Graceful Kayak Paddles

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Abstract

A kayak paddle is a graph made of two cycles joined by a path. We can define $KP(r, s, l)$ as two cycles of lengths r and s joined by a path of length l. If a graph G has m vertices and n edges, then a general vertex labeling of the graph is a one-to-one mapping of the vertex set of G into the set of all non-negative integers. If we have two vertices, say x and y joined by an edge xy, we define the edge length as $min\{x - y, y - x\}$, where the subtraction is performed in Z_{2n+1}. Two important types of vertex labelings are ρ- and β- labelings. In a ρ- or *rosy labeling* the vertices must be within the set $\{0, 1, \ldots, 2n\}$ and the set of the edge lengths must be equal to $\{1, 2, \ldots, n\}$. A β- or graceful labeling is a ρ-labeling where all the vertex labels must come from $\{0, 1, \ldots, n\}$, and the set of edge lengths must be equal to $\{1, \ldots, n\}$. If a graph can be labeled using either a rosy labeling or a β-labeling, then it can cyclically decompose K_{2n+1}. D. Froncek and L. Tollefson proved results for kayak paddles decomposing a complete graph using rosy labelings. In this thesis we investigate the existence of graceful labelings of kayak paddles.

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Chapter 1

Introduction

Throughout this thesis we will only be referring to finite graphs with no loops or multiple edges. We will investigate the existence of a certain type of graph labeling for an infinite class of graphs. There are several main types of graph labelings. One of them, often called "Rosa type labeling," is a type of vertex labeling. This type of labeling was introduced in 1967 by A. Rosa in his paper *On certain valuations of the vertices of a graph* [19]. The vertex labelings he introduced became very useful tools in graph decompositions. He called them valuations instead of labelings. The ones he introduced were ρ-, σ-, β-, and α-labelings. If a graph G has m vertices and n edges, then a general vertex labeling of the graph is a one-to-one mapping of the vertex set of G into the set of all non-negative integers. We will refer to the vertex by the number which it is labeled in its labeling. If we have two vertices, say x and y joined by an edge xy, then we can define the edge length as $min\{x-y, y-x\}$, where the subtraction is performed in Z_{2n+1}. The above mentioned graph labelings are based on how both the vertex labels and edge labels are restricted. All vertex and edge labels must be distinct. The ρ-labeling is the least restrictive. In a ρ- or *rosy labeling* the vertices must be within the set $\{0, 1, \ldots, 2n\}$ and the set of the edge lengths must be equal to $\{1, 2, \ldots, n\}$. A β-labeling is a ρ-labeling where all the vertex labels must come from $\{0, 1, \ldots, n\}$, and the set of edge lengths must be equal to $\{1, \ldots, n\}$. And in an α-labeling we restrict a β-labeling to a bipartite graph so that there is a number λ within the set $\{0, 1, \ldots, n\}$, such that for any given edge (x, y) either $f(x) \leq \lambda, f(y) > \lambda$ or $f(x) > \lambda, f(y) \leq \lambda$. So when there is an α-labeling, we have a bipartite graph where all of the vertex labels in one partite are less than or equal to λ, and in the other partite set all of the vertex labels are strictly greater than λ. We will sometimes refer to these as "small" and "large" vertices.

1.1 Graph decompositions

We say that a complete graph K_n has a *G-decomposition* if there are subgraphs $G_0, G_1, G_2, \ldots, G_s$ of K_n, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i. The decomposition is *cyclic* if $s = n-1$ and there exists an ordering $(x_1, x_2, \ldots, x_n)$ of the vertices of K_n and isomorphisms $\phi_i : G_0 \to G_i, i = 1, 2, \ldots, n-1$ such that $\phi_i(x_j) = x_{i+j}$ for every $j = 1, 2, \ldots, n$, where the subscripts are taken modulo n. We will repeatedly use the following Rosa's theorems [19] to cyclically decompose complete graphs:

Theorem 1. *A cyclic decomposition of the complete graph K_{2n+1} into subgraphs isomorphic to a given graph G with n edges exists if and only if there exists a ρ-labeling of the graph G.*

Theorem 2. *If a graph G with n edges has an α-labeling, then there exists a cyclic decomposition of the complete graph K_{2kn+1} into subgraphs isomorphic to G, where k is an arbitrary natural number.*

A kayak paddle is a graph made of two cycles joined by a path. We can define $KP(k, m, l)$ as two cycles of lengths k and m joined by a path of length l. The above theorems tell us that if a graph can be labeled using either a rosy labeling or a β-labeling, then it can cyclically decompose a complete graph. D. Froncek and L. Tollefson proved in [5] and [6] results for kayak paddles decomposing a complete graph using rosy labelings. While the existence of a ρ-labeling guarantees a decomposition, we do not want to stop there. We want to find labelings of kayak paddles using the more restrictive β-labeling. The β-labeling is also known as graceful labeling and is probably the most popular graph labeling.

1.2 Previous Results

A canoe paddle is a graph consisting of a path joined to a cycle. M. Truszczyński proved that all canoe paddles are graceful in [20]. He called them dragons instead of canoe paddles. They have also been called kites or tadpoles by other authors. Truszczyński also studied other classes of unicyclic graphs and proved they were graceful. In [20] in addition to proving dragons are graceful, he also proved some classes of graphs containing caterpillars are graceful. A *caterpillar* is a graph in which if all vertices of degree one are removed, we are left with a path. A *star graph* is a graph on m vertices where one vertex has degree $m-1$ and all others have degree one. The star graph is isomorphic to the complete bipartite graph $K_{1,m}$. Truszczyński proved that if H is a caterpillar then the graph $C_k \cup H$ with $k \equiv 0$ or $3 \pmod 4$ is graceful. He also proved that $S_k(m)$ is graceful for $k \geq 4$, $m \geq 1$, where by $S_k(m)$ we mean the graph obtained by identifying a vertex of C_k with the vertex of degree m in the star $K(1,m)$. Another class of graphs he found graceful is $S_k(m) \cup H$ where again H is a caterpillar. Truszczyński conjectured that all unicyclic graphs except for C_n with $n \equiv 1$ or $2 \pmod 4$ are graceful.

There are many other cycle-related graphs which have been studied and found graceful. A *wheel*, W_n, is a cycle C_n with one additional vertex with edges connecting to every vertex in the cycle. It was shown by R. Frucht in [7] and C. Hoede and H. Kuiper in [9] that all wheels are graceful. A *helm*, H_n, is a wheel in which a pendant edge is attached at each vertex of the n-cycle. It was shown that helms are graceful by J. Avel and O. Favaron in [3]. A graph called a *web* was defined by K. Koh, D. Rogers, H. Teo, and K. Yap in [10] by taking a helm and joining the pendant points to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. Later it was proved in [18] by Q. Kang, Z. Liang, Y. Gao, and G. Yang, that webs are in fact graceful. Yang defined a *generalized web* by iterating the process of creating a web by continuing to join the pendant points to form a new cycle and then adding pendant points to this cycle. The notation he gave for this is $W(t,n)$, the generalized web with t n-cycles. So the original web would be denoted as $W(2,n)$. Yang shows in [18] that $W(3,n)$ and $W(4,n)$ are graceful. V. Abhyanker and V. Bhat-Nayak found that $W(5,n)$ is also graceful in [1]. A *gear graph* is a modification of the wheel made by adding a vertex between every pair of adjacent vertices of the cycle. In [16], K. Ma and C. Feng proved that all gears are graceful. In [14], Y. Liu proved that the graph obtained from a gear graph by inserting two or more vertices between every pair of vertices of the outer cycle of a wheel is also graceful.

A *cycle with a chord* is a cycle in which two nonconsecutive vertices of the cycle are connected with an edge or "chord." It was shown by C. Delorme, M. Maheo, H. Thuillier, K. Koh, and H. Teo in [4] and Ma and Feng in [15] that any cycle with a chord is graceful. Following this, a *cycle with a* $P_k - chord$ was defined by Koh and Yap in [11] as a cycle with the path P_k joining two nonconsecutive vertices of the cycle. They conjectured that all cycles with a P_k-chord are graceful and proved the case when $k = 3$. The case $k \geq 4$ was proved by N. Punnim and N. Pabhapote in [17]. Refer to Gallian's survey on graph labelings [8] for even more cycle-related graceful graphs.

Chapter 2

Useful Results

There are many previous results that we will need to find graceful labelings of kayak paddles. The proofs are included since they are primarily constructive proofs and may be referred to when constructing the graceful kayak paddles.

Rosa proved in [19] the following theorem:

Theorem 3.

(a) An α-labeling of C_n exists if and only if $n \equiv 0 \pmod 4$

(b) A β-labeling of C_n exists if and only if $n \equiv 0$ *or* $3 \pmod 4$.

