High-energy amplitudes at the next-to-leading order

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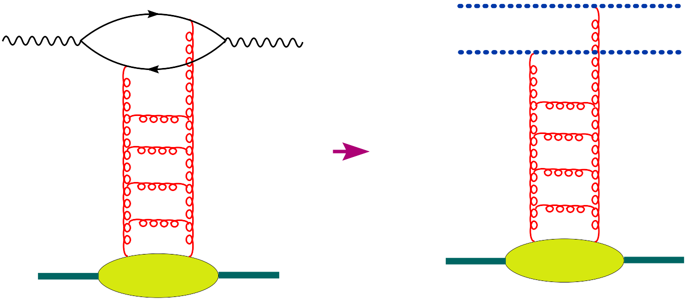
Outline

- High-energy OPE in Wilson lines. in the leading order.
- NLO amplitude in $\mathcal{N} = 4$ SYM.
- NLO evolution of color dipoles and NLO impact factor in QCD.
- Conclusions.

DIS at high energy

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp)\langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du n^\mu A_\mu(un + x_\perp)\Big] \qquad \text{Wilson line}$$

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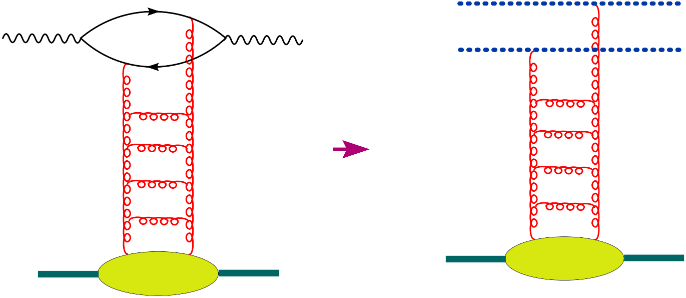

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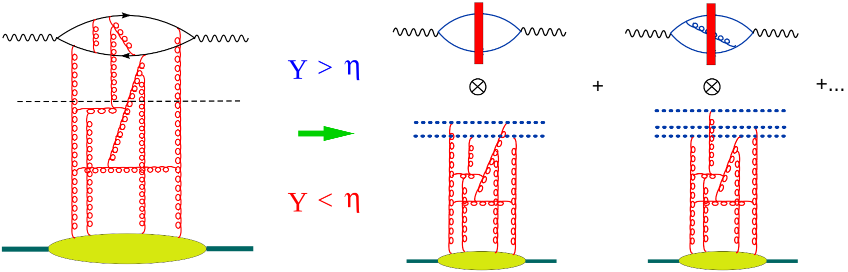
Formally, $\rightarrow$ means the operator expansion in Wilson lines

Four steps of an OPE

- To factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- To find the evolution equations of the operators with respect to factorization scale.
- To solve these evolution equations.
- To convolute the solution with the initial conditions for the evolution and get the amplitude

High-energy expansion in color dipoles



η - rapidity factorization scale

Rapidity Y > η - coefficient function (“impact factor”)

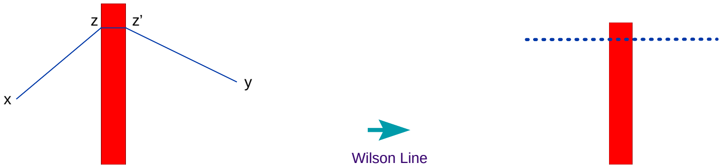
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U^\eta_x = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A^\eta_+(x_+, x_\perp)\Big]$$

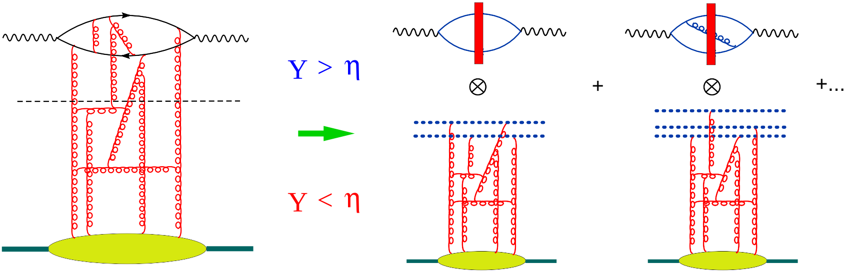
$$A^\eta_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

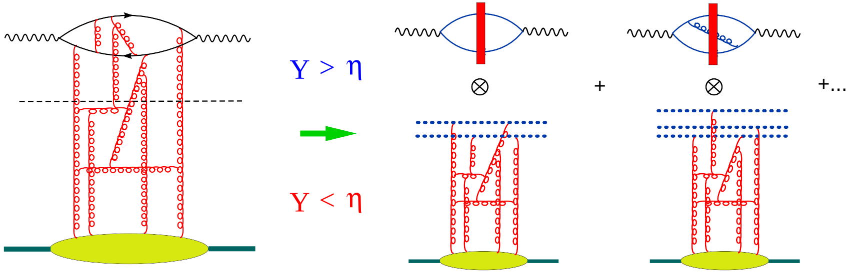


[$x \to z$: free propagation]$\times$
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]\times$
[$z \to y$: free propagation]



The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\ \text{NLO contribution}$$



η - rapidity factorization scale

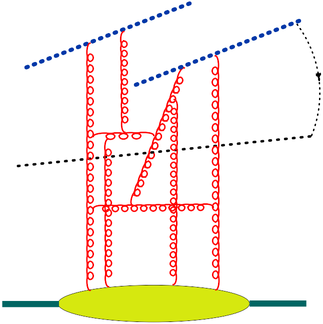
Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}$$
$$- N_c\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2)$$

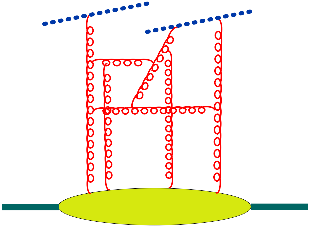
(Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).



