Confinement, RG flows in the complex coupling plane and Fisher’s zeros

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work done in part with Alexei Bazavov, Alan Denbleyker, Daping Du, Yuzhi “Louis” Liu and Haiyuan Zou
Content of the talk

- Running/walking couplings, discrete beta functions and confinement
- Renormalization Group (RG) flows in the complex plane
- Fisher’s zeros as separatrices in RG flows (sigma models)
- Numerical calculations of Fisher’s zeros in Lattice Gauge Theory
- Conclusions.
**Multiflavor QCD**

$N_c = 3$, $N_f$ fundamental Dirac fermions

$\beta_{2\text{-loop}}$ is universal and has an IR fixed point for $N_f$ large enough (Banks-Zaks)

Asymptotic freedom is lost

$N_f = 17 > 16.5$

Can we trust perturbation theory at large coupling?
Walking Technicolor

\[ \text{for a review: F. Sannino, arXiv 0911.0931} \]

\[ \text{m}_q \sim g_{\text{ETC}}^2 \frac{\langle \bar{Q}Q \rangle_{\text{ETC}}}{M_{\text{ETC}}^2} \]

\[ \langle \bar{Q}Q \rangle_{\text{ETC}} \sim \left( \frac{\Lambda_{\text{ETC}}}{\Lambda_{\text{TC}}} \right) \gamma(g) \]

\[ \langle \bar{Q}Q \rangle_{\text{TC}} \]

\[ \text{needs: } \gg \langle \bar{Q}Q \rangle_{\text{TC}} \sim \Lambda_{\text{TC}}^3 \]
Can we vary a parameter in such a way that a UV and a IR fixed point coalesce?

Kaplan, Lee, Son
Stephanov, Pineda
Moroz, Schmidt
A of Ph. 325

β

IR
UV

varying Nf, D, ...

2-zeros of β move to the complex plane

walking models are at the "edge" of the confining side of this family of models
Renormalization Group (RG)

(Illustrated for spin models)

Block spin.

$a \rightarrow 2a$ (discrete)

$\rightarrow \quad 2a$

$\rightarrow \quad \cdots$

Same large distance observables.

flow in Hamiltonian space

if after enough iterations the flow becomes

1-dimensional, we can match the asymptotic

flow for $(g, q)$ with $(g', 2a)$

iterated $n$ times

iterated $n-1$ times.
In lattice gauge theory

$$\beta \equiv \frac{2 N_c}{g^2} \quad \text{(not the \, \beta\,-\, function!)}$$

$$\not\beta = \beta_\text{latt.} - \beta'_\text{latt.} \quad \text{called the discrete beta function.}$$

$$\Delta \beta = \beta - \beta' \approx \frac{11 N_c^2}{24 \pi^2} \ln 2 \quad \text{for large} \, \beta$$

$$b=2 \quad \text{(change in linear scale)} \quad a \rightarrow ba$$

$$\left\{ \begin{array}{l}
\text{pure gauge.} \\
\text{large } \beta
\end{array} \right.$$
Relation between discrete and continuous RG

(based on Schrödinger conjugation, see (Utrinque & Zachos)
PRD 83 065019)

$$\left( \beta_{\text{lat}} \rightarrow \beta'_{\text{lat}}, \quad a \rightarrow 2a \right)$$

$$\beta_{\text{cs}} \left( \beta'_{\text{lat}} \right) = \frac{d}{d \beta_{\text{lat}}} \beta_{\text{cs}} \left( \beta_{\text{lat}} \right)$$

$$\beta_{\text{cs}} \left( \beta'_{\text{lat}} \right) = \prod_{\ell=1}^{n} \left( \frac{d}{d \beta_{\text{lat}}} \ell \right) \beta_{\text{cs}} \left( \beta_{\text{lat}} \right)$$