Proof. Let C_n be a cycle $v_1, v_2, \ldots, v_n, v_1$.

Let $n \equiv 0 \pmod 4$.

The labeling of C_n, in which vertex v_i has value a_i for $i \in \{1, 2, \ldots, n\}$:

$$a_i = \begin{cases} (i-1)/2 & i \text{ odd}, \\ n+1-i/2 & i \text{ even}, i \leq n/2, \\ n-i/2 & i \text{ even}, i > n/2, \end{cases} \tag{2.1}$$

is evidently an α-labeling of C_n.

Let $n \equiv 3 \pmod 4$. The labeling of C_n, in which the vertex v_i has the value a_i for $i \in \{1, 2, \ldots, n\}$:

$$a_i = \begin{cases} n+1-i/2 & i \text{ even}, \\ (i-1)/2 & i \text{ odd}, i \leq (n-1)/2, \\ (i+1)/2 & i \text{ odd}, i > (n-1)/2, \end{cases} \tag{2.2}$$

is evidently a β-labeling of C_n. □

Definition 1. *Let G be a graph with k edges and a graceful labeling, β, such that $\beta(x_0) = 0$ and $\beta(x_1) = k$ and H be a graph with an alpha-labeling, α, such that $\alpha(y_0) = \lambda$ and $\alpha(y_1) = \lambda + 1$. Then by $G \odot H$ we denote the graph arising from G and H by joining the vertex x_0 with y_0 or x_1 with y_1.*

The following result has been proved by many authors. We will use it to find graceful labelings for kayak paddles where one of the cycles is 0 (mod 4) and the other is either 0 or 3 (mod 4):

Theorem 4. *Let G and H be graphs with disjoint sets of vertices. If G has an alpha labeling, α, and H has a graceful labeling, β, then $G \odot H$ is graceful.*

Proof. Define λ as follows: For all edges $xy \in E(G)$, $\alpha(x) \leq \lambda < \alpha(y)$

Define the sets A and B as follows:

$A = \{x | \alpha(x) \leq \lambda\}$

$B = \{x | \alpha(x) > \lambda\}$

Then if $\alpha(x_0) = \lambda$ and $\beta(y_0) = 0$ and $|E(H)| = m$, the following function f on $V(H)$ gives a graceful labeling of $G \odot H$.

$$f(z) = \begin{cases} \alpha(z) & z \in A \\ \beta(z) + \lambda & z \in V(H) \\ \alpha(z) + m & z \in B. \end{cases} \tag{2.3}$$

And if $\alpha(x_0) = \lambda + 1$ and $\beta(y_0) = m$, the following function g on $V(H)$ gives a graceful labeling of $G \odot H$.

$$g(z) = \begin{cases} \alpha(z) & z \in A \\ \beta(z) + \lambda + 1 & z \in V(H) \\ \alpha(z) + m & z \in B. \end{cases} \tag{2.4}$$

□

Another previous result we need is for labeling kayak paddles with both cycles of length 2 (mod 4). When the two cycles are equal, we use a result from [13] by A. Kotzig. In this paper they refer to a $Q(r, s)$ graph as a graph consisting of r components where each component is an s-cycle. So the one which helps us in this case is the $Q(2, s)$ graph, which is a graph with two cycles, and we will need $s = 4k + 2$, so in our notation the graph $C_{4k+2} \cup C_{4k+2}$.

In [12], Kotzig proved the following, which is referenced in the next theorem.

Lemma 5. *Let G be bipartite graph having a β-labeling and each of its components is Eulerian. Then $|E(G)| \equiv 0 \pmod 4$*

The proof of this theorem provides us with the graceful labeling of two $2 \pmod 4$ cycles:

Theorem 6. *$C_s \cup C_s$ has an α-labeling if and only if s is even and $s \geq 4$.*

Proof. By Lemma 5, because in a graph $G = C_s \cup C_s$ we have $|V(G)| = |E(G)| = 2s$, and if $C_s \cup C_s$ has an α-labeling, then s is even and $2s \equiv 0 \pmod 4$. So necessity follows from $2s \equiv 0 \pmod 4$ and from the fact that every even cycle contains at least 4 vertices. For odd s, the graph is not bipartite and it is known that only bipartite graphs have α-labelings. The construction of an α-labeling of $C_s \cup C_s$ with $s = 4k$ is given in Figure 1 and Figure 2 and for $s = 4k + 2$ in Figure 1 and Figure 3. The sequences of labels describing the order in which vertices can be encountered if we travel around the first cycle, are the following ($k > 1$):

(i) If $s = 4k : (0, 8k, 1, 8k - 1, \ldots, k - 1, 7k + 1; k; 5k + 1, 3k - 1, 5k + 2, 3k - 2, ..., 5k + k - 1, \ldots, 2k + 1, 6k)$

(ii) If $s = 4k + 2 : (0, 8k + 4, 2k + 2, 6k + 3, 2k + 3, 6k + 2, 2k + 4, \ldots, 5k + 5, 3k + 1; 7k + 3, k, 7k + 4, k - 1, \ldots, 7k + k + 2, 1, 8k + 3)$.

The labeling of the second cycle of $C_s \cup C_s$ follows from symmetry (see Figure 2 and Figure 3). If $s = 4k$ we can describe the symmetry as follows. Let C be the first cycle and C' be the second cycle. If we call the vertices in C $x_1, x_2, \ldots, x_s$ and the vertices in C' $y_1, y_2, \ldots, y_s$, then we can find the vertex labels of the second cycle using the first cycle:

$$f(y_i) = \begin{cases} 2k - f(x_i) & x_i \text{ is in lower part of cycle} \\ 6k + 1 - f(x_i) & x_i \text{ is in upper part of cycle} \end{cases} \tag{2.5}$$

When $s = 4k + 2$, the second cycle will be labeled $4k + 2, 4k + 3, 2k, 6k + 4, 2k - 1, 6k + 5, \ldots, 7k + 2, k + 1, 5k + 4, 3k + 2, 5k + 3, 3k + 3, 5k + 2, \ldots, 4k + 1, 4k + 4$.

In a very elementary way one can easily find that the sequences of edge-values have the required property. This proves the theorem. □

When the two cycles are both 2 (mod 4) but they are not equal lengths, we use a result from [2] by J. Abraham and A. Kotzig. We use the proof of the following theorem which gives us constructions for two 2 (mod 4) cycles, C_{4k+2} and C_{4m+2} when $k \neq m$. In the proof they refer to an α_k-*labeling* of P_n which can be defined as follows:

Definition 2. *If k, n are integers, $0 \leq k < n$, and the end vertices (i.e., the two vertices of degree 1) of P_n are w, z, then an α-labeling ψ of P_n will be called an α_k-labeling of P_n if $min\{\psi(w), \psi(z)\} = k$.*

Theorem 7. *Let k, m be positive integers. Then the graph $C_{4k+2} \cup C_{4m+2}$ has an α-labeling.*

Proof. For $k = m$, the proposition was again proved by Kotzig in [13]. Without any loss of generality we can and will therefore assume that $k < m$. We will distinguish two cases according to whether $k < m \leq 2k$ or $m \geq 2k + 1$.

Case 1: Let $m \geq 2k + 1$. The values of the successive vertices of C_{4k+2} are given in the following sequence: $3m+4k+4, m, 3m+4k+3, m+1, 3m+4k+2, m+2, \ldots, 3m+3k+4, m+k, 3m+3k+2, m+k+2, 3m+3k+1, m+k+3, \ldots, 3m+2k+4, m+2k, 3m+2k+3, m+2k+1$. The resulting values of the edges are $2m + 4k + 4, 2m + 4k + 3, 2m + 4k + 2, \ldots, 2m + 2k + 4, 2m + 2k + 2, 2m + 2k, 2m + 2k - 1, \ldots, 2m + 4, 2m + 3, 2m + 2, 2m + 2k + 3$.

To construct the labeling of the vertices of C_{4m+2} we will use two auxiliary paths and their α-labeling: an α_{m-k-2}-labeling of P_{2m-1} and an α_k-labeling of P_{2m}. The end vertices of P_{2m-1}

have the values $m-k-2, 2m-k-2$ (the first value is small, the second one is large). The end vertices of P_{2m} have the values k and $m-k$ (both small). We will now modify the labelings of these two paths as follows: In the labeling of P_{2m-1} we will leave the small values unchanged, and increase each large value by $4k+2m+5$. In the labeling of P_{2m}, we will increase the value of each vertex by $2k+m+2$. Now, the end vertices of P_{2m-1} have the values $m-k-2, 4m+3k+3$, and the end vertices of P_{2m} will have the values $m+3k+2$ and $2m+k+2$. The reader will observe that the value $3k+3m+3$ has not been assigned to any vertex so far, so we assign it to a vertex (isolated at this moment) and join this vertex with one end vertex of each of the two paths under consideration—i.e., to the vertices labeled $m-k-2$ and $m+3k+2$. This combines P_{2k-1} and P_{2m} into one path with the end vertices labeled $4m+3k+3$ and $2m+k+2$. Now we join these two vertices and obtain C_{4m+2}; the reader can easily verify that the values assigned to the vertices of C_{4m+2}, together with those assigned to the vertices of C_{4k+2}, constitute an α-labeling of $C_{4k+2} \cup C_{4m+2}$.

