Integral Equation Methods for the Simulation of Viscoelastic Ultrasound Vibro-Acoustography

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Dedication

To my whole family which supported me over the years.
Abstract

This thesis work originated from a collaborative project with J. Greenleaf and M. Fatemi at the Ultrasound Research Laboratory at the Mayo Clinic. The main objective of the project was to develop a full simulation infrastructure for the assessment and design of Ultrasound Vibro-Acoustography (UVA) systems. While significant results had already been obtained for simplified (purely acoustic) models of UVA (where the human tissue is modeled as fluid-like material), there were certainly possibilities for further improvements. In this connection, the work in this thesis focuses on the derivation of more realistic viscoelastic UVA models that extend the acoustic models, and on the development of efficient numerical algorithms to resolve the resulting mathematical problems.

The first part of the thesis reviews the fundamental background of acoustic and viscoelastic wave propagation, followed by a detailed description of UVA models. A variety of numerical schemes for resolving the mathematical models are briefly discussed and their advantages and limitations are reviewed. This review suggests that accelerated integral equation methods might constitute the most suitable approach to the problem at hand.

The second part of the thesis presents the derivation of the relevant boundary integral equations (BIEs) formulation and introduces an efficient numerical scheme with high order accuracy for solving the problem. Numerical experiments are provided to demonstrate the validity of the proposed scheme. The thesis ends with some conclusions and suggestions for future work.
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Chapter 1

Introduction

This thesis is motivated from a collaborative project with scientists at the Ultrasound Research Laboratory at the Mayo clinic. We worked with J. Greenleaf and M. Fatemi who first proposed the idea of ultrasound stimulated vibro-acoustic spectrography. The main objective of the project was to develop a full simulation infrastructure for the assessment and design of Ultrasound Vibro-Acoustography (UVA) systems [26, 27].

It is of practical interest in medical diagnosis that one is able to detect anomalies in soft tissue [58]. For example, early detection of tumors or calcification can have a dramatic effect on treatment and prognosis. Researchers from different fields, from medicine and biomechanics to materials science and applied mathematics, have been working on advancing the capabilities of diagnostic tests for many years. In spite of significant advances in imaging capabilities, early detection of tissue abnormalities remains a challenge.

Perhaps the earliest mode of examination is the classical technique of “palpation” where a medical practitioner literally feels by touch possible changes in tissue stiffness; such changes are then correlated to possible pathologies [1]. Clearly, this method is of limited applicability as it cannot assess anomalies that are deep inside the body or are too small to detect by hand. Moreover palpation does not provide a quantitative assessment, and it also greatly relies on the abilities of the practitioner. These limitations have provided great impetus for the search for alternative, quantitative tools which, while relying on the same principles as the palpation technique, replace the human interpretation with an automated mechanism based on accurate mathematical models.
These mechanisms should allow for the determination of the mechanical properties (e.g., stiffness) of (possibly deep) tissue from remote measurements and, as such, they fall in the category of what has come to be known as “Elastography”. The general technique of elastic imaging seeks to reconstruct the mechanical properties of soft tissue based on measurements of the response to external forces. These forces can be quasi-static or dynamic. For example, static elastography was introduced in [58] (see also [57, 61]) and has been applied to breast and prostate imaging [17, 45]. On the other hand, dynamic elastography [44], using external sources to vibrate soft tissue, can be categorized as magnetic resonance elastography [50, 55, 56, 72] and sonoelastography [30, 46, 48, 49, 59] depending on how the resulting displacement is measured.

A major limitation of these techniques relates to their accuracy as they rely on non-localized external forces. To overcome this, one may utilize the acoustic “radiation force” generated by an ultrasound source to mimic the application of an internal force. The acoustic radiation force is a unidirectional force along the wave propagation path that results from the transfer of the wave momentum to the propagation medium [76]. In [74], Sugimoto proposed a technique for measuring tissue hardness with the use of radiation force produced by a single focused ultrasound beam. This method, however, is limited by the fact that the beam still exerts forces around a large area along its entire path [26].

A significant advance in this direction resulted from the work of Fatemi and Greenleaf, who proposed a new imaging modality based on ultrasound-stimulated low frequency response, also known as Ultrasound Vibro-Acoustography (UVA) [27]. In UVA, a spherical confocal transducer is used to focus two ultrasound beams with slightly different frequencies (typically in the kHz range) at each point of an object. The nonlinear interference in the focal region results in a highly localized dynamic radiation force causing part of the object to oscillate at the difference frequency. The low frequency acoustic field (usually referred to as the “acoustic emission”) is recorded by using a hydrophone, and an image is formed by scanning different focal points; see Figure 2.1. The new proposed UVA model has an advantage over the previous imaging techniques because it utilizes two confocal ultrasound beams resulting localized forces and therefore images of higher resolution can be obtained.

While significant results have been already achieved for the existing UVA [51] (where
the human tissue is modeled as fluid-like material), there are certainly possibilities for further improvements. This thesis is devoted to the search for a suitable viscoelastic UVA model as an extension of the acoustic model, and to the development of efficient algorithms to resolve such problems.

A wide variety of techniques are available for the numerical simulation of viscoelastic wave propagation. Among these, finite difference methods are probably the easiest to implement. For instance, Robertsson et al. [64] proposed a staggered scheme of second-order accuracy in time and fourth-order accuracy in space for 2-D and 3-D applications from shallow marine environments. Numerous extensions of such finite difference methods have been also introduced, such as the rotated staggered grid (RSG) approach [68], and parallel versions have also been implemented [9]. Finite element methods, on the other hand, are arguably the most flexible. Some of the robust methodologies include EVSS [63], space-time/Galerkin Least squares with discontinuous stress [4], DEVSS [32], and hybrid FE/FV [81]. In addition to finite difference/finite element schemes, asymptotic methods based on geometrical optics are also popular for computations of 3-D acoustic/viscoelastic wave fields. Ray theory in viscoelastic wave propagation problems has been intensively studied, mostly in the context of seismic applications [14, 35, 75, 77, 78].

However, each of the numerical schemes above has its own limitations. For instance, finite differences methods are difficult to adapt to complex geometrical arrangements due to their reliance on structured grids, while the versatility of finite elements methods can lead to sub-optimal computation costs. More importantly, perhaps, for the solution of scattering problems, the use of finite differences/elements requires the design and implementation of accurate and efficient “absorbing boundary conditions” to terminate the computational domain. Asymptotic methods, on the other hand, are generally inapplicable in the UVA context as the wavelength of the high-frequency fields is actually comparable to the size of the tissue abnormalities to be detected.

These considerations suggest that a viable alternative may be provided by methods based on the solution of (surface) integral equation formulations. Indeed, in contrast with asymptotic approaches, these formulations provide rigorous solutions to the underlying differential models; and, unlike finite-elements or finite-differences, they do not require the introduction of artificial boundaries, as the very formulation encodes the
correct behavior of outgoing waves. Moreover, surface integral equation methods discretize only the boundary of the inhomogeneities, instead of their volume as required in finite difference/element methods, which can result in a reduction of computational cost; see Section 4.3.

Numerical algorithms based on integral equation methods for viscoelastic stress analysis and wave scattering problems have been intensively researched; see e.g. [3, 16, 47, 60, 70] and references therein. However, these and every other former implementation are not well suited for dynamic, computationally intensive applications such as those related to UVA. For example, the boundary integral equation formulation in [3, 15, 53, 71] was derived specifically to solve quasi-static viscoelastic problems, where only Kelvin’s fundamental solution is required. The formulation for dynamic viscoelasticity can be framed in the frequency/Laplace domain ([2, 20, 28, 40, 60]) or in the time domain ([16, 21, 31, 47, 70]). Formulations in frequency/Laplace domain are generally simpler to implement than those in the time domain, though they require the use of inverse transforms if transient responses are of interest. On the other hand, formulations in the time domain require the knowledge of viscoelastic fundamental solutions which are not completely developed for general viscoelastodynamics [70]. Regardless of the framework, however, every one of the above referenced integral-equation solvers display low-order rates of convergence with the consequent increase in computational cost to attain a desired accuracy. Moreover, none solve the resulting discretized problem in an “accelerated” manner, leaving their cost beyond that of domain-based (finite difference/finite element) methods, in spite of the aforementioned potential advantages of surface formulations. This thesis seeks to remedy this situation with the development of an accelerated, high-order integral-equation viscoelastic solver. The model we shall develop extends that Malcolm et al. [52] to fully viscoelastic materials. The solver, in turn, is inspired by the work of Bruno and Kunyansky in acoustic scattering [11].

The rest of this thesis is organized as follows. In Chapter 2, we first introduced some preliminary physical background on wave propagation and the scattering problem. In addition, a detailed description of the mathematical models of UVA is provided. Chapter 3 then presents a brief survey of numerical algorithms that are applicable in this context. In particular, we derive the boundary integral equations for the UVA models; an efficient high-order method for its resolution is discussed in Chapter 4. Numerical results are
then included in Chapter 5, which exemplify the high-order accuracy of the approach, as well as the possibility for its accelerated evaluation. Finally, our conclusions are summarized in Chapter 6.
Chapter 2

Preliminaries

Before going into the mathematical formulation of the ultrasound vibro-acoustography model, we first present some fundamentals on (acoustic and viscoelastic) wave propagation theory in Section 2.1 and 2.2. The precise mathematical formulation for scattering problems is also included here in Section 2.3 and, finally, the UVA model is described in Section 2.4. More precisely, Section 2.4.1 includes a discussion of an acoustic UVA model, and Section 2.4.2 extends this to a (more realistic) mathematical model that incorporates the viscoelastic properties of the inhomogeneities.

2.1 Acoustic wave propagation

The acoustic wave equation can be derived based on the conservation of mass and conservation of momentum. To begin, we notice that the conservation of mass for a fluid (or gas) can be written as

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2.1.1) \]

where \( \rho \) is the mass density and \( \mathbf{v} \) is the velocity of particles in the medium. The equation for the linear momentum balance, also known as Euler’s equation, is provided by

\[ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \nabla p, \quad (2.1.2) \]

where \( p \) denotes pressure and it relates to the density via the state equation

\[ p = p(\rho), \quad (2.1.3) \]
For small deviations from an equilibrium state \( v_0, \rho_0, p_0 \), a perturbation analysis can be carried out, which readily delivers that the first order correction to the pressure satisfies

\[ \partial_t^2 p - c^2 \Delta p = 0, \quad (2.1.4) \]

where the speed of sound \( c \) is defined by

\[ c^2 = \frac{d p}{d \rho}(\rho_0). \quad (2.1.5) \]

Equation (2.1.4) describes the wave propagation in an acoustic source-free medium.

### 2.2 Viscoelastic wave propagation

In this section, we describe the mathematical equations that govern viscoelastic waves based on a simple Voigt model. To begin, we note that equation (2.1.2) should be replaced by the more general Navier equations

\[ \partial_t v + (v \cdot \nabla)v = \frac{1}{\rho} \nabla \cdot T, \quad (2.2.1) \]

where \( T \) is the Cauchy stress tensor. If we assume the background to be homogeneous, the Voigt stress-strain relation is given by [79]

\[ T = \int_{-\infty}^{t} g_1(t-t') \text{tr} D(t')dt' + 2 \int_{-\infty}^{t} g_2(t-t') D(t')dt', \quad (2.2.2) \]

where \( D \) is the rate of deformation tensor (the symmetric part of the velocity gradient tensor), and \( g_1, g_2 \) are relaxation functions. More precisely,

\[
D = \frac{\nabla v + (\nabla v)^T}{2}, \\
g_1(s) = (K - \frac{2}{3} \mu) H(s) + (\xi - \frac{2}{3} \eta) \delta(s), \\
g_2(s) = \mu H(s) + \eta \delta(s),
\]

where \( K, \mu \) are elastic bulk and shear moduli, \( \xi, \eta \) are viscous bulk and shear moduli, and \( H(s), \delta(s) \) denote the Heaviside step function and the Dirac distribution.

