

Variational Methods and the Orbits with Collisions in the
N-body Problem

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Dedication

To my parents, Da-Chin Huang and Lin-Chu Chen, and my wife, Meng-Chiao Huang

Abstract

The objective of this thesis is to study variational methods in the Newtonian N -body problem.

Chapter 1 contains the introduction of Chapters 2 and 3. Chapter 2 is devoted to the variational proof of the Schubart-like orbits in the collinear four-body problem. In Chapter 3, we discuss the limiting configurations of the action-minimizing triple collision orbits in the planar three body problem.

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Chapter 1

Introduction

1.1 The N -Body Problem

The Newtonian N -body problem describes the motion of N particles in a d -dimensional Euclidean space according to the Newton's gravitation law. We consider N point particles with positive masses m_i and positions $x_i \in \mathbb{R}^d$. The equations of motion can be written as

$$m_i \ddot{x}_i = \frac{m_i m_j (x_j - x_i)}{|x_j - x_i|^3} = \frac{\partial}{\partial x_i} U(x_1, \dots, x_N), \quad i = 1, \dots, N, \quad (1.1)$$

where $U(x_1, \dots, x_N)$ is the potential function,

$$U(x_1, \dots, x_N) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|x_i - x_j|}.$$

The equations (1.1) are Euler-Lagrange equations of the Lagrangian action functional. The least action principle suggests that the solutions of (1.1) are extremal for the Lagrangian action functional. However, variational methods had not been successfully applied to the Newtonian N -body problem until Chenciner and Montgomery's work[1]. The main difficulty is to avoid unnecessary collisions occurred in the minimizers of the Lagrangian action functional. Collisions make the potential functional become infinite and the action may still be finite. They minimized the action functional on a suitable space to yield the remarkable figure-eight orbits of the three-body problem with equal masses. Since then, the periodic and the quasiperiodic solutions

of the N -body problem were studied extensively by variational methods. We refer to [2, 3, 4, 5, 6, 7, 8] and the references therein. The Euler-Lagrange solutions corresponding to action minimizers is called the *minimizing solutions*. Maderna and Venturelli [9] also used the variational method to prove the existence of the parabolic motions in the N -body problem.

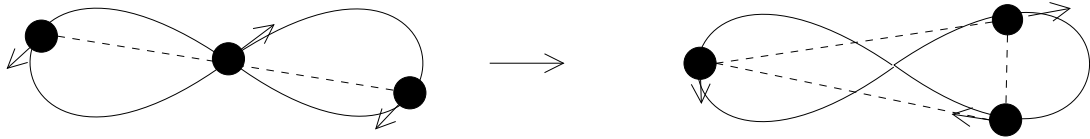


Figure 1.1: Figure-eight orbit in the three-body problem[2]

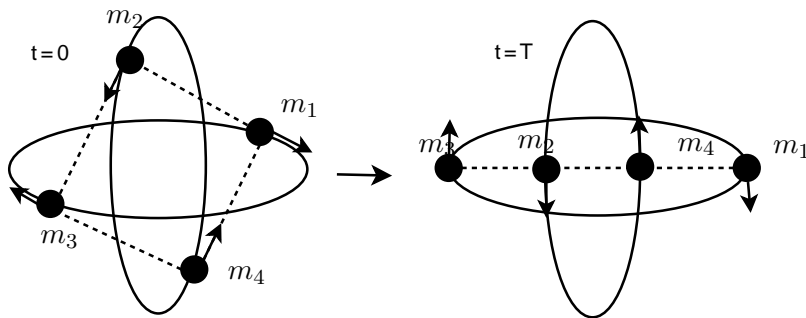


Figure 1.2: Chen's orbit in the four-body problem[3]

1.2 Schubart-like Orbits in the Four-Body Problem

The Schubart-like orbits (Fig.1.4) in the collinear four-body problem perform a motion similar to the one (Fig.1.3) in the collinear three-body problem discovered by Schubart[10] numerically. Schubart-like orbits are periodic solutions having exactly two binary collisions and a simultaneous binary collision per period. In Chapter 2, the proof of the existence of these orbits is based on the direct method in Calculus of Variations. We exploit the variational structure of the problem, and show that the minimizers of Lagrangian action functional in a well-chosen space have the desired properties.

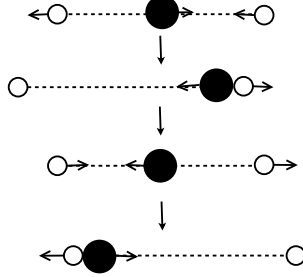


Figure 1.3: A Schubart orbit in the three-body problem

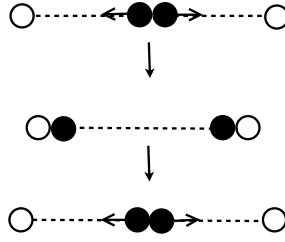


Figure 1.4: A Schubart-like orbit in the four-body problem

1.3 Triple Collision Orbits in the Planar Three-Body Problem

Assume that an orbit of the N -body problem collides at $t = T$, then the size of this orbit has the order $(T - t)^{2/3}$ near the collision time. We try to observe the asymptotic behavior of the total collision orbits by rescale it by $(T - t)^{2/3}$. We call the limit of the rescaled configurations of the total collision orbits when t approaches T the *limiting configurations*. We assume that the particles collide simultaneously at $t = T$, and use the following transformation:

$$\Phi : (x, t) \rightarrow (X, u) \text{ via } X_i = \frac{x_i}{(T - t)^{2/3}}, \quad u = \ln(T - t), \quad i = 1, \dots, N. \quad (1.2)$$

We denote prime be differentiation with respect to u . Then (1.1) becomes

$$X_i'' + \frac{1}{3}X_i' = \frac{2}{9}X_i + \frac{1}{m_i} \frac{\partial U(X_1, \dots, X_N)}{\partial X_i}, \quad i = 1, \dots, N. \quad (1.3)$$

A classical result showed that if $x_i(t) \rightarrow 0$ as $t \rightarrow T$, $\forall i = 1, \dots, N$, then X'_i and $X''_i \rightarrow 0$ as $u \rightarrow -\infty$, $\forall i = 1, \dots, N$. This shows that X approaches the *central configuration* set

$$\mathcal{CC}_N = \{Q \in (\mathbb{R}^2)^N \mid \frac{2}{9}Q_i + \sum_{j=1, \dots, N, j \neq i} \frac{m_j(Q_i - Q_j)}{|Q_i - Q_j|^3} = 0, i = 1 \dots N. \} \quad (1.4)$$

as $u \rightarrow -\infty$. Let \mathbf{C}^* be a central configuration in \mathcal{CC}_N . We denote $\mathcal{M}_{\mathbf{C}^*}$ be the set of all possible rotation of the configuration \mathbf{C}^* . In the planar three-body problem,

$$\mathcal{CC}_3 = \mathcal{M}_{L^+} \cup \mathcal{M}_{L^-} \cup \mathcal{M}_{E_1} \cup \mathcal{M}_{E_2} \cup \mathcal{M}_{E_3}.$$

Here, L^+ and L^- are called Lagrangian (equilateral) points and E_i ($i=1,2,3$) are called Euler (collinear) points.

For the planar three-body problem, Siegel[11] showed that $\lim_{u \rightarrow -\infty} X(u)$ must be some point in \mathcal{CC}_3 . **A natural question is which central configuration $X(u)$ approaches.** We answer this question for the minimizing orbits in the three-body problem. In Chapter 3, we showed that if the initial configuration is not collinear, the configuration of a minimizing orbit keeps its orientation and the limiting configuration is a Lagrangian point. As for collinear initial configuration, with a technical assumption (see (3.8)), there exist two minimizing orbits. The limiting configuration of these orbits are the Lagrangian points.

Chapter 2

Schubart-like Orbits in the Newtonian Collinear Four-Body Problem

The Schubart-like orbits in the collinear four-body problem perform a motion similar to the one in the collinear three-body problem discovered by Schubart[10] numerically. Schubart-like orbits are periodic solutions having exactly two binary collisions and a simultaneous binary collision per period. The proof of the existence of these orbits given in this chapter is based on the direct method in Calculus of Variations. We exploit the variational structure of the problem, and show that the minimizers of Lagrangian action functional in a well-chosen space have the desired properties.

2.1 Introduction

We consider the Newtonian four-body problem on the real line. Let x_i and m_i be the position and the mass of the i -th body, respectively ($i = 1, \dots, 4$). The equations of motion are written as

$$\ddot{x}_i = - \sum_{j \neq i, j=1, \dots, 4} m_j \frac{(x_i - x_j)}{|x_i - x_j|^3}, i = 1, \dots, 4. \quad (2.1)$$

The numerical study of the periodic solutions with binary collisions and simultaneous binary collisions in the symmetric collinear four-body problem was done by

Sweatman[12, 13] and Sekiguchi with Tanikawa [14] independently. The solutions have two binary collisions and a simultaneous binary collision per period. In [13, 14], both of them consider the case when the masses of outer two bodies and inner two bodies can be different ($m_1 = m_4$ and $m_2 = m_3$). Ouyang and Yan[15] used topological methods to show the existence of such periodic solutions. Recently, the linear stability of these solutions is established in [16]. The proof of the linear stability of these solutions is based on the analytic-numerical method of Roberts[17].

The periodic solutions of the four-body problem in [12, 13, 14, 15] are similar to the one found numerically by Schubart[10] in the collinear three-body problem with equal masses. We call the solutions found by Schubart the *Schubart orbits*. The Schubart orbits have two binary collisions per period. Subsequently, Hénon extended this result to the case $m_1 = m_3$. Moeckel[18] gave a rigorous proof based on the shooting argument. After that, Venturelli[6] gave a variational proof. In [18] and [6], they consider the case $m_1 = m_3$. Hereafter, we call the solutions we study in this paper the *Schubart-like orbits* in the collinear four-body problem.

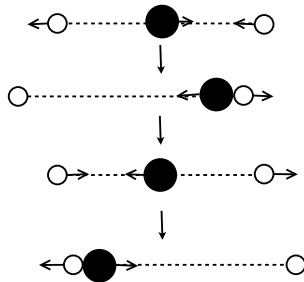


Figure 2.1: A Schubart orbit in the three-body problem

To describe the Schubart-like orbits, we fix the center of mass at the origin and let $2T$ be the period of the orbits. We impose the symmetry on the positions of the four bodies that the positions of the bodies 1 and 2 are the reflection of the positions of the bodies 3 and 4, respectively, i.e.

$$x_1(t) = -x_4(t), x_2(t) = -x_3(t), \quad \text{for } t \in \mathbb{R}.$$

We assume that $x = (x_1, x_2, x_3, x_4)$ has the order $x_1 \geq x_2 \geq x_3 \geq x_4$. At time $t = 0$, a binary collision in x occurs while the positions of the bodies 1 and 4 are on the positive

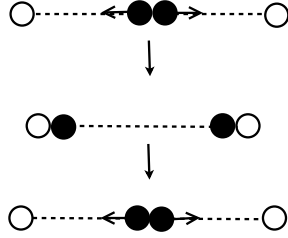


Figure 2.2: A Schubart-like orbit in the four-body problem

half-line and on the negative half-line, respectively, with zero velocity, i.e.

$$x_2(0) = x_3(0), x_1(0) = -x_4(0) > 0, \dot{x}_1(0) = \dot{x}_4(0) = 0. \quad (2.2)$$

When $t \in [0, T]$, the positions of the bodies 1 and 3 decrease, while the positions of the bodies 2 and 4 increase, i.e.

$$\dot{x}_1(t) = -\dot{x}_4(t) < 0 \text{ and } \dot{x}_2(t) = -\dot{x}_3(t) > 0, \text{ for } t \in (0, T). \quad (2.3)$$

At time $t = T$, a simultaneous binary collision in x occurs, i.e.

$$x_1(T) = x_2(T) = -x_3(T) = -x_4(T) > 0. \quad (2.4)$$

When $t \in [T, 2T]$, the orbits are obtained by $x_i(t) = x_i(2T - t)$, $i = 1, \dots, 4$. For $t < 0$, the solutions are thus obtained by $x_i(t) = x_i(-t)$, $i = 1, \dots, 4$. The singularities due to binary collisions and simultaneous binary collisions can be removed with suitable regularization transformation (see Section 2.3). Thus, we can view $x = (x_1(t), x_2(t), x_3(t), x_4(t))$ a periodic solution of (2.1).

In [15], Ouyang and Yan fixed the energy equal -1 and showed that if $x_1(0)$ is less than some constant, there exists a periodic solution of (2.1). The following theorem is the main result in this chapter. The proof is based on variational methods.

Theorem 2.1. *If $m_2 = m_3$, and $m_1 = m_4$, there exists a $2T$ -periodic ($T > 0$) solution $x = (x_1, x_2, x_3, x_4)$ to (2.1), which satisfies the following:*

- (a) $x_1(t) = -x_4(t)$, $x_2(t) = -x_3(t)$ and $(x_1, x_2, x_3, x_4)(t) \neq (0, 0, 0, 0)$ for all $t \in \mathbb{R}$.
 $x_2(t) = x_3(t) = 0$ if and only if $t = 0 \pmod{2T}$ and $x_1(t) = x_2(t) > 0$ if and only if $t = T \pmod{2T}$.

(b) x has the following symmetry properties.

$$x_i(t) = x_i(-t), \quad x_i(2T - t) = x_i(t), \quad i = 1, \dots, 4, \quad \forall t \in \mathbb{R}.$$

(c) $\dot{x}_1(t) = \dot{x}_4(t) = 0$ if and only if $t = 0 \pmod{2T}$.

(d) $\dot{x}_1 = -\dot{x}_4 < 0$ on $(0, T)$, and $\dot{x}_2 = -\dot{x}_3 > 0$ on $(0, T)$; $\dot{x}_1 = -\dot{x}_4 > 0$ on $(T, 2T)$, and $\dot{x}_2 = -\dot{x}_3 < 0$ on $(T, 2T) \pmod{2T}$.