Strong

Czyngz

Numerically
Calculable

Analytical
Interpolation
Out Large
Czyngz

(Small
Czyngz)
Continua \rightarrow discrete:

\[
\int_{g}^{g'} \frac{dg'}{\beta_{cs}(g')} = \ln \Lambda \bigg|_{\Lambda}^{\Lambda/2} = -\ln z
\]

\[
g = g(\Lambda)
\]

\[
g' = g(\Lambda/2)
\]

\[
\beta_{cs} < 0 \quad \Rightarrow \quad \frac{g - g'}{g'} = -\ln z \quad \Delta \beta < \frac{1}{g - g'} > 0
\]

\[
\beta_{cs} > 0 \quad \Rightarrow \quad \frac{\ln z}{g' - g} \quad \Delta \beta < 0
\]
3D Ising : running of $\beta$ as $\omega$ Kadanoff RMP

(Note: UV and IR are relative to other fixed points; here we are on the unstable direction of WF fixed point)
up ↔ down
left ↔ right

Opposite signs in opposite orders.
Models Considered

• $2D \, O(N)$ non-linear sigma models at large $N$ (with Haiyuan Zou).

• Ising hierarchical model $D = 2$ (no transition!) and 3 (usual Wilson fixed point). The probability distribution for the total spin in blocks of any size can be calculated exactly (with Yuzhi Liu).

• $U(1)$ and $SU(2)$ 4D LGT (Fisher’s zeros at different volume; no RG flows yet; with A. Bazavov, A. Denbelyker and D. Du)).

• Models with fermions, in progress.

Note: at finite volume, these models have partition functions analytical in the entire complex $\beta$ plane.
The question of confinement

- How can we maintain a long scale confining behavior for $SU(N)$ gauge theories down to arbitrary small bare coupling (where we reach the continuum limit)?

- The Renormalization Group (RG) method should provide a quantitative answer to this question. Terry Tomboulis has constructed upper and lower bounds on free energies and Wilson loops calculated by successive decimations (MPLA 34 2717). Approximate numerical implementations, for instance with the MCRG method, would be interesting.

- The results that we will present suggest that this can be done by “analytical continuations” in the complex coupling plane starting from the weak coupling region.
Fisher's zeros

\[ Z(\beta) = 0 \quad \text{(Lee-Yang zeros } Z(e^H) = 0) \]

1D Ising: \[ e^{4\beta'} = \cosh^2(2\beta) \quad \text{Nelson-Fisher} \]
\[ \text{A. of Phys 91} \]

decimation

Zeros of \( Z_{2L} \) \( \downarrow \) Zeros of \( Z_L \)

\[ \text{Dawsonard} \]
\[ \text{Heller NP 84/10} \]
In general, the singular part of the free energy has a homogeneous transform:

\[ f_{\text{sing.}}(\beta) = b^{-D} f_{\text{sing.}}(\beta') \]

so if \( \beta \) is a Fisher zero

\( \beta' \) is also a Fisher zero.

\[ f = - \frac{\ln Z}{V} \]
Conformal Scenario

Im $\beta$

$\text{Fischler's zeros}$

Symmetric phase

Critical point

Broken symmetry phase

Re $\beta$
Complex RG flows

- Losing conformality = confinement = mass gap = complex fixed points

- New picture: Fisher’s zeros as boundary and gates for complex RG flows

- Numerical methods used for real numbers can be extended to complex numbers. For details see:
  
  Overview of the group work: arXiv:1005.1993; PRL 104 251601;
arXiv:1103.4846 (Hierarchical model with Yuzhi Liu, PRD in press)
2D $O(N)$ non-linear sigma model

$$Z = \int \prod_x d^N \phi_x \delta(\vec{\phi}_x \cdot \vec{\phi}_x - 1) e^{-\left(\frac{1}{g_0^2}\right) \sum_{x,e} \left(1 - \vec{\phi}_x \cdot \vec{\phi}_x + e\right)}$$

$$\beta \equiv \frac{1}{(g_0^2 N)} \text{ (inverse 't Hooft coupling)},$$

$$M \equiv \frac{m_{\text{gap}}}{\Lambda_{\text{UV}}}$$

Large $N$: $\beta(M^2) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{d^2 k}{(2\pi)^2} \frac{1}{2(2 - \cos(k_1) - \cos(k_2) + M^2)}$

Complex RG I: $m_{\text{gap}} = \epsilon e^{i\theta} \text{ (small circle around 0)}, \Lambda_{\text{UV}} \rightarrow \Lambda_{\text{UV}} / b$

