Topological Methods in Symplectic Geometry

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Abstract

In the first part, we study existence of the Lefschetz decomposition for de Rham cohomology, which is characterized by the strong Lefschetz property. A new spectral sequence for symplectic manifolds is also defined. In the second part, we show that the Lagrangian Luttinger surgery preserves the Kodaira dimension. Some constraints on Lagrangian tori in symplectic four manifolds with non-positive Kodaira dimension are also derived.
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Chapter 1

Introduction

The object of this thesis is the topological properties of symplectic manifolds, especially in dimension four.

In the first part, we will analyze the relation of primitive cohomologies and the strong Lefschetz property for symplectic manifolds. Let \((M, \omega)\) be a symplectic manifold of dimension \(2n\). The symplectic structure \(\omega\) induces the Lefschetz operator

\[
L : \Omega^k(M) \rightarrow \Omega^{k+2}(M), \; \alpha \mapsto \omega \wedge \alpha
\]

It is well known that \(L\) together with its dual \(\Lambda\) and a projection \(H\) give a \(sl(2)\) representation on \(\Omega^*(M)\):

\[
\begin{align*}
[\Lambda, L] &= H, \\
[H, \Lambda] &= 2\Lambda, \\
[H, L] &= -2L
\end{align*}
\]

If \(I^k\) is the direct sum of irreducible representation of dimension \(k + 1\), \(\Omega^*(M)\) can be decomposed to the vector subspace

\[
\mathcal{L}^{p,q} = \Omega^{2p+q}(M) \cap I^{n-q}
\]

For each \(\alpha \in \Omega^k(M)\), there is a unique decomposition

\[
\alpha = \sum_{r \geq \max\{k-n,0\}} \frac{1}{r!} L^r B^\alpha_{k-2r}, \quad B^\alpha_{k-2r} \in \mathcal{L}^{0,k-2r}.
\]

This decomposition is called the Lefschetz decomposition of differential forms. The elements of highest weight for the \(sl(2)\) representation are called primitive forms. The
Lefschetz decomposition shows that each differential form can be generated by primitive forms and the operator $L$. An interesting fact about the decomposition is that it decomposes the exterior derivative to

$$d : \mathcal{L}^{p,q} \to \mathcal{L}^{p,q+1} + \mathcal{L}^{p+1,q-1}$$

and there are differential operators

$$\partial_+ : \mathcal{L}^{p,q} \to \mathcal{L}^{p,q+1}, \partial_- : \mathcal{L}^{p,q} \to \mathcal{L}^{p,q-1}$$

which commute with $L$ and satisfy $d = \partial_+ + L\partial_-$. If we restrict our attention to the primitive forms $\mathcal{P}^k = \mathcal{L}^{0,k}$, the complexes $(\mathcal{P}^*, \partial_\pm)$ define two cohomologies

$$\mathcal{P}H^k_+(M) = \ker \partial_+ \cap \mathcal{P}^k, \mathcal{P}H^k_-(M) = \ker \partial_- \cap \mathcal{P}^k$$

**Question 1.0.1.** Can each de Rham cohomology class be decomposed as direct sum of primitive cohomology classes and their wedge product with $\omega$?

Unfortunately, this is not the case for any symplectic manifold. We show that these question is characterized by the strong Lefschetz property, which states that the natural homomorphism

$$\mathcal{L}^k : H^{n-k}(M) \to H^{n+k}(M), [\alpha] \mapsto [\alpha \wedge \omega^k]$$

is an isomorphism for $0 \leq k \leq n$.

**Theorem 1.0.2.** There is a canonical isomorphism

$$\sum_{r \geq \max\{k-n,0\}} L^r \mathcal{P}H^k_+ - 2r(M) \to H^k(M)$$

for any $k$ if and only if $M$ satisfies the strong Lefschetz property.

The actions of the differential $d$ and the Koszul-Brylinski operator $d^\Lambda = [d, \Lambda]$ give other descriptions for the strong Lefschetz property. A smooth form $\alpha$ is called symplectic harmonic if $d\alpha = d^\Lambda \alpha = 0$. It is proved by Cavalcanti [13], Mathieu [35], Merkulov [39] and Yan [54] that the following conditions are equivalent:

1. $M$ satisfies the strong Lefschetz property.
2. \( \text{Im} d \cap \ker d^\Lambda = \text{Im}d^\Lambda \cap \ker d = \text{Im}dd^\Lambda \).

3. Each cohomology class of \( H^*(M) \) has a symplectic harmonic representative.

The second condition is called the \( dd^\Lambda \)-lemma. There are similar concepts for \( dd^\Lambda \)-lemma and symplectic harmonic forms in primitive forms, which we call \( \partial_+ \partial_- \)-lemma and primitive harmonic forms respectively.

We show that

**Proposition 1.0.3.**  
1. The \( dd^\Lambda \)-lemma holds for \( M \) if and only if \( M \) has the \( \partial_+ \partial_- \)-lemma.

2. \( \alpha \) is symplectic harmonic if and only if each primitive term of its Lefschetz decomposition is primitive harmonic.

Hence, the strong Lefschetz property has equivalent conditions in primitive forms.

**Theorem 1.0.4.** The following properties are equivalent for a symplectic manifold \( M \):

1. \( M \) satisfies the strong Lefschetz property.

2. The \( \partial_+ \partial_- \)-lemma holds for \( M \).

3. Each cohomology class of \( H^*(M) \) has a primitive harmonic representative.

We use more elementary techniques to prove Theorem 1.0.4 directly.

**Question 1.0.5.** How far is \( M \) from being a strong Lefschetz manifold if it is not?

When the strong Lefschetz property fails, we can estimate the discrepancy between the dimension of \( H^{n-k}(M) \) and the rank of \( \mathcal{L}^k \). It turns out that \( \text{rank}(\mathcal{L}^k) \) is bounded by the dimension of symplectic cohomologies defined by Tseng and Yau in [17]. They consider the operators \( d \) and \( d^\Lambda = [d, \Lambda] \) together to define cohomologies

\[
H^k_{d+d^\Lambda}(M) = \frac{\ker(d + d^\Lambda) \cap \Omega^k(M)}{\text{Im}d^\Lambda \cap \Omega^k(M)}
\]

\[
H^k_{dd^\Lambda}(M) = \frac{\ker dd^\Lambda \cap \Omega^k(M)}{(\text{Im}d + \text{Im}d^\Lambda) \cap \Omega^k(M)}
\]

\[
H^k_{d\cap d^\Lambda}(M) = \frac{\ker(d + d^\Lambda) \cap \Omega^k(M)}{d\Omega^{k-1}(M) + d^\Lambda \Omega^{k+1}(M)}
\]
where $\tilde{\Omega}^\ast(M)$ is the space of $ddL$-closed forms. Using the corresponding operators for primitive forms, there is a cohomology for primitive forms analogous to each of the above cohomologies. It turns out that these cohomologies have the Lefschetz decomposition.

**Theorem 1.0.6.** [47] The cohomology $H_{d+dL}$ has the Lefschetz decomposition:

$$H^k_{d+dL}(M) = \bigoplus_r L^r P H^{k-2r}_{d+dL}(M)$$

Similarly for $H_{ddL}$ and $H_{d\cap dL}$.

Tseng and Yau also show that $H^k_{d+dL}(M)$ is canonically isomorphic to $H^k(M)$ for any $k$ if and only if $M$ satisfies the strong Lefschetz property. We prove that

**Theorem 1.0.7.** $\dim H^{n-k}_{d\cap dL}(M) \leq \text{rank} L^k \leq \dim H^{n-k}(M)$.

In the low dimensional case, the bound is sharp:

**Proposition 1.0.8.** Assume $M$ is a symplectic 4-manifold. Then

$$\dim H^{n-k}_{d\cap dL}(M) = \text{rank} L^k.$$

We notice that the $ddL$-lemma strongly relates to the degeneration of spectral sequences. In [12], Brylinski used the canonical complex to construct a spectral sequence for manifolds with Poisson structure and showed that it degenerates at $E_1$ for symplectic manifolds. Motivated by the Frölicher spectral sequence for complex manifolds, we construct a spectral sequence $E_{p,q}^r$ from the double complex $(\Omega^\ast(M), \partial_+, L\partial_-)$. Moreover, we give a complete description of the degeneration for symplectic 4-manifolds.

We use the famous Kodaira-Thurston manifold to show that this spectral sequence is not necessarily degenerating at $E_1$. In the computation, we also raise the issue in the spirit of the Nomizu theorem. It says that the computation of cohomology for nilmanifolds can be reduced to the Chevalley-Eilenberg cohomology for the corresponding nilpotent Lie algebra. The coincidence of the new cohomologies for the Kodaira-Thurston manifold and the nilpotent Lie algebra $ch_3$ suggests the validity of the Nomizu type theorem.

Furthermore, it is known that the current of Lagrangian submanifolds is primitive. In symplectic 4-manifold, we can use the Luttinger surgery along a Lagrangian torus to construct new symplectic manifolds. Hence we can compare the primitive cohomologies.
of two symplectic 4-manifolds connected by the Luttinger surgery. In the second part of this thesis, we focus on the discussion of the Luttinger surgery and its applications.

Let \((X, \omega)\) be a symplectic 4-manifold with a Lagrangian torus \(L\). It was discovered by Luttinger in [34] that there is a family of surgeries along \(L\) that produce symplectic 4-manifolds. This family is countable and indexed by the pairs \(([\gamma], k)\), where \([\gamma]\) is an isotopy class of simple closed curves on \(L\) and \(k\) is an integer. When \(X = \mathbb{R}^4\) and \(\omega\) is the standard symplectic form \(\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2\), he also applied Gromov’s celebrated work in [23] to show that, for any Lagrangian torus \(L\), all the resulting symplectic manifolds are symplectomorphic to \((\mathbb{R}^4, \omega_0)\). This does not occur in general; a Luttinger surgery often fails even to preserve homology. As a matter of fact, many new exotic small manifolds are constructed via this surgery. We observe that the Luttinger surgery preserves one basic invariant:

**Theorem 1.0.9.** The Luttinger surgery preserves the symplectic Kodaira dimension.

The symplectic Kodaira dimension of a symplectic 4-manifold \((X, \omega)\) is defined by the products \(K_\omega^2\) and \(K_\omega \cdot [\omega]\), where \(K_\omega\) is the symplectic canonical class; if \((X, \omega)\) is minimal, then

\[
\kappa(X, \omega) = \begin{cases} 
-\infty & K_\omega^2 < 0 \text{ or } K_\omega \cdot [\omega] < 0 \\
0 & K_\omega^2 = 0 \text{ and } K_\omega \cdot [\omega] = 0 \\
1 & K_\omega^2 = 0 \text{ and } K_\omega \cdot [\omega] > 0 \\
2 & K_\omega^2 > 0 \text{ and } K_\omega \cdot [\omega] > 0 
\end{cases}
\]

For a general symplectic 4-manifold, the Kodaira dimension is defined as the Kodaira dimension of any of its minimal models. According to [31], \(\kappa(X, \omega)\) is independent of the choice of symplectic form \(\omega\) and hence is denoted by \(\kappa(X)\).

Theorem 1.0.9 is related to a question of Auroux in [6] (see Remark 5.3.4). Furthermore, together with the elementary analysis of the homology change, the invariance of \(\kappa\) implies that

**Theorem 1.0.10.** Let \((X, \omega)\) be a symplectic 4-manifold with \(\kappa(X) = -\infty\) and \((\tilde{X}, \tilde{\omega})\) be constructed from \((X, \omega)\) via a Luttinger surgery. Then \((\tilde{X}, \tilde{\omega})\) is symplectomorphic to \((X, \omega)\).
For minimal symplectic manifolds of Kodaira dimension zero, i.e., symplectic Calabi- Yau surfaces, we conclude that the Luttinger surgery is a symplectic CY surgery. Moreover, together with the homology classification of such manifolds in [32], we have

**Theorem 1.0.11.** Suppose \((X, \omega)\) is a symplectic 4-manifold with \(\kappa(X) = 0\) and \(\chi(X) > 0\). If \((\tilde{X}, \tilde{\omega})\) is constructed from \((X, \omega)\) under a Luttinger surgery, then \(X\) and \(\tilde{X}\) have the same integral homology type.

In fact, we conjecture that \(\tilde{X}\) and \(X\) in Theorem 1.0.11 are diffeomorphic to each other. For symplectic CY surfaces with \(\chi = 0\), the only known examples are torus bundles over torus. We conjecture that they all can be obtained from \(T^4\) via Luttinger surgeries (Conjecture 5.4.9).

Theorems 1.0.10 and 1.0.11 provide topological constraints, phrased in terms of topological preferred framings (see Definition 6.1.1), on the existence of exotic Lagrangian tori in such manifolds.

**Theorem 1.0.12.** Let \(L\) be a Lagrangian torus in \((X, \omega)\). If \(\kappa(X) = -\infty\), or \(L\) is null-homologous, \(\kappa(X) = 0\) and \(\chi(X) > 0\), then the Lagrangian framing of \(L\) is topological preferred. In particular, the invariant \(\lambda(L)\) in [17] vanishes whenever it is defined.

The organization of this thesis is as follows. In Chapter 2, we first introduce some basic concepts regarding linear operators. It includes the Lefschetz operator and its dual together with the symplectic star operator. Then we introduce the operator \(\partial_+\), \(\partial_-\) and primitive cohomology.

Chapter 3 focuses on the strong Lefschetz property. We first define the concepts of \(\partial_+\partial_-\)-lemma and primitive harmonic forms. Then we use more elementary technique to prove the equivalent relations of the strong Lefschetz property. We also show that the strong Lefschetz property can be characterized by primitive forms and primitive cohomology can realize normal cohomology theories when the strong Lefschetz property holds. In the last section, we introduce symplectic cohomologies and discuss their compatible properties with the Lefschetz operator. Eventually, we show that for symplectic manifolds without strong Lefschetz property, the symplectic cohomologies can be used to detect the rank of the Lefschetz operator.

In Chapter 4, we focus on the concept of spectral sequences. After reviewing the definition briefly, we discuss the relation between \(d\delta\)-lemma and the degeneration of
spectral sequences. We also find elements in the double complex which can measure the degeneration of the spectral sequence. Then we construct the symplectic-de Rham spectral sequence and show that it has different properties from Brylinski’s canonical spectral sequence. The last section includes the computation of spectral sequences for some symplectic nilmanifolds.

We move on to the Luttinger surgery and its implications in Chapters 5 and 6. In Chapter 5, we first introduce the concept of the symplectic Kodaira dimension. Then we review the construction of the Luttinger surgery and show that the Luttinger surgery does not change the Kodaira dimension for symplectic 4-manifold. The effect of the Luttinger surgery for symplectic Calabi-Yau 4-manifolds is discussed in detail.

In Chapter 6, we generalize the Luttinger surgery and introduce the torus surgery construction. We focus on the choice of framings and give the constraints on related homology classes.
Chapter 2

Primitive cohomology

In this chapter, we define the primitive cohomology induced by the Lefschetz decomposition. We first recall several linear operators for smooth differential forms in Section 2.1. They include the Lefschetz operator and the symplectic star operator given by the symplectic structure. Section 2.2 devotes to the sl(2) representation structure for the Lefschetz operator and the definition of primitive forms. The Lefschetz decomposition of differential forms is also mentioned. Section 2.3 introduces a decomposition of the differential $d$. In Section 2.4, we define the primitive cohomologies. The explicit computation of primitive cohomology for the Kodaira-Thurston manifold is given in Section 2.5.

2.1 Linear operators

Let $V$ be a vector space. A symplectic structure on $V$ is given by a nondegenerate 2-form $\omega \in \bigwedge^2 V^*$. $\omega$ defines a map $\bar{\omega} : V \to V^*$ as

$$\bar{\omega}(X) = \omega(X, \cdot)$$

Since $\omega$ is nondegenerate, $\bar{\omega}$ is an isomorphism of vector spaces. The inverse $\bar{\omega}^{-1} : V^* \to V$ can be extended to a multilinear map in the exterior algebra

$$\bar{\omega}^{-1} : \bigwedge^k V^* \to \bigwedge^k V, \quad \bar{\omega}^{-1}(\varphi_1 \wedge \cdots \wedge \varphi_k) = \bar{\omega}^{-1}(\varphi_1) \wedge \cdots \wedge \bar{\omega}^{-1}(\varphi_k)$$
The symplectic structure naturally induces a linear operator on the exterior algebra of $V$:

**Definition 2.1.1.** Let $(V, \omega)$ be a symplectic vector space. The Lefschetz operator $L : \bigwedge^k V^* \to \bigwedge^{k+2} V^*$ is defined as

$$L(\varphi) = \omega \wedge \varphi.$$ 

The Lefschetz operator $L$ has a dual operator defined by the interior product. Let $\varphi \in \bigwedge^k V^*$ and $X \in \bigwedge^l V$ with $2n \geq k \geq l \geq 1$. The interior product

$$\iota : \bigwedge^l V \times \bigwedge^k V^* \to \bigwedge^{k-l} V^*$$

is given by

$$\iota_X \varphi(Z) = \varphi(X \wedge Z), \forall Z \in \bigwedge^l V$$

We also define $\iota_c \varphi := c \varphi$ for $X = c \in \mathbb{R} \cong \bigwedge^0 V$. It is easy to show that

$$\iota_X \iota_Y = \iota_Y \iota_X = (-1)^{|X||Y|} \iota_{X \wedge Y}$$

**Definition 2.1.2.** Let $(V, \omega)$ be a symplectic vector space. The dual Lefschetz operator $\Lambda : \bigwedge^k V^* \to \bigwedge^{k+2} V^*$ is defined as

$$\Lambda \varphi = \iota_{\omega^{-1}(\varphi)}.$$ 

**Remark 2.1.3.** Assume $V^*$ is endowed with a metric $(\ , \ )$ and $e_1, e_2, \ldots, e_{2n}$ is an orthonormal basis of $V^*$. It induces a metric on $\bigwedge^* V^* := \bigoplus_{k \in \mathbb{Z}} \bigwedge^k V^*$ with orthonormal basis

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} | 1 \leq i_1 < i_2 < \cdots < i_k \leq 2n\}$$

The dual Lefschetz operator $\Lambda$ also can be defined as the adjoint operator satisfying

$$(L \alpha, \beta) = (\alpha, \Lambda \beta)$$

for any $\alpha, \beta \in \bigwedge^* V^*$.

The other important linear operator is the symplectic star operator $*_{\omega} : \Omega^p \to \Omega^{2n-p}$. It is defined as

$$*_{\omega} \varphi = \iota_{\omega^{-1}(\varphi)} \frac{\omega^n}{n!}.$$
It is easy to show that
\[ *_{s}(\varphi \wedge \varrho) = t_{\bar{\omega}^{-1}((\varphi \wedge \varrho))} \omega^{n} = (-1)^{pq} t_{\bar{\omega}^{-1}(\varrho)} (t_{\bar{\omega}^{-1}(\varphi)}(*_{s} \varphi)) \]
for any \( \varphi \in \bigwedge^{p} V^{*} \) and \( \varrho \in \bigwedge^{q} V^{*} \).

**Canonical basis**

A canonical basis of a symplectic vector space \((V, \omega)\) is a basis \(X_{1}, Y_{1}, \ldots, X_{n}, Y_{n}\) of \(V\) with dual basis \(x_{1}, y_{1}, \ldots, x_{n}, y_{n}\) \(\in V^{*}\) satisfying
\[ x_{i}(Y_{j}) = 0, \quad x_{i}(X_{j}) = y_{i}(Y_{j}) = \delta_{ij} \]
such that \(\omega = \sum x_{i} \wedge y_{i}\). It is an easy exercise that
\[ \frac{\omega^{n}}{n!} = x_{1} \wedge y_{1} \wedge x_{2} \wedge y_{2} \wedge \cdots \wedge x_{n} \wedge y_{n} =: \prod x \wedge y \]
and
\[ \bar{\omega}^{-1}(y_{i}) = X_{i}, \quad \bar{\omega}^{-1}(x_{i}) = -Y_{i}, \quad \bar{\omega}^{-1}(\omega) = \sum X_{i} \wedge Y_{i}. \]

Since any symplectic vector space possesses a canonical basis, it is useful to give an explicit description of the linear operators in canonical basis.

Let \(\mathcal{I} := \{1, 2, \ldots, n\}\). For any \(I \subset \mathcal{I}\), we use \(I^{c} := \mathcal{I} - I\) to denote the complement. For \(I = \{i_{1}, i_{2}, \ldots, i_{k}\}\), we define
\[ (x \wedge y)_{I} := x_{i_{1}} \wedge y_{i_{1}} \wedge \cdots \wedge x_{i_{k}} \wedge y_{i_{k}}, \]
\[ x_{I} := x_{i_{1}} \wedge \cdots \wedge x_{i_{k}}, \]
\[ y_{I} := y_{i_{1}} \wedge \cdots \wedge y_{i_{k}}. \]

Given mutually disjoint subset \(I, I', I'' \subset \mathcal{I}\), we define \(J = I \cup I' \cup I''\) and use the following notation to denote the monomial
\[ c_{I, I', I''} := (x \wedge y)_{I} \wedge x_{I'} \wedge y_{I''} \]
It is clear that \(\{c_{I, I', I''}\}\) generates the vector space \(\bigwedge^{*} V^{*}\). Hence we can use it to show that
Lemma 2.1.4. For any $\alpha \in \wedge^q V^*$,
\[(\Lambda L - L\Lambda)\alpha = (n - q)\alpha.\] (2.1)

Proof. It is enough to show the identity for monomials when $\omega$ is canonical. Let $q = |c_{I',I''}| := \deg(c_{I',I''}) = 2|I| + |I'| + |I''|$. From the following relations
\[
Lc_{I',I''} = \omega \wedge c_{I',I''} = \sum_{j \in J^c} (x \wedge y)_{I + \{j\}} \wedge x' \wedge y''
\]
\[
\Lambda c_{I',I''} = \sum_{k \in I} (x \wedge y)_{I - \{k\}} \wedge x' \wedge y''
\]
we have
\[
\Lambda Lc_{I',I''} = \sum_{j \in J^c} \sum_{k \in (I + \{j\})^c} (x \wedge y)_{I + \{j\} - \{k\}} \wedge x' \wedge y''
\]
\[
= \sum_{k \in I, j \in J^c} (x \wedge y)_{I - \{k\} + \{j\}} \wedge x' \wedge y'' + |J^c|c_{I',I''}
\]
\[
L\Lambda c_{I',I''} = \sum_{k \in I} \sum_{j \in (I - \{k\})^c} (x \wedge y)_{I - \{k\} + \{j\}} \wedge x' \wedge y''
\]
\[
= \sum_{k \in I, j \in J^c} (x \wedge y)_{I - \{k\} + \{j\}} \wedge x' \wedge y'' + |I|c_{I',I''}
\]

So
\[
(\Lambda L - L\Lambda)c_{I',I''} = (|J^c| - |I|)c_{I',I''} = (n - |I| - |I'| - |I''| - |I|)c_{I',I''} = (n - q)c_{I',I''}
\]

We introduce another basic elements of $\wedge^* V^*$ now. Assume $I = (i_1, i_2, \cdots, i_{2k})$, $i_j \in (I' \cup I'')^c$ is an ordered set with even elements. We define
\[
a_{p,(i_1,i_2),\cdots,(i_{2k-1},i_{2k}),I',I''} := \sum_{M \cap J = \emptyset, |M| = p} (x \wedge y)_M \wedge \left(\bigwedge_{j=1}^{k} (x_{i_{2j-1}} \wedge y_{i_{2j-1}} - x_{i_{2j}} \wedge y_{i_{2j}})\right) \wedge x' \wedge y''
\]
and
\[
B := \{a_{p,(i_1,i_2),\cdots,(i_{2k-1},i_{2k}),I',I''}, I', I'' \subset I \text{ mutually disjoint, } |I| : \text{ even}\}
For instance,

\[ a_{0,(1,3),(2,5),(4),(6)} = (x_1 \wedge y_1 - x_3 \wedge y_3) \wedge (x_2 \wedge y_2 - x_5 \wedge y_5) \wedge x_4 \wedge y_6 \]

If \( \varphi = a_{0,(i_1,i_2),\cdots,(i_{2k-1},i_{2k}),I',I''} \), we also use \( a_{p,\varphi} \) to denote \( a_{p,(i_1,i_2),\cdots,(i_{2k-1},i_{2k}),I',I''} \) and sometimes \( I, I' \) and \( I'' \) are denoted as \( I_{\varphi}, I'_{\varphi}, I''_{\varphi} \) to emphasize their relations. Note that \( a_{0,\varphi} \) is defined for \( 0 \leq p \leq n - |\varphi| \).

**Lemma 2.1.5.** Let \( (V,\omega) \) be a symplectic vector space of dimension \( 2n \) with a canonical basis \( X_1,Y_1,\cdots,X_n,Y_n \) and a dual basis \( x_1,y_1,\cdots,x_n,y_n \). Let \( \varphi = a_{0,\varphi} \in B \) and \( q = |\varphi| = |I_{\varphi}| + |I'_{\varphi}| + |I''_{\varphi}| = |J| \). We also define \( a_{-1,\varphi} = a_{n-q+1,\varphi} = 0 \).

(1) \[
L a_{p,\varphi} = (p + 1)a_{p+1,\varphi} \\
A a_{p,\varphi} = (n - p - q + 1)a_{p-1,\varphi}
\]

In particular, \( a_{p,\varphi} = \frac{1}{p!} L^p \varphi \).