On substituting (2.2.2) into equation (2.2.1) we obtain

\[ \rho (\partial_t v + (v \cdot \nabla)v) = (K + \frac{4}{3} \mu) \nabla \nabla \cdot u + (\xi + \frac{4}{3} \eta) \nabla \nabla \cdot v - \mu \nabla \times \nabla \times u - \eta \nabla \times \nabla \times v, \quad (2.2.3) \]
where \( u \) is the displacement field. The first order approximation then yields

\[
\partial_t^2 u - c_l^2 \nabla \cdot u + c_t^2 \nabla \times \nabla \times u - \sigma_l \nabla \nabla \cdot \partial_t u + \sigma_t \nabla \times \nabla \times \partial_t u = 0, \tag{2.2.4}
\]

where the velocities of compression and shear waves are defined by

\[ c_l = \sqrt{(K + 4\mu)/\rho_0} \]

and \( c_t = \sqrt{\mu/\rho_0} \), and the kinematic compression and shear viscosity are defined by

\[ \sigma_l = (\xi + 4\eta)/\rho_0 \quad \text{and} \quad \sigma_t = \eta/\rho_0. \]

Equation (2.2.4) describes the wave propagation in a viscoelastic source-free medium.

### 2.3 The scattering problem

In a variety of applications, the equations derived in Sections 2.1 and 2.2 conform part of boundary value problems (BVPs) (see e.g. [38, 82]), where they are simply supplemented with boundary conditions on a bounded (interior) domain. In contrast, however, in the context of UVA, the relevant problem is the (exterior) scattering problem. In this connection, and for the sake of a clearer presentation, we shall first review the acoustic scattering problem formulation in the frequency domain, i.e. the scattering problem for the Helmholtz equation.

To begin, we first express the total field \( p \) (defined everywhere in \( \mathbb{R}^3 \)) as the sum of the “incident” wave \( p^{inc} \) and the “scattered” wave \( p^s \)

\[
p = p^{inc} + p^s. \tag{2.3.1}
\]

Notice that the scattered field is generated due to the change in sound speed at (bounded) inhomogeneities generically denoted by a domain \( D \). The incident wave propagates assuming no inhomogeneities and therefore satisfies the Helmholtz equation

\[
\Delta p^{inc} + \kappa^2 p^{inc} = 0, \tag{2.3.2}
\]

where the wave number \( \kappa \) is given by \( \kappa^2 = \omega^2/c^2 \). Depending on the physical characteristics of the scatterer, different boundary conditions can be imposed on the surface of the inhomogeneity \( D \). For example, for a sound-soft obstacle, the total pressure must vanish on the surface, i.e.

\[
p = 0 \text{ on } \partial D, \tag{2.3.3}
\]
which is known as the Dirichlet condition. For a sound-hard obstacle, the normal component of the velocity of the total field must be zero on the boundary, corresponding to the Neumann condition

\[
\frac{\partial p}{\partial n} = 0 \text{ on } \partial D. \tag{2.3.4}
\]

Depending on the context, other boundary conditions may be relevant, e.g. the “impedance” or “Robin” condition

\[
\frac{\partial p}{\partial n} + \mu p = 0 \text{ on } \partial D. \tag{2.3.5}
\]

To ensure the scattered field is outgoing (no energy should be radiated in from infinity), it is necessary to impose a condition at infinity. This condition (which ensures unique solvability) is known as the “Sommerfeld radiation condition” and (in three space dimensions) it takes on the form

\[
\lim_{r \to \infty} r \left( \frac{\partial p^s}{\partial r} - ikp^s \right) = 0, \tag{2.3.6}
\]

where \( r = |\mathbf{x}| \) and the limit is assumed to hold uniformly in all directions.

For the viscoelastic scattering problem, the scattered field \( p^s \) admits the Helmholtz decomposition,

\[
p^s = -\nabla \omega + \nabla \times \zeta, \tag{2.3.7}
\]

where the scalar potential \( \omega \) satisfies the Sommerfeld radiation condition while the vector field \( \zeta \) satisfies the Silver-M"uller radiation condition

\[
\lim_{r \to \infty} r (\nabla \times \zeta \times \mathbf{x} - ik\zeta) = 0. \tag{2.3.8}
\]

\section{2.4 Ultrasound Vibro-Acoustography}

As mentioned in the Introduction, ultrasound vibro-acoustography is a medical imaging technique that relies on the low frequency acoustic emission due to the nonlinear interaction of high frequency fields. A diagram of the experimental setup is shown in Fig. 2.1.

The model we present can be divided into three distinct stages, namely: 1) high frequency propagation of the two ultrasound beams; 2) interaction of the two beams in 1) around the focal point; and 3) low frequency propagation resulting from the acoustic emission in 2); see Fig. 2.2 for the illustration of these three stages.
Figure 2.1: Schematic of experimental setup of UVA model. The radiation force produced by the confocal transducer causes the object to vibrate. The resulting acoustic emission is then recorded by a hydrophone. The final image is formed by scanning different points inside the object [26].

Figure 2.2: Illustration of the three stages of UVA model. 1) high frequency propagation of the two ultrasound beams; 2) interaction of the two beams in 1) around the focal point; and 3) low frequency propagation resulting from the acoustic emission in 2).
Depending on the different characteristics of the media, UVA can be modeled as an acoustic wave propagation problem (described in Section 2.4.1) or as a viscoelastic one (described in Section 2.4.2).

### 2.4.1 Acoustic UVA model

We first derive the mathematical model for a UVA system in the acoustic case. In this scenario, the two wave fields are propagating through a fluid-like background medium and therefore, according to Section 2.1, they satisfy the scalar wave equation

\[ \partial_t^2 \phi_1 - c^2 \Delta \phi_1 = 0, \quad (2.4.1) \]

where \( \phi_1 \) is the velocity potential (\( \mathbf{v} = -\nabla \phi_1 \), and the subscript 1 means first order effects). The wave speed \( c \) is given by

\[ c^2 = \frac{dp}{d\rho}(\rho_0) = \frac{p_0 \gamma}{\rho_0} \]

where \( \gamma \) is the specific heat ratio in the polytropic state equation

\[ \frac{p}{p_0} = \left( \frac{\rho}{\rho_0} \right)^\gamma. \quad (2.4.2) \]

To derive the equation for the second order velocity potential \( \phi_2 \), we start by writing the quantities to second order as

\[
\begin{align*}
\mathbf{v} &= \epsilon \mathbf{v}_1 + \epsilon^2 \mathbf{v}_2 + O(\epsilon^3), \\
\phi &= \epsilon \phi_1 + \epsilon^2 \phi_2 + O(\epsilon^3), \\
\rho &= \rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + O(\epsilon^3), \\
p &= p_0 + \epsilon p_1 + \epsilon^2 p_2 + O(\epsilon^3).
\end{align*}
\]

(2.4.3) (2.4.4) (2.4.5) (2.4.6)

Then, we expand equation (2.4.2) around \( \rho_0 \) to obtain

\[
\begin{align*}
p &= p_0 + \frac{p_0 \gamma}{\rho_0}(\rho - \rho_0) + \frac{p_0 \gamma}{\rho_0} \cdot \frac{\gamma - 1}{2 \rho_0} (\rho - \rho_0)^2 + O(\epsilon^3) \\
&= p_0 + \epsilon^2 (\epsilon \rho_1 + \epsilon^2 \rho_2) + c^2 \frac{\gamma - 1}{2 \rho_0} (\epsilon \rho_1 + \epsilon^2 \rho_2)^2 + O(\epsilon^3) \\
&= p_0 + \epsilon (c^2 \rho_1) + \epsilon^2 (c^2 \rho_2 + \frac{c^2 \gamma - 1}{2 \rho_0} \rho_1^2) + O(\epsilon^3).
\end{align*}
\]
From this, it follows that

\[ p_1 = c^2 \rho_1, \]  
\[ p_2 = c^2 \rho_2 + c^2 \gamma \rho_1^2, \]

and \( \rho_1 \) is then obtained from (2.4.7) and (2.1.2)

\[ \rho_1 = p_1 \frac{c^2}{c^2} = \rho_0 \frac{\partial_t \phi_1}{c^2}. \]

Next we note that the second order terms in equations (2.1.2) and (2.1.1) satisfy

\[ \rho_0 \partial_t v_2 + \rho_1 \partial_t v_1 + \rho_0 (v_1 \cdot \nabla) v_1 = -\nabla p_2, \]

and

\[ \partial_t \rho_2 + \rho_0 \nabla \cdot v_2 + \nabla \rho_1 \cdot v_1 + \rho_1 \nabla \cdot v_1 = 0. \]

Finally, taking the time derivative of \( \nabla p_2 \) and the spatial gradient of \( \partial_t \rho_2 \) and using (2.4.1)–(2.4.8), we obtain the equation for the second order potential

\[ \partial_t^2 \phi_2 - c^2 \Delta \phi_2 = \partial_t \left( |\nabla \phi_1|^2 + \frac{\gamma - 1}{2c^2} |\partial_t \phi_1|^2 \right). \]

Further details of this derivation can be found in [36].

In the case of the UVA experiment, we have

\[ \phi_1 = \phi_1^1(x) e^{-i\omega_1 t} + \phi_1^2(x) e^{-i\omega_2 t}, \]

where the superscripts denote the fields at two slightly different frequencies \( \omega_1 \) and \( \omega_2 \). Therefore, waves with a time dependence (oscillating at the difference frequency), \( e^{-i(\omega_1 - \omega_2) t} \) and \( e^{-i(\omega_2 - \omega_1) t} \) will occur in the quadratic terms on the right hand side of equation (2.4.10). The radiation force, which can be derived from the right hand side of equation (2.4.10), causes part of the tissue to vibrate at this difference frequency \( \Delta \omega := \omega_1 - \omega_2 \). The low frequency acoustic response is then recorded by a hydrophone.

Equations (2.4.1) and (2.4.10) describe the full mathematical model of UVA in the acoustic case.

Simulations of the acoustic model as described above have been carried out in [51, 52]. For instance, in Fig. 2.3 we display the results obtained therein for the interaction
Figure 2.3: The top two figures show the real parts of the two high frequency fields (at 3MHz and 3.05MHz); The bottom figure shows the sum of these two fields.

of the high-frequency fields propagated to the focal point; Fig. 2.4, on the other hand, shows an image of the right hand side of equation (2.4.10); finally, Fig. 2.5 describes a comparison between real laboratory data and the corresponding simulated image for a specific (two-dimensional) experiment on an aluminium cylinder of circular cross-section (see [51, 52] for further details).

2.4.2 Viscoelastic UVA model

In the above discussion, only compressional waves are considered. However, shear waves should also be taken into account due to the intrinsic viscoelastic characteristics of soft tissue [51]. In fact, while the bulk moduli are typically several orders of magnitude larger than shear moduli, the variation of the latter can be several orders of magnitude more pronounced than that of the former. For this reason, the use of shear moduli to characterize tissue has great potential in medical applications [69]. In what follows, we
Figure 2.4: The imaginary part of the right hand side of equation (2.4.10) computed for the same model as the fields shown in Fig. 2.3.

generalize the acoustic UVA model to the linear viscoelastic case.