Case 2: Let $k < m \leq 2k$. First we will describe the construction of the labeling of the vertices of C_{4k+2}. We take an arbitrary α_{2k-m}-labeling of the path P_{4k}; the values of its end vertices will be $2k-m$, m. In this path, we leave the small values $(0, 1, \ldots, 2k)$ unchanged, and add $4m+4$ to each large value. Finally, we join both end vertices of P_{4k} with the vertex labeled by $3m+2k+4$. This closes C_{4k+2}; its edges have the values $4m+5, 4m+6, \ldots, 4m+4k+4, 4m+4, 2k+2m+4$. In the labeling of C_{4m+2}, the consecutive vertices will be labeled as follows: $2k+1, 2k+4m+4, 2k+2, 2k+4m+3, \ldots, k+m, 3k+3m+5, k+m+2, 3k+3m+4, k+m+3, 3k+3m+3, \ldots, 2k+m+1, 2k+3m+5, 2k+m+2, 2k+3m+3, 2k+m+3, 2k+3m+2, \ldots, 2k+2m+1, 2k+2m+4, 2k+2m+2, 2k+2m+3$. The values of the edges are then $4m+3, 4m+2, \ldots, 2k+2m+5, 2k+2m+3, 2k+2m+2, \ldots, 2m+4, 2m+3, 2m+1, 2m, \ldots, 2, 1, 2m+2$. □

In their results about rosy kayak paddles [5] and [6], D. Froncek and L. Tollefson defined a *gap graceful labeling*, denoted $\ddot{\beta}$. In this labeling the vertices are labeled by elements in the set $\{0, 1, 2, \ldots, n, n+1\}$ and the set of edge lengths is an n-element subset of $\{1, 2, \ldots, n, n+1\}$. If we are missing the edge length p, then it can be called a *p-gap graceful labeling* and denoted by $\ddot{\beta}_p$. We will call this labeling a *p-gap α-labeling* and denote by $\ddot{\alpha}_p$ if it also has the alpha-like property that there exists a number α_0 such that for every edge $e = xy$ in G with $\ddot{\alpha}_p(x) < \ddot{\alpha}_p(y)$ it holds that $\ddot{\alpha}_p(x) \leq \alpha_0 < \ddot{\alpha}_p(y)$. This gap-graceful labeling will help us in a few cases of finding graceful kayak paddles.

We will need the following gap graceful labeling found in [6]. We postpone the proof until Section 3.3, as we will need to refer to it in the proof of Theorem 21.

Lemma 8. *The cycle C_{4m+1} has a $(2i-1)$-gap graceful labeling for every $m > 0$ and $1 \leq i \leq m$ or $m + 2 \leq i \leq 2m + 1$.*

Chapter 3

Results

A kayak paddle $KP(k, m, l)$ consists of two cycles C_k, C_m which are joined by a path of length l. Looking at cases depending on the the two cycles modulo four, there are a total of 16 cases. The first two cases we will look at are $k, m \equiv 0 \pmod 4$ and $k \equiv 0 \pmod 4$, $m \equiv 3 \pmod 4$. The graceful labelings for these two cases are very simple and straightforward when compared to other cases.

3.1 $k \equiv 0 \pmod 4$, $m \equiv 0$ or $3 \pmod 4$

Theorem 9. *The graph $KP(k, m, l)$ can be gracefully labelled when $k \equiv 0 \pmod 4$ and $m \equiv 0$ or $3 \pmod 4$.*

Proof. Case 1: l is even. Start by labeling the cycle C_m with $m \equiv 0$ or $3 \pmod 4$ cycle, using the β-labeling from [19]. This gives us edge lengths of 1 through m. We then add the path of length l onto the highest vertex, m. Label the path: $(m)[m+1](-1)[m+2](m+1)[m+3](-2)\ldots$ Since l is even, the path will end with $\ldots(-\frac{l}{2})[m+l](m+\frac{l}{2})$. Now increase all vertices in our graph $C_m \odot P_{l+1}$ by $\frac{l}{2}$, and the last vertex of the path will be $m+l$. This graph is now gracefully labeled.

Next we need to add the other cycle, C_k, where k must be $0 \pmod 4$. This is because we need a bipartite α-labeling, which is again found in [19]. We will now stretch the graph by increasing all of the large vertices by $m+l$ and leaving the small ones the same. This gives us the edges from $m+l+1$ up to $m+l+k$ and vertices $0 \leq v < \frac{k}{2}$ and $m+l+\frac{k}{2} \leq v \leq m+l+k$.

Last we need to attach C_k to $C_m \odot P_{l+1}$, so we increase all vertices in $C_m \odot P_{l+1}$ by $\frac{k}{2}$ so we still have edge lengths $1, ..., m+l$, but instead have vertices $\frac{k}{2}, \ldots, m+l+\frac{k}{2}$. Since l is even, the last vertex of our path is now $m+l+k$, which is in C_k, so this is where it will be attached to complete the graceful labeling of $KP(k, m, l)$.

Case 2: l is odd. We will start by labeling the cycle C_m with $m \equiv 0$ or $3 \pmod 4$ cycle, using the β-labeling from [19]. This gives us edge lengths of 1 through m. We then add the path of length l onto the highest vertex, m. Label the path: $(m)[m+1](-1)[m+2](m+1)[m+3](-2)\ldots$ If l is odd, then the path will end with $\ldots(m+\frac{l-1}{2})[m+l](-\frac{l+1}{2})$. Now increase all vertices in our graph $C_m \odot P_{l+1}$ by $\frac{l+1}{2}$, and the last vertex of the path will be 0. This graph is gracefully labeled.

Next we need to add the other cycle, C_k, where k must be $0 \pmod 4$. This is because we need a bipartite α-labeling. In this case instead of using the α-labeling in [19], we will need to use a modified α-labeling, in which the small partite has vertices $0, 1, \ldots, \frac{k}{2}$ instead of $0, 1, \ldots, \frac{k}{2}-1$. We will now stretch the graph by increasing all of the large vertices by $m+l$ and leaving the small ones the same. This gives us the edges from $m+l+1$ up to $m+l+k$ and vertices $0 \leq v < \frac{k}{2}$ and $m+l+\frac{k}{2} \leq v \leq m+l+k$. Last we need to attach C_k to $C_m \odot P_{l+1}$, so we increase all vertices in $C_m \odot P_{l+1}$ by $\frac{k}{2}$ so we still have edge lengths $1, ..., m+l$, but instead have vertices $\frac{k}{2}, \ldots, m+l+\frac{k}{2}$. Since l is odd, the last vertex of our path is now k, which is in C_k, so this is where it will be attached to complete the graceful labeling of $KP(k, m, l)$. □

In this proof we used the method of combining a graceful graph with an alpha labeled graph by "stretching" the bipartite alpha graph and then connecting the two previously disjoint graphs at a common vertex so the resulting graph gracefully labeled. The proof uses the method shown in Theorem 4 two times, first to connect the path of length l to the cycle C_m and then to connect to the other cycle C_k. So $G = C_k$, $H = C_m \odot P_{l+1}$. We choose between functions f and g depending on whether the path is odd or even.

3.2 $k \equiv m \equiv 2 \pmod 4$

For kayak paddles with $k \equiv m \equiv 2 \pmod 4$, the method from [2] listed on page 7 in the Useful Results section gives an α-labeling of the graph $C_{4k+2} \cup C_{4m+2}$. There are three separate cases: $2k < m$, $k < m \leq 2k$, and $k = m$.

3.2.1 $2k < m$ or $k < m \leq 2k$

Lemma 10. *If $k \neq m$, then the graph $KP(4k+2, 4m+2, l)$ can be gracefully labeled for $l > 2m$ if l is even and $l \geq 2m+3$ if l is odd.*

Proof. Case 1: $2k < m$.