Collision singularity in the N -body problem means that two or more particles occupy the same position. This makes the equations of motion fail. The extension of a collision in a orbit has been studied in different ways. In the collision singularity of the two-body problem, Levi-Civita and Moser used different transformations to extend the collision of the solution as an elastic bounce in the planar and k -dimensional ($k > 2$) case, respectively. Sundman discovered that a solution ending in a collision singularity at time t^* , can be written as a convergent power series in $(t - t^*)^{\frac{1}{3}}$ near $t = t^*$. Such an extension is called *branch regularization*. For the flows near the collision orbits, Easton[19] showed a homeomorphism from collision and near collision orbits to ejection and near ejection orbits by surgery. This means that orbits nearby the collision orbit behave like the collision orbit and do not experience collisions. Easton called this *block regularization*. As for the triple collisions, McGehee[20] showed that, with some values of the masses, the triple collision in the collinear three body problem is neither branch regularizable nor block regularizable.

For the simultaneous binary collisions in the planar N -body problem, Simó and Lacomba[21] showed that a block regularization exists which also preserves time parametrization. They called it *time continuation regularization*. In the collinear N -body problem, Elbiały[22] showed that the simultaneous binary collisions are C^1 -block regularizable. Martínez and Simó [23] showed that the degree of differentiability of the regularization of the simultaneous binary collisions in the planar four-body problem is strictly less than $8/3$.

For the Schubart-like orbits in the four-body problem, only binary collisions and simultaneous binary collisions can happen. One can use Aarseth-Zaere type regularization ([12], or Section 2.3) to show that the collisions in the Schubart-like orbits can be viewed as elastic bounces. From the viewpoint of block regularization, the nearby

orbits of the regularizable collision orbit behave like the collision orbit. By [22] and [21], the Schubart-like orbits in this paper are time continuation regularizable and C^1 -block regularizable. Therefore, the existence of such orbits helps us understand the behavior of the nearby orbits.

In this chapter, we use variational methods to prove the existence of the Schubart-like orbits. In Section 2.2, we show the existence of minimizers of Lagrangian action functional \mathcal{A} in a well-chosen space Σ . In Section 2.3, we use Aarseth-Zaere-type regularization to define the periodic collision solutions in the collinear four-body problem. In Sections 2.4, we show that the minimizers of \mathcal{A} in Σ are free of extra binary and simultaneous binary collisions. In Section 2.5, we show that the minimizers of \mathcal{A} in Σ are free of total collisions. The precise definition of \mathcal{A} and Σ is given in the next section.

2.2 Existence of a minimizer

Without loss of generality, we may assume that $m_1 = m_4 = 1$ and $m_2 = m_3 = m$. Let

$$x_1 = \xi, x_2 = \eta, x_3 = -\eta, x_4 = -\xi$$

in the space

$$\Sigma = \{(\xi, \eta) \in (H^1[0, T], \mathbb{R}^2) \mid \eta(0) = 0, \xi(T) = \eta(T) \text{ and } \xi(t) \geq \eta(t), \text{ for } t \in [0, T]\},$$

where H^1 is Sobolev space. We remark that Σ is a weakly closed subset of a reflexive Banach space with the norm, $\|(\xi, \eta)\| = \sqrt{\|\xi\|_{H^1}^2 + \|\eta\|_{H^1}^2}$. The total collision space, the binary collision space and the simultaneous binary collision space are defined by

$$\Sigma_{TC} = \{(\xi, \eta) \in \Sigma \mid \xi(\tau) = \eta(\tau) = 0 \text{ for some } \tau \in [0, T]\},$$

$$\Sigma_{BC} = \{(\xi, \eta) \mid \xi > 0, \eta = 0.\}$$

and

$$\Sigma_{SBC} = \{(\xi, \eta) \mid \xi = \eta > 0.\}$$

respectively.

(2.1) can be written as follows:

$$\begin{aligned} \ddot{\xi} &= -\frac{1}{4\xi^2} - \frac{m}{(\xi - \eta)^2} - \frac{m}{(\xi + \eta)^2} \\ \ddot{\eta} &= \frac{1}{(\xi - \eta)^2} - \frac{1}{(\xi + \eta)^2} - \frac{m}{4\eta^2}. \end{aligned} \tag{2.5}$$

The corresponding Lagrangian action functional on the half period is defined by

$$\mathcal{A} = \int_0^T \frac{1}{2}(2\dot{\xi}^2 + 2m\dot{\eta}^2) + \frac{1}{2\xi} + \frac{2m}{\xi - \eta} + \frac{2m}{\xi + \eta} + \frac{m^2}{2\eta} dt. \quad (2.6)$$

Here we denote the kinetic energy

$$K = \dot{\xi}^2 + m\dot{\eta}^2$$

and the potential

$$U = \frac{1}{2\xi} + \frac{2m}{\xi - \eta} + \frac{2m}{\xi + \eta} + \frac{m^2}{2\eta}.$$

We need the following theorem to ensure the existence of the minimizers of the action functional \mathcal{A} in Σ .

Theorem 2.2. [24] *Let X be reflexive Banach space and $M \subset X$ a weakly closed subset. If*

- (a) $f : M \rightarrow \mathbb{R}$ is weakly lower semicontinuous,
- (b) f is coercive,

then f attain its infimum on M .

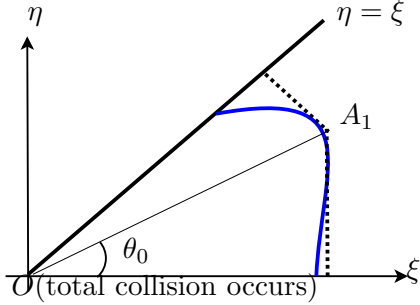
Here, f is (weakly) lower semicontinuous if for any sequence $\{x_n\}$ (weakly) converging to x in X , we have

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n);$$

f is coercive, if $f(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

Since the H^1 -weak convergence means uniform convergence, the integral of the potential is lower semicontinuous by Fatou's lemma. By the weakly lower semicontinuity of the norm $\|\cdot\|$, we thus know that the action functional \mathcal{A} is weakly lower semicontinuous in Σ (we refer to [3, 6] for more details). The minimum of the action functional \mathcal{A} is not achieved in H^1 . For example, we can consider $(\xi_n, \eta_n) = (n, 2n)$, then $\mathcal{A}(\xi_n, \eta_n) \rightarrow 0$ as $n \rightarrow \infty$. We need to show that the restriction of the orbits to Σ ensures the coercivity of the action functional \mathcal{A} . With the following proposition and Theorem 3.44, we prove that the action \mathcal{A} has a minimizer in Σ .

Proposition 2.1. $\mathcal{A}|_{\Sigma}$ is coercive.

Figure 2.3: A path in Σ

Proof. The same argument is used in [6]. Let $t_1 \in [0, T]$ be the time such that

$$\max_{t \in [0, T]} \sqrt{\xi^2 + \eta^2}(t) = \sqrt{\xi^2 + \eta^2}(t_1).$$

Let θ_0 be the angle between $\overrightarrow{OA_1}$ and the ξ -axis, where O is the origin and $A_1 = (\xi(t_1), \eta(t_1))$. It follows that $\frac{\pi}{4} - \theta_0$ is the angle of the two lines, $\overrightarrow{OA_1}$ and $\xi = \eta$. Because the sum of the distance from A_1 to the ξ -axis and the distance from A_1 to $\xi = \eta$ is less than the length of the curve connecting ξ -axis and $\xi = \eta$ (see Fig. 2.2), we thus have

$$\sin \theta_0 \sqrt{\xi^2 + \eta^2}(t_1) + \sin\left(\frac{\pi}{4} - \theta_0\right) \sqrt{\xi^2 + \eta^2}(t_1) \leq \int_0^T \sqrt{\dot{\xi}^2 + \dot{\eta}^2} dt.$$

The minimum of $\sin(\theta_0) + \sin(\frac{\pi}{4} - \theta_0)$ for $\theta_0 \in (0, \pi/4)$ is achieved when $\theta_0 = 0$ or $\theta_0 = \pi/4$, and is equal to $1/\sqrt{2}$. It follows from

$$\sin(\theta_0) + \sin\left(\frac{\pi}{4} - \theta_0\right) \geq 1/\sqrt{2} \quad \text{for } \theta_0 \in (0, \pi/4)$$

and the Hölder inequality that

$$\frac{1}{\sqrt{2}} \sqrt{\xi^2 + \eta^2}(t_1) \leq \int_0^T \sqrt{\dot{\xi}^2 + \dot{\eta}^2} dt \leq T^{1/2} \left[\int_0^T \dot{\xi}^2 + \dot{\eta}^2 dt \right]^{1/2}. \quad (2.7)$$

We obtain

$$\int_0^T \dot{\xi}^2 + \dot{\eta}^2 dt \leq T \left[\sqrt{\xi^2 + \eta^2}(t_1) \right]^2 \leq 2T^2 \int_0^T \dot{\xi}^2 + \dot{\eta}^2 dt \leq \frac{2T^2}{\min\{1, m\}} \mathcal{A}. \quad (2.8)$$

It follows that $\|(\xi, \eta)\|^2 \leq c\mathcal{A}(\xi, \eta)$ where c is a constant dependent on m and T . \square

Proposition 2.2. *If (ξ, η) is a minimizer of $\mathcal{A}|_{\Sigma}$, it is a solution of (2.5) in every time interval without collision times.*

We omit the proof of Proposition 2.2, because the argument is standard (see, e.g., [6]). We remark that the set of collision times is a closed subset with zero measure of $[0, T]$.

2.3 Regularization of minimizers

In this section, we introduce the Aarseth-Zare-type regularization, so that we have a rigorous definition of periodic collision solutions of the collinear four-body problem. The binary collisions and simultaneous binary collisions in (2.5) can be viewed as elastic bounces after regularization of these collisions.

We use two transformations to obtain the regularized Hamiltonian. For (ξ, η) , the system has Hamiltonian

$$H = \frac{1}{4}u^2 + \frac{1}{4m}v^2 - \frac{1}{2\xi} - \frac{m^2}{2\eta} - \frac{2m}{\xi + \eta} - \frac{2m}{\xi - \eta}$$

where u and v are the conjugate momentum to ξ and η respectively, i.e.

$$u = 2\dot{\xi} \quad \text{and} \quad v = 2m\dot{\eta}.$$

We now apply canonical transform to the coordinate (ξ, η, u, v) by

$$q_1 = \xi - \eta, \quad q_2 = 2\eta, \quad p_1 = u, \quad p_2 = \frac{1}{2}(u + v).$$

The new Hamiltonian is

$$H = \left(1 + \frac{1}{m}\right)\frac{p_1^2}{4} - \frac{p_1 p_2}{m} + \frac{p_2^2}{m} - \frac{2m}{q_1} - \frac{m^2}{q_2} - \frac{2m}{q_1 + q_2} - \frac{1}{2q_1 + q_2}. \quad (2.9)$$

We then use another canonical transform and rescale the time t by

$$q_i = Q_i^2, \quad p_i = \frac{P_i}{2Q_i} \quad (i = 1, 2) \quad \text{and} \quad dt = q_1 q_2 ds.$$

The regularized Hamiltonian is

$$\begin{aligned} \Gamma &= \frac{dt}{ds}(H - E) \\ &= \frac{(m+1)Q_2^2 P_1^2 - 4Q_1 Q_2 P_1 P_2 + 4Q_1^2 P_2^2}{16m} - m^2 Q_1^2 - 2m Q_2^2 \\ &\quad - \frac{2Q_1^2 Q_2^2}{Q_1^2 + Q_2^2} - \frac{2m Q_1^2 Q_2^2}{2Q_1^2 + Q_2^2} - Q_1^2 Q_2^2 E \end{aligned}$$

where E is the energy, and the equation of motion for the regularized Hamiltonian Γ are

$$\begin{cases} \frac{d}{ds}Q_1 &= \frac{1+m}{8m}Q_2^2P_1 - \frac{1}{4m}Q_1Q_2P_2 \\ \frac{d}{ds}Q_2 &= \frac{1}{2m}Q_1^2P_2 - \frac{1}{4m}Q_1Q_2P_1 \\ \frac{d}{ds}P_1 &= \frac{1}{4m}P_1P_2Q_2 - \frac{1}{2m}Q_1P_2^2 + 2m^2Q_1 + \frac{4mQ_1Q_2^4}{(Q_1^2+Q_2^2)^2} + \frac{2Q_1Q_2^4}{(Q_1^2+Q_2^2)^2} + 2Q_1Q_2^2E \\ \frac{d}{ds}P_2 &= \frac{1}{4m}P_1P_2Q_1 - \frac{1+m}{8m}Q_2P_1^2 + 4mQ_2 + \frac{4mQ_1^4Q_2}{(Q_1^2+Q_2^2)^2} + \frac{4Q_1^4Q_2}{(Q_1^2+Q_2^2)^2} + 2Q_1^2Q_2E. \end{cases} \quad (2.10)$$

We remark here that the map $(Q_1, Q_2, P_1, P_2) \rightarrow (q_1, q_2, p_1, p_2)$ is a covering map. The regularized equations (2.10) have physical meaning only when $\Gamma = 0$. It means that the projection of (Q_1, Q_2, P_1, P_2) are the solutions of (2.5) up to time scale when $\Gamma = 0$.

Definition 2.1. *A continuous map $(\xi, \eta) : \mathbb{R} \rightarrow \mathcal{X}$ is called a periodic collision solution of (2.5) if the following conditions are satisfied.*

- (a) *Only binary collisions or simultaneous binary collisions occur.*
- (b) *Binary and simultaneous binary collision times are isolated.*
- (c) *$(\xi, \eta)(t)$ are solutions of (2.5) in every time interval without collisions.*
- (d) *Binary and simultaneous binary collisions are regularized in the following sense.*
 - (1) *The total energy E is constant.*
 - (2) *After time reparametrization, the map (Q_1, Q_2, P_1, P_2) is a periodic solution of (2.10).*

Let (ξ, η) be a minimizer of $\mathcal{A}|_\Sigma$ and (u, v) be conjugate momentum to (ξ, η) . We show that (xi, η) satisfy Definition 2.1 (d). Assume that $\dot{\xi}(0) = 0$ and $\dot{\xi}(T) + m\dot{\eta}(T) = 0$ (We will show that the minimizers of $\mathcal{A}|_\Sigma$ satisfy this property in Section 6). Let (O_1, Q_2, P_1, P_2) be the lift of (ξ, η, u, v) . Then they satisfy

$$Q_1(0) = A, Q_2(0) = 0, P_1(0) = 0, P_2(0) = 2 \text{ for some } A > 0, \quad (2.11)$$

and

$$Q_1(S) = 0, Q_2(S) = B, P_1(S) = -4, P_2(S) = 0 \text{ for some } B > 0, \quad (2.12)$$

where $S = \int_0^T \frac{dt}{q_1(t)q_2(t)}$.

