Modeling the Traffic Flow Evolution Process after a Network Disruption

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
Doctor Of Philosophy

December, 2010

ABSTRACT

Major disruption to a transportation network can disturb traffic flow patterns significantly. To deploy effective and efficient traffic restoration projects, a good prediction of the traffic flow pattern under network disruption is vital. Although traffic flow evolution processes have been modeled in various ways in the literature, very limited attention has been paid to the traffic flow evolution process after an unexpected network disruption. In fact, due to the lack of data, none of the existing day-to-day traffic assignment models have been compared against reality, and thus their quality has not yet been verified. There clearly exists a gap between day-to-day traffic flow evolution modeling and their practical applications, especially under network disruption scenarios that are of great interest to traffic management authorities. This doctoral research is dedicated to bridging that gap by developing and validating innovative new models for deterministic day-to-day traffic assignment problem.

The first innovation is the development of a link-based traffic dynamic model for studying traffic evolution. Existing deterministic day-to-day traffic assignment models were all built upon path flow variables. Most path-based models, however, suffer two essential shortcomings. One is that their application requires a given initial path flow pattern, which is typically unidentifiable, i.e., mathematically nonunique and practically unobservable. In addition, different initial path flow patterns constituting the same link flow pattern generally gives different day-to-day link flow evolutions. The second shortcoming is the path overlapping problem, whereby the interdependence of paths is ignored, leading to unreasonable results for networks with overlapping paths. The proposed link-based day-to-day traffic dynamic model avoids the two shortcomings, and captures travelers' cost-minimization behavior in their path finding as well as their

inertia. The stable point of the link-based dynamical system is rigorously proven to be the classic Wardrop user equilibrium. Its asymptotic stability is guaranteed under mild conditions.

Our second innovation is the establishment of a "prediction-correction" framework for modeling traffic evolution after an unexpected network disruption. By studying actual behavioral changes of drivers after the collapse of the I-35W Mississippi River Bridge in Minneapolis, we found that most existing day-to-day traffic assignment models would not be suitable for modeling traffic evolution under network disruption, because they assume that drivers' travel cost perception depends solely on their experience from previous days. They do not recognize that, when a significant network change occurs unexpectedly, travelers' past experience on a traffic network may not be entirely useful if the disturbance to traffic patterns is extensive. To remedy this, this research proposes a "prediction-correction" model to describe the traffic equilibration process, in which travelers predict traffic patterns after network changes and gradually correct their predictions according to their new travel experience. We also prove rigorously that, under mild assumptions, the proposed "prediction-correction" process has the Wardrop user equilibrium flow pattern as a globally attractive point.

Most importantly, this doctoral research verifies the proposed models against a real network disruption scenario. The proposed models are calibrated and validated with field data collected after the collapse of the I-35W Bridge. This study bridges the gap between theoretical modeling and practical applications of day-to-day traffic equilibration approaches and promotes a further understanding of traffic equilibration processes after an unexpected network disruption.

Acknowledgements

This dissertation has benefitted immensely from the guidance and assistance of many people. I first want to express my sincere gratitude to my advisor, Dr. Henry Liu, for his valuable guidance in my academic research and generous support throughout my graduate study. His serious dedication, rigorous and diligent working attitude set an example to my career. I am also grateful to Dr. Gary Davis, Dr. David Levinson and Dr. Jason Cao, not only for their precious time serving in the exam committee, but for their constructive suggestions and comments on this research and teaching courses which have been helpful to me. Their encouragement, patience, and enthusiasm have been of great value in the completion of this dissertation.

Thanks should also go to all colleagues in our research group for their collaboration, and in particular to Dr. Xiaolei Guo, Saif Eddin Jabari, and Xuan Di for the inspiring discussions on the link-based traffic dynamic model and the stability analysis. I also appreciate the help from Adam Danczyk, Xinkai Wu, and Shangjiang Zhu for dealing with the loop detector data and survey data.

Last, but not least, my family has been incredibly supportive throughout my life. I want to acknowledge my wife, Wenqi Li, and my parents Gensheng He and Li Ma. Their everlasting love and spiritual support has been tremendous and greatly appreciated.

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List of Notation

^{T}	Transpose operator
$[\cdot]_+$	Projection of nonnegativity, i.e., $[x]_+ = \max\{0, x\}$
$\|\cdot\|$	The Euclidean norm, i.e., $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$
$< \cdot, \cdot >$	Inner product of two vectors
a	Link index
c_a	Link cost on link $a \in \mathcal{L}$
$\mathbf{c}$	Link cost vector
$c(\cdot)$	Link performance function
d	Demand
f	Path flow
$\hat{f}$	Predicted path flow
p_i^t	Cordon i's volume percentage w.r.t. the first day's cordon volume
r	Route index
t	Time index
u_i^t	Hourly volume entering cordon i on day t
w	Origin-destination pair index
x	Link flow
$\hat{x}$	Predicted link flow
C_r	Route cost on route $r \in \mathcal{R}$

$\mathbf{C}$	Route cost vector
$D(\cdot,\cdot)$	Distance measure function
$\mathcal{G}$	Fully-connected directed graph
L	Lipschitz constant
$\mathcal{L}$	Link set
$\mathcal{N}$	Node set
$\mathbf{P}$	Travelers' perceived travel cost vector
$\hat{\mathbf{P}}$	Travelers' predicted travel cost vector
$\mathrm{Pr}_{\Omega}(\cdot)$	Projection of a vector into the convex set Ω
$\mathcal{R}$	Route set
$\mathbb{R}^n$	$n-$dimentional real space
$\mathbb{R}^n_+$	Positive orthant of $\mathbb{R}^n$
T	Studying time interval
$\mathcal{W}$	Origin-destination pair set
α	Perception update weight
β	Cost sensitivity
λ	Prediction dumping parameter
κ	Strong monotonicity constant
ν	Flow change rate in continuous traffic dynamic
θ	Parameter vector to be estimated
Δ	Link-path incidence matrix
Φ	OD-path incidence matrix
Ω	Feasible link flow set

Chapter 1

Introduction

Prediction of day-to-day traffic flow evolution is vital for the prioritization of traffic restoration projects after a network disruption, such as the unexpected collapse of the I-35W Mississippi River Bridge in Minneapolis in 2007. From a traffic management perspective, effective planning for traffic restoration after unexpected disruption requires understanding how traffic patterns evolve from a disequilibrium state toward a new equilibrium. Therefore, the focus here is on the transition process, rather than the final equilibrium state.

Day-to-day (or inter-periodic) traffic modeling methods are believed to be most appropriate for analyzing traffic equilibration processes. As noted by Watling and Hazelton (2003), the most appealing feature of day-to-day approaches is the great flexibility they provided to researchers and practitioners with their wide range of behavior rules, levels of aggregation, and traffic modes that can be synthesized into a unified framework. This equilibration paradigm helps transportation planning and management in modeling the evolution of traffic states and the trajectories of that evolution.

1.1 Problem Statement

The objective of day-to-day traffic modeling differs from that of conventional static traffic assignment, which distributes traffic flow in a network such that a predefined goal, e.g., user equilibrium, is achieved. This approach is usually adopted for long-term transportation planning purposes. However, in the case of an unexpected network disruption, transportation planners and engineers need a dynamic traffic assignment model to describe how traffic flow patterns evolve to the equilibrium state so they can effectively deploy the resources to reduce traffic congestion and improve efficiency. Thus, day-to-day traffic assignment models focus more on the disequilibrium states of a traffic network. It requires appropriate methodology to answer the questions associated with traffic evolution characteristics, such as how a disturbed flow pattern re-equilibrates to a new stable pattern after network disruption or whether an equilibrium flow pattern can eventually be reached.

A day-to-day traffic evolution model provides a process to adjust network flows from one day to another based on a description of how travelers change their daily behaviors (in the presence of different kinds of information) following a disequilibrating structural change in the underlying travel demand, physical infrastructure, or associated technology (Friesz et al. 1994). The focus of day-to-day traffic assignment modeling is a transient procedure due to the changes of supply, and thus it differs from another category of dynamic traffic assignment (e.g., Jayakrishnan et al. 1995; Ran and Boyce 1996; Peeta and Ziliaskopoulos 2001) where the problem focuses on the within-day dynamics. The output of day-to-day traffic assignment models is a description of flow variations or flow evolution trajectories. With a good understanding of flow variations (or flow trajectories), dynamic traffic control strategies can effectively react to perturbations after an unexpected network disruption.

Although day-to-day assignment modeling has much appeal as a research topic, as pointed out by Mahmassani (1990), it is difficult to obtain real-world observational evidence to verify a day-to-day traffic assignment model. Most studies on day-to-day dynamic traffic modeling (e.g., Mahmassani et al. 1986; Jotisankasa and Polak 2005) rely on experimental approaches rather than field data. Recently, Zhu et al. (2010) reviewed the existing studies on planned and unplanned network disruptions. These studies, (including Chang and Nojima 2001; Hunt et al. 2002; Dimitriou et al. 2006; Lo and Hall 2006) provided empirical observations of traffic fluctuation under network disruption; however, none of the above studies modeled these observations mathematically. Due to the lack of real world data, very few studies have compared the outcomes of a day-to-day traffic assignment model against reality, and thus the quality of existing models has not been verified. As noted by Friesz and Shah (2001), the urgent need for day-to-day traffic dynamic modeling is neither to establish delicate mathematical formula nor to develop computational tools, but rather "to gather data which allows construction and calibration of day-to-day adjustment dynamics". There clearly exists a gap between day-to-day traffic flow evolution models and their practical applications, especially for disrupted networks which are of great concern to traffic management authorities.

This doctoral research is devoted to bridging the gap by developing and validating new deterministic day-to-day traffic assignment models, based on observational data collected after the collapse of I-35W Mississippi River Bridge on August 1, 2007. Since the I-35W highway bridge is a major artery in the Twin Cities highway network, its collapse constituted a significant disruption to trip-making patterns in the region. While catastrophic to those affected in terms of fatalities, injuries, and loss of personal property, such event provides a rare opportunity for transportation scientists to observe the traffic flow evolution process empirically.

In addition to the lack of validation, most existing deterministic day-to-day assignment models suffer from two shortcomings that are inherent in path-based modeling mechanism when they are applied to real-world scenarios. One shortcoming of path-based models is the path flow non-uniqueness. The application of existing models requires a given initial path flow pattern, which is typically unidentifiable, i.e., mathematically non-unique and practically unobservable. In particular, for the path-based models, different initial path flow patterns constituting the same link flow pattern generally gives different day-to-day link flow evolutions. The second shortcoming of the path-based models is the path overlapping problem. That is, the path-based models ignore the interdependence among paths and thus can give very unreasonable results for networks with paths that overlap. These two shortcomings were first pointed out in our recent paper He et al. (2010) and since then, it has attracted other researchers' interest to develop new models to overcome the unrealism of path-based models, e.g., the splitting rate method proposed in Smith (2010).

Finally, based on the empirical evidence, we found that existing deterministic day-to-day traffic assignment models would not be suitable for modeling traffic evolution after an unexpected network disruption, because they assume that drivers' cost perception depends solely on their experiences from previous days. Such cost updating mechanisms can be considered as cost correction processes, but they cannot be applied directly to a traffic network with a significant unexpected disruption. Under such conditions, travelers' past experience on a traffic network may not be entirely useful because the traffic patterns they knew previously may no longer exist.

1.2 Research Objectives and Scope

The main goal of this research is to gain insights into the process of traffic flow evolution by developing mathematical models which can account for real-world traffic behavior, such as that observed after the collapse of the I-35W Mississippi River Bridge. Therefore, the scope of this research can be illustrated from both theoretical and empirical perspectives, as shown in Figure 1.1.

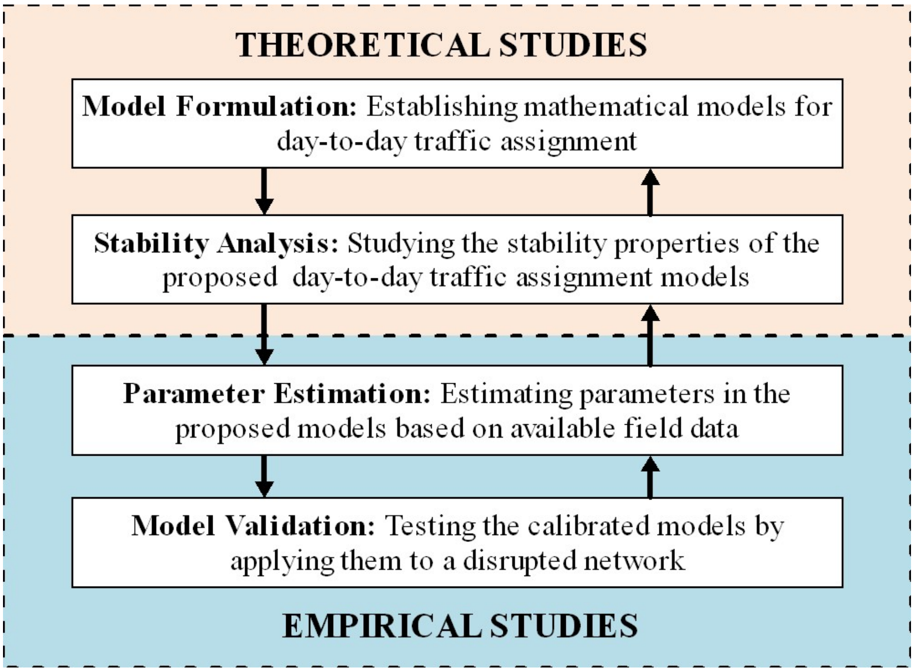


Figure 1.1: Research Components

On the theoretical side, to avoid the two shortcomings with path-based methods, we propose a day-to-day traffic assignment model that exploits the features of link flows in the network. By directly dealing with link flow variables, our model captures travelers cost-minimization behavior as well as their inertia. The fixed point of our link-based dynamical system is the classic user equilibrium (UE) flow. Comparing with the path-based models, the link-based models also reduce the computational complexity associated with path enumeration, therefore enable us to calibrate and validate in a large network such as the Twin Cities.

The dissertation also devotes significant theoretical effort to analyzing the stability of the proposed models. Stability is an important mathematical property of day-to-day assignment models that is beneficial for real-world applications. A stable system minimizes deviation from the desired flow patterns. Without the stability property, day-to-day assignment results may lead to increased unpredictability and volatility, making them useless for traffic planning. Many factors can impact the stability of a traffic system, including network disruption, traffic controls, vehicle incidents, demand variations, and travel information provision. A comprehensive stability analysis would allow traffic engineers and planners to apply the proposed models appropriately.

On the empirical side, the collapse of the I-35W Bridge provides us a natural setting to verify the proposed day-to-day traffic assignment models. Thus, our objective is to calibrate and validate the proposed day-to-day traffic assignment models, such that they replicate real-world traffic dynamics after an unexpected network disruption. Two primary sources of information are used after the I-35W Bridge collapse: travel behavior surveys and loop detector data from the Twin Cities freeway system.

In summary, the research objectives of this study are focused on:

- Developing mathematical formulations to model the traffic evolution process after an unexpected network disruption;
- Studying the stability properties of the proposed models;
- Calibrating the proposed day-to-day traffic assignment models;
- Validating the proposed models against field data collected after the I-35W Bridge collapse.

In light of these objectives, the scope of this research will be restricted to deterministic models with fixed demand and a single user class. Here, the deterministic model is built upon the "perfect information" assumption, i.e., travelers have no perception errors on the actual link costs in the past for their route choice decision-making. Fixed-demand refers to the assumption that the travel demand for each origin-destination

(OD) pair is known and fixed for the entire study period. "Single-user-class" means that drivers' route choice behaviors are assumed homogeneous in the model.

In the literature, day-to-day traffic assignment models are broadly classified as either deterministic or stochastic, and as working in either discrete or continuous temporal spaces. Although the proposed day-to-day evolution models are deterministic, they can easily be extended in many directions. Figure 1.2 depicts the categories of day-to-day traffic assignment models. The shaded boxes in the figure indicate the features considered in this research.

Specifically, this research investigates the mathematical formulations for both continuous time and discrete time models. The continuous-time flow dynamic provides a good theoretical foundation for studying day-to-day traffic fluctuation. The adjustment process derived from continuous time space is generally presented as an ordinary differential equation (ODE), the solutions to which can be used to characterize the trajectories of evolving traffic flow. However, trip adjustment based on continuous time is implausible in the real world and requires the addition of appropriate discrete-time approximations. Therefore, while the term "day" t is used as a continuous variable in the continuous-time model, our discrete-time day-to-day model uses t to denote an index of days.

To clarify the research scope, this dissertation will not explicitly study the link performance function or origin-destination demands. Instead, we assume these are given and need not concern us now. Our focus is deterministic day-to-day traffic assignment models with fixed demand and single user class. Such models can be easily extended in the future to stochastic, or elastic demand, or multi-user-class cases without too much difficulty. Concentrating on the basic form of our models allows us to put more efforts into validating the proposed models.

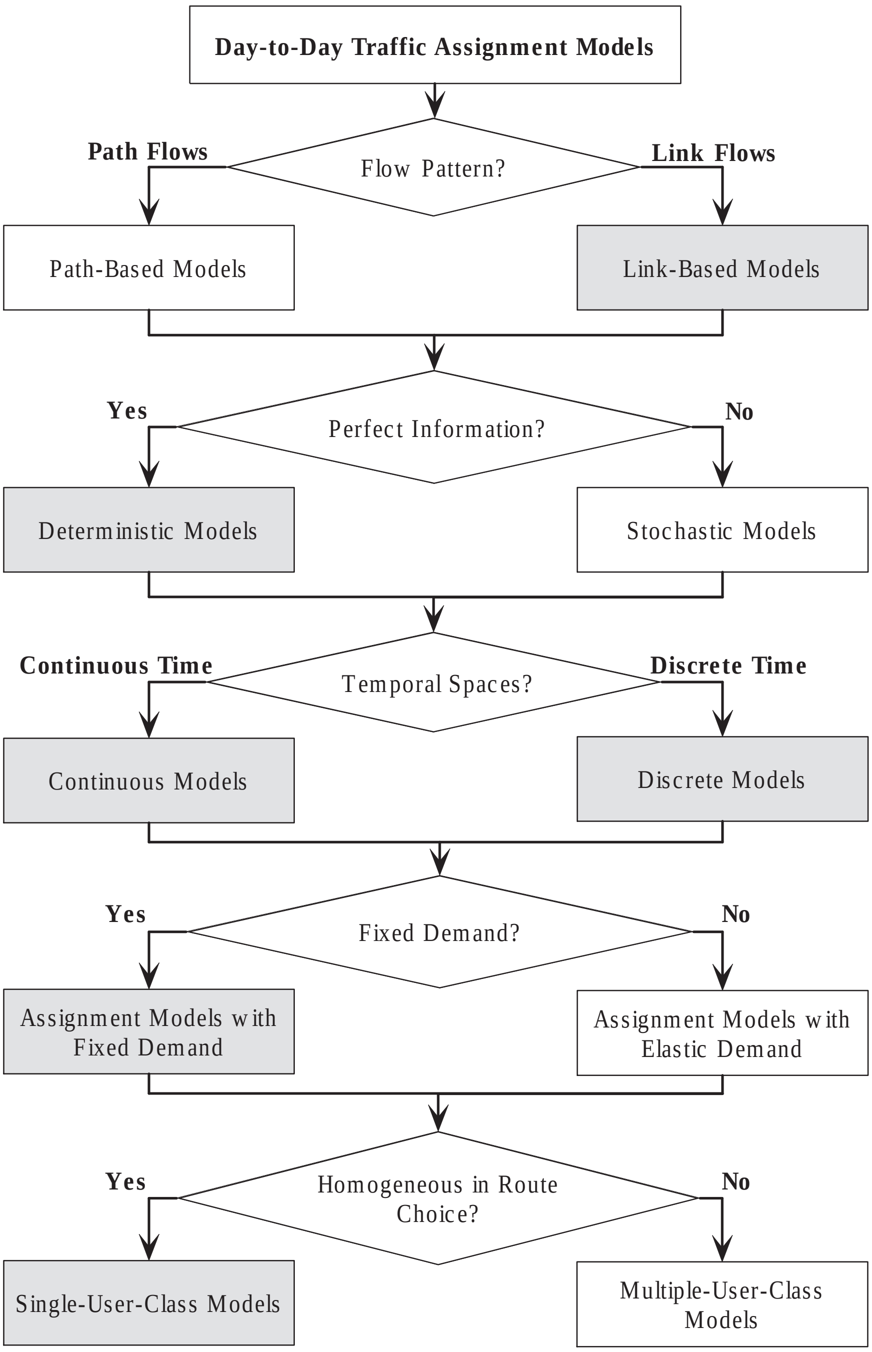


Figure 1.2: Categories of Day-to-Day Traffic Assignment Models (Shaded boxes indicate features of the models to be proposed)

1.3 Research Contributions

Overall, this dissertation not only contributes significantly to the theoretical understanding and modeling of traffic flow equilibration process, but also provides traffic management agencies a practical tool for predicting traffic flow patterns after an unexpected network disruption. The proposed day-to-day traffic assignment models differ from traditional models from two perspectives:

(1) *Link-Flow-Based Model*

The proposed models are based on link flows, which circumvent two shortcomings inherent to path-based traffic dynamic models, namely the path flow non-uniqueness problem and the path-overlapping problem. We establish a continuous-time link flow dynamic model to describe the traffic evolution process, and further provide its discrete-time approximation model to represent the day-to-day flow pattern changes. Stability properties for both continuous-time and discrete-time models are studied comprehensively. The asymptotic stability of the continuous-time model combined with the global attractiveness of the fixed point in the discrete-time model provide great tractability of both models and thus allow transportation planners and engineers apply these models effectively.

