DIRAC COMPLEX, DOLBEAULT COMPLEX, AND SUPERSYMMETRIC QUANTUM MECHANICS

or

ONE MORE PROOF OF THE ATIYAH-SINGER THEOREM

and

NON-KAHLERIAN WONDERS

based on [E.Ivanov + A.S., arXiv:1012.2069]

Minneapolis, May 13, 2011

ATIYAH-SINGER THEOREM

(High school version)

- Consider the motion of a massless electron on the plane in external magnetic field $B(x,y)$.

Dirac operator:

$$\not{\mathcal{D}} \;=\; \sigma_j(\partial_j - iA_j), \qquad\qquad j = 1,2$$

- $\{\not{\mathcal{D}}, \sigma_3\} = 0 \;\; \rightarrow$

Double degeneracy of all excited level $\equiv$ supersymmetry

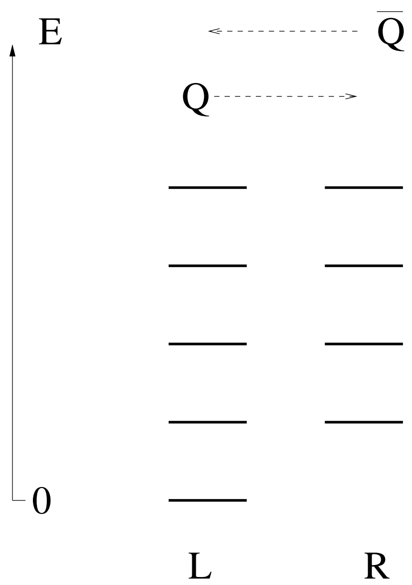

Figure 1: Landau levels

Supercharges: $Q = i\not{D}\frac{1+\sigma^3}{2},\ \bar{Q} = i\not{D}\frac{1-\sigma^3}{2}$

SUSY algebra: $Q^2 = \bar{Q}^2 = 0,\ \{Q, \bar{Q}\} = -\not{D}^2$

The index and its integral representation

$$I_{\not{D}} = n_L^0 - n_R^0 \; = \; \sum_n \langle n|\sigma^3|n\rangle e^{\beta\langle n|\not{D}^2|n\rangle/2}$$
$$= \mathrm{Tr}_{\mathrm{functional}} \left\{ \sigma_3 e^{\beta \not{D}^2/2} \right\}$$

- Heat kernel calculations

From matrices to Grassmann variables
Mapping $\psi \to \sigma^+, \bar{\psi} \to \sigma^-$
such that $\psi^2 = \bar{\psi}^2 = 0, \; \{\psi, \bar{\psi}\} = 1.$

•Dirac index $\equiv$ Witten index of a SQM system with the Hamiltonian

$$H = \frac{1}{2}(P_j - A_j)^2 + \frac{1}{2} B[\psi, \bar{\psi}]$$

$$I = \int \prod_\tau \frac{d\bar\pi(\tau)d\bar z(\tau)d\pi(\tau)dz(\tau)}{(2\pi)^2} d\bar\psi(\tau)d\psi(\tau)$$

$$\exp\left\{\int_0^\beta d\tau\,\left[i\pi\dot z + i\bar\pi\dot{\bar z} + i\dot{\bar\psi}\psi - H(\pi,\bar\pi,z,\bar z;\bar\psi,\psi)\right]\right\}\ ,$$

with periodic boundary conditions.

- For small β, this gives (?) an ordinary integral

$$I = \int \frac{d\pi dz d\bar\pi d\bar z}{4\pi^2} d\psi d\bar\psi \exp\{-\beta H\} = \frac{1}{2\pi}\int B(x,y)\,dxdy\,.$$

in words:

DIRAC INDEX = MAGNETIC FLUX

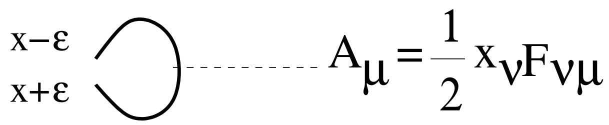

Figure 2: Schwinger-splitted ferm. propagator in external field

- Proof via Anomalous divergence

$$\partial_\mu J_\mu \;=\; \frac{1}{4\pi}\epsilon_{\alpha\beta}F_{\alpha\beta}\ .$$