By the symmetry of (2.10), $(Q_1, Q_2, P_1, P_2)(s)$ satisfy

$$\begin{aligned} Q_1(-s) &= Q_1(s), \quad Q_2(-s) = -Q_2(s), \quad Q_1(2S - s) = -Q_1(s), \quad Q_2(2S - s) = Q_2(s), \\ P_1(-s) &= -P_1(s), \quad P_2(-s) = P_2(s), \quad P_1(2S - s) = P_1(s), \quad P_2(2S - s) = -P_2(s). \end{aligned}$$

We thus obtain a periodic solution to (2.10). Let (ξ, η, u, v) be the projection of (Q_1, Q_2, P_1, P_2) for $s \in \mathbb{R}$. (ξ, η) satisfies

$$\xi(-t) = \xi(2T - t) = \xi(t), \quad \eta(-t) = \eta(2T - t) = \eta(t). \quad (2.13)$$

Since (ξ, η) satisfies the above symmetry, the total energy is constant.

2.4 Absence of extra binary and simultaneous binary collisions

In this section, we show that the minimizers of $\mathcal{A}|_\Sigma$ have no extra binary or simultaneous binary collisions. Let

$$\xi = \frac{r}{2}\sqrt{1 + \sin \theta}, \quad \eta = \frac{r}{2}\sqrt{\frac{1 - \sin \theta}{m}}, \quad (2.14)$$

where $r = \sqrt{2(\xi^2 + m\eta^2)}$, and θ satisfies

$$\cos \theta = \frac{-4\sqrt{m}\xi\eta}{r^2} \quad \text{and} \quad \sin \theta = \frac{2(\xi^2 - m\eta^2)}{r^2}.^1$$

In the (r, θ) -coordinates, the binary collision space

$$\Sigma_{BC} = \{(\xi, \eta) | \xi > 0, \eta = 0\} = \{(r, \theta) | r > 0, \theta = \frac{\pi}{2}\},$$

and the simultaneous binary collision space

$$\Sigma_{SBC} = \{(\xi, \eta) | \xi = \eta > 0\} = \{(r, \theta) | r > 0, \theta = \theta_m\}.$$

Here $\theta_m \in (\pi/2, 3\pi/2)$ is such that $\sin \theta_m = \frac{1-m}{1+m}$.

The kinetic energy and the potential in the (r, θ) -coordinates can be written as follows.

¹ The choice of (r, θ) is motivated from the combination of Jacobi coordinates and the Hopf map. We refer to [1, 3] for more details.

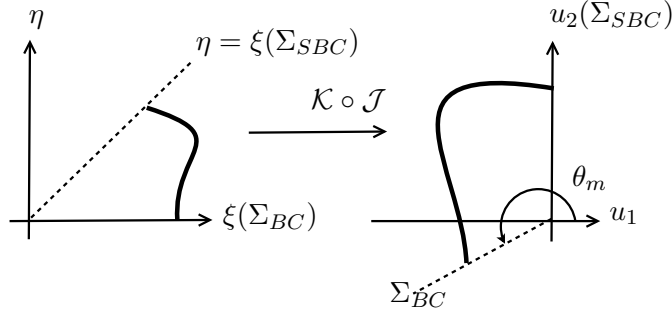


Figure 2.4: (ξ, η) and (u_1, u_2) -plane in the case of $m > 1$.

Lemma 2.1. (*[1, 3]*)

1. The kinetic energy K in (r, θ) is given by

$$K = \dot{r}^2 + \frac{r^2}{4}\dot{\theta}^2.$$

2. The potential U in (r, θ) is given by

$$\begin{aligned} U(r, \theta) &= \frac{1}{r} \left(\frac{1}{\sqrt{1 + \sin \theta}} + \frac{4m}{\sqrt{1 + \sin \theta} - \sqrt{\frac{1 - \sin \theta}{m}}} + \frac{4m}{\sqrt{1 + \sin \theta} + \sqrt{\frac{1 - \sin \theta}{m}}} + \frac{m^{5/2}}{\sqrt{1 - \sin \theta}} \right) \\ &:= \frac{1}{r} U_0(\theta), \quad \theta \in \left[\frac{\pi}{2}, \theta_m \right]. \end{aligned} \tag{2.15}$$

Differentiating $U_0(\theta)$ with respect to θ , we have

$$\begin{aligned} U_0'(\theta) &= -\frac{\cos \theta}{2(1 + \sin \theta)^{3/2}} - 2m \frac{\frac{\cos \theta}{\sqrt{1 + \sin \theta}} + \frac{\cos \theta}{\sqrt{m(1 - \sin \theta)}}}{\left(\sqrt{1 + \sin \theta} - \sqrt{\frac{1 - \sin \theta}{m}} \right)^2} \\ &\quad - 2m \frac{\frac{\cos \theta}{\sqrt{1 + \sin \theta}} - \frac{\cos \theta}{\sqrt{m(1 - \sin \theta)}}}{\left(\sqrt{1 + \sin \theta} + \sqrt{\frac{1 - \sin \theta}{m}} \right)^2} + \frac{m^{5/2} \cos \theta}{2(1 - \sin \theta)^{3/2}}. \end{aligned} \tag{2.16}$$

When θ is close to $(\frac{\pi}{2})^+$, the fourth term of (2.16) is close to negative infinity and the other terms in $U_0'(\theta)$ are bounded. We conclude that $U_0(\theta)$ is strictly decreasing in $[\frac{\pi}{2}, \frac{\pi}{2} + \epsilon_1]$ for some $\epsilon_1 > 0$. Similarly, $U_0(\theta)$ is strictly decreasing in $[\theta_m - \epsilon_2, \theta_m]$ for some $\epsilon_2 > 0$.

Theorem 2.3. *The minimizers of $\mathcal{A}|_\Sigma$ have no extra binary or simultaneous binary collisions, i.e. $\xi = \eta$ only when $t = T$ and $\eta = 0$ only when $t = 0$.*

Proof. Let $(\xi(t), \eta(t))$ be a minimizer of $\mathcal{A}|_\Sigma$. Assume that $(\xi(t), \eta(t))$ has an extra binary collision. Let (ξ, η) be represented by the (r, θ) -coordinates, we thus have

$$\theta(\tau) = \frac{\pi}{2} \text{ for some } \tau \in (0, T).$$

Given $\epsilon > 0$ sufficiently small, we choose $\delta_1, \delta_2 > 0$, such that

$$\theta(\tau - \delta_1) = \frac{\pi}{2} + \epsilon \text{ and } \theta(\tau + \delta_2) = \frac{\pi}{2} + \epsilon,$$

where

$$\tau - \delta_1 = \max\{t \mid \theta(t) = \frac{\pi}{2} + \epsilon \text{ and } t < \tau\}$$

and

$$\tau + \delta_2 = \min\{t \mid \theta(t) = \frac{\pi}{2} + \epsilon \text{ and } t > \tau\}.$$

We consider a deformation of (r, θ) by

$$\begin{aligned} \bar{r}(t) &= r(t), \quad t \in [0, T] \\ \bar{\theta}(t) &= \begin{cases} \theta(t), & t \in [0, \tau - \delta] \cup [\tau + \delta, T], \\ \theta(\tau), & t \in (\tau - \delta_1, \tau + \delta_2) \end{cases} \end{aligned}$$

By the property of $U_0(\theta)$, we have $U_0(\bar{\theta}(t)) \leq U_0(\theta(t))$ for $t \in (\tau - \delta_1, \tau + \delta_2)$. It follows that $\mathcal{A}(\bar{r}(t), \bar{\theta}(t)) < \mathcal{A}(r(t), \theta(t))$, a contradiction to the minimality of the solution. We can show that $\mathcal{A}|_\Sigma$ has no extra simultaneous binary collisions($\theta = \theta_m$) by the same argument. □

2.5 Absence of total collisions

We use the local deformation method to show that the minimizers of $\mathcal{A}|_\Sigma$ admit no total collisions. This method is due to Venturelli[25, 6]. We use big-oh and little-oh notation for asymptotic statements in this section.

Theorem 2.4. *The minimizers of $\mathcal{A}|_\Sigma$ are free of total collisions.*

Proof. We prove the statement by contradiction. Let $(\xi(t), \eta(t))$ be a minimizer of $\mathcal{A}|_{\Sigma}$, and assume that $\xi(\tau) = \eta(\tau) = 0$ for some $\tau \in [0, T]$. Combining Euler-Moulton Theorem[26] and Sundman's estimates[27], we have

$$\begin{aligned} \xi(t) &= \xi_0 |t - \tau|^{2/3} + o(|t - \tau|^{2/3}), & \dot{\xi}(t) &= \frac{2}{3} \xi_0 |t - \tau|^{-1/3} + o(|t - \tau|^{-1/3}), \\ \eta(t) &= \eta_0 |t - \tau|^{2/3} + o(|t - \tau|^{2/3}), & \dot{\eta}(t) &= \frac{2}{3} \eta_0 |t - \tau|^{-1/3} + o(|t - \tau|^{-1/3}), \end{aligned} \quad (2.17)$$

as $t \rightarrow \tau$, where $\xi_0 > \eta_0 > 0$ satisfying

$$\frac{4}{9}(\xi_0^2 + m\eta_0^2) = \frac{1}{2\xi_0} + \frac{2m}{\xi_0 - \eta_0} + \frac{2m}{\xi_0 + \eta_0} + \frac{m^2}{2\eta_0}.$$

Let the function f_ε be defined by

$$f_\varepsilon(t) = \begin{cases} 1 & \text{if } t \in [-\varepsilon^{3/2}, \varepsilon^{3/2}], \\ \frac{\varepsilon^{3/2} + \varepsilon - t}{\varepsilon} & \text{if } t \in [\varepsilon^{3/2}, \varepsilon^{3/2} + \varepsilon], \\ \frac{t + \varepsilon^{3/2} + \varepsilon}{\varepsilon} & \text{if } t \in [-\varepsilon^{3/2} - \varepsilon, -\varepsilon^{3/2}], \\ 0 & \text{if } t \in \mathbb{R} \setminus [-\varepsilon^{3/2} - \varepsilon, \varepsilon^{3/2} + \varepsilon]. \end{cases} \quad (2.18)$$

If $\varepsilon > 0$ is sufficiently small, let $(\xi_\varepsilon, \eta_\varepsilon)$ be a deformation of (ξ, η) defined by

$$\xi_\varepsilon = \xi + \varepsilon \tilde{\xi} f_\varepsilon(t - \tau), \quad \eta_\varepsilon = \eta + \varepsilon \tilde{\eta} f_\varepsilon, \quad (2.19)$$

where

$$\begin{cases} \tilde{\xi} > 0, \tilde{\eta} = 0, & \text{if } \tau = 0 \\ \tilde{\xi} = \xi_0, \tilde{\eta} = \eta_0, & \text{if } \tau \in (0, T) \\ \tilde{\xi} = \tilde{\eta} > 0, & \text{if } \tau = T. \end{cases} \quad (2.20)$$

We remark here that $(\xi_\varepsilon, \eta_\varepsilon)$ are in Σ . We want to show that

$$\Delta \mathcal{A} = \mathcal{A}(\xi_\varepsilon, \eta_\varepsilon) - \mathcal{A}(\xi, \eta) < 0, \quad (2.21)$$

for $\varepsilon > 0$ sufficiently small. We only prove the case of $\tau \in (0, T)$, and the other two cases can be handled by the same technique.

$\Delta \mathcal{A}$ can be decomposed as follows.

$$\Delta \mathcal{A}(\varepsilon) = \mathcal{A}_1^+(\varepsilon) + \mathcal{A}_2^+(\varepsilon) + \mathcal{A}_3^+(\varepsilon) + \mathcal{A}_1^-(\varepsilon) + \mathcal{A}_2^-(\varepsilon) + \mathcal{A}_3^-(\varepsilon), \quad (2.22)$$

where

$$\begin{aligned}\mathcal{A}_1^+(\varepsilon) &= \int_{\tau+\varepsilon^{3/2}}^{\tau+\varepsilon^{3/2}+\varepsilon} \left[\dot{\xi}_\varepsilon^2 + m\dot{\eta}_\varepsilon^2 \right] - \left[\dot{\xi}^2 + m\dot{\eta}^2 \right] dt, \\ \mathcal{A}_2^+(\varepsilon) &= \int_{\tau+\varepsilon^{3/2}}^{\tau+\varepsilon^{3/2}+\varepsilon} \left[U(\xi_\varepsilon, \eta_\varepsilon) - U(\xi, \eta) \right] dt, \mathcal{A}_3^+(\varepsilon) = \int_{\tau}^{\tau+\varepsilon} \left[U(\xi_\varepsilon, \eta_\varepsilon) - U(\xi, \eta) \right] dt, \\ \mathcal{A}_1^-(\varepsilon) &= \int_{\tau-\varepsilon^{3/2}-\varepsilon}^{\tau-\varepsilon^{3/2}} \left[\dot{\xi}_\varepsilon^2 + m\dot{\eta}_\varepsilon^2 \right] - \left[\dot{\xi}^2 + m\dot{\eta}^2 \right] dt, \\ \mathcal{A}_2^-(\varepsilon) &= \int_{\tau-\varepsilon^{3/2}-\varepsilon}^{\tau-\varepsilon^{3/2}} \left[U(\xi_\varepsilon, \eta_\varepsilon) - U(\xi, \eta) \right] dt, \mathcal{A}_3^-(\varepsilon) = \int_{\tau-\varepsilon}^{\tau} \left[U(\xi_\varepsilon, \eta_\varepsilon) - U(\xi, \eta) \right] dt.\end{aligned}$$

We estimate the term $\mathcal{A}_1^+(\varepsilon)$. By definition of $(\xi_\varepsilon, \eta_\varepsilon)$,

$$\mathcal{A}_1^+(\varepsilon) = \int_{\tau+\varepsilon^{3/2}}^{\tau+\varepsilon^{3/2}+\varepsilon} \left[-2(\xi_0\dot{\xi} + m\eta_0\dot{\eta}) + \xi_0^2 + m\eta_0^2 \right] dt \quad (2.23)$$