Infinite volume, small coupling (AF): $\beta(M^2) \simeq 1/(4\pi) \ln(1/M^2)$

$$\ln(e^{i\theta} \Lambda) = i\theta + \ln\Lambda \text{ (asymptotically flat complex flows)}$$
Figure 1: Infinite $L$ RG flows (arrows). The blending blue crosses are the $\beta$ images of two lines of points located very close above and below the $[-8, 0]$ cut of $\beta(M^2)$ in the $M^2$ plane.
Singular points appear at the end of lines of zeros

Figure 2: Zeros of partition function (blue) for $L = 6$ (left) and $L = 8$ (right) and $N = 2$. The singular points appear in red.
Complex RG II: Two-lattice matching

We consider the sums of the spins in four $L/2 \times L/2$ blocks $B$; $NB$ is a nearest neighbor block of $B$. We define (possibly by reweighting):

$$R(\beta, L) \equiv \frac{\langle (\sum_{x \in B} \vec{\phi}_x)(\sum_{y \in NB} \vec{\phi}_y) \rangle_\beta}{\langle (\sum_{x \in B} \vec{\phi}_x)(\sum_{y \in B} \vec{\phi}_y) \rangle_\beta}.$$ 

A discrete RG transformation mapping $\beta$ into $\beta'$ while the lattice spacing changes from $a$ to $2a$ is obtained by matching: $R(\beta, L) = R(\beta', L/2)$.

Search with Newton's method: ambiguity $\equiv |\beta - \beta_{\text{closest}}|/|\beta - \beta_{\text{2d.closest}}|$
Figure 3: RG flows for the 2-lattice matching between $8 \times 8$ and $4 \times 4$ lattices. Circles and triangles are the singular points for $L = 4$ and $L = 8$. 
Figure 4: Discrete $\beta_{CS}$ function: $b(\Lambda) - b(\Lambda/2) \simeq \ln 2/2\pi$ at large volume and large $b$ (weak coupling) for the rescaling (left) and two-lattice matching (right) methods. Note: $b$ is the inverse 't Hooft coupling and was denoted $\beta$ before.
Hierarchical Model (review in JPA 40)

- It is a lattice model with block interactions depending on the details of the block configurations in a minimal way. The Local Potential Approximation is exact.

- Its recursion formula is related to Wilson’s approximate recursion formula (that allowed the first numerical RG calculations) but the exponents are different. (JPA 29)

- The hierarchical approximation is in principle improvable, but it has never been tried beyond one dimension.
The Hamiltonian of the hierarchical model with volume $V = 2^{n_{\text{max}}}$ is defined as:

$$H = -\frac{1}{2} \sum_{n=1}^{n_{\text{max}}} \left( \frac{c}{4} \right)^n e_n \sum_{B^{(n)}} \left( \sum_{x \in B^{(n)}} \phi_x \right)^2$$

where $B^{(n)}$ denotes hierarchically nested blocks of size $2^n$. The $e_n$ are parameters that are set to 1 in the conventional Hamiltonian but will be changed at the end of this article later to consider “deformations” of the model. The parameter $c$ controls the strength of the interaction and how it decays with the size of the blocks. From the scaling of a free massless Gaussian field under a change of the lattice spacing by a scaling factor $b$, we can include a dimension $D$ through the relation $c/4 = b^{2-D}$ with $b^D = 2$. 

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Figure 5: Unambiguous RG flows for the hierarchical model in the complex $\beta = 1/kT$ plane obtained by the two lattice method. The crosses and open boxes are at the Fisher’s zeros for $2^4$ and $2^5$ sites.
Figure 6: RG flows for the $D = 2$ hierarchical model in the complex $\beta$ plane obtained by the two lattice method. Circles and triangles are at the Fisher’s zeros for $2^4$ and $2^5$ sites. Darker=more ambiguous.
Figure 7: Discrete $\beta$ function for $D=3$, 2 and 1.7
Figure 8: RG flows in the complex $\beta$ plane for $D = 3$ (top), 2 (middle) and 1.7 (bottom) and Fisher’s zeros for $n_{max} = 3$ and 4. A darker background indicates an ambiguous solution.
Deformations