(2) \[
\ast_s c_{I',I'',I'} = (-1)^{\frac{(|I'| + |I''|)(|I'| + |I''| - 1)}{2}} c_{J_{\varphi},I',I''} \\
\ast_s a_{p,\varphi} = (-1)^{\frac{q(q-1)}{2}} a_{n-p-q,\varphi}
\]

**Proof.** (1) \[
L a_{p,\varphi} = \sum_{M \cap J = \emptyset, |M| = p} \omega \wedge (x \wedge y)_M \wedge \varphi \\
= \sum_{M \cap J = \emptyset, |M| = p} \sum_{k \in (M \cup J)^c} (x \wedge y)_k \wedge (x \wedge y)_M \wedge \varphi \\
= \sum_{M_1 \cap J = \emptyset, |M_1| = p+1} (p + 1)(x \wedge y)_{M_1} \wedge \varphi \\
= (p + 1)a_{p,\varphi}
\]
By (2.1),
\[
\Lambda L^p \varphi = (LAL)^{p-1} \varphi + (n - 2p + 2 - q)L^{p-1} \varphi
\]
= \[L(LAL)^{p-2} \varphi + (n - 2p + 4 - q)L^{p-2} \varphi\] + \[\cdots\] + \[(n - 2p + 2 - q)L^{p-1} \varphi\]
= \[L^p \Lambda \varphi + (n - q)L^{p-1} \varphi + (n - q - 2)L^{p-1} \varphi + \cdots + (n - q - 2p + 2)L^{p-1} \varphi\]
= \[(pn - pq - p(p - 1))L^{p-1} \varphi\]

So
\[
\Lambda a_{p, \varphi} = \frac{1}{p!} \Lambda L^p \varphi = \frac{n - q - p + 1}{(p - 1)!} L^{p-1} \varphi = (n - p - q + 1)L^{p-1} \varphi
\]

(2) It is easy to show that \(\omega^{-1}(c_{I', I''}) = (-1)^{|I'|}(X \wedge Y)_I \wedge Y_{I'} \wedge X_{I''}\).

\[
*_{s}c_{I', I''}((X \wedge Y)_{J_c} \wedge X_{I'} \wedge Y_{I''})
\]
= \[\omega^{-1}(c_{I', I''}) \prod x \wedge y((X \wedge Y)_{J_c} \wedge X_{I'} \wedge Y_{I''})\]
= \[(-1)^{|I'|} \prod x \wedge y((X \wedge Y)_I \wedge Y_{I'} \wedge X_{I''} \wedge (X \wedge Y)_{J_c} \wedge X_{I'} \wedge Y_{I''})\]
= \[(-1)^{|I'|} \prod x \wedge y((X \wedge Y)_{J_c+I} \wedge X_{I'} \wedge Y_{I''} + (X \wedge Y)_{J_c} \wedge X_{I'} \wedge Y_{I''})\]
= \[(-1)^{|I'| + |I''| + |J_c| + |I'|-1} \frac{|I'| + |I''| - 1}{2} \prod x \wedge y((X \wedge Y)_J \wedge (X \wedge Y)_{I'} \wedge (X \wedge Y)_{I''})\]
= \[(-1)^{|I'| + |I''| + |J_c| + |I'|-1} \frac{|I'| + |I''| - 1}{2} c_{J_c, I', I''}\]

Since \(\omega^n\) is a monomial, we can conclude that
\[
*_{s}(c_{I', I''}) = (-1)^{|I'| + |I''| + |J_c| + |I'|-1} \frac{|I'| + |I''| - 1}{2} \prod x \wedge y((X \wedge Y)_{J_c \wedge X_{I'} \wedge Y_{I''}} = (-1)^{|I'| + |I''| + |J_c| + |I'|-1} \frac{|I'| + |I''| - 1}{2} c_{J_c, I', I''}\]

If we express \(a_{0, \varphi}\) as a linear combination of \(c_{I', I''}\) and choose all \(M \subset J^c\) with \(|M| = p\), we can have
\[
*_{s}(a_{p, \varphi}) = (-1)^{|I'| + |I''| + |J_c| + |I'|-1 + |I|} \frac{a_{n-p-q, \varphi}}{2} = (-1)^{|I'| + |I''| + |J_c| + |I'|-1 + |I|} \frac{|I'| + |I''| - 1 + |I|}{2} a_{n-p-q, \varphi}\]

When \(|I|\) is even, we have \((-1)^{|I'| + |I''| + |J_c| + |I'|-1 + |I|} = (-1)^{\frac{|I'| + |I''| - 1 + |I|}{2}}\). Hence the second assertion is proved.
Assertion (2) can be summarized in the following diagram:

\[
\begin{align*}
L : a_0,\phi & \xrightarrow{1} \ldots \xrightarrow{p-1} a_{p-1},\phi \xrightarrow{p} a_p,\phi \xrightarrow{p+1} a_{p+1},\phi \xrightarrow{p+2} \ldots \xrightarrow{n-q} a_{n-q},\phi \\
\Lambda : a_0,\phi & \xleftarrow{n-q} \ldots \xleftarrow{n-p-q+2} a_{p-1},\phi \xleftarrow{n-p-q+1} a_p,\phi \xleftarrow{n-p-q} a_{p+1},\phi \xleftarrow{n-p-q-1} \ldots \xleftarrow{1} a_{n-q},\phi
\end{align*}
\]

Corollary 2.1.6. 1.

\[
\Lambda^s L^s (a_{p,\phi}) = \frac{(p+s)!}{p!} \cdot \frac{(n-p-q)!}{(n-p-q-s)!} (a_{p,\phi}) \quad (2.2)
\]

\[
L^a \Lambda^s (a_{p,\phi}) = \frac{s!}{(p-s)!} \cdot \frac{(n-p-q+s)!}{(n-p-q)!} (a_{p,\phi}) \quad (2.3)
\]

2.

\[
*_{s,s} = 1. \quad (2.4)
\]

3. \( *_{s} \Lambda = L *_{s} \) or \( \Lambda = *_{s} L *_{s} \).

From Lemma 2.1.5 and Corollary 2.1.6, we can see that the operators \( L, \Lambda \) and \( *_{s} \) have simple formulas in \( B \). If we can show that \( B \) generates \( \wedge^* V^* \), then we have given explicit formulas for these operators.

Lemma 2.1.7. \( \wedge^* V^* \) is generated by \( B \) as a vector space.

Proof. It is enough to show that any \( c_{I,I',I''} \) is a linear combination of elements in \( B \). We will prove it by induction on \( |I| \). Without loss of generality, we can assume \( I' = I'' = \emptyset \) and \( I = \{1, 2, \ldots, k\} \).

If \( I = \emptyset \), we have \( c_{\emptyset,I',I''} = a_{0,I',I''} \). When \( I = \{1\} \),

\[
c_{\{1\},\emptyset,\emptyset} = x_1 \land y_1 = \frac{1}{n} (\omega + \sum_{i=2}^{n} (x_1 \land y_1 - x_i \land y_i)).
\]

Assume the statement is true for \( |I| \leq k \) and

\[
c_{\{2,3,\ldots,k+1\},\emptyset,\emptyset} = \sum_{p,\phi} b_{p,\phi} a_{p,\phi}.
\]

When \( I = \{1, 2, \ldots, k + 1\} \),

\[
c_{I,\emptyset,\emptyset} = (x \land y)_1 \land (\sum_{p,\phi} b_{p,\phi} a_{p,\phi}).
\]
We will use the following equality

\[
(x \land y)_1 = \frac{1}{t} \left( \sum_{j \in I_\varphi} ((x \land y)_1 - (x \land y)_j) + \omega - \omega_\varphi \right)
\]  

(2.5)

where \(\omega_\varphi = \sum_{i \in I_\varphi} (x \land y)_i\) and \(t := n - |I_\varphi|\). Note that (2.5) holds whether \(1 \in I_\varphi\) or not. It is easy to show that \(\varphi \land \omega_\varphi = 0\). If \(1 \notin I_\varphi\), then

\[
\varphi \land (x \land y)_1 = \frac{1}{t} \varphi \land (\sum_{j \notin I_\varphi} ((x \land y)_1 - (x \land y)_j) + \omega - \omega_\varphi)
\]

and \(\varphi \land ((x \land y)_1 - (x \land y)_j) \in \mathcal{B}\). When \(1 \in I_\varphi\) and \((x \land y)_j - (x \land y)_1\) is a factor of \(\varphi\), we use the notation \(\varphi = \varphi' \land ((x \land y)_j - (x \land y)_1)\). Then

\[
\varphi \land (x \land y)_1 = \varphi' \land (x \land y)_1 \land (x \land y)_j.
\]

We can assume \(j = 2\). By the following equality

\[
\begin{align*}
(x \land y)_1 \land (x \land y)_2 &= \frac{1}{t(t+1)} \sum_{i,j \notin I_\varphi, i \neq j} ((x \land y)_1 - (x \land y)_i) \land ((x \land y)_2 - (x \land y)_j) \\
&- \frac{1}{t(t+1)} (\omega - \omega_\varphi')^2 + \frac{1}{t} (\omega - \omega_\varphi') \land ((x \land y)_1 + (x \land y)_2),
\end{align*}
\]

we have

\[
\begin{align*}
\varphi' \land (x \land y)_1 \land (x \land y)_2 &= \frac{1}{t(t+1)} \sum_{i,j \notin I_\varphi, i \neq j} \varphi' \land ((x \land y)_1 - (x \land y)_i) \land ((x \land y)_2 - (x \land y)_j) \\
&- \frac{1}{t(t+1)} \varphi' \land \omega^2 + \frac{1}{t} \varphi' \land \omega \land ((x \land y)_1 + (x \land y)_2).
\end{align*}
\]

Using induction assumption and the result for \(1 \notin I_\varphi\), the assertion is proved. \(\square\)

If \((M, \omega)\) is a symplectic manifold with dimension \(2n\), we can set \(V = T_xM\) and the linear operators discussed in this section are defined on \(\Omega^k(M, \mathbb{R})\). All lemmas above hold for \(\Omega^k(M, \mathbb{R})\) as well.
2.2 Lefschetz decomposition and primitive forms

In the previous section, we show that the set $B$ generates $\wedge^* V^*$. In this section, we will give a decomposition of $\wedge^* V^*$ and show that $B$ is compatible with this decomposition.

Let $\prod_q : \wedge^* V^* \to \wedge^q V^*$ be the projection and $H = \oplus (n - q) \prod_q$. $L, \Lambda$ and $H$ give a representation of the Lie algebra $sl(2)$ acting on $\wedge^* V^*$:

**Lemma 2.2.1.**

\[
[\Lambda, L] = H, \quad (2.6)
\]
\[
[H, \Lambda] = 2\Lambda, \quad (2.7)
\]
\[
[H, L] = -2L \quad (2.8)
\]

**Proof.** The first assertion is proved in Lemma 2.1.4. The other two identities can be shown straightforwardly:

\[
(\Lambda L - LH)c_{I, I', I''} = (n - q)\Lambda c_{I, I', I''} - (n - q + 2)\Lambda c_{I, I', I''} = 2\Lambda c_{I, I', I''}
\]
\[
(HL - LH)c_{I, I', I''} = (n - q - 2)L c_{I, I', I''} - (n - q) L c_{I, I', I''} = -2L c_{I, I', I''}
\]

It gives the other way to decompose $\wedge^* V^*$. Let $I^k$ be the direct sum of irreducible $sl(2)$-representation of dimension $k + 1$. Then $\wedge^* V^* = \bigoplus I^k$. For $a_{0, \varphi} \in B$, Lemma 2.1.5 and Corollary 2.1.6 show that

\[
\Lambda a_{0, \varphi} = 0, L^{n-|\varphi|+1} a_{0, \varphi} = 0, \Lambda^{n-|\varphi|} L^{n-|\varphi|} a_{0, \varphi} = ((n - |\varphi|)!)^2 a_{0, \varphi}
\]

Hence $a_{0, \varphi}, a_{1, \varphi}, \cdots a_{n-|\varphi|, \varphi} \in I^{n-|\varphi|}$. Let

\[
\mathcal{L}^{p, q} = \text{Span}\{a_{p, \varphi} ||\varphi| = q\}
\]

Hence

\[
I^k = \sum_{0 \leq p \leq n-k} \mathcal{L}^{p, k}
\]

The elements of the $sl(2)$ representation with highest weight play an important role in our study of differential forms and their cohomologies.
Definition 2.2.2. $B_q \in \bigwedge^q V^*$ is called primitive if $\Lambda B_q = 0$. Define

$$\mathcal{P} := \{ \varphi \in \bigwedge^* V^* | \varphi : \text{primitive} \}$$

and $\mathcal{P}^q = \mathcal{P} \cap \bigwedge^q V^*$.

Proposition 2.2.3. (1) $\mathcal{P}^k = L^{0,k}$. In particular, $\mathcal{P}^k = 0$ if $k < 0$ or $k > n$.

(2) Each $\alpha \in \bigwedge^k V^*$ can be decomposed uniquely as

$$\alpha = \sum_{r \geq \max\{k-n,0\}} \frac{1}{r!} L^r B^\alpha_{k-2r}$$

for some primitive elements $B^\alpha_{k-2r} \in \mathcal{P}^{k-2r}$.

(3)

$$\Lambda^a L^a|_{L^{p,q}} = \frac{(p+a)!}{p!} \cdot \frac{(n-p-q)!}{(n-p-q-a)!} \text{Id}$$

$$L^a \Lambda^a|_{L^{p,q}} = \frac{p!}{(p-a)!} \cdot \frac{(n-p-q+a)!}{(n-p-q)!} \text{Id}$$

(4) $* : L^{p,q} \to L^{n-p-q,q}$ is an isomorphism and, for $\varphi \in L^{0,q}$,

$$* \varphi = (-1)^{\frac{q(q-1)}{2}} \frac{p!}{(n-p-q)!} L^{n-p-q} \varphi$$

(5) When $p \leq \frac{n-q}{2}$,

$$* = (-1)^{\frac{q(q-1)}{2}} \frac{p!}{(n-p-q)!} L^{n-2p-q}$$

If $p > \frac{n-q}{2}$

$$* = (-1)^{\frac{q(q-1)}{2}} \frac{(n-p-q)!}{p!} \Lambda^{2p-n+q}.$$

This proposition can be proved by Lemma 2.1.5, Corollary 2.1.6 and the theory of $sl(2)$ representations. The decomposition (2.9) is called the Lefschetz decomposition of differential forms.
Since $\dim L^{p,q} = \dim L^{p',q}, 0 \leq p, p' \leq n - q$, we have
\[
\dim L^{p,q} = \dim P^q = \dim \bigwedge^q V^* - \dim \bigwedge^{q-2} V^* = \left(\frac{2n}{q}\right) - \left(\frac{2n}{q-2}\right)
\]

Now we want to generalize this theory to symplectic manifolds. For any point $x$ in a symplectic manifold $(M, \omega)$, the Darboux theorem says that there is a neighborhood $U$ of $x$ symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. Although the generators $c_{I, I', I''}$ and $a_{p, \alpha}$ depend on the choice of local chart, the subspace $L^{p,q}$ and the Lefschetz decomposition are well defined. Hence we can define the Lefschetz decomposition for any smooth forms on $\Omega^*(M)$ and all the above lemmas and propositions hold as well.

### 2.3 $\partial_+, \partial_-$ and $d^A$ operator

In this section, we first give a decomposition of the differential operator $d$ which is compatible with the Lefschetz decomposition. Then we compare the decomposed operators with the symplectic operator $d^A$ studied by Brylinski in [12].

Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. For any primitive form $\alpha \in \Omega^0(M)$, we have
\( \mathcal{P}^q(M) \), the \((q + 1)\)-form \( d\alpha \) has the Lefschetz decomposition (2.9)

\[
d\alpha = B_{q+1}^{d\alpha} + LB_{q-1}^{d\alpha} + \frac{1}{2} L^2 B_{q-2}^{d\alpha} + \cdots.
\]

The decomposition of \( d \)-exact forms has only the first two terms:

**Proposition 2.3.1.** Assume \( \alpha \in \mathcal{P}^q \), then

\[
d\alpha = B_{q+1}^{d\alpha} + \omega \wedge B_{q-1}^{d\alpha}.
\]

Moreover, for \( L^p\alpha \in \mathcal{L}^{p,q} \),

\[
d\alpha = L^p B_{q+1}^{d\alpha} + L^{p+1} B_{q-1}^{d\alpha}.
\]

**Proof.** It is enough to show it locally. Assume a canonical basis in a local chart is given and consider the set of generators \( \mathcal{B} \). If \( \alpha = f a_{0,\varphi} \), then \( d\alpha \) is a \( \mathcal{C}^\infty \)-linear combination of \( a_{0,\varphi} \) and \( x_i \wedge y_i \wedge a_{0,\varphi} \). Applying \( \Lambda \) to each component, we can see that \( \Lambda^2 d\alpha = 0 \). By Corollary 2.1.6, the Lefschetz decomposition of \( d\alpha \) has only the first two terms. The second assertion follows from the easy fact \([d,L] = 0 \). \( \square \)

**Definition 2.3.2.** The maps

\[
\partial_+ : \mathcal{L}^{p,q} \to \mathcal{L}^{p,q+1}, \partial_- : \mathcal{L}^{p,q} \to \mathcal{P}^{p,q-1}
\]

are defined as \( \partial_+(L^p\alpha) = L^p B_{q+1}^{d\alpha} \) and \( \partial_-(L^p\alpha) = L^p B_{q-1}^{d\alpha} \).

It is clear that \( d = \partial_+ + L \partial_- \) and \([\partial_+, L] = [\partial_-, L] = 0 \). From the equation

\[
0 = d^2 = (\partial_+ + L \partial_-)(\partial_+ + L \partial_-)
\]

\[
= \partial_+^2 + L \partial_- \partial_+ + \partial_+ L \partial_- + L \partial_- L \partial_-
\]

\[
= \partial_+^2 + L(\partial_- \partial_+ + \partial_+ \partial_-) + L^2 \partial_-^2
\]

we have \( \partial_+^2 = \partial_-^2 = 0 \) and \( L(\partial_- \partial_+ + \partial_+ \partial_-) = 0 \).

**Remark 2.3.3.** The relation \( L(\partial_- \partial_+ + \partial_+ \partial_-) = 0 \) can be simplified to \( \partial_- \partial_+ + \partial_+ \partial_- = 0 \) except the highest term \( p = n - q \).
In [12], Brylinski defines the operator
\[ d^\Lambda := [d, \Lambda]. \]

\( d^\Lambda \) depends on the symplectic structure and is a differential operator of degree \(-1\).

It is shown in [12] that \((d^\Lambda)^2 = 0\) and \((\Omega^*(M), d^\Lambda)\) defines a cohomology \(H^*_\Lambda(M)\).

Proposition 2.3.1 implies that
\[ d^\Lambda(\mathcal{L}^{p,q}) \subset \mathcal{L}^{p-1,q+1} \oplus \mathcal{L}^{p,q-1} \]

Hence \( d^\Lambda \) is decomposed as \( d^\Lambda = \partial^\Lambda_+ + \partial^\Lambda_- \) with
\[ \partial^\Lambda_+: \mathcal{L}^{p,q} \to \mathcal{L}^{p-1,q+1}, \partial^\Lambda_-: \mathcal{L}^{p,q} \to \mathcal{L}^{p,q-1} \]

The decompositions of \( d \) and \( d^\Lambda \) can be summarized in the following diagram:

\[
\begin{array}{ccc}
\mathcal{L}^{p,q+1} & \to & \mathcal{L}^{p+1,q-1} \\
\downarrow \partial_+ & & \uparrow \partial_- \\
\mathcal{L}^{p,q} & \to & \mathcal{L}^{p-1,q+1} \\
\uparrow \partial^\Lambda_+ & & \downarrow \partial^\Lambda_- \\
\mathcal{L}^{p-1,q+1} & \to & \mathcal{L}^{p,q-1} 
\end{array}
\]

Actually, \( d^\Lambda \) can be described by \( \partial_+, \partial_-, L \) and \( \Lambda \).

**Proposition 2.3.4.** On \( \mathcal{L}^{p,q} \),

1. \( d^\Lambda = (-1)^q * s d * s \).

2. \([d^\Lambda, L] = d, [d^\Lambda, \Lambda] = 0, [d^\Lambda, H] = -d^\Lambda.\)

3. \[
\begin{align*}
\partial^\Lambda_+ & = \frac{1}{n-p-q} \Lambda \partial_+ & 0 \leq p < n - q \\
\partial^\Lambda_+ L^p \varphi & = p L^{p-1} \partial_+ \varphi & p = n - q
\end{align*}
\]

and
\[ \partial^\Lambda_- = (p + q - n - 1) \partial_- . \]
Proof. (1) Assume \( \varphi \) is primitive and \( 0 < p \leq n - q \).

\[
(d \Lambda - \Lambda d)L^p \varphi = d \Lambda L^p \varphi - \Lambda dL^p \varphi \\
= p(n - p - q + 1)dL^{p-1} \varphi - \Lambda L^p \partial_+ \varphi - \Lambda L^{p+1} \partial_- \varphi \\
= p(n - p - q + 1)L^{p-1} \partial_+ \varphi + p(n - p - q + 1)L^p \partial_- \varphi \\
- p(n - p - q)L^{p-1}\partial_+ \varphi - (p + 1)(n - p - q + 1)L^p \partial_- \varphi \\
= pL^{p-1} \partial_+ \varphi + (p + q - n - 1)L^p \partial_- \varphi
\]

On the other hand,

\[
s_1 \Lambda = (d \Lambda - \Lambda d)L^p \varphi = (d \Lambda - \Lambda d)\left( \frac{p!}{(n - p - q)!} L^{n-p-q} \varphi \right) \\
= \left( -1 \right)^{\frac{q(q-1)}{2}} \frac{p!}{(n - p - q)!} L^{n-p-q} \partial_+ \varphi + \left( -1 \right)^{\frac{q(q+1)}{2}} \frac{(n - p - q)!}{(p - 1)!} L^{p-1} \partial_+ \varphi \\
+ \left( -1 \right)^{\frac{q(q-1)(q-2)}{2}} \frac{(n - p - q + 1)!}{p!} L^p \partial_- \varphi \\
= \left( -1 \right)^q pL^{p-1} \partial_+ \varphi + \left( -1 \right)^q L^p \partial_- \varphi \\
= \left( -1 \right)^q \left( pL^{p-1} \partial_+ \varphi + (p + q - n - 1)L^p \partial_- \varphi \right)
\]

When \( p = 0 \), the assertion is given by ignoring all \( \partial_+ \) terms in the above equalities.

(2)

\[
[d^\Lambda, L] = (d \Lambda - \Lambda d)L - L(d \Lambda L - \Lambda d) \\
= d \Lambda L - \Lambda dL - Ld \Lambda + L \Lambda d \\
= d(H + \Lambda) - \Lambda L d - Ld \Lambda + L \Lambda d \\
= dH + d \Lambda L - Ld \Lambda + (L \Lambda - \Lambda L)d \\
= dH - Hd \\
= [d, H]
\]
\[ [d^A, A] = (-1)^{q+1} \ast_s d \ast_s L \ast_s - (-1)^{q+1} \ast_s L \ast_s \ast_s d \ast_s = (-1)^{q+1} \ast_s (dL - Ld) \ast_s = 0 \]

The third one is obvious.

(3) From the proof of part (1), we have

\[
d^A L^p \varphi = \begin{cases} 
  pL^{p-1} \partial_+ \varphi + (p + q - n - 1)L \partial_- \varphi & 0 < p \leq n - q \\
  (q - n - 1)\partial_- \varphi & p = 0
\end{cases}
\]

So the assertion is true for \( \partial_+^A \) with \( p = 0, n - q \) and for \( \partial_-^A \). When \( 0 < p < n - q \), by Proposition 2.2.3(3),

\[
pL^{p-1} \partial_+ \varphi = \frac{1}{p(n-p-q)} \Lambda L \Lambda L^{p-1} \partial_+ \varphi = \frac{1}{(n-p-q)} \Lambda \partial_+ L^p \varphi
\]

\[ \square \]

**Example 2.3.5.** Let \( M = \mathbb{R}^4 / \mathbb{Z}^4 \) be a 4-torus with standard symplectic structure \( \omega = x_1 \wedge y_1 + x_2 \wedge y_2 \). The primitive forms of \( M \) are

\[
\mathcal{P}^0 = \langle 1 \rangle, \quad \mathcal{P}^1 = \langle dx_1, dy_1, dx_2, dy_2 \rangle,
\]

\[
\mathcal{P}^2 = \langle dx_1 \wedge dx_2, dx_1 \wedge dy_2, dy_1 \wedge x_2, dy_1 \wedge dy_2, \sigma \rangle
\]

where \( \sigma = dx_1 \wedge dy_1 - dx_2 \wedge dy_2 \). For \( f \in C^\infty(M) \),

\[
d(fdx_1) = f_{y_1} dy_1 \wedge dx_1 + f_{x_2} dx_2 \wedge dx_1 + f_{y_2} dy_2 \wedge dx_1
\]

\[
= -f_{x_2} dx_1 \wedge dx_2 - f_{y_2} dx_1 \wedge dy_2 - f_{y_1} \left( \frac{1}{2} \omega + \frac{1}{2} \sigma \right)
\]

So

\[
\partial_+(fdx_1) = -f_{x_2} dx_1 \wedge dx_2 - f_{y_2} dx_1 \wedge dy_2 - \frac{1}{2} f_{y_1} \sigma,
\]

\[
\partial_-(fdx_1) = -\frac{1}{2} f_{y_1}.
\]

### 2.4 Primitive cohomologies

If we focus on primitive forms, we have two complexes

\[ 0 \to \mathcal{P}^0 \overset{\partial_+}{\to} \mathcal{P}^1 \overset{\partial_+}{\to} \cdots \overset{\partial_+}{\to} \mathcal{P}^{n-1} \overset{\partial_+}{\to} \mathcal{P}^n \overset{\partial_+}{\to} 0 \]
The primitive cohomologies are defined as
\[ \mathcal{P}H^k(M) = \frac{\ker \partial^+ \cap \mathcal{P}^k}{\partial^+ \mathcal{P}^{k-1}}, \mathcal{P}H^k(M) = \frac{\ker \partial^+ \cap \mathcal{P}^k}{\partial^- \mathcal{P}^{k+1}} \]
for \(0 \leq k < n\). \(\mathcal{P}H^+\) and \(\mathcal{P}H^-\) has the following properties

**Proposition 2.4.1.** \cite{48}

1. \(\dim \mathcal{P}H^k_{\pm} \leq \infty\) for \(0 \leq k \leq n-1\).

2. For \(k = 0, 1\), \(\mathcal{P}H^k_+(M) \cong H^k(M)\) and \(\mathcal{P}H^k_-(M) \cong H^k_d(M)\).

3. There is a natural isomorphism \(\mathcal{P}H^k_{\pm} \cong \mathcal{P}H^k_{\mp}, \ 0 \leq k \leq n\).

### 2.5 Example: Kodaira-Thurston manifold

The Kodaira-Thurston manifold is the first example of symplectic manifold which is not Kähler. In this section, we will compute some primitive cohomology of this manifold.

The Kodaira-Thurston manifold \(M\) can be viewed as a quotient space \(M = \mathbb{R}^4/\Gamma\) where \(\Gamma\) is a discrete subgroup of \(Diff(\mathbb{R}^4)\). If \((x_1, y_1, x_2, y_2)\) is the coordinate of \(\mathbb{R}^4\), \(\Gamma\) is the group generated by the diffeomorphisms:

\[
\begin{align*}
(x_1, y_1, x_2, y_2) & \rightarrow (x_1 + 1, y_1, x_2, y_2 + \lambda x_2) \\
(x_1, y_1, x_2, y_2) & \rightarrow (x_1, y_1 + 1, x_2, y_2) \\
(x_1, y_1, x_2, y_2) & \rightarrow (x_1, y_1, x_2 + 1, y_2) \\
(x_1, y_1, x_2, y_2) & \rightarrow (x_1, y_1, x_2, y_2 + 1)
\end{align*}
\]

for a fixed \(\lambda \in \mathbb{N}\). A smooth function \(f\) on \(M\) is equivalent to a \(\Gamma\)-invariant function on \(\mathbb{R}^4\) and the differential forms on \(M\) correspond to the \(\Gamma\)-invariant differential forms on \(\mathbb{R}^4\). Note that \(f_{x_1}, f_{y_1}, f_{x_2} + \lambda x_1 f_{y_2}\) and \(f_{y_2}\) are also \(\Gamma\)-invariant. It is easy to see that the forms

\[
dx_1, dy_1, dx_2, dy_2 - \lambda x_1 dx_2
\]

are invariant under \(\Gamma\) and their wedge products generate all differential forms on \(M\).