Since $\dot{\xi}(t)$ and $\dot{\eta}(t) > 0$ near $t = \tau$, and $\xi_0 > \eta_0 > 0$, we have $\xi_0\dot{\xi}(t) + m\eta_0\dot{\eta}(t) > 0$ near $t = \tau$. We obtain

$$\mathcal{A}_1^+(\varepsilon) \leq (\xi_0^2 + m\eta_0^2)\varepsilon.$$

Since $\xi_\varepsilon \geq \xi$, $\eta_\varepsilon \geq \eta$, $\xi_\varepsilon - \eta_\varepsilon \geq \xi - \eta$, and $\xi_\varepsilon + \eta_\varepsilon \geq \xi + \eta$, for ε sufficiently small, we have

$$U(\xi_\varepsilon, \eta_\varepsilon) \leq U(\xi, \eta).$$

It follows that $\mathcal{A}_2^+(\varepsilon) \leq 0$. Although $\mathcal{A}_3^+(\varepsilon) \leq 0$ by the same argument, we need a more precise estimate of $\mathcal{A}_3^+(\varepsilon)$. We decompose $\mathcal{A}_3^+(\varepsilon)$ as follows.

$$\begin{aligned}\mathcal{A}_3^+(\varepsilon) &= \int_{\tau}^{\tau+\varepsilon^{3/2}} \left[\left(\frac{1}{2\xi_\varepsilon} - \frac{1}{2\xi} \right) + \left(\frac{2m}{\xi_\varepsilon - \eta_\varepsilon} - \frac{2m}{\xi - \eta} \right) + \left(\frac{2m}{\xi_\varepsilon + \eta_\varepsilon} - \frac{2m}{\xi + \eta} \right) \right. \\ &\quad \left. + \left(\frac{m^2}{2\eta_\varepsilon} - \frac{m^2}{2\eta} \right) \right] dt.\end{aligned}$$

By Sundman's estimates and the change of variables $t = \tau + (\varepsilon s)^{3/2}$, we have

$$\begin{aligned}& \int_{\tau}^{\tau+\varepsilon^{3/2}} \frac{1}{2\xi_\varepsilon} - \frac{1}{2\xi} dt \\ &= \frac{1}{2} \int_{\tau}^{\tau+\varepsilon^{3/2}} \frac{1}{\xi_0|t - \tau|^{2/3} + \varepsilon\xi_0 + o(|t - \tau|^{2/3})} - \frac{1}{\xi_0|t - \tau|^{2/3} + o(|t - \tau|^{2/3})} dt \\ &= \frac{3\varepsilon^{3/2}}{4} \int_0^1 \left[\frac{1}{\varepsilon\xi_0 s + \varepsilon\xi_0 + o(\varepsilon s)} - \frac{1}{\varepsilon\xi_0 s + o(\varepsilon s)} \right] s^{1/2} ds \\ &= \frac{3\varepsilon^{1/2}}{4} \int_0^1 \left[\frac{1}{\xi_0 s + \xi_0} - \frac{1}{\xi_0 s} \right] s^{1/2} ds + O(\varepsilon).\end{aligned}$$

Denoting $C_1 = \frac{3}{4} \int_0^1 \left[\frac{1}{\xi_0 s} - \frac{1}{\xi_0 s + \xi_0} \right] s^{\frac{1}{2}} ds > 0$. We obtain

$$\int_{\tau}^{\tau + \varepsilon^{3/2}} \frac{1}{2\xi_{\varepsilon}} - \frac{1}{2\xi} dt = -C_1 \varepsilon^{\frac{1}{3}} + O(\varepsilon).$$

The other three terms in $\mathcal{A}_3^+(\varepsilon)$ have similar estimates. We have

$$\mathcal{A}_3^+(\varepsilon) = -C_2 \varepsilon^{\frac{1}{2}} + O(\varepsilon) \text{ for some } C_2 > 0.$$

Combining the estimates of $\mathcal{A}_1^+(\varepsilon)$, $\mathcal{A}_2^+(\varepsilon)$ and $\mathcal{A}_3^+(\varepsilon)$, we have

$$\mathcal{A}_1^+(\varepsilon) + \mathcal{A}_2^+(\varepsilon) + \mathcal{A}_3^+(\varepsilon) \leq -C_2 \varepsilon^{\frac{1}{2}} + O(\varepsilon). \quad (2.24)$$

In a similar way, we obtain the same estimate for $\mathcal{A}_1^-(\varepsilon) + \mathcal{A}_2^-(\varepsilon) + \mathcal{A}_3^-(\varepsilon)$ which is negative for ε sufficiently small. We find a contradiction since (ξ, η) is a minimizer of $\mathcal{A}|_{\Sigma}$.

□

2.6 End of the proof of Theorem 1

Proposition 2.3. *Let (ξ, η) be a minimizer of $\mathcal{A}|_{\Sigma}$. Then $\dot{\xi}(0) = 0$ and $(\dot{\xi} + m\dot{\eta})(T) = 0$.*

Proof. Let $(\hat{\xi}_{\varepsilon}(t), \hat{\eta}_{\varepsilon}(t))$ be a variation of (ξ, η) defined by

$$\hat{\xi}_{\varepsilon}(t) = \xi(t) + \varepsilon \delta \xi(t), \quad \hat{\eta}_{\varepsilon}(t) = \eta(t) + \varepsilon \delta \eta(t),$$

where $\varepsilon \in \mathbb{R}$ and $(\delta \xi(t), \delta \eta(t)) \in H^1([0, T], \mathbb{R}^2)$. With the restriction of $(\hat{\xi}_{\varepsilon}(t), \hat{\eta}_{\varepsilon}(t)) \in \Sigma$, we have $\delta \eta(0) = 0$ and $\delta \xi(T) = \delta \eta(T)$.

The map $\varepsilon \rightarrow \mathcal{A}|_{\Sigma}((\hat{\xi}_{\varepsilon}, \hat{\eta}_{\varepsilon}))$ is differentiable at $\varepsilon = 0$, we thus have

$$\int_0^T \left[2\dot{\xi} \delta \dot{\xi} + 2m\dot{\eta} \delta \dot{\eta} + \frac{\partial U(\xi, \eta)}{\partial \xi} \delta \xi + \frac{\partial U(\xi, \eta)}{\partial \eta} \delta \eta \right] dt = 0. \quad (2.25)$$

After integration by parts, we obtain

$$\begin{aligned} & \lim_{a \rightarrow T^-} 2 \left[\dot{\xi}(a) \delta \xi(a) + m\dot{\eta}(a) \delta \eta(a) \right] - 2 \left[\dot{\xi}(0) \delta \xi(0) + m\dot{\eta}(0) \delta \eta(0) \right] \\ & + \int_0^T \left[-2\ddot{\xi} \delta \xi - 2m\ddot{\eta} \delta \eta + \frac{\partial U(\xi, \eta)}{\partial \xi} \delta \xi + \frac{\partial U(\xi, \eta)}{\partial \eta} \delta \eta \right] dt \\ & = \lim_{a \rightarrow T^-} 2 \left[\dot{\xi}(a) \delta \xi(a) + m\dot{\eta}(a) \delta \eta(a) \right] - 2\dot{\xi}(0) \delta \xi(0) = 0 \end{aligned} \quad (2.26)$$

Since $\delta\xi(0)$, $\delta\xi(T)$, and $\delta\eta(T)$ are arbitrary, and $\delta\xi(T) = \delta\eta(T)$, we have

$$\lim_{a \rightarrow T^-} \dot{\xi}(a) + m\dot{\eta}(a) = 0 \text{ and } \dot{\xi}(0) = 0.$$

We thus define $\dot{\xi}(T) + m\dot{\eta}(T) = 0$.

□

We now prove Theorem 1 (d). Let (ξ, η) be a minimizer of $\mathcal{A}|_{\Sigma}$. By

$$\dot{\xi}(0) = 0 \text{ and } \ddot{\xi} = -\frac{1}{4\xi^2} - \frac{m}{(\xi - \eta)^2} - \frac{m}{(\xi + \eta)^2} < 0,$$

we have

$$\dot{\xi}(t) < 0 \text{ on } (0, T]. \quad (2.27)$$

By (2.13), we have $\dot{\xi}(t) > 0$ on $[T, 2T)$. Similarly, by

$$(\dot{\xi} + m\dot{\eta})(T) = 0 \text{ and } \ddot{\xi} + m\ddot{\eta} = -\frac{1}{4\xi^2} - \frac{2m}{(\xi + \eta)^2} - \frac{m^2}{4\eta^2} < 0,$$

we have

$$\dot{\xi} + m\dot{\eta} < 0 \text{ on } (T, 2T]. \quad (2.28)$$

It follows from $\dot{\xi}|_{[T, 2T)} > 0$ and (2.28) that $m\dot{\eta} < 0$ on $(T, 2T]$. By (2.13), we have

$$\begin{aligned} (-1)^{|n|} \dot{\xi}(t) &< 0 \text{ when } t \in (nT, (n+1)T), \\ (-1)^{|n|} \dot{\eta}(t) &> 0 \text{ when } t \in (nT, (n+1)T), \end{aligned} \quad (2.29)$$

where $n \in \mathbb{Z}$.

Chapter 3

Minimizing Triple Collision Orbits in the Three-Body Problem

In this chapter, we study the minimizing triple collision orbits in the planar Newtonian three-body problem with arbitrary masses. We show that for a given non-collinear initial configuration, the minimizing triple collision orbit is collision-free until a simultaneous collision, and its limiting configuration is the Lagrangian configuration with the same orientation as the initial configuration. For the collinear initial configuration, under a certain technical assumption, there exist two minimizing orbits. The limiting configurations of these orbits are the two opposite Lagrangian configurations.

3.1 Introduction

The purpose of this chapter is to analyze the asymptotic behavior of the minimizing triple collision orbits of the planar Newtonian three-body problem. Let x_i and m_i be the position and the mass of the i -th body respectively where $x_i \in \mathbb{R}^2$ and $i = 1, 2, 3$. The equations of motion of the Newtonian three-body problem are

$$m_i \ddot{x}_i = \frac{\partial U(x)}{\partial x_i}, \quad i = 1, 2, 3. \quad (3.1)$$

Here, $U(x)$ is the potential function, $U(x) = U(x_1, x_2, x_3) = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|x_i - x_j|}$. We assume that the particles collide simultaneously at $t = T$ and use the following transformation[27]:

$$\Phi : (x, t) \rightarrow (X, u) \text{ via } X_i = \frac{x_i}{(T-t)^{2/3}}, \quad u = \ln(T-t), \quad i = 1, 2, 3. \quad (3.2)$$

We use the prime notation to represent differentiation with respect to u . (3.1) becomes

$$X_i'' + \frac{1}{3}X_i' = \frac{2}{9}X_i + \frac{1}{m_i} \frac{\partial U(X_1, X_2, X_3)}{\partial X_i}, \quad i = 1, 2, 3. \quad (3.3)$$

We can see that $u \rightarrow -\infty$ as $t \rightarrow T^-$. Winter[27] showed that if $x_i(t) \rightarrow 0$ as $t \rightarrow T^-$, for $i = 1, 2, 3$, then X_i' and $X_i'' \rightarrow 0$ as $u \rightarrow -\infty$, for $i = 1, 2, 3$. This shows that $X = (X_1, X_2, X_3)$ approaches the *central configuration* set

$$\mathcal{CC}_3 = \{Q \in (\mathbb{R}^2)^3 \mid \frac{2}{9}Q_i + \sum_{j=1,2,3,j \neq i} \frac{m_j(Q_i - Q_j)}{|Q_i - Q_j|^3} = 0, \quad i = 1, 2, 3\} \quad (3.4)$$

as $u \rightarrow -\infty$. \mathcal{CC}_3 is the set of the critical points of $\frac{1}{9} \sum_{i=1}^3 m_i X_i^2 + U(X)$. In particular,

$$\mathcal{CC}_3 = \mathcal{M}_{L^+} \cup \mathcal{M}_{L^-} \cup_{i=1}^3 \mathcal{M}_{E_i}.$$

Here, we call L^i ($i = +, -$) the Lagrangian(equilateral) configurations and E_i ($i = 1, 2, 3$) the Euler(collinear) configurations. The set $\mathcal{M}_{\{\cdot\}}$ corresponds to all possible rotations of these configuration in \mathbb{R}^2 . In [11], Siegel showed that $\lim_{u \rightarrow -\infty} X(u)$ must be some point in \mathcal{CC}_3 . **A natural question is which central configuration $X(u)$ approaches.** We answer this question for the minimizing orbits in the three-body problem.

The equations (3.1) are Euler-Lagrange equations of the Lagrangian action functional. The least action principle suggests that the solutions of (3.1) are extremal for the Lagrangian action functional. However, variational methods had not been successfully applied to the Newtonian N -body problem until the work of Chenciner and Montgomery's [1]. They minimized the action functional on a suitable space to yield the remarkable figure-eight orbits of the three-body problem with equal masses. Since then, the periodic and the quasiperiodic solutions of the N -body problem were studied extensively by variational methods. We refer to [2, 3, 4, 5, 6, 7, 8] and the references therein. The Euler-Lagrange solutions corresponding to action minimizers are called the *minimizing solutions*. Maderna and Venturelli [9] also used the variational method to prove the existence of the parabolic motions in the N -body problem.