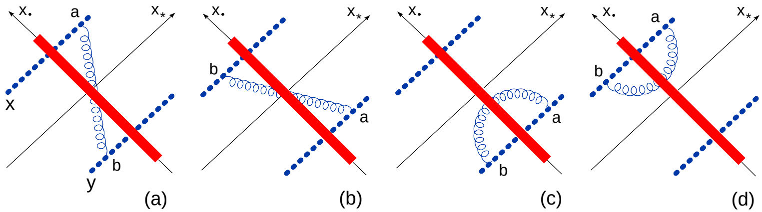
$$\alpha_s(\eta_1 - \eta_2) K_{\text{evol}} \quad \otimes$$



Evolution equation in the leading order

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non-linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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LLA for DIS in pQCD $\Rightarrow$ BFKL eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Why NLO correction?

- To check that high-energy OPE works at the NLO level.
- To determine the argument of the coupling constant.
- To get the region of application of the leading order evolution equation.
- To check conformal invariance (in $\mathcal{N}$=4 SYM)

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_{-} \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_{+} \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_{\pm}] = \pm\hat{S}_{\pm}, \quad \frac{1}{2}[\hat{S}_{+}, \hat{S}_{-}] = \hat{S}_0,$$

$$[\hat{S}_{-}, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_{+}, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

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Conformal (Möbius) invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}\mathrm{Tr}\{U_x U_y^\dagger\}]$$
$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

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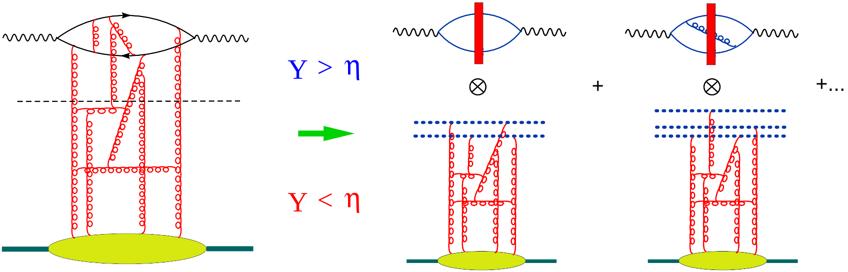
In the leading order - OK. In the NLO - ?

Expansion of the amplitude in color dipoles in the NLO



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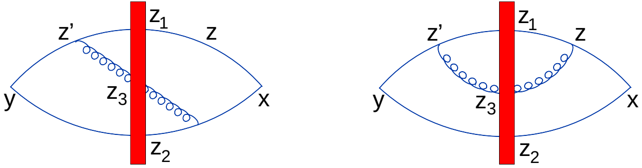
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order - conf. invariant impact factor

$$I_{\rm LO} = \frac{x_+^{-2}y_+^{-2}}{\pi^2\mathcal{Z}_1^2\mathcal{Z}_2^2}, \qquad \mathcal{Z}_i \equiv \frac{(x-z_i)^2_\perp}{x_+} - \frac{(y-z_i)^2_\perp}{y_+} \qquad CCP, 2007$$

NLO impact factor



(a) (b)

$$I^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \; = \; -\, I^{\rm LO}\times\frac{\lambda}{\pi^2}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3 - \frac{i\pi}{2} + C\Big]$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} \; = \; \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$

$$+\,\frac{\lambda}{2\pi^2}\int d^2z_3\;\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{a z_{12}^2}{z_{13}^2 z_{23}^2} \; + \; O(\lambda^2)$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}(z_1,z_2){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$$

$$+ \int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO} = -I^{\rm LO}\frac{\lambda}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 - i\pi + 2C\Big]$$

The new NLO impact factor is conformally invariant
$\Rightarrow$ ${\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$ is Möbius invariant

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.

Definition of the NLO evolution kernel

Operator equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

$$\Rightarrow \quad \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle + O(\alpha_s^3)$$

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Subtraction of the (LO) contribution (with the rigid rapidity cutoff)

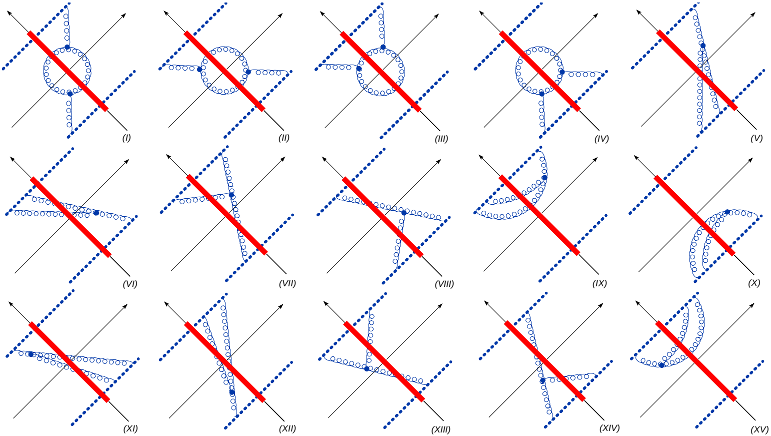
$\Rightarrow\quad \left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