A variety of models have been proposed to capture the viscoelastic properties of soft tissue. Among the simplest we encounter the Maxwell model which relies on the analogy of a spring and a dashpot in series, and the Voigt model based on a parallel arrangement of these elements (see Fig. 2.6 for an illustration). More complex linear models can be constructed by the combination of the Maxwell model and the Voigt model. For example, Buchthal [12] proposed a model using a combination of an infinite number of Voigt and Maxwell elements (“continuous relaxation spectrum model”). A richer class of models can be obtained by allowing for nonlinear behavior. For example, Fung [29] proposed a quasi-linear viscoelastic theory of soft tissue, which would be applicable to large deformations.

In what follows, we specialize the discussion to a mathematical representation of UVA experiments that allow for viscoelastic behavior (as is discussed in Section 2.2) based on a simple Voigt model. More general models can be easily adopted without much difficulty.
Figure 2.5: A cylindrical aluminium rod long in the y-direction was submerged in water and an image was made of this rod using a transducer with an aperture of $\pi/6$. Top left: simulated image. Top right: experimental image. Bottom: a comparison of a slice of the image for fixed $z = 0$.

Figure 2.6: Left: Maxwell model (a spring and a dashpot in series). Right: Voigt Model (a spring and a dashpot in parallel).
With the similar treatment as in the acoustic UVA model, the high frequency displacement wave fields generated by the transducer obey the vector wave equation (cf. equation (2.4.1))

\[
\partial_t^2 \mathbf{u}_1 - c_l^2 \nabla \cdot \mathbf{u}_1 + c_l^2 \nabla \times \nabla \times \mathbf{u}_1 - \sigma_l \nabla \nabla \cdot \partial_t \mathbf{u}_1 + \sigma_t \nabla \times \nabla \times \partial_t \mathbf{u}_1 = 0.
\tag{2.4.11}
\]

To get the equation for the second order correction, we first expand the left hand side of the equation (2.2.3) as

\[
\left( \rho_0 + \rho_1 \epsilon + \rho_2 \epsilon^2 \right) \left( \partial_t \mathbf{v}_1 \epsilon + \partial_t \mathbf{v}_2 \epsilon^2 + (\mathbf{v}_1 \epsilon \cdot \nabla) \mathbf{v}_1 \epsilon \right).
\]

Moving the term \( \rho_0 (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1 \epsilon^2 + \rho_1 \partial_t \mathbf{v}_1 \epsilon^2 \) to the right hand side and equating the second order terms yields (cf. equation (2.4.10))

\[
\partial_t^2 \mathbf{u}_2 - c_l^2 \nabla \cdot \mathbf{u}_2 + c_l^2 \nabla \times \nabla \times \mathbf{u}_2 - \sigma_l \nabla \nabla \cdot \partial_t \mathbf{u}_2 + \sigma_t \nabla \times \nabla \times \partial_t \mathbf{u}_2 = -\mathbf{v}_1 \cdot \nabla \mathbf{v}_1 - \frac{\rho_1}{\rho_0} \partial_t^2 \mathbf{u}_1.
\tag{2.4.12}
\]

Again here, in the case of UVA we have

\[
\mathbf{u}_1 = \mathbf{u}_1^1(\mathbf{x}) e^{-i \omega_1 t} + \mathbf{u}_1^2(\mathbf{x}) e^{-i \omega_2 t},
\]

from which the radiation force and resulting emission at the difference frequency \( \Delta \omega \) can be derived. Equations (2.4.11) and (2.4.12) describe the full mathematical model of UVA in the viscoelastic case.

The next two chapters relate to the design and implementation of numerical methods for simulations based on the model described above. In the final chapter (Chapter 5), in turn, we shall exhibit numerical results garnered from our formulation of choice based on a potential theoretic formulation. As we shall show, the high-order, unaccelerated version of the resulting codes provides an effective tool for the simulation of the low-frequency emission. The acceleration strategy that we develop in Chapter 4, on the other hand, should also allow for simulations of the (more computationally intensive) high-frequency propagation.
Chapter 3

Numerical methods

Wave propagation/scattering problems are pervasive in a large spectrum of engineering and industrial applications. For instance, applications are found in fields as varied as ultrasound imaging, sonar, geophysical prospection, to name but a few. The technological relevance of these applications has provided significant impetus to the design of advanced numerical simulators for problems of wave propagation in the last few decades. These include most notably finite differences, finite elements, methods based on asymptotic approximations, and integral equation methods. In the following sections, we briefly review the advantages and limitations of each of these schemes. In particular, we justify our choice of integral equation methods for the UVA model described in Section 2.4.2.

3.1 Finite difference methods

There are a number of finite difference time domain (FDTD) schemes available in literature (e.g. [7, 43, 54, 64, 80]) for the simulation of viscoelastic wave propagation. The most popular approach is the centered difference scheme on a staggered grid [80, 43], wherein the time derivatives are approximated by using the leapfrog method, thus providing a second order accurate solution to the problem; see Fig. 3.1. Extensions of these finite difference methods, such as the “rotated staggered grid” (RSG) approach [68] have also been introduced and parallel versions have been implemented [9].

While finite difference schemes are probably the easiest to implement and lead to
Figure 3.1: Spatial grid cells in the conventional and staggered grid. All displacement vector components $U,V$ and $W$ are located at each grid position in the conventional grid. Displacement components $U,V$ and $W$ are located at different grid positions as well as stress-tensor components $T^{xx}, T^{yy}, T^{zz}, T^{xy}, T^{yz},$ and $T^{zx}$ are in the staggered grid [54].

sparse matrix equations, these characteristics arise largely due to their reliance on structured grids which, in turn, makes them difficult to adapt to complex geometrical arrangements. Moreover, finite difference schemes suffer from significant numerical dispersion and dissipation errors [5]. More importantly, the use of finite difference methods requires the design and implementation of accurate and efficient “absorbing boundary conditions (ABC)” to truncate the computational domain. In spite of significant advances in the development of such conditions [19, 22, 24, 25, 41, 67], they still remain a main source of errors in finite difference simulations.
3.2 Finite element methods

Finite element methods (FEM) seek to find an approximate solution (in a finite element space) to the variational formulation of the relevant partial differential equations. To solve the problem

\[ Lu = f, \]  

(3.2.1)

an equivalent weak formulation is constructed

\[ B(u, v) = \langle f, v \rangle, \forall v \in V, \]  

(3.2.2)

where \( v \) is the testing function. Next, the infinite dimensional problem is approximated by a finite dimensional version

\[ B(u_h, v_h) = \langle f, v_h \rangle, \forall v_h \in V_h. \]  

(3.2.3)

After choosing a suitable basis for \( V_h \) (e.g. piecewise linear polynomials), the solution can be written as a linear combination of the basis functions and the associated coefficients can be obtained by solving the resulting (structured) linear system.

Finite element methods have been widely used for the numerical simulation of wave propagation in viscoelastic media (see e.g. [6, 34, 37, 39] and the references therein). While finite element methods are arguably among the most flexible, this versatility can lead to sub-optimal computational cost (e.g. in surface scattering simulations). In addition, FEM share some of the challenges associated with their finite difference counterparts, e.g. the necessity for absorbing boundary conditions and numerical dispersion and dissipation errors, although the latter can be largely alleviated by resorting to higher order schemes.

3.3 Asymptotic methods

In addition to finite difference and finite element methods, asymptotic methods based on “ray theory” constitute popular approaches for computations of 3D acoustic or viscoelastic wave fields. These methods rely on approximations that are valid in specific regimes where the underlying equations can be simplified leading to more tractable models.
Asymptotic methods for viscoelastic wave propagation problems have been intensively studied, particularly in the context of seismic applications. For instance, complex space-time ray tracing algorithms for dispersive and attenuating media were discussed in [18, 75, 42, 35]. On the other hand, Vavryčuk [77, 78] proposed the real ray tracing theory that is applicable to smoothly inhomogeneous viscoelastic media.

However, these methods are only asymptotically valid as the frequency becomes infinite. By this very nature, they are particularly effective at high frequencies, compared with full wave methodologies, but they can incur in significant errors a low to moderate frequencies and, in particular, in the “resonance domain” wherein the size of the wavelength is comparable to that of the geometric features of scattering objects. As a result, asymptotic methods are inapplicable in UVA simulations as the wavelength of the high-frequency fields is similar to the size of the tissue abnormalities to be detected.

3.4 Integral equations method

The limitations of the numerical methods mentioned above suggest that a flexible alternative may be provided by methods based on the solution of (surface) integral equation formulations. Indeed, in contrast with asymptotic approaches, these formulations provide rigorous solutions to the underlying differential models; and, unlike finite-elements or finite-differences, they do not require the introduction of artificial boundaries, as the very formulation encodes the correct behavior of outgoing waves. Moreover, the surface integral equation methods discretize only the boundary of the problem, instead of its volume as required in finite difference/element methods, which can result in a reduction of computational cost.

Numerical algorithms based on integral equation methods for viscoelastic stress analysis and wave scattering problems have been intensively researched (see [3, 16, 47, 60, 70] and references therein). However, none of these schemes incorporate suitable speed-up strategies that will enable realistic simulations, such as high-order accuracy and appropriate acceleration techniques. In an effort to remedy this situation, in what follows we present a new numerical procedure based on the solution of the (frequency-domain) integral-equation formulation of viscoelastic scattering that displays high order convergence and accelerated evaluations of interaction between degrees of freedom.
3.4.1 Integral equations formulation

In the UVA model, one (or more) scatterers or obstacles are embedded in a homogeneous background medium. The material parameters in both the scatterer and the medium are (different) constants. The derivation of the integral equations is based on the application of Green’s theorem in the interior (denoted $D$) and the exterior (denoted $D^C$) of the scatterer(s) $D$. In the discussions below, we assume the wave fields to be time harmonic with time dependence $e^{-i\omega t}$. Therefore, we could rewrite equation (2.2.4) in the frequency domain as

$$\mathcal{L}(u) = \omega^2 u + c_t^2 \nabla(\nabla \cdot u) - c_t^2 \nabla \times (\nabla \times u) = 0,$$

where

$$c_t^2 = c_t^2 - i\omega \sigma, \quad c_t^2 = c_t^2 - i\omega \sigma.$$ 

In order to get the boundary integral equations, we need Green function for the displacement and the traction. We first define the displacement Green function $G^u(y, x)$ by

$$\mathcal{L}G^u(y, x) = -\delta(y - x)I,$$

where $x, y \in \mathbb{R}^3$ and $\delta(\cdot)$ is the 3-dimensional Dirac function. Letting

$$\lambda^* = \lambda - i\omega \lambda_v \quad \text{and} \quad \mu^* = \mu - i\omega \eta,$$

where $\lambda_v = \xi - \frac{2}{3} \eta$ and $\xi, \eta$ are the bulk and shear viscosity, on the other hand, the traction Green function $G^t(y, x)$ can be obtained by applying the operator $t$, defined by

$$t(u) = [\lambda^*(\nabla \cdot u)I + \mu^*(\nabla u + \nabla u^T)] \cdot n,$$

to the displacement Green’s function $G^u(y, x)$ and the normal $n$ is evaluated at point $y$, i.e. $G^t(y, x) = t^y(G^u(y, x))$. Notice that the traction Green functions are different when the operator $t$ is applied to the interior and exterior displacement Green function as these regions are characterized by different material parameters.