- May be derived by Schwinger splitting

$$J_\mu \to J_\mu(\epsilon) = \bar{\psi}(x+\epsilon)\gamma_\mu\gamma^5\psi(x-\epsilon)$$

Gen. even-dimensional manifold with Ab. gauge field

$$I_{\not D} = \int e^{\mathcal{F}/2\pi} \det{}^{-1/2} \left[\frac{\sin \frac{\mathcal{R}}{4\pi}}{\frac{\mathcal{R}}{4\pi}} \right] ,$$

with

$$\mathcal{F} = F_{MN}\, dx^M \wedge dx^N , \qquad \mathcal{R}_{MN} = \frac{1}{2} R_{MNPQ}\, dx^P \wedge dx^Q .$$

- Heat kernel proof — Atiyah + Singer 1968,1971

- Functional integral proof — Alvarez-Gaumé, 1983; Friedan + Windey, 1984.
  (based on the standard susy structure $\{\not D(1 \pm \sigma_3); \not D^2\}$)

- This talk: an alternative proof based on an alternative susy structure for Kähler manifolds.

A SQM MODEL

- Consider the chiral (antichiral) superfields

$$Z^j(t_L,\theta) = z^j(t_L)+\sqrt{2}\theta\psi^j, \quad \bar{Z}^{\bar{i}}(t_R,\bar{\theta}) = \bar{z}^{\bar{j}}-\sqrt{2}\bar{\theta}\bar{\psi}^{\bar{j}}\ .$$

$(t_{L,R} = t \mp i\theta\bar{\theta})$

- Consider the action

$$S = \int dt d^2\theta\, (\mathcal{L}_\sigma + \mathcal{L}_{\text{gauge}})\,,$$

$$\mathcal{L}_\sigma = -\frac{1}{4} h_{i\bar{j}}(Z,\bar{Z})\, DZ^{\,i}\bar{D}\bar{Z}^{\,\bar{j}}\,, \quad \mathcal{L}_{\text{gauge}} = W(Z,\bar{Z})$$

with

$$D = \frac{\partial}{\partial\theta} - i\bar{\theta}\partial_t\,, \bar{D} = -\frac{\partial}{\partial\bar{\theta}} + i\theta\partial_t$$

- In components:

$$S=\int dt\mathcal{L}\;=\int dt\Big\{h_{i\bar{j}}\left[\dot{z}^{\,i}\dot{\bar{z}}^{\,\bar{j}}+\frac{i}{2}\left(\psi^{\,i}\dot{\bar{\psi}}^{\,\bar{j}}-\dot{\psi}^{\,i}\bar{\psi}^{\,\bar{j}}\right)\right]-$$
$$\frac{i}{2}\left[\left(2\partial_t h_{i\bar{j}}-\partial_i h_{t\bar{j}}\right)\dot{z}^{\,i}-\left(2\partial_{\bar{j}}h_{t\bar{i}}-\partial_{\bar{i}}h_{t\bar{j}}\right)\dot{\bar{z}}^{\,\bar{i}}\right]\psi^{\,t}\bar{\psi}^{\,\bar{j}}+\left(\partial_t\partial_{\bar{l}}h_{i\bar{k}}\right)\psi^{\,t}\psi^{\,i}\bar{\psi}^{\,\bar{l}}\bar{\psi}^{\,\bar{k}}$$
$$+\left[\partial_i\partial_{\bar{k}}W\,\psi^{\,i}\bar{\psi}^{\,\bar{k}}-\frac{i}{2}\left(\partial_i W\dot{z}^{\,i}-\partial_{\bar{i}}W\dot{\bar{z}}^{\,\bar{i}}\right)\right]\Big\}.$$

- $h_{i\bar{j}}$ - the metric.
- the terms in the second line are expressed via Christoffel symbols, spin connections, and torsions.