Typical integral

$$\int_0^1 dv\,\frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction



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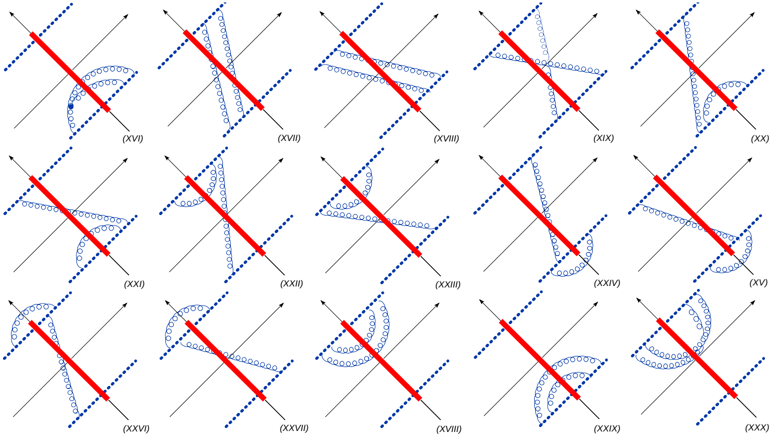

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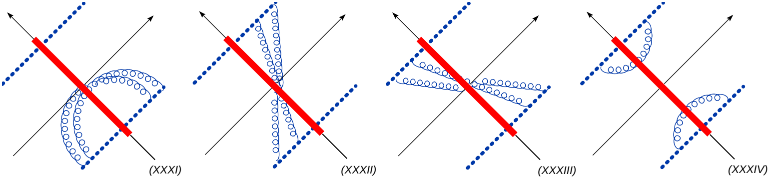
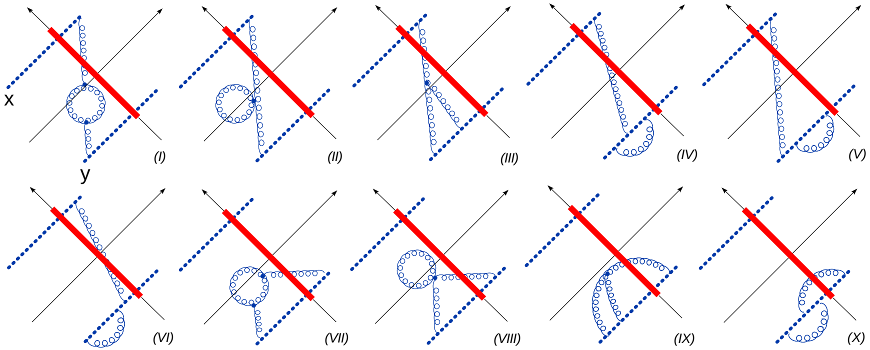
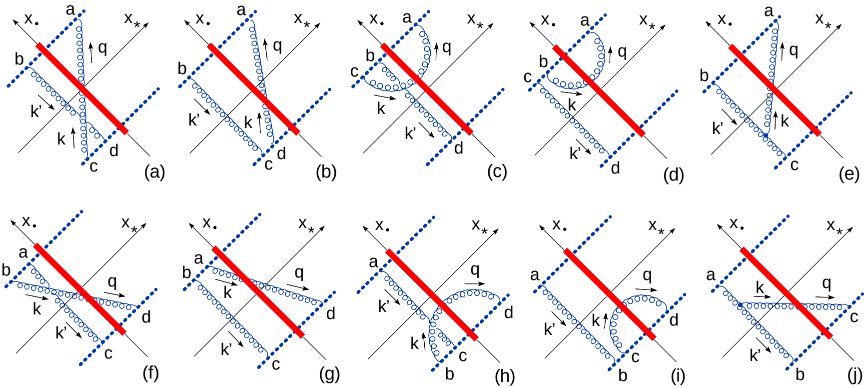


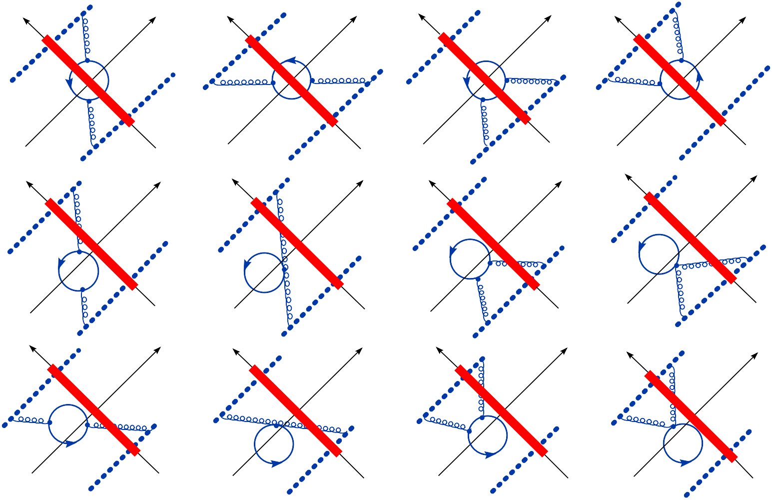
"Running coupling" diagrams



$1 \rightarrow 2$ dipole transition diagrams



$\mathcal{N}=4$ SYM diagrams (scalar and gluino loops)



Evolution equation for color dipole in $\mathcal{N}=4$

(Giovanni A. Chirilli and I.B.)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\,[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\,\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

Evolution equation for color dipole in $\mathcal{N}=4$

(Giovanni A. Chirilli and I.B.)