By Marchal's theorem[28], for a given time $T > 0$ and configuration $x_0 \in (\mathbb{R}^2)^3$, the minimizers of the Lagrangian action functional

$$\mathcal{L}_1(x) = \int_0^T \frac{1}{2} \sum_{i=1}^3 m_i \dot{x}_i^2 + U(x) dt \quad (3.5)$$

in the function space

$$\gamma_{x_0,0;T} = \{x \in H^1((0, T), \mathbb{R}^6) | x(0) = x_0, x(T) = 0 \text{ and } \mathcal{L}_1(x) < +\infty\} \quad (3.6)$$

are collision-free in $(0, T)$. Here, $H^1((0, T), \mathbb{R}^6)$ is a Sobolev space. Furthermore, (3.3) corresponds to the weighted action functional

$$\mathcal{L}_2(X) = \int_{-\infty}^{\ln T} \left\{ \frac{1}{2} \sum_{i=1}^3 m_i X_i'^2 + \frac{1}{9} \sum_{i=1}^3 m_i X_i^2 + U(X_1, X_2, X_3) \right\} e^{\frac{1}{3}u} du.$$

It is natural to search for the minimizers of \mathcal{L}_2 in the function space

$$\Gamma_{x_0; \ln T} = \{X \in H_{loc}^1((-\infty, \ln T), \mathbb{R}^6) | X(\ln T) = \frac{x_0}{T^{2/3}} \text{ and } \mathcal{L}_2(X) < +\infty\}. \quad (3.7)$$

We transform the problem of minimizing the action functional \mathcal{L}_1 with two fixed ends to that of minimizing the action functional \mathcal{L}_2 with one fixed end. In Section 3.2, we show that a minimizer of \mathcal{L}_2 in $\Gamma_{x_0; \ln T}$ corresponds to a minimizer of \mathcal{L}_1 in $\gamma_{x_0,0;T}$. It follows that a minimizer of \mathcal{L}_2 in $\Gamma_{x_0; \ln T}$ is collision-free in $(-\infty, \ln T)$. We use the variational structure of \mathcal{L}_2 to show that the Euler configurations with certain mass distribution cannot be the limiting configuration of the minimizing triple collision orbits in the three-body problem. \mathcal{L}_2 is also convenient to study certain kinds of central configurations of the N -body problem cannot be the limiting configurations of the minimizing total collision orbits.

Our main results in this chapter are about the limiting configurations of the minimizing triple collision orbits in the planar three-body problem. We discuss the limiting configurations of the minimizing triple collision orbits for two possible cases of initial configurations: non-collinear initial configurations and collinear initial configurations. Since the equations of motion (3.1) and (3.3) have rotational symmetry, we study the solution in the shape space. The shape space of the planar three-body problem is the space of oriented congruence classes of triangles. It is homeomorphic to \mathbb{R}^3 . Montgomery[29]

gave good spherical coordinates (R, ϕ, θ) to describe it (see Section 3.3). The origin in the shape space corresponds to the triple collision and each ray from the origin corresponds to the class of the similar triangles. In particular, the positive z -axis and the negative z -axis correspond to two opposite equilateral triangle. In Theorem 3.1, we show that if the initial configuration of the three bodies projects to a point in the upper half-space of the shape space, the projection of the configuration of the minimizing triple collision orbit stays in the upper half-space of the shape space and approaches the origin along the positive z -axis.

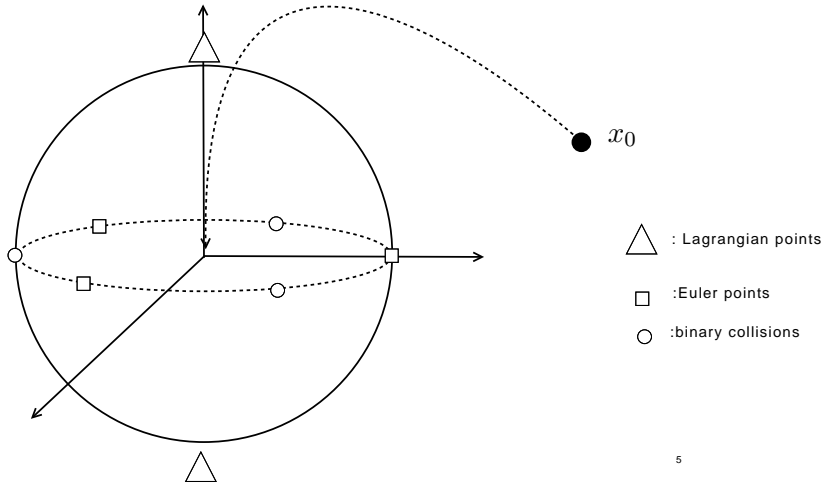


Figure 3.1: A minimizing triple collision orbit x in the shape space.

Theorem 3.1. *For a given non-collinear initial configuration x_0 in the planar three-body problem and time $T > 0$, there exists a minimizing solution of (3.1) that maintains its orientation and has a triple collision at $t = T$. The minimizing triple collision orbit is collision-free in $(0, T)$. Furthermore,*

$$\frac{x}{(T-t)^{2/3}} \rightarrow x^* \text{ as } t \rightarrow T^- \text{ (or } X = \Phi(x) \rightarrow x^* \text{ as } u \rightarrow -\infty),$$

where $x^* \in \mathcal{CC}_3$ is a Lagrangian configuration which has the same orientation as the initial configuration.

For the case of the collinear initial condition, with certain mass distribution condition (see (3.8)), there exist at least two minimizing triple collision orbits. In Theorem 3.2,

we show that if the initial configuration of the three bodies projects to a point in the xy plane of the shape space, the projection of the configuration of the minimizing triple collision orbit either stays in the upper half-space of the shape space and approaches the origin along the positive z -axis or stays in the lower half-space of the shape space and approaches the origin along the negative z -axis

Theorem 3.2. *Let $x_0 = (x_{01}, x_{02}, x_{03}) \in (\mathbb{R}^2)^3$ be a collinear initial configuration. Assume that $x_{0i} = (a_{0i}, 0)$, $i = 1, 2, 3$, and $a_{03} < a_{02} < a_{01}$. Let $(R_*, 0, \theta_*)$ be the normalized spherical coordinates (see Section 3.3) of the Euler configuration with the same order as x_0 , and let $\lambda_* = \lambda(0, \theta_*)$, where λ is defined in Proposition 1. If $(R_*, 0, \theta_*)$ satisfies*

$$R_*^2 \lambda_*^2 + 72 \frac{\partial^2 U}{\partial \phi^2}(R_*, 0, \theta_*) < 0, \quad (3.8)$$

then for any time $T > 0$, there are two solutions of (3.1), called x^+ and x^- , with $x^\pm(0) = x_0$ and $x^\pm(T) = 0$, which are collision-free in $(0, T)$. Furthermore, $\Phi(x^+)$ and $\Phi(x^-)$ approach some point in \mathcal{M}_{L^+} and \mathcal{M}_{L^-} , respectively, as $u \rightarrow -\infty$.

McGehee's blow-up technique is a standard way to study the qualitative behavior of the orbits in the three-body problem while they pass the neighborhood of the triple collision singularity. We refer to [20] for the case of the collinear three-body problem and [30, 31, 32, 33] for the case of the isosceles three-body problem. They studied the reduced system on the triple-collision manifold. The flows on the triple-collision manifold have no physical reality but they can determine the behavior of the orbits close to the manifold. The triple-collision manifold \mathcal{TC} of the planar three-body problem is a six-dimensional manifold embedded in a twelve-dimensional phase space. The equilibrium points of the McGehee flow are normally-hyperbolic and form five circles on \mathcal{TC} . We can eliminate the rotational symmetry of \mathcal{TC} to reduce its dimensionality to five, but it is still not easy to analysis the behavior of the flow near the equilibrium points on \mathcal{TC} because of dimensionality.

We can interpret Theorems 1 and 2 in terms of the McGehee flow and triple-collision manifold \mathcal{TC} with elimination of the rotation symmetry. The minimizing orbits $x(t)$ in Theorems 1 and 2 are in the stable set of the Lagrangian equilibrium points. Let X_0 be a Lagrangian configuration with inertia equal to 1 and $V_0 = \sqrt{2U(X_0)}$; then $(X_0, V_0 X_0)$

is an equilibrium point of the McGehee flow in the collision manifold. Theorems 1 and 2 suggest the following corollary.

Corollary 3.1. *The stable set of $(X_0, V_0 X_0)$ projects on at least half of the shape space.*

This chapter is organized as follows. In Section 3.2, we present the variational approach to the minimizing total collision orbits in the planar N -body problem. Section 3.3 is devoted to the proofs of Theorem 3.1 and Theorem 3.2.

3.2 Total Collision Orbits in the Planar N -Body Problem

Let x_i , $v_i = \dot{x}_i$, and m_i be the position, the velocity, and the mass of the i -th particle, respectively, for $i = 1, \dots, N$. Assume that the center of mass is located at the origin. Set

$$x \in \mathcal{X} = \{x = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N \mid m_1 x_1 + \dots + m_N x_N = 0\}, \quad (3.9)$$

where \mathcal{X} is called the *configuration space*. We define the collision space

$$\Delta = \{x \in \mathcal{X} \mid x_i = x_j \text{ for some } i \neq j, 1 \leq i \leq N, 1 \leq j \leq N\}.$$

The equations of motion of the Newtonian N -body problem are

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}, \quad i = 1, \dots, N. \quad (3.10)$$

Here, $U(x_1, \dots, x_N) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|x_i - x_j|}$ is the potential function. The Hamiltonian, the angular momentum, and the moment of inertia are

$$H = \frac{1}{2} \sum_{i=1}^N m_i \dot{x}_i^2 - U(x_1, \dots, x_N), \quad (3.11)$$

$$\omega = \sum_{i=1}^N m_i x_i \times \dot{x}_i, \quad (3.12)$$

and

$$I = \sum_{i=1}^N m_i x_i^2, \quad (3.13)$$

respectively.

Let $x_0 \in \mathcal{X} \setminus \Delta$ be the initial configuration. Assume that the solution contains a simultaneous collision at $t = T$. We define the transformation $\Phi : (x, t) \rightarrow (X, u)$ by

$$X = \frac{x}{(T-t)^{\frac{2}{3}}} \quad \text{and} \quad u = \ln(T-t), \quad (3.14)$$

and denote by Φ^* its pull-back. We use the dot and prime notation for differentiation with respect to t and u , respectively. Note that X is also in \mathcal{X} . The equations of motion of $X = (X_1, \dots, X_N)$ with respect to the new time variable are

$$X_i'' + \frac{1}{3}X_i' = \frac{2}{9}X_i + \frac{1}{m_i} \frac{\partial U(X_1, \dots, X_N)}{\partial X_i}, \quad i = 1, \dots, N. \quad (3.15)$$

We can see that $u \rightarrow -\infty$ as $t \rightarrow 0$, i.e., $u \rightarrow -\infty$ corresponds to the limit for the total collision.

A classical result concerning the total collision orbits can be found in Wintner's book (Chapter V, [27]). We summarize the result as follows.

Theorem 3.3. [27] *Let $x = (x_1, \dots, x_N)$ be a solution of (3.1) containing a simultaneous collision at $t = T$. Then there exists $\mu_0 > 0$, such that*

(a) $\omega = 0$.

(b) $I \sim (\frac{3}{2}\mu_0^2)^{\frac{4}{3}}(T-t)^{\frac{4}{3}}$, $\dot{I} \sim (12\mu_0^2)^{\frac{1}{3}}(T-t)^{\frac{1}{3}}$ and $\ddot{I} = (\frac{2}{3}\mu_0)^{\frac{2}{3}}(T-t)^{-\frac{2}{3}}$ as $t \rightarrow T^-$.

(c) $\frac{2}{9}\mathbf{I} - \mathbf{U} \rightarrow 0$ and $(T-t)^{-\frac{2}{3}}|x_i - x_k| > \text{constant} > 0$ as $t \rightarrow T^-$, where $\mathbf{I} = (T-t)^{-\frac{4}{3}}I$ and $\mathbf{U} = (T-t)^{\frac{2}{3}}U$.

Furthermore, let $(X, u) = \Phi(x, t)$; then

(d) $X_i' \rightarrow 0$, $X_i'' \rightarrow 0$, $|X_i''''| < \text{constant}$ and $|X_i| < \text{constant}$ as $u \rightarrow -\infty$, for $i = 1, \dots, N$.

It follows that X approaches the set

$$\mathcal{CC}_N = \{Q = (Q_1, \dots, Q_N) \in \mathcal{X} \mid \frac{2}{9}Q_i + \frac{1}{m_i} \frac{\partial U(Q)}{\partial Q_i} = 0, i = 1, \dots, N\} \quad (3.16)$$

as $u \rightarrow -\infty$. An element in \mathcal{CC}_N is called a central configuration. In general, if $Q = (Q_1, \dots, Q_N) \in \mathcal{X} \setminus \Delta$ satisfies

$$\lambda Q_i + \frac{1}{m_i} \frac{\partial U(Q)}{\partial Q_i} = 0, \quad i = 1, \dots, N \quad (3.17)$$

for some λ , we call such Q a central configuration. The value of λ only affects the size of the central configuration. For convenience, the central configurations mentioned in this chapter are normalized to have $\lambda = \frac{2}{9}$, as such central configurations are the rest points of (3.15).

From now on, we consider only the planar N -body problem. Let $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$ in \mathbb{R}^{2N} . We denote the mass inner product and mass wedge product in \mathbb{R}^{2N} , respectively, as follows:

$$x \cdot y = \sum_{i=1}^N m_i(x_i, y_i) \tag{3.18}$$

and

$$\omega(x, y) = \sum_{i=1}^N m_i x_i \times y_i. \tag{3.19}$$

The norm of $x \in \mathbb{R}^{2N}$ is denoted by $\|x\|$, where $\|x\| = \sqrt{x \cdot x}$.

We identify two elements in \mathcal{CC}_N if one can be transformed to the other via a rotation. There are only $N!/2$ collinear central configurations in the N -body problem [26]. For $N = 3$, the central configurations were found by Euler and Lagrange. There are three collinear configurations and two equilateral triangles in the planar case (see Fig. 3.2). These two equilateral triangles are equivalent in \mathbb{R}^3 . Therefore, there are four central configurations in \mathbb{R}^3 . For $N = 4$, Hampton and Moeckel[34] showed that there are only finitely many central configurations. Recently Alain Albouy and Vadim Kaloshin have announced the finiteness for the planar five-body problem, for all positive masses except possibly those within an algebraic closed subset of codimension 2. However, it is not known whether the number of central configurations is finite for any given positive masses when $N > 5$.

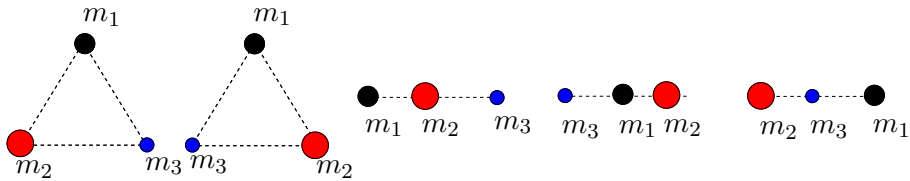


Figure 3.2: Five central configurations in the planar three-body problem.