Assume \(M\) has a symplectic structure

\[
\omega = dx_1 \wedge dy_1 + dx_2 \wedge (dy_2 - \lambda dx_2) = dx_1 \wedge dy_1 + dx_2 \wedge dy_2
\]
$\mathcal{P}^2$ is generated by

$$dx_1 \wedge dx_2, dx_1 \wedge dy_2, dy_1 \wedge dx_2, dy_1 \wedge dy_2, \sigma = dx_1 \wedge dy_1 - dx_2 \wedge dy_2$$

So the $\partial_\pm$ operator is

$$\partial_+(Pdx_1) = -P_{x_2}dx_1 \wedge dx_2 - P_{y_2}dx_1 \wedge dy_2 - \frac{1}{2}P_y \sigma$$

$$= -(P_{x_2} + \lambda x_1 P_{y_2})dx_1 \wedge dx_2 - P_{y_2}dx_1 \wedge (dy_2 - \lambda y_1 dx_2) - \frac{1}{2}P_y \sigma$$

$$\partial_+(Qdy_1) = -Q_{x_2}dy_1 \wedge dx_2 - Q_{y_2}dy_1 \wedge dy_2 + \frac{1}{2}Q_x \sigma$$

$$= -(Q_{x_2} + \lambda x_1 Q_{y_2})dy_1 \wedge dx_2 - Q_{y_2}dy_1 \wedge (dy_2 - \lambda y_1 dx_2) + \frac{1}{2}Q_x \sigma$$

$$\partial_+(Rdx_2) = R_{x_1}dx_1 \wedge dx_2 + R_{y_1}dy_1 \wedge dx_2 + \frac{1}{2}R_y \sigma$$

$$\partial_+(S(dy_2 - \lambda x_1 dx_2)) = -\lambda Sdx_1 \wedge dx_2 + S_{x_1}dx_1 \wedge (dy_2 - \lambda x_1 dx_2)$$

$$+ S_{y_1}dy_1 \wedge (dy_2 - \lambda x_1 dx_2) - \frac{1}{2}(S_{x_2} + \lambda x_1 S_{y_2}) \sigma$$

$$\partial_-(Pdx_1) = -\frac{1}{2}P_y \sigma$$

$$\partial_-(Qdy_1) = \frac{1}{2}Q_x \sigma$$

$$\partial_-(Rdx_2) = -\frac{1}{2}R_y \sigma$$

$$\partial_-(S(dy_2 - \lambda x_1 dx_2)) = \frac{1}{2}(S_{x_2} + \lambda x_1 S_{y_2})$$

$$\partial_-(Adx_1 \wedge dx_2) = A_{y_2}dx_1 - A_{x_1}dx_2$$

$$\partial_-(Bdx_1 \wedge (dy_2 - \lambda x_1 dx_2)) = -(B_{x_2} + \lambda x_1 B_{y_2})dx_1 - B_{y_1}(dy_2 - \lambda x_1 dx_2)$$

$$\partial_-(Cdy_1 \wedge dx_2) = C_{y_2}dy_1 + C_{x_1}dx_2$$

$$\partial_-(Ddy_1 \wedge (dy_2 - \lambda x_1 dx_2)) = -(D_{x_2} + \lambda x_1 D_{y_2})dy_1 - \lambda Ddx_2 + D_{x_1}(dy_2 - \lambda x_1 dx_2)$$

$$\partial_-(E\sigma) = -E_{x_1}dx_1 - E_{y_1}dy_1 + (E_{x_2} + \lambda x_1 E_{y_2})dx_2 + E_{y_2}(dy_2 - \lambda x_1 dx_2)$$

If $\Psi = Adx_1 \wedge dx_2 + Bdx_1 \wedge (dy_2 - \lambda x_1 dx_2) + Cdy_1 \wedge dx_2 + Ddy_1 \wedge (dy_2 - \lambda x_1 dx_2) + E\sigma$,.
then

\[ \partial_- \Psi = (A_{y_2} - (B_{x_2} + \lambda x_1 B_{y_2}) - E_{x_1}) dx_1 \]  
\[ + (C_{y_2} - (D_{x_2} + \lambda x_1 D_{y_2}) - E_{y_1}) dy_1 \]  
\[ + (-A_{y_1} + C_{x_1} - \lambda D + (E_{x_2} + \lambda x_1 E_{y_2})) dx_2 \]  
\[ + (-B_{y_1} + D_{x_1} + E_{y_2})(dy_2 - \lambda x_1 dx_2) \]  

(2.11)

(2.12)

(2.13)

(2.14)

The computation of \( PH^* (M) \) is similar to the computation of de Rham cohomologies. The main idea is to reduce the number of variables and to use induction. Let \( f \in C^\infty (\mathbb{R}^4) \) be a \( \Gamma \)-invariant function. We define

\[ \bar{f}(x_1, x_2, y_2) := \int_0^1 f(x_1, t, x_2, y_2) dt, \]

and

\[ g(x_1, y_1, x_2, y_2) = \int_0^{y_1} f(x_1, t, x_2, y_2) - \bar{f}(x_1, x_2, y_2) dt \]

Then \( g_{y_1} = f - \bar{f} \). Moreover, \( g \) is \( \Gamma \)-invariant and \( \bar{f} \) is independent of \( y_1 \). So \( \partial_- (-2g dx_1) = f - \bar{f} \) and \([f] = [\bar{f}] \in PH^0 (M) \). Hence we can assume \( f = f(x_1, x_2, y_2) \) depends on \( x_1, x_2 \) and \( y_2 \) only. Now we consider another description of \( M \). If \( N \) is a smooth manifold and \( \phi : N \to N \) is a diffeomorphism, the mapping torus of \( N \) is the quotient space

\[ N_\phi := N \times [0, 1] / (0, x) = (1, \phi(x)). \]

So \( M \) can be viewed as a mapping torus of \( T^3 \) with diffeomorphism \( \phi(y_1, x_2, y_2) = (y_1, x_2, y_2 + \lambda x_2) \). Let \( \rho : [0, 1] \to [0, 1] \) be a smooth increasing function whose value is 0 near 0 and 1 near 1. Now we define

\[ \hat{f}(x_1, x_2) := \int_0^1 f(x_1, x_2, s) ds \]

\[ h_1(x_1, x_2, y_2) := \int_0^{y_2} \left( f(x_1, x_2, t) - \hat{f}(x_1, x_2) \right) dt \]

\[ h_2(x_1, x_2, y_2) := \int_0^{y_2} \left( f(x_1, x_2, t) - \hat{f}(x_1, x_2) \right) dt \]

and

\[ h(x_1, x_2, y_2) = (1 - \rho(x_1)) h_1(x_1, x_2, y_2) + \rho(x_1) h_2(x_1, x_2, y_2) \]
on $M$. It is easy to see that $h_{1,y_2} = h_{2,y_2} = h_{y_2} = f - \hat{f}$. So we can assume that $f$ is independent of $y_2$. Similar to the reduction of $y_1$, we can eliminate $x_1, x_2$ and assume that $f = c$ is a constant. Hence $\mathcal{P}H^0(M) \cong \mathbb{R}$.

Note that the elimination of $x_1$ and $x_2$ works when $f$ is independent of $y_2$. Let $Pdx_1 + Qdy_1 + Rdx_2 + S(dy_2 - \lambda x_1 dx_2)$ be a $\partial$-closed 1-form where $P, Q, R$ and $S$ are $\Gamma$-invariant functions of $x_1, y_1, x_2, y_2$. So it satisfies

$$-P_{y_1} + Q_{x_1} - R_{y_2} + S_{x_2} + \lambda x_1 S_{y_2} = 0 \quad (2.15)$$

We want to reduce the variables of these functions. Here we will write down the variables to emphasize the dependence of variables and function. For example, $P = P(x_1, y_2)$ means that $P$ is independent of $y_1$ and $x_2$. We consider the formula (2.11), (2.12), (2.13), (2.14).

1. Use $A_{y_2}, C_{y_2}, E_{y_2}$ to eliminate $y_2$ in $P, Q, S$. By (2.15), $R_{y_2}$ is independent of $y_2$ and so is $R$.

2. Use $B_{y_1}, D_{x_1}$ to get $S = S(x_2)$.

3. Use $E_{y_1}$ to get $Q = Q(x_1, x_2)$.

4. Use $A_{y_1}$ to get $R = R(x_1, x_2)$. By (2.15), $P_{y_1}$ and $P$ are independent of $y_1$ and $P = P(x_1, x_2)$.

5. Use $E_{x_1}$ to get $P = P(x_2)$.

6. Use $C_{x_1}$ to get $R = R(x_2)$. By (2.15), $Q_{x_1}$ and $Q$ are independent of $x_1$ and $Q = Q(x_2)$. So (2.15) becomes $S_{x_2} = 0$ and $S =$ constant.

7. Use $B_{x_2}$ to get $P =$ constant.

8. Use $D_{x_2}$ to get $Q =$ constant.

9. Use $E_{x_2}$ to get $R =$ constant. Then use $\lambda D$ to eliminate $R$.

Hence $\mathcal{P}H^1_{\perp}(M) = \mathbb{R}^3$. 

Chapter 3

Strong Lefschetz property

The Lefschetz operator $L$ gives an isomorphism

$$L^k : \Omega^{n-k}(M) \to \Omega^{n+k}(M)$$

$$\alpha \mapsto \alpha \wedge \omega^k$$

Moreover, the commutativity of $d$ and $L$ induces a map

$$\mathcal{L}^k : H^{n-k}(M) \to H^{n+k}(M)$$

$$[\alpha] \mapsto [\alpha \wedge \omega^k]$$

By Poincare duality, $H^{n-k}(M)$ and $H^{n+k}(M)$ have the same dimension. If $\mathcal{L}^k$ is an isomorphism for any $k$, we say that $M$ satisfies the strong Lefschetz property. It is well known that this property holds for K"ahler manifolds:

**Proposition 3.0.1** (Hard Lefschetz property). *If $M$ is a K"ahler manifold and $\omega$ is the K"ahler form, then the map

$$\mathcal{L}^k : H^{n-k}(M) \to H^{n+k}(M)$$

$$[\alpha] \mapsto [\alpha \wedge \omega^k]$$

is an isomorphism for any $0 \leq k \leq n$.*

In general, the strong Lefschetz property does not hold for symplectic manifolds. Actually, it is one of the main tool to show that a symplectic manifold is not K"ahler.
There are several concepts strongly relating to the strong Lefschetz property. The properties of symplectic and primitive harmonic forms are given in Section 3.1. Section 3.2 discusses the $dd^\Lambda$-lemma and $\partial_+\partial_-$-lemma. In Section 3.3 we reprove that the strong Lefschetz property has equivalent conditions given in the first two sections. In Section 3.4 we show that the existence of Lefschetz decomposition for de Rham cohomology is characterized by the strong Lefschetz property. In Section 3.5 we use the symplectic cohomologies to discuss the failure of the strong Lefschetz property.

### 3.1 Symplectic and primitive harmonic forms

In Hodge theory, one key ingredient is the existence of harmonic forms. In a symplectic manifold, a smooth form $\alpha$ is called symplectic harmonic if $d\alpha = d^\Lambda \alpha = 0$. A symplectic harmonic form can be characterized by its Lefschetz decomposition:

**Proposition 3.1.1.** For any $\alpha \in \Omega^k(M)$, the following conditions are equivalent:

(i) $\alpha$ is symplectic harmonic;

(ii) each primitive term of the Lefschetz decomposition of $\alpha$ is $\partial_+\partial_-$ and $\partial_+\partial_-$-closed.

**Proof.** (ii)$\Rightarrow$(i) is obvious. So we only need to show (i)$\Rightarrow$(ii). Assume $\alpha$ is symplectic harmonic and

$$\alpha = \alpha_0 + L\alpha_1 + \cdots, \alpha_r \in \mathcal{P}^{k-2r}$$

is the Lefschetz decomposition of $\alpha$. Note that $\alpha_0 = \alpha_1 = \cdots = \alpha_{k-n-1} = 0$ when $k > n$. The Lefschetz decomposition of $d\alpha$ is

$$d\alpha = (\partial_+ + L\partial_-)(\alpha_0 + L\alpha_1 + \cdots) = \partial_+\alpha_0 + L(\partial_-\alpha_0 + \partial_+\alpha_1) + L^2(\partial_-\alpha_1 + \partial_+\alpha_2) + \cdots$$

$\alpha$ is $d$-closed implies that

$$\partial_+\alpha_0 = 0, \partial_-\alpha_{r-1} = -\partial_+\alpha_r, r = 1, 2, \cdots.$$ 

On the other hand, $d^\Lambda \alpha = 0$ implies that

$$\partial_+^L\alpha_0 = 0, \partial_+^L L_r^{-1} \alpha_{r-1} = -\partial_+^L L_r^\alpha_r, r = 1, 2, \cdots.$$
When $k > n$ and $r = k - n$, the first term of $\alpha$ is $L^{k-n} \alpha_{k-n}$ and $\Lambda^+ L^{k-n} \alpha_{k-n} = 0$. It implies that $\partial_+ \alpha_{k-n} = 0$. In general,

$$\Lambda^+ L^{r-1} \alpha_{r-1} = ((r - 1) + (k - 2r + 2) - n - 1) \partial_- L^{r-1} \alpha_{r-1} = (n - k + r) L^{r-1} \partial_+ \alpha_r$$

and

$$-\partial_+ L^r \alpha_r = -\frac{1}{n - r - (k - 2r)} \Lambda \partial_+ L^r \alpha_r$$

$$= -\frac{(r - 1 + 1)(n - (r - 1) - (k - 2r + 1))}{n - k + r} \partial_+ L^{r-1} \alpha_r$$

$$= -\frac{r(n - k + r)}{n - k + r} \partial_+ L^{r-1} \alpha_r$$

So $\partial_- \alpha_{r-1} = -\partial_+ \alpha_r$ can be nonzero only when $n - k + r = -r$ or $r = \frac{k - n}{2}$, which is impossible.

This equivalent condition play an important role in this chapter.

**Definition 3.1.2.** $\alpha \in \mathcal{P}^k(M)$ is called primitive harmonic if $\partial_+ \alpha = \partial_- \alpha = 0$. In general, $\alpha \in \Omega^k(M)$ is primitive harmonic if each primitive term of its Lefschetz decomposition is primitive harmonic.

### 3.1 $dd^\Lambda$-lemma and $\partial_- \partial_-$-lemma

In symplectic geometry, there is a property similar to the $\partial \bar{\partial}$-lemma in complex geometry.

**Definition 3.2.1.** $M$ has the $dd^\Lambda$-lemma if

$$\text{Im} d \cap \ker d^\Lambda = \text{Im} d^\Lambda \cap \ker d = \text{Im} dd^\Lambda$$

Similarly, we can define the primitive version of $dd^\Lambda$-lemma in the space of primitive forms.

**Definition 3.2.2.** $M$ has the $\partial_+ \partial_-$-lemma if

$$\text{Im} \partial_+ \cap \ker \partial_- \cap \mathcal{P}^k = \text{Im} \partial_- \cap \partial_+ \cap \mathcal{P}^k = \text{Im} \partial_+ \partial_- \cap \mathcal{P}^k, \quad 1 \leq k \leq n - 1 \quad (3.1)$$

$$\text{Im} \partial_+ \cap \ker \partial_- \cap \mathcal{P}^0 = \text{Im} \partial_+ \partial_- \cap \mathcal{P}^0 \quad (3.2)$$

$$\text{Im} \partial_- \cap \ker \partial_+ \cap \mathcal{P}^n = \text{Im} \partial_+ \partial_- \cap \mathcal{P}^n \quad (3.3)$$
These two lemmas hold simultaneously:

**Proposition 3.2.3.** Let \((M, \omega)\) be a symplectic manifold. Then \(M\) has the \(dd^\Lambda\)-lemma if and only if \(M\) has the \(\partial_+ \partial_-\)-lemma.

**Proof.** \(dd^\Lambda\)-lemma \(\Rightarrow\) \(\partial_+ \partial_-\)-lemma:

Assume \(\alpha\) is a primitive harmonic primitive \(k\)-form, i.e. \(\alpha \in \mathcal{P}^k\) and \(\partial_+ \alpha = \partial_- \alpha = 0\). Then \(d\alpha = d^\Lambda \alpha = 0\).

If \(\alpha = \partial_- \gamma = \frac{1}{k-n-1} d^\Lambda \gamma\), then \(\alpha = dd^\Lambda c\) for some \(c = c_0 + Lc_1, c_0, c_1 \in \mathcal{P}. \Rightarrow \alpha = (q - n - 1) \partial_+ \partial_- c_0\).

If \(\alpha = \partial_+ \beta\), then \(d^\Lambda d\beta = d^\Lambda (\alpha + L \partial_- \beta) = 0\). So \(d\beta = dd^\Lambda c\) for some \(c = c_0 + Lc_1 + L^2 c_2\) and \(\partial_- \partial_+ c_0 = \partial_+ \beta = \alpha\).

\(\partial_+ \partial_-\)-lemma \(\Rightarrow\) \(dd^\Lambda\)-lemma:

If \(\alpha = \alpha_0 + L \alpha_1 + \cdots\) and \(d\alpha = d^\Lambda \alpha = 0\), then \(\partial_+ \alpha_i = \partial_- \alpha_i = 0\) for any \(i\). Assume \(\gamma = \gamma_0 + L \gamma_1 + \cdots\) and \(\alpha = d\gamma = (\partial_- \gamma_0 + \partial_+ \gamma_1) + L(\partial_- \gamma_1 + \partial_+ \gamma_2) + \cdots\). Applying \(\partial_+\) to \(\alpha_0\), we have

\[\partial_+ \alpha_0 = \partial_+ \partial_- \gamma_0 = 0\]

So \(\partial_- \gamma_0 \in \text{Im} \partial_- \cap \ker \partial_+\) and \(\partial_- \gamma_0 = \partial_+ \partial_- c_0\). Similarly, we can find \(c_1, c_2, \cdots\) such that \(\partial_- \gamma_i = \partial_+ \partial_- c_i\). Replacing \(c_i\) by \(kc_i\) for some appropriate constant \(k\), we have \(d^\Lambda d(c_0 + Lc_1 + \cdots) = \alpha\).

The argument is the same when \(\alpha = d^\Lambda \gamma\). \(\square\)

**Remark 3.2.4.** The condition (3.2) holds for any compact symplectic manifolds (c.f. proof of Theorem 3.3.2).

### 3.3 Strong Lefschetz property, primitive harmonic representatives and \(\partial_+ \partial_-\)-lemma

In this section, we want to use primitive forms to describe the strong Lefschetz property.

We have the following well known fact

**Theorem 3.3.1.** \([13, 35, 39, 54]\) The following properties are equivalent for a symplectic manifold \(M\):

- \(\text{(a)}\) \(\partial_+ \partial_-\)-lemma
- \(\text{(b)}\) \(dd^\Lambda\)-lemma
- \(\text{(c)}\) The strong Lefschetz property
- \(\text{(d)}\) The primitive harmonic representation property

\(\square\)
1. $M$ satisfies the strong Lefschetz property;
2. The $dd^\Lambda$-lemma holds for $M$;
3. Each cohomology class of $H^*(M)$ has a symplectic harmonic representative.

According to Proposition 3.1.1 and 3.2.3, this theorem can be rephrased as

**Theorem 3.3.2.** The following properties are equivalent for a symplectic manifold $M$:

1. $M$ satisfies the strong Lefschetz property (SLP).
2. The $\partial_+\partial_-$-lemma holds for $M$.
3. Each cohomology class of $H^*(M)$ has a primitive harmonic representative (PH).

In the proofs of Theorem 3.3.1 they use the language of quivers, superalgebras and generalized complex geometry. Here we use more elementary techniques to prove Theorem 3.3.2 directly.

**Proof. $\partial_+\partial_-$-lemma $\Rightarrow$ PH**

Assume $\alpha = \alpha_0 + L\alpha_1 + \cdots$ is closed:

$$0 = d\alpha = \partial_+\alpha_0 + L(\partial_-\alpha_0 + \partial_+\alpha_1) + L^2(\partial_-\alpha_1 + \partial_+\alpha_2) + \cdots$$

Let $\varepsilon_i = \partial_+\alpha_i = -\partial_-\alpha_{i-1}$. By $\partial_+\partial_-$-lemma, there exists $\gamma_i$ such that $\partial_+\partial_-\gamma_i = \varepsilon_i$. Let $\gamma = \gamma_1 + L\gamma_2 + \cdots$ and $\bar{\alpha} = \alpha - d\gamma$. The Lefschetz decomposition of $\bar{\alpha}$ is

$$\bar{\alpha} = \alpha_0 + L\alpha_1 + \cdots - d(\gamma_1 + L\gamma_2 + \cdots)$$

$$= (\alpha_0 - \partial_+\gamma_1) + L(\alpha_1 - \partial_-\gamma_1 - \partial_+\gamma_2) + \cdots + L^i(\alpha_i - \partial_-\gamma_i - \partial_+\gamma_{i+1}) + \cdots$$

Since

$$\partial_+(\alpha_i - \partial_-\gamma_i - \partial_+\gamma_{i+1}) = \partial_+\alpha_i - \partial_+\partial_-\gamma_{i+1} = \varepsilon_i - \varepsilon_i = 0$$

$$\partial_-(\alpha_i - \partial_-\gamma_i - \partial_+\gamma_{i+1}) = \partial_-\alpha_i - \partial_-\partial_+\gamma_i = \varepsilon_{i+1} - \varepsilon_{i+1} = 0$$

$\bar{\alpha}$ is primitive harmonic and $[\bar{\alpha}] = [\alpha - d\gamma] = [\alpha].$
\textbf{PH} \Rightarrow \textbf{SLP}

For any \( \sigma \in H^{2n-k}(M) \), there exists a primitive harmonic representative

\[
\alpha = L^{n-k} \alpha_0 + L^{n-k+1} \alpha_1 + \cdots, \quad \alpha_i \in \mathcal{P}^{k-2i}
\]

So \( \alpha_0 + L\alpha_1 + \cdots \) is closed and \( \sigma = L^{n-k} [\alpha_0 + L\alpha_1 + \cdots] \).

\textbf{SLP} \Rightarrow \partial_+ \partial_-\text{-lemma}

We want to prove it by induction at \( k \). In a symplectic local chart with canonical basis \( (x_1, y_1, \cdots, x_n, y_n) \), the \( \partial_- \) part of \( d\Omega^1(M) \) is contributed by \( dx_i \wedge dy_i \) terms. Note that

\[
ndx_i \wedge dy_i - \omega = (dx_i \wedge dy_i - dx_1 \wedge dy_1) + \cdots + (dx_i \wedge dy_i - dx_n \wedge dy_n)
\]

or

\[
dx_i \wedge dy_i = \frac{1}{n} (\omega + (dx_i \wedge dy_i - dx_1 \wedge dy_1) + \cdots + (dx_i \wedge dy_i - dx_n \wedge dy_n))
\]

If \( \sigma = \sum (p_i dx_i + q_i dy_i) \), then

\[
d\sigma = \sum_i \left( \frac{\partial q_i}{\partial x_i} dx_i \wedge dy_i \right) + \text{other terms}
\]

and

\[
\partial_- \sigma = \sum_i \frac{1}{n} \left( \frac{\partial q_i}{\partial x_i} - \frac{\partial p_i}{\partial y_i} \right).
\]

For any \( f \in \mathcal{P}^0(M) = \Omega^0(M) \), locally,

\[
\partial_- \partial_+ f = \partial_- \left( \sum_i \left( \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial y_i} dy_i \right) \right) = \frac{1}{n} \sum_i \left( \frac{\partial^2 f}{\partial x_i \partial y_i} - \frac{\partial^2 f}{\partial y_i \partial x_i} \right) = 0
\]

So \( \partial_- \partial_+ (\mathcal{P}^0) = 0 \). We also know that

\[
\ker \partial_+ \cap \mathcal{P}^0 = \{ f = c | c : \text{constant} \}
\]

\[
\text{Im} \partial_- \cap \mathcal{P}^0 = \text{Im} d^A \cap \mathcal{P}^0 = \text{Im} \ast_s d \ast_s \cap \mathcal{P}^0 = \ast_s (\text{Im} d \cap \Omega^{2n}(M)) = \{ f \neq \text{constant} \}
\]

Hence

\[
\text{Im} \partial_- \cap \ker \partial_+ \cap \mathcal{P}^0 = 0 = \text{Im} \partial_+ \partial_- \cap \mathcal{P}^0
\]

and the \( \partial_+ \partial_-\text{-lemma} \) holds for \( \mathcal{P}^0(M) \).
Using \( \partial_- \partial_+ (\mathcal{P}^0) = 0 \) again, we have \( \partial_+ \Omega^0(M) \subset \ker \partial_- \) and
\[
\text{Im} \partial_+ \cap \ker \partial_- \cap \mathcal{P}^1 = \partial_+ \Omega^0(M) = \text{Im} \partial_+ \partial_- \cap \mathcal{P}^1
\]
On the other hand, if \( \sigma \in \text{Im} \partial_- \cap \ker \partial_+ \cap \mathcal{P}^1 \) and \( \sigma = \partial_- \eta \), then \( [\partial_- \sigma] = [dL^{n-1} \eta] = 0 \in H^{n-1}(M) \) and SLP implies that \( [\sigma] = 0 \) and \( \sigma \in d\Omega^0(M) \). So \( \partial_+ \partial_- \)-lemma holds for \( \mathcal{P}^1(M) \).

Now assume \( \partial_+ \partial_- \)-lemma holds for \( \mathcal{P}^i(M) \), \( i < k \) and \( \varepsilon \in \mathcal{P}^k \), \( \partial_+ \varepsilon = \partial_- \varepsilon = 0 \).

If \( \varepsilon = \partial_+ \gamma \), then \( [\varepsilon] = [\varepsilon - d\gamma] = [-L \partial_- \gamma] \) and \( \partial_+ \partial_- \gamma = -\partial_- \partial_+ \gamma = -\partial_- \varepsilon = 0 \). By hypothesis, \( \partial_- \gamma = \partial_+ \partial_- \varepsilon \) for some \( c \in \mathcal{P}^{k-2} \). So \( \gamma + \partial_+ c \) satisfies
\[
\partial_- (\gamma + \partial_+ c) = 0, \partial_+ (\gamma + \partial_+ c) = \varepsilon
\]
and \( L^{n-k+1}(\gamma + \partial_+ c) \) is closed. SLP implies that \( [L^{n-k+1}(\gamma + \partial_+ c)] = \mathcal{L}^{n-k+1}[\beta] \) for some closed \( k \)-form \( \beta = \beta_0 + L \beta_1 + \cdots \). Hence \( L^{n-k+1}(\gamma + \partial_+ c - \beta) = dL^{n-k} \xi \) for some \( \xi = \xi_0 + L \xi_1 + \cdots \in \Omega^k(M) \), \( \xi_i \in \mathcal{P}^{k-2i} \). In particular, \( \partial_- \xi_0 + \partial_+ \xi_1 = \gamma + \partial_+ c - \beta_0 \) and \( \partial_+ \partial_- \xi_0 = \partial_+ \gamma = \varepsilon \).