To begin, then, we recall that scattered field $u^s$ and incident field $u^{inc}$ satisfy the
following equations

\[ \mathcal{L}u_i^s = 0, \quad x \in D, \]
\[ \mathcal{L}u_i^{inc} = -f, \quad x \in D, \]
\[ \mathcal{L}u_e^s = 0, \quad x \in D^c, \]
\[ \mathcal{L}u_e^{inc} = -f, \quad x \in D^c, \]

where the subscripts \( i \) and \( e \) correspond to the interior and exterior fields. Moreover, the scattered fields should also satisfy the radiation conditions.

From Green’s representation theorem, we have the following equations

\[
\begin{align*}
  u_i^s(x) &= \int_{\partial D} G_i^u(y, x) t_i^s(y) - G_i^f(y, x) u_i^s(y) d\sigma_y, \quad x \in D, \\
-u_i^s(x) &= \int_{\partial D} G_i^u(y, x) t_i^s(y) - G_i^f(y, x) u_i^s(y) d\sigma_y, \quad x \in D^c, \tag{3.4.3} \\
0 &= \int_{\partial D} G_i^{inc}(y, x) t_i^{inc}(y) - G_i^f(y, x) u_i^{inc}(y) d\sigma_y, \quad x \in D, \\
0 &= \int_{\partial D} G_e^{inc}(y, x) t_e^{inc}(y) - G_e^f(y, x) u_e^{inc}(y) d\sigma_y, \quad x \in D^c,
\end{align*}
\]

where we denote the traction vector \( t(y) = t(u(y)) \) and subscripts and superscripts have the same meaning as before. Upon using the jump conditions for the single- and double-layer potentials

\[
\lim_{x \to \partial D} \int_{\partial D} G_i^u(y, x) t(u)(y) d\sigma_y = \int_{\partial D} G_i^u(y, x) t(u)(y) d\sigma_y,
\]
\[
\lim_{x \to \partial D} \int_{\partial D} G_i^f(y, x) u_i(y) d\sigma_y = \int_{\partial D} G_i^f(y, x) u_i(y) d\sigma_y - \frac{1}{2} u_i(x),
\]
\[
\lim_{x \to \partial D} \int_{\partial D} G_e^f(y, x) u_e(y) d\sigma_y = \int_{\partial D} G_e^f(y, x) u_e(y) d\sigma_y + \frac{1}{2} u_e(x),
\]

we obtain

\[
\begin{align*}
  \frac{1}{2} u_i^s(x) &= \int_{\partial D} G_i^u(y, x) t_i^s(y) - G_i^f(y, x) u_i^s(y) d\sigma_y, \quad x \in \partial D, \\
-\frac{1}{2} u_i^s(x) &= \int_{\partial D} G_i^u(y, x) t_i^s(y) - G_i^f(y, x) u_i^s(y) d\sigma_y, \quad x \in \partial D, \\
-\frac{1}{2} u_i^{inc}(x) &= \int_{\partial D} G_i^{inc}(y, x) t_i^{inc}(y) - G_i^f(y, x) u_i^{inc}(y) d\sigma_y, \quad x \in \partial D, \\
\frac{1}{2} u_i^{inc}(x) &= \int_{\partial D} G_i^{inc}(y, x) t_i^{inc}(y) - G_i^f(y, x) u_i^{inc}(y) d\sigma_y, \quad x \in \partial D.
\end{align*}
\]
Summing all of the above equations and applying boundary conditions

\[ u_i^s(x) + u_i^{inc}(x) = u_e^s(x) + u_e^{inc}(x), \]
\[ t_i^s(x) + t_i^{inc}(x) = t_e^s(x) + t_e^{inc}(x), \]
we get the boundary integral equations for displacement and traction

\[ u_e(x) = u_i^{inc}(x) + \int_{\partial D} (G_i^u - G_e^u) t_e(y) - (G_i^t - G_e^t) u_e(y) d\sigma_y, \tag{3.4.4} \]
\[ t_e(x) = t_i^{inc}(x) + \int_{\partial D} (t_i^u G_i^u - t_i^e G_e^u) t_e(y) - (t_i^u G_i^t - t_i^e G_e^t) u_e(y) d\sigma_y, \tag{3.4.5} \]
where

\[ u_e(x) = u_i^s(x) + u_i^{inc}(x), \]
\[ t_e(x) = t_i^s(x) + t_i^{inc}(x), \]
\[ u_i^{inc}(x) = u_i^{inc}(x) + u_i^{inc}(x), \]
\[ t_i^{inc}(x) = t_i^{inc}(x) + t_i^{inc}(x). \]

Thus, defining the kernels

\[ U = G_i^u - G_e^u, \quad W = t_i^u G_i^u - t_i^e G_e^u, \]
\[ T = G_i^t - G_e^t, \quad V = t_i^u G_i^t - t_i^e G_e^t, \]
we can rewrite equations (3.4.4) and (3.4.5) as

\[ u_e(x) = u_i^{inc}(x) + \int_{\partial D} G(y, x) t_e(y) - T(y, x) u_e(y) d\sigma_y, \tag{3.4.6} \]
\[ t_e(x) = t_i^{inc}(x) + \int_{\partial D} W(y, x) t_e(y) - V(y, x) u_e(y) d\sigma_y. \tag{3.4.7} \]

It can be shown that the kernel \( U \) is weakly-singular of \( O(r^{-1}) \) (\( r = |y - x| \)), \( T, W \) are strongly singular of \( O(r^{-2}) \), and \( V \) is hypersingular of \( O(r^{-3}) \). So the singular integral in (3.4.6) exists in the Cauchy principal value sense, while the singular integral in (3.4.7) exists in the Hadamard finite part sense. In Appendix A, we give the explicit formulas for these kernels.

The coupled system of surface integral equations (3.4.6) and (3.4.7) provides a complete mathematical formulation of the low-frequency wave propagation of the viscoelastic UVA model. Indeed, the low-frequency field at any other points can be evaluated
through the use of the representation formulas in (3.4.3) above. In the next chapter, we introduce an efficient high-order numerical scheme to resolve these surface integral equations.
Chapter 4

An efficient high-order integral equation solver

While, as we mentioned, integral equation formulations offer significant advantages from a numerical perspective, they also give rise to substantial challenges. These arise from two main difficulties, namely: 1) the accurate evaluation of the singular integrals that result from the use of the Green function; and 2) the efficient computation of the (nonsingular) interaction between degrees of freedom on $\partial D$. A variety of techniques have been proposed to deal with each of these impediments. In connection with 1), for instance, ideas of singularity extraction [62] and local corrections [13] have been advanced; in relation with 2), on the other hand, fast multipole methods (FMM) [23, 65, 73] and adaptive integral methods (AIM) [8] provide accelerated techniques to evaluate the full matrix-vector products that arise from discretization of integral equations such as those above. While these techniques provide significant gains, they also present certain limitations, such as constraints in the attainable convergence orders, and/or instabilities in certain regimes (e.g. the “subwavelength” breakdown problem in FMM).

A significant advance in this direction was initiated with the work in [11] (see also [10] and the references therein), where a new high-order integration scheme was proposed and combined with an acceleration method that improves upon the standard AIM technique in the context of sound-soft acoustic scattering experiments (or electromagnetics in the case of [10]). In this approach, high-order accuracy is achieved through a complete
analytical resolution of kernel singularities and acceleration is attained through the use of equivalent sources (much as in AIM) though these are suitably restricted to provide further gains in computational complexity.

In this chapter, we provide the details of an extension of this approach that is applicable to penetrable and viscoelastic inhomogeneities. Additional difficulties arise in this context in both the resolution of the singularities, which can be “strongly” and even “hyper-” singular, and in the selection of the nature of the “equivalent sources”. In the case of the former, as we have stated, the accurate treatment of singularities demands their interpretation in the sense of Hadamard [33], and the incorporation of subtle integration rules to retain high-order accuracy. The details are presented in section 4.2. The appropriate implementation of the acceleration scheme, on the other hand, can be attained by appealing to the representation of displacements and tractions in terms of suitable potentials. The details are included in Section 4.3.

4.1 The numerical scheme

The scheme is based on the iterative solution of a discretized version of equations (3.4.6) and (3.4.7). As such, it suffices to describe a (high-order, accelerated) integration procedure that can efficiently provide the values of the right-hand sides for any given surface displacements and tractions. To this end, we begin with a covering of the boundary \( \partial D \) by a number \( K \) of overlapping patches \( P^k \) and choose a partition of unity \( w^k \) subordinated to the covering of the boundary. Now, we can evaluate the singular integral with kernel \( K \)

\[
I(x) = \int_{\partial D} K(x, y) \phi(y) dy
\]
through the use of parameterizations (local to each patch) and a splitting of the form

\[ I(x) = \sum_{k=1}^{K} \int_{x \in P_k} \int_0^1 \int_0^1 K(x, y(u^k, v^k)) \phi(y(u^k, v^k)) w^k(y(u^k, v^k)) J(u^k, v^k) du^k dv^k \]  

\[ (4.1.1) \]

\[ = \sum_{k=1}^{K} \int_{y \in D \cap N_x} \int_0^1 \int_0^1 K(x, y(u^k, v^k)) \phi(y(u^k, v^k)) w^k(y(u^k, v^k)) J(u^k, v^k) du^k dv^k \]

\[ + \sum_{k=1}^{K} \int_{y \in N_x \setminus D} \int_0^1 \int_0^1 K(x, y(u^k, v^k)) \phi(y(u^k, v^k)) w^k(y(u^k, v^k)) J(u^k, v^k) du^k dv^k, \]  

\[ (4.1.2) \]

where \( N_x \) stands for the neighborhood of \( x \) and \( J \) denotes the Jacobian of the parameterization. The calculation of the adjacent interactions (the second term in (4.1.2)), requires a specialized quadrature rule to achieve high-order accuracy. The details are included in Section 4.2. On the other hand, when the integration point \( y \) does not lie in the neighborhood \( N_x \), the first term in (4.1.2) is nonsingular and therefore can be evaluated through using of the trapezoidal rule. In this case, super-algebraic convergence is obtained since we can treat the integrand as a periodic function with compact support due to the partition of unity. The difficulty here lies in the attainment of efficiency in this situation. Section 4.3 provides a possible strategy to this end.

### 4.2 Accuracy: Adjacent interactions

As we mentioned, when the field point \( x \) is not in patch \( P_k \), the adjacent interactions are non-singular and therefore can be evaluated through use of trapezoidal rule. However, if \( x \) lies in \( P_k \), we need a more careful treatment. To begin, we introduce another partition of unity to localize the singularity. To be more specific, we write

\[ \sum_{k=1}^{K} \int_{y \in N_x} \int_{x \in P_k} \int_0^1 \int_0^1 K(x, y(u^k, v^k)) \phi(y(u^k, v^k)) w^k(y(u^k, v^k)) J(u^k, v^k) du^k dv^k \]

\[ = \sum_{k=1}^{K} \int_{y \in N_x} \int_0^1 \int_0^1 \cdots (1 - \eta(r)) + \sum_{k=1}^{K} \int_{y \in N_x} \int_0^1 \int_0^1 \cdots \eta(r), \]  

\[ (4.2.1) \]
where \( \eta(r) = 1 \) if the distance between \( x \) and \( y \) in the parameter plane is less than \( r_0 \) and \( \eta(r) \) vanishes if the distance is greater than or equal to \( r_1 \). Therefore, the integrand in the first term of (4.2.1) is a smooth periodic function and, once again, the trapezoidal rule can be used to evaluate the integral accurately. To compute each integral in the second term of the right-hand side of (4.2.1), we first introduce the polar transformation

\[
\begin{align*}
    u &= u_0 + \rho \cos \theta \\
    v &= v_0 + \rho \sin \theta,
\end{align*}
\]

where \((u_0, v_0)\) are the coordinates of the point \( x \) in parameter space. Now we can write each integral in the second term of the right-hand side of (4.2.1) as

\[
I_s = \int_0^{2\pi} \int_0^{r_1} K(x(u_0, v_0), y(u, v))\phi(y(u, v))w(y(u, v))J(u, v)\eta(\rho)d\rho d\theta
\]

where \( F(\rho, \theta) = K\phi w J \eta \rho \) displays a singularity no stronger than \( \rho^{-2} \). By using a Laurent series expansion with respect to \( \rho \)

\[
F(\rho, \theta) = \frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} + O(1), \tag{4.2.2}
\]

and attaching the correct interpretation to the integrals as described in [33], we obtain a final formula for the evaluation of the singular integrals

\[
I_s = \int_0^{2\pi} \int_0^{r_1} \left\{ F(\rho, \theta) - \left[ \frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} \right] \right\} d\rho d\theta - \frac{1}{r_1} \int_0^{2\pi} F_{-2}(\theta)d\theta. \tag{4.2.3}
\]

Note that, if the kernel is strongly singular, then \( F_{-2}(\theta) = 0 \). If the kernel is weakly singular, then both \( F_{-2}(\theta) \) and \( F_{-1}(\theta) \) vanish. Since both integrands in (4.2.3) are regular, we can use a Gaussian quadrature to evaluate the integrals.