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}$$
$$\times\ [\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$
$$-\ \frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$$
$$\times\ \mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N}=4$

$$\frac{d}{d\eta}\big[\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$

$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\;\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$

$$-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}$$

$$\times\,\mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}[(\hat{U}^{\eta}_{z_3})^{aa'}(\hat{U}^{\eta}_{z_4})^{bb'}-(z_4\to z_3)]$$

Now Möbius invariant!

NLO Amplitude in $\mathcal{N}$=4 SYM theory

The pomeron contribution to a 4-point correlation function in $\mathcal{N}=4$ SYM can be represented as $\lambda \equiv g^2 N_c$

$$(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$
$$= \frac{i}{8\pi^2}\int d\nu\ \tilde{f}_+(\nu)\tanh\pi\nu\frac{\sin\nu\rho}{\sinh\rho}F(\nu,\lambda)R^{\frac{1}{2}\omega(\nu,\lambda)}$$

Cornalba(2007)

$\omega(\nu,\lambda)=\frac{\lambda}{\pi}\chi(\nu)+\lambda^2\omega_1(\nu)+...$ is the pomeron intercept,
$\tilde{f}_+(\omega)=(e^{i\pi\omega}-1)/\sin\pi\omega$ is the signature factor.
$F(\nu,\lambda)=F_0(\nu)+\lambda F_1(\nu)+...$ is the "pomeron residue".

R and r are two conformal ratios:

$$R = \frac{(x-x')(y-y')^2}{(x-y)^2(x'-y')^2},\quad r = R\Big[1-\frac{(x-y')^2(y-x')^2}{(x-x')^2(y-y')^2}+\frac{1}{R}\Big]^2,\quad \cosh\rho=\frac{\sqrt{r}}{2}$$

In the Regge limit $s\to\infty$ the ratio R scales as s while r does not depend on energy.

The pomeron contribution to a 4-point correlation function in $\mathcal{N} = 4$ SYM can be represented as $\lambda \equiv g^2 N_c$

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Cornalba(2007)

$\omega(\nu,\lambda) = \frac{\lambda}{\pi}\chi(\nu) + \lambda^2\omega_1(\nu) + ...$ is the pomeron intercept,
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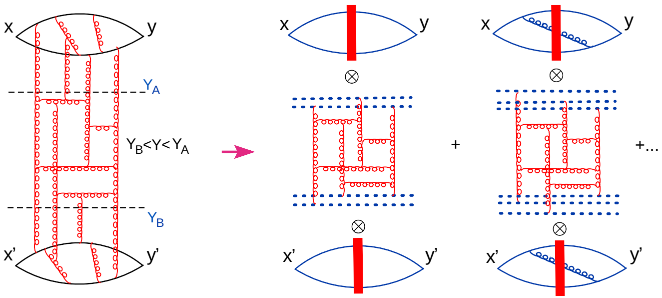
In the Regge limit $s\to\infty$ the ratio R scales as s while r does not depend on energy.

$\omega_0(\nu)$, $\omega_1(\nu)$ and $F_0(\nu)$ were known.

We reproduced $\omega_1(\nu)$ (Lipatov & Kotikov, 2000) and found $F_1(\nu)$

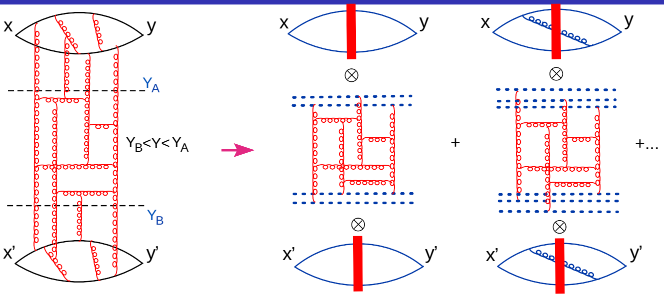
NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0 = \frac{x_+y_+}{(x-y)^2}$, $b_0 = \frac{x'_-y'_-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy

$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$ $(a_0b_0 = R)$



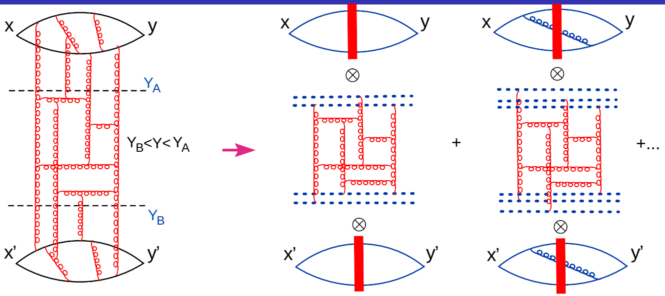
$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Dipole-dipole scattering

$$\chi(\gamma)\equiv 2C-\psi(\gamma)-\psi(1-\gamma)$$

$$[DD] = \int d\nu\int dz_0\,\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac{1}{2}+i\nu}\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac{1}{2}-i\nu}D(\frac{1}{2}+i\nu;\lambda)R^{\omega(\nu)/2}$$