- For Kähler manifolds,

$$h_{i\bar{k}}(Z,\bar{Z}) = \partial_i \partial_{\bar{k}} K(Z,\bar{Z}) \ ,$$

the torsion terms vanish, and things simplify.

Classical supercharges and hamiltonian

$$\begin{aligned}
Q^K_{cl} &= \sqrt{2}\left[\Pi_k - i\bar{\psi}^{\,\bar{a}}\psi^{\,b}\omega_{k,\bar{a}b}\right] e^k_c \psi^{\,c}, \\
\bar{Q}^K_{cl} &= \sqrt{2} e^{\bar{k}}_{\bar{c}} \bar{\psi}^{\,\bar{c}} \left[\bar{\Pi}_{\bar{k}} + i\bar{\psi}^{\,\bar{a}}\psi^{\,d}\bar{\omega}_{\bar{k},d\bar{a}}\right].
\end{aligned}$$

$$\begin{aligned}
H^K_{cl} &= g^{i\bar{k}} \left(\Pi_i - i\omega_{i,\bar{b}a}\,\bar{\psi}^{\,\bar{b}}\psi^{\,a}\right)\left(\bar{\Pi}_{\bar{k}} + i\bar{\omega}_{\bar{k},a\bar{b}}\,\bar{\psi}^{\,\bar{b}}\psi^{\,a}\right) \\
&\quad -2e^i_a e^{\bar{k}}_{\bar{b}} \partial_i \partial_{\bar{k}} W \psi^{\,a} \bar{\psi}^{\,\bar{b}} \ ,
\end{aligned}$$

where $\Pi_k = P_k + (i/2)\partial_k W$ and $\omega_{j,\bar{b}a} = e^{\bar{k}}_{\bar{b}} \partial_j e^{\bar{a}}_{\bar{k}}$ are Kähler spin connections.

Quantization

• Ordering ambiguities. Want to keep supersymmetry at quantum level.

• Universal recipe (A.S., 1987):

a) Weyl ordering of classical supercharges gives "flat" supercharges

flat $\equiv$ acting in the Hilbert space with the "flat" measure $\int \prod dp\, dx \ldots$

b) covariant supercharges are obtained by a similarity transformation $Q \to (\det h)^{-1/2} Q (\det h)^{1/2}$.

$$Q^{cov} = \sqrt{2}\psi^c e_c^k \left[\Pi_k - \frac{i}{2}\partial_k(\ln\det\bar{e}) + i\psi^b\bar{\psi}^{\bar{a}}\omega_{k,\bar{a}b}\right]$$

$$\bar{Q}^{cov} = \sqrt{2}\bar{\psi}^{\bar{c}} e_{\bar{c}}^{\bar{k}} \left[\bar{\Pi}_{\bar{k}} - \frac{i}{2}\partial_{\bar{k}}(\ln\det e) + i\bar{\psi}^{\bar{a}}\psi^d\bar{\omega}_{\bar{k},d\bar{a}}\right],$$

COMPLETION TO EXTENDED SUSY KAHLER MODEL

(the one obtained by reduction from 2 dimensions)

- The Lagrangian $\mathcal{L}_\sigma$ can be reduced to

$$\mathcal{L}_\sigma^K \;=\; -\frac{i}{2}\,\dot{Z}^{\,k}\partial_k\,K$$

(K - Kähler potential)

- Introduce chiral fermionic superfields $\Phi^j, \bar{\Phi}^{\bar{k}}$ and write

$$\tilde{\mathcal{L}}^K = \mathcal{L}_\sigma^K + \frac{1}{4}\,h_{i\bar{k}}\,\Phi^i\,\bar{\Phi}^{\bar{k}}$$

Bingo !

Geometric interpretation

1. Dolbeault

- Choose

$$W = \frac{1}{2}(\ln \det h)$$

($\partial_k W$ is called a canonical or determinant or tautological bundle) and assume $\det \bar{e} = \det\, e = \sqrt{\det\ \mathrm{h}}$.

Then

a) Π_k is reduced to a holomorphic derivative and

b) The action of $\hat{Q}$ on the wave functions is isomorphic to the action of the external holomorphic derivatives ∂ on the holom. $(p, 0)$ - forms.

c) $\hat{\bar{Q}}$ maps to $\partial^\dagger$.