If \( \varepsilon = \partial_- \eta \), then \( [L^{n-k} \varepsilon] = [dL^{n-k-1} \eta] = 0 \) and SLP implies that \( [\varepsilon] = 0 \) as well. So \( \varepsilon = \partial_+ \gamma \) for some \( \gamma \) again.

### 3.4 Lefschetz decomposition for de-Rham cohomology

We are motivated by the Lefschetz decomposition of forms and ask whenever \( H^*(M) \) has a similar structure. The primitive cohomology \( \mathcal{P}H^*_\pm(M) \) seems to be a good fit to play the role of primitive forms in the Lefschetz decomposition. Unfortunately, the situation is not so simple for cohomology. It even does not have a natural map between \( H^*(M) \) and \( \mathcal{P}H^*_\pm(M) \). Let
\[
\Omega^k_h = \{ \alpha \in \Omega^k | d\alpha = d^\Lambda \alpha = 0 \}
\]
and
\[
\Omega^k_{ph} = \{ \alpha \in \Omega^k | \alpha \text{ is primitive harmonic} \}.
\]
By Proposition \ref{prop:3.1.1}, \( \Omega^k_h = \Omega^k_{ph} \). Consider the natural maps
\[
\phi : \Omega^k_h \longrightarrow H^K(M)
\]
\[ \phi_\pm : \Omega^k_{ph} \to H^k(M) \]

Theorem 3.3.2 implies that

\[ SLP \iff \phi \text{ is surjective} \]

Assume the strong Lefschetz property holds now. We define a map

\[ \Phi_+ : \bigoplus_i L^i \mathcal{P}H^{k-2i}_+(M) \to H^k(M) \]

by \( \Phi_+(\sigma) = \phi(\alpha) \) for some \( \alpha \in \Omega^k_{ph} \) such that \( \phi_+(\alpha) = \sigma \). Similar to Theorem 3.3.2, we can show that SLP implies that \( \phi_+ \) is surjective. If \( \alpha = \alpha_0 + L\alpha_1 + \cdots \in \Omega^k_{ph} \) satisfies \( \phi_+(\alpha) = 0 \), there exists \( \gamma_i \in \mathcal{P}^* \) such that \( \partial_+\partial_- \gamma_i = \alpha_i \). Then \( \alpha = d\partial_- (c_0 + Lc_1 + \cdots) \) and \( \phi(\alpha) = 0 \in H^k(M) \). Hence \( \Phi_+ \) is well defined.

On the other hand, if \( \phi(\alpha) = 0 \), there exists \( \gamma = \gamma_0 + L\gamma_1 + \cdots \) such that \( d\gamma = \alpha \), i.e.

\[ \alpha_0 = \partial_+\gamma_0, \quad \alpha_i = \partial_-\gamma_{i-1} + \partial_+\gamma_i. \]

So \( \partial_+\partial_-\gamma_{i-1} = \partial_+\alpha_i - \partial_+\partial_-\gamma_i = 0 \) and \( [\alpha_i] = [\partial_-\gamma_{i-1}] \in \mathcal{P}H^*_+(M) \). By \( \partial_+\partial_- \)-lemma, \( \partial_-\gamma_{i-1} \in \text{Im} \partial_+\partial_- \) and \( \partial_-\gamma_{i-1} = \partial_+\partial_-c_i \) for some \( c_i \in \mathcal{P}^* \). So \( [\alpha_i] = [\partial_+\partial_-c_i] = 0 \in \mathcal{P}H^*_+(M) \) and \( \Phi \) is injective.

This argument works for \( \mathcal{P}H^*_-(M) \) and we have shown:

**Theorem 3.4.1.** Assume a symplectic manifold \((M, \omega)\) satisfies the strong Lefschetz property. Then there are canonical isomorphisms

\[ \Phi_+ : \bigoplus_i L^i \mathcal{P}H^{k-2i}_+(M) \to H^k(M) \]

and

\[ \Phi_- : \bigoplus_i L^i \mathcal{P}H^{k-2i}_-(M) \to H^k(M). \]

### 3.5 Symplectic cohomologies

There are other symplectic cohomology given by the combination of \( d \) and \( d^\Lambda \).
Definition 3.5.1. \[47\]

\[ H^k_{d+d^\Lambda}(M) = \frac{\ker(d + d^\Lambda) \cap \Omega^k(M)}{Imdd^\Lambda \cap \Omega^k(M)} \]

\[ H^k_{dd^\Lambda}(M) = \frac{\ker dd^\Lambda \cap \Omega^k(M)}{(Imd + Imd^\Lambda) \cap \Omega^k(M)} \]

\[ H^k_{d\Lambda}(M) = \frac{\ker(\partial + d^\Lambda) \cap \Omega^k(M)}{\partial \Omega^{k-1}(M) + d^\Lambda \Omega^{k+1}(M)} \]

where \(\tilde{\Omega}^*(M)\) is the space of \(dd^\Lambda\)-closed forms.

In [47], it is showed that all the cohomologies \(H_{d^\Lambda}, H_{d+d^\Lambda}, H_{dd^\Lambda}, H_{d\Lambda}\) are finite dimensional. Furthermore,

**Proposition 3.5.2.** \(H_{d\Lambda}(M) \cong H_{d^\Lambda}(M) \cap H_d(M) \cong H_{d+d^\Lambda}(M) \cap H_{dd^\Lambda}(M)\)

Similarly, we can define the primitive cohomologies for other symplectic cohomologies:

**Definition 3.5.3.** \[47\]

\[ \mathcal{P}H^k_{d+d^\Lambda}(M) = \frac{\ker(\partial_+ + \partial_-) \cap \mathcal{P}^k(M)}{Imd_+d_- \cap \mathcal{P}^k(M)} \]

\[ \mathcal{P}H^k_{dd^\Lambda}(M) = \frac{\ker \partial_+ \partial_- \cap \mathcal{P}^k(M)}{\partial_+ \mathcal{P}^{k-1}(M) + \partial_- \mathcal{P}^{k+1}(M)} \]

\[ \mathcal{P}H^k_{d\Lambda}(M) = \frac{\ker(\partial_+ + \partial_-) \cap \mathcal{P}^k(M)}{\partial_+ \mathcal{P}^{k-1}(M) + \partial_- \mathcal{P}^{k+1}(M)} \]

where \(\tilde{\mathcal{P}}^*(M)\) is the space of \(\partial_+ \partial_-\)-closed forms.

Unlike \(\mathcal{P}H_+\) and \(\mathcal{P}H_-\), they are equal to the ”primitive” part of their corresponding cohomologies:

**Proposition 3.5.4.**

\[ \mathcal{P}H^k_{d+d^\Lambda}(M) \cong \frac{\ker(d + d^\Lambda) \cap \mathcal{P}^k(M)}{Imdd^\Lambda \cap \mathcal{P}^k(M)} \]

\[ \mathcal{P}H^k_{dd^\Lambda}(M) \cong \frac{\ker dd^\Lambda \cap \mathcal{P}^k(M)}{(Imd + Imd^\Lambda) \cap \mathcal{P}^k(M)} \]

\[ \mathcal{P}H^k_{d\Lambda}(M) \cong \mathcal{P}H^k_{d+d^\Lambda}(M) \cap \mathcal{P}H^k_{dd^\Lambda}(M) \]
Hence they have Lefschetz decomposition:

**Theorem 3.5.5.** [47] The cohomologies $H^d + \Lambda$, $H^d \Lambda$ and $H^d \cap \Lambda$ have Lefschetz decomposition:

$$H^k_\ast(M) = \oplus_r L^r \mathcal{P} H^{k-2r}_\ast(M)$$

When we consider other symplectic cohomologies, Theorem 3.5.5 implies that

**Theorem 3.5.6.** [47] The cohomologies $H^d + \Lambda$, $H^d \Lambda$ and $H^d \cap \Lambda$ satisfies the strong Lefschetz properties:

$$L^k : H^{n-k}_\ast(M) \rightarrow H^{n+k}_\ast(M)$$

According to this theorem, it is natural to expect that these cohomologies can detect properties of the symplectic structure related to the strong Lefschetz property. If strong Lefschetz property does not hold, we have the following lemma

**Lemma 3.5.7.** $[\alpha] \in \ker L^k$ if and only if there exist $\varphi \in \partial_\ast \mathcal{P}^{n-k+1} \cap \ker d$ such that $[\varphi] = [\alpha]$.

**Proof.** It is obvious for $k = 0$ and we assume $k > 0$. Let $\alpha = \alpha_0 + L \alpha_1 + L^2 \alpha_2 + \cdots, \alpha_i \in \mathcal{P}^{n-k-2i}$, be the Lefschetz decomposition of $\alpha$. $L^k([\alpha]) = 0$ implies that there exists $\eta = \eta_0 + L \eta_1 + L^2 \eta_2 + \cdots, \eta_i \in \mathcal{P}^{n-k+1-2i}$ such that $dL^{n-k-1}\eta = L^{n-k}\alpha$. We define $\eta' := \eta_1 + L \eta_2 + L^2 \eta_3 + \cdots$ and $\varphi = \alpha - d\eta'$. Then $[\varphi] = [\alpha]$ and it is easy to show that $\varphi = \alpha_0 - \partial_+ \eta_1 = \partial_- \eta_0$.

Conversely, if $\alpha = \partial_- \eta$ for some $\eta \in \mathcal{P}^{n-k+1}$, then $L^k \alpha = L^k \partial_- \eta = dL^{k-1}\eta$ is exact.

The main result of this section is

**Theorem 3.5.8.** Let $(M, \omega)$ be a symplectic manifold. Then $\dim H^k_{d+d\Lambda}(M) \leq \text{rank} \mathcal{L}^{n-k}$.

**Proof.** Define

$$\tilde{H}^k(M) := \{ \alpha \in H^k_d(M) | \alpha \text{ has a harmonic representative} \}$$

Then

$$\tilde{H}^k(M) = \frac{\ker(d + d\Lambda) \cap \Omega^k(M)}{I \tau d \cap \ker(d + d\Lambda) \cap \Omega^k(M)}$$
If $d\alpha \in \text{Im}(d + \Lambda) \cap \Omega^k(M)$, then $0 = (d + \Lambda)d\alpha = dd^\Lambda\alpha$ and $\alpha \in \Omega^{k-1}$. So there is a surjective map
\[ p : \tilde{H}^k(M) \rightarrow H^k_{\text{dr}+\Lambda}(M). \]
Moreover, Lemma 3.5.7 implies that $\ker\mathcal{L}^{n-k} \subset \tilde{H}^k(M)$ and there is a projection
\[ \pi : \tilde{H}^k(M) \rightarrow \tilde{H}^k(M)/\ker\mathcal{L}^{n-k}. \]
For $\alpha \in \ker\mathcal{L}^{n-k}$, by Lemma 3.5.7, we can find $\varphi \in \mathcal{P}^k$ and $\eta \in \mathcal{P}^{k+1}$ such that $[\varphi] = \alpha$ and $\partial_-\eta = \varphi$. So $\eta \in \Omega^{k+1}(M)$ and $p(\alpha) = 0$. Hence there is a map
\[ h : \tilde{H}^k(M)/\ker\mathcal{L}^{n-k} \rightarrow H^k_{\text{dr}+\Lambda}(M) \]
such that $p = h \circ \pi$. $p$ is surjective implies that $h$ is also surjective and
\[ \dim H^k_{\text{dr}+\Lambda}(M) \leq \dim(\tilde{H}^k(M)/\ker\mathcal{L}^{n-k}) \]
So
\[ \dim H^k_{\text{dr}+\Lambda}(M) \leq \dim(\tilde{H}^k(M)/\ker\mathcal{L}^{n-k}) \leq \dim(H^k(M)/\ker\mathcal{L}^{n-k}) = \text{rank}\mathcal{L}^{n-k}. \]
\[ \square \]

This lower bound is sharp for 4-manifolds:

**Corollary 3.5.9.** Let $(M, \omega)$ be a symplectic 4-manifold. Then $\dim H^k_{\text{dr}+\Lambda}(M) = \text{rank}\mathcal{L}^{n-k}$.

**Proof.** It is obvious for $\mathcal{L}^2$. When $k = 1$, $\ker d \cap \Omega^1 = \ker(d + \Lambda) \cap \Omega^1$ implies that $H^1(M) = \tilde{H}^1(M)$. It is clear that $[d\Omega^0] = 0$ in $\tilde{H}^1(M)$ and $[d\Lambda\Omega^2] \subset \ker\mathcal{L}^1$. So $h$ is injective and
\[ \dim H^1_{\text{dr}+\Lambda}(M) = \dim(\tilde{H}^1(M)/\ker\mathcal{L}^1) = \dim H^1(M)/\ker\mathcal{L}^1 = \text{rank}\mathcal{L}^1. \]
To prove it for $\mathcal{L}^0$, we first note that
\[ \ker(d + \Lambda) \cap \Omega^2(M) = \partial_-\mathcal{P}^2 \oplus \mathbb{R}\omega \]
and $(\ker d \cap \ker(d + \Lambda)) \cap \Omega^2(M) = S$ where
\[ S = \{\alpha_0 + L\alpha_1 | \alpha_0 \in \mathcal{P}^2, \alpha_1 \in \mathcal{P}^0, \partial_-\alpha_0 = -\partial_+\alpha_1 \neq 0\} \]
Recall that $\partial_\Omega^1 = \Omega^0 - \mathbb{R}\omega$ and $\mathbb{R}\omega = \ker d \cap \Omega^0$. So $\tilde{\Omega}^1 = 0, \tilde{\Omega}^3 = 0$. and 

$$H^1_{d\cap d^\Lambda}(M) \cong \partial_- \mathcal{P}^2 \oplus \mathbb{R}\omega.$$ 

In the composition of maps 

$$\ker(d + d^\Lambda) \cap \Omega^2 \xrightarrow{p_1} \ker d \cap \Omega^2 \xrightarrow{p_2} H^2(M),$$

$p_1$ is injective and $p_2$ is surjective. Because $d\Omega^1 \subset S$, $p_2 \circ p_1$ is injective. For any $\alpha_0 + L\alpha_1 \in S, \alpha_1 = \partial_- \gamma$ for some $\gamma \in \Omega^1$. Then 

$$[\alpha_0 + L\alpha_1] = [\alpha_0 + L\alpha_1 - d\gamma] = [\alpha_0 - \partial_+ \gamma], \alpha_0 - \partial_+ \gamma \in \ker \partial_- \mathcal{P}^2.$$ 

Hence $p_2 \circ p_1$ is surjective and gives an isomorphism between $H^1_{d\cap d^\Lambda}(M)$ and $H^1(M)$. □
Chapter 4

Symplectic-de Rham spectral sequence

The main objective of this chapter is to focus on spectral sequences. Spectral sequences were first introduced by Leray in 40’s. Since then it has become an important tool in computing homology or cohomology of geometric or algebraic objects. In Section 4.1, we briefly review the theory of spectral sequences and some basic properties.

In complex geometry, Frölicher ([19]) discovered a spectral sequence for any complex manifold which relates the Dolbeault cohomology of complex structures to the de Rham cohomology of smooth manifolds. More precisely, the Frölicher spectral sequence $E_{p,q}^1$ is created by the double complex $(\Omega^\ast(M), \partial, \bar{\partial})$ for a complex manifold $M$. It is well known that $E_{1}^{p,q}$ is isomorphic to $H^p(M, \Omega^q)$ and the spectral sequence $E_{r}^{p,q}$ converges to $H^\ast(M, \mathbb{R})$. The convergence of $E_{r}^{p,q}$ shows that the de Rham cohomology can be decomposed as the direct sum of the Dolbeault cohomology of a complex structure. One special feature is the degeneration of $E_{r}^{p,q}$ in Kähler manifolds. If $M$ is Kähler, the Frölicher spectral sequence degenerates at $E_1$. This is proven by using the Hodge decomposition of differential forms. The other important property of Kähler manifolds is the $\partial\bar{\partial}$-lemma. It states that a $\partial$-closed and $\bar{\partial}$-exact form is also $\partial\bar{\partial}$-exact. We believe that the property can be described purely algebraically in the frame of spectral sequences, which is the main topic in Section 4.2.

Using the operator $d$ and $d^A$, Brylinski defined the canonical spectral sequence for
any manifold with Poisson structures. He showed that this spectral sequence degenerates at $E_1$ when the manifold is symplectic. In Section 4.3, we construct a new spectral sequence for symplectic manifolds and show that it has more subtle degeneration properties than the canonical spectral sequence.

In Section 4.4, we analyze the spectral sequence of the Kodaira-Thurston manifold and a symplectic 6-dimensional nilmanifold.

### 4.1 Spectral sequences and degeneration

In this section, we review the theory of spectral sequences and some of their properties.

**Definition 4.1.1.** A spectral sequence is a collection of differential bi-graded modules $(E^r_{s,t}, d_r)$ such that $d_r$ is of degree $(r, 1-r)$ and $E^r_{s+1,t}$ is isomorphic to $H^p_q(E^r_{s,t}, d_r)$.

The most general approach to construct a spectral sequence is the method of exact couples. For our purpose, we will only consider the spectral sequences which arise from filtered complexes.

**Definition 4.1.2.** A filtered differential graded module is an $\mathbb{N}$-graded module $A = \oplus_{n=0}^{\infty} A^n$ over a field $k$, endowed with a filtration $F$ and a linear map $D : A \to A$ satisfying

1. $d$ is of degree 1: $d(A^n) \subset A^{n+1}$;
2. $d \circ d = 0$;
3. The filtered structure is descending:
   $$A = F^0 A \supseteq F^1 A \supseteq \cdots \supseteq F^p A \supseteq F^{p+1} A \supseteq \cdots$$
4. The map $d$ preserves the filtered structure: $d(F^p A) \subset F^p A$ for any $p$.

Furthermore, if there exists a function $t : \mathbb{N} \to \mathbb{N}$ such that $F^{t(n)} A^n = 0$, $A^*$ is called a bounded filtered differential graded module.

**Theorem 4.1.3.** $[36] (A, d, F)$ defines a spectral sequence $\{E^r_{s,t}, d_r\}$ with $d_r$ of bi-degree $(r, 1-r)$. Moreover,

$$E^s_t \cong H^{s+t}(F^s A/F^{s+1} A)$$

and $E^r_{s,t}$ converges to $H^*(A, d)$ if $(A, d, F)$ is bounded.
More precisely, we have

\[ Z^{s,t} = \{ \varphi \in F^s A^{s+t} | d\varphi \in F^{s+r} A^{s+t+1} \} \]

\[ B^{s,t} = F^s A^{s+t} \cap dF^{s-r} A^{s+t-1} \]

\[ Z^{s,t}_\infty = F^s A^{s+t} \cap \ker d \]

\[ B^{s,t}_\infty = F^s A^{s+t} \cap \text{Im} d \]

\[ E^{s,t}_r = \frac{Z^{s,t}}{Z^{s+1,t-1} + B^{s,t}} \]

(let \( Z^{s+1,t-1} = F^{s+1} A^{s+t} \) and \( B^{s,t}_r = dF^{s+1} A^{s+t+1} \)) and \( d_r : E^{s,t}_r \to E^{s+r,t-r+1}_r \) is induced by \( d : Z^{s,t}_r \to Z^{s+r,t-r+1}_r \).

**Example 4.1.4** (double complexes). Let \( A = \bigoplus A^{s,t} \) be a bi-graded \( \mathbb{C} \)-or \( \mathbb{R} \)-module and \( A^k = \bigoplus_{s+t=k} A^{s,t} \). A filtration on \( A \) is given as

\[ F^p A^q = \{ \varphi = \sum_{s+t=q} \varphi^{s,t} | \varphi^{s,t} = 0 \text{ for } s < p \} \]

Assume there are two maps \( d' : A^{s,t} \to A^{s+1,t} \) and \( d'' : A^{s,t} \to A^{s,t+1} \) satisfying \( d'^2 = d''^2 = 0 \) and \( d'd'' + d''d' = 0 \). The double complex \((A, d', d'')\) defines a filtered complex \((A, d' + d'', F)\) and the corresponding spectral sequence \( \{ E^{s,t}_r \} \).

\[
\begin{array}{ccc}
\ast & \ast \\
\alpha_1 & \overset{d'}{\rightarrow} & \ast \\
\alpha_2 & \overset{d'}{\rightarrow} & \ast \\
\end{array}
\]

In the rest of this section, we assume \( \{ E^{s,t}_r \} \) is given by a double complex \((A, d', d'')\). We are interested in the degeneration of \( E^{s,t}_r \). In particular, we want to explore the relation between \( E^{s,t}_r \) and \( E^{s,t}_{r+1} \).

Consider the natural inclusions

\[ B^{s,t}_0 \subset B^{s,t}_1 \subset \cdots \subset B^{s,t}_\infty \subset Z^{s,t}_\infty \subset \cdots \subset Z^{s,t}_1 \subset Z^{s,t}_0, \]

the inclusion map \( Z^{s,t}_{r+1} \to Z^{s,t}_r \) induces a homomorphism

\[ \alpha_{s,t,r} : \frac{Z^{s,t}_{r+1}}{Z^{s+1,t-1}_{r+1}} \to \frac{Z^{s,t}_r}{Z^{s+1,t-1}_r} \]
Lemma 4.1.5.  

(1) \( \alpha = \alpha_{s,t,r} \) is injective.

(2) If \( \alpha \) is not surjective, then there exists

\[ \xi = \xi_0 + \xi_1 + \cdots + \xi_{r-1} \in Z_{r+1}^{s,t}, \xi_i \in A_{s+i,t-i} \]

such that

(a) \( \xi_0 \neq 0; \)

(b) \( \xi_0 \) is not the leading term of any d-closed form;

(c) \( \xi \) can not extend to the right, i.e. \( \partial \xi_r \in A_{s+r,t-r}, d'' \xi_r = d' \xi_{r-1}. \)

Remark 4.1.6.  

1. (b) implies that \( d' \xi_i \neq 0, i = 0, 1, \ldots, r-1. \)

2. Since \( r - 1 \geq 0, \) we automatically assume \( r \geq 1. \)

Proof. Let \( \xi = \xi_0 + \xi_1 + \cdots \in F^s A^{s+, \xi_i \in A^{s+i,t-i}.} \)

(1) Assume \( \xi \in Z_{r+1}^{s,t}. \) So \( d(\xi_0 + \cdots + \xi_r) = d' \xi_r \in A^{s+r,1,t-r}. \) If \( \alpha([\xi]) = 0, \) then \( \xi \in Z_{r+1}^{s+1,t-1}. \) It means that \( \xi_0 = 0 \) and \( d(\xi_0 + \cdots + \xi_{r-1}) = d' \xi_{r-1} \in A^{s+r,1,t-r+1}. \)

The second condition is true for any \( \xi \in Z_{r+1}^{s,t}. \) The first condition implies that \( \xi \in Z_{r+1}^{s+1,t-1} \) and \( [\xi] = 0. \)

(2) Consider the following decomposition of \( \alpha: \)

\[
\frac{Z_{r+1}^{s,t}}{Z_r^{s+1,t-1}} \xrightarrow{\alpha_1} \frac{Z_r^{s,t}}{Z_r^{s+1,t-1}} \xrightarrow{\alpha_2} \frac{Z_r^{s,t}}{Z_{r-1}^{s+1,t-1}}
\]

It is clear that \( \alpha_1 \) is injective and \( \alpha_2 \) is surjective. If \( \alpha \) is not surjective, \( \alpha_1 \) is not surjective and there exists \( \xi \in Z_{r+1}^{s,t} \) such that \( [\xi] = ([\xi] \mod Z_r^{s+1,t-1}, \alpha_2([\xi]) \notin I \alpha. \) Since \( [\xi] = [\xi_0 + \cdots + \xi_r] \) in \( Z_{r+1}^{s,t}, \) we can assume \( \xi = \xi_0 + \cdots + \xi_r. \)

First note that \( [\xi] \neq 0 \) and \( \xi_0 \neq 0. \) \( \alpha_2([\xi]) \notin I \alpha \) implies that \( [\xi] \notin I \alpha_1, \) which is equivalent to the condition that \( \xi \notin Z_{r+1}^{s,t} \) and \( d \xi \) has a nonzero term at \( A^{s+r,1,t-r+1}, \) i.e.

\[ d \xi = (d' \xi_{r-1} + d'' \xi_r) + d' \xi_r, d' \xi_{r-1} + d'' \xi_r \neq 0 \]

If \( d' \xi_{r-1} = -d'' \xi_r' \) for some \( \xi_r' \in A^{s+r,1,t-r}, \) then

\[ d(\xi - \xi_r + \xi_r') = d' \xi_{r-1} + d'' \xi_r + d' \xi_r' = d' \xi_r \in A^{s+r,1,t-r} \]
and \( \xi - \xi_r + \xi_r' \in Z_{r+1}^{s,t} \). So \( \xi = (\xi - \xi_r + \xi_r') + (\xi_r - \xi_r') \in Z_{r+1}^{s,t} + Z_{r-1}^{s+1,t-1} \) and

\[
\alpha_2([\xi]) = \alpha_2([\xi - \xi_r + \xi_r']) + \alpha_2([\xi_r - \xi_r']) = \alpha_2([\xi - \xi_r + \xi_r']) = \alpha([\xi - \xi_r + \xi_r'])
\]

which is a contradiction.

If there exists a \( d \)-closed form \( \xi' = \xi'_0 + \xi'_1 + \cdots \in F^sA^{s+t}, \xi_i \in A^{s+i,t-i} \) with \( \xi'_0 = \xi_0 \), then \( \xi = (\xi - \xi') + \xi', \xi - \xi' \in Z_{r-1}^{s+1,t-1}, \xi' \in Z_{r+1}^{s,t} \). So

\[
\alpha_2([\xi]) = \alpha_2([\xi - \xi']) + \alpha_2([\xi']) = \alpha_2([\xi']) = \alpha([\xi'])
\]

which is a contradiction again. Hence \( \xi \) satisfies conditions (a)-(c).

The converse of Lemma 4.1.5(2) may not be true, i.e. a homogeneous form \( \xi = \xi_0 + \cdots + \xi_{r-1} \in Z_r^{s,t} \) satisfying the conditions (a)-(c) does not imply that \( (\xi \mod Z_{r-1}^{s+1,t-1}) \notin \text{Im} \alpha \). Suppose \( \xi' = \xi'_0 + \xi'_1 + \cdots \in Z_{r+1}^{s,t} \) and \( \alpha([\xi']) = (\xi \mod Z_{r-1}^{s+1,t-1}) \). Let \( \bar{\xi} = \xi - (\xi'_0 + \cdots + \xi'_{r-1}) \). Then

\[
d\bar{\xi} = d'\xi_{r-1} - d'\xi'_{r-1}.
\]

Since \( d'\xi'_{r-1} = -d''\xi_i' \), \( \xi \) can not extend to the right if and only if \( \bar{\xi} \) can not extend to the right. \( \alpha([\xi']) = (\xi \mod Z_{r-1}^{s+1,t-1}) \) implies that \( \xi'_0 = \xi_0 \). So the first nonzero term of \( \bar{\xi} \) is \( \xi_i - \xi'_i \neq 0 \) for some \( i > 0 \) (such term exists since \( \xi_{r-1} - \xi'_{r-1} \neq 0 \)) and \( \bar{\xi} \) can not extend to the right, i.e. \( \bar{\xi} \) satisfies (a)(c) (with different indexes \( s,t,r \)). If the leading term of \( \bar{\xi} \) is the leading term of some closed form \( \xi'' \), we can consider \( \bar{\xi} - \xi'' \) and have a larger index \( i \). From (c), this process will stop at some \( i \leq r-1 \). Note that \( \bar{\xi} \in Z_{r-i}^{s+i,t-i} \).