Marchal[28] used variational methods to show that for any given two configurations and time $T > 0$, the minimizing solution of (3.1) joining these two configurations is

collision-free in $(0, T)$. The existence of the minimizing solutions can be found in [25]. Combining these two results, we have the following theorem.

Theorem 3.4. *Given two configurations $x^i = (r_1, \dots, r_N)$ and $x^f = (s_1, \dots, s_N) \in (\mathbb{R}^2)^N$ and time $T > 0$, let*

$$\mathcal{L}_1(x) = \int_0^T \frac{1}{2} \sum_{i=1}^N m_i \dot{x}_i^2 + U(x_1, \dots, x_N) dt \quad (3.20)$$

be the action functional of (3.1) in the space

$$\begin{aligned} \gamma_{x_i, x_f; T} = \{q \in H^1((0, T), \mathcal{X}) \mid q = (q_1, \dots, q_N), q(0) = x_i, q(T) = x_f, \\ \mathcal{L}_1(q) < +\infty\}. \end{aligned} \quad (3.21)$$

Then there is an action minimizing path joining x_i to x_f in time T that is collision-free for $t \in (0, T)$.

By Theorem 3.4, the restriction of \mathcal{L}_1 to

$$\begin{aligned} \gamma_{x_0, 0; T} = \{q \in H^1((0, T), \mathcal{X}) \mid q = (q_1, \dots, q_N), q(0) = x_0, q(T) = 0, \\ \mathcal{L}_1(q) < +\infty\} \end{aligned} \quad (3.22)$$

attains its minimum in $\gamma_{x_0, 0; T}$, and the minimizer is collision-free in $(0, T)$.

Now, we consider the weighted action functional of (3.15),

$$\mathcal{L}_2(X) = \int_{-\infty}^{\ln T} \left\{ \frac{1}{2} \sum_{i=1}^N m_i X_i'^2 + \frac{1}{9} \sum_{i=1}^N m_i X_i^2 + U(X_1, \dots, X_N) \right\} e^{\frac{1}{3}u} du,$$

restricted to

$$\begin{aligned} \Gamma_{x_0; \ln T} = \{Q \in H_{loc}^1((-\infty, \ln T), \mathcal{X}) \mid Q = (Q_1, \dots, Q_N), Q(\ln T) = \frac{x_0}{T^{2/3}}, \\ \mathcal{L}_2(Q) < +\infty\}. \end{aligned} \quad (3.23)$$

Fife and McLeod[?] used this type of weighted action functional to study traveling waves in reaction-diffusion systems. We naturally ask whether the minimizers of $\mathcal{L}_2|_{\Gamma_{x_0; \ln T}}$ minimize $\mathcal{L}_1|_{\gamma_{x_0, 0; T}}$ and the minimizers of $\mathcal{L}_1|_{\gamma_{x_0, 0; T}}$ minimize $\mathcal{L}_2|_{\Gamma_{x_0; \ln T}}$. This is not obvious: because the term $e^{\frac{1}{3}u}$ in \mathcal{L}_2 approaches 0 as $u \rightarrow -\infty$, the unboundedness of $X(u)$ near $-\infty$ may not affect the finiteness of the action \mathcal{L}_2 . We show the difference of the infima of \mathcal{L}_1 and \mathcal{L}_2 is a constant that depends on the initial configuration and time T .

Theorem 3.5. *Given an initial configuration $x_0 \in \mathcal{X}$ and time $T > 0$, let X be a path in $\Gamma_{x_0; \ln T}$. Then*

$$\mathcal{L}_2(X) = -\frac{\|x_0\|^2}{3T} + \mathcal{L}_1(x),$$

where $(x, t) = \Phi^*(X, u)$.

Proof. Recall that the prime and dot denote differentiation with respect to u and t , respectively, and $du = \frac{-1}{(T-t)}dt$. By change of variables, we have

$$X' = -\dot{x}(t)(T-t)^{1/3} - \frac{2}{3}(T-t)^{-2/3}x(t)$$

and

$$\begin{aligned} \mathcal{L}_2(X) &= \int_{-\infty}^{\ln T} \left(\frac{1}{2} \|X'\|^2 + \frac{1}{9} \|X\|^2 + U(X) \right) e^{\frac{1}{3}u} du \\ &= \int_0^T \left[\frac{1}{2} \|\dot{x}\|^2 (T-t)^{2/3} + \frac{4}{9} \|x\|^2 (T-t)^{-4/3} + \frac{4}{3} \frac{x \cdot \dot{x}}{(T-t)^{1/3}} \right] + \\ &\quad \frac{1}{9} \frac{\|x\|^2}{(T-t)^{4/3}} + U(x)(T-t)^{2/3} \Big] \frac{dt}{(T-t)^{2/3}} \\ &= \int_0^T \left[\frac{1}{2} \|\dot{x}\|^2 + \frac{1}{3} \frac{\|x\|^2}{(T-t)^2} + \frac{1}{3} \frac{d}{dt} \frac{\|x\|^2}{T-t} + U(x) \right] dt \\ &= \frac{\|x(t)\|^2}{3(T-t)} \Big|_{t=0}^T + \mathcal{L}_1(x). \end{aligned}$$

To complete the proof, we need to show that $\frac{\|x(t)\|^2}{T-t} \rightarrow 0$ as $t \rightarrow T^-$. This is done in Lemma 3.1. □

Lemma 3.1. *Let X be a path in $\Gamma_{x_0; \ln T}$ and $(x, t) = \Phi^*(X, u)$; then*

$$\frac{\|x(t)\|^2}{T-t} \rightarrow 0 \text{ as } t \rightarrow T^-.$$

Proof. By change of variables, we have

$$\int_{-\infty}^{\ln T} \|X\|^2 e^{\frac{1}{3}u} du = \int_0^T \frac{\|x\|^2}{(T-t)^{\frac{4}{3}}} (T-t)^{\frac{1}{3}} \frac{1}{(T-t)} dt = \int_0^T \frac{\|x\|^2}{(T-t)^2} dt. \quad (3.24)$$

Because $\int_0^T \frac{\|x\|^2}{(T-t)^2} dt$ is bounded, there is an increasing sequence $\{\delta_m\}_{m=1}^\infty \in (0, T)$ such that $\frac{\|x(\delta_m)\|^2}{T-\delta_m} \rightarrow 0$ as $\delta_m \rightarrow T^-$. Otherwise, if $\liminf_{t \rightarrow T^-} \frac{\|x\|^2}{T-t} = 2c_1$ for some $c_1 > 0$,

then

$$\frac{\|x\|^2}{(T-t)^2} \geq \frac{c_1}{T-t} \quad \text{in } (T-c_2, T) \text{ for some } c_2 > 0. \quad (3.25)$$

(3.25) contradicts the boundedness of $\int_0^T \frac{\|x\|^2}{(T-t)^2} dt$.

As in (3.24), we have

$$\begin{aligned} & \int_{-\infty}^{\ln T} \|X'\|^2 e^{\frac{1}{3}u} du \\ &= \int_0^T \left\| \frac{\dot{x}(T-t)^{2/3} + \frac{2}{3}x(T-t)^{-1/3}}{(T-t)^{4/3}} (T-t) \right\|^2 (T-t)^{\frac{1}{3}} \frac{1}{(T-t)} dt \\ &= \int_0^T \left\| \dot{x} + \frac{2}{3} \frac{x}{(T-t)} \right\|^2 dt. \end{aligned} \quad (3.26)$$

By Hölder's inequality, we have

$$\int_0^T \|\dot{x}\|^2 dt \leq 2 \int_0^T \left\| \dot{x} + \frac{2}{3} \frac{x}{(T-t)} \right\|^2 dt + 2 \int_0^T \left\| \frac{2}{3} \frac{x}{(T-t)} \right\|^2 dt. \quad (3.27)$$

By (3.24), (3.26), and (3.27), $\int_0^T \|\dot{x}\|^2 dt$ is bounded.

We now prove the lemma by contradiction. Assume that there is an increasing sequence $\{\delta'_m\}_{m=1}^\infty \in (0, T)$ such that

$$\frac{\|x(\delta'_m)\|^2}{T - \delta'_m} \rightarrow c_3 \text{ for some } c_3 > 0 \text{ as } \delta'_m \rightarrow T^-.$$

Without loss of generality, we may assume that $\delta_m > \delta'_m, \forall m$. For m sufficiently large, we have

$$\begin{aligned} \frac{1}{2}c_3 &\leq \frac{\|x(\delta_m)\|^2}{T - \delta_m} - \frac{\|x(\delta'_m)\|^2}{T - \delta'_m} \\ &= \int_{\delta'_m}^{\delta_m} \frac{d\|x(t)\|^2}{dt (T-t)} dt \\ &= \int_{\delta'_m}^{\delta_m} \frac{2x \cdot \dot{x}}{(T-t)} + \frac{\|x\|^2}{(T-t)^2} dt \\ &\leq 2 \left[\int_{\delta'_m}^{\delta_m} \frac{\|x\|^2}{(T-t)^2} dt \right]^{\frac{1}{2}} \left[\int_{\delta'_m}^{\delta_m} \|\dot{x}\|^2 dt \right]^{\frac{1}{2}} + \int_{\delta'_m}^{\delta_m} \frac{\|x\|^2}{(T-t)^2} dt. \end{aligned}$$

Because $\frac{\|x\|^2}{(T-t)^2}$ and $\|\dot{x}\|^2$ are in $L^1((0, T), \mathbb{R})$, the right-hand side of the above inequality approaches 0 as $\delta_m, \delta'_m \rightarrow T^-$. This is a contradiction. \square

Theorem 3.6. x minimizes

$$\mathcal{L}_1(x) = \int_0^T \frac{1}{2} \|\dot{x}\|^2 + U(x) dt$$

in

$$\begin{aligned} \gamma_{x_0,0;T} = \{q \in H^1((0, T), \mathcal{X}) \mid q = (q_1, \dots, q_N), q(0) = x_0, q(T) = 0, \\ \mathcal{L}_1(q) < +\infty\} \end{aligned} \quad (3.28)$$

if and only if $X = \Phi(x)$ minimizes

$$\mathcal{L}_2(X) = \int_{-\infty}^{\ln T} \left\{ \frac{1}{2} \|X'\|^2 + \frac{1}{9} \|X\|^2 + U(X) \right\} e^{\frac{1}{3}u} du$$

in

$$\begin{aligned} \Gamma_{x_0;\ln T} = \{Q \in H_{loc}^1((-\infty, \ln T), \mathcal{X}) \mid Q = (Q_1, \dots, Q_N), Q(\ln T) = \frac{x_0}{T^{2/3}}, \\ \mathcal{L}_2(Q) < +\infty\}. \end{aligned} \quad (3.29)$$

Proof. By Lemma 3.1, we see that

$$\Phi^* \Gamma_{x_0;\ln T} \subset \gamma_{x_0,0;T}.$$

By (3.14) and Theorem 3.3, we know that

$$x \in \Phi^* \Gamma_{x_0;\ln T}. \quad (3.30)$$

Therefore, we have

$$\min_{q \in \gamma_{x_0,0;T}} \mathcal{L}_1(q) = \min_{Q \in \Gamma_{x_0;\ln T}} \mathcal{L}_2(Q) - \frac{\|x_0\|^2}{3T}.$$

□

As a consequence of Theorem 3.4 and Theorem 3.6, we know that a minimizer of $\mathcal{L}_2|_{\Gamma_{x_0;\ln T}}$ is collision-free in $(-\infty, \ln T)$ and is a classical solution of (3.15) with the initial configuration $X(\ln T) = \frac{x_0}{T^{2/3}}$.

The angular momentum $\omega(x, \dot{x})$ of (3.1) is zero along the total collision orbit. However, the angular momentum $\omega(X, X')$ of (3.15) may not be identically zero. The following lemma shows the behavior of the angular momentum of (3.15).

Lemma 3.2. *If X is a solution of (3.15) on $(-\infty, \ln T]$ for some $T > 0$, then $\omega(X, X')$ is either identically zero or increasing to infinity as $u \rightarrow -\infty$. In particular, if $\Phi^*(X)$ is a total collision orbit with a simultaneous collision at $t = T$, then $\omega(X, X') \equiv 0$.*

Proof. Let $C(u) = \sum_{i=1}^N m_i X_i \times X'_i$. Then

$$\begin{aligned} C'(u) &= \sum_{i=1}^N m_i X'_i \times X'_i + \sum_{i=1}^N m_i X_i \times X''_i \\ &= \sum_{i=1}^N m_i X_i \times \left(-\frac{1}{3}X'_i + \frac{2}{9}X_i + m_i^{-1} \frac{\partial U}{\partial X_i}\right) \\ &= \sum_{i=1}^N m_i X_i \times -\frac{1}{3}X'_i \\ &= -\frac{1}{3}C(u). \end{aligned}$$

It follows that $C(u)$ either vanishes for all $u \in (-\infty, \ln T]$ or grows to infinity as $u \rightarrow -\infty$.

If $\Phi^*(X)$ is a total collision orbit with a simultaneous collision at $t = T$, then $X' \rightarrow 0$ and X is bounded as $u \rightarrow -\infty$. It follows that $\omega(X, X')$ vanishes for all u . \square

3.3 Proofs of Theorems 3.1 and 3.2

The shape space for the planar three-body problem was described by Moeckel[35]. It represents the configuration of the three bodies in \mathbb{R}^3 . In Chenciner and Montgomery's[1] paper on the figure-eight shape orbits in the three-body problem with equal masses, the main device is the shape sphere of the planar three-body problem. Montgomery[29] continued to study the shape space of the planar three-body problem with arbitrary mass distribution and gave a *good coordinate* to describe it. He showed that any bounded solutions of the three-body problem have infinitely many syzygies provided that they do not have a triple collision and their angular momentum is zero.