- ∂ and $\partial^\dagger$ form the Dolbeault complex.

- $W = -\frac{1}{2}(\ln \det h)$.

In this case,

- $\bar{\Pi}_{\bar{k}}$ is reduced to the antiholomorphic derivative
- $\hat{\bar{Q}}$ is mapped to $\bar{\partial}$ and $\hat{Q}$ to $\bar{\partial}^{\dagger}$.
- We obtain the antiholomorphic Dolbeault complex.
- Generic $W \quad \longrightarrow$ twisted Dolbeault and/or anti-Dolbeault complex.

2. Dirac

• Let $W = 0$. Then

$$Q = \sqrt{2}\psi^b e_b^k \left[\partial_k + \frac{1}{2}\omega_{k,\bar{a}d}(\bar{\psi}^{\,\bar{a}}\psi^{\,d} - \psi^{\,d}\bar{\psi}^{\,\bar{a}})\right] .$$

• Map fermion variables to γ - matrices: $\sqrt{2}\psi^a \equiv \gamma^a$, $\sqrt{2}\bar{\psi}^{\bar{a}} \equiv \gamma^{\bar{a}}$. Then

$$Q+\bar{Q} \equiv \not{\mathcal{D}} = \gamma^A e_A^M \left(\partial_M + \frac{1}{4}\omega_{M,BC}\gamma^B\gamma^C\right) \equiv \gamma^A \mathcal{D}_A .$$

• Another real supercharge

$$S = i\left[Q - \bar{Q}\right] = \gamma^A I_A^B \mathcal{D}_B ,$$

where I_A^B ($I^2 = -1$) is the matrix of complex structure, $I = \mathrm{diag}(i\sigma_2, \ldots, i\sigma_2)$

• Noticed before by Kirschberg + Lange + Wipf, 2005.

• $W \neq 0 \quad \longrightarrow$ Re[Q] is the twisted Dirac operator (with external gauge field)

• Two different supersymmetry structures including $\not{D}$ for Kähler manifolds: (i) $\not{D} + S$ and (ii) $\not{D} + \not{D}\gamma^5$.

CONCLUSION:

For Kähler manifolds, the Dirac complex, twisted by a bundle proportional to the tautological bundle $\partial_k \ln \det h$, is equivalent to a twisted holomorphic or antiholomorphic Dolbeault complex.

THE INDEX: EXPLICIT CALCULATION

• small β limit; functional integral $\to$ ordinary integral,

$$I \; = \; \left(\frac{1}{2\pi}\right)^n \int \prod_j dz^j d\bar{z}^{\bar{j}} \, \det \|h_{i\bar{k}}\| \, \det \|\mathcal{F}_{a\bar{b}}\| \; ,$$

with $\mathcal{F}_{a\bar{b}} = e^i_a e^{\bar{k}}_{\bar{b}} \partial_i \partial_{\bar{k}} W$ (generalized magnetic field strength).

• For CP^n with $W = \frac{q}{2(n+1)} \ln \det h$, this gives

$$I_{CP^n} \; \stackrel{?}{=} \; \frac{q^n}{n!} \; .$$

• Not integer and strange. Does not take into account the curvature.

- The correct result:

$$I_{CP^n} = \binom{q+(n-1)/2}{n} ,$$

is integer if q is integer (odd n) or half-integer (even n)

Resolution of the paradox : one cannot neglect higher Fourrier modes.
One should instead expand

$$z^j(\tau) = z^{j(0)} + \sum_{m\neq 0} z^{j(m)} e^{2\pi i m\tau/\beta} ,$$

etc. and integrate over $\prod_{jm} dz^{j(m)} \cdots$ in the Gaussian approximation.

$$\text{grav. factor} \;=\; \det\|h_{i\bar{k}}\| \prod_{m=1}^{\infty} \frac{\Omega_m^{2n}}{\det\|\Omega_m^2 \delta_j^q + R_j^s R_s^q\|}$$

with $\Omega_m = 2\pi m/\beta, \quad R_j^q = h^{\bar{k}q} R_{j\bar{k}l\bar{p}} \psi^l \bar{\psi}^{\bar{p}}.$