In summary,

**Lemma 4.1.7.** If \( \xi = \xi_0 + \cdots + \xi_{r-1} \in Z_r^{s,t} \) satisfying the conditions in (2), then for some \( r - 1 \geq i \geq 0 \), the map

\[
\alpha_{s+i,t-i,r-i} : Z_{r-i}^{s+i,t-i} \rightarrow Z_{r-i}^{s+i,t-i}
\]

is not surjective.

Such forms also can be used to detect the surjectivity of the inclusion of boundaries:
Lemma 4.1.8. If

$$\xi = \xi_0 + \xi_1 + \cdots + \xi_{r-1} \in \mathbb{Z}^{s,t}, \xi_i \in A^{s+i,t-i}$$

such that

(a) $$\xi_0 \neq 0$$;

(b) $$\xi$$ is not the leading term of any $$d$$-closed form;

then the inclusion map $$\beta_{s+r,t-r+1,r} : B_{s+r,t-r+1}^{s+r,t-r+1} \to B_r^{s+r,t-r+1}$$ is not surjective.

Proof. $$d\xi = d'\xi_{r-1} \in A^{s+r,t-r+1} \cap B_{s+r,t-r+1}^{s+r,t-r+1}$$ by definition. Assume $$d'\xi_{r-1} \in B_{s+r,t-r+1}^{s+r,t-r+1}$$, then there exists $$\xi' = \xi'_1 + \xi'_2 + \cdots, \xi'_i \in A^{s+i,t-i}$$ such that $$d\xi' = d'\xi_{r-1}$$. So $$\xi - \xi' = \xi_0 + (\xi_1 - \xi'_1) + \cdots$$ and $$d(\xi - \xi') = 0$$. Hence $$\xi - \xi'$$ is a $$d$$-closed form whose leading term is the same as $$\xi$$, which is a contradiction. \qed

Definition 4.1.9. Let $$\eta = \eta_0 + \eta_1 + \eta_2 + \cdots + \eta_r \in A^{p+q}, \eta_i \in A^{p+i,q-i}$$. $$\eta$$ is call unstable indecomposable if

1. $$d''\eta_0 = 0, d''\eta_{i-1} = -d''\eta_i \neq 0, 1 \leq i \leq r$$

In other words, $$d\eta = d'\eta_r \in A^{p+r+1,q-r}$$ (so $$\eta \in Z_{r+1}^{p,q}$$ and $$d\eta \in B_{r+1}^{p+r+1,q-r}$$).

2. $$\eta_0$$ is not the leading term of any $$d$$-closed form.

Moreover, $$\eta$$ is called maximal if $$\eta + \eta_{r+1}$$ is not unstable indecomposable for any $$\eta_{r+1} \neq 0 \in A^{p+r,q-r}$$. 

These lemmas show that the existence of unstable indecomposable elements can detect when the spectral sequence degenerates.

**Theorem 4.1.10.** The spectral sequence induced by a double complex \((A, d', d'')\) degenerates at \(E_1\) if and only if there are no unstable indecomposable elements.

**Proof.** Assume there exists an unstable indecomposable element 

\[
\xi = \xi_0 + \xi_1 + \cdots + \xi_{r-1} \in \mathbb{Z}^{s,t}, \xi_i \in A^{s+i,t-i}
\]

By Lemma 4.1.7, \(\alpha_{s+i,t-i,r-i}^{s+i,t-i}\) is not surjective for some \(r - 1 \geq i \geq 0\). Hence \(E_{r-i+1}^{s+i,t-i} \not\cong E_{r-i}^{s+i,t-i}\) and \(1 \leq r - i \leq r\).

Conversely, if \(E_r \not\cong E_{r+1}\), then either \(\alpha\) or \(\beta\) is not surjective. If \(\alpha_{s,t,r}\) is not surjective, the existence of unstable indecomposable elements is proven by Lemma 4.1.5.

When \(\beta_{s,t,r}\) is not surjective, there exists \(\xi = \xi_0 + \xi_1 + \cdots, \xi_i \in A^{s-r+i,t+(r-1)-i}\) such that \(d\xi \in F^sA^{s+t}\) and \(d\xi \not\in B_{r-1}^{s,t}\). \(\xi_0 \neq 0\) by assumption. If \(\xi_0\) is the leading term of a closed form \(\xi' = \xi'_0 + \xi'_1 + \cdots\), then \(d(\xi - \xi') = d\xi\) and \(d\xi \in B_{r-1}^{s,t}\), which is a contradiction. So \(\xi\) is an unstable indecomposable element. \(\square\)
4.2 \( d'd''\)-lemma

In Chapter 2, we discuss the concepts of \( \partial_+\partial_-\)-lemma and primitive harmonic forms for symplectic manifolds. They can be generalized to any double complex.

**Definition 4.2.1.** A double complex \((A,d',d'')\) satisfies the \( d'd''\)-lemma if

\[
\text{Im} d' \cap \ker d'' = \text{Im} d'' \cap \ker d' = \text{Im} d'd''
\]

**Definition 4.2.2.** An element \( \varphi \in A^{p,q} \) is called harmonic if \( d' \varphi = d'' \varphi = 0 \).

The existence of harmonic forms can be characterized by the \( d'd''\)-lemma:

**Lemma 4.2.3.** If \((A,d',d'')\) satisfies \( d'd''\)-lemma and \( D = d' + d'' \), then each class in \( H_D(A) \) has a harmonic representative.

**Proof.** Assume \( \alpha = \sum_{i=k}^l \alpha_i \in A^n, \alpha_i \in A^{i,n-i} \) is a \( D \)-closed form, i.e.

\[
d''\alpha_k = 0, d'\alpha_k = -d''\alpha_{k+1}, \ldots, d'\alpha_{l-1} = -d''\alpha_l, d'\alpha_l = 0
\]

Let \( \varepsilon_i = d'\alpha_i = d''\alpha_{i+1} \in A^{i+1,n-i}, i = k, k+1, \ldots, l-1 \). Then \( \varepsilon_i \in \text{Im}d' \cap \ker d'' \cap \text{Im}d'' \cap \ker d' \). By \( d'd''\)-lemma, \( \varepsilon_i = d'd''\gamma_i \) for some \( \gamma_i \in A^{i,n-i-1} \). Consider \( \alpha' = \alpha - D \sum_{i=k}^{l-1} \gamma_i = \sum_{i=k}^{l-1} (\alpha_i - D\gamma_i) + \alpha_l \). It is clear that \([\alpha] = [\alpha'] \in H_D(A)\). Moreover,

\[
d'(\alpha_i - D\gamma_i) = d'\alpha_i - d'd''\gamma_i = \varepsilon_i - \varepsilon_i = 0
\]

\[
d'\alpha_l = 0
\]

\[
d''\alpha_k - D\gamma_k = 0 + \varepsilon_k
\]

\[
d''(\alpha_i - D\gamma_i) = d''\alpha_i - d'd''\gamma_i = -\varepsilon_{i-1} + \varepsilon_i
\]

\[
d'\alpha_l = -\varepsilon_{l-1}
\]

So \( d'\alpha' = 0 \) and

\[
d''\alpha' = \varepsilon_k + (-\varepsilon_k + \varepsilon_{k+1}) + (-\varepsilon_{k+1} + \varepsilon_{k+2}) + \cdots + (-\varepsilon_{l-2} + \varepsilon_{l-1}) - \varepsilon_{l-1} = 0
\]

Hence \( \alpha' \) is a harmonic form in the class \([\alpha]\). \( \square \)

**Theorem 4.2.4.** If \( A \) satisfies \( d'd''\)-lemma, then the spectral sequence of \( A \) degenerates at \( E_1 \).
Proof. It is enough to show that \( d_1 = 0 \). Assume \( \alpha \in A^{p,q} \) is a \( d'' \)-closed element. We want to show that \( d_1[\alpha] = [d'\alpha] = 0 \). Since \( d''d'\alpha = -d'd''\alpha = 0, \alpha \in Im d' \cap \ker d'' \). By \( d'd'' \)-lemma, \( d'\alpha = d'\gamma \) for some \( \gamma \). So \( [\alpha] = [\alpha - d''\gamma] \in H_{p'}(A) \) and \( d_1[\alpha] = [d'(\alpha - d''\gamma)] = 0 \).

4.3 Symplectic-de Rham spectral sequence

Using the Lefschetz decomposition, we can construct a spectral sequence similar to the Frölicher spectral sequence for complex manifolds. Let \( A^{s,t} = \mathcal{L}^{s,t} - s \), i.e. \( s = p, t = p+q \). \( \Omega^*(M) \) is a graded algebra:

\[
A^k = \bigoplus_{s+t=k} A^{s,t} = \bigoplus_{s+t=k} \mathcal{L}^{s,k-2s} = \bigoplus_{2p+q=k} \mathcal{L}^{p,q} = \Omega^k(M)
\]

A filtration on \( \Omega^*(M) \) is given as

\[
F_x^s A = \{ \varphi = \sum \varphi^{s,t} | \varphi^{s,t} = 0 \text{ for } s \geq x \}
\]

The map \( d = \partial_+ + L\partial_- : A^{s,t} \to A^{s,t+1} \oplus A^{s+1,t} \) is a differential operator with degree 1.

**Theorem 4.3.1.** The spectral sequence defined by the double complex \( (A^{s,t}, L\partial_-, \partial_+) \) satisfies:

\[
E_{s,t}^0 = A^{s,t}
\]

\[
E_{s,t}^1 = H_{\partial_+}^{s,t-s}(M) := L^s(\mathcal{P}H_{\partial_+}^{1-s}(M))
\]

In particular,

\[
E_{0,t}^0 \cong E_{0,t+1}^1 \cong \ldots \cong E_{n-t,n}^n
\]

and

\[
E_{1,t}^0 \cong E_{1,t+1}^1 \cong \ldots \cong E_{1,n-t-1,n-1}^n
\]

**Proof.** This is an easy consequence of the definition.
In [12], Brylinski constructed the canonical complexes for Poisson manifolds and showed that the induced spectral sequence degenerates at $E_1$ for symplectic manifolds. By Theorem 4.2.4 we already see that the symplectic-de Rham spectral sequence have different properties from the canonical spectral sequence of Brylinski. When $M$ is a 4-manifold, the degeneration is completely determined by the $d'd''$-lemma or the strong Lefschetz property.

**Proposition 4.3.2.** For a symplectic 4-manifold $M$, the strong Lefschetz property holds if and only if the symplectic-de Rham spectral sequence degenerates at $E_1$.

**Proof.** If $M$ satisfies strong Lefschetz property, the $\partial_+\partial_-$-lemma also holds by Theorem 3.3.1, which implies that the symplectic-de Rham spectral sequence degenerates at $E_1$ by Theorem 4.2.4.

Conversely, assume the strong Lefschetz fails and there is a 1-form $\eta \in \mathcal{P}^1$ such that $[\eta] \neq 0 \in H^1(M)$ but $[L\eta] = 0 \in H_3(M)$. So there exist $\varphi_2 \in \mathcal{P}^2$, $\varphi_0 \in \mathcal{P}^0$ with $L\eta = d(\varphi_2 + L\varphi_0) = L(\partial_-\varphi_2 + \partial_+\varphi_0)$. We may replace $\eta$ by $\eta - \partial_+\varphi_0$ and assume $\varphi = \partial_-\varphi_2$. Since $\partial_+\varphi_2 = 0$ and $\eta \notin \partial_+\mathcal{P}^0$, $\varphi_2$ is an unstable indecomposable element. So the spectral sequence does not degenerate at $E_1$ by Theorem 4.1.10.

For higher dimensional symplectic manifolds, we hope that the $\partial_+\partial_-$-lemma is also equivalent to the $E_1$-degeneration of the symplectic-de Rham spectral sequence.

### 4.4 Examples

In general, the computation of de Rham cohomology is very complicated. We usually compute other cohomology and use equivalent conditions or dualities to show that they are isomorphic to each other. For nilmanifolds, Nomizu’s theorem shows that their de Rham cohomology is isomorphic to the Chevalley-Eilenberg cohomology of their corresponding nilpotent Lie algebra.

In this section, we consider two nilmanifolds and analyze some cohomologies and symplectic-de Rham spectral sequence given by the Chevalley-Eilenberg complex. In our cases, they are equivalent to the corresponding cohomologies and symplectic-de Rham spectral sequence of the manifolds. For general nilmanifolds, we need to find a Nomizu type theorem for these cohomologies. It will be discussed elsewhere.
4.4.1 Four dimensional nilpotent Lie algebra

Consider the nilpotent Lie algebra $\mathfrak{ch}_3$ in [11]. It is given by the formulas

$$de_1 = de_2 = de_4 = 0, de_3 = e_{14}$$

So $de_{23} = -e_{124}$ and the symplectic-de Rham spectral sequence for $\mathfrak{ch}_3$ is

$$E_0 : \begin{array}{c}
e_{13}, e_{14}, e_{23}, e_{24}, \sigma \\
e_1, e_2, e_3, e_4 \\
L^2 = \omega^2 \\
L_1 = \omega \\
1 \end{array}$$

$$E_1 : \begin{array}{c}
e_{13}, e_{23}, e_{24}, \sigma \\
e_1, e_2, e_4 \\
L^2 = \omega^2 \\
L_1 = \omega \\
1 \end{array}$$

$$E_2 : \begin{array}{c}
e_{13}, e_{24}, \sigma \\
e_1, e_2, e_4 \\
L^2 = \omega^2 \\
L_1 = \omega \\
1 \end{array}$$

4.4.2 Six-dimensional nilmanifolds

The fine properties of spectral sequence can be seen in high dimension. We consider the same 6-dimensional example mentioned in [18]. $\mathfrak{g}$ is the Lie algebra satisfying

$$de_1 = de_3 = de_6 = 0$$

$$de_4 = e_{15}$$

$$de_5 = e_{13}$$

$$de_2 = e_{14} + e_{35} + e_{36}$$

$\mathcal{P}^2$ is generated by

$$e_{13}, e_{14}, e_{15}, e_{16}, e_{23}, e_{24}, e_{25}, e_{26}, e_{35}, e_{36}, e_{45}, e_{46}, \sigma_1 = e_{12} - e_{34}, \sigma_2 = e_{34} - e_{56}$$
\( \mathcal{P}^3 \) is generated by

\[ e_{135}, e_{136}, e_{145}, e_{146}, e_{235}, e_{236}, e_{245}, e_{246}, \sigma_1 \wedge e_5, \sigma_1 \wedge e_6, \sigma_2 \wedge e_1, (\sigma_1 + \sigma_2) \wedge e_3, (\sigma_1 + \sigma_2) \wedge e_4 \]

The differentials are

\[
\begin{align*}
    de_{13} &= de_{14} = de_{15} = de_{16} = de_{35} = de_{36} = 0 \\
    de_{23} &= -e_{134} = -\frac{1}{2} \sigma_2 \wedge e_1 - L(\frac{1}{2} e_1) \\
    de_{24} &= e_{125} - e_{345} - e_{346} = \sigma_1 \wedge (e_5 + \frac{1}{2} e_6) - L(\frac{1}{2} e_6) \\
    de_{25} &= e_{123} + e_{145} - e_{356} = e_{145} + (\sigma_1 + \sigma_2) \wedge e_3 \\
    de_{26} &= e_{146} + e_{356} = e_{146} - \frac{1}{2} (\sigma_1 + \sigma_2) \wedge e_3 + L(\frac{1}{2} e_3) \\
    de_{45} &= -e_{134} = -\frac{1}{2} \sigma_2 \wedge e_1 - L(\frac{1}{2} e_1) \\
    de_{46} &= e_{156} = -\frac{1}{2} \sigma_2 \wedge e_1 + L(\frac{1}{2} e_1) \\
    d\sigma_1 &= -2e_{135} - e_{136} \\
    d\sigma_2 &= e_{135} - e_{136}
\end{align*}
\]

\[
\begin{align*}
    de_{135} &= de_{136} = de_{145} = de_{146} = d((\sigma_1 + \sigma_2) \wedge e_3) = d(\sigma_2 \wedge e_1) = 0 \\
    de_{235} &= e_{1235} - e_{1345} = L(e_{35} - e_{15}) \\
    de_{236} &= -e_{1346} = L(-e_{16}) \\
    de_{245} &= -e_{1234} + e_{3456} = L(-\sigma_1 - \sigma_2) \\
    de_{246} &= e_{1256} - e_{3456} = L(\sigma_1) \\
    d\sigma_1 \wedge e_5 &= e_{1356} = L(e_{13}) \\
    d\sigma_1 \wedge e_6 &= -2e_{1356} = L(-2e_{13}) \\
    d\sigma_2 \wedge e_2 &= e_{1235} - e_{1236} = L(e_{35} - e_{36}) \\
    d(\sigma_1 + \sigma_2) \wedge e_4 &= e_{1345} + 2e_{1346} = L(e_{15} + 2e_{16})
\end{align*}
\]

Hence we have

\[
\begin{align*}
    E_{1,1}^0 &= \mathcal{PH}_+^1 = \langle [e_1], [e_3], [e_6] \rangle > \\
    E_{1,2}^0 &= \mathcal{PH}_+^2 = \langle [e_{16}], [e_{14} - e_{36}], [e_{35} - e_{36}], [e_{23} - e_{45}], [e_{45} - e_{46}] \rangle > \\
    E_{1,3}^0 &= \mathcal{PH}_+^3 = \langle [e_{235}], [e_{236}], [e_{245}], [e_{246}], [\sigma_1 \wedge e_5], [\sigma_2 \wedge e_2], [(\sigma_1 + \sigma_2) \wedge e_3], [(\sigma_1 + \sigma_2) \wedge e_4] \rangle >
\end{align*}
\]

(Note that \([\sigma_1 \wedge e_5] = -\frac{1}{2} [\sigma_1 \wedge e_6], [(\sigma_1 + \sigma_2) \wedge e_3] = -[e_{145}] = 2[e_{146}]\).

\[
E_{1,3}^1 = L(\mathcal{PH}_+^2) \oplus L < [e_{23} + e_{46}], [e_{24}], [e_{25}], [e_{26}], [\sigma_1], [\sigma_2] >
\]
For $E_2$, we have

$$Z_2^{0,3}/Z_1^{1,2} = <e_{135}, e_{136}, e_{145}, e_{146}, (\sigma_1 + \sigma_2) \wedge e_3, \sigma_2 \wedge e_1, \sigma_1 \wedge (2e_5 + e_6), Le_4 - (\sigma_1 + \sigma_2) \wedge e_4 - 2e_{236}>$$

$$B_1^{0,3} = <e_{135}, e_{136}, Le_1, \sigma_2 \wedge e_1, \sigma_1 \wedge (2e_5 + e_6) - Le_6, e_{145} + (\sigma_1 + \sigma_2) \wedge e_3, e_{146} - \frac{1}{2}(\sigma_1 + \sigma_2) \wedge e_3 + L\left(\frac{1}{2}e_3\right) >$$

$$\Rightarrow E_2^{0,3} = [e_{145}, [\sigma_1 \wedge (2e_5 + e_6)], [Le_4 - (\sigma_1 + \sigma_2) \wedge e_4 - 2e_{236}]] >$$

$$Z_2^{1,3}/Z_1^{2,2} = L < e_{13}, e_{14}, e_{15}, e_{16}, e_{25}, e_{35}, e_{36}, e_{23} - e_{45}, e_{45} + e_{46}, \sigma_1, \sigma_2 >$$

$$B_2^{1,3} = L < e_{13}, e_{15}, e_{16}, e_{35}, e_{36}, \sigma_1, \sigma_2 >$$

$$E_2^{1,3} = L < [e_{14}], [e_{25}], [e_{23} - e_{45}], [e_{45} + e_{46}] >$$

$$E_2^{0,2} = < e_{14}, e_{16}, e_{35}, e_{36}, e_{23} - e_{45} > / (e_{14} + e_{35} + e_{36})$$

$$E_2^{2,3} = L^2 < [e_2], [e_4], [e_5] >$$

The degrees of $E_r$ are:

<table>
<thead>
<tr>
<th>$E_0$ :</th>
<th>14 14 6 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>14 6 1</td>
</tr>
<tr>
<td></td>
<td>6 1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$E_1$ :</td>
<td>8 11 6 1</td>
</tr>
<tr>
<td></td>
<td>5 3 1</td>
</tr>
<tr>
<td></td>
<td>3 1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$E_2 = E_\infty$ :</td>
<td>3 4 3 1</td>
</tr>
<tr>
<td></td>
<td>4 3 1</td>
</tr>
<tr>
<td></td>
<td>3 1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
If we consider the other nonsymplectomorphic symplectic structure, we can change the basis as \((1, 2, 3, 4, 5, 6) \rightarrow (1, 4, 3, 5, 6, 2)\) and have

\[
de_1 = de_2 = de_3 = 0
\]
\[
de_4 = e_{15} - e_{23} + e_{36}
\]
\[
de_5 = e_{16}
\]
\[
de_6 = e_{13}
\]

The differentials are

\[
d e_{13} = d e_{15} = d e_{16} = d e_{23} = d e_{36} = 0
\]
\[
d e_{14} = e_{123} - e_{136} = -e_{136} + \frac{1}{2}(\sigma_1 + \sigma_2) \wedge e_3 + L(\frac{3}{2}e_3)
\]
\[
d e_{24} = e_{125} - e_{236} = -e_{236} + \frac{1}{2}\sigma_1 \wedge e_5 + L(\frac{3}{2}e_5)
\]
\[
d e_{25} = e_{126} = \frac{1}{2}\sigma_1 \wedge e_6 + L(\frac{3}{2}e_6)
\]
\[
d e_{26} = e_{123} = \frac{1}{2}(\sigma_1 + \sigma_2) \wedge e_3 + L(\frac{3}{2}e_3)
\]
\[
d e_{35} = e_{136}
\]
\[
d e_{45} = e_{146} - e_{356} - e_{235} = e_{146} - e_{235} + \frac{1}{2}(\sigma_1 + \sigma_2) \wedge e_3 - L(\frac{1}{2}e_3)
\]
\[
d e_{46} = -e_{134} + e_{156} - e_{236} = -e_{236} - \sigma_2 \wedge e_1
\]
\[
d \sigma_1 = -e_{135}
\]
\[
d \sigma_2 = 2e_{135}
\]
\[
\varphi_0 d e_{135} = d e_{136} = d e_{236} = d(\sigma_2 \wedge e_1) = d((\sigma_1 + \sigma_2) \wedge e_3) = 0
\]
\[
d e_{145} = e_{1235} + e_{1356} = L(e_{35} + e_{13})
\]
\[
d e_{146} = e_{1236} = L(e_{36})
\]
\[
d e_{235} = e_{1236} = L(e_{36})
\]
\[
d e_{245} = e_{1246} - e_{2356} = L(e_{46} - e_{23})
\]
\[
d e_{246} = e_{1256} - e_{1234} = L(-\sigma_2)
\]
\[
d \sigma_1 \wedge e_5 = -e_{1346} = L(-e_{16})
\]
\[
d \sigma_1 \wedge e_6 = -e_{1356} = L(-e_{13})
\]
\[
d \sigma_2 \wedge e_2 = 2e_{1235} = L(2e_{35})
\]
\[
d(\sigma_1 + \sigma_2) \wedge e_4 = -e_{1345} + e_{1236} + e_{2356} = L(-e_{15} + e_{36} + e_{23})
\]

Hence we have

\[
E_1^{0,1} = \mathcal{P}H_+ = \langle [e_1], [e_2], [e_3] \rangle
\]
\[ E_{1,0}^0 = \mathcal{PH}_+^2 = \langle [e_{15} + e_{23}], [e_{23} + e_{36}], [e_{14} + e_{35} - e_{26}], [2\sigma_1 + \sigma_2] \rangle \]

\[ E_{1,0}^{0,3} = \mathcal{PH}_+^2 = \langle [e_{145}], [e_{146}], [e_{236}], [e_{245}], [e_{246}], [\sigma_2 \land e_2], [(\sigma_1 + \sigma_2) \land e_4] \rangle \]

(Note that \([e_{146}] = [e_{235}], [e_{236}] = \frac{1}{2}[\sigma_1 \land e_5] = -[\sigma_2 \land e_1].\)

\[ E_{1,3}^1 = L(\mathcal{PH}_+^2) \oplus L \langle [e_{14}], [e_{24}], [e_{25}], [e_{35}], [e_{45}], [e_{46}], [\sigma_1 - \sigma_2] \rangle \]

For \(E_2\), we have

\[ Z_{2,0}^{0,3} / Z_{1,1}^{1,2} = \langle e_{135}, e_{136}, e_{236}, e_{146} - e_{235}, (\sigma_1 + \sigma_2) \land e_3, \sigma_2 \land e_1, \sigma_1 \land e_5 + Le_5, \sigma_1 \land e_6 + Le_6, \sigma_2 \land e_2 - 2e_{145} + Le_6, (\sigma_1 + \sigma_2) \land e_4 - 2e_{146} + Le_4 \rangle \]

\[ B_{1,0}^{0,3} = \langle e_{135}, e_{136}, -e_{236} + \frac{1}{2}\sigma_1 \land e_5 + L(\frac{1}{2}e_5), \frac{1}{2}\sigma_1 \land e_6 + L(\frac{1}{2}e_6), \frac{1}{2}(\sigma_1 + \sigma_2) \land e_3 + \frac{1}{2}e_3, e_{146} - e_{235} + (\sigma_1 + \sigma_2) \land e_3, -e_{236} - \sigma_2 \land e_1 \rangle \]

\[ \Rightarrow E_{2,0}^{0,3} = \langle e_{236}, [\sigma_2 \land e_2 - 2e_{145} + Le_6], [(\sigma_1 + \sigma_2) \land e_4 - 2e_{146} + Le_4] \rangle \]

\[ Z_{2,0}^{1,3} / Z_{1,1}^{2,2} = L < e_{13}, e_{14} - e_{26}, e_{14} + e_{25}, e_{15}, e_{23}, e_{35}, e_{36}, e_{46}, \sigma_1, \sigma_2 > \]

\[ B_{2,1}^{1,3} = L < e_{13}, e_{16}, e_{35}, e_{36}, e_{46} - e_{23}, e_{15} - e_{23}, \sigma_2 > \]

\[ E_{2,1}^{1,3} = L < e_{15} + e_{23} + e_{46}, [e_{14} - e_{26}], [e_{14} + e_{45}], [\sigma_1] > \]

\[ E_{2,0}^{0,3} = \langle e_{15}, e_{23}, e_{36}, e_{14} - e_{26} + e_{35}, 2\sigma_1 + \sigma_2 > (e_{15} - e_{23} + e_{36}) \rangle \]

\[ E_{2,0}^{2,3} = L^2 < [e_1], [e_2], [e_4] > \]

The degrees of \(E_r\) are:

<table>
<thead>
<tr>
<th></th>
<th>14</th>
<th>14</th>
<th>6</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_0)</td>
<td>14</td>
<td>6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(       )</td>
<td>6</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(       )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Using this example, we can observe that the information revealed by the symplectic-de Rham spectral sequence depends on the choice of symplectic structure. We believe that

**Conjecture 4.4.1.** If $\omega$ and $\omega'$ are symplectic structures of $M$, then the symplectic-de Rham spectral sequence for $(M, \omega)$ and $(M, \omega')$ will degenerate at the same step.
Chapter 5

Luttinger surgery and Kodaira dimension

The main focus of this chapter is on Kodaira dimension and Luttinger surgery. The concept of symplectic Kodaira dimension is reviewed in Section 5.1. We also briefly discuss the classification problems for each Kodaira dimension. In Section 5.2, we define the Luttinger surgery and show some applications. In Section 5.3, we prove the main theorem of this chapter, which shows that Luttinger surgery preserves the symplectic Kodaira dimension. In Section 5.4, we use the classification of symplectic manifolds for each Kodaira dimension to discuss the effect of Luttinger surgery.