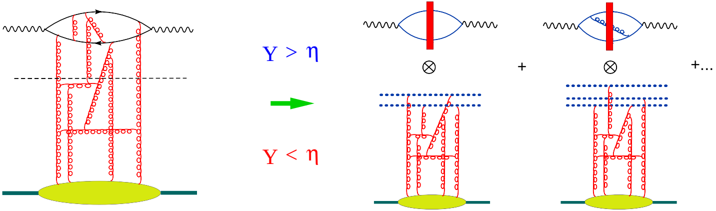
$$D(\gamma;\lambda) = \frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)}\Big\{1-\frac{\lambda}{4\pi^2}\Big[\frac{\chi(\gamma)}{\gamma(1-\gamma)}-\frac{\pi^2}{3}\Big]+O(\lambda^2)\Big\}$$



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Result : (G.A. Chirilli and I.B.)

$$F(\nu) = \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu}$$
$$\times\Big\{1+\frac{\alpha_sN_c}{\pi}\Big[-2\psi'\big(\frac{1}{2}+i\nu\big)-2\psi'\big(\frac{1}{2}-i\nu\big)+\frac{\pi^2}{2}-\frac{8}{1+4\nu^2}\Big]+O(\alpha_s^2)\Big\}$$



DIS structure function $F_2(x)$: photon impact factor + evolution of color dipoles+ initial conditions for the small-x evolution

Photon impact factor in the LO

$$(x-y)^4 T\{\hat{\bar{\psi}}(x)\gamma^\mu\hat{\psi}(x)\hat{\bar{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} I^{\rm LO}_{\mu\nu}(z_1,z_2)\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$I^{\rm LO}_{\mu\nu}(z_1,z_2) = \frac{\mathcal{R}^2}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)-\frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\Big].$$

$$\kappa \equiv \frac{\sqrt{s}}{2x_*}\Big(\frac{p_1}{s}-x^2p_2+x_\perp\Big)-\frac{\sqrt{s}}{2y_*}\Big(\frac{p_1}{s}-y^2p_2+y_\perp\Big)$$

$$\zeta_i \equiv \Big(\frac{p_1}{s}+z_{i\perp}^2p_2+z_{i\perp}\Big), \qquad \mathcal{R} \equiv \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

Composite "conformal" dipole $[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}$ - same as in $\mathcal{N}=4$ case.

$$
\begin{aligned}
&(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\}\\
&= \int\frac{d^2z_1d^2z_2}{z_{12}^4}\Big\{I^{\mu\nu}_{\mathrm{LO}}(z_1,z_2)\big[1+\frac{\alpha_s}{\pi}\big][\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}\\
&+\int d^2z_3\Big[\frac{\alpha_s}{4\pi^2}\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big(\ln\frac{\kappa^2(\zeta_1\cdot\zeta_3)(\zeta_1\cdot\zeta_3)}{2(\kappa\cdot\zeta_3)^2(\zeta_1\cdot\zeta_2)}-2C\Big)I^{\mu\nu}_{\mathrm{LO}}+\ I^{\mu\nu}_2\Big]\\
&\qquad\times[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\}-N_c\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
(I_2)_{\mu\nu}(z_1,z_2,z_3) \ =\ &\frac{\alpha_s}{16\pi^8}\frac{\mathcal{R}^2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big\{\frac{(\kappa\cdot\zeta_2)}{(\kappa\cdot\zeta_3)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[-\frac{(\kappa\cdot\zeta_1)^2}{(\zeta_1\cdot\zeta_3)}\\
&+\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}+\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)(\zeta_1\cdot\zeta_2)}{(\zeta_1\cdot\zeta_3)(\zeta_2\cdot\zeta_3)}-\frac{\kappa^2(\zeta_1\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}\Big]\\
&+\frac{(\kappa\cdot\zeta_2)^2}{(\kappa\cdot\zeta_3)^2}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)}{(\zeta_2\cdot\zeta_3)}-\frac{\kappa^2(\zeta_1\cdot\zeta_3)}{2(\zeta_2\cdot\zeta_3)}\Big]+(\zeta_1\leftrightarrow\zeta_2)\Big\}
\end{aligned}
$$