Subcase 1.1: $l > 2m$ is even. Use the α-labeling of $C_{4k+2} \cup C_{4m+2}$. The largest value of a vertex on $C_{4k+2} \cup C_{4m+2}$ is $4k+4m+4$ which is a part of C_{4m+2}. Now attach a path of length $l-1$ here and label the vertices $-1, 4k+4m+5, -2, 4k+4m+6, \ldots$ Now we have all edge lengths besides $4k+4m+4+l-1$=$4k+4m+3+l$. Since there is nowhere to get this edge, we instead skip a "smaller" edge length d in the path by modifying vertex lengths, which will result in all edge labels after the skipped edge being one larger. We will then have all edge lengths except for one which will be called d. Then we will add the last edge of missing length d to connect back to the vertex labeled $3m+4k+4$, which is the largest vertex in C_{4k+2}, therefore connecting the two cycles. If l is even then the last vertex in the path of length $l-1$ will be $-\frac{l}{2}$. The edge length we want to skip in this path is $d = 3m+4k+4+\frac{l}{2}$. Now skipping d while leaving all other edge lengths the same will cause a shift in some of the vertex labels of the path, so it may force the vertex originally labeled $-\frac{l}{2}$ to be relabeled $-(\frac{l}{2}+1)$, depending on whether d is even or odd. If d is even, then the vertex remains $-\frac{l}{2}$ and we can add edge length d to connect back to the vertex labeled $3m+4k+4$, which is part of C_{4k+2}. If d is odd then the last vertex in the path will become $-(\frac{l}{2}+1)$, and we can add edge length d to connect back to the vertex labeled $3m+4k+3$, which is also part of C_{4k+2}.

Subcase 1.2: l is odd, $l \geq 2m+3$. First add the path of length $l-1$ and label just as above in the even case. The last vertex of the path will be $4k+4m+4+\lfloor\frac{l}{2}\rfloor$. Now we can attach back to the smallest vertex value in C_{4k+2}, which will have the vertex value m, which will make $d = 3m+4k+4+\lfloor\frac{l}{2}\rfloor$. Now when d is skipped in the path if it forces vertex $4k+4m+4+\lfloor\frac{l}{2}\rfloor$ to become $4k+4m+4+\lfloor\frac{l}{2}\rfloor+1$, then since it is one greater adding an edge of length d will bring us to vertex $m+1$ and $m+1$ will always be still be part of C_{4k+2}.

Case 2: $k < m \leq 2k$. We can use the same method as in Case 1, but the bounds will be different because the labeling of the two cycles is different. Here the largest vertex in C_{4k+2} is $4k+4m+4$ and the largest vertex on C_{4m+2} is $2k+4m+4$.

Subcase 2.1: l is even, $l > 4k$. Add the path of length $l-1$ and end at $-\frac{l}{2}$, now d will be $2k+4m+4+\frac{l}{2}$, so if skipping d causes it to increase, we can attach to $2k+4m+3$ which will always still be part of C_{4m+2}. Since $k < m \leq 2k$, $l > 4k$ is sufficient for $l > 2m$.

Subcase 2.2: l is odd, $l \geq 4k+5$. Adding the path of length $l-1$ will again end at a vertex labeled $4k+4m+4+\lfloor\frac{l}{2}\rfloor$, and this time the smallest vertex value in C_{4m+2} is $2k+1$ so the edge length needed is $d = 2k+4m+3+\lfloor\frac{l}{2}\rfloor$ and if skipping d in the path forces the last vertex to become $4k+4m+5+\lfloor\frac{l}{2}\rfloor$, then use d to connect back to $2k+2$, which is part of C_{4m+2}, instead. □

The following is a labeling of the graph $C_{4k+2} \cup C_{4m+2}$ with $k < m \leq 2k$ from [2]. We will refer to this labeling as γ:

First we will describe the construction of the labeling of the vertices of C_{4k+2}. We take an arbitrary α_{2k-m}-labeling of the path P_{4k}; the values of its end vertices will be $2k-m$, m. In this path, we leave the small values $0, 1, \ldots, 2k$ unchanged, and add $4m+4$ to each large value. Finally, we join both end vertices of P_{4k} with the vertex labeled by $3m+2k+4$. This closes C_{4k+2}; its edges have the values $4m+5, 4m+6, \ldots, 4m+4k+4, 4m+4, 2k+2m+4$. In the labeling of C_{4m+2}, the consecutive vertices will be labeled as follows: $2k+1, 2k+4m+4, 2k+2, 2k+4m+3, \ldots, k+m, 3k+3m+5, k+m+2, 3k+3m+4, k+m+3, 3k+3m+3, \ldots, 2k+m+1, 2k+3m+5, 2k+m+2, 2k+3m+3, 2k+m+3, 2k+3m+2, \ldots, 2k+2m+1, 2k+2m+4, 2k+2m+2, 2k+2m+3$. The values of the edges are then $4m+3, 4m+2, \ldots, 2k+2m+5, 2k+2m+3, 2k+2m+2, \ldots, .2m+4, 2m+3, 2m+1, 2m, \ldots, 2, 1, 2m+2$.

Lemma 11. *Let the graph $C_{4k+2} \cup C_{4m+2}$ with $k < m \leq 2k$ have labeling γ. Then there exist two labelings g_1^- and g_2^- such that they have all edge lengths $1, 2, \ldots, 4k+4m+5$ except for $2m+3$ in g_1^- and $4m+5$ in g_2^-.*

Proof. It can be seen that these labelings exist by looking at the construction of the γ labeling. In the g_2^- labeling it is possible to skip the edge label $4m+5$ because it connects the vertices $2k$ and $2k+4m+5$. $2k+4m+5$ is the smallest "large" vertex in a modified α path that forms the cycle, so if we increase all vertices greater or equal than $2k+4m+5$ by one, then all edge lengths greater or equal than $4m+5$ will also be increased by one. So now we have all edge lengths $1, 2, \ldots, 4k+4m+4, 4k+4m+5$ except for the skipped edge $4m+5$.

In g_1^- the edge label $2m+3$ is skipped instead by increasing all vertices greater or equal $2k+3m+5$. This is possible because vertex $2k+3m+5$ is the smallest vertex of this section of C_{4m+2}, which behaves like an α-labeling and successfully stretches all edges greater than or equal to $2m+3$ up to the highest edge in the cycle. In cycle C_{4k+2}, all of the large vertices are greater than $2k+3m+5$, so that the cycle will have all of its large vertices increased by one which will increase all of the edge lengths by one. So now we have modified vertices $2k+3m+6, \ldots, 4k+4m+5$. And the edge lengths in the graph are $1, 2, \ldots, 4k+4m+4, 4k+4m+5$, except for the skipped edge $2m+3$. □

Lemma 12. *$KP(4k+2, 4m+2, l)$ with $k < m \leq 2k$ is graceful for $l \leq 4k$ if l is even and $l \leq 4k+5$ if l is odd.*

Proof. We have three cases: $l = 1$, $l \geq 3$ is odd, and l is even. In all three cases we will use either g_1^- or g_2^- and add the path of length l onto the largest vertex $4k+4m+5$ which is part of C_{4k+2}, and then connect back to the other cycle C_{4m+2} using the skipped edge, which is $2m+3$ in g_1^- and $4m+5$ in g_2^-. When connecting back we can connect to any vertex between $k+m$ and $2k+4m+4$ with two exceptions. The first is $k+m+1$ because it was skipped in the labeling. The second is $2k+3m+4$ because it is part of C_{4k+2}. Also if we are connecting back using $2m+3$ then we will be using g_1^-, so all vertices greater or equal than $2k+3m+5$ are increased by one including some in C_{4m+2}.

Case 1: $l = 1$, then we are connecting directly from the relabeled vertex $4k+4m+5$ back to cycle C_{4m+2}. There are two cases. First if $2k \neq m+3$, we will start with the g_1^- labeling, so all vertices greater or equal than $2k+3m+5$ are increased by one and the vertex we connect back to is $4k+2m+2$. The second case if $2k = m+3$ is when $4k+2m+2$ is equal to $2k+3m+5$ which is a forbidden vertex using g_1^-, so we must use g_2^- and connect back with edge length $4m+5$ which will bring us to vertex $4k$ which belongs to C_{4m+2}.

Case 2: $l \geq 3$ is odd, $l \leq 4k+5$: Start by the labeling g_2^- for the two cycles. First add a path of length $l-1$ onto our relabeled vertex $4m+4k+5$, and vertex labels will be $-1, 4m+4k+6, -2, 4m+4k+7, \ldots, 4m+4k+5+\frac{l-1}{2}$. Now we need to connect from this vertex $4m+4k+\frac{l-1}{2}$ back to the cycle C_{4m+2}, using the skipped vertex $4m+5$. Adding a path of length $4m+5$ from the vertex $4m+4k+5+\frac{l-1}{2}$ brings us to the vertex $4k+\frac{l-1}{2}$. This is always greater than $k+m+1$, so we avoid the skipped vertex. We just need to make sure that $4k+\frac{l-1}{2}$ is less than the vertex $2k+3m+4$, which it is if $l < 4m+9$, and since $k < m$, then this method certainly works for $l \leq 4k+5$, which is all we need since the previous method covers anything greater than that.

