2\left(-p^2+\frac{\partial^2}{\partial\sigma^2}\right)+j(j+1)\right). \tag{13}$$

- Solving the associated equation

$$\left[-\sigma^2\left(-p^2+\partial_\sigma^2\right)\pm p\sigma+j^2-\frac{1}{4}\right]\theta_j^{\pm p}(\sigma)=0 \tag{14}$$

- In the interval $[\epsilon, R]$ with initial conditions

$$\theta_j^{\pm p}(\epsilon)=0 \qquad \partial_\sigma\theta_j^{\pm p}(\epsilon)=1 \tag{15}$$

The (preliminary) Result

$$
\begin{aligned}
\ln Z^{(j)} = \frac{1}{2}\sum_p \Big(& -5\,(2j+1)\ln\left[K_{j+1/2}(p\epsilon)\right] - (2j+3)\ln\left[K_{j+3/2}(p\epsilon)\right] \\
& - (2j+1)\ln\left[K_{j+1/2}(p\epsilon)\right] - (2j-1)\ln\left[K_{j-1/2}(p\epsilon)\right] \\
& + 4\,(j+1)\ln\left[\frac{K^2_{j+3/2}(p\epsilon) - K^2_{j+1/2}(p\epsilon)}{2(j+1)}\right] \\
& + 4j\ln\left[\frac{K^2_{j+1/2}(p\epsilon) - K^2_{j-1/2}(p\epsilon)}{2j}\right] + 4\,(2j+1)\ln\left[p\epsilon\right]\Big)
\end{aligned} \tag{16}
$$

Divergences

$$\ln Z^{(j)} = \frac{T}{\epsilon}\left(-\left(2j+1\right)\ln\left(\frac{\Lambda\epsilon}{T}\right)+g\left(j,\frac{\Lambda\epsilon}{T}\right)\right) \quad (17)$$

- Where $g(j,x)$ is finite as $x\to\infty$.
- -T/ϵ is the Euler characteristic of AdS_2 with a cutoff ϵ
- The Euler characteristic is not Weyl invariant [Alvarez].
- The D-brane is a string? Not really.

The Matrix Model

$$\langle F(X)\rangle = \frac{1}{Z}\int \prod_{i=1}^{N} dx_i\, \Delta^2(x) F(x) \exp\left(-\frac{2N}{\lambda}\sum_{i=1}^{N} x_i^2\right) \tag{18}$$

where

$$\Delta(x) = \det\left(x_i^{j-1}\right) \tag{19}$$

- The Vandermonde determinant.
- P_j is any polynomial of order j.

$$\langle F(X)\rangle = \frac{1}{Z}\int \prod_{i=1}^{N} dx_i\, \det\left[P_{j-1}(x_i)\right]^2 F\left(\sqrt{\frac{\lambda}{2N}}x\right) \exp\left(-\sum_{i=1}^{N} x_i^2\right) \tag{20}$$

The Fundamental WL: Exact Result

$$\langle Tre^{M} \rangle = \frac{1}{N} L^1_{N-1}(-4N\lambda ?) \exp[2N\lambda ?] \quad (21)$$

- L^1_{N-1} Associated Laguerre polynomial.
- You probably know this in the large N limit:

$$\lim_{N\to\infty} \frac{1}{N} L^1_{N-1} \to (2/\sqrt{\lambda}) I_1(\sqrt{\lambda}) \quad (22)$$

k-wound string versus symmetric representation

- The k-wound string is computed by one integral:

$$F(X) = m_{(k)}(X) \to N e^{k x_1} \tag{23}$$

- For the k-symmetric representation:

$$F(X) = m_{(k)}(X) + m_{(k-1,1)}(X) + \cdots \tag{24}$$

$$\to N e^{k x_1} + N(N-1) e^{(k-1)x_1 + x_2} + \cdots \tag{25}$$

- Master integral:

$$F(X) = \exp\left(\sum_{i=1}^{N} w_i x_i\right) \qquad \sum_{i=1}^{N} w_i = k \tag{26}$$

Formal Answer

- Defining

$$I_{ij}(y) = \int_{-\infty}^{\infty} dx\, e^{-(x-y)^2} P_{i-1}(x) P_{j-1}(x) \tag{27}$$

$$\left\langle \exp\left(\sum_{i=1}^{N} w_i x_i\right)\right\rangle = \frac{1}{Z}\exp\left(\sum_{i=1}^{N} y_i^2\right) \tag{28}$$

$$\sum_{\substack{i_1,\ldots,i_N\\ j_1,\ldots,j_N}} \epsilon^{i_1\cdots i_N}\epsilon^{j_1\cdots j_N} I_{i_1 j_1}(y_1)\cdots I_{i_N j_N}(y_N) \tag{29}$$

where

$$y_i = w_i\sqrt{\frac{\lambda}{8N}} \tag{30}$$

Comments

- We get a nice picture for F1 and D3 in terms of $OSp(4^*|4)$ representations. The D5 seems to be consistent with this.
- Finding the 1-loop correction to the WL expectation value amounts to computing functional determinants.
- Similar calculation for the D5.
- Study other less symmetric WL.
- The cusp anomalous dimension in the fundamental from spinning string(Tseytlin, Roiban, Tirziu).
- Anti-symmetric representation of the cusp anomalous dimension [classical solution is known] (Armoni).