To get the expressions for \( F_{-2}(\theta) \) and \( F_{-1}(\theta) \), the following Taylor expansions for \( x_i - y_i, r^n \) and \( \gamma_i \) are needed. To this end, it is convenient to define

\[
\begin{align*}
    E(\theta) &= \left\{ \sum_{k=1}^{3} [E_k(\theta)]^2 \right\}^{1/2}, \tag{4.2.4} \\
    G(\theta) &= \sum_{k=1}^{3} E_k(\theta) F_k(\theta), \tag{4.2.5}
\end{align*}
\]
where

\[ E_i(\theta) = \frac{\partial y_i}{\partial u}(u_0, v_0) \cos \theta + \frac{\partial y_i}{\partial v}(u_0, v_0) \sin \theta, \]

\[ F_i(\theta) = \frac{\partial^2 y_i}{\partial u^2}(u_0, v_0) \frac{\cos^2 \theta}{2} + \frac{\partial^2 y_i}{\partial u \partial v}(u_0, v_0) \cos \theta \sin \theta + \frac{\partial^2 y_i}{\partial v^2}(u_0, v_0) \frac{\sin^2 \theta}{2}. \]

Then, it can be shown that

\[ y_i - x_i = \rho E_i(\theta) + \rho^2 F_i(\theta) + O(\rho^3), \]

\[ r^n = \rho^n E^n \left(1 + n\rho \frac{G}{E^2}\right) + O(\rho^{n+2}), \]

\[ r_s = \frac{E_i}{E} + \rho \left(\frac{F_i}{E} - \frac{E_i G}{E^3}\right) + O(\rho^2). \]

In addition, we need the Taylor series expansion for the density function \( \phi(y(u, v)) w(y(u, v)) \) at \((u_0, v_0)\), i.e., both the function value and the derivatives at \((u_0, v_0)\). This can be done by using FFTs for any given values of the surface displacements and tractions on the discretized grid. Note here that the polar change of variables results in the need for such values off the grid, which can be evaluated by suitable interpolation strategies. Specifically, for this we

1. Obtain the Fourier coefficients along the grid lines via FFT;
2. Obtain the function values and derivatives on a refined grid via FFT;
3. Construct a cubic interpolating polynomial on each interval of the refined grid.

With these provision, in Appendix B, we derive the explicit formulas of \( F_{-1}(\theta) \) and \( F_{-2}(\theta) \) for the kernels \( T, W, \) and \( V \).

### 4.3 Efficiency: Non-adjacent interactions

In the previous section, we discussed a numerical scheme for evaluating adjacent interactions to high order accuracy. To compute non-adjacent interactions, direct use of standard quadratures rules would result in an \( O(N^2) \) algorithm which can quickly become prohibitive as the number of grid points \( N \) increases. In this section, we introduce an acceleration scheme, based on equivalent sources on Cartesian grids, that is analogous to that used in [11] in the context of acoustic scattering.

Let us begin by assuming that the inhomogeneity \( D \) is contained in a cube with side \( S \), and that this cube is divided into a number \( L^3 \) of identical smaller cubic cells \( c_i \). On
every cell $c_i$ that intersects the boundary of $D$, we can think of the evaluation of the corresponding portion of the first term in (4.1.2) as that of the potential due to a few monopoles and dipoles (henceforth referred to as the “true sources”) that correspond to the discretized version of the integral. The basic idea of the approach (much as in AIM [8]) is to replace the true sources (located on $\partial D$) by “equivalent sources” located on $\partial c_i$. The essential requirement is that the equivalent sources produce (to any desired accuracy) the same field as that of the true sources outside the “super-cell” $S_i$ comprised of $c_i$ and its “adjacent” 26 nearest neighboring cells. Once these equivalent sources are computed, their interaction can be evaluated efficiently through FFTs, and readily transferred back onto $\partial D$. More precisely, the procedure can be divided into three steps, namely:

1. **Representation by equivalent sources.** For each pair of parallel faces of a cell $c_i$, $F^l_i, l = 1, 2, 3$, we seek a representation of the quantity $\int_{\partial D \in c_i} G(y, x) t(y) d\sigma_y$ generated by the true sources in $c_i$ by both monopoles $G(x, x^l_{i,j}) \xi_{G,i,j}^l$ and dipoles $T(x, x^l_{i,j}) \zeta_{G,i,j}^l$ at points $x^l_{i,j}, j = 1, 2, \ldots, M^{eq}/2$ on $F^l_i$, where the subscript $G$ corresponds to the kernel $G$ in equation (3.4.6). The field generated by these equivalent sources is given by (cf. equation (3.4.6))

$$
\psi_{G,na,eq}^{c_i,l}(x) = \frac{1}{2} M^{eq} \sum_{j=1}^{M^{eq}} \left[ G(x, x^l_{i,j}) \xi_{G,i,j}^l + T(x, x^l_{i,j}) \zeta_{G,i,j}^l \right],
$$

(4.3.1)

In order to determine the intensities $\xi_{G,i,j}^l$ and $\zeta_{G,i,j}^l$ we equate the field (4.3.1) to $\psi_{G,na,true}^{c_i,l}$, the field generated by true sources in $c_i$, on $\partial S_i$. To this end, and for the sake of numerical stability, this problem is cast as one of least-squares approximation wherein the quantity

$$
||\psi_{G,na,eq}^{c_i,l} - \psi_{G,na,true}^{c_i,l}||_2
$$

is minimized over a number $n^{coll} \approx 2 M^{eq}$ of collocation points on $\partial S_i$. Specifically, we seek the least square solution of the (over-determined) linear system

$$
A \begin{bmatrix} \xi \\ \zeta \end{bmatrix} = b,
$$

(4.3.2)

where the $n^{coll} \times M^{eq}$ matrix $A$ corresponds to the evaluation of $\psi_{na,eq}^{c_i,l}$ and the vector $b$ to that of the field generated by true sources. Note that, since all the cells are identical, the QR decomposition of $A$ need only be computed once.
Similarly, we can obtain the equivalent source representation of the field associated with other kernels, i.e. (cf. equation (3.4.7))

\[
\int_{\partial D \in c_i} T(y, x) u(y) d\sigma_y, \int_{\partial D \in c_i} W(y, x) t(y) d\sigma_y, \text{ and } \int_{\partial D \in c_i} V(y, x) u(y) d\sigma_y,
\]

by writing

\[
\psi_{c_i,l}^{T,na,eq}(x) = \sum_{j=1}^{1/2 M_{eq}} \left[ G(x, x_{i,j}^l) \xi_{T,i,j}^l + T(x, x_{i,j}^l) \zeta_{T,i,j}^l \right], \quad (4.3.3)
\]

\[
\chi_{c_i,l}^{W,na,eq}(x) = \sum_{j=1}^{1/2 M_{eq}} \left[ W(x, x_{i,j}^l) \xi_{W,i,j}^l + V(x, x_{i,j}^l) \zeta_{W,i,j}^l \right], \quad (4.3.4)
\]

\[
\chi_{c_i,l}^{V,na,eq}(x) = \sum_{j=1}^{1/2 M_{eq}} \left[ W(x, x_{i,j}^l) \xi_{V,i,j}^l + V(x, x_{i,j}^l) \zeta_{V,i,j}^l \right]. \quad (4.3.5)
\]

The unknown intensities can be obtained by solving the least square problem in a similar fashion as above.

In Chapter 5, we present a variety of numerical tests that confirm the validity of the representations in (4.3.1), (4.3.3)-(4.3.5).

2. Evaluation of non-adjacent interactions on the Cartesian grids. After the intensities on the Cartesian grids \(x_{i,j}^l\) are computed, for a fixed \(l(l = 1, 2, 3)\), the evaluation of the non-adjacent field values at the equivalent source locations on any cell \(c_m\) can be written as

\[
\sum_{i=1}^{L^3} \psi_{na,eq}^{c_i,l}(x_{m,j}^l) - \sum_{\{i|c_i \subseteq S_m\}} \psi_{eq}^{c_i,l}(x_{m,j}^l), \quad (4.3.6)
\]

and

\[
\sum_{i=1}^{L^3} \chi_{na,eq}^{c_i,l}(x_{m,j}^l) - \sum_{\{i|c_i \subseteq S_m\}} \chi_{eq}^{c_i,l}(x_{m,j}^l). \quad (4.3.7)
\]

A main point here relates to the fact that these two sums can be efficiently computed via fast Fourier transforms as they are exact convolutions on Cartesian grids. And this, in turn, results in an accurate approximation for the field \(\psi_{na,true}\) and \(\chi_{na,true}\) due to non-adjacent interactions throughout the boundary of each cell \(c_m\).
3. Evaluation of the surface values of non-adjacent interactions. Finally, to find the field values at the true source locations (on $\partial D$) from values at the boundary of $c_m$, a Dirichlet problem for the displacement must be solved. A choice for the size of each cell can be made to ensure that these problems can be solved in a unique and stable manner (away from resonances). To this end, a discretized spherical wave expansion is used and equated to the boundary values of the displacement field on $c_m$. Again here, for numerical stability the coefficients are sought in the form of a least-squares solution to an over-determined system.
Chapter 5

Numerical results

In this section we present a variety of numerical results to validate the procedure described in Chapter 4. To this end, we begin, in Section 5.1, with a comparison of results obtained from an implementation of the numerical algorithm to the exact analytical solution for a (penetrable) viscoelastic sphere [83]. Section 5.2, in turn, is devoted to a presentation of results corresponding to a “kidney-like” shape for which an analytical solution is unavailable. Finally, in Section 5.3 we provide numerical evidence of the realizability of the equivalent source identification as described above in Section 4.3.