5.1 Symplectic Kodaira dimension

Let \((M, \omega)\) be a symplectic manifold. An almost complex structure on \(M\) is a linear map \(J \in TM \otimes T^*M\) such that \(J^2 = -1\). An almost complex structure \(J\) on symplectic 4-manifold \((M, \omega)\) is tamed by \(\omega\) if \(\omega(v, Jv) > 0\) for any \(x \in M\) and \(v \neq 0 \in T_x M\). \(J\) is compatible with \(\omega\) if it is tamed by \(\omega\) and satisfies \(\omega(Ju, Jv) = \omega(u, v)\) for any \(u, v \in T_x M\). Since the space \(\mathcal{J}(M)\) of \(\omega\)-compatible almost complex structures is nonempty and contractible, we can define the first Chern class \(c_1(M, \omega)\) as \(c_1(M, J)\) by choosing any \(J \in \mathcal{J}(M)\). A codimension 2 smooth submanifold \(N\) of \((M, \omega)\) is a symplectic submanifold if \((N, \omega|_N)\) is also a symplectic manifold. A symplectic 4-manifold \(M\) is called symplectically minimal if there is no symplectic sphere \(N\) embedded
in $M$ such that $N^2 = -1$. If $M$ is not symplectically minimal and $S$ is an embedded symplectic $-1$-sphere in $M$, it is shown in [37] that $M$ can be blown down along $S$ to get another symplectic manifold. Since the procedure of blowing down reduces $b_2$ by 1, a minimal symplectic 4-manifold $M'$ will be achieved in finite many steps. $M'$ is called a symplectic minimal model of $M$. If $(M, \omega)$ is minimal, the Kodaira dimension of $M$ is defined by

$$\kappa(M, \omega) = \begin{cases} 
-\infty & K^2_\omega < 0 \text{ or } K_\omega \cdot \omega < 0 \\
0 & K^2_\omega = 0 \text{ and } K_\omega \cdot \omega = 0 \\
1 & K^2_\omega = 0 \text{ and } K_\omega \cdot \omega > 0 \\
2 & K^2_\omega > 0 \text{ and } K_\omega \cdot \omega > 0 
\end{cases}$$

In general, the Kodaira dimension of a non-minimal symplectic manifold is defined as the Kodaira dimension of its minimal model.

**Remark 5.1.1.**

1. The case $K^2_\omega > 0$, $K_\omega \cdot \omega = 0$ is not contained in the list. From Lemma 2.5 in [31], if $(M, \omega)$ is minimal with $K \cdot \omega = 0$ and $K^2_\omega \geq 0$, then $K_\omega$ is a torsion class and hence $K^2_\omega = 0$. We can also observe that $K_\omega$ is a torsion class implies that $K^2_\omega = 0$ and $K_\omega \cdot \omega = 0$. Hence a symplectic 4-manifold $M$ has $\kappa = 0$ exactly when its canonical class is a torsion class.

2. According to [31], the Kodaira dimension of minimal model is well defined. Actually, the minimal model is uniquely determined up to diffeomorphism when $\kappa(M, \omega) \geq 0$. If $M$ has minimal model diffeomorphic to rational or ruled surfaces, the minimal model is not unique. But they still have the same $\kappa$.

3. It is also shown in [31] that $\kappa(M, \omega)$ is independent of choice of symplectic form $\omega$.

4. If $\omega$ is a Kähler form on $M$, the Kodaira dimension of $M$ as a complex surface coincides with the Kodaira dimension $\kappa(M, \omega)$ of $M$ for symplectic 4-manifolds.

5.1.1 $\kappa = -\infty$

A rational symplectic 4-manifold is a symplectic 4-manifold whose underlying smooth manifold is $S^2 \times S^2$ or $\mathbb{C}P^2 \sharp k\overline{\mathbb{C}P^2}$, $k \geq 0$. A ruled symplectic 4-manifold is a symplectic
4-manifold whose underlying smooth 4-manifold is either a sphere bundle over Riemann surface Σ_g, g ≥ 1 or blowups of such manifolds.

**Theorem 5.1.2 ([33]).** A symplectic 4-manifold \((M, \omega)\) has \(\kappa = -\infty\) if and only if it is rational or ruled.

There are some other characterization of such manifolds.

**Theorem 5.1.3 ([37]).** A symplectic 4-manifold \((M, \omega)\) is rational or ruled if and only if \(M\) has an embedded symplectic sphere with non-negative square.

### 5.1.2 \(\kappa = 0\)

As we have mentioned, a minimal symplectic 4-manifold \((M, \omega)\) has \(\kappa = 0\) if and only if \(K_\omega\) is a torsion class. There are some general properties of such symplectic manifolds.

**Theorem 5.1.4 ([30]).**

1. \(2\chi + 3\sigma = 0\)
2. \(M\) has even intersection form.
3. \(K_\omega\) is either trivial, or of order two, which only occurs when \(M\) is an integral homology Enriques surface.
4. \(M\) is spin except when \(M\) is an integral homology Enriques surface.

Kähler surfaces with \(\kappa = 0\), which also have (holomorphic) Kodaira dimension 0, have been classified: K3 surfaces, Enriques surfaces, hyperelliptic surfaces and torus \(T^4\). The first non-Kähler symplectic manifold, the Kodaira-Thurston manifold, also has \(\kappa = 0\). It is an example of \(T^2\) bundle over \(T^2\). In fact, we know that

**Lemma 5.1.5.** Every oriented \(T^2\) bundle over \(T^2\) admits a symplectic form \(\omega\) such that \(K_\omega\) is a torsion class. Moreover, all such manifolds are minimal and with \(\kappa = 0\).

Since \(M\) is a \(K(\pi, 1)\) manifold, there is no homotopy nontrivial sphere in \(M\). Hence \(M\) is minimal. The existence of symplectic structures is given in [20]. If the fiber is homology essential, Thurston’s construction gives rise to a symplectic form such that \(M\) is a symplectic fibration. On the other hand, if the fundamental class of fibers is zero in \(H_2(M, \mathbb{Z})\), \(M\) is a total space of \(S^1\) bundle over a 3-manifold which in turn is a principal \(S^1\) bundle over \(T^2\), and thus admits a symplectic structure by the construction in [15].
No other minimal symplectic structure 4-manifolds are constructed so far. We believe that the following conjecture is true.

**Conjecture 5.1.6.** The only minimal symplectic 4-manifolds with \(\kappa = 0\) are K3 surfaces, Enriques surfaces and \(T^2\) bundles over \(T^2\).

Although we can not answer the question now, some topological properties in our situation are found. Here is the table of symplectic surfaces with \(\kappa = 0\) which we know so far:

<table>
<thead>
<tr>
<th>(b^+)</th>
<th>(b_1)</th>
<th>(\chi)</th>
<th>(\sigma)</th>
<th>(b^-)</th>
<th>manifolds</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>24</td>
<td>-16</td>
<td>19</td>
<td>K3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4-torus</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>Kodaira surface</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>12</td>
<td>-8</td>
<td>9</td>
<td>Enriques surface</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>hyperelliptic surface</td>
</tr>
</tbody>
</table>

We can observe that the Euler number of these manifolds are 0, 12 or 24 and satisfy the bound condition:

\[(\ast) \ b_1 \leq 4, \ b^+ \leq 3, \ b^- \leq 19\]

It is conjectured in [31] that a minimal symplectic 4-manifold with \(\kappa = 0\) satisfies \((\ast)\), which is proved in [32].

**Theorem 5.1.7.** If \(M\) is a minimal symplectic 4-manifold with \(\kappa = 0\), then the rational homology of \(M\) is the same as that of K3 surface, Enriques surface, or a \(T^2\) bundle over \(T^2\). In particular, the Euler number of \(M\) is 0, 12, or 24, the signature of \(M\) is 0, -8, or -16, and the Betti numbers of \(M\) satisfy \(b_1 \leq 4, \ b^+ \leq 3 \text{ and } b^- \leq 19\).

This theorem offers some evidence for the conjecture.

**5.1.3 \(\kappa = 1\)**

In [21], Gompf showed that any finitely presented group is the fundamental group of a symplectic 4-manifold with \(\kappa = 1\) (and \(\kappa = 2\)). So it is impossible to classify symplectic 4-manifolds of such types. But we can ask what kind of invariants can be realized by...
these symplectic manifolds. Gompf also asked whether a non-ruled symplectic four-manifold necessarily has non-negative Euler characteristic. When $\kappa = 0$, this statement is equivalent to $\sigma \leq 0$. Since

$$0 = c_1^2 = 2\chi + 3\sigma = 4 + 4b^+ - 4b_1 + \sigma$$

We can conclude that

$$b_1 = 1 + b^+ + \frac{\sigma}{4} \geq 2 + \frac{\sigma}{4} \ (M \text{ is symplectic, so } b^+ \geq 1.)$$

On the other hand, Noether’s formula implies that

$$\chi + \sigma \equiv 0 (mod\ 4) \text{ or } b^+ - b_1 \text{ is odd}$$

Hence

$$\sigma = -2\chi - 2\sigma \equiv 0 (mod\ 8)$$

A pair of integer $(a, b)$ corresponding to $(\sigma, b_1)$ is called admissible if $a = 8k$ for some non-positive integer $k$ and $b_1 \geq \max\{0, 2 + \frac{a}{4}\}$. In simply connected case, $\sigma = 8k$ with $k < 0$. The pairs $(8k, 0)$ are realized by Dolgachev surfaces and elliptic surfaces $E(-k)$ with $k \leq 2$.

A symplectic 4-manifold $(M, \omega)$ induces a natural map $\cup[\omega] : H^1(M, \mathbb{R}) \to H^3(M, \mathbb{R})$. The rank of the kernel is called the degeneracy of $(M, \omega)$ and denoted as $d(M, \omega)$. A triple $(a, b, c) \in \mathbb{Z}^3$ corresponding to $(\sigma, b_1, d)$ is called Lefschetz admissible if it satisfies

1. $a = 8k$ where $k$ is a non-positive integer,
2. $b \geq \max\{0, 2 + \frac{a}{4}\}$,
3. $0 \leq c \leq b$ and $b - c$ is even.

The symplectic realization of such triples are shown in [1]:

**Theorem 5.1.8.** For any Lefschetz admissible triple $(a, b, c)$, there exists a minimal symplectic 4-manifold $(M, \omega)$ with $\kappa = 1$ and

$$(\sigma(M), b_1(M), d(M, \omega)) = (a, b, c)$$
The other similar problem is the nullity question. For any $\alpha \in H^1(M, \mathbb{R})$ and $i = 1, 2$, consider the map $i_\alpha = \cup \alpha : H^i(M, \mathbb{R}) \to H^{i+1}(M, \mathbb{R})$. The dimension of the linear space $\{ \alpha \in H^1(M, \mathbb{R}) | i_\alpha = 0 \}$, denoted by $n_i(M)$, is called the $i$-nullity of $M$. By Poincaré duality, $n_1(M) = n_2(M)$. Hence we can just say nullity $n(M)$ of $M$. It also follows from Poincaré duality that nullity is a lower bound of degeneracy: $n(M) \leq d(M, \omega)$. A triple $(a, b, c)$ corresponding to $(\sigma, b_1, n)$ is called null admissible if it satisfies

1. $a = 8k$ where $k$ is a non-positive integer,
2. $b \geq \max\{0, 2 + \frac{a}{4}\}$,
3. $0 \leq c \leq b$ and $c \neq b - 1$.

It is shown in [9] that many null admissible triples can be realized by symplectic 4-manifolds with $\kappa = 0$.

5.1.4 $\kappa = 2$

As we mention in previous section, it is also impossible to classify symplectic 4-manifolds with $\kappa = 2$. On the other hand, not many manifolds in this situation are known. But we are interested in whether these manifolds satisfy the Bogomolov-Miyaoka-Yau inequality $\chi \geq 3\sigma$.

Fintushel and Stern conjectured that minimal symplectic 4-manifolds with $\kappa = 2$ satisfy the symplectic Noether inequality:

$$K_\omega^2 \geq \frac{\chi(M) + \sigma(M)}{4} - (2 + c(K_\omega))$$

Here $c(K_\omega)$ is the maximal number of connected components of an embedded symplectic representative of $K_\omega$. When $\chi$ and $\sigma$ are replaced by $b^+$ and $b_1$ as $[\chi(M) + \sigma(M)]/4 = [b^+(M) - b_1(M) + 1]/2$, after dropping the $b_1$ term, it induces a stronger inequality

$$K_\omega^2 \geq \frac{b^+(M) - 3}{2} - c(K_\omega)$$

We believe that any symplectic manifold with $\kappa \geq 0$ satisfies this inequality.
5.2 Luttinger surgery

In this section, we describe the Luttinger surgery following [1]. Applications to Lagrangian fibrations are also discussed. We assume all manifolds are oriented.

5.2.1 Construction

Topologically, Luttinger surgery is a framed torus surgery. We start with a general description of framed torus surgeries. Let \( X \) be a smooth 4-manifold and \( L \subset X \) an embedded 2-torus with trivial normal bundle. Then let \( U \) be a tubular neighborhood of \( L \). If we assume \( Y = X - U \) is the complement of \( U \), \( Z = \partial Y = \partial U \) and \( g : Z \rightarrow Z \) is a diffeomorphism, a new manifold \( \tilde{X} \) can be constructed by cutting \( U \) out of \( X \) and gluing it back to \( Y \) along \( Z \) via \( g \):

\[
\tilde{X} = Y \cup_g U. \quad (5.1)
\]

Such surgery is called a torus surgery.

It is often more explicit to describe this process via a framing of \( L \).

**Definition 5.2.1.** Let \( X, L, U \) and \( Z \) be given as above. A diffeomorphism \( \varphi : U \rightarrow T^2 \times D^2 \) is called a framing of \( L \) if \( \varphi^{-1}(T^2 \times 0) = L \). Let \( \pi_1 : T^2 \times D^2 \rightarrow T^2 \) be the projection. For any \( \gamma \subset L \) and \( z \in \partial D^2 \), the lift \( \gamma_\varphi = \varphi^{-1}(\pi_1(\gamma) \times z) \) of \( \gamma \) in \( Z \) is called a longitudinal curve of \( \varphi \). Let

\[
\partial \varphi : Z \rightarrow \partial(T^2 \times D^2) \cong T^2 \times S^1
\]

be the induced map. Two framings \( \varphi_1, \varphi_2 : U \rightarrow T^2 \times D^2 \) are smoothly isotopic to each other if the map

\[
\partial \varphi_2 \circ (\partial \varphi_1)^{-1} : T^2 \times S^1 \rightarrow T^2 \times S^1
\]

is homotopic to the identity map.

\( \partial \varphi \) induces a \( S^1 \)-bundle structure on \( Z \). A positive oriented fiber \( \mu \) of \( Z \) is called a meridian of \( L \). For \( \tilde{X} \) in (5.1), we will use \( \tilde{L} \) to denote the torus \( L \subset U \subset \tilde{X} \). Notice that \( \tilde{L} \) also inherits a framing \( \tilde{\varphi} \) and its meridian \( \tilde{\mu} \subset Z \) satisfies

\[
[\tilde{\mu}] = p[\mu] + k[\gamma_\varphi],
\]
in $H_1(Z;\mathbb{Z})$. Here $\gamma$ is a longitudinal curve of $\varphi$ and $p,k$ are coprime integers. The diffeomorphism type of $\tilde{X}$ only depends on the class $[\tilde{\mu}]$. It is called a generalized logarithmic transform of $X$ along $(L,\varphi,\gamma)$ with multiplicity $p$ and auxiliary multiplicity $k$, or of type $(p,k)$ (see [22]), and denoted as $X_{(L,\varphi,\gamma,p,k)}$. For brevity, we will call it a $(p,k)$-surgery.

If $X$ is a symplectic 4-manifold, Weinstein’s theorem states that there is a canonical framing for any Lagrangian torus of $X$.

**Definition 5.2.2.** Let $X$ be a symplectic 4-manifold and $L$ a Lagrangian torus of $X$. A framing $\varphi$ of $L$ is called a Lagrangian framing if $\varphi^{-1}(T^2 \times z)$ is a Lagrangian submanifold of $X$ for any $z \in D^2$.

Topologically, a Luttinger surgery is a $(1,k)$-surgery with respect to a Lagrangian framing. In order to deal with the symplectic structure, it is more convenient to use a square neighborhood rather than the disk neighborhood of $L$ as above.

Express the cotangent bundle $T^*T^2$ as

$$\{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \}/(x_1 = x_1 + 1, x_2 = x_2 + 1)$$

equipped with the canonical 2-form

$$\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$

and let

$$U_r = \{(x_1, x_2, y_1, y_2) \in T^*T^2 \mid -r < y_1 < r, -r < y_2 < r\},$$

There exists a tubular neighborhood $U$ of $L$ and a symplectomorphism $\varphi : (U,\omega) \to (U_r,\omega_0)$ for small $r$ which satisfies

$$\varphi(L) = T^2 \times (0,0).$$

In addition, given a simple closed curve $\gamma$ on $L$, we can choose the coordinates $x_1, x_2$ of $T^2$ such that

$$\varphi(\gamma) = \{(x_1,0,0,0) \mid x_1 \in \mathbb{R}/\mathbb{Z}\}.$$
integer $k$, we can define a diffeomorphism $h_k$:

\[
A_{r,\frac{r}{2}} \rightarrow A_{r,\frac{r}{2}} \\
(x_1, x_2, y_1, y_2) \mapsto \begin{cases} 
(x_1 + kf(y_1), x_2, y_1, y_2) & \text{if } y_2 \geq \frac{r}{2} \\
(x_1, x_2, y_1, y_2) & \text{otherwise}
\end{cases}
\]

Observe that

\[h_k^*(\omega_0) = \omega_0, \quad (5.2)\]

which follows from the relation

\[(dx_1 + kf'(y_1)dy_1) \wedge dy_1 + dx_2 \wedge dy_2 = \omega_0\]

for $y_2 \geq \frac{r}{2}$.

Let $X_L = X - \varphi^{-1}(U_{\frac{r}{2}})$ and define $g_k = \varphi^{-1} \circ h_k \circ \varphi$ via the following diagram

\[
\begin{array}{ccc}
X_L & \supset U & \supset U - \varphi^{-1}(U_{\frac{r}{2}}) \\
\downarrow \varphi & \Downarrow g_k & \Downarrow \varphi \\
A_{r,\frac{r}{2}} & \xrightarrow{h_k} & A_{r,\frac{r}{2}}
\end{array}
\]

then we can construct a new smooth manifold

\[\tilde{X} := X_L \cup g_k U.\]

Notice that, by (5.2), we have $g_k^*(\omega) = \omega$. Thus $\tilde{X}$ carries a symplectic form $\tilde{\omega}$ induced by $\omega$. This process is called a Luttinger surgery (along the Lagrangian torus $L$).

We know that

**Lemma 5.2.3.** [17] Any two Lagrangian framings of a Lagrangian torus are smoothly isotopic to each other.

Hence the symplectomorphism type of $(\tilde{X}, \tilde{\omega})$ only depends on the Lagrangian isotopy class of $L$, the isotopy class of $\gamma$ in $L$, and the integer $k$. Therefore, $\tilde{X}$ is also denoted as $X(L, \gamma, k)$.

It is worth mentioning that a Luttinger surgery can be reversed. Let $\tilde{L}, \tilde{\gamma}$ be the subsets $\varphi^{-1}(T^2 \times (0, 0))$ and $\varphi^{-1}(\mathbb{R}/\mathbb{Z} \times (0, 0, 0))$ of $\tilde{X}$. We can apply the Luttinger surgery to $X(L, \gamma, k), \tilde{L}, \tilde{\gamma}$ with coefficient $-k$ to recover $X$. 
5.2.2 Lagrangian fibrations

One natural source of Lagrangian tori is smooth fibers of Lagrangian fibrations.

**Definition 5.2.4.** Let \((X, \omega)\) be a symplectic 4-manifold, and let \(B\) be a 2-manifold (with boundary or vertices). A smooth map \(\pi : X \to B\) is called a Lagrangian fibration if there exists an open dense subset \(B_0 \subset B\) such that \(\pi^{-1}(b)\) is a compact Lagrangian submanifold of \(X\) for any \(b \in B_0\). \(X\) is called Lagrangian fibered if such a structure exists.

It is easy to see that any smooth fiber of a Lagrangian fibration must be a torus. Moreover, we have

**Lemma 5.2.5.** A Luttinger surgery along a Lagrangian fiber preserves the Lagrangian fibration structure.

**Proof.** Let \(\pi : X \to B\) be a Lagrangian fibration and \(L = \pi^{-1}(b) \subset X\) a generic fiber. Using notations from section 2.1, it is shown in [40] that there is a neighborhood \(B_r\) of \(b\) and \(U = \pi^{-1}(B_r)\) with local charts \(\varphi : (U, \omega) \to (U_r, \omega_0)\) and \(\varphi_0 : B_r \to D_r = (-r, r) \times (-r, r)\) such that the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\varphi} & U_r \\
\pi \downarrow & & \downarrow \pi_0 \\
B_r & \xrightarrow{\varphi_0} & D_r
\end{array}
\]

commutes. Here \(\pi_0\) is the projection \((x_1, x_2, y_1, y_2) \mapsto (y_1, y_2)\).

If \(\tilde{X} = X_L \cup_{g_k} U\) is obtained by performing Luttinger surgery along \(L\) (indexed by \(\gamma \subset L\) and \(k \in \mathbb{Z}\)), we can define a map \(\tilde{\pi} : \tilde{X} \to B\) as \(\tilde{\pi}(\tilde{x}) = \pi(x)\). Since \(\pi_0 \circ h_k = \pi_0\), we have

\[
\pi \circ g_k = \pi \circ \varphi^{-1} \circ h_k \circ \varphi = \varphi_0^{-1} \circ \pi_0 \circ h_k \circ \varphi = \varphi_0^{-1} \circ \pi_0 \circ \varphi = \pi
\]

So \(\tilde{\pi}\) is well-defined. It is clear that \(\tilde{\pi}\) is Lagrangian and \(\tilde{X}\) also possesses a Lagrangian fibration structure.

Lagrangian fibrations appear widely in toric geometry, integral systems and mirror symmetry. We will discuss almost toric fibration introduced by Symington in some detail.
Definition 5.2.6. An almost toric fibration of a symplectic 4-manifold \((X, \omega)\) is a Lagrangian fibration \(\pi : X \to B\) with the following properties: for any critical point \(x\) of \(\pi\), there exists a local coordinate \((x_1, x_2, y_1, y_2)\) near \(x\) such that \(x = (0, 0, 0, 0)\), \(\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2\), and \(\pi\) has one of the forms

\[
(x_1, x_2, y_1, y_2) \rightarrow \begin{cases} 
(x_1^2 + y_1^2, x_2 + y_2^2) \\
(x_1^2 + y_1^2, x_2) \\
(x_1^2 - y_1^2, x_2)
\end{cases}
\]

An almost toric 4-manifold is a symplectic 4-manifold equipped with an almost toric fibration.

The base \(B\) of an almost toric fibration has an affine structure with boundary and vertices. Moreover, these three types of critical points project to vertices, edges and interior of \(B\) respectively. Almost toric fibrations are classified by Leung and Symington:

Theorem 5.2.7. [29] Let \((X, \omega)\) be a closed almost toric 4-manifold. There are seven types of almost toric fibrations according to the homeomorphism type of the base \(B\).

1. \(\mathbb{CP}^2 \sharp n \mathbb{CP}^2\) or \(S^2 \times S^2\), \(B\) is (homeomorphic to) a disk;
2. \((S^2 \times T^2) \sharp n \mathbb{CP}^2\) or \((S^2 \tilde{\times} T^2) \sharp n \mathbb{CP}^2\), \(B\) is a cylinder;
3. \((S^2 \times T^2) \sharp n \mathbb{CP}^2\) or \((S^2 \tilde{\times} T^2) \sharp n \mathbb{CP}^2\), \(B\) is a Möbius band;
4. the K3 surface, \(B\) is a sphere;
5. the Enriques surface, \(B\) is \(\mathbb{RP}^2\);
6. a torus bundle over torus with monodromy

\[
\left\{ I, \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \right\}, m \in \mathbb{Z}
\]

\(B\) is a torus;
7. a torus bundle over the Klein bottle with monodromy

\[
\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \right\}, m \in \mathbb{Z}
\]

\(B\) is a Klein bottle.
An immediate consequence of this classification is the calculation of the symplectic Kodaira dimension.

**Proposition 5.2.8.** If \((X, \omega) \to B\) is an almost toric fibration, then \(\kappa(X) \leq 0\). Moreover, \(\kappa(X) = 0\) if and only if the base \(B\) is closed.

The effect of Luttinger surgeries on almost toric fibrations is also easy to describe.

**Proposition 5.2.9.** Suppose \((X, \omega) \to B\) is an almost toric fibration and \((\tilde{X}, \tilde{\omega})\) is obtained from \((X, \omega)\) by performing a Luttinger surgery along a smooth fiber \(L\), then \((\tilde{X}, \tilde{\omega})\) retains an almost toric fibration structure with the same base. Moreover, \(\tilde{X}\) is diffeomorphic to \(X\) if \(\chi(B) > 0\).

**Proof.** The first statement is given by Lemma 5.2.5. If \(\chi(B) > 0\), \(X\) and \(\tilde{X}\) are in one of the types (1)-(5) in Theorem 5.2.7. In each of them, the list of manifolds are distinguished by the type of intersection forms and Euler numbers. So the second result for types (1)-(3) follows from Proposition 5.4.4 and the fact that homology classes of Lagrangian tori in manifolds with \(b^+ = 1\) are torsion. It is clear for (4) and (5) from the classification.