With two-gluon (NLO BFKL) accuracy

$$\frac{1}{N_c}(x-y)^4 T\{\hat{\bar{\psi}}(x)\gamma^\mu\hat{\psi}(x)\hat{\bar{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \frac{\partial\kappa^\alpha}{\partial x^\mu}\frac{\partial\kappa^\beta}{\partial y^\nu}\int\frac{dz_1 dz_2}{z_{12}^4}\,\hat{\mathcal{U}}_{a_0}(z_1,z_2)\big[\mathcal{I}^{\rm LO}_{\alpha\beta}\big(1+\frac{\alpha_s}{\pi}\big)+\mathcal{I}^{\rm NLO}_{\alpha\beta}\big]$$

$$\mathcal{I}^{\alpha\beta}_{\rm LO}(x,y;z_1,z_2) = \mathcal{R}^2\frac{g^{\alpha\beta}(\zeta_1\cdot\zeta_2)-\zeta_1^\alpha\zeta_2^\beta-\zeta_2^\alpha\zeta_1^\beta}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

$$\begin{aligned}\mathcal{I}^{\alpha\beta}_{\rm NLO}(x,y;z_1,z_2) = \frac{\alpha_s N_c}{4\pi^7}\mathcal{R}^2\Big\{&\frac{\zeta_1^\alpha\zeta_2^\beta+\zeta_1\leftrightarrow\zeta_2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big[4\mathrm{Li}_2(1-\mathcal{R})-\frac{2\pi^2}{3}+\frac{2\ln\mathcal{R}}{1-\mathcal{R}}+\frac{\ln\mathcal{R}}{\mathcal{R}}\\ &-4\ln\mathcal{R}+\frac{1}{2\mathcal{R}}-2+2\big(\ln\frac{1}{\mathcal{R}}+\frac{1}{\mathcal{R}}-2\big)\big(\ln\frac{1}{\mathcal{R}}+2C\big)-4C-\frac{2C}{\mathcal{R}}\Big]\\ &+\Big(\frac{\zeta_1^\alpha\zeta_1^\beta}{(\kappa\cdot\zeta_1)^2}+\zeta_1\leftrightarrow\zeta_2\Big)\Big[\frac{\ln\mathcal{R}}{\mathcal{R}}-\frac{2C}{\mathcal{R}}+2\frac{\ln\mathcal{R}}{1-\mathcal{R}}-\frac{1}{2\mathcal{R}}\Big]-\frac{2}{\kappa^2}\Big(g^{\alpha\beta}-2\frac{\kappa^\alpha\kappa^\beta}{\kappa^2}\Big)\\ &+\Big[\frac{\zeta_1^\alpha\kappa^\beta+\zeta_1^\beta\kappa^\alpha}{(\kappa\cdot\zeta_1)\kappa^2}+\zeta_1\leftrightarrow\zeta_2\Big]\Big[-2\frac{\ln\mathcal{R}}{1-\mathcal{R}}-\frac{\ln\mathcal{R}}{\mathcal{R}}+\ln\mathcal{R}-\frac{3}{2\mathcal{R}}+\frac{5}{2}+2C+\frac{2C}{\mathcal{R}}\Big]\\ &+\frac{g^{\alpha\beta}(\zeta_1\cdot\zeta_2)}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big[\frac{2\pi^2}{3}-4\mathrm{Li}_2(1-\mathcal{R})\\ &-2\big(\ln\frac{1}{\mathcal{R}}+\frac{1}{\mathcal{R}}+\frac{1}{2\mathcal{R}^2}-3\big)\big(\ln\frac{1}{\mathcal{R}}+2C\big)+6\ln\mathcal{R}-\frac{2}{\mathcal{R}}+2+\frac{3}{2\mathcal{R}^2}\Big]\Big\}\end{aligned}$$

5 tensor structures (CCP, 2009)

Conformal vectors

$$\kappa^\mu = \frac{\sqrt{s}}{2x_*}\Big(\frac{p_1^\mu}{s} - x^2 p_2^\mu + x_\perp^\mu\Big) - \frac{\sqrt{s}}{2y_*}\Big(\frac{p_1^\mu}{s} - y^2 p_2^\mu + y_\perp^\mu\Big)$$

$$\zeta_1^\mu = \Big(\frac{p_1^\mu}{s} + z_{1\perp}^2 p_2^\mu + z_{1\perp}^\mu\Big), \qquad \zeta_2^\mu = \Big(\frac{p_1^\mu}{s} + z_{2\perp}^2 p_2^\mu + z_{2\perp}^\mu\Big)$$

Photon Impact Factor at NLO

Conformal vectors

$$\kappa^\mu = \frac{\sqrt{s}}{2x_*}\left(\frac{p_1^\mu}{s} - x^2 p_2^\mu + x_\perp^\mu\right) - \frac{\sqrt{s}}{2y_*}\left(\frac{p_1^\mu}{s} - y^2 p_2^\mu + y_\perp^\mu\right)$$

$$\zeta_1^\mu = \left(\frac{p_1^\mu}{s} + z_{1\perp}^2 p_2^\mu + z_{1\perp}^\mu\right), \qquad \zeta_2^\mu = \left(\frac{p_1^\mu}{s} + z_{2\perp}^2 p_2^\mu + z_{2\perp}^\mu\right)$$

DIS photon impact factor is a linear combination of the following tensor basis

$$\mathcal{I}_1^{\mu\nu} = g^{\mu\nu} \qquad \mathcal{I}_2^{\mu\nu} = \frac{\kappa^\mu\kappa^\nu}{\kappa^2}$$

$$\mathcal{I}_3^{\mu\nu} = \frac{\kappa^\mu\zeta_1^\nu + \kappa^\nu\zeta_1^\mu}{\kappa\cdot\zeta_1} + \frac{\kappa^\mu\zeta_2^\nu + \kappa^\nu\zeta_2^\mu}{\kappa\cdot\zeta_2}$$

$$\mathcal{I}_4^{\mu\nu} = \frac{\kappa^2\zeta_1^\mu\zeta_1^\nu}{(\kappa\cdot\zeta_1)^2} + \frac{\kappa^2\zeta_2^\mu\zeta_2^\nu}{(\kappa\cdot\zeta_2)^2} \qquad \mathcal{I}_5^{\mu\nu} = \frac{\zeta_1^\mu\zeta_2^\nu + \zeta_2^\mu\zeta_1^\nu}{\zeta_1\cdot\zeta_2}$$