(2) *Prediction-Correction Framework*

Another distinctive feature of the proposed day-to-day assignment models is that the cost updating process considers drivers' forward-looking behavior responding to network disruption. We assume that drivers make predictions of future traffic conditions due to network topology changes. The forward-looking responses are modeled by introducing a predictive component into cost perception, such that the perceived cost pattern involves drivers' anticipation of congestion resulting from network disruption. The predictive flow patterns are then gradually corrected with driver's actual experiences. We call this cost updating procedure a "prediction-correction" process. Comparisons of the assignment results against field data observed before and after the I-35W Bridge collapse demonstrate that the prediction component in the model plays a crucial role in replicating the characteristic of traffic recovery after an unexpected network disruption.

Additionally, to the best of our knowledge, this is the first time that day-to-day traffic assignment models are validated against real-world evolutionary dynamics. Existing approaches of traffic evolution modeling, while well-grounded in theory, have not been validated in real-world disrupted networks. When dealing with real-world networks, their sizes usually make this type of analysis computationally burdensome. Since the proposed models directly deal with link flows, we are able to apply a column generation technique to avoid path enumeration, and thereby develop an efficient solution approach to solve day-to-day traffic assignment for large networks.

1.4 Thesis Organization

This dissertation is organized as follows. Chapter 2 provides a brief overview of traffic assignment models. Some background of the study is provided including a literature review of static traffic assignment modeling, since static traffic assignment has a close connection to day-to-day traffic assignment models. The review of day-to-day traffic assignment mainly focuses on existing deterministic models, to which the proposed models belong. Finally, the literature review also involves the studies about the stability properties of equilibration models, since stability is one of the most important properties of traffic evolution models.

Chapter 3 establishes a continuous-time link flow dynamic model to describe the evolution of aggregate day-to-day route flows. Unlike traditional path-based models, the proposed link flow dynamic model successfully avoids the two shortcomings: path-flow non-uniqueness and path overlapping problem. This chapter also presents a stability analysis and discusses the underlying relationship between the proposed link-based dynamic and other models.

Chapter 4 extends the continuous-time model to discrete-time day-to-day traffic assignment. We first discuss the empirical findings of drivers' route choices after the I-35W Bridge collapse, and explain why the predicted congestion of drivers plays an importance role in their route choices. In order to more realistically reflect drivers' abilities to predict and adapt traffic congestion after an unexpected network disruption, we include this forward-looking behavior in the perception updating process. Stability properties of the discrete-time model are also discussed in this chapter. Comparisons between models with and without the prediction process demonstrate the flexibility of the proposed "prediction-correction" structure.

Chapter 5 calibrates and validates the proposed models using field data collected after the I-35W Bridge collapse. The numerical results shown in this chapter demonstrate that the prediction component plays a key role in replicating the traffic restoration process after an unexpected network disruption.

Chapter 6 summarizes the findings of the dissertation and suggests ways to extend the research in the future.

Chapter 2

Literature Review

The traffic assignment problem (TAP) is to distribute the trips on the links of a transportation network, given the traffic demand, the relationship between link flows and link travel time, and the route choice principle of travelers. Traffic assignment is important in the transportation planning process because it enables transportation planners and engineers to evaluate the performance of a transportation network, and helps them to analyze alternative designs to better serve network demand.

The day-to-day traffic assignment problem is a specific type of TAP which examines traffic flows as they evolve dynamically towards equilibrium. Unlike the static traffic assignment models, which concern with the final equilibrium state, the day-to-day traffic assignment models focus more on the trajectory of evolution, from a disequilibrium to an equilibrium or from an equilibrium to another equilibrium. Since these two types of problems are closely related, in this chapter, we first provide an overview of static traffic assignment models, including the mathematical programming approach, the variational inequality (VI) approach, complementarity approach, and the fixed point approach.

Following the overview of static traffic assignment, a brief review of existing day-to-day traffic assignment studies is given. Because this research focuses on deterministic day-to-day traffic evolution modeling, the three prevailing continuous-time deterministic models are emphasized. From the review, we can clearly see the close relationship between continuous day-to-day traffic assignment modeling and static TAP modeling. Finally, we review the literature on the stability analysis of day-to-day assignment models in this chapter.

The following notation will be used throughout this dissertation. The day-to-day traffic assignment modeling is considered on a fully connected and directed graph, denoted by $\mathcal{G}(\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{L}$ is the set of links. A subset of $\mathcal{N}$ serve as *origin* nodes, from which traffic flows will be generated; and some others serve as *destination* nodes, to which traffic flows will be finally destined. All origin and destination nodes constitute the network centroids and all origin-destination (OD) pairs constitute a set denoted by $\mathcal{W}$. Since we are dealing with dynamic (or day-to-day) traffic conditions, we denote t as the time instant in the study period $[0, T]$. t represents the continuous time in continuous day-to-day traffic assignment modeling while it represents day t in discrete day-to-day traffic assignment models. A set of paths, denoted by $\mathcal{R}^w$, connect one OD pair indexed by w. The total number of trips (or demand) for OD pair w on day t is represented by d_t^w. The demand vector for all OD pairs is denoted by $\mathbf{d}$. Throughout this research, we assume that day-to-day traffic demands are constant for each OD pair. The flow on path $r \in \mathcal{R}^w$ at time instant (or on day) t is denoted by f_r^t. All path flows at time instant (or on day) t are represented by the vector $\mathbf{f}^t$. Let x_a^t represent flow on link $a \in \mathcal{L}$ at time instant (or on day) t. The vector of link flows is denoted by $\mathbf{x}^t$. Let $\Delta = (\delta_{ar})$ represent the arc-path incidence matrix, then $\mathbf{x}^t = \Delta \mathbf{f}^t$. Let $\Phi = (\phi_{ap})$ represent the OD-path incidence matrix, then $\mathbf{d}^t = \Phi \mathbf{f}^t$. Let $\mathbf{c}^t$ denote the link cost vector at time instant (or on day) t, with individual link cost c_a^t. If we assume that the link costs are functions of the whole link flow vector, then $\mathbf{c}^t = c\left(\mathbf{x}^t\right)$, where $c\left(\cdot\right)$ is a given deterministic function. Finally, denote driver pre-trip perceived link cost at time instant (or on day) t by $\mathbf{P}^t$.

2.1 Static Traffic Assignment Models

Most static travel choice models are formulated based on Wardrop's first principle (Wardrop 1952), usually known as *Wardrop equilibrium*, or *user equilibrium* (UE):

> *The journey times in all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route.*

The estimation of user equilibrium has been typically achieved through solving optimization, complementarity, variational inequality, or fixed point problems. Beckmann et al. (1956) were the first to propose an nonlinear programming (NLP) formulation for the UE principle. It follows that:

$$\min z(x) = \sum_{a\in\mathcal{L}} \int_0^{x_a} c_a(\omega)\mathrm{d}\omega \tag{2.1a}$$

$$\text{s.t.} \sum_k f_k^w = d_w \qquad \forall w \in \mathcal{W} \tag{2.1b}$$

$$f_k^w \geq 0 \qquad \forall k \in \mathcal{R}^w, w \in \mathcal{W} \tag{2.1c}$$

$$x_a = \sum_w \sum_k f_k^w \delta_{ak}^w \qquad \forall a \in \mathcal{L}. \tag{2.1d}$$

This NLP formulation (2.1) remains the basic and most important model for static and deterministic traffic assignment problems. The existence, uniqueness, and optimality conditions have been investigated in Beckmann et al. (1956). However, Smith (1979) and Dafermos (1980) pointed out that Beckman's formulation is only valid for symmetric traffic assignment in which travel cost for a link is only determined by its own flows. When traffic interactions (e.g., traffic at intersections) are considered in the link cost function, the NLP formulation may not be suitable for modeling traffic equilibrium.

In the literature, many algorithms exist for solving the NLP (2.1). The Frank-Wolfe algorithm is one popularly used link-based algorithm for solving Beckman's basic model (LeBlanc et al. 1975). The gradient projection (GP) method (Bertsekas 1976; Bertsekas and Gallager 1992; Jayakrishnan et al. 1994; Chen et al. 2002) and the disaggregated simplicial decomposition (DSD) algorithm (Larsson and Patriksson 1992) are path-based approaches. Recently, an origin-based algorithm was developed by Bar-Gera (1999, 2002) based upon the seminal work by Gallager (1977) and Bertsekas (1979).

Another popular formulation for static traffic assignment is variational inequality (VI) approach. Smith (1979) was the first to point out that the path flow vector $\mathbf{f}$ follows a deterministic user equilibrium if and only if it satisfies demand feasibility, i.e., $\mathbf{d} = \Phi\mathbf{f}$ and $\mathbf{f} \geq 0$, and all used routes have minimum cost at equilibrium flow levels:

$$f_r > 0 \Rightarrow C_r(\mathbf{f}) \leq C_s(\mathbf{f}) \qquad \forall r, s \in \mathcal{R}^w. \tag{2.2}$$

He further showed that the equilibrium condition (2.2) can be formulated as an equivalent system, as to find a vector $\mathbf{f}^* \in \Omega_f$, such that

$$(\mathbf{f} - \mathbf{f}^*)^T C(\mathbf{f}^*) \geq 0, \qquad \forall \mathbf{f} \in \Omega_f, \tag{2.3}$$

where $\mathbf{f}$ denotes a vector of path flows and $\Omega_f = \{\mathbf{f}|\Phi\mathbf{f} = \mathbf{d}, \mathbf{f} \geq 0\}$ denotes the feasible path flow set. Notice the relationship between link flows and path flows: $\mathbf{x} = \Delta\mathbf{f}$ and the relationship between path costs and link costs: $\mathbf{C} = \Delta^T\mathbf{c}$; then the path-based inequality system (2.3) is equivalent to a link-based inequality system, defined as to find a vector $\mathbf{x}^* \in \Omega$ such that:

$$(\mathbf{x} - \mathbf{x}^*)^T c(\mathbf{f}^*) \geq 0, \qquad \forall \mathbf{x} \in \Omega, \tag{2.4}$$

where $\mathbf{x}$ denotes a vector of link flows and $\Omega = \{\mathbf{x}|\mathbf{x} = \Delta\mathbf{f}, \Phi\mathbf{f} = \mathbf{d}, \mathbf{f} \geq 0\}$ denotes the feasible link flow set. The inequality system (2.4) is also referred to a finite-dimensional variational inequality problem, usually abbreviated as VI(Ω, c).

The existence and uniqueness of VI solution have been investigated under mild conditions. Smith (1979) proved that a VI solution exists, if the feasible set Ω is closed and convex and link cost function $c(\cdot)$ is continuous; in addition, if $c(\cdot)$ is strictly monotone on Ω, the solution is unique. Dafermos (1980) proved the existence and uniqueness of VI formula for the traffic equilibrium problem under a strong monotonicity condition of link cost function.

The VI approach has a close relationship with the NLP approach for modeling the UE principle. Nagurney (1999) has shown that, when the link performance function $c(\cdot)$ is continuous and differentiable on Ω and its Jacobian matrix $\nabla c(\mathbf{x})$, is symmetric and positive definite, there exist a real-valued convex function $G : \Omega \to \mathbb{R}$ satisfying $\nabla G(\mathbf{x}) = c(\mathbf{x})$ with $\mathbf{x}^*$ the solution of VI(Ω, c) also being the solution of the optimization problem (2.1).

For more general asymmetric traffic networks, i.e., the Jacobian matrix $\nabla c(\mathbf{x})$ is asymmetric, it is usually hard to find a corresponding optimization problem regarding the VI problem (2.4). Under such condition, gap function plays an important role in connecting VI problems to optimization problems. In the literature, Van Vliet (1987) established the connection between Frank-Wolfe algorithm and variational inequalities. This relationship was implicitly analyzed by Hearn (1982) in which the inequality system was described as gap functions. Later, Larsson and Patriksson (1994) established a unified reformulations of VI using primal and dual gap functions. Lo and Chen (2000) applied a gap function proposed by Fischer (1992) to convert the VI problem to an equivalent unconstrained optimization problem.

Nonlinear complementarity problem (NCP) is another approach to model UE conditions. The NCP is defined as follows. Let $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ be continuous. The nonlinear complementarity problem is to find an $\mathbf{x}^* \in \mathbb{R}^n$ such that

$$F(\mathbf{x})^T\mathbf{x} = 0, \tag{2.5a}$$

$$F(\mathbf{x}) \geq 0, \tag{2.5b}$$

$$\mathbf{x} \geq 0. \tag{2.5c}$$

Let τ_w^* represent the minimal travel cost connecting OD pair w at equilibrium. For any route $r \in \mathcal{R}^w$, the UE condition states that

$$C_r(\mathbf{f}^*) - \tau_w^* \begin{cases} = 0 \quad if \quad f_r^* > 0 \\ \geq 0 \quad if \quad f_r^* = 0. \end{cases} \tag{2.6}$$

That is, when the travel cost on path r is larger than the minimum travel cost, the traffic flow on that route must be zero. This cost-flow relationship actually introduces a route adjustment behavior rule that will be discussed later. Let $F(\mathbf{f}) \doteq C_r(\mathbf{f}) - \tau_w^*$. Then F is a continuous mapping defined on $\mathbb{R}_+^n$; and the path flow $\mathbf{f} \geq 0$. Therefore, above UE conditions are equivalent to the NCP problem (2.5). Aashtiani (1979) established a relationship between NCP and elastic demand traffic assignment problem. Patriksson (1994) reviewed the NCP for modeling traffic equilibrium. Chen et al. (2001) proposed a self-adaptive projection and contraction algorithm to solve the equilibrium problem with NCP formulation.

The NCP model (2.6) for traffic assignment problem has a close relationship with the VI model (2.3). Actually, the NCP model is a special VI problem. In other words, the NCP (2.6) is a VI in which $\Omega = \mathbb{R}_+^n$. If $\mathbf{f}^*$ is a solution to (2.6), then $F(\mathbf{f}^*) = C(\mathbf{f}^*) - \tau^* \geq 0$. For any $\mathbf{f} \geq 0$, it follows that $\mathbf{f}^T F(\mathbf{f}^*) \geq 0$, and hence

$$(\mathbf{f} - \mathbf{f}^*)^T F(\mathbf{f}^*) = \mathbf{f}^T F(\mathbf{f}^*) \geq 0. \tag{2.7}$$

Thus, $\mathbf{f}^*$ is a solution to the above VI problem. The relationship between VI and NCP can also be found in Nagurney (1999).

The UE conditions can also be formulated as a fixed point problem. As mentioned by Patriksson (1994), fixed point formulations for traffic assignment arise in two different ways. One is from the reformulation of a VI or NCP formulation. Thanks to the early work of Eaves (1971), the variational inequality problem $\mathrm{VI}(\Omega, c)$ is known to be equivalent to finding a fixed point of the following projection equation:

$$\mathbf{x} = \mathrm{Pr}_{\Omega}[\mathbf{x} - c(\mathbf{x})], \tag{2.8}$$

where Pr_{Ω} is the norm projection on convex set Ω, defined as:

$$\mathrm{Pr}_{\Omega}(x) = \arg\min_{y \in \Omega} \|x - y\|. \tag{2.9}$$

Solving the fixed point problem (2.8) is equivalent to finding the zero point of the nonsmooth equation:

$$e(\mathbf{x}) = \mathbf{x} - \mathrm{Pr}_{\Omega}[\mathbf{x} - c(\mathbf{x})]. \tag{2.10}$$

The equation $e(\cdot)$ is also recognized as the residual function of VI (2.4). This fixed point approach is typically used to study the mathematical properties of a traffic dynamical system or the convergence properties of an iterative solution algorithm for a traffic assignment problem.

Another fixed point approach for modeling traffic equilibrium is due to Fisk and Nguyen (1981). They proposed a fixed point model for traffic equilibrium with elastic demand. For a given demand vector $\mathbf{d} \geq 0$, let $\tau(\mathbf{d})$ be the vector of minimum travel costs obtained when assigning the demand onto the network according to the UE principle. Let g be a demand function of minimum travel cost. Then, the equilibrium conditions can be represented as to find $\mathbf{d}^*$ such that

$$g(\tau(\mathbf{d}^*)) = \mathbf{d}^*. \tag{2.11}$$

Some other approaches are also available to formulate and solve the static traffic equilibrium assignment problems, including optimal control theory (Ran et al. 1993) and mixed complimentary problem (Wie et al. 2002). However, the inherent dynamic nature of traffic flow is ignored by the static models, which leads to poor predictions of short-term traffic status, especially under network disruption scenarios. As a result, another type of traffic assignment models have been proposed in the literature.

2.2 Day-to-Day Traffic Assignment Models

In the transportation literature, Watling (1999) classified day-to-day traffic assignment models, by whether they are continuous or discrete, and deterministic or stochastic. In the following, we will first review the deterministic continuous-time day-to-day traffic evolution models, followed by a brief review of stochastic day-to-day traffic evolution models.

Continuous-time day-to-day traffic dynamics can be established when time steps are "sufficiently" small. Existing continuous-time day-to-day dynamics employ differential equations to describe traffic evolution. Recently, Yang and Zhang (2009) classified deterministic day-to-day traffic dynamical systems into five major categories: simplex gravity flow dynamics (Smith 1983), proportional-switch adjustment process (e.g., Smith 1984; Smith and Wisten 1995; Huang and Lam 2002; Peeta and Yang 2003), network tatonnement adjustment (e.g., Friesz et al. 1994), projected dynamical system (e.g., Zhang and Nagurney 1996; Nagurney and Zhang 1997), and evolutionary traffic dynamics (e.g., Sandholm 2001; Yang 2005). All these day-to-day dynamics are path-based models, i.e., they all explicitly use path flow variables and give explicit path flow evolution trajectories. A common property of these models is that the stationary point or fixed point is the classic UE flow pattern. Yang and Zhang (2009) further showed that all five categories of traffic dynamical systems are established upon the same underlying behavioral assumptions, which they call the *Rational Behavior Adjustment Process* (RBAP). Among the RBAP category, proportional-switch adjustment process (PAP) (Smith 1984), network tatonnement adjustment (NTA) (e.g., Friesz et al. 1994), and projected dynamical system (PDS) (Zhang and Nagurney 1996) are three dynamical systems that have been most intensively studied in the transportation literature.

Smith (1984) was the first researcher who proposed a continuous-time route choice adjustment process under fixed demand. His model was based on the simple assumption that drivers would consider switching their routes in order to take advantage of lower costs in a continuous-time system. The flow switching rate is determined as a proportion of the product of the flow on the higher cost route and the difference of the route travel costs between these two routes. Apply notation $[x]_+ = \max\{x, 0\}$. For a non-equilibrium flow pattern f, the traffic flow changes from route r to route s at a rate $f_r[C_r(f) - C_s(f)]_+$. Then the dynamic of route flow (i.e., the changing rate of route flow f_r) can easily be derived as the difference between the total inflow switching from any route s to route r and outflow switching from route r to any other route s.

$$\frac{\mathrm{d}f_r(t)}{\mathrm{d}t} = -\sum_s \left(f_r[C_r(f) - C_s(f)]_+ - f_s[C_s(f) - C_r(f)]_+\right). \tag{2.12}$$

We can clearly see the connection between PAP (2.12) and NCP (2.6): At equilibrium, we have $f_r^* \geq 0$ and $[C_r(f^*) - C_s(f^*)]_+ \geq 0$, while $f_r^*[C_r(f^*) - C_s(f^*)]_+ = 0, \forall r, s$, therefore, $\frac{\mathrm{d}f_r(t)}{\mathrm{d}t} = 0$.

Friesz et al. (1994) introduced the concept of network tatonnement adjustment (abbreviated as NTA) from microeconomics to modeling day-to-day traffic flow dynamics. Their proposed tatonnement model considers the evolution processes of reported path costs and path flows simultaneously. Specifically, the change rate of the reported path cost is proportional to the excess travel demand; while the change rate of path flow is proportional to the negative excess travel cost. When we are concerned only with nonnegativity of costs and flows, their tatonnement adjustment model is shown as:

$$\begin{aligned}
\frac{\mathrm{d}u_w(t)}{\mathrm{d}t} &= \kappa_w\{[u_w(t) + \alpha(v_w(u(t)) - \sum_{r\in\mathcal{R}} f_r(t))]_+ - u_w(t)\}, && \text{(2.13a)}\\
\frac{\mathrm{d}f_r(t)}{\mathrm{d}t} &= \eta_r\{[f_r(t) - \beta(c_r(t) - u_w(t))]_+ - f_r(t)\}, && \text{(2.13b)}\\
u_w(0) &= u_w^0, && \text{(2.13c)}\\
f_r(0) &= f_r^0, && \text{(2.13d)}
\end{aligned}$$

where $u_w(t)$ denotes the reported travel cost for OD pair w at time t, and $v_w(\cdot)$ denotes the travel demand function for OD pair w. The value of $v_w(u(t)) - \sum_{r\in\mathcal{R}} f_r(t)$ implies the excess travel demand for OD pair w; while $c_r(t) - u_w(t)$ represents the excess travel cost for route r.