5.1 Scattering by a penetrable ball

Suppose that a penetrable ball characterized by Lamé constants $\lambda_i, \mu_i$ and $\rho_i$ is embedded in an isotropically viscoelastic medium with corresponding constants $\lambda_e, \mu_e, \rho_e$. To resolve the boundary integral equations (3.4.6) and (3.4.7), we shall assume that the incident wave takes the form

$$u^{inc} = z_1 e^{i k_e z},$$

(5.1.1)

where $z_1 = (0, 0, 1)^T$ and $k_e = \sqrt{(\lambda_e + 2\mu_e)/\rho_e}$; this is a plane longitudinal wave with unit amplitude. In the case of scattering by a spherical obstacle, the analytical solution can be obtained by expressing the wave fields (both interior and exterior) through the use of their spherical expansions and applying the boundary conditions [83]. Here we compare this analytical solution to results obtained from an iterative solution of the integral equations that uses the GMRES [66] procedure in complex arithmetic.
The boundary of the penetrable obstacle, i.e. a sphere with radius $a = 1.00$, is covered by four overlapping patches (see Fig. 5.1). The parameters used in the simulations for the low frequency case $\omega = 1.00$ are as follows: for the exterior domain, $\lambda_e^* = 4.00 - 3.00i, \mu_e^* = 2.00 - 2.00i$ and $\rho_e = 8.00$; for the interior domain, in turn, $\lambda_i^* = 12.00 - 4.00i, \mu_i^* = 2.00 - 3.00i$ and $\rho_i = 8.00$ (see equation (3.4.2)). Notice that these values result in propagation constants $k_e$ and $k_i$ that satisfy $k_e a = 0.81 + 0.31i$ (exterior) and $k_i a = 0.63 + 0.18i$ (interior) for the longitudinal direction and $k_e a = 1.55 + 0.64i$ and $k_i a = 1.31 + 0.70i$ for the transverse direction. The computed scattered field (z-component of the displacement field) together with the total field and the incident field are displayed in Fig. 5.2. Images of three orthogonal slices are also provided in the same figure. Table 5.1 shows a quantitative comparison with the exact solution, which provides clear evidence of high-order convergence.

![Figure 5.1: Four overlapped patches for the unit sphere.](image-url)
Figure 5.2: The z-component of the displacement field ($\omega = 1$). Left top: Incident field. Right top: Total field. Bottom left: Scatter field. Bottom right: 3 orthogonal slices of the total field.

<table>
<thead>
<tr>
<th>Number of unknowns</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4 \times 16 \times 16$</td>
<td>$5.64 \times 10^{-2}$</td>
</tr>
<tr>
<td>$4 \times 32 \times 32$</td>
<td>$4.10 \times 10^{-3}$</td>
</tr>
<tr>
<td>$4 \times 64 \times 64$</td>
<td>$5.45 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 5.1: $\omega = 1$. Relative error (calculated against the exact analytical solution) in the total displacement on the boundary of the ball of radius $a = 1$ with the configuration as in Figure 5.2.
The next set of examples correspond to simulations at a significantly higher frequency $\omega = 5.00$. The incoming wave is again a longitudinal plane wave. The parameters here are as follows: for the exterior domain, $\lambda_e^* = 4.00 - 3.00i$, $\mu_e^* = 2.00 - 2.00i$ and $\rho_e = 8.00$; while for the interior domain we have $\lambda_i^* = 12.00 - 4.00i$, $\mu_i^* = 2.00 - 3.00i$ and $\rho_i = 8.00$ (see equation (3.4.2)). Under these conditions, the wavenumbers satisfy $k_e a = 4.06 + 1.53i$ (exterior) and $k_i a = 3.13 + 0.90i$ (interior) in the longitudinal direction, and $k_e a = 7.77 + 3.22i$ and $k_i a = 6.56 + 3.51i$ in the transverse direction.

Figure 5.3: The $z$-component of the displacement field ($\omega = 5$). Left top: Incident field. Right top: Total field. Bottom left: Scatter field. Bottom right: 3 orthogonal slices of the total field.
Table 5.2: $\omega = 5$. Relative error (calculated against the exact analytical solution) in the total displacement on the boundary of the ball of radius $a = 1$ with the configuration as in Figure 5.3.

<table>
<thead>
<tr>
<th>Number of unknowns</th>
<th>Relative error</th>
</tr>
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<td>4 $\times$ 16 $\times$ 16</td>
<td>$2.95 \times 10^{-2}$</td>
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<tr>
<td>4 $\times$ 32 $\times$ 32</td>
<td>$1.82 \times 10^{-3}$</td>
</tr>
<tr>
<td>4 $\times$ 64 $\times$ 64</td>
<td>$1.18 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

The computed scattered field ($z$-component of the displacement field) together with the total field and the incident field are given in Fig. 5.3. The total field on three orthogonal slices are also shown in the same figure.

For the sake of completeness finalize this section with experiments that replicate those above, but wherein the material constants possess no imaginary part, that is, for a purely elastic arrangements. Figures 5.4 and 5.5 display the fields at frequencies $\omega = 1$ and 5, respectively. Tables 5.3 and 5.4 show the corresponding errors when compared to the exact solution in these cases.
Figure 5.4: The z-component of the displacement field ($\omega = 1$). Left top: Incident field. Right top: Total field. Bottom left: Scatter field. Bottom right: 3 orthogonal slices of the total field. (Purely elastic case)

<table>
<thead>
<tr>
<th>Number of unknowns</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
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<td>$1.50 \times 10^{-2}$</td>
</tr>
<tr>
<td>$4 \times 32 \times 32$</td>
<td>$1.62 \times 10^{-3}$</td>
</tr>
<tr>
<td>$4 \times 64 \times 64$</td>
<td>$5.65 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Table 5.3: $\omega = 1$. Relative error (calculated against the exact analytical solution) in the total displacement on the boundary of the ball of radius $a = 1$ with the configuration as in Figure 5.4.
Figure 5.5: The $z$-component of the displacement field ($\omega = 5$). Left top: Incident field. Right top: Total field. Bottom left: Scatter field. Bottom right: 3 slices of the total field. (Purely elastic case)

<table>
<thead>
<tr>
<th>Number of unknowns</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4 \times 16 \times 16$</td>
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<td>$4 \times 32 \times 32$</td>
<td>$1.61 \times 10^{-3}$</td>
</tr>
<tr>
<td>$4 \times 64 \times 64$</td>
<td>$5.41 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 5.4: $\omega = 5$. Relative error (calculated against the exact analytical solution) in the total displacement on the boundary of the ball of radius $a = 1$ with the configuration as in Figure 5.5.
5.2 Scattering by a “kidney-like” shape

In this section, we show simulation results for a penetrable kidney-shaped obstacle. The boundary of the obstacle, given by

\[ \frac{x^2}{a^2} + \frac{(\alpha_1 \cos(\pi x) + y)^2}{b^2(1 - \alpha_2 \cos(\pi x))} + \frac{z^2}{c^2(1 - \alpha_3 \cos(\pi x))} = 1 \]  

(5.2.1)

is covered by four patches (see Fig. 5.6). Here, we choose \( a = b = c = 1.00 \) and \( \alpha_1 = 0.30, \alpha_2 = 0.40, \alpha_3 = 0.10 \). The incident wave is taken to be a longitudinal plane wave with the incident direction \((1, 1, 1)^T\). The parameters are as follows: for the exterior domain \( \lambda_e^* = 4.00 - 0.20i, \mu_e^* = 2.00 - 0.10i \) and \( \rho_e = 8.00 \), and for the interior domain we set these to be \( \lambda_i^* = 12.00 - 0.30i, \mu_i^* = 2.00 - 0.20i \) and \( \rho_i = 8.00 \). As a result, for the longitudinal direction, \( Re(k_e)a = 5.00 \) (exterior) and \( Re(k_i)a = 3.53 \) (interior); for the transverse direction, \( Re(k_e)a = 10.00 \) and \( Re(k_i)a = 9.96 \). The computed scattered field (z-component of the displacement field) together with the total field and the incident field are given in Fig. 5.7. The total fields on three orthogonal slices are also shown in the same figure.
Figure 5.6: Four overlapped patches of the boundary of a kidney-shaped obstacle.
5.3 Equivalent source identification

In Section 4.3, we discussed the accelerated version of the non-adjacent interactions. The main task of the acceleration consists of (efficiently) finding the equivalent sources on two parallel faces of each cell in a manner such that the non-adjacent field generated by the equivalent sources is equal, outside the “supercell” $S_i$, to the field generated by the true sources inside that cell in the least-squares sense. In this section, we present the numerical experiments that demonstrate the feasibility of such equivalent source identification.

To begin, we shall be concerned with the equivalent source identification for the displacement field (c.f. equation (4.3.1)). Suppose we have a box of size $a = 2$ and a number $N$ of equivalent source points on each of the two faces parallel to the $y – z$ plane.
A number $M$ of the evaluation points are located on the boundary of the supercell, a box of size $3a$. There are 5 true sources with random values inside the box; see Fig. 5.8 for an illustration.

Figure 5.8: Illustration of the equivalent source identification. Blue circles: evaluation points. Magenta crosses: equivalent source points. Red stars: true source points.

The relative errors $\epsilon_{G,\infty}$ and $\epsilon_{T,\infty}$ of the true field $\psi^\text{true}_G$ and the approximated field $\psi^\text{appr}_G$, defined by

$$
\epsilon_{G,\infty} = \frac{||\psi^\text{true}_G - \psi^\text{appr}_G||}{||\psi^\text{true}_G||}, \quad \epsilon_{T,\infty} = \frac{||\psi^\text{true}_T - \psi^\text{appr}_T||}{||\psi^\text{true}_T||},
$$

and obtained through the procedure outlined in Section 4.3 are shown in Table 5.5.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$N$</th>
<th>$\epsilon_{G,\infty}$</th>
<th>$M$</th>
<th>$N$</th>
<th>$\epsilon_{T,\infty}$</th>
</tr>
</thead>
<tbody>
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<td>$4 \times 4 \times 4$</td>
<td>$4 \times 4 \times 2$</td>
<td>$1.02 \times 10^{-2}$</td>
<td>$4 \times 4 \times 4$</td>
<td>$4 \times 4 \times 2$</td>
<td>$1.05 \times 10^{-2}$</td>
</tr>
<tr>
<td>$8 \times 8 \times 8$</td>
<td>$8 \times 8 \times 2$</td>
<td>$1.16 \times 10^{-5}$</td>
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<td>$1.86 \times 10^{-9}$</td>
<td>$16 \times 16 \times 16$</td>
<td>$16 \times 16 \times 2$</td>
<td>$6.74 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 5.5: Equivalent source identification associated with kernel $G$ and $T$. 
Finally we present the numerical tests for the equivalent source identification associated with kernels $W$ and $V$. We define the relative errors by

$$\varepsilon_{W,\infty} = \frac{||\chi_{W}^{true} - \chi_{W}^{appr}||_{\infty}}{||\chi_{W}^{true}||_{\infty}}, \quad \varepsilon_{T,\infty} = \frac{||\chi_{T}^{true} - \chi_{T}^{appr}||_{\infty}}{||\chi_{T}^{true}||_{\infty}},$$

and the results are shown in Table 5.6.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$N$</th>
<th>$\varepsilon_{W,\infty}$</th>
<th>$M$</th>
<th>$N$</th>
<th>$\varepsilon_{V,\infty}$</th>
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</thead>
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<td>$4 \times 4 \times 2$</td>
<td>$1.05 \times 10^{-2}$</td>
<td>$4 \times 4 \times 4$</td>
<td>$4 \times 4 \times 2$</td>
<td>$2.05 \times 10^{-2}$</td>
</tr>
<tr>
<td>$8 \times 8 \times 8$</td>
<td>$8 \times 8 \times 2$</td>
<td>$4.04 \times 10^{-5}$</td>
<td>$8 \times 8 \times 8$</td>
<td>$8 \times 8 \times 2$</td>
<td>$4.60 \times 10^{-4}$</td>
</tr>
<tr>
<td>$16 \times 16 \times 16$</td>
<td>$16 \times 16 \times 2$</td>
<td>$2.17 \times 10^{-9}$</td>
<td>$16 \times 16 \times 16$</td>
<td>$16 \times 16 \times 2$</td>
<td>$6.94 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 5.6: Equivalent source identification associated with kernel $W$ and $V$.