Cornalba, Costa, Penedones (2010)

Mellin representation of the LO impact factor

$$\int \frac{d^2z_1 d^2z_2}{z_{12}^2} I_{LO}^{\mu\nu}(z_1, z_2)\Big(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\Big)^{\gamma} = \frac{1}{\pi^4} B(1-\gamma)\Gamma(\gamma+2)\Gamma(3-\gamma)$$
$$\times \Big\{\frac{\gamma(1-\gamma)D_1}{12(1+\gamma)(2-\gamma)} + \frac{D_2}{2(1+\gamma)(2-\gamma)} - \frac{D_3^{\mu\nu}}{8(1+\gamma)(2-\gamma)}$$
$$-\frac{\gamma(1-\gamma)D_4^{\mu\nu}}{16(1+2\gamma)(3-2\gamma)(1+\gamma)(2-\gamma)} - \frac{D_1^{\mu\nu} + D_2^{\mu\nu}}{8}\Big\}_{\mu\nu}\Big(\frac{\kappa^2}{(\kappa\cdot\zeta_0)^2}\Big)^{\gamma}$$

Mellin representation of the LO impact factor

$$\int \frac{d^2z_1 d^2z_2}{z_{12}^2} I_{LO}^{\mu\nu}(z_1,z_2)\Big(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\Big)^\gamma = \frac{1}{\pi^4}B(1-\gamma)\Gamma(\gamma+2)\Gamma(3-\gamma)$$
$$\times\Big\{\frac{\gamma(1-\gamma)D_1^{\mu\nu}}{12(1+\gamma)(2-\gamma)} + \frac{D_2^{\mu\nu}}{2(1+\gamma)(2-\gamma)} - \frac{D_3^{\mu\nu}}{8(1+\gamma)(2-\gamma)}$$
$$-\frac{\gamma(1-\gamma)D_4}{16(1+2\gamma)(3-2\gamma)(1+\gamma)(2-\gamma)} - \frac{D_1^{\mu\nu}+D_2^{\mu\nu}}{8}\Big\}_{\mu\nu}\Big(\frac{\kappa^2}{(\kappa\cdot\zeta_0)^2}\Big)^\gamma$$

where

$$(D_1+D_2)^{\mu\nu} = -2\Delta^2 x_* y_* \kappa^{-2}\partial_x^\mu\partial_y^\nu\kappa^2$$
$$D_2^{\mu\nu} = -\Delta^2 x_* y_* \partial_x^\mu(\ln\kappa^2)\partial_y^\nu\ln\kappa^2$$
$$D_3^{\mu\nu} = 4\gamma\Delta^2 x_* y_*\big[(\partial_x^\mu\ln\kappa^2)\partial_\nu^y\ln(\kappa\cdot\zeta_0) + (\partial_y^\nu\ln\kappa^2)\partial_\mu^x\ln(\kappa\cdot\zeta_0) - (\partial_x^\mu\ln\kappa^2)\partial_y^\nu\ln\kappa^2\big]$$
$$D_4^{\mu\nu} = 4\gamma(1+2\gamma)\Delta^2 x_* y_*\big[-\frac{1}{3}\partial_x^\mu\partial_y^\nu\ln\kappa^2 - \partial_x^\mu(\ln\kappa^2)\partial_y^\nu\ln\kappa^2$$
$$+(\partial_x^\mu\ln\kappa^2)\partial_\nu^y\ln(\kappa\cdot\zeta_0) + (\partial_y^\nu\ln\kappa^2)\partial_\mu^x\ln(\kappa\cdot\zeta_0) - 2\partial_\mu^x\ln(\kappa\cdot\zeta_0)\partial_\nu^y\ln(\kappa\cdot\zeta_0)\big]$$

$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $C=-\psi(1)$ is the Euler constant, and $\psi'(a)=\frac{d}{da}\ln\Gamma(a)$

Mellin representation of the NLO impact factor

$$\int \frac{d^2z_1 d^2z_2}{z_{12}^4} I^{\mu\nu}_{NLO}(z_1,z_2)\Big(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\Big)^\gamma = \frac{\alpha_s N_c}{4\pi}\frac{B(1-\gamma)\Gamma(3-\gamma)\Gamma(2+\gamma)}{\pi^4(2-\gamma)(1+\gamma)} \times$$

$$\Big\{ \frac{D_1^{\mu\nu}}{3\sin^2(\gamma\pi)}\Big[(1-\cos(2\gamma\pi))\left(\chi-\gamma(1-\gamma)\left(C\chi-\frac{1}{2}\right)\right) - 1-\gamma(1-\gamma)\frac{\pi^2}{3}(5+\cos(2\gamma\pi)\Big]$$