Friesz et al. (1994) pointed out that the NTA model (2.13) is a *global projective dynamical system*, since "under mild regularity conditions, its trajectories remain feasible throughout time". Later, Smith et al. (1997) rigorously showed that the disequilibrium trajectories depicted by (2.13) tend to be entirely within the feasible region Ω and do not follow constraint boundaries.

The third popular deterministic day-to-day evolution model was proposed by Zhang and Nagurney (1996). The methodology they applied to model route flow dynamics is the projected dynamical systems (PDS) theory, introduced by Dupuis and Nagurney (1993). This PDS adjusts day-to-day route flows by a minimum norm projection operator, rather than the proportional switching in Smith's work.

The PDS describes the traffic dynamic as follows. If a route travel time exceeds the equilibrium cost, then the route flow decreases; if the cost is less than the equilibrium cost, then the route flow increases. Of course the route flows cannot be decreased if they are currently zero. The formulation of the dynamic is represented as:

$$\frac{\mathrm{d}f_r(t)}{\mathrm{d}t} = \begin{cases} \alpha(\hat{\lambda}_w(f) - C_r(f)) & \text{if} \quad f_r(t) > 0 \\ \Big[\alpha(\hat{\lambda}_w(f) - C_r(f))\Big]_+ & \text{if} \quad f_r(t) = 0, \end{cases} \tag{2.14}$$

where $\hat{\lambda}_w(f)$ denotes the minimal route travel cost for OD pair w. The ordinary differential equation representation of the PDS is:

$$\frac{\mathrm{d}f_r(t)}{\mathrm{d}t} = \Pi_\Omega(f, -C(f)), \tag{2.15}$$

where the projection operator Π_Ω is defined as:

$$\Pi_\Omega(f, v) = \lim_{\varepsilon\to 0} \frac{\mathrm{Pr}_\Omega(f + \varepsilon v) - f}{\varepsilon}, \tag{2.16}$$

with Pr_Ω being the norm projection on convex set Ω, defined as (2.9). As we can see, the right hand side of the ordinary differential equation (2.14) is discontinuous, due to the existence of projection operator. Zhang and Nagurney (1995) indicated that the formulation of PDS (2.15) represents the Gateaux directional derivative of projection operator Pr_Ω. The PDS proposed by Zhang and Nagurney (1996) is also known as a *local projected dynamical system*, in contrast with the global projective dynamical system proposed by Friesz et al. (1994).

Both NTA and PDS models have a close relationship with the VI problem (2.3). In the NTA model (2.13), the right hand side of the dynamical system represents the value of a residual function (2.10) of the VI problem. The projection operator Π_Ω in PDS measures the directional derivative of the residual function (2.10) of the VI problem at current path flow pattern f.

Despite some debates existing between these approaches, overall the continuous-time models provide some suggestive information on link flow evolution and equilibrium state (Watling and Hazelton, 2003). Although these models have good mathematical properties for describing traffic evolution, the continuous-time trip adjustments are not plausible in reality since drivers' route choice is not continuously adjustable in time. Therefore, in accordance with the daily changes in traffic flows, all continuous-time day-to-day traffic evolution models have discrete-time approximations to avoid the unrealistic continuous-time trip adjustment. It is believed that discrete versions of day-to-day traffic equilibration models are more suited for traffic systems with day-to-day fluctuation.

In discrete-time day-to-day traffic dynamical systems, travelers are assumed to use the same rule to make daily route choices, in accordance with changes in traffic flows. Specifically, Friesz et al. (1994) employed a projection-type discretization algorithm, given by Bertsekas and Gafni (1982), to approximate the continuous traffic trajectories in the NTA model. Nagurney and Zhang (1997) specified their continuous model in a discrete temporal space with fixed demand and applied Euler's method to solve the PDS. Other discrete-time day-to-day dynamics were formulated as difference equations (e.g., Horowitz 1984) or simulation models (Mahmassani et al. 1986).

As we have mentioned, day-to-day traffic evolution models can be classified into deterministic processes and stochastic processes. So far the day-to-day assignment models reviewed are deterministic processes. Due to the random nature of transportation systems, stochasticity has been introduced into day-to-day assignment approaches. Differing from deterministic processes, stochastic day-to-day assignment models consider drivers' perception errors.

Most existing stochastic day-to-day assignment models follow Markov processes; examples include models proposed in Cascetta (1989) and Hazelton and Watling (2004), where the computation of transition matrices depends only on the previous traffic state. The transition probability matrix could be specified and leads to approximations of system mean (or deterministic) dynamics, as shown by Daganzo and Sheffi (1977), Davis and Nihan (1993), and Yang and Liu (2007b). Other stochastic approaches, e.g., Horowitz (1984), Cantarella and Cascetta (1995), Watling (1999), adopted the assumption of memory length, assuming that route choice probabilities depend on weighted averages of experienced travel times. To solve stochastic models, Davis and Nihan (1993) provided a particular Gaussian multi-variant autoregressive process and Hazelton et al. (1996) proposed a Markov Chain Monte Carlo (MCMC) method.

Many stochastic day-to-day dynamics have the stochastic user equilibrium (SUE) as their fixed point, when they assume the route choice fractions follow a logit model. For example, Cantarella and Cascetta (1995) applied fixed point formulation (2.11) to model traffic equilibrium in a dynamical system, whose fixed point is equivalent to the SUE flow pattern; in the logit assignment based model of Watling (1999), the logit-based

SUE is the fixed point.

Although above-mentioned studies represent decades of research on traffic flow evolution, very limited attention has been paid to modeling traffic flow evolution process for a disrupted network. Existing studies on day-to-day traffic flow evolution are mainly concerned with the properties of the final state and generally ignore the trajectory of traffic stabilization, that itself is important for studying traffic flow evolution under unexpected disruption. In fact, due to the lack of data, very few studies have compared day-to-day traffic evolution models against reality, and thus the quality of currently existing models has not been verified yet. In contrast, our study will use the field data collected after the I-35W Bridge collapse to verify the proposed models.

2.3 Stability Analysis of Traffic Assignment Models

We may believe that network flow would converge to the equilibrium if sufficient time has elapsed under the same traffic condition. However, Beckmann et al. (1956) showed in static cases that equilibrium pattern will not always be reached from an arbitrary initial state, even if equilibrium exists uniquely. For this reason, stability is an important issue for day-to-day traffic assignment.

In addition to existence and uniqueness properties of traffic dynamic models, stability critically impacts the application of day-to-day traffic evolution models, mainly because it is a property associated with system perturbation. Unstable equilibrium is sensitive to perturbation and makes it difficult to provide useful information in practice. The stability property involves characterizing the "attractiveness" of an equilibrium solution following a perturbation to the flows, where the attractiveness is governed by the relationship between link costs and link flows, and the rules governing traveler route choice behavior. A detailed summary of stability analysis in the context of deterministic evolution models is provided in Yang (2001).

Many studies associated with network equilibration modeling address the stability (e.g., Smith 1984) and attraction domain (e.g., Bie and Lo 2010) of various traffic adjustment processes. Smith (1984) applied a Lyapunov theorem to show that the proportional-switch adjustment process (2.12) is globally stable under the assumption that route cost function is continuously differentiable, monotone, with no explicit capacity restrictions. And he also proved that the stable point of this dynamic actually satisfies Wardrop's equilibrium.

The stability property for the tatonnement adjustment process (2.13) was also analyzed by applying the Lyapunov theorem in Friesz et al. (1994). The authors provided the regularity conditions in order to develop a Lyapunov function for the tatonnement process. The asymptotic stability of Wardrop's equilibrium at elastic demand is assured under the assumption that the route cost function and the negative inverse demand function are both continuous and strictly monotone. They also discussed the stability properties under complete or incomplete information provision.

Regarding the stability of projected dynamical system (2.15), Zhang and Nagurney (1995) proposed regularity conditions to ensure local and global convergence to stationary points. Zhang and Nagurney (1996) further proposed regularity conditions for performing local stability analysis in a traffic assignment problem. Their study was based on a distance measure function which satisfies the requirements of Lyapunov function under regularity conditions. In addition, they also addressed how to evaluate asymptotic stability on the equilibrium state in the absence of the monotonicity assumption, which was first studied by Bernstein (1990). Later, Nagurney and Zhang (1997) applied the global stability specifically to day-to-day traffic evolution. The notion of asymptotic stability was introduced in these papers by considering the attractiveness of the stationary points.

Differing from previous stability studies, Bie and Lo (2010) applied the attraction domain to consider the asymptotic stability, where the attraction domain of an equilibrium state is defined as the set of all states that will dynamically evolve to the equilibrium. They developed different computation schemes to estimate the range of the attraction domain for an asymptotically stable equilibrium. Their study is useful for developing appropriate transportation network designs to achieve stable equilibria, or helping an instable system to shift from an unstable equilibrium to a stable equilibrium.

For stochastic cases, Horowitz (1984) considered the stability property by applying three route cost perception updating processes: (1) weighted averages of measured travel costs, (2) weighted averages of perceived route costs, and (3) unchanged perceived costs unless actual usage. He showed that, even though the existence and uniqueness conditions hold at the equilibrium, flow trajectories may exhibit three different presentations under reasonable learning mechanisms: oscillating about the equilibrium, evolving periodically converging to an attractor but not equilibrium, or stabilizing to equilibrium.

Besides, Cantarella and Cascetta (1995) considered the stability of the stochastic equilibrium for general networks. In that study, traffic dynamics are formulated by a recurrence function of the vector of perceived costs. Equilibrium stability was analyzed by linearizing the dynamical system around the equilibrium. Stability in their research requires that the corresponding Jacobian matrix, when evaluated at the equilibrium point, has eigenvalues (real or complex) all within the unit circle. Later, Cantarella and Velona (2003) showed that the stability of equilibrium cannot be given for granted. By studying the eigenvalues of the Jacobian matrix of the transition function and applying the bifurcation analysis, they demonstrated that a fixed point of a SUE model may not be stable even for a simple three-link network. The bifurcation analysis offers another way to study the stability of the equilibrium flow pattern.

In this study, we perform stability analysis following the definitions of stability provided in Zhang and Nagurney (1995) and Bie and Lo (2010). We show that the proposed link-based traffic dynamic has the classic user equilibrium link flow as the stable point and that it is asymptotically stable under mild conditions.

Chapter 3

A Continuous-Time Link-Based Day-to-Day Traffic Dynamic

Day-to-day traffic dynamic models aim to capture daily traffic evolution. It focuses on the evolution trajectory itself, rather than the final (static) equilibrium state. To describe the trajectory, when time steps are "sufficiently" small, day-to-day traffic dynamics can be established on continuous temporal space, therefore differential equations can be employed. Existing continuous-time traffic dynamics employ differential equations to describe traffic evolution. Similar to most of the deterministic day-to-day dynamics, we assume that travelers have perfect perception on the actual travel costs occurring in the past.

This chapter will first investigate path-based day-to-day traffic assignment models, since most of the existing deterministic day-to-day traffic assignment models were all built upon path flow variables. As will be demonstrated in this chapter, path-based models suffer two major shortcomings, i.e., the path-flow non-uniqueness and the path-overlapping problem. To address these shortcomings, a link-based day-to-day traffic dynamic model is developed, whose stability property will be discussed. The proposed link-based traffic dynamic model (He et al., 2010) has the classic user equilibrium as its fixed point.

3.1 Shortcomings of Path-Based Day-to-Day Traffic Flow Dynamics

In this section we demonstrate two essential shortcomings of path-based day-to-day traffic dynamics. The two shortcomings become most manifest when the models are used rather than just theoretically discussed, thus we shall apply a path-based dynamic to a simple yet illustrative network. We adopt a discrete-time approach in this section because it suits real application better than a continuous-time approach.

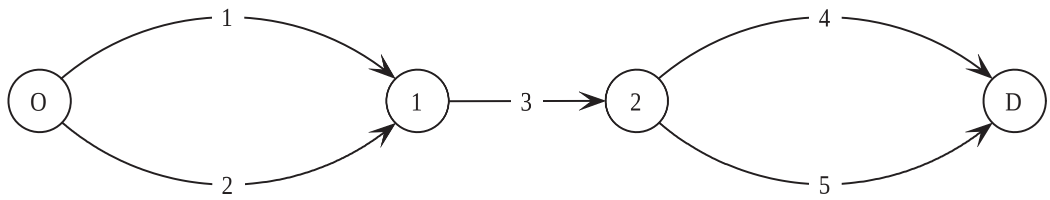


Figure 3.1: A Network to Demonstrate the Shortcomings of Path-Based Dynamics

We begin with the simple network shown in Figure 3.1, consisting of 4 nodes and 5 links. There is one origin-destination (OD) pair from Node O to Node D connected by four paths numbered as below:

Path 1 Link sequence $1 \longrightarrow 3 \longrightarrow 4$;

Path 2 Link sequence $1 \longrightarrow 3 \longrightarrow 5$;

Path 3 Link sequence $2 \longrightarrow 3 \longrightarrow 4$;

Path 4 Link sequence $2 \longrightarrow 3 \longrightarrow 5$.

Assume that the network flow in Figure 3.1 is initially at user equilibrium (UE) and that a capacity reduction on Link 4 takes place at day 0. Let $\tilde{\mathbf{x}}$ be the initial UE link flow pattern, and assume that $\tilde{\mathbf{x}} > 0$, i.e., each link initially has some flow. Let $\tilde{C}$ be the initial UE path travel cost, i.e., all four paths have the same travel cost $\tilde{C}$ before the capacity reduction takes place. When the capacity reduction happens at day 0, the link flow pattern is still $\tilde{\mathbf{x}}$, and thus it is clear that the path costs of Path 2 and Path 4 do not change (Link 4 not included in these paths), while the path costs of Path 1 and Path 3 increase to the same level (due to the same cost increase of Link 4) denoted as C^0 such that $C^0 > \tilde{C}$.

3.1.1 The path-flow-nonuniqueness problem

Now we have an observation of the network condition at day 0: link flow pattern $\mathbf{x}^0 = \tilde{\mathbf{x}}$, which gives link cost vector $\mathbf{c}(\tilde{\mathbf{x}})$ and path cost vector $\mathbf{C} = (C^0, \tilde{C}, C^0, \tilde{C})'$. Suppose we have at hand a well-established day-to-day traffic assignment model, then we should be able to predict the day-to-day traffic equilibration process after day 0. However, to apply any path-based model, we need a given initial path flow $\mathbf{f}^0$. Without further assumptions, all that we know about $\mathbf{f}^0$ is that

$$\boldsymbol{\Delta}\mathbf{f}^0 = \mathbf{x}^0, \tag{3.1}$$

and it is clear that $\mathbf{f}^0$ satisfying (3.1) is not unique for the example here.

It should be mentioned that the path flow pattern corresponding to a given link flow pattern is generally nonunique and unobservable. In particular, the nonuniqueness of the UE path flow pattern is well known and is typically not considered as a problem of the UE solution because the UE link flows, and thus the UE system performance, are unique (under mild technical conditions). In the same spirit, neither should the nonuniqueness of $\mathbf{f}^0$ be regarded as a problem of the path-based day-to-day dynamics if different $\mathbf{f}^0$ (constituting the same $\mathbf{x}^0$) can give the same day-to-day evolution of the link flow pattern. Unfortunately, such an ideal property generally does not hold in real world settings, and thus the nonuniqueness of $\mathbf{f}^0$ is indeed a problem of path-based day-to-day dynamics. We shall demonstrate this by our small example as well as with some general derivations.

To show that different $\mathbf{f}^0$ will give different link flow evolution results, it suffices to calculate the link flow pattern at day 1, i.e., to show that $\mathbf{x}^1$ takes different values for different $\mathbf{f}^0$. According to Yang and Liu (2007a), we can apply to our example a discrete-time proportional-switch adjustment, defined by

$$f_r^t = f_r^{t-1} + \frac{1}{T_w} \sum_{s \in \mathcal{R}_w} \left(f_r^{t-1} [C_r^{t-1}(f) - C_s^{t-1}(f)]_+ - f_s^{t-1} [C_s^{t-1}(f) - C_r^{t-1}(f)]_+ \right), \quad (3.2)$$

where

$$T_w = \sum_{r \in \mathcal{R}_w} \sum_{s \in \mathcal{R}_w} [C_r(f) - C_s(f)]_+ + M \quad (3.3)$$

and M in (3.3) represents drivers' reluctance to change, i.e., a larger M indicates that more travelers prefer maintaining previous choices.

Then we obtain $\mathbf{f}^1$ as follows

$$\mathbf{f}^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \end{pmatrix} = \begin{pmatrix} (1-\rho) f_1^0 \\ f_2^0 + 0.5\rho x_4^0 \\ (1-\rho) f_3^0 \\ f_4^0 + 0.5\rho x_4^0 \end{pmatrix}, \quad (3.4)$$

where $\rho > 0$ is a coefficient given by

$$\rho = \frac{2(C^0 - \tilde{C})}{4(C^0 - \tilde{C}) + M}. \quad (3.5)$$

M in (3.5) is the reluctance parameter in the discrete-time PAP model (3.2)-(3.3). Then, from $\mathbf{x}^1 = \mathbf{\Delta f}^1$ we have

$$\mathbf{x}^1 = \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \\ x_5^1 \end{pmatrix} = \begin{pmatrix} x_1^0 + 0.5\rho x_4^0 - \rho f_1^0 \\ x_2^0 + 0.5\rho x_4^0 - \rho f_3^0 \\ x_3^0 \\ (1-\rho)x_4^0 \\ x_5^0 + \rho x_4^0 \end{pmatrix}. \tag{3.6}$$

It can be seen clearly from (3.6) that different $\mathbf{f}^0$ will give different $\mathbf{x}^1$, which means that the day-to-day evolution of link flow depends on the (nonunique and unobservable) initial path flow pattern. This problem actually exists generally for many path-based day-to-day dynamics, not just for the PAP dynamic applied to our small example. To see this, we look into the general case.

For a general network, assuming an observed initial link flow pattern $\mathbf{x}^0$, then the initial path flow pattern $\mathbf{f}^0$ solves the linear equation system (3.1). Let n_p be the number of paths. When $n_p > \text{rank}(\mathbf{\Delta})$, the $\mathbf{f}^0$ solution has a degree of freedom $DF = n_p - \text{rank}(\mathbf{\Delta})$ and can be written in the form of

$$\mathbf{f}^0 = \mathbf{Z}\xi + \mathbf{f}^*, \tag{3.7}$$

where $\mathbf{Z}$ is a constant $n_p \times DF$ matrix with full column rank satisfying $\mathbf{\Delta Z} = 0$, $\mathbf{f}^*$ is a constant vector, and ξ is an *indeterminate* $DF \times 1$ vector. It is clear that different values of ξ represent different $\mathbf{f}^0$. Observe that $\mathbf{\Delta Z} = 0$ guarantees that $\mathbf{x}^0 = \mathbf{\Delta f}^0$ does not depend on ξ, i.e., different $\mathbf{f}^0$ constitute the same $\mathbf{x}^0$.

Consider a path-based day-to-day dynamic such that the path flow at day 1 is given by

$$\mathbf{f}^1 = \mathbf{U}\mathbf{f}^0 = \mathbf{U}\mathbf{Z}\xi + \mathbf{U}\mathbf{f}^*, \tag{3.8}$$

where $\mathbf{U}$ is the path flow updating matrix determined by the specific day-to-day dynamic. Then the link flow at day 1 is given by

$$\mathbf{x}^1 = \mathbf{\Delta f}^1 = \mathbf{\Delta U Z}\xi + \mathbf{\Delta U f}^*, \tag{3.9}$$

Note that it generally holds that $\mathbf{\Delta U Z} \neq 0$, because $\mathbf{U}$ is essentially given by the day-to-day dynamic, typically related to cost and flow, not determined by the link-path incidence matrix $\mathbf{\Delta}$. Then it can be seen from (3.9) that $\mathbf{x}^1$ depends on ξ, which simply means that $\mathbf{x}^1$ depends on $\mathbf{f}^0$. Thus it is demonstrated that different initial path flow patterns generally give different link flow evolutions.

We summarize our point formally in Remark 3.1.

Remark 3.1 *For a path-based day-to-day dynamic, different initial path flow patterns constituting the same initial link flow pattern generally lead to different link flow day-to-day evolutions.*

According to this observation, if a path-based day-to-day dynamic is to be applied, it is necessary to first identify the initial path flow pattern. However, the path flow pattern is typically nonunique and unobservable given a link flow pattern. This shortcoming of many path-based models is referred to as the *path-flow-nonuniqueness* problem in this research.