We follow Montgomery's work[29] to introduce the shape space for the planar three-body problem. We refer to [29] for more details. We use a two step transformation to show that the configuration of the planar three-body problem can be visualized in \mathbb{R}^3 . Let x_1, x_2 and $x_3 \in \mathbb{R}^2$ be the positions of the three bodies and m_1, m_2 and $m_3 \in \mathbb{R}_+$

be their masses. We assume that the center of mass is at the origin, i.e., that

$$x = (x_1, x_2, x_3) \in \mathcal{V} = \{x = (x_1, x_2, x_3) \in (\mathbb{R}^2)^3 \mid \sum_{i=1}^3 m_i x_i = 0\}.$$

The equations of motion, kinetic energy, and potential are

$$\ddot{x}_i = \frac{\partial U}{\partial x_i}, \quad i = 1, 2, 3, \quad K = \frac{1}{2} \sum_{i=1}^3 m_i \dot{x}_i^2, \quad \text{and} \quad U = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|x_i - x_j|}, \quad (3.31)$$

respectively. The Jacobi coordinates (ξ_1, ξ_2) for (x_1, x_2, x_3) are

$$\begin{cases} \xi_1 = x_2 - x_1, \\ \xi_2 = x_3 - \left(\frac{m_1}{m_1 + m_2} x_1 + \frac{m_2}{m_1 + m_2} x_2 \right). \end{cases} \quad (3.32)$$

The kinetic energy and the potential can be written in terms of (ξ_1, ξ_2) as

$$K(\dot{x}) = K(\dot{\xi}_1, \dot{\xi}_2) = \frac{1}{2} (M_1 |\dot{\xi}_1|^2 + M_2 |\dot{\xi}_2|^2) \quad (3.33)$$

and

$$U(\xi_1, \xi_2) = \frac{m_1 m_2}{|\xi_1|} + \frac{m_2 m_3}{\left| \xi_2 - \frac{m_1}{m_1 + m_2} \xi_1 \right|} + \frac{m_1 m_3}{\left| \xi_2 + \frac{m_2}{m_1 + m_2} \xi_1 \right|}. \quad (3.34)$$

We normalize the Jacobi coordinates by

$$(z_1, z_2) := (\sqrt{M_1} \xi_1, \sqrt{M_2} \xi_2), \quad (3.35)$$

where $M_1 = \frac{m_1 m_2}{m_1 + m_2}$ and $M_2 = \frac{(m_1 + m_2) m_3}{m_1 + m_2 + m_3}$. The kinetic energy and the potential can be written in terms of (z_1, z_2) as

$$K(\dot{z}_1, \dot{z}_2) = \frac{1}{2} (|\dot{z}_1|^2 + |\dot{z}_2|^2) \quad (3.36)$$

and

$$U(z_1, z_2) = \frac{m_1 m_2 \sqrt{M_1}}{|z_1|} + \frac{m_2 m_3 \sqrt{M_2}}{\left| z_2 - \frac{\sqrt{M_1 M_2}}{m_2} z_1 \right|} + \frac{m_1 m_2 \sqrt{M_2}}{\left| z_2 + \frac{\sqrt{M_1 M_2}}{m_1} z_1 \right|}. \quad (3.37)$$

After the Jacobi transformation, the configuration space \mathcal{V} can be parametrized by (z_1, z_2) , and it is obvious that we can identify \mathcal{V} with \mathbb{C}^2 . In summary, the Jacobi transformation maps \mathcal{V} to \mathbb{C}^2 by

$$\mathcal{J}_m(x_1, x_2, x_3) = (z_1, z_2). \quad (3.38)$$

We take the quotient by the rotational symmetry about the angular momentum from the configuration space \mathcal{V} to yield the reduced configuration \mathcal{V}' . We say that two points (z_1, z_2) and (z'_1, z'_2) are equivalent if there is $e^{2\pi i\theta}$, $\theta \in \mathbb{R}/\mathbb{Z}$, such that $(z_1, z_2) = e^{2\pi i\theta}(z'_1, z'_2)$. We identify the reduced configuration \mathcal{V}' with \mathbb{R}^3 via the Hopf fibration

$$\mathcal{H}(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2\operatorname{Re}(\bar{z}_1 z_2), 2\operatorname{Im}(\bar{z}_1 z_2)) := (w_1, w_2, w_3) = w.$$

The spherical coordinates (χ, ψ) in the shape space are defined by

$$\frac{w}{\|w\|} = (\cos(\psi) \cos(\chi), \sin(\psi) \cos(\chi), \sin(\chi)). \quad (3.39)$$

The momentum of inertia in these coordinates is

$$I_m = \sum_{i=1}^3 m_i |x_i|^2 = \mu_1 |\xi_1|^2 + \mu_2 |\xi_2|^2 = 4(|w_1|^2 + |w_2|^2 + |w_3|^2).$$

The subscript m in \mathcal{J}_m and I_m denotes the parametric dependence on the mass distribution.

Proposition 3.1. [29]

(a) *The kinetic energy with zero angular momentum in spherical coordinates is*

$$K = \frac{1}{2}(\dot{R}^2 + \frac{R^2}{4}(\dot{\chi}^2 + \cos^2(\chi)\dot{\psi}^2)), \quad (3.40)$$

where $R = \sqrt{I_m}$.

(b) *Let $m = (m_1, m_2, m_3)$ and $m' = (m'_1, m'_2, m'_3) \in \mathbb{R}_+^3$ be two different mass distributions. Then the corresponding metric d^2s_m and $d^2s_{m'}$ on shape sphere are related by the formula*

$$\frac{m_1 + m_2 + m_3}{m_1 m_2 m_3} I_m^2 ds_m^2 = \frac{m'_1 + m'_2 + m'_3}{m'_1 m'_2 m'_3} I_{m'}^2 ds_{m'}^2.$$

(c) Let (ϕ, θ) be the spherical coordinates obtained by the combination of the Jacobi transformation and Hopf fibration, where the masses are all 1, $\mathcal{H} \circ \mathcal{J}_{(1,1,1)}$; then

$$d^2 s_m = \lambda(\phi, \theta)^2 (d\phi^2 + \cos^2(\phi)d\theta^2) \text{ where } \lambda = \sqrt{\frac{3m_1 m_2 m_3}{M} \frac{I_{(1,1,1)}}{I_m}}. \quad (3.41)$$

Remark 3.1. We call such spherical coordinates obtained in Proposition 3.1(c) the normalize spherical coordinates in the shape space. In Proposition 3.1, Montgomery artificially set all the masses equal to 1 in order to yield the normalized spherical coordinate. Assuming that $R = 1$, in the normalized spherical coordinates $L^+ = (0, 0, 1)$ and $L^- = (0, 0, -1)$ are the Lagrangian points (see Fig. 3.1). Furthermore, in the normalized spherical coordinates, $\lambda(\phi, \theta)$ and $U(R, \phi, \theta)$ satisfy the following lemma, which helps us to manipulate these functions in the proofs of Theorem 1 and Theorem 2.

Lemma 3.3. [29]

1.

$$\left(1 - \frac{\cos(\phi)}{\sin(\phi)} \frac{1}{\lambda} \frac{\partial \lambda}{\partial \phi}\right) = c\lambda, \quad (3.42)$$

where

$$c = \frac{m_1 m_2 + m_3 m_1 + m_2 m_3}{M} \sqrt{\frac{M}{3m_1 m_2 m_3}} \text{ and } M = m_1 + m_2 + m_3.$$

2.

$$\frac{\cos \phi}{\sin \phi} \frac{\partial U}{\partial \phi} < 0 \quad (3.43)$$

as long as $\phi \neq \pm \frac{\pi}{2}$ and U is finite.

3. U is an even function of ϕ .

3.3.1 Non-Collinear Initial Configurations

We show that the coordinate ϕ of the minimizing orbits (equation (3.39)) satisfies the following monotonicity property. The fact in Lemma 3.3 that the Lagrangian in normalized spherical coordinates decreases with respect to ϕ ($0 \leq \phi \leq \pi/2$) and increases with respect to ϕ ($-\pi/2 \leq \phi \leq 0$) will be used in the proof of the following lemma.

Lemma 3.4. *Let $x = (x_1, x_2, x_3)$ be a minimizer of \mathcal{L}_1 in $\gamma_{x_0, 0; T}$, and (R, ϕ, θ) be the normalized spherical coordinates of (x_1, x_2, x_3) . Assuming that $\phi(0) \in (0, \pi/2)$, then*

$$\phi \text{ increases to } \frac{\pi}{2} \text{ in } [0, T].$$

Similarly, if $\phi(0) \in (-\pi/2, 0)$, then

$$\phi \text{ decreases to } -\frac{\pi}{2} \text{ in } [0, T].$$

Proof. In the normalized spherical coordinates, the action functional \mathcal{L}_1 can be written as

$$\mathcal{L}_1(x) = \int_0^T \frac{1}{2}(\dot{R}^2 + \frac{R^2}{4}\lambda^2(\phi, \theta)(\dot{\phi}^2 + \cos^2(\phi)\dot{\theta}^2)) + U(\phi, \theta, R)dt. \quad (3.44)$$

If the coordinate ϕ of (R, ϕ, θ) does not satisfy the monotonicity property, we shall show that there exists a deformation of (R, ϕ, θ) whose action is lower than that of (R, ϕ, θ) .

Remark that $U(R, \cdot, \theta)$ attains its minimum at $\phi = \frac{\pi}{2}$. By Lemma 3.3, we have

$$\frac{\partial \lambda^2(\phi, \theta) \cos^2(\phi)}{\partial \phi} < 0 \text{ and } \frac{\partial U}{\partial \phi} < 0, \quad \forall \phi \in (0, \frac{\pi}{2}). \quad (3.45)$$

First, assume that $\phi(t) \in (0, \pi/2)$, $\forall t \in (0, T)$ and ϕ is not increasing to $\pi/2$ in $[0, T]$. There exist a, b , and t_1 with $a < t_1 < b$ such that $\phi(t_1) < \phi(t)$, $\forall t \in (a, b) \setminus \{t_1\}$ and $\phi(a) = \phi(b)$. Let $\bar{R}(t) = R(t)$ and $\bar{\theta}(t) = \theta(t)$ for $t \in [0, T]$, and

$$\bar{\phi}(t) = \begin{cases} \phi(t), & t \in [0, a] \cup [b, T], \\ \phi(b), & t \in (a, b). \end{cases}$$

By (3.44), we have $\mathcal{L}_1(\bar{R}, \bar{\phi}, \bar{\theta}) < \mathcal{L}_1(R, \phi, \theta)$. It follows that ϕ is increasing to $\pi/2$ if $\phi(0) \in (0, \pi/2)$ and $\phi(t) \in (0, \pi/2]$ for $t \in (0, T]$.

Note that $\mathcal{P} = \{\phi = -\frac{\pi}{2}, 0, \frac{\pi}{2}\}$ is invariant. If ϕ intersects \mathcal{P} at t_2 , then $\dot{\phi}(t_2) \neq 0$. We can reflect $\phi|_{[0, t_2]}$ over \mathcal{P} and freeze $\phi|_{(t_2, T]}$, R and θ . We thus obtain a new orbit minimizing the action, which contradicts the fact that action minimizers correspond to solutions of the Euler-Lagrange equations. □

Proof of Theorem 3.1

Let $x(t)$ be a minimizing triple collision orbit in Theorem 1. Assume that L^+ is the

Lagrangian configuration with the same orientation as $x(0)$. By Lemma 3.4 and [11] (Chapter 1 §13),

$$\lim_{t \rightarrow T^-} \frac{x}{(T-t)^{2/3}} = x^* \text{ for some } x^* \in \mathcal{M}_{L^+}.$$

□

Let x be a minimizing triple collision orbit in the planar three-body problem with collinear initial configuration. In the normalized shape space, one can see that the limiting configuration of x must be one of two Lagrangian points or three Euler points. Theorem 3.4 suggests that a minimizing triple collision orbit is collision-free until a simultaneous collision. If the configuration of x is collinear for all time, then we can exclude the other two Euler points because it cannot pass through the binary collision points to end at the Euler point with a different order. If the configuration of x is not collinear at $t = \tau$, then by Lemma 4 (or the reflection principle in [1]), it has to retain the same orientation until the simultaneous collision. This means that if (R, ϕ, θ) are the normalized spherical coordinates of x , then ϕ is either negative or positive after $t = \tau$. We thus obtain another orbit by mapping $(R, \phi, \theta) \rightarrow (R, -\phi, \theta)$.

Corollary 3.2. *For a given collinear initial configuration $x_0 = (x_{01}, x_{02}, x_{03})$ in the planar three-body problem and time $T > 0$, there exists a minimizing solution x of (3.1) that has a simultaneous collision at $t = T$ and is collision-free in $(0, T)$.*

- (a) *Suppose that the configuration of x is not collinear at $t = \tau$, for some $\tau \in (0, T)$, then there exist two minimizing triple collision orbits that are collision-free in $(0, T)$. The limiting configurations of these two orbits are two opposite orientations of Lagrangian points.*
- (b) *Suppose that the configuration of x is collinear for $t \in [0, T)$ and the initial configuration has the order $x_{0i} < x_{0j} < x_{0k}$ (after a suitable rotation), where $\{i, j, k\}$ is the permutation of $\{1, 2, 3\}$; then the limiting central configuration of x is an Euler point with the same order.*

3.3.2 Collinear Initial Configurations

In this subsection, we discuss a sufficient condition that the limiting configurations of the minimizing triple collision orbits are the Lagrangian points when their initial configurations are collinear. In [36], Barutello and Secchi calculated the variational Morse-like

index of colliding solutions to the N -body problem. They considered the N -body problem with $-\alpha$ -homogeneous potential ($0 < \alpha < 2$) and used McGehee-type variables to study the action of the colliding solution. We have a similar result to that of [36] in the case of $\alpha = 1$. The sufficient condition in Theorem 3.2 is sharper than the analogous one in [36]. The main idea is to show that if the limiting configuration of an orbit is the Euler point, then it cannot minimize $\mathcal{L}_2(X)$ in $\Gamma_{x_0; \ln T}$. Our argument also can be carried out in the case of a general $-\alpha$ -homogeneous potential ($0 < \alpha < 2$).