Case 3: l is even, $l \leq 4k$: Follow the same method as in Case 2 by adding on the path of length $l-1$ to the highest vertex, $4m+4k+5$, and it will end with a vertex labeled $-\frac{l}{2}$. Now we will try to connect back using the edge length $4m+5$ which will bring us to vertex $4m+5-\frac{l}{2}$. If this happens to be equal to forbidden vertex value $2k+3m+4$ then we can instead use the g_1^- labeling. Then using our skipped edge length $2m+3$, we will connect back to vertex $2m+3-\frac{l}{2}$. Now we also have to make sure that the vertex we connect back to is greater that our other forbidden vertex, $k+m+1$, so if we need it greater or equal than $k+m+2$ then $l < 4m+6$, so it definitely works for all $l \leq 4k$ which is all we need since the previous method covers everything greater than this. So depending on what we used to connect back, we used either g_1^- or g_2^-.

Now once we choose either the g_1^- or g_2^- labeling, we can increase all of the edge lengths in the new graph by the negative of the lowest vertex in the graph, since we went into negative vertex labeled. Call the smallest vertex $-b$. This will bring all vertices up by b and the highest vertex will now be $4m+4k+5+b$, and we will have all edge lengths $1, 2, \ldots, 4m+4k+5+b$ so the graph is gracefully labeled. When we increase these vertices the new labeling will be just called g. In the case where $l = 1$ negative vertices were not used, so $b = 0$. □

The following is a missing case:

Open Case 13. *$KP(4k+2, 4m+2, l)$ is gracefully labeled for $l \leq 2m$ if l is even and $l < 2m+3$ if l is odd and $2k < m$.*

3.2.2 $m = k$

In [13], A. Kotzig proved that the graph $C_{4k+2} \cup C_{4m+2}$ with $m = k$ can be labeled gracefully. The labeling of the vertices in the first cycle if $k > 1$ is $0, 8k+4, 2k+2, 6k+3, 2k+3, 6k+2, 2k+4, \ldots, 5k+5, 3k+1; 7k+3, k, 7k+4, k-1, \ldots, 7k+k+2, 1, 8k+3$. The labeling of the second cycle is $4k+2, 4k+3, 2k, 6k+4, 2k-1, 6k+5, \ldots, 7k+2, k+1, 5k+4, 3k+2, 5k+3, 3k+3, 5k+2, \ldots, 4k+1, 4k+4$.

We will need the following lemma:

Lemma 14. *The graph $C_{4k+2} \cup C_{4k+2}$ has a p-gap graceful labeling for $1 \leq p \leq 2k+1$.*

Proof. We start with the graceful labeling of $C_{4k+2} \cup C_{4m+2}$ with $m = k$, described in [13].
If we increase all vertices greater or equal to $3k+3$ then the edge length $2k+1$ will be skipped.
If we increase all vertices greater or equal to $3k+4$ then the edge length $2k-1$ will be skipped.
...
If we increase all vertices greater or equal to $4k+2$ then the edge length 3 will be skipped.
If we increase all vertices greater or equal to $4k+3$ then the edge length 1 will be skipped.
So we can skip any odd edge between 1 and $2k+1$ by increasing the appropriate vertices. $\square$

Here is the list of labels for the graceful labeling of $C_{4k+2} \cup C_{4k+2}$, where the symbols $<>$ and $\langle\rangle$ indicate vertices of the first or second cycle, respectively.

$< 0, \ldots, k > \langle k+1, \ldots, 2k \rangle < 2k+2, \ldots, 3k+1 > \langle 3k+2, \ldots, 5k+4 \rangle < 5k+5, \ldots, 6k+3 > \langle 6k+4, \ldots, 7k+2 \rangle < 7k+3, \ldots, 8k+4 >$

Lemma 15. *The graph $KP(4k+2, 4k+2, l)$ can be gracefully labeled for $l \geq 2$ and $k \geq 1$ when $l \leq 2k+2$, and $k \geq 3$ when $l \geq 2k+4$.*

Proof. There are three main cases:

Case 1: $2 \leq l \leq 2k$

In this case we must start by using Lemma 14 to skip an edge length, which will depend on which edge we need to connect back to our second cycle. So once the edge is skipped in the gap labeling of $C_{4k+2} \cup C_{4k+2}$, the highest vertex will be labeled $8k+5$, part of the second cycle. Next add the path of length $l-1$ to this vertex and label as follows: $(8k+5)[8k+6](-1)[8k+7](8k+6)[8k+8](-2)\ldots(-\frac{l}{2})$. So now we need to connect back to the first cycle using our skipped edge length p. We want to connect back to a vertex r, $k+1 \leq r \leq 2k$, and $r = -\frac{l}{2} + p$.

Therefore $k+1+\frac{l}{2} \leq p \leq 2k+\frac{l}{2}$. If $l=2$ then $k+2 \leq p \leq 2k+1$ and if $l=2k$ then $2k+1 \leq p \leq 3k$, so we can always use $p=2k+1$ to connect back.

Case 2: $l = 2k+2$

There are three subcases:

Case 2.1: $k=1$, so $l=4$. Stretch large vertices, and label path: (12)[2](10)[3](7)[1](8)[4](4) 1 is in second cycle and 4 is in first cycle.

Case 2.2: $k=2$, so $l=6$. Stretch large vertices, and label path: (21)[5](16)[4](12)[3](15)[2] (13)[1](14)[6](8) 21 is in first cycle and 8 is in second cycle.

Case 2.3: $k \geq 3$

Since the labeling of the two cycles is an alpha labeling, we can stretch all of the larger vertices to skip small edge lengths between 1 and l. So since there are $8k+4$ total vertices, we will increase all of the vertices greater than $4k+2$ by $l=2k+2$. So now we have vertices $0,1,2,\ldots,4k+2,6k+4,\ldots,10k+6$ and edge lengths $2k+3,\ldots,10k+6$. So we still need edge lengths $1,\ldots,2k+2$ and have the vertices $4k+3,...,6k+3$ available. We will start by attaching the first edge of the path onto the vertex $8k+4$ which is in the first cycle. We will label the path as follows: $(8k+4)[2k+1](6k+3)[2k](4k+3)[2k-1](6k+2)[2k-2](4k+4)\ldots(5k+4)[2](5k+2)[1](5k+3)[2k+2](7k+5)$. The last edge will bring us back to our second cycle.

Case 3: $2k+4 \leq l$

There are two subcases, $k=2$ and $k \neq 2$. We solve the latter here. The former remains open and is listed separately.

In the case $k \neq 2$ we will add the path onto the highest vertex of the graph $C_{4k+2} \cup C_{4k+2}$ which is $8k+4$ and is part of the first cycle. Then start with a path labeled as follows: $(8k+4)[8k+5](-1)[8k+6](8k+6)[8k+7](-2)\ldots(-\frac{l}{2})$. Now we will want to stretch and skip an odd edge length that is greater or equal than 8k+5, so we can use it to connect back to the other cycle. We will connect back to the vertex with the label $7k+2$ (or $7k+1$), and after stretching the last vertex in the path will have the label $-\frac{l}{2}-1$, since we are skipping an odd vertex. So the vertex we will want to skip is whichever of $7k+3+\frac{l}{2}$ or $7k+2+\frac{l}{2}$ is odd. And since $l \geq 2k+4$, this vertex will be greater or equal to 8k+5, which is the first vertex we can skip by stretching within the path. Last we increase all vertices by $-\frac{l}{2}$ so our vertices fall in the set $\{0, 1, \ldots, 8k+4+l\}$, and the graph is gracefully labeled.

The following case is missing:

Open Case 16. *The graph $KP(4k+2, 4k+2, l)$ can be gracefully labeled for $k = 2$ when $l \geq 2k+4$ is even.*

Lemma 17. *The graph $KP(4k+2, 4k+2, l)$ can be gracefully labeled for $l \geq 3$ and $k \geq 3$ when l is odd.*

Proof. We have five cases:

Case 1: $1 \leq l \leq 2k-3$

In this case we must start by using Lemma 14 to skip an edge length, which will depend on which edge we need to connect back to our second cycle in the last step. So once the edge is skipped in the gap labeling of $C_{4k+2} \cup C_{4k+2}$, the highest vertex will be labeled $8k+5$, part of the second cycle. Next add the path of length $l-1$ to this vertex and label as follows: $(8k+5)[8k+6](-1)[8k+7](8k+6)[8k+8](-2)\ldots(8k+5+\frac{l-1}{2})$. Now we need to connect back to the first cycle using the skipped edge length p. We want to connect back to a vertex r, $6k+4 \leq r \leq 7k+2$. We can use $p = 2k+1$ to connect back when $1 \leq l \leq 2k-3$. The last vertex in the path of length l will be $8k+5+\frac{l-1}{2}$, so adding the edge $p = 2k+1$ will connect back to $r = 6k+4+\frac{l-1}{2}$. If $l = 1$ we connect straight from vertex $8k+5$ to vertex r using edge $2k+1$.