Several comments can be made about the path-flow-nonuniqueness problem. First, a possible solution is to use some kind of estimation method to obtain an estimation of the initial path flow pattern. For example, the most likely path flow estimation method (e.g., Larsson et al. 2001 and Bar-Gera 2006), could be adopted. Such methods receive limited attention in this study, however, because there is another problem associated with path-based day-to-day traffic dynamics which can not be solved by estimation methodologies, as we will discuss in the next section. Second, because the path-flow-nonuniqueness problem is actually caused by the difficulty of identifying the initial path flow pattern, arguably it is not a theoretical shortcoming of the path-based models. That is to say, if the initial path flow pattern could somehow be identified, then it becomes a simple matter to apply path-based models. Indeed, it is theoretically possible to identify the path flow pattern of a network if we had a practical means to trace all vehicle paths. In this sense, the path-flow-nonuniqueness problem could be viewed as a technology or cost problem rather than a modeling problem.

Last but not least, although existing day-to-day traffic assignment models are all built upon path flow variables, not all have this path-flow-nonuniqueness problem. For example, the models that essentially conduct a logit stochastic user equilibrium (SUE) assignment for each day (e.g., Watling 1999) do not have this problem, because under the SUE setting, a path flow pattern is uniquely determined once path costs are given. Also, for the stochastic models that focus on the probability distribution of flow states and/or the expected flow state (e.g., Cascetta 1989), the specific flow evolution trajectory is not a concern (as the trajectory is essentially a realization of a stochastic process). Such models can initialize their simulation processes by drawing a random flow pattern from the stationary distribution, and thus render the nonuniqueness of path flow irrelevant. In general, the path-flow-nonuniqueness problem exists for most (if not all) existing deterministic day-to-day dynamics whose stable points are the classic Wardrop UE, i.e., those summarized by Yang and Zhang (2009).

3.1.2 The path-overlapping problem

While the nonuniqueness problem is in some sense not a modeling problem but only a cost or technology problem for path-based day-to-day dynamics, there does exist another more intractable problem which is rooted inherently in the path-based methodology. To see this problem, let us revisit the small example presented earlier. By examining the network shown in Figure 3.1, we can see that the network is “separable”, that is, the subnetwork from Node O to Node 1 and the subnetwork from Node 2 to Node D are totally independent of each other. As a result, a capacity reduction on Link 4, as we saw before, should not affect the flow split between Links 1 and 2. More rigorously, it could be stated as below:

> *For a network of the type shown in Figure 3.1 with fixed travel demand and separable link cost functions (i.e., no spillback effect), and assuming that the network flow is originally at stable equilibrium, a capacity reduction on Link 4 should not change or affect the stability of the flow split between Link 1 and Link 2.*

This statement is a reasonable and logical "expectation" about the network shown in Figure 3.1, and a model that violates this expectation is, at the very least, not amenable to such a network. Unfortunately, many existing path-based day-to-day dynamic models do violate this expectation, i.e., they unreasonably predict a flow fluctuation between Links 1 and 2 subsequent to a capacity reduction on Link 4. As an illustration, equation (3.6) shows that the PAP dynamic generally gives $x_1^1 \neq x_1^0$ and $x_2^1 \neq x_2^0$, i.e., there is a change of the flow split between Links 1 and 2 when a capacity reduction on Link 4 happens. One may argue that if we set a particular value for $\mathbf{f}^0$, i.e., let $f_1^0 = f_3^0 = 0.5x_4^0$, then we have $x_1^1 = x_1^0$ and $x_2^1 = x_2^0$. However, $f_1^0 = f_3^0 = 0.5x_4^0$ could easily be infeasible, not to mention how arbitrary it is to set such a value. For example, if $x_1^0 = 1$ and $x_4^0 = 3$, then $f_1^0 = f_3^0 = 0.5x_4^0$ would give $f_1^0 = 1.5 > x_1^0$, which is obviously infeasible.

Path-based day-to-day dynamics do not apply to the small network shown in Figure 3.1 because these models do not consider path interdependence. That is, users of Path 1 are modeled to be indifferent to Path 2 and Path 4 when they consider route switching, because Path 2 and Path 4 have equal path costs. In reality, however, users of Path 1 probably prefer Path 2 to Path 4 because Path 2 overlaps more with their current path. Such a path overlapping effect is not accounted for by existing path-based models, in which path cost is considered as the only driving factor of day-to-day traffic evolution. Our example shows that, a path-based day-to-day dynamic that ignores the path-overlapping effect could give very unreasonable predictions on day-to-day traffic equilibration. This shortcoming of the path-based models is referred to as the *path-overlapping problem* in this research.

While we chose a small example for convenience, the actual path-overlapping problem could be much more general. That is, even if a particular path-based day-to-day traffic model does not violate the intuition of our small network, it may still exhibit the path-overlapping problem. In general, as long as a model (implicitly) assumes that users are indifferent to two paths with the same cost, then the path-overlapping problem is likely to exist, especially for deterministic models which provide explicit flow evolution trajectories. For the stochastic models dealing with the probability distribution of flow states and/or the expected flow state rather than the flow evolution trajectory, the path-overlapping effect may also exist, due to the assumptions the models make regarding the randomness of the system. To keep this study more focused, we limit our attention to the deterministic models.

The path-overlapping problem is actually a common problem for all path-based models dealing with travelers' route choice behaviors. One well known example in the context of static traffic assignment is the logit-based SUE model. In the literature many studies (e.g., Cascetta et al. 1996; Vovsha and Bekhor 1998; Ben-Akiva and Bierlaire 2003; Frejinger and Bierlaire 2007) have been devoted to overcoming the path-overlapping problem of the logit SUE model, typically by introducing some measure of path overlap or capturing the nested hierarchy of networks.

Here, for the path-based day-to-day dynamics, it is possible to solve (or alleviate) the path-overlapping problem using similar methods. For example, we may identify one specific path flow set to avoid these shortcomings by using the proportionality (i.e., an approximation of entropy maximization) property (Bar-Gera and Boyce, 1999; Bar-Gera, 2010). By doing so, the unreasonable flow changes, as illustrated in the previous small example, would not happen. However, we must notice that enumerating paths is a difficult task for a large network. In this dissertation we do not take this effort because we have two problems with the path-based day-to-day dynamics, the path-flow-nonuniqueness problem and the path-overlapping problem. Although we have suggested possible solutions to each of the two problems, we conjecture that to handle the two problems simultaneously within the path-based methodology would be difficult and might end up with a rather complicated model. Therefore, we shall resort to a different approach, the link based methodology.

3.2 Mathematical Formulation

To avoid the two problems with path-based methods, we propose a day-to-day traffic assignment model that exploits the features of link flows in the network. By directly dealing with link flow variables, our model captures travelers' cost-minimization behavior as well as their inertia. The fixed point of our link-based dynamical system is the classic user equilibrium (UE) flow. In formal terms, we describe our day-to-day traffic assignment model as follows.

The general form of our link-based day-to-day dynamic in continuous temporal space is presented as:

$$\dot{\mathbf{x}} = \nu(\mathbf{y} - \mathbf{x}) \tag{3.10a}$$

$$\mathbf{y} = \arg\min_{y \in \Omega} \beta c(\mathbf{x})^T \mathbf{y} + (1-\beta) D(\mathbf{x}, \mathbf{y}), \tag{3.10b}$$

where ν is a positive constant parameter determining the flow changing rate, and $(\mathbf{y}-\mathbf{x})$ provides a flow changing trend. Dynamic (3.10a) refers to the idea that, on any day, the (link) flow pattern tends to move from the current flow pattern $\mathbf{x}$ towards a "target" flow pattern $\mathbf{y}$ based on the current day's network situation. Thus the model is essentially determined by how the "target" flow pattern $\mathbf{y}$ is defined. In this research we let $\mathbf{y}$ solve the minimization problem (3.10b) given current link flow $\mathbf{x}$ and network topology, where $\Omega = \{\mathbf{x} \mid \mathbf{x} = \Delta\mathbf{f}, \mathbf{d} = \Phi\mathbf{f}, \mathbf{f} \geq 0\}$ denotes the feasible link flow set following the flow conservation constraints; $0 < \beta < 1$ is a positive scalar, representing the weight of pursuing the shortest path, as will be seen later. (Note that we do not allow $\beta = 0$ or $\beta = 1$.) $D(\mathbf{x}, \mathbf{y})$ in (3.10b) is a measure of the distance between the "target" flow $\mathbf{y}$ and the current flow $\mathbf{x}$. A distance measure $D(\mathbf{x}, \mathbf{y})$ here is defined as follows.

Definition 3.1 *A mapping $D : \Omega \times \Omega \to \mathbb{R}$ is said to be a distance measure function of $\mathbf{x} \in \Omega \subseteq \mathbb{R}^n$ and $\mathrm{y} \in \Omega \subseteq \mathbb{R}^n$ if*

(1) $D(\mathbf{x}, \mathbf{y})$ is continuously differentiable;

(2) $D(\mathbf{x}, \mathbf{y})$ is strictly convex;

(3) $D(\mathbf{x}, \mathbf{y})$ is non-negative;

(4) $D(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{y} = \mathbf{x}$.

Notice that the strict convexity of $D(\mathbf{x}, \mathbf{y})$ ensures the uniqueness of the "target" flow determination (3.10b). The distance function $D(\mathbf{x}, \mathbf{y})$ may have various representations following Definition 3.1. For example, $D(\mathbf{x}, \mathbf{y})$ could take the form of the Euclidean distance $D(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^2$. Different representations of $D(\mathbf{x}, \mathbf{y})$ provide distinct mathematical properties for the proposed day-to-day dynamic (3.10). We will discuss the specification of $D(\mathbf{x}, \mathbf{y})$ later.

It is clear that problem (3.10b) is a weighted summation of two problems, namely the linear programming (LP) problem

$$\min_{\mathbf{y} \in \Omega} c(\mathbf{x})^T \mathbf{y}, \tag{3.11}$$

and the following minimization problem

$$\min_{\mathbf{y}\in\Omega} D(\mathbf{x},\mathbf{y}). \tag{3.12}$$

We shall look into LP (3.11) and problem (3.12) to obtain a better understanding of the proposed day-to-day dynamic (3.10). LP (3.11) optimizes the link flow pattern in terms of cost minimization under a given fixed link cost vector $c(\mathbf{x})$. Because link cost vector $c(\mathbf{x})$ is given and fixed, LP (3.11) essentially finds the shortest path for each OD pair. That is, the optimal solution to LP (3.11) corresponds to an all-or-nothing traffic assignment, i.e., all the travel demand will be assigned to the shortest path for each OD pair. Therefore, if we let the target flow in dynamic (3.10a) solve LP (3.11), then the dynamic would be translated as, *at any day-to-day time, travelers tend to switch to the current shortest path.* This behavioral interpretation is reasonable for our purposes: like most day-to-day dynamics, it is a means to capture travelers' cost-minimization behavior.

However, the optimal solution to LP (3.11) is generally not unique, which means that simply letting $\mathbf{y}$ solve LP (3.11) would make dynamic (3.10a) indeterminate. This problem exists because there may be multiple shortest paths between OD pairs at equilibrium, as demonstrated in Bar-Gera and Boyce (2005). To illustrate, let us first revisit the example studied in Section 3.1. When a capacity reduction on Link 4 happens to the network shown in Figure 3.1, there are two equally shortest paths, namely Path 2 and Path 4. Then, any arbitrary demand split between Path 2 and Path 4 will give an optimal solution to LP (3.11) for the given current flow $\mathbf{x} = \tilde{\mathbf{x}}$. For a more general example, not related to a specific network topology, let us assume that the current flow is at UE, i.e., $\mathbf{x}$ is the UE link flow. In this case, it is obvious that the UE shortest path between each OD pair is typically not unique, and neither is the optimal solution to LP (3.11). Observe that, in this case, $\mathbf{y} = \mathbf{x}$ is one of the optimal solutions to LP (3.11).

The fact that LP (3.11) has multiple optimal solutions implies that we need to pick one from its optimal solution set to be the target flow $\mathbf{y}$ in dynamic (3.10a). For the example in Figure 3.1, we should choose the one such that the flow split between Links 1 and 2 remains unchanged, as we have discussed in the previous section. If the current flow is at UE, obviously we should simply pick $\mathbf{y} = \mathbf{x}$, i.e., the UE flow pattern remains the same. For both examples, when we select a target flow $\mathbf{y}$ that is closest to the current flow $\mathbf{x}$, we are in essence capturing *travelers' inertia* or *reluctance to change*, i.e., travelers do not make unnecessary changes when they seek to minimize their travel costs based on the current situation. In the example of Figure 3.1, a change between Links 1 and 2 would be "unnecessary" because it would not reduce a traveler's travel cost based on the current day situation. In the same spirit, any link flow change would be "unnecessary" when the current flow is already at UE. Mathematically, this inertia effect is best captured by problem (3.12), that is, a simple minimization of the distance $D(\mathbf{x}, \mathbf{y})$ between the target flow $\mathbf{y}$ and the current flow $\mathbf{x}$.

At this point, we have built our model by combining LP (3.11) and problem (3.12), along with a behavioral interpretation of the proposed day-to-day dynamic (3.10), namely that the first term of (3.10b) captures travelers' cost-minimization behaviors and the second term reflects travelers' inertia. Having this link-based day-to-day traffic model, we now shall revisit the two path-based problems mentioned in previous section and see how well they are addressed by the link-based model. It is obvious that the path-flow-nonuniqueness problem no longer exists because our link-based model requires only an initial link flow pattern, which is practically observable and mathematically unique (under mild technical conditions).

The path-overlapping problem is a bit more complicated. In general terms, we know that a perfect day-to-day dynamic model needs to capture how travelers account for network hierarchy when making their route switching decisions. We cannot claim that our link flow dynamic model (3.10) achieves this goal without further empirical study.

Nevertheless, narrowly speaking, how paths overlap with each other is an irrelevant question in our link-based model, and thus we do not have to make assumptions related to path overlapping. In other words, unlike the path-based models, our link-based model does not assume that travelers are necessarily indifferent to two paths with equal costs. In this sense, the path-overlapping problem is avoided by our link-based model. This is best illustrated by the small network shown in Figure 3.1, where, if we apply our model, the second term of problem (3.10b) ensures that a capacity reduction on Link 4 will not cause flow fluctuation between Links 1 and 2. That is because that the solution of problem (3.10b) would always select a pattern without flow changes on Links 1 and 2 to minimize the distance function $D(\mathbf{x}, \mathbf{y})$, implying that drivers only consider their route choice at node 2. By contrast, most existing path-based models fail to achieve this reasonable and intuitive prediction, since the resulting path flow variations would directly change link flows. Of course, for models applying the proportionality, path-flow-nonuniqueness and path-overlapping problems can be avoided as well. However, as we discussed, enumerating path set is burdensome and the generated path set is difficult to verify.

After discussing how the proposed model avoids the two shortcomings inherent in path-based models, we are ready to evaluate the stability of the model. We perform the stability analysis on a specific version of our link-based day-to-day traffic model given by (3.10) with the squared Euclidean distance measure, i.e., $D(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|^2$.

3.3 Stability Analysis

Stability is an important issue for dynamic traffic networks. Associated with network perturbations, stability property addresses the problem of whether and how an equilibrium is achieved. Unstable equilibrium is sensitive to perturbation and makes it difficult to provide useful information. For this reason, the stability property of the proposed model critically impacts its application in practice.

In the following we will first prove that the fixed point of the day-to-day dynamic (3.10), with $D(\mathbf{x}, \mathbf{y})$ specified as the squared Euclidean distance, is the UE link flow pattern. The proof indicates that whenever the proposed traffic dynamic is stable, its stable link flow pattern always satisfies the UE principle. We shall start with the following lemmas.

Lemma 3.1 *A link flow pattern* $\mathbf{y}$ *solves LP (3.11) for given* $\mathbf{x}$ *if and only if* $\mathbf{y}$ *assigns all travel demand to the shortest paths determined by link cost vector* $c(\mathbf{x})$*.* □

Lemma 1 is self-evident, and directly leads to the following lemma.

Lemma 3.2 *A link flow pattern* $\mathbf{x}$ *is the UE link flow if and only if* $\mathbf{y} = \mathbf{x}$ *solves LP (3.11) for given* $\mathbf{x}$*.* □

By the definition of UE, a link flow pattern $\mathbf{x}$ is the UE link flow if and only if $\mathbf{x}$ assigns all travel demand to the shortest paths determined by link cost vector $c(\mathbf{x})$. Thus Lemma 3.2 is a simple derivation of Lemma 3.1.

Lemma 3.3 *Let* $D(\mathbf{x}, \mathbf{y})$ *be a squared Euclidean distance function, then, for a given link flow pattern* $\mathbf{x}$*,* $\mathbf{y} = \mathbf{x}$ *solves problem (3.10b) if and only if* $\mathbf{y} = \mathbf{x}$ *solves LP (3.11).*

Proof. *Necessity:* It is readily seen that $\mathbf{y} = \mathbf{x}$ always minimizes the second term of problem (3.10b). When $y = x$ solves LP (3.11), it also minimizes the first term of problem (3.10b). Thus, if $\mathbf{y} = \mathbf{x}$ solves LP (3.11), then it can be said that $\mathbf{y} = \mathbf{x}$ minimizes both terms of problem (3.10b) and thus solves problem (3.10b).

Sufficiency: Let us rewrite the objective function of problem (3.10b) as

$$Z(\mathbf{y}) = \beta Z_1(\mathbf{y}) + (1 - \beta) Z_2(\mathbf{y}), \tag{3.13}$$

where $0 < \beta < 1$, $Z_1(\mathbf{y}) = c(\mathbf{x})^T\mathbf{y}$, and $Z_2(\mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|^2$. If we suppose that $\mathbf{y} = \mathbf{x}$ solves problem (3.10b) but does not solve LP (3.11), then there is a feasible direction $\mathbf{x}$ at $\mathbf{y} = \mathbf{x}$ such that

$$\nabla Z_1(\mathbf{y})^T\mathbf{z} < 0, \tag{3.14}$$

where $\nabla Z_1(\mathbf{y})$ is the gradient of $Z_1(\mathbf{y})$ at $\mathbf{y} = \mathbf{x}$. On the other hand, it holds readily $\nabla Z_2(\mathbf{y}) = 0$ at $\mathbf{y} = \mathbf{x}$, and thus we have

$$\nabla Z_2(\mathbf{y})^T\mathbf{z} = 0. \tag{3.15}$$

Combining (3.14) and (3.15) gives $\nabla Z(\mathbf{y})^T\mathbf{z} < 0$ at $\mathbf{y} = \mathbf{x}$, which contradicts that $\mathbf{y} = \mathbf{x}$ solves the minimization problem (3.10b). Thus, $\mathbf{y} = \mathbf{x}$ solves LP (3.11) is necessary for $\mathbf{y} = \mathbf{x}$ to solve the problem (3.10b). This completes the proof. □

The proposed link-based day-to-day dynamic (3.10) always has the classical UE flow pattern as its stable point, as summarized by the following theorem.

Theorem 3.1 *With $D(\mathbf{x}, \mathbf{y})$ defined by Definition 3.1, a link flow pattern $\mathbf{x}$ is the stable point of dynamic (3.10) if and only if $\mathbf{x}$ is the UE link flow.*

Proof. The stable point of dynamic (3.10), i.e., the link flow $\mathbf{x}$ that gives $\dot{\mathbf{x}} = 0$, is clearly the $\mathbf{x}$ such that $\mathbf{y} = \mathbf{x}$ solves problem (3.10b). Thus, combining Lemma 3.2 and Lemma 3.3 immediately derives the theorem. □

This theorem clarifies that the stable link flow pattern of the proposed model satisfies UE principle, that is an important property of the proposed day-to-day dynamic (3.10) in a continuous temporal space.

In the following, we analyze the stability of the proposed link flow dynamic (3.10), in order to better understand its convergence property and track the continuous trajectories of the system. Our stability analysis starts with two definitions of stability, similar to those defined in Nagurney and Zhang (1997) and Bie and Lo (2010) but on a space of link flows.

Definition 3.2 (Stability of the System) *The link flow dynamic (3.10) is stable if for every initial link flow pattern* $\mathbf{x}^0$ *and every equilibrium link flow pattern,* $\mathbf{x}^*$*, the Euclidean distance* $\|\mathbf{x}^0(t) - \mathbf{x}^*\|$ *is a monotone non-increasing function of time* t*.*

Definition 3.3 (Asymptotic Stability of the System) *The link flow dynamic (3.10) is asymptotically stable if it is stable, and for every initial link flow pattern* $\mathbf{x}^0$*, there exists some equilibrium flow pattern* $\mathbf{x}^*$*, such that*

$$\|\mathbf{x}^0(t) - \mathbf{x}^*\| \to 0, \qquad \text{as} \qquad t \to \infty, \tag{3.16}$$

where $\mathbf{x}^0(t)$ *solves (3.10) with* $\mathbf{x}^0(0) = 0$*.*

A number of researchers (e.g., Smith 1984, Friesz et al. 1994, and Zhu and Marcotte 1995) have defined stability in the Lyapunov sense. Actually, the Euclidean distance function $\|\mathbf{x}^0(t) - \mathbf{x}^*\|$ is a Lyapunov function if the right-hand-side of the link dynamic (3.10a) is continuously differentiable.

The following definitions of *Lipschitz Continuity* and *Strong Monotonicity* are necessary for establishing the stability and asymptotic stability of the proposed link flow dynamic.