Propositions 5.2.8 and 5.2.9 provide examples of Luttinger surgeries preserving the symplectic Kodaira dimension. In the next section, we will show that it is true for any Luttinger surgery.

### 5.3 Preservation of Kodaira dimension

In this section, we prove Theorem 1.0.9. To proceed, we must first prove the invariance of minimality under Luttinger surgery.

#### 5.3.1 Minimality

A symplectic (smooth) \(-1\) class is a degree 2 homology class represented by an embedded symplectic (smooth) sphere with self-intersection \(-1\). A symplectic 4-manifold is called *symplectically (smoothly) minimal* if it does not have any symplectic (smooth) \(-1\) class. The symplectic minimality is actually equivalent to smooth minimality.
Proposition 5.3.1. The Luttinger surgery preserves the minimality.

Proof. Since a Luttinger surgery can be reversed and the reverse operation is also a Luttinger surgery, it suffices to show that, if we start with a non-minimal symplectic 4-manifold, then after a Luttinger surgery, the resulting symplectic manifold is still non-minimal. But this is a direct consequence of the following fact in [53]:

Theorem 5.3.2. Given a Lagrangian torus $L$ and a symplectic $-1$ class, there is an embedded symplectic $-1$ sphere in that class which is disjoint from $L$.

5.3.2 Kodaira dimension

Now, we analyze the effect of Luttinger surgery on the symplectic canonical class $K_\omega$ and the symplectic class $[\omega]$. Recall that $X_L$ is an open submanifold of both $X$ and $\tilde{X}$, and let $\nu : X_L \to X$ and $\tilde{\nu} : X_L \to \tilde{X}$ be the inclusions.

To prepare for the following lemma, we use the notations from Section 5.2.1. For the sake of simplicity, we will identify any object in $X$ with their image of $\varphi$ and $(x_1, x_2, y_1, y_2), (x'_1, x'_2, y'_1, y'_2)$ will denote the coordinates of $A_{r, \frac{r}{2}}$ on $X_L$ and $U$ respectively.

Lemma 5.3.3. There exists a 2-dimensional submanifold $S \subset X_L$ such that $\nu_*([S]) = PD(K_\omega) \in H_2(X)$ and $\tilde{\nu}_*([S]) = PD(K_\tilde{\omega}) \in H_2(\tilde{X})$.

Proof. Let $J$ be a $\omega$-tamed almost complex structure in $X$ which induces a complex structure on $T^*U$ as

$$J(dx_1) = -dy_1, \quad J(dx_2) = -dy_2$$

Assume $\rho : (-r, r) \to [0, 1]$ is a continuous increasing function satisfying

$$\rho(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > \frac{r}{3} \end{cases}.$$

Another almost complex structure $J'$ in $T^*U$ is defined as

$$J'(dx'_1) = -krf'(y_1)\rho(y_2)dx'_1 - (k^2f^2(y_1)\rho(y_2) + 1)dy'_1, \quad J'(dx'_2) = -dy'_2$$

It is easy to check that $J'$ is $\omega$-tamed and $(g_k)_*(J) = J'$ in $X_L \cap U$. 
Let \( \pi : \mathcal{L} \to X \) and \( \tilde{\pi} : \tilde{\mathcal{L}} \to \tilde{X} \) be the canonical bundles of \( X \) and \( \tilde{X} \), respectively, and let \( s : X \to \mathcal{L} \) and \( \tilde{s} : \tilde{X} \to \tilde{\mathcal{L}} \) denote the corresponding embeddings of zero sections. Since \( \mathcal{L} \) is trivial on \( U \), we can find a global section \( \sigma \) of \( \mathcal{L} \) and a Thom class \( \Phi \in H^2_{cv}(\mathcal{L}) \) such that \( \sigma = (dx_1 + idy_1) \wedge (dx_2 + idy_2) \) in \( X_L \cap U \) and \( \Phi = 0 \) in \( s(U) \). Another nonzero \((2,0)\)-form in \( U \) is constructed as

\[
\sigma' = (dx'_1 + iJ'(dx'_1)) \wedge (dx'_2 + iJ'(dx'_2))
\]

In \( X_L \cap U \), we have

\[
g_k^*(\sigma') = g_k^*((dx'_1 + iJ'(dx'_1)) \wedge (dx'_2 + iJ'(dx'_2)))
\]

\[
= (dx_1 + kJ'(y_1)dy_1 + i(-kJ'(y_1)dx_1 + dy_1)) \wedge (dx_2 + idy_2)
\]

\[
= (1 - ikf'(y_1))(dx_1 + idy_1) \wedge (dx_2 + idy_2)
\]

\[
= (1 - ikf'(y_1))\sigma
\]

\( \sigma \) and \( \sigma' \) give two local trivializations of \( \tilde{\pi}^{-1}(X_L \cap U) \) with transition function \( \theta = 1 - ikf'(y_1) \). Since \( -\frac{\pi}{2} < \arg(\theta) < \frac{\pi}{2} \), we can normalize the frame of \( \tilde{\pi}^{-1}(U) \) such that \( \theta = 1 \). Hence \( \Phi |_{\tilde{\pi}^{-1}(X_L)} \) can be extended to \( \tilde{\mathcal{L}} \) via constant function and form a Thom class \( \tilde{\Phi} \) satisfying

1. \( \tilde{\Phi} = \Phi \) in \( \mathcal{L} \mid_{X_L} \cong \tilde{\mathcal{L}} \mid_{X_L} (= \pi^{-1}(X_L)) \).

2. \( \tilde{\Phi} \) is independent of the coordinates \((x'_1, x'_2, y'_1, y'_2)\) in \( \tilde{\pi}^{-1}(U) \). In particular, \( \tilde{\Phi} = 0 \) in \( \tilde{s}(U) \).

It is clear that these \(2\)-forms \( e = s^*(\Phi)\) and \( \tilde{e} = \tilde{s}^*(\tilde{\Phi}) \) are equivalent in \( X_L \) and vanish in \( U \subset X \) and \( U \subset \tilde{X} \) respectively. Using these representatives, we can find a \(2\)-submanifold \( S \subset \text{supp}(e) \subset X_L \) which is Poincaré dual to \( K_\omega \) in \( X \), and dual to \( K_\omega \) in \( \tilde{X} \).

The main theorem can be proved now.

**Proof of Theorem 1.0.3** Suppose \( \tilde{X} \) is obtained from \( X \) by applying a Luttinger surgery along \( L \). Let us first consider the case in which \( X \) is minimal. By Proposition 5.3.1, \( \tilde{X} \)
is also minimal. Let $K_\omega$ and $K_\tilde{\omega}$ denote the canonical classes of $X$ and $\tilde{X}$ respectively. By Lemma 5.3.3, there exists a submanifold $S \subset X_L$ such that $\nu_*(\lfloor S \rfloor) = PD(K_\omega)$ and $\tilde{\nu}_*(\lfloor S \rfloor) = PD(K_\tilde{\omega})$. We also know that $\omega = \tilde{\omega}$ in $X_L$. So

$$K_\omega^2 = \int_S K_\omega = \int_S K_\tilde{\omega} = K_\tilde{\omega}^2$$

$$K_\omega \cdot \lfloor \omega \rfloor = \int_S \omega = \int_S \tilde{\omega} = K_\tilde{\omega} \cdot \lfloor \tilde{\omega} \rfloor$$

Thus the Kodaira dimensions of $X$ and $\tilde{X}$ coincide.

If $X$ is not minimal, we can blow down $X$ along symplectic $-1$ spheres disjoint from $L$ to a minimal model. These spheres are contained in $X_L$ and the same procedure can be applied to $\tilde{X}$, so we can argue as above.

Theorem 1.0.9 can be used to distinguish non-diffeomorphic manifolds. In [2, 3, 4, 8, 18], several symplectic manifolds homeomorphic but not diffeomorphic to non-minimal rational surfaces are constructed. With $\kappa = 2$ for the building blocks, it also easily follows from Theorem 1.0.9 that they are exotic.

**Remark 5.3.4.** 1. The main theorem is proved based on the invariance of $K_\omega \cdot \lfloor \omega \rfloor$ and $K_\omega^2$. Actually, the class $\lfloor \omega \rfloor^2$ is also preserved since the volume is invariant under a Luttinger surgery. Theorem 1.0.9 is expected, in light of Auroux’s Question 2.6 in [6]:

Let $X_1, X_2$ be two integral compact symplectic 4-manifolds with the same $(K^2, \chi, K_\omega \cdot \lfloor \omega \rfloor, \lfloor \omega \rfloor^2)$. Is it always possible to obtain $X_2$ from $X_1$ by a sequence of Luttinger surgeries?

2. It is well known that the Dolgachev surfaces $S(p, q)$ obtained by performing two logarithmic transforms with multiplicities $p > 1, q > 1$ to $\mathbb{CP}^2 \# 9 \mathbb{CP}^2$ have $\kappa = 1$. So a generalized logarithmic transform may not preserve $\kappa$ (see a related discussion in [11]).
5.4 Manifolds with non-positive $\kappa$

In this section we apply Theorem 1.0.9 to study the effect of Luttinger surgeries on symplectic 4-manifolds with $\kappa \leq 0$.

5.4.1 Torus surgery and homology

We start by analyzing how homology changes under a general torus surgery. Suppose $X$ is a smooth 4-manifold and $L \subset X$ is an embedded 2-torus with trivial normal bundle. Moreover, $U, Y, Z, g, \tilde{X}$ are defined as in Section 5.2.1.

To compare the homology of $X$ and $\tilde{X}$, we need to compare both of them with the homology of $Y$. The inclusion $i : Z \to Y$ induces homomorphisms

$$i_k^\mathbb{Z} : H_k(Z; \mathbb{Z}) \to H_k(Y; \mathbb{Z}) \quad (5.3)$$

and

$$i_k^\mathbb{Q} : H_k(Z; \mathbb{Q}) \to H_k(Y; \mathbb{Q}) \quad (5.4)$$

in homology. We often use $i_k$ to denote $i_k^\mathbb{Q}$ and $H_k(-)$ to denote $H_k(-, \mathbb{Q})$. We also use $r(A)$ to denote the dimension of any $\mathbb{Q}$-vector space $A$.

The following lemma is a well know fact, for which we offer a geometric argument.

**Lemma 5.4.1.** $[\mu] \in \ker i_1$ if and only if $[L] \neq 0$ in $H_2(X)$.

**Proof.** Suppose $i_1[\mu] = 0$ in $H_1(Y)$, i.e. $li_1^\mathbb{Z}[\mu] = 0$ in $H_1(Y; \mathbb{Z})$ for some positive integer $l$. Thus $l$ copies of $\mu$ bounds an oriented surface $A$ in $Y$. Extend $A$ by $l$ normal disks inside the tubular neighborhood to obtain a closed oriented surface $A'$ intersecting with $L$ at $l$ points with the same sign. This implies in particular that $[L] \neq 0$ in $H_2(X)$.

Conversely, suppose $[L] \neq 0$ in $H_2(X)$, then there exists a closed oriented surface $B$ in $X$ intersecting $L$ with nonzero algebraic intersection numbers, say $l$. We may assume that the intersection is transverse with $l + b$ positive intersection points and $b$
negative intersection points. We can further assume that \( B \) intersects the closure of \( U \) at \( l + 2b \) normal disks, \( l + b \) of those having positive orientations, the remaining \( b \) disks having negative orientations. This implies that the complement of those disks in \( B \) is an oriented surface in \( Y \), whose boundary is homologous to \( l\mu \), and thus \( i_1[\mu] \) is zero.

When we consider the integral homology, Lemma 5.4.1 immediately implies

**Corollary 5.4.2.** If \( [L] = 0 \) in \( H_2(X; \mathbb{Z}) \), then \( i_1^Z[\mu] \) is a non-torsion class in \( H_1(Y; \mathbb{Z}) \).

Consider the Mayer-Vietoris sequence

\[
\cdots \to H_{k+1}(Z) \xrightarrow{\partial_{k+1}} H_k(Y) \oplus H_k(U) \xrightarrow{\nu_k} H_k(X) \xrightarrow{\partial_k} H_{k-1}(Z) \xrightarrow{\rho_{k-1}} \cdots
\]

(5.5)

where \( \rho_k = (i_k, j_k) \) and \( \nu_k = \nu'_k \oplus (-\nu''_k) \) with \( i_k, j_k, \nu'_k, \nu''_k \) induced by inclusions.

**Lemma 5.4.3.**

1. \( \partial_1 = 0 \) and \( \nu'_1 : H_1(Y) \to H_1(X) \) is surjective.

2. \[
\begin{align*}
r(\text{Im}\rho_1) &= \begin{cases} 3 & \text{if } i_1[\mu] \neq 0 \\ 2 & \text{if } i_1[\mu] = 0 \end{cases} \\
	ext{and } \rho_1 \text{ is injective if and only if } [\mu] \notin \ker i_1.
\end{align*}
\]

3. \[
H_1(X; \mathbb{Z}) \cong H_1(Y; \mathbb{Z})/\langle i_1^Z[\mu] \rangle
\]

(5.6)

4. If \( [L] = 0 \in H_2(X) \), then \( \nu'_2 : H_2(Y) \to H_2(X) \) is surjective.

**Proof.**

1. It is clear that any class \( a \) in \( H_1(X; \mathbb{Z}) \) can be represented by a 1-cycle \( C \) disjoint from \( L \). \( C \) is also disjoint from \( Z \) if the neighborhood \( U \) is small enough. So \( \partial_1 a = [C \cap Z] = 0 \). \( C \subset Y \) implies that \( \nu'_1 \) is surjective.

2. We know \( \ker \rho_1 = \ker i_1 \cap \ker j_1 \subset \ker j_1 \). Since \( \ker j_1 = \langle [\mu] \rangle \),

\[
\ker \rho_1 = \begin{cases} 0 & \text{if } i_1[\mu] \neq 0 \\ \langle [\mu] \rangle & \text{if } i_1[\mu] = 0 \end{cases}
\]

and \( \rho_1 \) is injective if and only if \( i_1[\mu] \neq 0 \). The rank of \( \text{Im}\rho_1 \) is given from \( r(\text{Im}\rho_1) = r(H_1(Z)) - r(\ker \rho_1) \).
3. The sequence \([5.5]\) induces a short exact sequence

\[
0 \rightarrow H_1(Y; \mathbb{Z}) \oplus H_1(U; \mathbb{Z})/ \ker \nu_1 \rightarrow H_1(X; \mathbb{Z}) \rightarrow \text{Im} \partial_1 \rightarrow 0
\]

\(\partial_1 = 0\) implies that

\[
H_1(Y; \mathbb{Z}) \oplus H_1(U; \mathbb{Z})/ \ker \nu_1 \cong H_1(X; \mathbb{Z})
\]

Because \(\nu'_1\) is surjective, we also have

\[
H_1(X; \mathbb{Z}) = H_1(Y; \mathbb{Z})/ \ker \nu'_1
\]

If \(\{\mu, \gamma_1, \gamma_2\}\) is a basis of \(H_1(Z; \mathbb{Z})\), then \(\text{Im} \rho_1 = \langle ([\mu], 0), (\gamma_1, \gamma_1), (\gamma_2, \gamma_2) \rangle\) and \(\gamma_1, \gamma_2 \neq 0 \in H_1(U; \mathbb{Z})\). For \(a \in H_1(Y; \mathbb{Z})\), \(a \in \ker \nu'_1\) if and only if \((a, 0) \in \ker \nu_1 = \text{Im} \rho_1\), or \(a = ki_1^2[\mu]\) for some \(k \in \mathbb{Z}\). So \(\ker \nu'_1 = \langle i_1^2[\mu] \rangle\) and

\[
H_1(X; \mathbb{Z}) = H_1(Y; \mathbb{Z})/ \langle i_1^2[\mu] \rangle
\]

4.

\([L] = 0 \in H_2(X) \iff [\mu] \neq 0 \in H_1(Y)\) (by Lemma \([5.4.1]\))

\[\iff \rho_1 \text{ injective} \quad \text{(by part(2))}\]

\[\iff \partial_2 = 0 \quad \text{(exactness)}\]

\[\iff \nu_2 \text{ surjective} \quad \text{(exactness)}\]

Since \([L] = 0\) also implies \(\nu''_2 = 0\), \(\nu'_2\) has to be surjective.

All the results hold if we replace \(X, L, \mu\) by \(\tilde{X}, \tilde{L}\) and \(\tilde{\mu}\). Now, we are ready to compare \(X\) and \(\tilde{X}\).

**Comparing \(H_*(X)\) and \(H_*(\tilde{X})\)**

Lemma \([5.4.3]\) applied to torus surgeries, gives

**Proposition 5.4.4.** If \(\tilde{X}\) is obtained from \(X\) via a torus surgery, then

1. \(\chi(\tilde{X}) = \chi(X), \sigma(\tilde{X}) = \sigma(X)\).
2. 

\[ b_1(\tilde{X}) - b_1(X) = \begin{cases} 
0 & \text{if } i_1[\mu] = 0 = i_1[\tilde{\mu}] \text{ or } i_1[\mu] \neq 0 \neq i_1[\tilde{\mu}] \\
-1 & \text{if } i_1[\mu] = 0 \text{ and } i_1[\tilde{\mu}] \neq 0 \\
1 & \text{if } i_1[\mu] \neq 0 \text{ and } i_1[\tilde{\mu}] = 0 
\end{cases} \]

3. \[ |b_1(\tilde{X}) - b_1(X)| \leq 1 \text{ and } |b_2(\tilde{X}) - b_2(X)| \leq 2. \]

**Proof.**

1. Obvious.

2. Since \( \partial_1 = 0 \), we can conclude that

\[ b_1(X) = b_1(Y) + 2 - r(\text{Im}\rho_1) = \begin{cases} 
b_1(Y) - 1 & \text{if } i_1[\mu] \neq 0 \\
b_1(Y) & \text{if } i_1[\mu] = 0. \end{cases} \]

The same is true for \( b_1(\tilde{X}) \) with \( i_1[\mu] \) replaced by \( i_1[\tilde{\mu}] \). The proof is finished by comparing \( b_1(X) \) and \( b_1(\tilde{X}) \).

3. The first inequality is given by part (2). The second inequality follows from part (1) and the first inequality.

The next result concerns with the intersection forms.

**Proposition 5.4.5.** Suppose \( X \) and \( \tilde{X} \) are defined as above. If \([L]\) is a torsion class in \( H_2(X;\mathbb{Z})\) and the intersection form \( Q(X) \) is odd, then \( Q(\tilde{X}) \) is odd as well. In particular, if both \([L]\) and \([\tilde{L}]\) are torsion, then \( \tilde{X} \) and \( X \) have the same intersection form.

**Proof.** Since \( Q(X) \) is odd, there exists a closed oriented surface \( S \) in \( X \) such that \( S \cdot S \) is odd. By Lemma 5.4.3(4), \([S]\) \( \in \text{Im}\rho \) and \( S \) can be chosen such that \( S \subset Y \). Thus, \( S \) is contained in \( \tilde{X} \), and hence, \( Q(\tilde{X}) \) is also odd.

\[ \square \]

**5.4.2 \( \kappa = -\infty \)**

By Proposition 5.2.9, if a symplectic manifold \((X,\omega)\) with \( \kappa(X) = -\infty \) has an almost toric structure \( \pi : X \to B \) and if we apply a Luttinger surgery along a smooth fiber of \( \pi \), the new manifold \((\tilde{X},\tilde{\omega})\) is diffeomorphic to \((X,\omega)\). Such phenomenon is still true for any 4-manifold with \( \kappa = -\infty \). Moreover, we have the stronger Theorem 1.0.10.
Proof of Theorem 1.0.10. Proposition 5.3.1 allows us to reduce to the case where \((X, \omega)\) is minimal.

We first show that \(\tilde{X}\) is diffeomorphic to \(X\). Observe that the diffeomorphism types of minimal manifolds with \(\kappa = -\infty\) are distinguished by their Euler numbers and intersection forms. Since such manifolds have \(b^+ = 1\), the homology classes of Lagrangian tori are torsion. Thus, both quantities are preserved by Proposition 5.4.4 and 5.4.5.

To show further that \((\tilde{X}, \tilde{\omega})\) and \((X, \omega)\) are symplectomorphic to each other, it is enough to show that \(\omega\) is cohomologous to \(\tilde{\omega}\) (36). If \(X\) is diffeomorphic to \(\mathbb{C}P^2\), the symplectic structure is determined by the volume \([\omega]^2\), which is preserved by Remark 5.3.4(1).

When \(X\) is ruled, \(H^2(X)\) is either generated by \(K_\omega\) and the Poincaré dual to the homology class of a fiber \(F = S^2\), or by \(K_\omega\) and \([\omega]\). Hence the class of \(\omega\) is determined by \(K_\omega \cdot [\omega], [\omega]^2\) and \([\omega](F)\). As mentioned above, the first two quantities are preserved. By 53 the fiber sphere can be chosen to be disjoint from \(L\), so it follows that the last quantity is also preserved.

\[\square\]

5.4.3 Luttinger surgery as a symplectic CY surgery

A symplectic CY surface is a symplectic 4-manifold with torsion canonical class, or equivalently, a minimal symplectic 4-manifold with \(\kappa = 0\).

By Theorem 1.0.9 and Proposition 5.3.1 we have

Proposition 5.4.6. A Luttinger surgery is a symplectic CY surgery in dimension four.

There is a homological classification of symplectic CY surfaces in 32 and 12.

Theorem 5.4.7. A symplectic CY surface is an integral homology K3, an integral homology Enriques surface or a rational homology torus bundle over torus.

The table in p.71 lists possible rational homological invariants of symplectic CY surfaces.

Proof of Theorem 1.0.11. It follows from Propositions 5.4.4, 5.4.6, Theorem 5.4.7 and the table above. \[\square\]
It is also speculated that a symplectic CY surface is diffeomorphic to the K3 surface, the Enriques surface or a torus bundle over torus. Thus we make the following

**Conjecture 5.4.8.** If \( X \) is a K3 surface, or an Enriques surface, then under a Luttinger surgery along any embedded Lagrangian torus, \( \tilde{X} \) is diffeomorphic to \( X \).

As for torus bundles over torus, we have

**Conjecture 5.4.9.** Any smooth oriented torus bundle \( X \) over torus possesses a symplectic structure \( \omega \) such that \((X, \omega)\) can be obtained by applying Luttinger surgeries to \((T^4, \omega_{\text{std}})\).

In the list of torus bundles over torus in [20], any manifold in classes (a), (b) and (d) has a Lagrangian bundle structure. For any such manifold, it is not hard to verify Conjecture 5.4.9 via Luttinger surgery along Lagrangian fibers.

**Remark 5.4.10.**

1. Conjectures 5.4.8 and 5.4.9 are clearly related to Question 2.6 in [6].

2. There is another symplectic CY surgery in dimension six, the symplectic conifold transition. If \((M^6, \omega)\) with \(K_\omega = 0\) contains disjoint Lagrangian spheres \(S_1, \ldots, S_n\) with homology relations generated by \(\sum_{i=1}^{n} \lambda_i[S_i] = 0\) with all \(\lambda_i \neq 0\), Smith, Thomas, Yau ([45]) construct from \((M^6, \omega)\) a new symplectic manifold \((M', \omega')\) with \(K_{\omega'} = 0\) and smaller \(b_3\).

### 5.4.4 Luttinger surgery in high dimension

In this section, we construct a symplectic surgery in high dimension, which can be viewed as a parametrized Luttinger surgery. In the following, the definition of \(U_r, \omega_0, A_{k,t}\) and \(h_k\) are given in Section 5.2.1.

Let \((X, \omega)\) be a symplectic manifold of dimension \(2n\) and \(L \subset X\) a coisotropic submanifold of codimension two. Moreover, we assume that \(L\) possesses a local product structure. This means that there is a tubular neighborhood \(U\) of \(L\), a symplectic manifold \((Y, \omega_Y)\) of dimension \(2n - 4\) and a symplectomorphism

\[\varphi : (U, \omega) \to (Y \times U_r, \omega_Y + \omega_0)\]
such that $\varphi(L) = Y \times T^2 \times (0,0)$. Let $X_L = X - \varphi^{-1}(Y \times U_\tau)$ and define $g_k = \varphi^{-1} \circ (id \times h_k) \circ \varphi$ via the following diagram

\[
\begin{array}{ccc}
X_L \supset U - \varphi^{-1}(Y \times U_\tau) & \xrightarrow[g_k]{\cong} & U - \varphi^{-1}(Y \times U_\tau) \subset U \\
\downarrow \varphi & & \downarrow \varphi \\
Y \times A_{r, \frac{\tau}{2}} & \xrightarrow{id \times h_k} & Y \times A_{r, \frac{\tau}{2}}
\end{array}
\]

then we can construct a new smooth manifold

$$\tilde{X} := X_L \cup g_k U.$$ 

Similar to the Luttinger surgery, the construction satisfies $g_k^*(\omega) = \omega$ and $\tilde{X}$ is a symplectic manifold. We want to compare the value of $K^k_\omega \cdot \Omega^{n-k}$ and $K^k_{\omega'} \cdot \omega'$. But the computation is more subtle and we will discuss it elsewhere.
Chapter 6

Torus surgery and framings

In this chapter, we discuss some topological properties of embedded tori in 4–manifolds. In Section 5.4.9, the torus surgery for an 2-torus with trivial normal bundle embedded in a smooth 4-manifold is defined. This is the topological generalization of Luttinger surgery. We also discuss the effect of torus surgery in homology. Section 6.1.2 defines several local framings for torus surgery and their constraints. The proofs of some lemmas are postponed to Section ??.

6.1 Topological preferred framing and Lagrangian framing

In this section, we will introduce topological preferred framings and compare them with the Lagrangian framing for Lagrangian tori in $\kappa \leq 0$ symplectic 4-manifolds.

Suppose $X$ is a smooth 4-manifold and $L \subset X$ is an embedded 2-torus with trivial normal bundle. Recall that a framing is a diffeomorphism $\varphi : U \to T^2 \times D^2$ for a tubular neighborhood $U$ of $L$ such that $\varphi^{-1}(T^2 \times 0) = L$ and a longitudinal curve of $\varphi$ is a lift $\gamma_\varphi$ of some simple closed curve $\gamma \subset L$ in $Z$. Let

$$H_{1,\varphi} := < [\gamma_\varphi], \gamma_\varphi : \text{longitudinal curve of } \varphi >$$

$H_{1,\varphi}$ is a subgroup of $H_1(Z; \mathbb{Z})$ and it induces a decomposition of $H_1(Z; \mathbb{Z})$:

$$H_1(Z; \mathbb{Z}) = < [\mu] > \oplus H_{1,\varphi}.$$  

Conversely, any rank 2 subgroup $V$ of $H_1(Z; \mathbb{Z})$ such that $[\mu]$ and $V$ generate $H_1(Z; \mathbb{Z})$ corresponds to a framing of $L$.  

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In [34], Luttinger introduced a version of topological preferred framings of Lagrangian tori in $\mathbb{R}^4$. It requires that $H_1, \phi$ is in the kernel of $i_1^Z$. On the other hand, Fintushel and Stern ([17]) defined null-homologous framings for a null-homologous torus via $i_2^Z$ (seemingly, under the assumption that $H_1(X;\mathbb{Z})$ vanishes, though not explicitly mentioned).