Case 2: $l = 2k-1$

Start by stretching in the middle, by increasing all vertices greater or equal to $4k+2$ by $l = 2k-1$. So now we have available vertices $4k+2, \ldots, 6k$ and need edges $1, \ldots, 2k-1$. Attach the first edge of length $k+1$ to the vertex $3k+1$, which is part of the first cycle, and continue as follows: (3k+1)[k+1](4k+2)[2k-2](6k)[2k-3](4k+3)…

Now somewhere between edges $4k+2$ and 1, we will get a second edge $k+1$, so we will skip this edge. This is done by increasing all vertices greater or equal to a, where a is the larger of the two vertices that had the edge length of $k+1$ between them. This will cause the end of the attached path of length $l-1$ to be either $5k+1$ or $5k+2$ depending on where the stretch occurred. Now we will add the final edge $2k-1$ to attach back to the second cycle, bringing us to vertex $7k$ or $7k+1$.

Case 3: $l = 2k+1$

Start by stretching all of the large vertices by increasing all the vertices greater or equal to $4k+3$ by $l = 2k+1$. We now have vertices $0, 1, 2, \ldots, 4k+2, 6k+3, \ldots, 10k+5$, and edge lengths $2k+2, \ldots, 10k+5$. We still need edge lengths $1, \ldots, 2k+1$ and have the vertices $4k+2, \ldots, 6k+2$ available. We will start by attaching the first edge of the path onto the vertex $7k+5$ which is in the second cycle. Then label the path as follows: $(7k+5)[2k](5k+5)[2k-1](3k+6)\ldots(4k+4)[2](4k+6)[1](4k+5)[2k+1](2k+4)$. So the last edge will bring us back to the first cycle.

Case 4: $l = 2k+3$

Stretch all the large vertices by increasing all the vertices greater or equal to $4k+3$ by $l = 2k+3$. Now we have vertices $0, 1, 2, \ldots, 4k+2, 6k+5, \ldots, 10k+7$, and edge lengths $2k+4, \ldots, 10k+7$. We still need edge lengths $1, \ldots, 2k+3$ and have the vertices $4k+3, \ldots, 6k+4$ available. We will start by attaching the first edge of the path onto the vertex $6k+6$, which is in the second cycle. Then label the path: $(6k+6)[2k+2](4k+4)[2k+1](6k+5)[2k](4k+5)\ldots(5k+6)[2](5k+4)[1](5k+5)[2k+3](7k+8)$. So the last edge will bring us back to the first cycle.

Case 5: $2k+5 \leq l$

In this case we will add the path onto the highest vertex of the graph $C_{4k+2} \cup C_{4k+2}$ which is $8k+4$ and is part of the first cycle. Then start with a path labeled as follows: $(8k+4)[8k+5](-1)[8k+6](8k+6)[8k+7](-2)\ldots(8k+4+\frac{l-1}{2})$. Now we will want to stretch and skip an odd edge length that is greater or equal than 8k+5, so we can use it to connect back to the other cycle. We will connect back to the vertex with the label $k+1$ (or $k+2$). Since we are skipping an odd vertex, stretching will only affect the negative vertices and not the last vertex. So the vertex we will want to skip is whichever of $7k+3+\frac{l-1}{2}$ or $7k+2+\frac{l-1}{2}$ is odd. And since $l \geq 2k+5$, this vertex will be greater or equal to 8k+5, which is the first vertex we can skip by stretching within the path. Last we increase all vertices by $\frac{l-1}{2}+1$ so our vertices fall in the set $\{0, 1, \ldots, 8k+4+l\}$, and the graph is gracefully labeled. □

Lemma 10, Lemma 12, Lemma 16, and Lemma 17 complete the proof of the following Theorem.

Theorem 18. *The graph $KP(k, m, l)$ can be gracefully labeled for $k \equiv m \equiv 2 \pmod 4$ for $k \geq 3$.*

3.3 $k \equiv 1 \pmod 4$, $m \equiv 3 \pmod 4$

By Theorem 4, if we have two disjoint graphs G and H, where G is gracefully labeled and H is α-labeled, then $G \odot H$ is gracefully labeled, where $G \odot H$ is the graph obtained by using the method in Theorem 4.

Lemma 19. *The C_{4k+3} has a $(2i)$-gap graceful labeling for $k > 0$ and $1 \leq i \leq 2k+1$.*

Proof. For $1 \leq i \leq k$, the vertex labels of the cycle are $\{0, 1, 2, ..., 4k+3, 4k+4\} \backslash \{2k+2-i, 3k+3\}$, the cycle is labeled:

$(0)[4k+4](4k+4)[4k+3](1)[4k+2](4k+3)\ldots$
$\ldots(k)[2k+4](3k+4)[2k+3](k+1)[2k+1](3k+2)\ldots$
$\ldots(2k+1-i)[2i+1](2k+2+i)[2i-1](2k+3-i)\ldots$
$\ldots(2k+4)[3](2k+1)[2](2k+3)[1](2k+2)$

For $k+1 \leq i \leq 2k+1$, the vertex labels of the cycle are $\{0, 1, 2, ..., 4k+3, 4k+4\} \backslash \{3k+2, 2k+2+i\}$ The first part of the cycle is labeled:

$(0)[4k+4](4k+4)[4k+3](1)[4k+2](4k+3)\ldots$
$\ldots(2k+1+i)[2i+1](2k-i)[2i-1](2k-1+i)\ldots$
$\ldots(3k+3)[2k+2](k+1)[2k](3k+1)\ldots$
$\ldots(2k+3)[3](2k)[2](2k+2)[1](2k+1).$

□

Lemma 20. *For $l \geq 1$ and $m \geq 1$ and $s = l + 4m$, $C_{4k+3} \odot P_s$ has a $(2i)$-gap graceful labeling for $1 \leq i \leq 2k + 2m + 1 + \lceil \frac{l}{2} \rceil$.*

Proof. For $1 \leq i \leq k+1$, it follows from the Lemma 19, by adding a path of length l onto the highest vertex labeled $4k+4$.

For $k+2 \leq i \leq 2k+2m+1+\lceil \frac{l}{2} \rceil$, we can break it into cases based on whether l is even or odd: In both cases we will use a modification of the following graceful labeling of C_{4k+3}:

$(0)[4k+3](4k+3)[4k+2](1)\ldots$

$\ldots(3k+3)[2k+2](k+1)[2k](3k+1)\ldots$

$\ldots(2k+3)[3](2k)[2](2k+2)[1](2k+1)$

Case 1: $l = 2r$. Start by labeling C_{4k+3} using the above graceful labeling, except everything is raised by $2m+r+1$:

$(r+2m+1)[4k+3](4k+2m+r+4)[4k+2](r+2m+2)\ldots$

$\ldots(3k+2m+r+4)[2k+2](k+2m+r+2)[2k](3k+2m+r+2)\ldots$

$\ldots(2k+2m+r+4)[3](2k+2m+r+1)[2](2k+2m+4+3)[1](2k+2m+r+2)$

Then we will add our path of length $l+4m = 2r+4m$ onto the highest vertex of the cycle, $4k+2m+r+4$, and we want to skip length $2i$, where $2k+2 \leq i \leq 2k+2m+r+1$. The path will be labeled:

$(4k+2m+r+4)[4k+4](2m+r)[4k+5](2m+r-1)\ldots$

$\ldots(2k+2m+r+3-i)[2i-1](2k+2m+r+2+i)[2i+1](2k+2m+r+1-i)\ldots$

$\ldots(1)[l+4m+4k+2](l+4m+4k+3)[l+4m+4k+3](0)[l+4m+4k+4](l+4m+4k+4)$

So we used all vertices except $2k+2m+r+2-i$.

Case 2: $l = 2r+1$: Start by labeling C_{4k+3} using the graceful labeling above, except everything is raised by $2m+r+2$:

$(r+2m+2)[4k+3](4k+2m+r+5)[4k+2](r+2m+3)\ldots$

$\ldots(3k+2m+r+5)[2k+2](k+2m+r+3)[2k](3k+2m+r+3)\ldots$

$\ldots(2k+2m+r+5)[3](2k+2m+r+2)[2](2k+2m+4+4)[1](2k+2m+r+3)$

Now we are adding on the path of length $l + 4m = 2r + 1 + 4m$ onto the highest vertex of the cycle, $4k + 2m + r + 5$ and we want to skip length $2i$, where $2k + 4 \leq i \leq 2k + 2m + r + 1$, so starting at the highest vertex label the path:

$$(4k + 2m + r + 5)[2k + 4](2m + r + 1)[2k + 5](4k + 2m + r + 5) \ldots$$
$$\ldots (2k + 2m + r - i + 4)[2i - 1](2k + 2m + r + i + 3)[2i + 1](2k + 2m + r + 2 - i) \ldots$$
$$\ldots (1)[4k + 4m + 3 + l](4k + 4m + 4 + l)[4k + 4m + 4 + l](0)$$

So we used all vertices except $2k + 2m + r + 2 - i$. □

As promised here is the proof of Lemma 8 in the Useful Results section. Recall **Lemma 8** was: The cycle C_{4m+1} has a $(2i - 1)$-gap graceful labeling for every $m > 0$ and $1 \leq i \leq m$ or $m + 2 \leq i \leq 2m + 1$.