Definition 3.4 (Lipschitz Continuity) *A mapping* F *is said to be Lipschitz continuous with constant* $L > 0$ *on the set* Ω *if, for each pair of points* $\mathbf{x}, \mathbf{y} \in \Omega$*, we have*

$$\|F(\mathbf{x}) - F(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|. \tag{3.17}$$

Definition 3.5 (Strong Monotonicity) *A mapping* F *is said to be strongly monotone on the set* Ω *if, for each pair of points* $\mathbf{x}, \mathbf{y} \in \Omega$*, there exists a constant* $\kappa > 0$ *whereby*

$$\langle \mathbf{x} - \mathbf{y}, F(\mathbf{x}) - F(\mathbf{y}) \rangle \geq \kappa\|\mathbf{x} - \mathbf{y}\|^2. \tag{3.18}$$

The relationship between link flow dynamic (3.10) and the projected dynamical system is shown in the following lemma.

Lemma 3.4 *Assume the distance measure function $D(\cdot,\cdot)$ is the squared Euclidean distance. The target flow determination (3.10b), represented as a minimization problem, projects a vector $\mathbf{x}-\gamma\mathbf{c}(\mathbf{x})$ to the current feasible link flow set Ω.*

Proof. When the distance function $D(\cdot,\cdot)$ is defined as the squared Euclidean distance, the objective function in Eq. (3.10b) is equivalent to

$$\begin{aligned}\beta c(\mathbf{x})^T\mathbf{y}+(1-\beta)D(\mathbf{x},\mathbf{y}) &= (1-\beta)\left(\frac{\beta}{1-\beta}c(\mathbf{x})^T\mathbf{y}+\|\mathbf{y}-\mathbf{x}\|^2\right)\\ &= (1-\beta)\left(2\gamma c(\mathbf{x})^T\mathbf{y}+\|\mathbf{y}-\mathbf{x}\|^2\right), \qquad (3.19)\end{aligned}$$

where $\gamma=\frac{\beta}{2(1-\beta)}$. Since $\mathbf{x}$ is a given link flow vector, then $c(\mathbf{x})$ becomes a known link cost vector. Therefore, the target flow determination is

$$\begin{aligned}\min_{\mathbf{y}\in\Omega}\beta c(\mathbf{x})^T\mathbf{y}+(1-\beta)D(\mathbf{x},\mathbf{y}) &\Leftrightarrow \min_{\mathbf{y}\in\Omega}2\gamma c(\mathbf{x})^T\mathbf{y}+\|\mathbf{y}-\mathbf{x}\|^2\\ &\Leftrightarrow \min_{\mathbf{y}\in\Omega}\|\mathbf{y}-\mathbf{x}\|^2+2\gamma c(\mathbf{x})^T\mathbf{y}-2\gamma c(\mathbf{x})^T\mathbf{x}+\gamma^2\|c(\mathbf{x})\|^2\\ &\Leftrightarrow \min_{\mathbf{y}\in\Omega}\|\mathbf{y}-\mathbf{x}+\gamma c(\mathbf{x})\|^2. \qquad (3.20)\end{aligned}$$

By the definition of the projection operator, $\mathrm{Pr}_\Omega(\mathbf{x})=\arg\min_{\mathbf{y}\in\Omega}\|\mathbf{y}-\mathbf{x}\|$, Eq. (3.20) shows that the target flow pattern $\mathbf{y}=\mathrm{Pr}_\Omega(\mathbf{x}-\gamma c(\mathbf{x}))$. □

In the literature, Fukushima (1992) established a connection between the gap minimization problem and asymmetric variational inequalities. In his paper, an objective function similar to the objective function in (3.20) is used to measure the gap between a current iterate and the optimal solution set. The same insight is applied here. The target flow determination (3.10b) implicitly provides a measurement of the distance between the current link flow pattern and the stable link flow set of day-to-day link dynamics.

With Lemma 3.4, the proposed day-to-day link flow dynamic (3.10) can be simply represented by:

$$\dot{\mathbf{x}}=\nu\left[\mathrm{Pr}_\Omega\left(\mathbf{x}-\gamma\mathbf{c}(\mathbf{x})\right)-\mathbf{x}\right], \qquad (3.21)$$

where γ and ν are small positive constants and Ω is a feasible link flow set which is a polyhedron in the link flow space.

Due to Theorem 3.1, the stable point of dynamic (3.10) $\mathbf{x}^*$ must be a user equilibrium flow pattern, which solves the variational inequality problem VI(Ω, c) defined as

$$\langle c(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle \geq 0, \qquad \forall \mathbf{x} \in \Omega, \tag{3.22}$$

where $c(\cdot)$ represents link performance function.

Based on the early work of Eaves (1971), a variational inequality problem VI(Ω, c) is equivalent to a fixed point problem $\mathbf{x} = \mathrm{Pr}_\Omega[\mathbf{x} - \gamma c(\mathbf{x})]$, for any $\gamma > 0$ (of course, in our study, $\gamma = \frac{\beta}{1-\beta} > 0$ when $0 < \beta < 1$). In other words, solving VI(Ω, c) is equivalent to finding a zero point of the residue vector defined as:

$$e(\mathbf{x}, \gamma) = \mathbf{x} - \mathrm{Pr}_\Omega[\mathbf{x} - \gamma c(\mathbf{x})]. \tag{3.23}$$

The magnitude of $\|e(\mathbf{x}, \gamma)\|$ depends on the value of γ for any given $\mathbf{x} \in \mathbb{R}^n$.

Let $\mathbf{x}^*$ be a UE flow pattern. Since $\mathrm{Pr}_\Omega[\mathbf{x} - \gamma c(\mathbf{x})] \in \Omega$, inequality (3.22) is satisfied, as

$$\langle \gamma c(\mathbf{x}^*), \mathrm{Pr}_\Omega[\mathbf{x} - \gamma c(\mathbf{x})] - \mathbf{x}^* \rangle \geq 0, \qquad \forall \mathbf{x} \in \mathbb{R}^n. \tag{3.24}$$

Before we state the next lemma, an important property of the projection operator Pr_Ω one convex set Ω is useful to analyze the stability of the link dynamic:

$$\langle v - \mathrm{Pr}_\Omega(v), \mathrm{Pr}_\Omega(v) - u \rangle \geq 0, \quad \forall v \in \mathbb{R}^n, u \in \Omega. \tag{3.25}$$

Lemma 3.5 *Assume that link performance functions $c(\cdot)$ are Lipschitz continuous and strongly monotone, then*

$$\langle \mathbf{x} - \mathbf{x}^*, e(\mathbf{x}, \gamma) \rangle \geq \gamma \left(\frac{\kappa}{L^2} - \frac{\gamma}{4} \right) \|c(\mathbf{x}) - c(\mathbf{x}^*)\|^2. \tag{3.26}$$

Proof. Let $v = \mathbf{x} - \gamma c(\mathbf{x})$ and $u = \mathbf{x}^*$ in inequality (3.25), then we have

$$\langle e(\mathbf{x},\gamma) - \gamma c(\mathbf{x}), \mathrm{Pr}_\Omega[\mathbf{x} - \gamma c(\mathbf{x})] - \mathbf{x}^* \rangle \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{3.27}$$

Adding (3.24) to (3.27), we have

$$\langle e(\mathbf{x},\gamma) - \gamma(c(\mathbf{x}) - c(\mathbf{x}^*)), \mathrm{Pr}_\Omega[\mathbf{x} - \gamma c(\mathbf{x})] - \mathbf{x}^* \rangle \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{3.28}$$

With the strong monotonicity and Lipschitz continuity of $c(\mathbf{x})$, it follows that

$$\begin{aligned}
0 &\leq \langle e(\mathbf{x},\gamma) - \gamma(c(\mathbf{x}) - c(\mathbf{x}^*)), \mathrm{Pr}_\Omega[\mathbf{x} - \gamma c(\mathbf{x})] - \mathbf{x}^* \rangle \qquad (3.29)\\
&= \langle e(\mathbf{x},\gamma) - \gamma(c(\mathbf{x}) - c(\mathbf{x}^*)), (\mathbf{x} - \mathbf{x}^*) - e(\mathbf{x},\gamma) \rangle \\
&= \langle e(\mathbf{x},\gamma), \mathbf{x} - \mathbf{x}^* \rangle - \|e(\mathbf{x},\gamma)\|^2 - \gamma\langle c(\mathbf{x}) - c(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle + \gamma\langle c(\mathbf{x}) - c(\mathbf{x}^*), e(\mathbf{x},\gamma)\rangle \\
&\leq \langle e(\mathbf{x},\gamma), \mathbf{x} - \mathbf{x}^* \rangle - \|e(\mathbf{x},\gamma)\|^2 - \gamma\kappa\|\mathbf{x} - \mathbf{x}^*\|^2 + \gamma\langle c(\mathbf{x}) - c(\mathbf{x}^*), e(\mathbf{x},\gamma)\rangle \\
&\leq \langle e(\mathbf{x},\gamma), \mathbf{x} - \mathbf{x}^* \rangle - \|e(\mathbf{x},\gamma)\|^2 - \frac{\gamma\kappa}{L^2}\|c(\mathbf{x}) - c(\mathbf{x}^*)\|^2 + \gamma\langle c(\mathbf{x}) - c(\mathbf{x}^*), e(\mathbf{x},\gamma)\rangle \\
&= \langle e(\mathbf{x},\gamma), \mathbf{x} - \mathbf{x}^* \rangle - \|e(\mathbf{x},\gamma) - \frac{\gamma}{2}(c(\mathbf{x}) - c(\mathbf{x}^*))\|^2 + \left(\frac{\gamma^2}{4} - \frac{\gamma\kappa}{L^2}\right)\|c(\mathbf{x}) - c(\mathbf{x}^*)\|^2.
\end{aligned}$$

This inequality implies (3.26). □

We can now establish the stability of the dynamic system (3.10).

Theorem 3.2 *Assume that link performance functions $c(\cdot)$ are Lipschitz continuous and strongly monotone on an open convex set $\mathcal{S} \supset \Omega$, and $\gamma < \frac{2\kappa}{L^2}$. Then, the link-based day-to-day dynamic system (3.10) is (1) stable and (2) asymptotically stable.*

Proof. (1) Define the distance function $D(\mathbf{x}^0, \mathbf{x}^*, t) := \frac{1}{2}\|\mathbf{x}^0(t) - \mathbf{x}^*\|^2$. Notice that the proposed link-based dynamic (3.10) can also be presented by projected dynamic (3.21). Thus, the derivative of $D(\mathbf{x}^0, \mathbf{x}^*, t)$ w.r.t time t can be shown as:

$$\dot{D}(\mathbf{x}^0, \mathbf{x}^*, t) = \frac{\mathrm{d}D}{\mathrm{d}t} = \frac{\mathrm{d}D}{\mathrm{d}\mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = (\mathbf{x}^0(t) - \mathbf{x}^*)^T \left\{\nu\left[\mathrm{Pr}_\Omega\left(\mathbf{x}^0(t) - \gamma c(\mathbf{x}^0(t))\right) - \mathbf{x}^0(t)\right]\right\}, \tag{3.30}$$

where $\mathbf{x}^0(t)$ solves the projected dynamic (3.21) with boundary condition $\mathbf{x}^0(0) = 0$. Due to Lemma 3.5 and $\mathbf{x}^0(t) \in \Omega$ for all t, it follows that

$$\begin{aligned} \dot{D}(\mathbf{x}^0, \mathbf{x}^*, t) &= -\nu(\mathbf{x}^0(t) - \mathbf{x}^*)^T e(\mathbf{x}^0(t), \gamma) \\ &\leq -\nu\gamma\left(\frac{\kappa}{L^2} - \frac{\gamma}{4}\right)\|c(\mathbf{x}^0(t)) - c(\mathbf{x}^*)\|^2. \end{aligned} \tag{3.31}$$

The condition $\gamma < \frac{2\kappa}{L^2}$ implies $(\frac{\kappa}{L^2} - \frac{\gamma}{4}) > 0$. Therefore, $\dot{D}(\mathbf{x}^0, \mathbf{x}^*, t) < 0$ for all $x^0(t)$, which means that the distance function $D(\mathbf{x}^0, \mathbf{x}^*, t)$ is monotonically non-increasing. By Definition 3.2, the link-based day-to-day dynamic system (3.10) is stable.

(2) Because of (1), the distance function $D(t)$ must be monotonically non-increasing and non-negative for all t. Therefore, we can let

$$\lim_{t\to\infty} D(t) = \eta. \tag{3.32}$$

If $\eta > 0$, then these must exist a sequence $\{t_n\}$, $t_n \to \infty$, as $n \to \infty$, such that

$$\lim_{n\to\infty} \dot{D}(t_n) = 0. \tag{3.33}$$

Suppose that this claim is false. Then there exist a $\delta > 0$ and a enough large $T > 0$, such that

$$\dot{D}(t) < -\delta, \qquad \forall t > T, \tag{3.34}$$

which contradicts the nonnegativity requirement of $D(t)$. Therefore, (3.33) is true.

The limitation (3.32) implies that the sequence $\mathbf{x}^0(t_n)$ is bounded. Therefore, there exists a subsequence $\{t_{n_j}\}$ such that

$$\lim_{j\to\infty} \mathbf{x}^0(t_{n_j}) = \bar{\mathbf{x}}. \tag{3.35}$$

By the definition of $D(t)$, we have

$$\lim_{j\to\infty} \frac{1}{2}\|\mathbf{x}^0(t_{n_j}) - \mathbf{x}^*\|^2 = \frac{1}{2}\|\bar{\mathbf{x}} - \mathbf{x}^*\|^2 = \eta > 0, \tag{3.36}$$

and hence $\bar{\mathbf{x}} \neq \mathbf{x}^*$. Substituting $\mathbf{x}^0(t_{n_j})$ in 3.31 yields

$$\dot{D}(\mathbf{x}^0(t_{n_j}), \mathbf{x}^*, t) \leq -\nu\gamma\left(\frac{\kappa}{L^2} - \frac{\gamma}{4}\right)\|c(\mathbf{x}^0(t_{n_j})) - c(\mathbf{x}^*)\|^2 < 0. \tag{3.37}$$

Because of (3.33), the left-hand-side of 3.37 converges to zero as $j \to \infty$. Conditions $\nu > 0$ and $\gamma < \frac{2\kappa}{L^2}$ imply $\nu\gamma(\frac{\kappa}{L^2} - \frac{\gamma}{4}) > 0$. And by (3.35), we have

$$\lim_{j\to\infty}\|c(\mathbf{x}^0(t_{n_j})) - c(\mathbf{x}^*)\|^2 = \|c(\bar{\mathbf{x}}) - c(\mathbf{x}^*)\|^2 = 0. \tag{3.38}$$

By the definition of strong monotonicity, (3.38) implies that $\bar{\mathbf{x}} = \mathbf{x}^*$ which is a contradiction to the earlier result from (3.36) that $\bar{\mathbf{x}} \neq \mathbf{x}^*$. This contradiction shows that $\eta = 0$. Therefore, for any $\mathbf{x}^0(0) \in \Omega$,

$$\|\mathbf{x}^0(t) - \mathbf{x}^*\|^2 \to 0, \qquad \text{as} \qquad t \to \infty. \tag{3.39}$$

By Definition 3.3, the link-based day-to-day dynamical system (3.10) is asymptotically stable. □

Remark 3.2 *Theorem 3.2 shows that the link-based day-to-day dynamical system (3.10) is globally asymptotically stable under the given sufficient conditions. The proof also shows that the solution trajectories of the proposed dynamic system (3.10) converge to equilibrium globally.*

Remark 3.3 *From the proof of Theorem 3.2, we can see that the larger the flow changing rate ν in (3.10a), the smaller the negative value of $\dot{D}(\mathbf{x}^0, \mathbf{x}^*, t)$. Therefore, the link-based day-to-day dynamical system (3.10) stabilizes faster with a larger ν, although the value of ν does not impact the proof of asymptotic stability.*

3.4 Distance Measure Function

Let us move on to look into the specification of the distance measure $D(\mathbf{x}, \mathbf{y})$ in the link flow dynamic (3.10). The most natural specification of $D(\mathbf{x}, \mathbf{y})$, as we noted earlier, is the squared Euclidean distance, i.e., $D(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|^2$. With this specification, the objective function of problem (3.10b) is a convex combination of a linear term and a quadratic term, making the minimization problem (3.10b) simply a strictly convex problem, i.e., a quadratic program (QP). This guarantees that the solution of problem (3.10b) $\mathbf{y}$ is unique for any given $\mathbf{x}$, and thus the dynamic is well-defined. The asymptotic stability of the proposed link-based traffic dynamic has been established based on the specification of squared Euclidean distance function.

Despite its perfect mathematical property and intuitive appeal, the Euclidean distance specification of $D(\mathbf{x}, \mathbf{y})$ has one deficiency that it is not robust to irrelevant changes to the network. To illustrate, let us assume a simple two-link network as shown in Figure 3.2(a), and then consider that a dummy node is added so that we obtain the network shown in Figure 3.2(b). The two networks are essentially the same network with the link cost functions shown in the figure, because adding a dummy node is an irrelevant change to the network. A robust model should be independent of this kind of "dummy node" effect. Unfortunately, the Euclidean distance formulation $D(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|^2$ is not robust in this respect: For the network of Figure 3.2(a) consisting of two links, we have

$$D(\mathbf{x}, \mathbf{y}) = (x_1 - y_1)^2 + (x_2 - y_2)^2, \tag{3.40}$$

while for the network of Figure 3.2(b) consisting of three links, we have

$$\begin{aligned} D(\mathbf{x}, \mathbf{y}) &= (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_2 - y_2)^2 \\ &= (x_1 - y_1)^2 + 2(x_2 - y_2)^2. \end{aligned} \tag{3.41}$$

Clearly, adding a dummy node to link 2 makes the flow (change) on link 2 have more impact on the Euclidean distance formulation. This reflects a very undesirable property of inconsistent distance values, since the two networks are actually the same.

Another natural specification of $D(\mathbf{x}, \mathbf{y})$ would be $D(\mathbf{x}, \mathbf{y}) = \|c(\mathbf{x}) - c(\mathbf{y})\|^2$, which

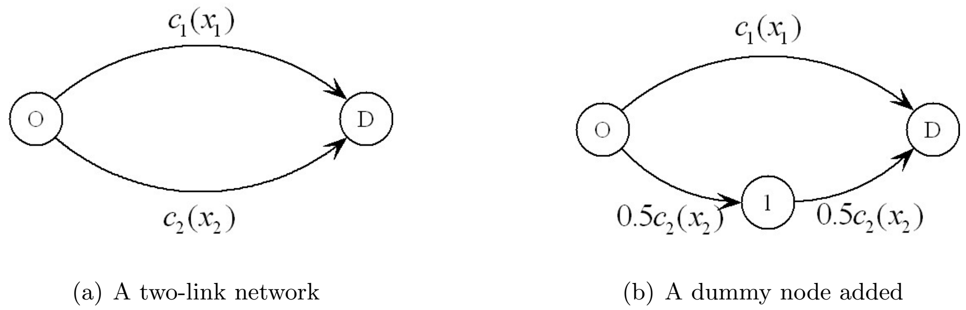


Figure 3.2: An Example of "Dummy Node" Effect on Distance Measure Function

is the squared Euclidean distance between the link cost vectors. It can be easily verified that this specification also has good mathematical properties, such as being strictly convex. However, again, the formulation is not robust to the "dummy node" effect: For the two-link network of Figure 3.2(a), we have

$$D(\mathbf{x}, \mathbf{y}) = [c_1(x_1) - c_1(y_1)]^2 + [c_2(x_2) - c_2(y_2)]^2, \tag{3.42}$$

while for the three-link network of Figure 3.2(b), we have

$$\begin{aligned} D(\mathbf{x}, \mathbf{y}) &= [c_1(x_1) - c_1(y_1)]^2 + [0.5c_2(x_2) - 0.5c_2(y_2)]^2 + [0.5c_2(x_2) - 0.5c_2(y_2)]^2 \\ &= [c_1(x_1) - c_1(y_1)]^2 + 0.5[c_2(x_2) - c_2(y_2)]^2. \end{aligned} \tag{3.43}$$

In this case, adding a dummy node to link 2 makes the flow (change) on link 2 have less impact on the formulation, further demonstrating that the specification of $D(\mathbf{x}, \mathbf{y})$ is not robust to the "dummy node" effect.

Although both of these natural specifications of $D(\mathbf{x}, \mathbf{y})$ suffer the same deficiency, i.e., lack of independence to irrelevant changes to the network, it is not difficult to construct other specifications of $D(\mathbf{x}, \mathbf{y})$ that do make our link-based model robust to irrelevant network changes (or, more specifically, robust to the "dummy node" effect). For example, it can be easily verified that both $D(\mathbf{x}, \mathbf{y}) = \sum_{a \in L}(x_a - y_a) \cdot (c_a(x_a) - c_a(y_a))$ and $D(\mathbf{x}, \mathbf{y}) = \sum_{a \in L} |c_a(x_a) - c_a(y_a)|$ measure the distance between $\mathbf{y}$ and $\mathbf{x}$ (i.e., their values increase as y_a moves away from x_a for each link $a \in \mathcal{L}$), and are robust to the "dummy node" effect. However, these specifications of $D(\mathbf{x}, \mathbf{y})$ are generally not strictly convex and thus do not guarantee the uniqueness of the solution to problem (3.10b), which means they may not serve as good definitions of distance measure function in our day-to-day link flow dynamic.