The following definition is essentially the same as in [17], but without assuming that $H_1(X;\mathbb{Z})$ vanishes.

**Definition 6.1.1.** Suppose $L$ is null-homologous, i.e., $[L] = 0$ in $H_2(X;\mathbb{Z})$. A framing $\phi$ is called a topological preferred framing if $[L, \phi] \in \ker i_2^Z$. Here, $L, \phi \subset Z$ is a longitudinal torus of $\phi$ given by $\phi^{-1}(T^2 \times z)$, $z \in \partial D^2$.

There is the following generalization when $[L]$ is a torsion class in $H_2(X;\mathbb{Z})$.

**Definition 6.1.2.** Assume $[L]$ is a torsion class in $H_2(X;\mathbb{Z})$. A framing $\phi$ is called a rational topological preferred framing if $[L, \phi] \in \ker i_2^Q$.

When $L$ is null-homologous, it is clear that a topological preferred framing is also a rational topological preferred framing.

### 6.1.1 Comparing $\ker i_1$ and $\ker i_2$

In order to compare various preferred framings and the Lagrangian framing, we need to investigate the relation of the maps $i_1$ and $i_2$ given by (5.3) and (5.4). Let $Y$ be a smooth oriented 4-manifold with boundary $Z = T^3$.

**Lemma 6.1.3.** The maps $i_1$ and $i_2$ satisfy the following properties

1. $r(\ker i_1) + r(\ker i_2) = 3$.

2. With the pairing

$$H_1(Z) \times H_2(Z) \to H_0(Z) \cong \mathbb{Q},$$

given by the cap product, $\ker i_2$ and $\ker i_1$ annihilate each other:

$$\ker i_2 = \text{ann}(\ker i_1) := \{ c \in H_2(Z) | a \cdot c = 0 \in H_0(Z) \text{ for any } a \in \ker i_1 \}$$

and $\ker i_1 = \text{ann}(\ker i_2)$. 
3. \( r(\ker i_1) > 0. \)

**Proof.** 1. Consider the exact sequence

\[
\cdots \xrightarrow{\partial_2} H_2(Z) \xrightarrow{i_2} H_2(Y) \xrightarrow{\partial_2} H_2(Y, Z) \xrightarrow{\partial_1} H_1(Z) \xrightarrow{i_1} H_1(Y) \xrightarrow{\partial_1} \cdots
\]

It induces a short exact sequence

\[
0 \longrightarrow H_2(Y)/\text{Im}i_2 \xrightarrow{\nu_2} H_2(Y, Z) \xrightarrow{\partial_1} \text{Im}\partial_1 = \ker i_1 \longrightarrow 0 \quad (6.1)
\]

By Lefschetz duality and universal coefficient theorem,

\[
H_2(Y, Z) \cong H^2(Y) \cong H_2(Y)
\]

and \( r(H_2(Y, Z)) = r(H_2(Y)) \). So \((6.1)\) implies that

\[
\begin{align*}
\quad r(\ker i_1) &= r(\text{Im}i_2) = r(H_2(Z)) - r(\ker i_2) \\
\end{align*}
\]

which is \((1)\).

2. Consider the dual pairing

\[
\begin{array}{cccc}
H_1(Y) & \times & H^1(Y) & \rightarrow & H_0(Y) \\
\uparrow i_1 & & \downarrow j & & \uparrow \cong \\
H_1(Z) & \times & H^1(Z) & \rightarrow & H_0(Z)
\end{array}
\]

Because the maps \( i_1 \) and \( j \) are induced by embedding and restriction, this pairing is natural, i.e., for \( a \in H_1(Z) \) and \( \alpha \in H^1(Y) \),

\[
< i_1(a), \alpha > = < a, j(\alpha) >
\]

There is an isomorphism of long exact sequences induced naturally by Lefschetz and Poincaré dualities:

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & H^1(Y, Z) & \rightarrow & H^1(Y) & \rightarrow & H^1(Z) & \xrightarrow{j} & H^2(Y, Z) & \rightarrow & \cdots \\
\downarrow \cap [Y] & & \downarrow \cap [Y] & & \downarrow \cap [Z] & & \downarrow \cap [Y] & & & & & \\
\cdots & \rightarrow & H_3(Y) & \xrightarrow{\delta_3} & H_3(Y, Z) & \xrightarrow{\partial_2} & H_2(Z) & \xrightarrow{i_2} & H_2(Y) & \xrightarrow{\delta_2} & \cdots
\end{array}
\]
Using this diagram, the dual pairing induces the intersection pairing:

\[
\begin{array}{c}
\downarrow \\
H_3(Y) \\
\downarrow \\
H_1(Y) \times H_3(Y, Z) \rightarrow H_0(Y) \\
\uparrow i_1 \\
\downarrow \partial = \partial_2 \\
H_1(Z) \times H_2(Z) \rightarrow H_0(Z) \\
\downarrow i_2 \\
H_2(Y) \\
\downarrow
\end{array}
\]

Given \( z_2 \in \ker i_2 \), there exists \( z_3 \in H_3(Y, Z) \) such that \( \partial z_3 = z_2 \). Let \( \beta \) be the Lefschetz dual of \( z_3 \) in \( H^1(Y) \). For any \( z_1 \in \ker i_1 \),

\[
z_1 \cdot z_2 = z_1 \cdot \partial z_3 = i_1(z_1) \cdot z_3 = \langle i_1(z_1), \beta \rangle = 0
\]

It shows that \( \ker i_2 \subset \ann(\ker i_1) \). From part (1),

\[
r(\ann(\ker i_1)) = r(H_2(Z)) - r(\ker i_1) = r(\ker i_2).
\]

So \( \ker i_2 = \ann(\ker i_1) \). Similar argument shows that \( \ker i_1 = \ann(\ker i_2) \).

3. Suppose \( r(\ker i_1) = 0 \), then \( r(\ker i_2) = 3 \) and \( i_2 \) is the zero map by part (1). Let \( T_1, T_2 \) be two nonisotopic embedded tori in \( Z \) intersecting in a curve \( C \) transversely. Since \( Z = T^3 \), \([T_1] \cap [T_2] = [C] \neq 0 \) in \( H_1(Z) \). Meanwhile, each \( T_i \) bounds a 3-manifold \( W_i \) in \( Y \). \( W_1 \cap W_2 \) is a 2-cycle whose boundary is \( C \) and \([C]\) is in the kernel of \( i_1 \), which contradicts the assumption that \( i_1 \) is injective.

Here is a geometric interpretation of this lemma. Assume \( z_2 \) is an integral class of \( \ker i_2 \) and \( C \) is a closed curve in \( Z \) such that \([C] \cdot z_2 \neq 0 \) in \( Z \). There exists a relative 3-cycle \( W \) in \((Y, Z)\) such that \([\partial W] = z_2 \). In particular, we can assume that \( W \) intersects \( Z \) transversely and \( \partial W \) intersects \( C \) transversely at \( a_1, \ldots, a_p \) and \( b_1, \ldots, b_n \) in \( Z \) with positive and negative intersections respectively. Furthermore, we can give a collar structure \( V \cong Z \times [0, \epsilon) \) near \( Z \) and assume \( W \cap V = \partial W \times [0, \epsilon) \). If we
push $C$ to $C' = C \times \frac{\epsilon}{2}$ in the interior of $Y$, then $C'$ and $W$ intersect transversely at $a_1 \times \frac{\epsilon}{2}, \cdots, a_p \times \frac{\epsilon}{2}$ and $b_1 \times \frac{\epsilon}{2}, \cdots, b_n \times \frac{\epsilon}{2}$ with positive and negative intersections respectively. Hence $[C'] = i_1([C])$ and

$$[C'] \cdot [W] = [C] \cdot [\partial W] = p - n = [C] \cdot z_2 \neq 0$$

So $[C]$ cannot be in $\ker i_1$.

**Remark 6.1.4.**

1. Part (1) of Lemma 6.1.3 is still true in arbitrary dimension. If $Y$ is a $(n+1)$-dimensional manifold with connected boundary $Z$ and $i_k : H_k(Z) \to H_k(Y)$ denotes the homomorphism induced by the inclusion $Z \to Y$, then

$$r(\ker i_{k-1}) + r(\ker i_k) = r(H_k(Z))$$

for $2 \leq k \leq n - 1$.

2. Part (3) of Lemma 6.1.3 is pointed out by Robert Gompf.

In the following, we give examples to illustrate Lemma 6.1.3 according to $r(\ker i_1)$.

1. Let $K_0$ be the trivial knot in $S^3$ and $X = S^1 \times S^3$. The complement of the torus $L = S^1 \times K_0$ is

$$Y = S^1 \times (S^3 - K_0) \cong S^1 \times (S^1 \times D^2)$$

If $t,m$ denote the isotopy classes of these two $S^1$ and $l = \partial D^2$, then $H_1(Z) = \langle [t], [m], [l] \rangle$ and $\ker i_1$ has rank 1 which is generated by $[l]$. On the other hand, $\ker i_2$ is generated by $[l \times t], [l \times m]$ and has rank 2.

In general, if $K$ is any knot in $S^3$ and $S$ is a Seifert surface with boundary $K$, we can define $t$ and $m$ as above and choose $l$ as the push-off of $K$ in $S$. Then $Y$ and $T^2 \times D^2$ have isomorphic homology groups and $\ker i_1$ is still generated by $l$, which bounds the surface $S$. Similarly, $\ker i_2$ has rank 2 and is generated by $[l \times t], [l \times m]$. They bound $S^1 \times S$ and $\{pt\} \times (S^3 - K)$ respectively. In [16], Fintushel and Stern use these manifolds as building blocks to define knot surgery in 4-manifolds.

In the next example, the results of Lemma 6.1.3 are not obvious. Let $\pi : X \to \Sigma_g$ be a ruled surface and the loop $\gamma \subset X$ be a lift of a loop in $\Sigma_g$. We can construct
a torus $L$ in $X$ as the product of $\gamma$ and some circle $b$ in the fiber. If $\mu \subset Z$ is a meridian of $L$ and $\pi(\gamma)$ is nontrivial in $\pi_1(\Sigma_g)$, it is easy to show that $\ker i_1$ is generated by a push-off of $b$. But $\ker i_2$ is not obvious even when $X = S^2 \times \Sigma_g$ is the trivial bundle. By Lemma 6.1.3 we know that $\ker i_2$ has rank 2 and is generated by $[\mu \times b]$ and $[\mu \times \gamma]$.

2. Let $L = a \times b$ be the Clifford torus embedded in the rational manifold $X = \mathbb{C}P^2$. The group $H_1(Z)$ is generated by $[a], [b]$ and the meridian $[\mu]$. It is easy to show that $\ker i_1 = \langle [a], [b] \rangle$ has rank 2 and $\ker i_2 = \langle [a \times b] \rangle$.

In general, if $X$ is simply connected and $L \subset X$ is a torus with trivial normal bundle, then $r(\ker i_1) = 2$ if and only if $[L] = 0$ in $H_2(X)$.

3. If $r(\ker i_1) = 3$, it follows from Lemma 5.4.3 that such surgery will not change the homology for any torus surgery.

There are similar results for Lemma 6.1.3 over $\mathbb{Z}$ if we consider $r(\cdot)$ as the rank of abelian groups. In particular, the following lemma is the analogue of 6.1.3(2).

**Lemma 6.1.5.** With the pairing

$$H_1(Z; \mathbb{Z}) \times H_2(Z; \mathbb{Z}) \to H_0(Z; \mathbb{Z}) \cong \mathbb{Z},$$

given by the cap product, $\ker i_2$ annihilates $\ker i_1$:

$$\ker i_2 \subset \text{ann}\,(\ker i_1) = \{ c \in H_2(Z; \mathbb{Z}) | a \cdot c = 0 \in H_0(Z; \mathbb{Z}) \text{ for any } a \in \ker i_1 \}.$$

**Proof.** By Lemma 6.1.3(2), $\ker i_1^Q = \text{ann}(\ker i_2^Q)$. If $H_1(Z; \mathbb{Z})$ is considered as the integral elements of $H_1(Z; \mathbb{Q})$, then $\ker i_1^Q = \ker i_1^P \otimes \mathbb{Q}$ and $\ker i_2^P \subset \ker i_1^Q$. So $\ker i_1^P \subset \text{ann}(\ker i_2^Q) = \text{ann}(\ker i_2^P)$. \qed

### 6.1.2 Preferred framings via $\ker i_1$

Now we characterize topological preferred framings via $i_1$. We first consider the rational ones.

**Proposition 6.1.6.** Assume $[L] = 0$ in $H_2(X; \mathbb{Q})$ and $\varphi$ is a framing of $L$. Then $\varphi$ is a rational topological preferred framing if and only if $\ker i_1^Q \subset H_1,\varphi \otimes \mathbb{Q}$. 
Proof.

\( \varphi \) is a rational topological preferred framing

\[ \Leftrightarrow [L_\varphi] \in \ker i_2^Q \]
\[ \Leftrightarrow < [L_\varphi] > \subset \ker i_2^Q \]
\[ \Leftrightarrow \ann (< [L_\varphi] >) \supset \ann (\ker i_2^Q) \]
\[ \Leftrightarrow \ker i_1^Q \subset H_{1, \varphi} \otimes Q. \text{(Lemma 6.1.3)} \]

\[ \square \]

In the integral cases, we have

**Proposition 6.1.7.** Suppose \( L \) is null-homologous and \( \varphi \) is a framing of \( L \). Then

1. \( L \) has topological preferred framings, and
2. \( \ker i_1^Z \subset H_{1, \varphi} \) if \( \varphi \) is a topological preferred framing.

Proof.

1. Since \([L] = 0 \) in \( H_2(X; \mathbb{Z}) \), there exists a 3-chain \( W \) such that \( \partial W = L \).

We can assume that \( W \) intersects \( Z \) transversely. In fact, we can choose a framing \( \varphi : U \rightarrow T^2 \times D^2 \) such that \( W \cap U = \varphi^{-1}(T^2 \times S_x) \), where \( S_x = \{(x, 0) \in D^2 | x \geq 0 \} \).

Then \( W \cap Z = \varphi^{-1}(T^2 \times (1, 0)) \) is a longitudinal torus of \( \varphi \) and \( W \cap Y \) is a relative 3-cycle of \((Y, Z)\) with \( \partial(W \cap Y) = W \cap Z \). So \([W \cap Z] \in \ker i_2^Z \) and \( \varphi \) is a topological preferred framing.

2. \([L_\varphi] \in \ker i_1^Z \) implies that \( \ann_Z([L_\varphi]) \supset \ann_Z(\ker i_2^Z) \). It is easy to observe that \( \ann_Z(< [L_\varphi] >) = H_{1, \varphi} \).

By Lemma 6.1.5,

\[ \ker i_1^Z \subset \ann_Z(\ker i_2^Z) \subset \ann_Z([L_\varphi]) = H_{1, \varphi}. \]

\[ \square \]

If \([L] \) is torsion in \( X \), Proposition 6.1.7(1) may fail in two situations. First, there may exist \( a \in H_1(Z; \mathbb{Z}) \) such that \( a \notin \ker i_1^Z \) but \( ka \in \ker i_1^Z \) for some nonzero integer \( k \). So we can only define rational topological preferred framings. Second, \([\mu] \) and \( \ker i_1^Z \) might not generate the group \( H_1(Z; \mathbb{Z}) \). In this case, rational topological preferred framings also do not exist.
Remark 6.1.8. 1. In knot theory, the notion of preferred framings is similar to that of Definition 6.1.1. Let $M$ be an integral homology 3-sphere and $K \subset M$ be a knot. If $V$ is a tubular neighborhood of $K$, a diffeomorphism $h : S^1 \times D^2 \to V$ satisfying $h(S^1 \times 0) = K$ is called a framing of $K$. Furthermore, $h$ is called a preferred framing if $h(S^1 \times a)$ is homologically trivial in $M - V$. For any knot $K$ in $M$, preferred framings exist and are unique up to isotopy ([43]).

2. It is easy to see from Proposition 6.1.7 that Luttinger's definition coincides with Definition 6.1.1 when $X = \mathbb{R}^4$.

3. In [17], an invariant $\lambda(L)$ is defined when $[L] = 0$ and $L$ has a unique topological preferred framing $\varphi_0$. Assume $\varphi_{\text{Lag}}$ is the Lagrangian preferred framing. Then $\varphi_{\text{Lag}} = \varphi_0$ if and only if $\lambda(L) = 0$. Otherwise, $\lambda(L)$ is the smallest positive integer $k$ such that $k[\mu] + [\gamma_{\varphi}] \in H_1,\varphi_{\text{Lag}}$ for some $[\gamma_{\varphi}] \in H_1,\varphi_0$.

6.1.3 (1, $k$)-surgeries and topological preferred framings

The following proposition relates rational topological preferred framings and (1, $k$)-surgeries.

Proposition 6.1.9. Suppose $X$ is a smooth 4-manifold and $L \subset X$ is a torus with trivial normal bundle such that $[L] = 0$ in $H_2(X;\mathbb{Q})$ and $\varphi$ is a framing of $L$. Let $\tilde{X} = X_{(L,\varphi,\gamma,1,k)}$ be constructed from $X$ via (1, $k$)-surgery along $(L, \varphi, \gamma)$.

1. If $\varphi$ is a rational topological preferred framing of $L$, then $\tilde{X}$ satisfies

$$r(H_1(\tilde{X})) = r(H_1(X))$$

for any $\gamma$ and $k$.

2. If $H_1(\tilde{X};\mathbb{Z}) \cong H_1(X;\mathbb{Z})$ for any $\gamma$ and $k$, then $\varphi$ is a rational topological preferred framing of $L$.

Proof. 1. By Lemma 5.4.3, we have

$$r(H_1(X)) = \begin{cases} r(H_1(Y)) - 1 & \text{if } i_{1}[\mu] \neq 0 \\ r(H_1(Y)) & \text{if } i_{1}[\mu] = 0. \end{cases}$$
Lemma 5.4.1 implies that \( i_1[\mu] \neq 0 \) in \( H_2(Y) \). Since \( \varphi \) is a rational topological preferred framing, Proposition 6.1.6 implies that \( [\mu] + k[\gamma_\varphi] \notin \ker i_1 \) for any integer \( k \) and \( [\gamma_\varphi] \in H_1,\varphi \). So

\[
r(H_1(\tilde{X})) = r(H_1(Y)) - 1 = r(H_1(X))
\]

if \( \tilde{X} \) is given via \((1,k)\)-surgery.

2. We first prove that any \( a \in \ker i_1 \) lies in \( H_1,\varphi \). Let \( a = s[\mu] + t[\gamma_\varphi] \) for some \( s, t \in \mathbb{Z} \) and \( \gamma \subset L \) (Recall that \( \gamma_\varphi \) is a lift of \( \gamma \)). For \( \tilde{X} = X(L,\varphi,\gamma,1,kt) \), the meridian of \( \tilde{L} \) in \( \tilde{X} \) satisfies

\[
[\tilde{\mu}] = [\mu] + kt[\gamma_\varphi] = [\mu] + k(a-s[\mu]) = (1-sk)[\mu] + ka \tag{6.2}
\]

So

\[
H_1(\tilde{X};\mathbb{Z}) = H_1(Y;\mathbb{Z})/\langle i_1^\mathbb{Z}((1-sk)[\mu] + ka) \rangle \quad \text{(by (5.6))}
\]

\[
= H_1(Y;\mathbb{Z})/\langle i_1^\mathbb{Z}((1-sk)[\mu]) \rangle \quad (a \in \ker i_1^\mathbb{Z})
\]

\[
\cong H_1(Y;\mathbb{Z})/\langle i_1^\mathbb{Z}[\mu] \rangle.
\]

Because \([\mu]\) is essential in \( Y \) and \( k \) is arbitrary, \( s \) should be zero and \( a \in H_1,\varphi \). Otherwise, \( H_1(X;\mathbb{Z}) \) has infinitely many torsion classes with different orders.

Tensoring with \( \mathbb{Q} \), we have

\[
\ker i_1 = \ker i_1^\mathbb{Z} \otimes \mathbb{Q} \subset H_1,\varphi \otimes \mathbb{Q}
\]

By Proposition 6.1.6 \( \varphi \) is a rational topological preferred framing.

\[\square\]

For integral cases, we have

**Proposition 6.1.10.** Suppose \( H_1(X;\mathbb{Z}) \) has no torsion and \( L \) is null-homologous. Then a framing \( \varphi \) of \( L \) is a topological preferred framing if and only if \( H_1(\tilde{X};\mathbb{Z}) \cong H_1(X;\mathbb{Z}) \) for any \( \tilde{X} = X(L,\varphi,\gamma,1,k) \) obtained from \( X \) via \((1,k)\)-surgery.

**Proof.** Assume \( \varphi \) is a topological preferred framing. Consider the 3-chain \( W \) given in the proof of Proposition 6.1.7. It is clear that a meridian \( \mu \) intersects \( W \) at one point. So \( i_1^\mathbb{Z}[\mu] \cdot [W] = \pm 1 \) and \( i_1^\mathbb{Z}[\mu] \) is a primitive class. Similarly, the simple closed curve \( \tilde{\mu} \)
has class \([\mu] + k[\gamma]\) and is homotopic to a curve intersecting \(W\) at one point. Hence \(i_1^Z[\mu]\) is also a primitive class for any \(\gamma, k\). Since \(H_1(Y; \mathbb{Z})\) is a free abelian group, we have \(H_1(Y, \mathbb{Z})/ < i_1^Z[\mu] > \cong H_1(Y, \mathbb{Z})/ < i_1^Z[\mu] > \). So \(H_1(X, \mathbb{Z}) \cong H_1(\tilde{X}, \mathbb{Z})\) by Lemma 5.4.3(3).

Conversely, if \(H_1(X, \mathbb{Z}) \cong H_1(\tilde{X}, \mathbb{Z})\) for any \(\tilde{X}\), the proof of Proposition 6.1.9(2) shows that \(\varphi\) is a rational topological preferred framing. The assumption that \(H_2(X, \mathbb{Z})\) has no torsion implies that \(\varphi\) is actually a topological preferred framing.

The knot surgery in [16] is an example of \((1, k)\)-surgeries. Suppose \(X_0 = S^3 \times S^1\), \(K\) is a knot in \(S^3\) and \(L = K \times S^1\). Let \(\mu\) be the meridian of \(K\) and \(b = S^1\). Then ker \(i_1^Z = < [a] >\) and ker \(i_2^Z = < [\mu \times a], [a \times b] >\).

Consider the framings \(\varphi_p\) of \(L\) where \(H_1, \varphi_p\) is generated by \(p[\mu] + [a]\) and \([b]\). It is clear that \(\varphi_p\) is a topological preferred framing of \(L\) exactly when \(p = 0\).

If \(K\) is the trivial knot and \([\gamma] = p[\mu] + [a]\), the resulting manifold of \((1, k)\)-surgery is

\[
S^3_{(L, \varphi_p, \gamma, 1, k)} \cong L(1 + pk, k) \times S^1
\]

and \(H_1(S^3_{(L, \varphi_p, \gamma, 1, k)}; \mathbb{Z}) \cong \mathbb{Z} + kp \oplus \mathbb{Z}\). If \(|p| > 1\), \(H_1(S^3_{(L, \varphi_p, \gamma, 1, k)}; \mathbb{Z})\) has rank 1 for any \(p, k\), but the torsion subgroups vary. Propositions 6.1.10 and 6.1.9 show that \(\varphi_p\) is not a rational topological preferred framing. Actually, such framings do not exist for \(L\).

### 6.1.4 Constraints for Lagrangian framings

Here we provide topological constraints on the isotopy classes of Lagrangian tori in many symplectic manifolds with non-positive Kodaira dimension. In particular, they imply that the invariant \(\lambda(L)\) of Fintushel and Stern (Remark 6.1.8(3)) is zero if the manifold has non-positive Kodaira dimension and vanishing integral \(H_1\). Recall that the Lagrangian framing for Lagrangian tori is defined in Section 5.2.1.

**Proposition 6.1.11.** Suppose \(L\) is a Lagrangian torus in \((X, \omega)\) and any Luttinger surgery along \(L\) preserves the integral homology.

1. If \(H_2(X; \mathbb{Z})\) is torsion free and \(L\) is null-homologous, then the Lagrangian framing of \(L\) is a topological preferred framing.
2. If $|L|$ is torsion then the Lagrangian framing of $L$ is a rational topological preferred framing.

Proof. The result follows directly from Propositions 6.1.10 and 6.1.9.

In particular, we have

**Corollary 6.1.12.** If $\kappa(X) = -\infty$ and $L$ is a Lagrangian torus in $X$, then the Lagrangian framing of $L$ is a topological preferred framing.

Proof. Since $b^+(X) = 1$ and $H_2(X; \mathbb{Z})$ has no torsion, $L$ is null-homologous. For any $\tilde{X}$ given from $X$ by applying Luttinger surgery along $L$, $\tilde{X}$ is diffeomorphic to $X$ by Theorem 1.0.10. Now the claim follows from Proposition 6.1.11.

In the case that $X$ is a symplectic CY surface, it is convenient to introduce

**Definition 6.1.13.** An embedded 2-torus $L$ of a 4-manifold $X$ is called essential if $[L] \neq 0$ in $H_2(X)$. Moreover, $L$ is called completely essential if $r(\ker i_1^Q) = 3$.

**Proposition 6.1.14.** If $\kappa(X) = 0$ and $X$ is an integral homology K3, then any Lagrangian torus $L$ satisfies one of the following conditions:

1. $L$ is null-homologous and the Lagrangian framing is a topological preferred framing.
2. $L$ is completely essential.

Proof. When $L$ is null-homologous, the claim follows from Theorem 1.0.11 and Proposition 6.1.11.

When $L$ is essential, $[\mu] \in \ker i_1^Q$ by Lemma 5.4.1. If $r(\ker i_1^Q) \neq 3$, there exists $\gamma \subset L$ such that $i_1^Q([\gamma]) \neq 0$. In the manifold $\hat{X} = X(L, \gamma, 1)$, the class $[\hat{\mu}] = [\mu] + [\gamma]$ is nonzero in $H_1(\hat{Y})$. By Proposition 5.4.4, $b_1(X) \neq b_1(\hat{X})$, which contradicts Theorem 1.0.11. So $r(\ker i_1^Q) = 3$ and $L$ is completely essential.

Similarly we have

**Proposition 6.1.15.** If $\kappa(X) = 0$ and $X$ is an integral homology Enriques surface, then any Lagrangian torus $L$ satisfies one of the following conditions:
1. $L$ is null-homologous and the Lagrangian framing is a topological preferred framing.

2. $[L]$ is torsion and the Lagrangian framing is a rational topological preferred framing.

Proof of Theorem 1.0.12. The first statement follows from Theorem 5.4.7, Corollary 6.1.12, Propositions 6.1.14(1) and 6.1.15(1). The last statement on $\lambda(L)$ follows from Remark 6.1.8(3).
References


[41] Ovando, Four dimensional symplectic Lie algebras