Proof. For $1 \leq i \leq k$, the vertex labels of the cycle are $\{0, 1, 2, ..., 4k+1, 4k+2\} \setminus \{2k+i, 3k+1\}$. The first part of the cycle is labeled:

$$(0)[4k + 2](4k + 1](1)[4k](4k + 1) \ldots$$
$$\ldots (3k + 3)[2k + 3](k)[2k + 2](3k + 2)[2k + 1](k + 1).$$

For $i = k$, we have a $(2k - 1)$-gap graceful labeling and the second part is labeled:

$$(k + 1)[2k - 2](3k - 1)[2k - 3](k + 2)[2k - 4](3k - 2) \ldots$$
$$\ldots (2k - 2)[4](2k + 2)[3](2k - 1)[2](2k + 1)[1](2k)[2k](0).$$

For $1 < i < k$, we have a $(2k - i)$-gap graceful labeling and the second part is labeled:

$$(k + 1)[2k - 1](3k)[2k - 2](k + 2)[2k - 3](3k - 1) \ldots$$
$$\ldots (2k - i)[2i + 1](2k + i + 1)[2i](2k - i + 1)[2i - 2](2k + i - 1) \ldots$$
$$\ldots (2k - 2)[4](2k + 2)[3](2k - 1)[2](2k + 1)[1](2k)[2k](0).$$

For $i = 1$, we have a 1-gap graceful labeling and the second part is labeled:

$(k+1)[2k-1](3k)[2k-2](k+2)[2k-3](3k-1)\ldots$
$\ldots(2k-2)[5](2k+3)[4](2k-1)[3](2k+2)[2](2k)[2k](0)$.

For $k+1 \leq i \leq 2k+1$, the vertex labels of the cycle are $\{0, 1, 2, \ldots, 4k+1, 4k+2\} \setminus \{2k-i+2, 3k+2\}$. The "lower" part of the cycle is labeled:

$(3k+3)[2k+2](k+1)[2k](3k+1)[2k-1](k+2)[2k-2](3k)\ldots$
$\ldots(2k+3)[3](2k)[2](2k+2)[1](2k+1)[2k+1](0)$

For $i = 2k+1$, the "upper" part of the cycle is labeled:

$(0)[4k+2](4k+2)[4k](2)[4k-1](4k+1)[4k-2](3)[4k-3](4k)\ldots$
$\ldots(k-1)[2k+5](3k+4)[2k+4](k)[2k+3](3k+3)$.

For $k+2 \leq i < 2k+1$, it is labeled:

$(0)[4k+2](4k+2)[4k+1](1)[4k](4k+1)[4k-1](2)[4k-2](4k)\ldots$
$\ldots(2k+i+2)[2i](2k-i+1)[2i-2](2k+i+1)[2i-3](2k-i+3)[2i-4]\ldots$
$\ldots(k-1)[2k+5](3k+4)[2k+4](k)[2k+3](3k+3)$.

For $i = k+1$, it is labeled:

$(0)[4k+2](4k+2)[4k+1](1)[4k](4k+1)[4k-1](2)[4k-2](4k)\ldots$
$\ldots(k-2)[2k+6](3k+4)[2k+5](k-1)[2k+4](3k+3)$. □

Theorem 21. *The graph $KP(4k+3, 4m+1, l)$ can be gracefully labeled for $l \geq 1$.*

Proof. Case 1: $m \leq k$ By Lemma 20, $C_{4k+3} \odot P_s$ has a $(2i)$-gap graceful labeling for $s = l + 4m$. Using this labeling with $i = m$, we have a $(2m)$-gap graceful labeling. We can then attach our one unused edge of length $2m$ from the end of the path P_s back to a vertex on the path which will close the cycle C_{4m+1}. If s is even (meaning l was even), then this last vertex is the highest vertex in the graph labeled $l + 4m + 4k + 4$, and we connect back to the vertex $l + 2m + 4k + 4$. If s is odd, then this last vertex is the lowest vertex, 0, and we connect back to vertex $2m$.

Case 2: $m > k$ By Lemma 8, C_{4m+1} has a $(2i - 1)$-gap graceful labeling for $1 \leq i \leq m$. Let $i = k + 1$, and label C_{4m+1} with a $(2k + 1)$-gap graceful labeling. Now we can add a path of length $l + 4k + 2$ and by Observation 1, since all paths are α-labeled, our graph will still be gap-gracefully labeled, with vertex labels $\{0, 1, \ldots, 4m + 4k + 4 + l\} \backslash \{2k + 1\}$. So now we have all but one edge $2k + 1$ and we can add this onto the last vertex in the path (labeled 0 if l is odd, or labeled $4m + 4k + 4 + l$ if l is even), and connect back to a previous vertex in the path which will close our cycle $4k + 3$. □

Chapter 4

Open Cases

In the case $k \equiv m \equiv 2 \pmod 4$ we are missing two cases:

Conjecture 1. *$KP(4k+2, 4m+2, l)$ with $2k < m$ is graceful when $l \leq 2m$ if l is even and $l \leq 2m+1$ if l is odd.*

Conjecture 2. *$KP(10, 10, l)$ is graceful when $l \geq 12$ is even.*

Open Problem 1. *It remains unknown whether the following kayak paddles are graceful:*

$KP(4k, 4m+1, l)$

$KP(4k, 4m+2, l)$

$KP(4k+1, 4m+1, l)$

$KP(4k+1, 4m+2, l)$

$KP(4k+2, 4m+3, l)$

$KP(4k+3, 4m+3, l)$

This table shows which cases are solved and which cases are open:

Cycles (mod 4)	$m \equiv 0$	$m \equiv 1$	$m \equiv 2$	$m \equiv 3$
$k \equiv 0$	Theorem 9	Open Problem	Open Problem	Theorem 9
$k \equiv 1$	Open Problem	Open Problem	Open Problem	Theorem 21
$k \equiv 2$	Open Problem	Open Problem	Theorem 18/Conjectures 1,2	Open Problem
$k \equiv 3$	Theorem 9	Theorem 21	Open Problem	Open Problem

Table 1

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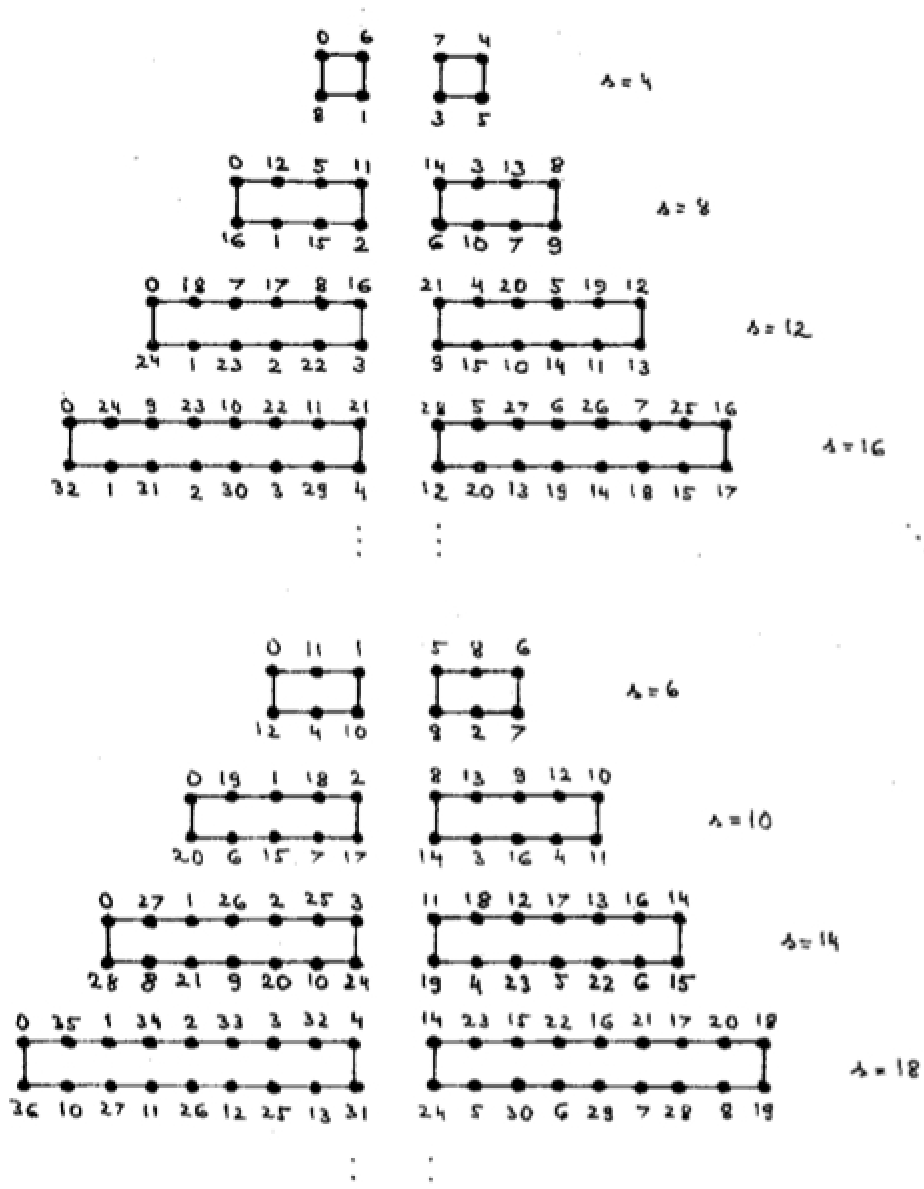
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7 4
3 5
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16 1 15 2
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32 1 31 2 30 3 29 4
28 5 27 6 26 7 25 16
12 20 13 19 14 18 15 17
s = 16
0 11 1
12 4 10
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11 18 12 17 13 16 14
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s = 18