This property tells us that, an alternative measure function needs to be developed for resolving the "dummy node" effect. To do so, we begin with a widely held assumption regarding link cost functions.

Assumption 3.1 *The link cost functions are separable, i.e.,* $c_a(\mathbf{x}) = c_a(x_a)$, *for all* $a \in \mathcal{L}$, *continuously differentiable and strictly monotonically increasing.*

Assumption 3.1 allows us to develop the following specification of the distance function:

$$D(\mathbf{x}, \mathbf{y}) = \sum_{a \in \mathcal{L}} \int_{x_a}^{y_a} (c_a(\omega) - c_a(x_a)) \mathrm{d}\omega. \tag{3.44}$$

It is not difficult to prove that $D(\mathbf{x}, \mathbf{y})$ as represented by (3.44) satisfies Definition 3.1.

Intuitively, $D(\mathbf{x}, \mathbf{y})$ given by (3.44) is a reasonable measure of the distance between the target flow $\mathbf{y}$ and the current flow $\mathbf{x}$, i.e., its value increases as y_a moves away from x_a for each link $a \in \mathcal{L}$. With Assumption 3.1, formulation (3.44) is a strictly convex function of $\mathbf{y}$ for any given $\mathbf{x}$, and it can be easily verified that this formulation is robust to the "dummy node" effect. Thus, we have a "good" specification of the distance measure $D(\mathbf{x}, \mathbf{y})$.

It should be noted that more complicated "good" specifications of $D(\mathbf{x}, \mathbf{y})$ certainly exist. For example, we can always change the specific formulation within the integral. From a more general viewpoint, how we specify $D(\mathbf{x}, \mathbf{y})$ actually reflects how we are modeling travelers' route switching behaviors (e.g., inertia, habit, consideration of the network hierarchy). Also note that, if the "dummy node" effect is not a concern for some networks, then the Euclidean distance formulation reminds us what are actually "good" specifications of $D(\mathbf{x}, \mathbf{y})$, and provides greater intuitive value than the formulation (3.44) proposed here.

3.5 Relationship with Other Models

Having a specific version of our proposed link-based dynamic (3.10) with the squared Euclidean distance measure, we can show that it has a close relationship with other dynamical systems in the literature. Specifically, we prove that the proposed model belongs to a general category of the rational behavior adjustment process. This relationship provides a better understanding of the proposed model in terms of the underlying drivers' choice behavior. Furthermore, this section shows that the proposed model also belongs to a broader class of dynamical systems, namely differential variational inequalities. This relationship offers distinct advantages in the modeling of traffic evolution characteristics, especially under network disruption scenarios.

3.5.1 Relationship with the Rational Behavior Adjustment Process

Recently, Yang and Zhang (2009) showed that existing deterministic traffic flow dynamic models share similar assumptions about travel adjustment behavior. Based on this behavioral adjustment principle, they defined the term of Rational Behavior Adjustment Process (RBAP) and proved that all five categories of deterministic day-to-day dynamical systems, including proportional-switch adjustment process and the projected dynamical system, can be classified as the RBAP. This broader principle of behavioral adjustment can serve as a general form for drivers' route choice adjustment processes, and help to develop equivalent stability properties for all adjustment processes belonging to this category.

For path flow dynamics, the RBAP is defined as follows (from Definition 2.1 in Yang and Zhang 2009).

Definition 3.6 *A day-to-day route choice adjustment process is called a RBAP with fixed travel demand if the aggregated travel cost of the entire network decreases based on the previous day's path travel costs when path flows change from day to day. Furthermore, if a path flow becomes stationary over days, then the flow is equivalent to the user equilibrium path flow. Mathematically, this can be expressed by*

$$\dot{\mathbf{f}}(t)\begin{cases} \in \Gamma & \text{if} \quad \Gamma \neq \emptyset \\ = 0 & \text{if} \quad \Gamma = \emptyset \end{cases} \qquad \text{where} \quad \Gamma = \left\{ \mathbf{z}(t) : \sum_{r\in\mathcal{R}^w} z_r^w(t) = 0, \mathbf{C}(t)^T\mathbf{z}(t) < 0 \right\}, \tag{3.45}$$

where $\dot{\mathbf{f}}(t)$ is the derivative of the path flow vector with respect to time t (in a day-to-day sense), $\mathbf{z}(t)$ is the vector of infinitesimal changes in path flows, and Γ represents the set of all feasible directions that reduce the aggregate travel cost based on the previous day's path costs $\mathbf{C}(t)$.

The constraint $\sum_{r\in\mathcal{R}^w} z_r^w(t) = 0$ is to ensure flow conservation of any path flow changes with fixed travel demand. Therefore (3.45) implies that at any time if there are any feasible path flow changes that will reduce the aggregate travel cost based on the previous day's path costs, then the dynamic motion $\dot{\mathbf{f}}(t)$ must be one of them, otherwise $\dot{\mathbf{f}}(t)$ should be zero. In other words, when there is no feasible route flow change pattern to further reduce the aggregated cost based on the previous day's path costs, a RBAP must be at its stationary point.

Notice that the RBAP given by Yang and Zhang (2009) is defined on day-to-day route flows. We can extend Definition 3.6 to the link flow pattern. First of all, notice the relationship between link flows and path flows: $\mathbf{x} = \Delta\mathbf{f}$, where Δ is the arc-path incidence matrix. From this, we can define a set of link flow change vectors $\Lambda = \Delta\Gamma$ for all feasible link flow patterns that reduce the aggregate travel cost as follows:

$$\Lambda = \Delta\Gamma = \left\{ \mathbf{y}(t) = \Delta\mathbf{z}(t) : \sum_{r\in\mathcal{R}^w} z_r^w(t) = 0, \mathbf{c}(t)^T\mathbf{y}(t) < 0 \right\}, \tag{3.46}$$

where $\mathbf{c}(t)$ denotes the link cost pattern which has a relationship with path cost pattern $\mathbf{C}(t)$ as $\mathbf{C}(t) = \Delta^T \mathbf{c}(t)$, and therefore $\mathbf{c}(t)^T \mathbf{y}(t) = \mathbf{c}(t)^T \Delta^T \mathbf{z}(t) = \mathbf{C}(t)^T \mathbf{z}(t)$. The constraint $\sum_{r \in \mathcal{R}^w} z_r^w(t) = 0$ is required in (3.46) to guarantee that the link flow change satisfies the flow conservation. We also have the relationship between link flow dynamics and path flow dynamics, $\dot{\mathbf{x}}(t) = \Delta \dot{\mathbf{f}}(t)$. In summary, we can define the RBAP on link flow dynamics as follows.

Definition 3.7 *A day-to-day link flow adjustment process is a RBAP with fixed travel demand if the aggregated travel cost of the entire network decreases based on the previous day's link cost pattern when link flows change from day to day. Furthermore, if a link flow pattern becomes stationary over days, then it is equivalent to the user equilibrium link flow pattern. In mathematical terms,*

$$\dot{\mathbf{x}}(t) \begin{cases} \in \Lambda & \text{if} \quad \Lambda \neq \emptyset \\ = 0 & \text{if} \quad \Lambda = \emptyset \end{cases} \quad \text{where} \quad \Lambda = \left\{ \mathbf{y}(t) = \Delta \mathbf{z}(t) : \sum_{r \in \mathcal{R}^w} z_r^w(t) = 0, \mathbf{c}(t)^T \mathbf{y}(t) < 0 \right\}, \tag{3.47}$$

where $\dot{\mathbf{x}}(t)$ is the derivative of the link flow vector with respect to time t (in a day-to-day sense), $\mathbf{y}(t)$ is the vector of infinitesimal changes in the link flows, and Λ is the set of all feasible directions that reduce the aggregate travel cost based on the previous day's link costs $\mathbf{c}(t)$.

Thus far we have extended the RBAP to link flows. Now we can prove the relationship between our proposed day-to-day link dynamic (3.10) and the RBAP. To complete our proof, we shall recall the relationship between the target flow determination and the projection operator $\mathrm{Pr}_\Omega(\cdot)$, as shown by Lemma 3.4. Based on formula (3.21), we can derive the relationship between the link based day-to-day traffic dynamic (3.10) and the RBAP defined by Definition 3.7.

Theorem 3.3 *The link based day-to-day traffic dynamic, defined as (3.21), is a RBAP.*

Proof. If $\mathbf{x}$ is in the interior of set Ω, i.e., $\mathbf{x} \in \Omega^0$, we have $\dot{\mathbf{x}} = -\gamma\mathbf{c}(\mathbf{x})$. Therefore,it readily shows that $\dot{\mathbf{x}}$ satisfies (3.47).

In the other case where $\mathbf{x}$ is on the boundary of set Ω (or $\mathbf{x} \in \partial\Omega$), the vector $\mathbf{x} - \gamma\mathbf{c}(\mathbf{x}) - \mathrm{Pr}_\Omega(\mathbf{x} - \gamma\mathbf{c}(\mathbf{x}))$ is in the normal cone of $\mathrm{Pr}_\Omega(\mathbf{x} - \gamma\mathbf{c}(\mathbf{x}))$ (see Section 1.1 in Facchinei and Pang (2003) for the definition of normal cone), i.e.,

$$\langle \mathbf{x} - \gamma\mathbf{c}(\mathbf{x}) - \mathrm{Pr}_\Omega(\mathbf{x} - \gamma\mathbf{c}(\mathbf{x})), \mathbf{x} - \mathrm{Pr}_\Omega(\mathbf{x} - \gamma\mathbf{c}(\mathbf{x}))\rangle \leq 0, \tag{3.48}$$

which is equivalent to

$$\langle -\gamma\mathbf{c}(\mathbf{x}), \mathbf{x} - \mathrm{Pr}_\Omega(\mathbf{x} - \gamma\mathbf{c}(\mathbf{x}))\rangle + \|\mathbf{x} - \mathrm{Pr}_\Omega(\mathbf{x} - \gamma\mathbf{c}(\mathbf{x}))\|^2 \leq 0, \tag{3.49}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. The second term in inequality (3.49) must be non-negative. By link dynamic definition (3.21), inequality (3.49) implies that

$$\langle -\gamma\mathbf{c}(\mathbf{x}), \mathbf{x} - \mathrm{Pr}_\Omega(\mathbf{x} - \gamma\mathbf{c}(\mathbf{x}))\rangle = \langle -\gamma\mathbf{c}(\mathbf{x}), -\frac{1}{\nu}\dot{\mathbf{x}}\rangle \leq 0. \tag{3.50}$$

In other words, $\mathbf{c}(\mathbf{x})^T\dot{\mathbf{x}} \leq 0$. The equality holds if and only if $-\mathbf{c}(\mathbf{x}^*)$ in the normal cone of $\mathbf{x}^*$, i.e., $\mathbf{x}^*$ is the UE link flow pattern. By Definition 3.7, link flow dynamic (3.21) is a RBAP. $\square$

3.5.2 Relationship with Differential Variational Inequality

The link-based day-to-day traffic dynamic, formulated as (3.10), also has a close relationship with the broader class of dynamical systems, known as Differential Variational Inequalities (DVI). DVI formally consists of an ordinary differential equation (ODE) parameterized by an algebraic vector that is required to be a solution of finite-dimensional state-dependent variational inequalities. Mathematically, given a time period $T > 0$, a closed convex set $U \subseteq \mathbb{R}^m$ and functions $f : \mathbb{R}^{1+n+m} \to \mathbb{R}^n$, $F : \mathbb{R}^{1+n+m} \to \mathbb{R}^m$, and $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^n$, the DVI is to find trajectories $x : [0, T] \to \mathbb{R}^n$ and $u : [0, T] \to \mathbb{R}^m$ such that for almost all $t \in [0, T]$

$$\dot{x}(t) \doteq \frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), u(t)) \tag{3.51a}$$

$$u(t) \in \mathrm{SOL}(U, F(t, x(t), \cdot)) \tag{3.51b}$$

$$0 = \Psi(x(0), x(T)). \tag{3.51c}$$

Eq. (3.51a) is a regular ODE presentation of system dynamics. Eq. (3.51b) presents a variational constraint, where $\mathrm{SOL}(U, F(t, x(t), \cdot))$ denotes the solution set of a variational inequality problem defined by $\mathrm{VI}(U, F)$; i.e., for any $u \in \mathrm{SOL}(U, F(t, x(t), \cdot))$, u satisfies $u \in U$ and $(u' - u)^T F(u) \geq 0 \ \forall u' \in U$. Eq. (3.51c) defines the initial and boundary conditions of the system.

DVI (3.51) is a broader class of dynamical systems, which has various presentations depending on how the variational constraint (3.51b) is presented. For example, if U is a cone, then the $\mathrm{VI}(U, F)$ is actually a nonlinear complementarity problem (NCP), defined as (2.5); and the DVI becomes a differential complementarity system. A survey of DVIs and other dynamical systems, solution approaches, and properties of DVIs can be found in Pang and Stewart (2008).

The non-traditional characteristics of DVI, including its algebraic inequalities and logical disjunctions, are appropriate to model systems with unilateral constraints and paradigm switches. Mathematically speaking, the DVI bridges classical smooth ordinary differential equations with contemporary mathematical programming; this new paradigm offers a broad, unifying framework for modeling many real-world disequilibrium problems. In the area of transportation, Friesz et al. (2001) was the first to use a DVI formulation to study the within-day dynamic user equilibrium problem. At that time, the authors called the parameterized ODE system a variational inequality control problem. In later studies, Friesz et al. (2006) and Friesz et al. (2007) successfully applied differential variational inequalities to dynamic oligopolistic network competition, network pricing, and resource allocation.

Although this broad class of dynamical system has been applied to the within-day dynamic user equilibrium problem, no one has studied day-to-day traffic dynamic as a DVI. Actually the DVI approach is a suitable methodology to model and solve day-to-day traffic flow evolution account for various route choice adjustment principles, by considering the external dynamic $\mathbf{u}(t)$ as the target flow $\mathbf{y}$ shown in (3.10).

By far we have presented the basic concept of DVI. Now we can move to address how our proposed link flow dynamic connects to the DVI. Expressing the objective function in (3.10b) in terms of its first-order optimality condition, we have:

$$\text{Find} \qquad \mathbf{y} \in \Omega \qquad \text{such that} \qquad \beta c(\mathbf{x}) + (1-\beta)\nabla_{\mathbf{y}} D(\mathbf{x}, \mathbf{y}) = 0. \tag{3.52}$$

Although the optimality condition is represented as an equation in Eq. (3.52), it can also be considered as a specific representation of a complementarity problem or variational inequalities (see Chapter 1 in Facchinei and Pang 2003). This tells us that model (3.10) is mathematically an instance of a DVI with linear dynamics and that the algebraic VIs are the optimal conditions of a strictly convex program that is parameterized by the state variable $\mathbf{x}$ of the system. Note that an equilibrium point of dynamic (3.10) is a user equilibrium as proved in Theorem 3.1.

The DVI framework appears to be a suitable tool to model and solve traffic evolution processes after network disruption. Compared with formulation (3.10), DVI (3.51) provides us more flexibility in modeling transient traffic dynamics under network disruption which involve sophisticated route-switching behaviors according to their adaptation and prediction capability. In our current version, the day-to-day traffic dynamic is captured by the minimization problem (3.10b). In real world conditions, day-to-day traffic dynamic could be impacted by a number of external forces, including traffic controls, information propagation, and psychological impacts from disaster. Such external forces can potentially be described by a DVI formulation (3.51) with external dynamics $\mathbf{u}(t)$.

Chapter 4

A Discrete-Time Day-to-Day Traffic Assignment Model for Disrupted Network

Chapter 3 developed a link-based traffic flow dynamic model in continuous temporal space, analyzed its stability properties, and presented the relationship with other continuous-time dynamical systems. The continuous-time model provides a fundamental description of the disequilibrium trajectories. Although continuous-time approaches have good mathematical properties, discrete versions of day-to-day traffic equilibration models are more appealing for practical applications, because of the realism of discrete-time trip adjustments.

In this chapter, we propose a general framework for discrete-time day-to-day traffic assignment models, which introduces an additional prediction component to capture drivers' adaptability and predictability. A detailed mathematical formulation is then presented based on the link-based traffic dynamic we proposed in Chapter 3. This chapter also analyzes the stability of the discrete-time model after the prediction component is introduced, and shows that the UE flow pattern is its globally attractive point under mild conditions. Some computational schemes are discussed, in order to address the challenges presented by solving large scale day-to-day traffic assignment problems. A small example demonstrates the flexibility of the proposed discrete-time model.

Before we discuss the discrete-time day-to-day traffic assignment model, this chapter first provides a presentation of the observed data collected after the I-35W Bridge collapse. The observed data collected after the I-35W Bridge collapse reflects many issues never previously considered in transportation models, for instance, the adaptability and predictability of drivers that will be considered in our proposed discrete-time model. Furthermore, the collapse of the I-35W Bridge also provides a natural experimental setting to verify the traffic dynamic models empirically. The research findings presented in this chapter are also provided in our recent paper, He and Liu (2010).

4.1 Observations from the I-35W Bridge Collapse

The I-35W Bridge in Minneapolis collapsed into the Mississippi River at 6:05 pm local time on August 1, 2007. Before the collapse, the forty-year old bridge was one of three primary freeway routes into downtown Minneapolis, carrying more than 140,000 total vehicles on a daily basis.

Compared to network disruptions considered in other studies, the event of I-35W Bridge collapse had unique features that significantly affected the vehicular traffic around the Twin Cities area. First, it was an unexpected incident. Unlike planned road closures for construction or maintenance, the bridge's sudden collapse made it difficult to predict traffic fluctuations after the incident. Travelers were forced to adapt a new daily traffic pattern in a very short time, and could no longer rely on their past experience to tell them which routes were the best. Second, the bridge was located at the center of the Twin Cities metropolitan area, and thus many travelers felt the impact of the collapse as they tried to access downtown Minneapolis and the University of Minnesota. Finally, this disruption happened on one of the most critical links of the impacted transportation network. Only one freeway alternative (I-94) is available in the adjacent area to provide accessibility for trips that traversed the river from the northeast.

The Bridge collapse and its aftermath raised many issues never previously considered in transportation models. We believe that understanding the impact of the disruption from a transportation standpoint requires a shift in perspective. It is no longer sufficient to view transportation systems as simply occupying an equilibrium state in terms of traveler' choices; instead, they need to be viewed as dynamically evolving into such fixed states.

The empirical data used in this study were collected from two sources: surveys of drivers potentially impacted by the collapse and loop detector data from the Twin Cities freeway system.

4.1.1 Survey Data

Two surveys were conducted after the I-35W bridge collapse. The first survey was conducted in the middle of September 2007, six weeks after the event. The survey questions were designed to elicit subjects' day-to-day travel choices for morning commutes and the factors underlying their choices, before and after the bridge collapse. The subjects of the survey were travelers who may have used the I-35W Bridge on a daily basis or who may have used routes that were affected when traffic was rerouted. We distributed 1000 questionnaires in downtown Minneapolis and on the Minneapolis campus of the University of Minnesota, two major communities in close proximity to the bridge. The total number of responses in the survey was 148. For a complete description of the survey, please refer to Zhu et al. (2010).

Results showed that traveler route choices changed significantly. By comparing the respondents' morning commute routes before and after the bridge collapse, we found that 31 respondents altered their daily morning commute routes on the day immediately after the bridge collapse. Interestingly, 14 out of these 31 respondents (about 45%), who changed their daily commuting routes, did not use the I-35W Bridge regularly before it collapsed, and the main reason for them to change route voluntarily was to reduce their travel times.

The second web based survey was conducted in November 2008. Its purpose was to gauge travel behavior change after the I-35W Bridge collapse and its reopening in September 2008. We sent 5000 invitations to random individuals within the Twin Cities seven-county metropolitan area and asked them to answer questions associated with their driving behavior changes due to the bridge collapse and reopening. From the survey results, we found that 52 out of 349 respondents changed their daily morning commute routes immediately after the bridge collapse. Similar to results of the first survey, more than half of those (28 out of 52) were not regular I-35W Bridge users. In addition, 21 of them clarified that "anticipation of worse traffic conditions" was one of the main reasons for changing their routes.

These survey data reveal that drivers' anticipation of traffic congestion was one major reason for changing their commute routes after the bridge collapse. These behavioral changes resulted in an interesting traffic recovery process that has been unveiled by Danczyk et al. (2010). The recovery process was reflected from freeway traffic volume variations recorded by loop detectors.

4.1.2 Traffic Volume Data

At an aggregate level, we studied traffic volume changes on the Twin Cities freeway network before and after the collapse using data collected by loop detectors. We set up three cordons, as provided in Danczyk et al. (2010) and illustrated by Figure 4.1, to define three study areas around the I-35W Bridge. The first cordon, shown as a solid circle, has a radius of about one half mile and covers the immediate adjacent area of the I-35W Bridge. Vehicles approaching the bridge have no alternative freeway routes inside this cordon, and thus have to exit. The second cordon, shown as a dash-dot circle, has a radius of two and a half miles. This cordon includes I-94 and Trunk Highway 280 (TH 280), which was designated as the detour route after the bridge collapse. The third cordon, shown as a 5.5-mile-radius dot circle, covers the Minneapolis Central Business District and the city of Minneapolis. Major freeway routes accessing into the Minneapolis Central Business District are included. Loop detectors placed around the cordon lines are shown as red dots in Figure 4.1.