Proof of Theorem 3.2

Let x be a minimizer of

$$\mathcal{L}_1(x) = \int_0^T \frac{1}{2} \sum_{i=1}^3 m_i \dot{x}_i^2 + U(x_1, x_2, x_3) dt$$

in the space

$$\begin{aligned} \gamma_{x_0, 0; T} = \{q \in H^1((0, T), \mathcal{V}) \mid q = (q_1, q_2, q_3), q(0) = x_0, q(T) = 0, \\ \mathcal{L}_1(q) < \infty\}. \end{aligned}$$

By Corollary 3.2, if the configuration of a minimizing solution is collinear for all time, the limiting configuration must be the Euler point with the same order as the initial configuration. We consider the weighted action functional of $X = \Phi(x)$ restricted in $\Gamma_{x_0; \ln T}$ in the normalized spherical coordinates

$$\mathcal{L}_2(X) = \int_{-\infty}^{\ln T} \left\{ \frac{1}{2} R'^2 + \frac{R^2}{8} \lambda^2(\phi, \theta) (\phi'^2 + \cos^2(\phi) \theta'^2) + \frac{1}{9} R^2 + U \right\} e^{\frac{1}{3}u} du.$$

If the limiting configuration of X is an Euler configuration, then by Corollary 3.2, we have

$$\phi \equiv 0 \text{ and } (\theta, R) \rightarrow (\theta_*, R_*) \text{ as } u \rightarrow -\infty,$$

where $(R_*, 0, \theta_*)$ are the normalized spherical coordinates of the Euler point with the same order as x_0 .

For a given $\varepsilon > 0$, there exists $\tau \ll 0$ such that

$$\begin{aligned} R(u) = R_* + O(\varepsilon), \lambda(0, \theta) = \lambda_* + O(\varepsilon), \theta(u) = \theta_* + O(\varepsilon), \text{ and} \\ \frac{\partial^2 U}{\partial \phi^2}(R, 0, \theta) = \frac{\partial^2 U}{\partial \phi^2}(R_*, 0, \theta_*) + O(\varepsilon), \quad \forall u < \tau. \end{aligned} \tag{3.46}$$

We consider a deformation of (R, ϕ, θ) by

$$\hat{\phi} = \begin{cases} \varepsilon \sin(R_*^a u - \tau + \pi) & \text{if } u \in [R_*^{-a}(\tau - \pi), R_*^{-a}\tau] \\ 0 & \text{otherwise} \end{cases}, \quad (3.47)$$

$$\hat{\theta} = \theta \text{ and } \hat{R} = R.$$

The constant a will be determined later.

We calculate the difference between the actions of (R, ϕ, θ) and $(\hat{R}, \hat{\phi}, \hat{\theta})$.

$$\begin{aligned} \Delta \mathcal{L} &= \mathcal{L}_2(\hat{R}, \hat{\phi}, \hat{\theta}) - \mathcal{L}_2(R, \phi, \theta) \\ &= \int_{-\infty}^{\ln T} \left\{ \frac{1}{2}(\hat{R}'^2 + \frac{\hat{R}^2}{4}\lambda^2(\hat{\phi}, \hat{\theta})\hat{\phi}'^2) + \frac{1}{9}\hat{R}^2 + U(\hat{R}, \hat{\phi}, \hat{\theta}) \right\} e^{\frac{1}{3}u} du \\ &\quad - \int_{-\infty}^{\ln T} \left\{ \frac{1}{2}R'^2 + \frac{1}{9}R^2 + U(R, \phi, \theta) \right\} e^{\frac{1}{3}u} du. \end{aligned}$$

By the definition of (R, ϕ, θ) and $(\hat{R}, \hat{\phi}, \hat{\theta})$, we have

$$\Delta \mathcal{L} = \int_{R_*^{-a}(\tau-\pi)}^{R_*^{-a}\tau} \left\{ \frac{\hat{R}^2}{8}\lambda^2(\hat{\phi}, \hat{\theta})\hat{\phi}'^2 + U(\hat{R}, \hat{\phi}, \hat{\theta}) - U(R, \phi, \theta) \right\} e^{\frac{1}{3}u} du.$$

The difference in the action can be decomposed as follows:

$$\Delta \mathcal{L} = \Delta \mathcal{L}^K + \Delta \mathcal{L}^U, \quad (3.48)$$

where

$$\Delta \mathcal{L}^K = \int_{R_*^{-a}(\tau-\pi)}^{R_*^{-a}\tau} \frac{\hat{R}^2}{8}\lambda^2(\hat{\phi}, \hat{\theta})\hat{\phi}'^2 e^{\frac{1}{3}u} du \quad (3.49)$$

and

$$\Delta \mathcal{L}^U = \int_{R_*^{-a}(\tau-\pi)}^{R_*^{-a}\tau} \{U(\hat{R}, \hat{\phi}, \hat{\theta}) - U(R, \phi, \theta)\} e^{\frac{1}{3}u} du. \quad (3.50)$$

Combining (3.47) and the change of variables $R_*^a u - \tau + \pi = v$, we have

$$\begin{aligned} \Delta \mathcal{L}^K &= \int_{R_*^{-a}(\tau-\pi)}^{R_*^{-a}\tau} \frac{1}{8}R^2\lambda^2(\hat{\phi}, \theta)\varepsilon^2 R_*^{2a} \cos^2(R_*^a u - \tau + \pi) e^{\frac{1}{3}u} du \\ &= \int_0^\pi \varepsilon^2 \frac{1}{8}R_*^{2+2a}\lambda_*^2 \cos^2(v) e^{\frac{1}{3}R_*^{-a}(v+\tau-\pi)} dv + \mathbf{e} \cdot O(\varepsilon^3) \\ &= \varepsilon^2 e^{\frac{1}{3}R_*^{-a}(\tau-\pi)} \frac{R_*^{2+a}\lambda_*^2}{8} \int_0^\pi \cos^2(v) e^{\frac{1}{3}R_*^{-a}v} dv + \mathbf{e} \cdot O(\varepsilon^3) \end{aligned} \quad (3.51)$$

where $\mathbf{e} = e^{\frac{1}{3}R_*^{-a}(\tau-\pi)}(e^{\frac{1}{3}R_*^{-a}\pi} - 1)$. Because U is an even function of the variable ϕ , it follows that $\frac{\partial U}{\partial \phi} = 0$ on $\phi = 0$. When ε is sufficiently small, we have

$$\begin{aligned} U(\hat{R}, \hat{\phi}, \hat{\theta}) &= U(\hat{R}, 0, \hat{\theta}) + \frac{\partial U}{\partial \phi}(\hat{R}, 0, \hat{\theta})\hat{\phi} + \frac{\partial^2 U}{\partial \phi^2}(\hat{R}, 0, \hat{\theta})\frac{\hat{\phi}^2}{2} + O(\hat{\phi}^3) \\ &= U(\hat{R}, 0, \hat{\theta}) + \varepsilon^2 \frac{\partial^2 U}{\partial \phi^2}(\hat{R}, 0, \hat{\theta}) \frac{\sin^2(R_*^a u - \tau + \pi)}{2} + O(\varepsilon^3). \end{aligned} \quad (3.52)$$

As in the estimate of $\Delta \mathcal{L}^K$ in (3.51), we have

$$\begin{aligned} \Delta \mathcal{L}^U &= \int_{R_*^{-a}(\tau-\pi)}^{R_*^{-a}\tau} \{U(R, \varepsilon \sin(R_*^a u - \tau + \pi), \theta) - U(0, \theta, R)\} e^{\frac{1}{3}u} du \\ &= \int_{R_*^{-a}(\tau-\pi)}^{R_*^{-a}\tau} \varepsilon^2 \frac{\partial^2 U}{\partial \phi^2}(R, 0, \theta) \frac{\sin^2(R_*^a u - \tau + \pi)}{2} e^{\frac{1}{3}u} du + \mathbf{e} \cdot O(\varepsilon^3) \\ &= \int_{R_*^{-a}(\tau-\pi)}^{R_*^{-a}\tau} \varepsilon^2 \frac{\partial^2 U}{\partial \phi^2}(R_*, 0, \theta_*) \frac{\sin^2(R_*^a u - \tau + \pi)}{2} e^{\frac{1}{3}u} du + \mathbf{e} \cdot O(\varepsilon^3) \\ &= \varepsilon^2 e^{\frac{1}{3}R_*^{-a}(\tau-\pi)} \frac{R_*^{-a}}{2} \frac{\partial^2 U}{\partial \phi^2}(R_*, 0, \theta_*) \int_0^\pi \sin^2(v) e^{\frac{1}{3}R_*^{-a}v} dv + \mathbf{e} \cdot O(\varepsilon^3). \end{aligned} \quad (3.53)$$

A simple calculation shows that

$$\int_0^\pi \cos^2(v) e^{\frac{1}{3}R_*^{-a}v} dv = \frac{3R_*^a}{2} \left(\frac{36R_*^{2a} + 2}{36R_*^{2a} + 1} \right) (e^{\frac{1}{3}R_*^{-a}\pi} - 1) \quad (3.54)$$

and

$$\int_0^\pi \sin^2(v) e^{\frac{1}{3}R_*^{-a}v} dv = \frac{3R_*^a}{2} \left(\frac{36R_*^{2a}}{36R_*^{2a} + 1} \right) (e^{\frac{1}{3}R_*^{-a}\pi} - 1). \quad (3.55)$$

Combining (3.51), (3.53), (3.54) and (3.55), we have the following estimate for $\Delta \mathcal{L}$.

$$\Delta \mathcal{L} = \mathbf{e} \varepsilon^2 \frac{3R_*^{2a}}{72R_*^{2a} + 2} \mathbf{F} + \mathbf{e} \cdot O(\varepsilon^3), \quad (3.56)$$

where

$$\mathbf{F} = \frac{18R_*^{2+2a} + R_*^2}{4} \lambda_*^2 + 18 \frac{\partial^2 U}{\partial \phi^2}(R_*, 0, \theta_*). \quad (3.57)$$

Choose a such that $R_*^{2a} = \varepsilon^{\frac{1}{2}}$. Observe that the first term on the right hand side of (3.56) still dominates (3.56), and the term R_*^{2+2a} in (3.57) can be neglected. It follows that $\Delta \mathcal{L} < 0$ if

$$R_*^2 \lambda_*^2 + 72 \frac{\partial^2 U}{\partial \phi^2}(R_*, 0, \theta_*) < 0. \quad (3.58)$$

The calculation of R_* , λ_* , and $\frac{\partial^2 U}{\partial \phi^2}(R_*, 0, \theta_*)$ in terms of m_1 , m_2 , and m_3 is discussed in the Appendix.

Appendix

We show how to calculate (3.58) in terms of m_1 , m_2 , and m_3 . The detail of the calculation can be found in [29]. By the hypotheses of Theorem 3.2, $x_{0i} = (a_{0i}, 0)$ and $a_{03} < a_{02} < a_{01}$, $i = 1, 2, 3$. We assume that $x^* = (x_1^*, x_2^*, x_3^*)$ is an Euler configuration with the same order as x_0 . Without loss of generality, we assume that $x_i^* = (a_i^*, 0)$, $i = 1, 2, 3$ and $a_3^* < a_2^* < a_1^*$. By [27], we have

$$\begin{aligned} a_1^* &= [m_2\rho + m_3(\rho + 1)]a, \\ a_2^* &= (m_3 - m_1\rho)a, \\ a_3^* &= -[m_1(\rho + 1) + m_2]a, \end{aligned} \tag{3.59}$$

where ρ is the only positive root of

$$\begin{aligned} (m_2 + m_3)y^5 + (2m_2 + 3m_3)y^4 + (m_2 + 3m_3)y^3 \\ - (3m_1 + m_2)y^2 - (3m_1 + 2m_2)y - (m_1 + m_2) = 0, \end{aligned} \tag{3.60}$$

and

$$a = \sqrt[3]{\frac{9}{2(m_2\rho + m_3(\rho + 1))} \left[\frac{m_2}{\rho^2 M^2} + \frac{m_3}{M^2(\rho + 1)^2} \right]},$$

where $M = m_1 + m_2 + m_3$.

By the equal-mass Hopf transformation, we have

$$\mathcal{H} \circ \mathcal{J}_{(1,1,1)}(x^*) = \left(\frac{1}{6}(\rho^2 - 2\rho - 2)M^2 a^2, \sqrt{\frac{1}{3}} \left(\frac{\rho^2}{2} + \rho \right) M^2 a^2, 0 \right). \tag{3.61}$$

By [29], we have

$$\frac{\partial^2 U}{\partial \phi^2}(0, \theta_*, R_*) = \frac{-M}{2R_* \hat{I}_*^{1/2}} \sum_{k=1}^3 p_k \hat{s}_k^{-3/2} \left(\left(\sum_{j=1}^3 p_j \gamma_j \right) (\gamma_k - 1) + \hat{I}_* \gamma_k \right) \tag{3.62}$$

where $\hat{I}_* = \sum_{k=1}^3 p_k (1 - \gamma_k)$.

$$\begin{aligned} p_1 &= \frac{m_2 m_3}{M} & \gamma_1 &= \frac{\rho^2 + \rho - \frac{1}{2}}{\rho^2 + \rho + 1} & \hat{s}_1 &= (1 - \gamma_1) \\ p_2 &= \frac{M}{m_1 m_3} & \gamma_2 &= \frac{-\frac{1}{2}\rho^2 - 2\rho - \frac{1}{2}}{\rho^2 + \rho + 1} & \hat{s}_2 &= (1 - \gamma_2) \\ p_3 &= \frac{M}{m_1 m_2} & \gamma_3 &= -\frac{\rho^2 - 2\rho - 2}{2(\rho^2 + \rho + 1)} & \hat{s}_3 &= (1 - \gamma_3) \end{aligned} \tag{3.63}$$

Example Consider an Euler configuration of $m_1 = m_3 = m$ and $m_2 = 1$. One can show that $\rho = 1$, $\gamma_1 = \gamma_3 = \frac{1}{2}$ and $\gamma_2 = -1$. After calculation and simplification, we have

$$R_*^2 \lambda_*^2 + 72 \frac{\partial^2 U}{\partial \phi^2}(R_*, 0, \theta_*) = \frac{27m}{8m+4} \left(\frac{9}{2} + \frac{9}{8}m\right)^{-\frac{1}{3}} (4 - 27m). \quad (3.64)$$

The above equation is negative whenever $m > \frac{4}{27}$.

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