Figure 1

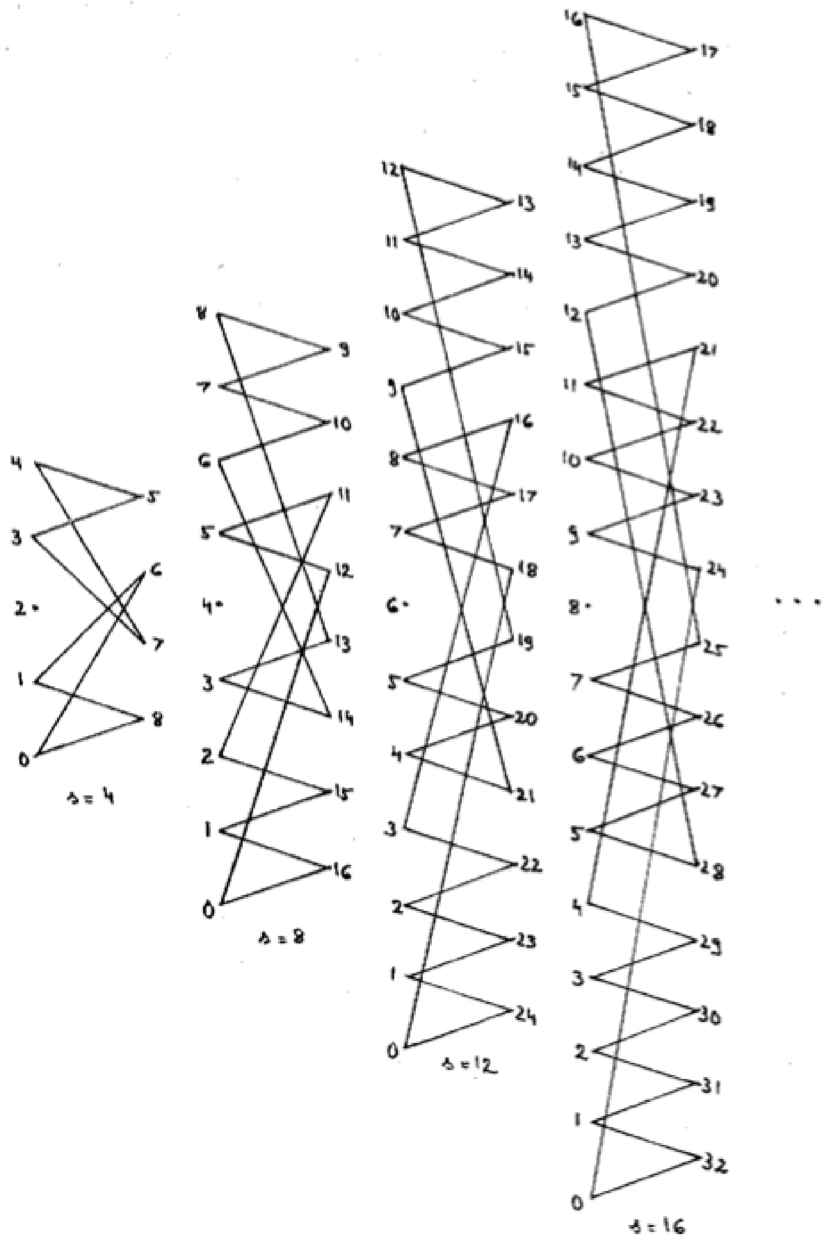

Figure 2

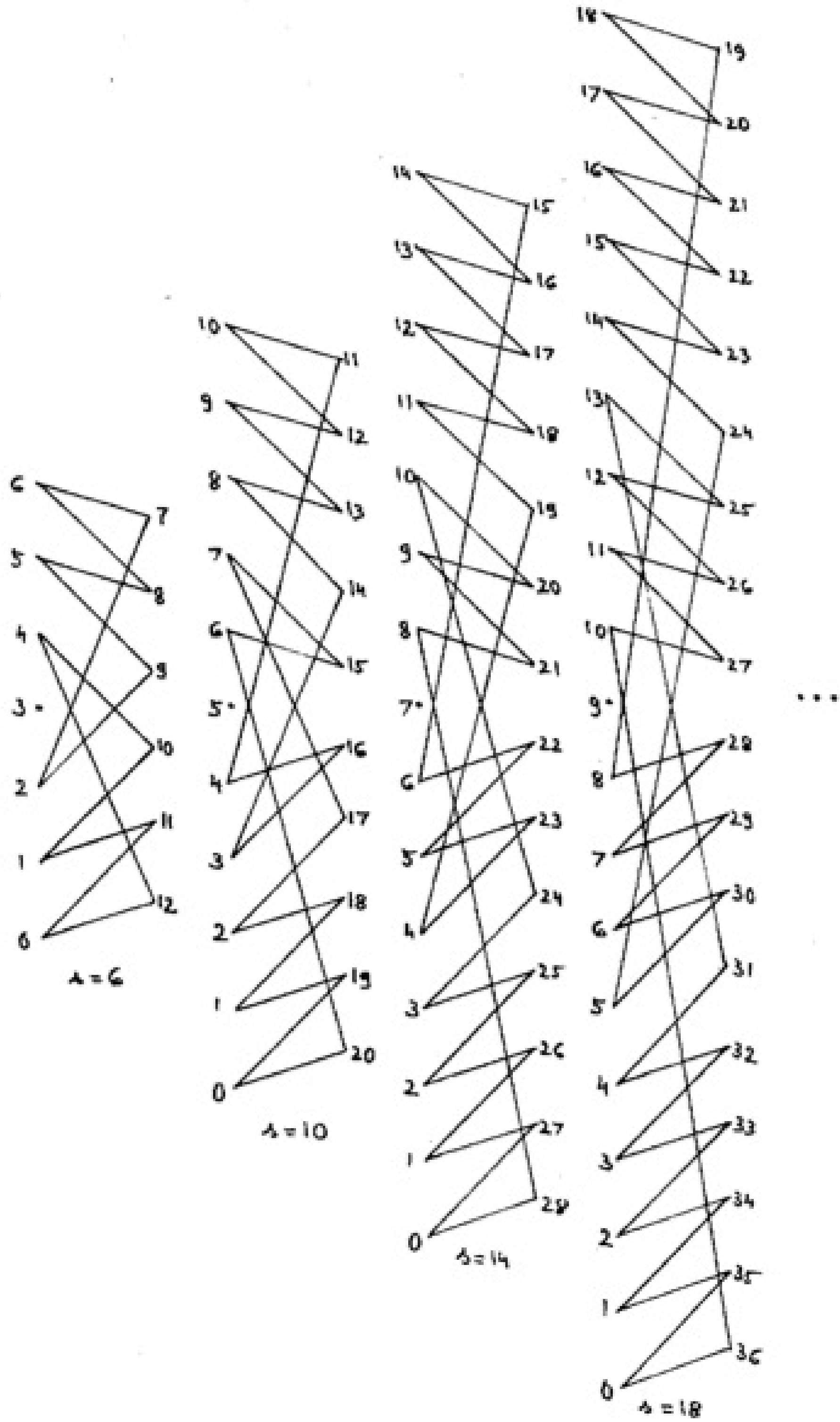

Figure 3