We aggregated the inbound traffic volumes from the loop detectors around the three cordons for the morning peak period (between 6 a.m. and 9 a.m. local time). Only non-holiday weekdays were considered. Figure 4.2 illustrates the daily traffic evolution between Monday, July 9th, 2007 and Friday, November 16th, 2007. As can be seen, the traffic counts on these three cordons are fairly stable prior to the collapse of the I-35W Bridge. However, immediately after the collapse, the traffic counts at the cordons drop significantly. The total number of vehicles entering Cordon 1 decreases to nearly zero, while the numbers of vehicles entering Cordons 2 and 3 decrease by 40 percent and 20 percent, respectively. In the weeks after the collapse, the traffic counts at these three cordons present a recovery process. Without alternative routes, the demand at Cordon 1 stabilizes to a point that is about 50 percent lower than the demand before the collapse. It takes about eight weeks for the demand to recover at Cordon 2 to the pre-collapse level. The full recovery of demand at Cordon 3 takes about four weeks which is much shorter than that at Cordon 2. Figure 4.2 demonstrates that the dramatic decreases in demand, or "shocks", diminish in intensity as alternatives become available, and

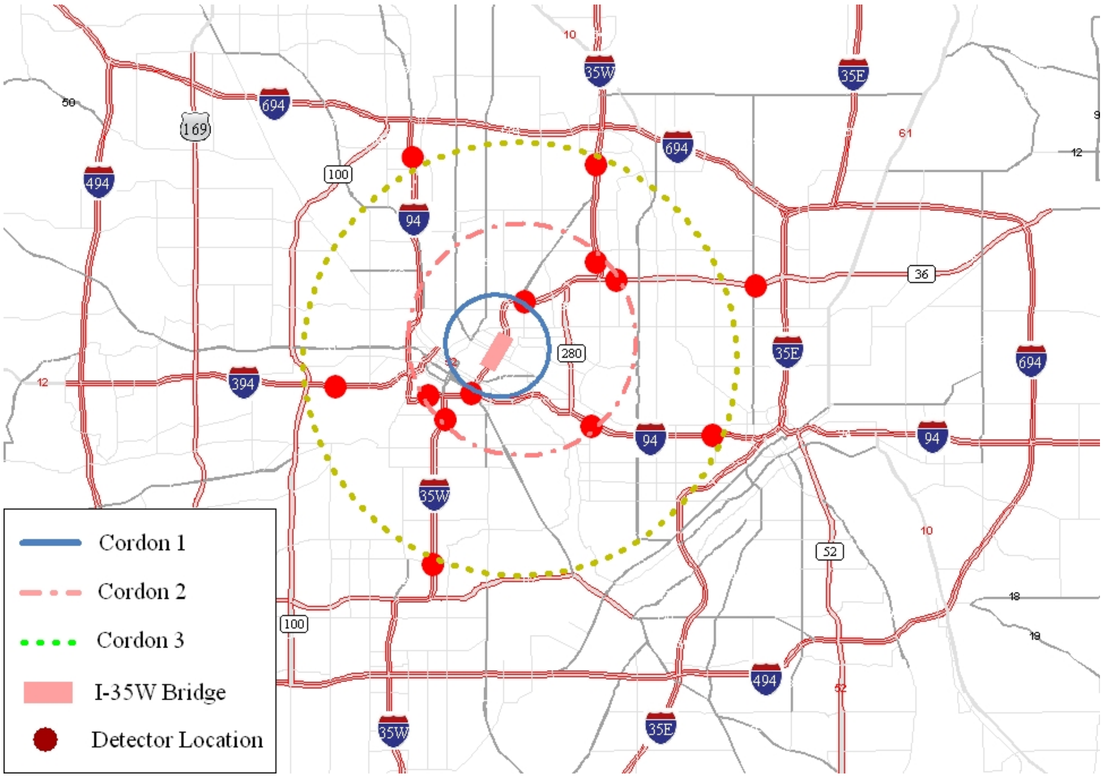


Figure 4.1: Three Cordon Circles around Minneapolis for the I-35W Bridge

generally corresponding to the distance of the cordon from the bridge.

We believe that the recovery of traffic volume on the cordons was mainly due to route adjustment made by drivers after the collapse, not to changes in origin-destination demand. By summing up the traffic counts from the loop detectors located on all the on-ramps in the Twin Cities area, we can analyze the change of daily demand before and after the bridge collapse. Figure 4.3 shows the daily demand entering the Twin Cities freeway system during morning peak period between July 23rd, 2007 and August 31st, 2007. As can be seen, freeway traffic demand in the days and weeks following the bridge collapse does not exhibit any drastic changes, fluctuating within the bounds of weekly variation. The lone exception is on Thursday, August 2nd, when a noticeable decrease

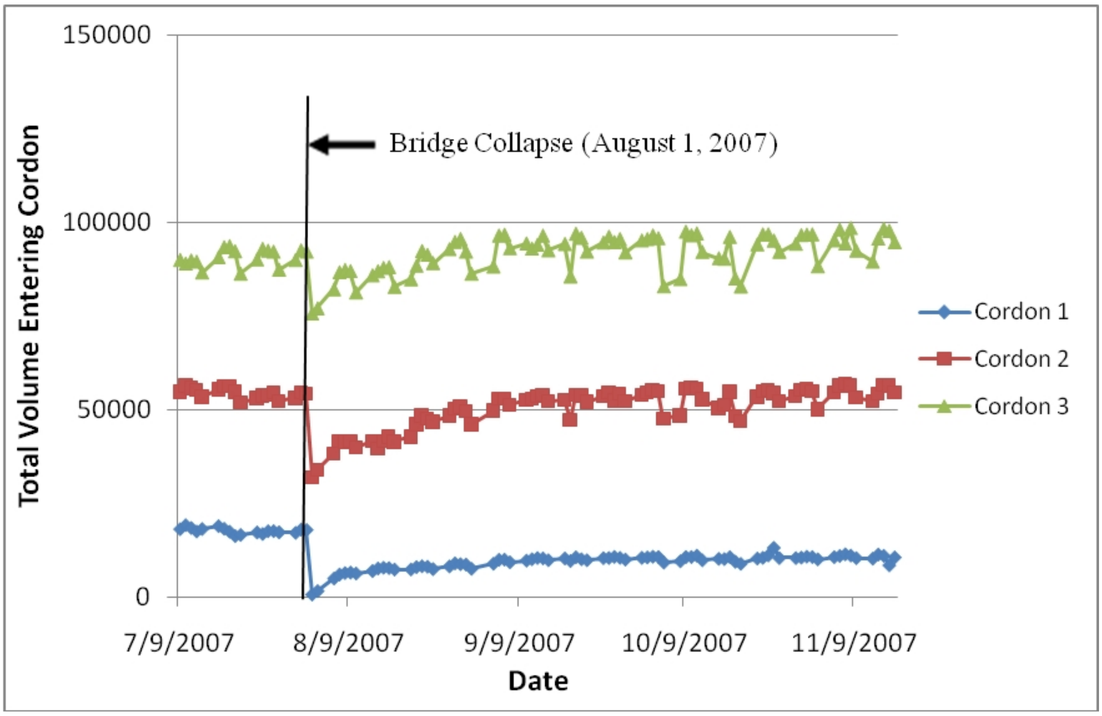


Figure 4.2: Inbound Traffic Volume Changes Crossing the Three Cordons

occurs when compared to other Thursdays; otherwise demand remains consistent with previous weeks. These results show that the bridge collapse did not influence freeway demand during the AM Peak on the system as a whole.

In summary, the empirical evidence from both the traveler surveys and the loop detector data suggests that, after an unexpected network disruption, travelers undergo a learning and exploration process of the new network. Following the I-35W Bridge collapse, drivers were observed to drastically avoid areas near the disruption site until the perceived congestion in that area had diminished. Even drivers who were not regular bridge users changed their daily routes because they anticipated worse traffic conditions on these routes due to the diverted traffic from the bridge. These observations provide strong evidence that drivers' predictions on a disrupted network play a crucial role in

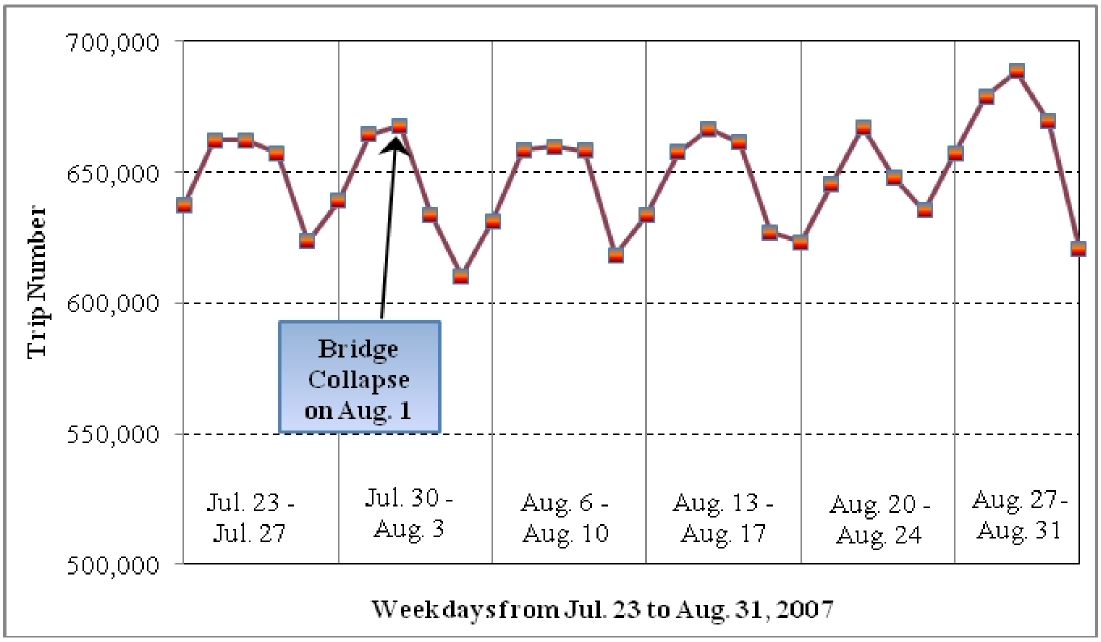


Figure 4.3: Daily Traffic Entering the Twin Cities Freeway Network via On-Ramps (6-9 a.m.)

their route choice decision-making and therefore, this factor cannot be ignored when we model the day-to-day traffic flow evolution process after network disruption.

4.2 Day-to-Day Traffic Assignment Framework

We have shown that drivers exhibited forward-looking behavior during the recovery period after the collapse of the I-35W Bridge as they attempted to avoid congestion. This type of forward-looking behavior significantly impacts the equilibration trajectory in modeling traffic flow evolution after network disruption; however, it has not been captured by traditional day-to-day traffic assignment models. This section solves this problem by introducing a prediction component into the driver perception update process.

4.2.1 Traditional Day-to-Day Traffic Assignment Framework

Traditional day-to-day traffic assignment models update drivers' daily cost perception by a "correction" process that combines drivers' most recent travel experience with their perceptions on previous days. In other words, the models rely entirely on historical information, for day-to-day traffic assignment. Figure 4.4 illustrates the basic structure of traditional "correction-only" assignment framework. As can be seen, network topology change is only considered during network loading, and travelers' cost perception variations are not impacted by topology changes before drivers make actual trips on a new network. When a network disruption happens on day t, traditional updating processes would not reflect cost perception change responding to the disruption before the model assigns traffic into the disrupted network on day $t + 1$. By contrast, our survey data showed that drivers do adjust their route choices under a disruption scenario, even though they do not have any experience yet with the new network.

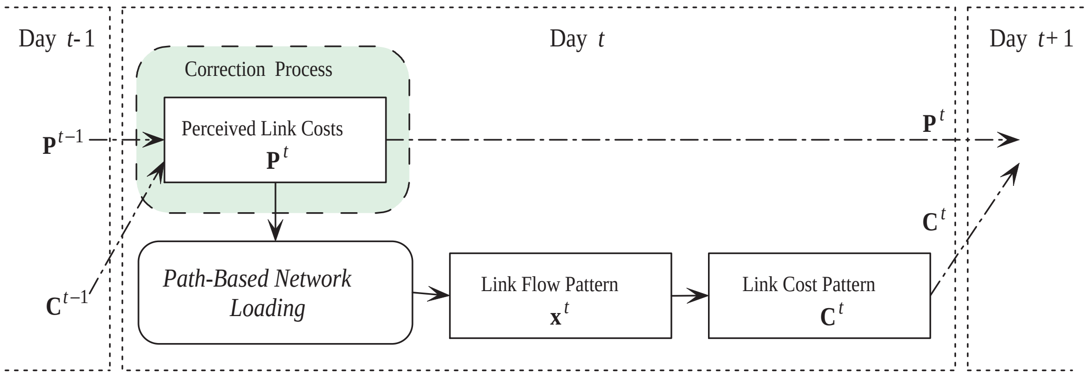


Figure 4.4: Traditional Framework for Day-to-Day Traffic Assignment

One feasible way to model the traffic restoration characteristics under network disruption is to assume drivers adjust their route choices based on perceptions that include their prediction of future traffic patterns as a consequence of network disruption. In fact, psychologists have already noticed that human behavior can depend on their predictions about their environment. As noted by Heath (2000), "*human behavior is clearly not random simply because people can predict to some extent how other people will behave*". Following this statement, we formulate drivers' anticipation of traffic conditions as a prediction process responding to a disruption, and integrate the prediction process with drivers' past experience. Note that, with the prediction component, traffic flows could be altered immediately after a disruption, even though drivers have yet to encounter congestion or even travel on the new network. This is a significant contribution of this study, since, in the existing day-to-day traffic assignment models, travelers cannot change their route choices until they experience congestion.

4.2.2 The "Prediction-Correction" Process

Here, we revise the traditional framework for day-to-day assignment models and propose a new framework, as illustrated by Figure 4.5, that differs in two ways. First, the process of updating travel cost perception now includes two major components: *Prediction* and *Correction*. The addition of a prediction process generates a predicted link flow pattern in the update process, which reflects drivers' anticipation of congestion resulting from network disruption.

The calculation of predicted link flows is illustrated by Figure 4.6. A network disruption will initiate a calculation of predicted traffic flow patterns. The initial predicted flow pattern is computed by redistributing all impacted traffic to paths with shortest distance. After the first day of network disruption, the update of predicted flow pattern proceeds by combining it with actual assigned link flows. We call this the prediction damping process, since the impact of the initial predicted flow pattern is gradually vanishing. With the prediction component, traffic flows will alter right after a disruption, even though drivers have not traveled on the new network yet. By introducing the

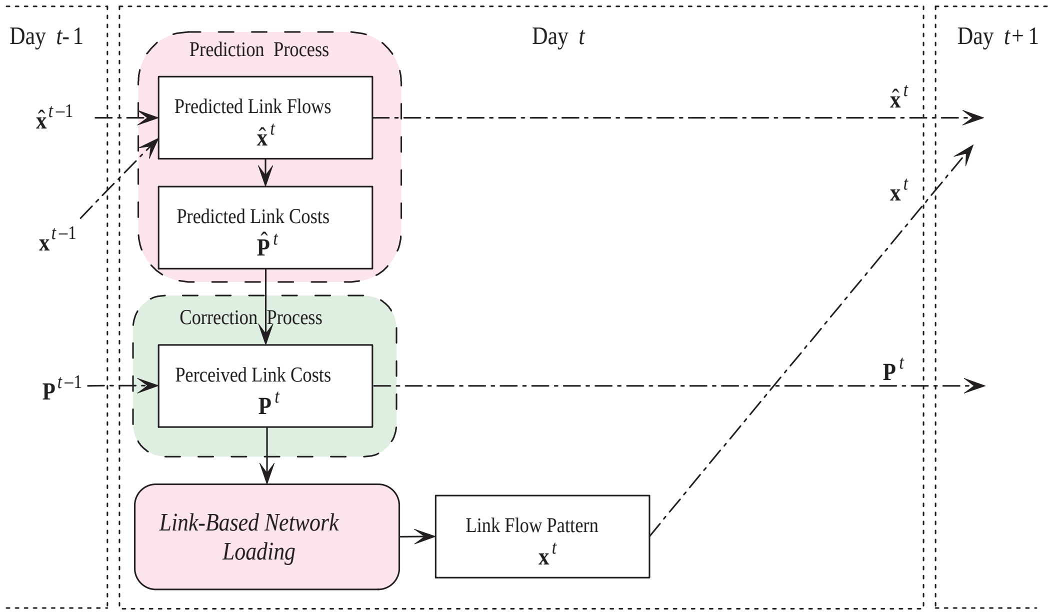


Figure 4.5: The Prediction-Correction Framework for Day-to-Day Traffic Assignment

prediction process, perceived travel cost patterns of drivers now includes a predicted future congestion pattern due to the disruption as well as historical cost patterns.

The second main feature of the proposed framework is its adoption of link-based network loading, instead of traditional path-based assignment processes. This link-based traffic dynamic model successfully avoids the shortcomings of path-based network loading approaches, as presented in Chapter 3.

4.3 Mathematical Formulation of the Discrete-Time Model

As in most of the studies on traffic equilibrium, we assume that drivers are rational in terms of route choice and they have perfect information about link costs from previous days. Since the proposed model involves drivers' predictions of future traffic, we further assume that drivers' prediction gradually vanishes if no additional network disruption

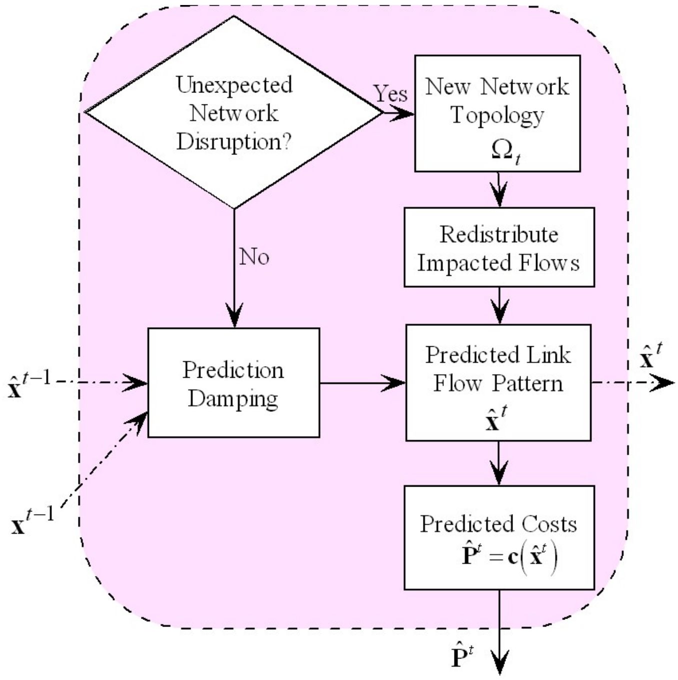


Figure 4.6: The Proposed Prediction Process

occurs. In other words, the "prediction-correction" framework proposed in this research eventually collapses into the traditional "correction-only" framework.

4.3.1 Formulation of Prediction

Following the proposed modeling framework in Section 4.2.2, we introduce a prediction component into the cost perception updating process to capture drivers' anticipation of traffic congestion. This prediction process has two main steps. The first step is to calculate the predicted flow pattern on the day when network disruption occurs. Assume a network disruption happens on day t_0. Then the initial predicted link flow pattern of the following day $\hat{\mathbf{x}}^{t_0+1}$ is a result of reassigning traffic flows impacted by the disruption. In the subsequent days after day $t_0 + 1$, the predicted link flow pattern $\hat{\mathbf{x}}^t$ becomes a weighted average between the previous day's predicted flow pattern $\hat{\mathbf{x}}^{t-1}$ and drivers' actually experienced flow pattern $\mathbf{x}^{t-1}$ (note that $t > t_0 + 1$).

Assume that only those drivers directly impacted by the disruption switch their current routes to the fastest path in the disrupted network. The initial predicted flows $\hat{\mathbf{x}}^{t_0+1}$ can be estimated by reassigning the impacted flows, which follows an all-or-nothing assignment. Mathematically, the predicted flow pattern $\hat{\mathbf{x}}^{t_0+1}$ on day $t_0 + 1$ is an optimal solution to the minimization problem:

$$\min_{\hat{\mathbf{x}} \in \Omega_{t_0+1}} \mathbf{c}_0^T \hat{\mathbf{x}}, \tag{4.1}$$

where $\mathbf{c}_0$ represents the free flow link travel cost vector, $\hat{\mathbf{x}}$ represents the impacted link flows due to the network topology changes, and Ω_{t_0+1} represents the feasible link flow set under the new network topology on day $t_0 + 1$.