It is interesting to consider deformations of the conventional model by taking $e_n = \log(n + 1)$ in the Hamiltonian. For $D = 2$, the strengthening of the block coupling is enough to restore order for $\beta > 2.35$ and create a discontinuity in the magnetization (Thouless effect). On the other hand, for $D < 2$, the $(c/4)^n$ factor is too weak to make the sum in theorem 5 of Dyson and there is no order at infinite volume. Interesting finite size effects appear for logarithmic corrections that become asymptotically unity, for instance $e_n = (\log(n + 1))^{1/n}$. As far the existence of an infinite volume long-range order concerned the situation is just like $e_n = 1$. 
Figure 9: Discrete $\beta$ function showing the merging of an IR and UV fixed point when $D$ is varied. The effect disappears at larger volume.
D=2, n=4 vs n=5

- RG flows
- Zeros n=4
- Zeros n=5

UV
IR
Figure 10: RG flows for the hierarchical model in the complex $\beta$ plane obtained by the two lattice method for $D = 3$ (top), 2 (middle) and 1.7 (bottom). The crosses and open boxes are at the Fishers zeros for $2^3$ and $2^4$ sites.
Fisher’s zeros in 4D LGT

Spectral decomposition: \( Z = \int_0^{S_{\text{max}}} dS n(S)e^{-\beta S} \)

\( n(S) \): density of states; \( \mathcal{N} \): number of plaquettes.

\( n(S)e^{-\beta \mathcal{N} s} = e^{\mathcal{N}(f(s)-\beta s)} = e^{\mathcal{N}(f(s_0)+(1/2)f''(s_0)(s-s_0)^2+...)} \)

with \( s = S/\mathcal{N} \) and \( f'(s_0) = \beta \). \( f(s) \) is a color entropy density.

If \( \text{Re} f''(s_0) < 0 \), the distribution becomes Gaussian in the infinite volume. Gaussian distributions have no complex zeros. The level curve \( \text{Re} f''(s_0) = 0 \) is the boundary of the region where Fisher’s zeros may appear.

In the \( U(1) \) case, conjugate pairs pinch the real axis, but for \( SU(2) \) a finite gap remains present.
Figure 11: $|\delta Z/Z|$ for $U(1)$ on $4^4$. 
Figure 12: Zeros of the Re (blue) and Im (red) part of $Z$ for $U(1)$ using the density of states for $4^4$. 
Figure 13: Images of the zeros of $f''(s)$ in the $\beta$ plane (open symbols) and Fisher’s zeros (filled symbols) for $U(1)$ on $4^4$ (squares) and $6^4$ (circles) lattices.
Figure 14: The lowest zeros for $4^4$, $6^4$ and $8^4$ (from left to right). They intersect the real axis near 1.01134(1). The diamonds are the values of $\beta_S$ where the plaquette distribution has double peaks of equal height. $\text{Im}\beta \propto L^{-3.06}$. 
Figure 15: Images of the zeros of $f''(s)$ in the $\beta$ plane (open symbols) and Fisher’s zeros (filled symbols) for $SU(2)$ on $4^4$ (squares) and $6^4$ (circles) lattices.
Figure 16: Effect of an adjoint term ($+0.5$), the lowest zero goes down by about 40 percent.
Figure 17: $SU(3)$ with three light flavors on a $4 \times 12^3$ lattice (using Don Sinclair plaquette distributions). $Im\beta \propto L^{-3}$. 
Conclusions

• It is possible to extend various RG flows to the complex $\beta$ plane.

• When the size of the system is comparable to the Compton wavelength of the gap, there is a strong scheme dependence.

• Fisher’s zeros control the global behavior of the RG flows (separatrices).

• Confinement=“open gate” (stable diagnostic).

• Modified perturbative methods that could be used to reproduce a confining behavior need to be developed.
• Plans: QED, $SU(3)$ with various $N_f$ (restoration of conformality at sufficiently large $N_f$ are signaled by zeros pinching the real axis).

• Thanks to the organizers!