It is to be noted that this high-order convergence is attainable only due to the correct choice of representative (equivalent) sources as described in (4.3.4) and (4.3.5). Indeed, for instance, neither

$$\int_{\partial D \in \mathcal{C}_{i}} W(y,x) t(y) d\sigma_{y}$$

nor

$$\int_{\partial D \in \mathcal{C}_{i}} V(y,x) u(y) d\sigma_{y}$$

can be approximated well by using monopoles $G(x, x_{i,j}^{l}) \xi_{l,i,j}^{i}$ and dipoles $T(x, x_{i,j}^{l}) \zeta_{l,i,j}^{i}$, as shown in Table 5.7.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$N$</th>
<th>$\varepsilon_{W,\infty}$</th>
<th>$M$</th>
<th>$N$</th>
<th>$\varepsilon_{V,\infty}$</th>
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<tbody>
<tr>
<td>$4 \times 4 \times 4$</td>
<td>$4 \times 4 \times 2$</td>
<td>$4.26 \times 10^{0}$</td>
<td>$4 \times 4 \times 4$</td>
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<td>$6.01 \times 10^{0}$</td>
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<td>$8 \times 8 \times 8$</td>
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<td>$6.20 \times 10^{0}$</td>
<td>$8 \times 8 \times 8$</td>
<td>$8 \times 8 \times 2$</td>
<td>$7.85 \times 10^{0}$</td>
</tr>
<tr>
<td>$16 \times 16 \times 16$</td>
<td>$16 \times 16 \times 2$</td>
<td>$2.28 \times 10^{0}$</td>
<td>$16 \times 16 \times 16$</td>
<td>$16 \times 16 \times 2$</td>
<td>$2.36 \times 10^{0}$</td>
</tr>
</tbody>
</table>

Table 5.7: Failure of equivalent source identification associated with kernel $W$ and $V$ by using monopoles $G(x, x_{i,j}^{l}) \xi_{l,i,j}^{i}$ and dipoles $T(x, x_{i,j}^{l}) \zeta_{l,i,j}^{i}$.
Chapter 6

Conclusions and future work

In this thesis we have presented a fast high-order integral equation solver for linear viscoelastic scattering problems, motivated by applications in ultrasound vibro-acoustography. The solver achieves its high-order accuracy by means of analytical resolutions of weak, strong and hyper-singular integrals based on local polar parameterizations. Its efficiency, on the other hand, results from the use of partitions of unity, and equivalent-source representations of displacements and tractions in terms of suitably chosen potentials on Cartesian grids, so as to allow for FFT-based acceleration. A variety of numerical experiments were presented to exemplify the high-order character of the resulting procedure, as well as the suitability of the equivalent source representation.

Immediate work for the future relates to the incorporation of the equivalent source representation into the overall scattering solver. As we have stated, this should allow for computations at higher frequencies and enable the simulation of the waves generated at the transducer, and of their nonlinear interaction in tissue. Longer term objectives include the implementation of geometries that will allow for comparison with experimental data; the extension to more complex viscoelastic models; and the use of the resulting codes in the development of an optimization scheme to aid in the virtual design of the next generation UVA systems.
References


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Appendix A

Derivation of kernels

It is well known that the displacement Green’s function is given, in the frequency domain, by
\[ G_{ij}^u(y, x) = \frac{e^{ikr}}{4\pi\rho_0(c_m^*)^2r} \left[ \gamma_i\gamma_j + (3\gamma_i\gamma_j - \delta_{ij})(-\frac{c_m^*}{i\omega r}) + (3\gamma_i\gamma_j - \delta_{ij})(-\frac{c_m^*}{i\omega r})^2 \right] 
- \frac{e^{iktr}}{4\pi\rho_0(c_t^*)^2r} \left[ (\gamma_i\gamma_j - \delta_{ij}) + (3\gamma_i\gamma_j - \delta_{ij})(-\frac{c_t^*}{i\omega r}) + (3\gamma_i\gamma_j - \delta_{ij})(-\frac{c_t^*}{i\omega r})^2 \right], \]
where
\[ r = |y - x|, \gamma_m = r, m = \frac{\partial}{\partial y_m} = \frac{y_m - x_m}{r}. \]
If we define
\[ \tau = \frac{i\omega r}{c_t^*}, \alpha = \frac{c_m^*}{c_t^*}, M(\tau) = \frac{e^{\alpha \tau}(1 - \alpha \tau) - e^\tau(1 - \tau)}{\tau^2}, \]
we can rewrite \( G_{ij}^u \) as
\[ G_{ij}^u = \frac{1}{4\pi\mu^*r} \{ \psi\delta_{ij} + \chi\gamma_i\gamma_j \}, \quad (A.0.1) \]
where
\[ \psi = e^\tau - M(\tau), \]
\[ \chi = e^{\alpha \tau} \alpha^2 - e^\tau + 3M(\tau). \]
To get the traction kernel, we need the strain kernel \( E_{ijk} \) and stress kernel \( \Sigma_{ijk} \). From the strain-displacement relationship, we have
\[ E_{ijk} = (G_{ij}^{uk} + G_{ij}^{ik})/2. \]
One can show that
\[ G^{ij,k}_u = \frac{1}{4\pi\mu^*r} \left\{ \left( \frac{\partial\psi}{\partial r} - \frac{\psi}{r} \right) \delta_{ij} \gamma_k + \left( \frac{\partial\chi}{\partial r} - \frac{3\chi}{r} \right) \gamma_i \gamma_j \gamma_k + \frac{\chi}{r} (\delta_{ik} \gamma_j + \delta_{jk} \gamma_i) \right\}. \]

Switching superscripts, we have
\[ G^{ik,j}_u = \frac{1}{4\pi\mu^*r} \left\{ \left( \frac{\partial\psi}{\partial r} - \frac{\psi}{r} \right) \delta_{ik} \gamma_j + \left( \frac{\partial\chi}{\partial r} - \frac{3\chi}{r} \right) \gamma_i \gamma_j \gamma_k + \frac{\chi}{r} (\delta_{ij} \gamma_k + \delta_{jk} \gamma_i) \right\}. \]

Therefore, we obtain the strain kernel
\[ E_{ijk} = \frac{1}{4\pi\mu^*r^2} \{ A_1 \gamma_i \gamma_j \gamma_k + A_2 (\delta_{ij} \gamma_k + \delta_{ik} \gamma_j) + \chi \delta_{jk} \gamma_i \}, \quad (A.0.2) \]
and the stress kernel
\[ \Sigma_{ijk} = \lambda^* \delta_{jk} E_{imm} + 2\mu^* E_{ijk} \]
\[ \Sigma_{ijk} = \frac{1}{2\pi r^2} \{ A_1 \gamma_i \gamma_j \gamma_k + A_2 (\delta_{ij} \gamma_k + \delta_{ik} \gamma_j) + A_3 \delta_{jk} \gamma_i \}, \quad (A.0.3) \]

where
\[ A_1 = \tau \chi' (\tau) - 3\chi, \]
\[ A_2 = (\tau \psi' (\tau) - \psi + \chi)/2, \]
\[ A_3 = \chi + \frac{1}{(2\alpha^2) - 1}(A_1 + 2A_2 + 3\chi). \]

Finally, the traction kernel
\[ T_{ij} = \Sigma_{ijk} n_k \]
is given by
\[ T_{ij} = \frac{1}{2\pi r^2} \left\{ (A_1 \gamma_i \gamma_j + A_2 \delta_{ij}) \gamma_m n^{(y)}_m + (A_2 \gamma_j n^{(y)}_i + A_3 \gamma_i n^{(y)}_j) \right\}. \quad (A.0.4) \]

Remark:
When \( \omega = 0, \sigma_l = \sigma_t = 0 \), we have
\[ \tau = 0, \alpha = \sqrt{\frac{1 - 2\nu}{2(1 - \nu)}}, \quad \text{and} \quad M(0) = \frac{1 - \alpha^2}{2}. \]
Moreover,
\[ \psi = 1 - M(0) = \frac{3 - 4\nu}{4(1 - \nu)}, \]
\[ \chi = \alpha^2 - 1 + 3M(0) = \frac{1}{4(1 - \nu)}, \]
\[ A_1 = -\frac{3}{4(1 - \nu)}, \]
\[ A_2 = -\frac{1 - 2\nu}{4(1 - \nu)}, \]
\[ A_3 = \frac{1 - 2\nu}{4(1 - \nu)}. \]

Therefore,
\[ G_{ij}^{(u)} = \frac{1}{16\pi \mu (1 - \nu) r} \left[ (3 - 4\nu) \delta_{ij} + \gamma_i \gamma_j \right], \] (A.0.5)
\[ T_{ij} = \frac{-1}{8\pi (1 - \nu) r^2} \left\{ (1 - 2\nu) (n_i^{(y)} \gamma_j - n_j^{(y)} \gamma_i) + [3 \gamma_i \gamma_j + (1 - 2\nu) \delta_{ij}] n_m^{(y)} \gamma_m \right\}, \] (A.0.6)
which agree with the known Kelvin fundamental solutions.

To get the other two kernels, we follow a similar procedure and obtain
\[ W_{ij} = \frac{-1}{2\pi r^2} \left\{ (A_1 \gamma_i \gamma_j + A_2 \delta_{ij}) \gamma_p n_p^{(x)} + (A_2 \gamma_i n_j^{(x)} + A_3 \gamma_j n_i^{(x)}) \right\}, \] (A.0.7)
and
\[ V_{ij} = \frac{-\mu^*}{\pi r^3} \left\{ \gamma_m n_m^{(y)} \left[ (B_1 \gamma_i \gamma_j + B_2 \delta_{ij}) \gamma_p n_p^{(x)} + B_2 \gamma_i n_j^{(x)} + B_3 \gamma_j n_i^{(x)} \right] \\
+ \left[ B_2 \gamma_j n_i^{(y)} + B_5 \gamma_i n_j^{(x)} \right] \gamma_p n_p^{(x)} + [B_2 \gamma_i \gamma_j + B_4 \delta_{ij}] n_q^{(x)} n_q^{(y)} \\
+ \left[ B_4 n_i^{(y)} n_j^{(x)} + B_6 n_j^{(y)} n_i^{(x)} \right] \right\}, \] (A.0.8)
where
\[ B_1 = A'_1 \tau - 5A_1, \]
\[ B_2 = (A'_2 \tau - 3A_2 + A_1)/2, \]
\[ B_3 = A_1 + (1/(2\alpha^2) - 1)(A'_1 \tau + 2A'_2 \tau - 6A_2), \]
\[ B_4 = A_2, \]
\[ B_5 = A'_3 \tau - 3A_3, \]
\[ B_6 = A_3 + (1/(2\alpha^2) - 1)(A'_3 \tau + 2A_2). \]
Appendix B