$$+D_2^{\mu\nu}\Big[-\frac{3}{\gamma(1-\gamma)}+2\chi\left(\frac{1}{\gamma(1-\gamma)-2C+1}\right)+\frac{4}{3}\pi^2(1-\frac{3}{\sin^2(\gamma\pi)})\Big]$$

$$+D_3^{\mu\nu}\Big[C\chi-\frac{1}{2}-\frac{1}{\gamma(1-\gamma)}-\frac{\chi}{4}\left(1-\frac{2}{\gamma(1-\gamma)}\right)-\frac{\pi^2}{3}(1-\frac{3}{\sin^2(\gamma\pi)})\Big]$$

$$+D_4^{\mu\nu}\Big[\frac{15+10\gamma-9\gamma^2-\gamma^3(2-\gamma)}{4\gamma[3+\gamma(1-4\gamma(2-\gamma)]}-\frac{1}{2(3+4\gamma(1-\gamma))}\left(\chi+\gamma(1-\gamma)(C\chi-\frac{\pi^2}{3}+\frac{\pi^2}{\sin^2(\gamma\pi)}\right)\Big]$$

$$+\frac{1}{4}(D_1^{\mu\nu}+D_2^{\mu\nu})(2-\gamma)(1+\gamma)\Big[4\psi'(1-\gamma)-4\psi'(3)+4C\chi(\gamma)+2\psi'(3-\gamma)+2\psi'(2+\gamma)-4\psi'(3)$$

$$+[\psi(3-\gamma)+\psi(2+\gamma)-2\psi(1)]^2-6-\frac{8}{(1+\gamma)(2-\gamma)}[\psi(2-\gamma)+\psi(1+\gamma)-2\psi(1)]$$

$$-\frac{4}{\gamma(1-\gamma)(1+\gamma)(2-\gamma)}\left(-\chi(\gamma)+\frac{3}{2}\right)-2\chi'(\gamma)-2\chi^2(\gamma)\Big]\Big\}\Big(\frac{\kappa^2}{(\kappa\cdot\zeta_0)^2}\Big)^\gamma$$

Mellin representation of the impact factor for polarized DIS

Contribution of spin 2 in t-channel:

$$\int \frac{d^2z_1 d^2z_2}{z_{12}^4}\big(I^{\mu\nu}_{LO}(z_1,z_2)+I^{\mu\nu}_{NLO}(z_1,z_2)\big)\Big(\frac{z_{12}}{z_{10}z_{20}}\Big)^{\gamma+1}\Big(\frac{\bar z_{12}}{\bar z_{10}\bar z_{20}}\Big)^{\gamma+1} = B(2-\gamma)\Gamma(3-\gamma)\Gamma(2+\gamma)$$
$$\times\Big[1+\frac{\alpha_s N_c}{4\pi}\Big\{4\psi'(1+\gamma)+4\psi'(2-\gamma)-8\psi'(3)-\frac{6\chi(2,\gamma)}{(1+\gamma)(2-\gamma)}+6+4C\chi(2,\gamma)$$
$$-\frac{6C}{(2-\gamma)(1+\gamma)}-\frac{6}{(1+\gamma)(2-\gamma)}\Big\}\Big]\Big(\frac{\kappa^2}{(\kappa\cdot\zeta_0)^2}\Big)^{\gamma}\Big(\partial^x_\mu\frac{X^2\bar Y-Y^2\bar X}{x_+y_+(\kappa\cdot\zeta_0)}\Big)\Big(\partial^y_\nu\frac{X^2\bar Y-Y^2\bar X}{x_+y_+(\kappa\cdot\zeta_0)}\Big)$$

$\chi(2,\gamma)=2\psi(1)-\psi(2-\gamma)-\psi(1+\gamma)$ $\qquad$ $X\equiv x-z_0,\ Y\equiv y-z_0$

Fourier transformation for the “forward case" (DIS) is straightforward.

I. B. and G. Chirilli

$$
\begin{aligned}
a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}}_a \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z_3\, \Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}}_a \\
& \times \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
& + \frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{z_{23}{}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2(z_{23}{}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{z_{23}{}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big] \\
& \times [\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4\to z_3)] \\
& + \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{23}{}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}{}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{23}{}^2z_{23}^2}\Big] \\
& \times [\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}
\end{aligned}
$$

$$b=\tfrac{11}{3}N_c-\tfrac{2}{3}n_f$$

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel.

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$

$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

Argument of coupling constant

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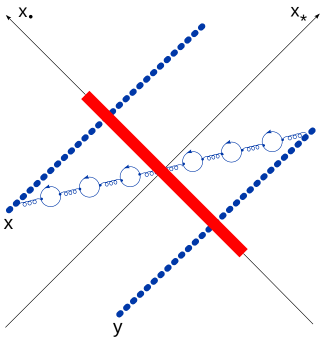
Renormalon-based approach: summation of quark bubbles



$$-\tfrac{2}{3}n_f \rightarrow b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

Argument of coupling constant

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{Tr}\{\hat U_{z_3}\hat U^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]$$
$$\times\Big[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|$$
$$\frac{\alpha_s(z_{13}^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$
$$\frac{\alpha_s(z_{23}^2)}{2\pi^2 z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL eigenvalues.
- The correlation function of four Z^2 operators is calculated at the NLO order.
- The analytic expression for the NLO photon impact factor is calculated.