• Doing integrals and going to real notation, we reproduce the known result

$$I \;=\; \int e^{\mathcal{F}/2\pi} \det{}^{-1/2} \left[\frac{\sin \frac{\mathcal{R}}{4\pi}}{\frac{\mathcal{R}}{4\pi}} \right] \;,$$

• Origin of $\sin[\cdots]$

$$\prod_{m=1}^{\infty} \frac{(2\pi m)^2}{(2\pi m)^2 + a^2} \;=\; \frac{a}{2\sinh(a/2)} \;.$$

• Higher loops are suppressed at small β.

NON-KAHLERIAN WONDERS

- Supersymmetric Hamiltonian

$$
\begin{aligned}
H_{qu}^{\rm cov} \;&=\; -\frac{1}{2}\triangle^{\rm cov} + \frac{1}{8}\left(R - \frac{1}{2}h^{\bar k j}h^{\bar l t}h^{\bar i n}C_{j\,t\,\bar i}\,C_{\bar k\,\bar l\,n}\right) \\
&\quad - 2\langle\psi^a\bar\psi^{\bar b}\rangle\, e^k_a e^{\bar l}_{\bar b}\partial_k\partial_{\bar l}W - \langle\psi^a\psi^c\bar\psi^{\bar b}\bar\psi^{\bar d}\rangle\, e^t_a e^j_c e^{\bar l}_{\bar b} e^{\bar k}_{\bar d}\,(\partial_t\partial_{\bar l}\,h_{j\bar k})\,.
\end{aligned}
$$

with

$$
C_{j\,t\,\bar i} \;=\; \partial_j h_{t\bar i} - \partial_t h_{j\bar i}\,,
$$

$$
-\triangle^{\rm cov} \;=\; h^{\bar k j}\left(\mathcal{P}_j\bar{\mathcal{P}}_{\bar k} + i\hat\Gamma^{\bar q}_{j\bar k}\bar{\mathcal{P}}_{\bar q} + \bar{\mathcal{P}}_{\bar k}\mathcal{P}_j + i\hat\Gamma^{s}_{\bar k j}\mathcal{P}_s\right)\,,
$$

$$
\begin{aligned}
\mathcal{P}_k \;&=\; -i\left(\frac{\partial}{\partial z^k} - \partial_k W\right) + i\hat\Omega_{j,\bar b a}\langle\psi^a\bar\psi^{\bar b}\rangle \\
\bar{\mathcal{P}}_{\bar k} \;&=\; -i\left(\frac{\partial}{\partial \bar z^{\,\bar k}} + \partial_{\bar k} W\right) - i\hat{\bar\Omega}_{\bar k,a\bar b}\langle\psi^a\bar\psi^{\bar b}\rangle\,.
\end{aligned}
$$

- Hatted $\hat\Gamma$ and $\hat\Omega$ include the torsion.

• For $W = \frac{1}{4} \ln \det h$, quantum supercharges are mapped to ∂ and $\partial^\dagger$.

• They are not reduced to the Dirac operator.

• No physical derivation for the Dolbeault non-Kählerian index is known.

Mathematicians say :

$$I_{\text{Dolbeault}} = \int \prod_{\alpha=1}^{d} \frac{\lambda_\alpha/2\pi}{e^{\lambda_\alpha/2\pi} - 1} ,$$

where λ_α are eigenvalues of the antisymmetric $2d \times 2d$ matrix $R_{MN} = \frac{1}{2} R_{MNPQ}\, dx^P \wedge dx^Q$.

• In Kähler case, $R_{MNPQ} \to R_{i\bar{j}k\bar{l}}$ and λ_α are the eigenvalues of the $d \times d$ matrix $R_{i\bar{j}k\bar{l}} dz^k \wedge d\bar{z}^{\bar{l}}$.

• Symmetric polynomials of λ_α are expressed via coefficients in a simple way.

• Generic complex manifold — ???