Derivation of $F_{-1}(\theta)$ and $F_{-2}(\theta)$

We first derive $F_{-1}(\theta)$ for the kernel $T$. If we define $\tilde{\phi}_j = \phi_j w$, where $w$ is the partition of unity, and notice that $J_k = J_{nk}$, we can rewrite $F(\rho, \theta)$ as

\[
F(\rho, \theta) = T_{ij} \tilde{\phi}_j J_{\eta \rho}
\]

\[
= \frac{1}{2\pi r^2} \left\{ \left( A_{1i} \gamma_i \gamma_j + A_{2i} \delta_{ij} \right) \gamma_m n_{m}^{(y)} + (A_{2j} \gamma_j n_{i}^{(y)} + A_{3j} \gamma_i n_{j}^{(y)}) \right\} \tilde{\phi}_j J_{\eta \rho}
\]

\[
= \frac{1}{2\pi r^2} \left\{ \left( A_{1i} \gamma_i \gamma_j + A_{2i} \delta_{ij} \right) \gamma_m J_{m}^{(y)} + (A_{2j} \gamma_j J_{i}^{(y)} + A_{3j} \gamma_i J_{j}^{(y)}) \right\} \tilde{\phi}_j J_{\eta}
\]

\[
= \frac{1}{2\pi} \left( \frac{1}{E^2 \rho} + O(1) \right) \left\{ \left( A_{1i} \gamma_i \gamma_j + A_{2i} \delta_{ij} \right) \gamma_m J_{m}^{(y)} + (A_{2j} \gamma_j J_{i}^{(y)} + A_{3j} \gamma_i J_{j}^{(y)}) \right\} \tilde{\phi}_j J_{\eta}.
\]

Therefore, $F_{-1}(\theta)$ is given by

\[
F_{-1}^T(\theta) = \frac{\tilde{\phi}_j^{(0)}}{2\pi E^2} \left\{ \left( A_{10} E_i E_j^{(x)} + A_{20} \delta_{ij} \right) \frac{E_m}{E} J_{m0}^{(x)} + A_{20} \frac{E_j}{E} J_{i0}^{(x)} + A_{30} \frac{E_i}{E} J_{j0}^{(x)} \right\},
\]

where

\[
\tilde{\phi}_j^{(0)} = \tilde{\phi}_j (u_0, v_0)
\]

\[
A_{i0} = \lim_{\tau \to 0} A_i(\tau)
\]

\[
J_{10}^{(x)} = \frac{\partial y_2}{\partial u} (u_0, v_0) \frac{\partial y_3}{\partial v} (u_0, v_0) - \frac{\partial y_3}{\partial u} (u_0, v_0) \frac{\partial y_2}{\partial v} (u_0, v_0)
\]

\[
J_{20}^{(x)} = \frac{\partial y_3}{\partial u} (u_0, v_0) \frac{\partial y_1}{\partial v} (u_0, v_0) - \frac{\partial y_1}{\partial u} (u_0, v_0) \frac{\partial y_3}{\partial v} (u_0, v_0)
\]

\[
J_{30}^{(x)} = \frac{\partial y_1}{\partial u} (u_0, v_0) \frac{\partial y_2}{\partial v} (u_0, v_0) - \frac{\partial y_2}{\partial u} (u_0, v_0) \frac{\partial y_1}{\partial v} (u_0, v_0).
\]
Similarly, $F_{-1}(\theta)$ for the kernel $W$ is given by

$$F_{-1}^{W}(\theta) = -\frac{\tilde{\phi}_{j0}}{2\pi E^2} \left\{ \left( A_{10} \frac{E_i E_j}{E^2} + A_{20} \delta_{ij} \right) \frac{E_p}{E} J^{(x)}_p + A_{20} \frac{E_i}{E} J^{(x)}_j + A_{30} \frac{E_j}{E} J^{(x)}_j \right\},$$

For the hypersingular kernel $V$, we need both $F_{-1}^{V}(\theta)$ and $F_{-2}^{V}(\theta)$.

$$F(\rho, \theta) = V_{ij} \tilde{\phi}_j J_{\eta \rho}$$
$$= -\frac{\mu^*}{\pi^3} \left\{ \left( (B_1 \gamma_i \gamma_j + B_2 \delta_{ij}) \gamma_p n^{(x)}_p + B_2 \gamma_j n^{(x)}_j + B_3 \gamma_j n^{(x)}_i \right) \right.$$  
$$+ \left[ B_2 \gamma_j n^{(y)}_i + B_3 \gamma_j n^{(y)}_j \right] \gamma_p n^{(x)}_p + B_2 \gamma_i \gamma_j + B_4 \delta_{ij} \right\} \tilde{\phi}_j J_{\eta \rho}$$
$$= -\frac{\mu^*}{\pi} \left\{ 1 \frac{E^3}{E^3 \rho^2} - \frac{3G}{E^3 \rho} + O(1) \right\} \times$$
$$\left\{ \left( (B_1 \gamma_i \gamma_j + B_2 \delta_{ij}) \gamma_p n^{(x)}_p + B_2 \gamma_i n^{(x)}_j + B_3 \gamma_j n^{(x)}_i \right) \right.$$  
$$+ \left[ B_2 \gamma_i J^{(y)}_i + B_3 \gamma_j J^{(y)}_j \right] \gamma_p n^{(x)}_p + B_2 \gamma_i \gamma_j + B_4 \delta_{ij} \right\} \tilde{\phi}_j \eta.$$  

Therefore, $F_{-2}^{V}(\theta)$ is given by

$$F_{-2}^{V}(\theta) = -\frac{\mu^*}{\pi E^3} \left\{ \left( B_{10} \frac{E_i E_j}{E^2} + B_{20} \delta_{ij} \right) \frac{E_p}{E} n^{(x)}_p + B_{20} \frac{E_i}{E} n^{(x)}_j + B_{30} \frac{E_j}{E} n^{(x)}_j \right.$$  
$$+ \left[ B_{20} \frac{E_j}{E} J^{(x)}_j + B_{30} \frac{E_i}{E} J^{(x)}_j \right] \frac{E_p}{E} n^{(x)}_p + B_{20} \frac{E_i}{E} n^{(x)}_j + B_{40} \delta_{ij} \right\} n^{(x)}_j J^{(x)}_{j0}$$
$$= -\frac{\mu^*}{\pi E^3} V_0,$$

and $F_{-1}^{V}(\theta)$ is given by

$$F_{-1}^{V}(\theta) = -\frac{\mu^*}{\pi} \left\{ -3G \tilde{\phi}_{j0} V_0 + \frac{1}{E^3} \left( \tilde{\phi}_{j1} V_0 + \tilde{\phi}_{j0} V_1 \right) \right\},$$
where

\[ \tilde{\phi}_{j1} = \frac{\partial \phi_j}{\partial u}(u_0, v_0) \cos \theta + \frac{\partial \phi_j}{\partial v}(u_0, v_0) \sin \theta \]

\[ V_1 = \gamma_{m1} J_{m0}^{(x)} \left[ B_{10} \frac{E_i E_j}{E^2} + B_{20} \delta_{ij} \right] \frac{E_p}{E} n_p^{(x)} + B_{20} \frac{E_i}{E} n_j^{(x)} + B_{30} \frac{E_j}{E} n_i^{(x)} \]

\[ + \frac{E_m}{E} J_{m1}^{(x)} \left[ B_{10} \frac{E_i E_j}{E^2} + B_{20} \delta_{ij} \right] \frac{E_p}{E} n_p^{(x)} + B_{20} \frac{E_i}{E} n_j^{(x)} + B_{30} \frac{E_j}{E} n_i^{(x)} \]

\[ + \frac{E_m}{E} J_{m0}^{(x)} \left[ B_{10} \left( \gamma_i E_j \frac{E_j}{E} + \gamma_j E_i \frac{E_i}{E} \right) \right] \frac{E_p}{E} n_p^{(x)} \]

\[ + \left( B_{10} \frac{E_i E_j}{E^2} + B_{20} \delta_{ij} \right) \gamma_p n_p^{(x)} + B_{20} \gamma_i n_j^{(x)} + B_{30} \gamma_j n_i^{(x)} \]

\[ + \left[ B_{20} \left( \gamma_i J_{j0}^{(x)} + \frac{E_j}{E} J_{i1}^{(x)} \right) + B_{50} \left( \gamma_i J_{j0}^{(x)} + \frac{E_j}{E} J_{i1}^{(x)} \right) \right] \frac{E_p}{E} n_p^{(x)} \]

\[ + \left[ B_{20} \frac{E_j}{E} J_{j0}^{(x)} + B_{50} \frac{E_j}{E} J_{j0}^{(x)} \right] \gamma_p n_p^{(x)} \]

\[ + \left( B_{20} \left( \gamma_i E_j \frac{E_j}{E} + \gamma_j E_i \frac{E_i}{E} \right) \right) n_q^{(x)} J_q + \left( B_{20} \frac{E_i E_j}{E^2} + B_{40} \delta_{ij} \right) n_q^{(x)} J_q + \]

\[ + B_{40} J_{i1}^{(x)} n_j^{(x)} + B_{60} J_{j1}^{(x)} n_i^{(x)} \]

and

\[ \gamma_{m1} = \frac{F_m}{E} - \frac{E_m G}{E^3} \]

\[ J_{m1}^{(x)} = \frac{\partial J_m}{\partial u}(u_0, v_0) \cos \theta + \frac{\partial J_m}{\partial v}(u_0, v_0) \sin \theta. \]

where

\[ \frac{\partial J_1}{\partial u} = \frac{\partial^2 y_2}{\partial u^2} \frac{\partial y_3}{\partial v} + \frac{\partial y_2}{\partial u} \frac{\partial^2 y_3}{\partial v^2} + \frac{\partial^2 y_3}{\partial u^2} \frac{\partial y_2}{\partial v} - \frac{\partial y_3}{\partial u} \frac{\partial^2 y_2}{\partial v^2} \]

\[ \frac{\partial J_2}{\partial u} = \frac{\partial^2 y_3}{\partial u^2} \frac{\partial y_1}{\partial v} + \frac{\partial y_3}{\partial u} \frac{\partial^2 y_1}{\partial v^2} + \frac{\partial^2 y_1}{\partial u^2} \frac{\partial y_3}{\partial v} - \frac{\partial y_1}{\partial u} \frac{\partial^2 y_3}{\partial v^2} \]

\[ \frac{\partial J_3}{\partial u} = \frac{\partial^2 y_1}{\partial u^2} \frac{\partial y_2}{\partial v} + \frac{\partial y_1}{\partial u} \frac{\partial^2 y_2}{\partial v^2} + \frac{\partial^2 y_2}{\partial u^2} \frac{\partial y_1}{\partial v} - \frac{\partial y_2}{\partial u} \frac{\partial^2 y_1}{\partial v^2} \]

\[ \frac{\partial J_1}{\partial v} = \frac{\partial^2 y_2}{\partial u \partial v} \frac{\partial y_3}{\partial v} + \frac{\partial y_2}{\partial u \partial v} \frac{\partial^2 y_3}{\partial v^2} + \frac{\partial^2 y_3}{\partial u \partial v} \frac{\partial y_2}{\partial v} - \frac{\partial y_3}{\partial u \partial v} \frac{\partial^2 y_2}{\partial v^2} \]

\[ \frac{\partial J_2}{\partial v} = \frac{\partial^2 y_3}{\partial u \partial v} \frac{\partial y_1}{\partial v} + \frac{\partial y_3}{\partial u \partial v} \frac{\partial^2 y_1}{\partial v^2} + \frac{\partial^2 y_1}{\partial u \partial v} \frac{\partial y_3}{\partial v} - \frac{\partial y_1}{\partial u \partial v} \frac{\partial^2 y_3}{\partial v^2} \]

\[ \frac{\partial J_3}{\partial v} = \frac{\partial^2 y_1}{\partial u \partial v} \frac{\partial y_2}{\partial v} + \frac{\partial y_1}{\partial u \partial v} \frac{\partial^2 y_2}{\partial v^2} + \frac{\partial^2 y_2}{\partial u \partial v} \frac{\partial y_1}{\partial v} - \frac{\partial y_2}{\partial u \partial v} \frac{\partial^2 y_1}{\partial v^2} \]