The second step updates drivers' predicted link flow pattern. For the days after day $t_0 + 1$, the proposed prediction process updates drivers' predicted link flow pattern by:

$$\hat{\mathbf{x}}^t = (1 - \lambda_t)\mathbf{x}^{t-1} + \lambda_t \hat{\mathbf{x}}^{t-1} \qquad \forall\ t > t_0 + 1, \tag{4.2}$$

where λ_t is the prediction damping parameter. Eq. (4.2) demonstrates that drivers' latest prediction is a weighted average of their experienced link flows and the previous prediction. This model requires $\lambda_t \to \infty$ as $t \to \infty$, such that it captures the vanishing impact of prediction in drivers' cost perception. The representation of λ_t can vary. In this study, we let $\lambda_t = \frac{1}{t-t_0}$ be an asymptotically decreasing function of time t. Other decreasing functions, for instance $\lambda_t = \lambda_0 \exp\left[-b\left(t - t_0\right)\right]$, can be applied as well. Given that λ_t is asymptotically decreasing, the predicted link flow will eventually converge to the latest actual link flow.

Given the predicted link flow pattern $\hat{\mathbf{x}}^t$, the predicted link cost $\hat{\mathbf{P}}^t$ is determined by $\hat{\mathbf{P}}^t = c\left(\hat{\mathbf{x}}^t\right)$. Note that the feasible link flow set $\Omega^t, \forall t > t_0$ differs from Ω^{t_0}, in that the network topology has been changed according to the disruption.

4.3.2 Formulation of Correction

In traditional traffic dynamic models, driver perceived travel costs $\mathbf{P}^t$ on day t is defined by a deterministic function of actual costs in the past. For example, Cascetta (1989) formulated $\mathbf{P}^t$ as a weighted average of costs experienced on days $t-1, t-2, ..., t-m$, for some finite number m, as: $\mathbf{P}^t = \sum_{i=1}^{m} b_i c^{t-i}$, where b_i is the weight of a driver's memory of travel cost on day $t-i$. More concisely, driver pre-trip perception is conditioned only on previous day's experience, or $\mathbf{P}^t = c^{t-1}$. Other studies, e.g., Davis and Nihan (1993), assume that drivers' perceived cost pattern is defined as a weighted average between perception and experience on a previous day:

$$\mathbf{P}^t = (1-\alpha)\mathbf{P}^{t-1} + \alpha \mathbf{c}^{t-1} \tag{4.3}$$

with $0 < \alpha < 1$ representing the cost perception updating weight. Eq. (4.3) implies an exponential weighted average of experienced costs over an infinite number of days in the past.

In our model, driver link cost perception is updated by:

$$\mathbf{P}^t = (1-\alpha)\mathbf{P}^{t-1} + \alpha \hat{\mathbf{P}}^t, \tag{4.4}$$

where the pre-trip link-cost prediction $\hat{\mathbf{P}}^t$ is determined by the prediction process described in the previous section. The value of $0 < \alpha \leq 1$ serves as a measure of the significance of prediction in drivers' perception. Note that the prediction process appears only after a disruption occurs. If no network changes occur, then the updating process Eq. (4.4) gradually collapses to the traditional correction process Eq. (4.3), due to $\hat{\mathbf{P}}^t \to \mathbf{c}^{t-1}$ when $t \to \infty$.

Compared with traditional models, drivers' perception here consists of both retrospective experience $\mathbf{P}^{t-1}$ and forward anticipation of traffic condition $\hat{\mathbf{P}}^t$ resulting from network disruption. The prediction process introduces a vanishing weight on the impact of prediction, using a decreasing weight function w.r.t. time t. As will be demonstrated in the small numerical examples in Section 4.6, different values of the parameters offer flexible representations of link flow dynamics.

4.3.3 Network Loading Process

After updating of travel cost perception in our day-to-day model, the network loading proceeds by a discrete-time link-based process. From the given continuous-time version of day-to-day dynamic (3.10), we can derive a discrete-time version, as:

$$\mathbf{x}^t - \mathbf{x}^{t-1} = \nu(\mathbf{y}^t - \mathbf{x}^{t-1}), \tag{4.5}$$

where $0 < \nu \leq 1$ is a positive constant parameter that determines the rate of link flow change. Eq. (4.5) means that, on day t, the link flow pattern tends to move from the current flow pattern $\mathbf{x}^{t-1}$ towards a "target" flow pattern $\mathbf{y}^t$, at a rate of ν.

For a given link flow $\mathbf{x}^{t-1}$ on day $t-1$, $\mathbf{y}^t$ solves an alternative minimization problem:

$$\min_{\mathbf{y} \in \Omega_t} \beta \left(\mathbf{P}^t\right)^T \mathbf{y} + (1-\beta) D(\mathbf{x}^{t-1}, \mathbf{y}). \tag{4.6}$$

The parameter $0 < \beta < 1$ indicates drivers' sensitivity to travel cost. A larger value of β represents that more drivers tend to use routes with lower travel costs. A smaller value of β implies drivers prefer to maintain the current pattern instead of seeking shorter routes. By solving minimization problem (4.6), we determine the "target" flow pattern $\mathbf{y}^t$, and traffic demand is now loaded into the network.

Unlike the continuous version (3.10b), which applies only an actual link cost vector $\mathbf{c}^{t-1}$, problem (4.6) applies the perception vector $\mathbf{P}^t$ combining the previous link cost perception vector $\mathbf{P}^{t-1}$ with a predicted link cost vector $\hat{\mathbf{P}}^t$, which converges to $\mathbf{c}^{t-1}$ with respect to time t, i.e., $\hat{\mathbf{P}}^t \to \mathbf{c}^{t-1}$ when $t \to \infty$.

Following the "prediction-correction" structure, the proposed discrete-time day-to-day traffic assignment contains four main steps as summarized in Table 4.1.

Table 4.1: Discrete-Time Day-to-Day Traffic Assignment with Prediction

Congestion Prediction:

$$\begin{cases} \hat{\mathbf{x}}^t = \arg\min_{\mathbf{x}\in\Omega_t} \mathbf{C}_0'\mathbf{x}; & \text{if} \quad t = t_0 + 1 \\ \hat{\mathbf{x}}^t = (1-\lambda_t)\mathbf{x}^{t-1} + \lambda_t \hat{\mathbf{x}}^{t-1}; & \text{if} \quad t > t_0 + 1 \end{cases} \qquad \hat{\mathbf{P}}^t = c\left(\hat{\mathbf{x}}^t\right) \tag{4.7a}$$

Perception Correction:

$$\mathbf{P}^t = (1-\alpha)\mathbf{P}^{t-1} + \alpha\hat{\mathbf{P}}^t \tag{4.7b}$$

Target Flow Determination:

$$\mathbf{y}^t = \arg\min_{\mathbf{y}\in\Omega_t} \beta \left(\mathbf{P}^t\right)' \mathbf{y} + (1-\beta)D(\mathbf{x}^{t-1}, \mathbf{y}) \tag{4.7c}$$

Network Loading:

$$\mathbf{x}^t - \mathbf{x}^{t-1} = \nu(\mathbf{y}^t - \mathbf{x}^{t-1}) \tag{4.7d}$$

In the most general form, our "prediction-correction" structure (4.7) requires four parameters: a prediction damping parameter λ_t in Eq. (4.7a), the cost updating weight α in Eq. (4.7b), the cost sensitivity parameter β in Eq. (4.7c), and the link flow changing rate ν in Eq. (4.7d). Under network disruption scenarios, link flow changing rate ν shall be set to 1, resulting in $\mathbf{x}^t = \mathbf{y}^t$; otherwise, some link flows will remain on removed links right after the disruption.

4.4 Stability Analysis

In the literature, many studies have focused on the stability properties of discrete-time traffic equilibrium. Watling and Hazelton (2003) in particular showed that a discrete-time day-to-day assignment model built upon the exponential-smoothing update process (4.4) would not guarantee that a user equilibrium (UE) solution is in any sense "attractive" or "stable". This statement raises a concern about the attractiveness of the user equilibrium flow pattern for our discrete-time model (4.7) containing a perception update process (4.4), despite the fact that the link-based day-to-day dynamic system (3.10) is asymptotically stable. Since this research puts the focus on the "prediction" component of perception updating, for simplicity, we set parameter α to 1 throughout the stability analysis to avoid the mathematical complexities that exponential-smoothing correction process would entail.

The definitions of stability used in this section are the same as Definition 3.2 and Definition 3.3 in Section 3.3. In addition, we define the attractiveness property for the fixed point of discrete-time model (4.7).

Definition 4.1 (Attractiveness of a system fixed point) *The fixed point x^* of link flow dynamic (4.7) is (globally) attractive if for any initial link flow pattern x^0, the Euclidean distance $\|x^t - x^*\| \to 0$ as $t \to \infty$.*

The attractiveness property is also called *convergence* in Bie and Lo (2010). According to Definition 3.3, a dynamical system is asymptotically stable if it is stable and its fixed point is attractive. However, the attractiveness property does not imply the stability of the system, since the attractiveness property does not require the Euclidean distance $\|x^t - x^*\|$ being a monotone decreasing function.

The target flow determination process (4.6) has a close relationship with projection operator $\mathrm{Pr}_\Omega(\cdot)$.

Lemma 4.1 *The target flow determination process (4.6) projects a vector $\mathbf{x}^{t-1} - \gamma \mathbf{P}^t$ to current feasible link flow set Ω_t, where $\gamma \doteq \frac{\beta}{2(1-\beta)}$ with β as the cost sensitivity parameter in Eq. (4.7c)*

Proof. The proof is same as Lemma 3.4. □

Notice that the perception pattern $\mathbf{P}^t$ is an implicit function of the most recent link flow pattern $\mathbf{x}^{t-1}$. Due to (4.2) and $\alpha = 1$, if no network changes occur after a network disruption that happens on day 0, then the prediction pattern $\hat{\mathbf{P}}^t$ converges to $c(\mathbf{x}^{t-1})$.

Denote the difference between predicted flow pattern and actual flow pattern by

$$\mathbf{z}^t = \hat{\mathbf{x}}^t - \mathbf{x}^t. \tag{4.8}$$

Using the notation $\mathbf{z}^t$ significantly simplifies the presentation in the following stability analysis. Due to (4.2),

$$\hat{\mathbf{x}}^{t+1} = \mathbf{x}^t + \lambda_t \mathbf{z}^t \tag{4.9}$$

and

$$\mathbf{P}^{t+1} = c(\hat{\mathbf{x}}^{t+1}) = c\left(\mathbf{x}^t + \lambda_t \mathbf{z}^t\right) \tag{4.10}$$

with $\lambda_t \to 0$ with $t \to \infty$. Then we define the residual function as:

$$e(\mathbf{x}^t, \mathbf{z}^t; \gamma) = \mathbf{x} - \mathrm{Pr}_\Omega[\mathbf{x} - \gamma c(\hat{\mathbf{x}}^{t+1})] = \mathbf{x} - \mathrm{Pr}_\Omega[\mathbf{x} - \gamma c(\mathbf{x}^t + \lambda_t \mathbf{z}^t)]. \tag{4.11}$$

If $c(\hat{\mathbf{x}}^{t+1}) = c(\mathbf{x}^t)$, which is true when $t = \infty$, then we have a simpler residual function:

$$e(\mathbf{x}; \gamma) = \mathbf{x} - \mathrm{Pr}_\Omega[\mathbf{x} - \gamma c(\mathbf{x})]. \tag{4.12}$$

The early work of Eaves (1971) has shown that a zero point of residual function (4.12) solves the variational inequality (VI) problem

$$(\mathbf{x} - \mathbf{x}^*)^T c(\mathbf{x}) \geq 0, \qquad \forall \mathbf{x} \in \Omega, \tag{4.13}$$

whose solution $\mathbf{x}^*$ satisfies the UE principle as shown by Patriksson (1994). In addition, Theorem 3.2 has shown that, if $\mathbf{x}^*$ is a stable point of link flow dynamic (4.7), $\mathbf{x}^*$ must be a UE flow pattern. The following lemma is important for proving the attractiveness property of the UE flow pattern in discrete-time model.

Lemma 4.2 *Assume that* $\mathbf{x}^*$ *is a UE flow pattern and link performance function* $c(\cdot)$ *is Lipschitz continuous and strongly monotone on an open convex set* $\mathcal{S} \supset \Omega$. *Let* L *denote* *the Lipschitz constant and κ denote the strong monotonicity constant. If $\gamma < \frac{4\kappa}{L^2}$, then any flow pattern $\mathbf{x}^t$ provided by the flow dynamic (4.7) satisfies*

$$(\mathbf{x}^t - \mathbf{x}^*)^T e(\mathbf{x}^t, \mathbf{z}^t; \gamma) \geq (1-\delta)\|e(\mathbf{x}^t, \mathbf{z}^t; \gamma)\|^2 - \|\lambda_t \mathbf{z}^t\|^2, \tag{4.14}$$

where $\delta > 0$ is a constant.

Proof. Let $v = \mathbf{x}^t - \gamma c(\hat{\mathbf{x}}^{t+1})$ and $u = \mathbf{x}^*$ in inequality (3.25), then we have

$$\left[e(\mathbf{x}^t, \mathbf{z}^t; \gamma) - \gamma c(\hat{\mathbf{x}}^{t+1})\right]^T \left(\mathrm{Pr}_\Omega[\mathbf{x}^t - \gamma c(\hat{\mathbf{x}}^{t+1})] - \mathbf{x}^*\right) \leq 0. \tag{4.15}$$

Notice that $\mathrm{Pr}_\Omega[\mathbf{x}^t - \gamma c(\hat{\mathbf{x}}^{t+1})] \in \Omega$. Based on (4.13), for the UE flow pattern $\mathbf{x}^*$, we have

$$\left(\mathrm{Pr}_\Omega[\mathbf{x}^t - \gamma c(\hat{\mathbf{x}}^{t+1})] - \mathbf{x}^*\right)^T \gamma c(\mathbf{x}^*) \leq 0. \tag{4.16}$$

Adding (4.15) to (4.16) gives us

$$\left\{e(\mathbf{x}^t, \mathbf{z}^t; \gamma) - \gamma[c(\hat{\mathbf{x}}^{t+1}) - c(\mathbf{x}^*)]\right\}^T \left(\mathrm{Pr}_\Omega[\mathbf{x}^t - \gamma c(\hat{\mathbf{x}}^{t+1})] - \mathbf{x}^*\right) \leq 0. \tag{4.17}$$

Applying the strong monotonicity and Lipschitz continuity of link performance function $c(\cdot)$, we have

$$\begin{aligned}
0 &\leq \left\{e(\mathbf{x}^t, \mathbf{z}^t; \gamma) - \gamma[c(\hat{\mathbf{x}}^{t+1}) - c(\mathbf{x}^*)]\right\}^T \left(\mathrm{Pr}_\Omega[\mathbf{x}^t - \gamma c(\hat{\mathbf{x}}^{t+1})] - \mathbf{x}^*\right) \\
&= \left\{e(\mathbf{x}^t, \mathbf{z}^t; \gamma) - \gamma[c(\hat{\mathbf{x}}^{t+1}) - c(\mathbf{x}^*)]\right\}^T \left[(\mathbf{x}^t - \mathbf{x}^*) - e(\mathbf{x}^t, \mathbf{z}^t; \gamma)\right] \\
&= (\mathbf{x}^t - \mathbf{x}^*)^T e(\mathbf{x}^t, \mathbf{z}^t; \gamma) - \|e(\mathbf{x}^t, \mathbf{z}^t; \gamma)\|^2 - \gamma(\mathbf{x}^t - \mathbf{x}^*)^T[c(\hat{\mathbf{x}}^{t+1}) - c(\mathbf{x}^*)] \\
&\qquad + \gamma[c(\hat{\mathbf{x}}^{t+1}) - c(\mathbf{x}^*)]^T e(\mathbf{x}^t, \mathbf{z}^t; \gamma) \\
&\leq (\mathbf{x}^t - \mathbf{x}^*)^T e(\mathbf{x}^t, \mathbf{z}^t; \gamma) - \|e(\mathbf{x}^t, \mathbf{z}^t; \gamma)\|^2 - \gamma\kappa\|\hat{\mathbf{x}}^{t+1} - \mathbf{x}^*\|^2 \\
&\qquad + \gamma(\lambda_t \mathbf{z}^t)^T[c(\hat{\mathbf{x}}^{t+1}) - c(\mathbf{x}^*)] + \gamma[c(\hat{\mathbf{x}}^{t+1}) - c(\mathbf{x}^*)]^T e(\mathbf{x}^t, \mathbf{z}^t; \gamma) \\
&\leq (\mathbf{x}^t - \mathbf{x}^*)^T e(\mathbf{x}^t, \mathbf{z}^t; \gamma) - \|e(\mathbf{x}^t, \mathbf{z}^t; \gamma)\|^2 - \gamma\kappa\|\hat{\mathbf{x}}^{t+1} - \mathbf{x}^*\|^2 \\
&\qquad + \frac{\gamma^2}{4}\|c(\hat{\mathbf{x}}^{t+1}) - c(\mathbf{x}^*)\|^2 + \|\lambda_t \mathbf{z}^t\|^2 + \gamma[c(\hat{\mathbf{x}}^{t+1}) - c(\mathbf{x}^*)]^T e(\mathbf{x}^t, \mathbf{z}^t; \gamma)
\end{aligned}$$

$$\begin{aligned}
&\leq (\mathbf{x}^t-\mathbf{x}^*)^T e(\mathbf{x}^t,\mathbf{z}^t;\gamma) - \|e(\mathbf{x}^t,\mathbf{z}^t;\gamma)\|^2 - \left(\frac{\gamma\kappa}{L^2}-\frac{\gamma^2}{4}\right)\|c(\hat{\mathbf{x}}^{t+1})-c(\mathbf{x}^*)\|^2 \\
&\qquad +\gamma[c(\hat{\mathbf{x}}^{t+1})-c(\mathbf{x}^*)]^T e(\mathbf{x}^t,\mathbf{z}^t;\gamma) + \|\lambda_t \mathbf{z}^t\|^2 \\
&\leq (\mathbf{x}^t-\mathbf{x}^*)^T e(\mathbf{x}^t,\mathbf{z}^t;\gamma) - \left(1-\frac{\gamma L^2}{4\kappa-\gamma L^2}\right)\|e(\mathbf{x}^t,\mathbf{z}^t;\gamma)\|^2 + \|\lambda_t \mathbf{z}^t\|^2 \\
&\qquad - \left\|\sqrt{\frac{\gamma L^2}{4\kappa-\gamma L^2}}e(\mathbf{x}^t,\mathbf{z}^t;\gamma) - \sqrt{\frac{\gamma\kappa}{L^2}-\frac{\gamma^2}{4}}[c(\hat{\mathbf{x}}^{t+1})-c(\mathbf{x}^*)]\right\|^2 \\
&\leq (\mathbf{x}^t-\mathbf{x}^*)^T e(\mathbf{x}^t,\mathbf{z}^t;\gamma) - \left(1-\frac{\gamma L^2}{4\kappa-\gamma L^2}\right)\|e(\mathbf{x}^t,\mathbf{z}^t;\gamma)\|^2 + \|\lambda_t \mathbf{z}^t\|^2.
\end{aligned}$$

This inequality implies inequality (4.14) with $\delta = \frac{\gamma L^2}{4\kappa-\gamma L^2} > 0$. □

We can now analyze the attractiveness property of the UE flow pattern.

Theorem 4.1 *Assume that $\mathbf{x}^*$ is a UE flow pattern and that link performance function $c(\cdot)$ is Lipschitz continuous and strongly monotone on an open convex set $\mathcal{S} \supset \Omega$. Let L denote the Lipschitz constant and κ denote the strong monotonicity constant. Also assume that $\sum_{t=0}^{\infty}\lambda_t^2 < \infty$ and $\gamma < \frac{4\kappa}{3L^2}$. Following the "prediction-correction" structure, the UE flow pattern is a globally attractive point of traffic dynamic (4.7).*

Proof. Because of Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned}
\|\mathbf{x}^{t+1}-\mathbf{x}^*\|^2 &= \|\mathrm{Pr}_\Omega[\mathbf{x}^t-\gamma c(\mathbf{x}^t+\lambda_t\mathbf{z}^t)]-\mathbf{x}^*\|^2 = \|(\mathbf{x}^t-\mathbf{x}^*)-e(\mathbf{x}^t,\mathbf{z}^t;\gamma)\|^2 \\
&= \|\mathbf{x}^t-\mathbf{x}^*\|^2 + \|e(\mathbf{x}^t,\mathbf{z}^t;\gamma)\|^2 - 2(\mathbf{x}^t-\mathbf{x}^*)^T e(\mathbf{x}^t,\mathbf{z}^t;\gamma) \\
&\leq \|\mathbf{x}^t-\mathbf{x}^*\|^2 - (1-2\delta)\|e(\mathbf{x}^t,\mathbf{z}^t;\gamma)\|^2 + \|\lambda_t\mathbf{z}^t\|^2. \qquad (4.18)
\end{aligned}$$

Since $\gamma < \frac{4\kappa}{3L^2}$, we have $1-2\delta > 0$. Notice that Ω is a bounded close set, and thus $\|\mathbf{z}^t\| = \|\hat{\mathbf{x}}^t - \mathbf{x}^t\|$ is also bounded for all t. Denote $a_0 = \sup_t \|\hat{\mathbf{x}}^t-\mathbf{x}^t\|^2$, then $\|\mathbf{z}^t\|^2 \leq a_0, \forall t$. Summing up (4.18) from $t=0$ to infinity, we have

$$\|\mathbf{x}^\infty-\mathbf{x}^*\|^2 \leq \|\mathbf{x}^0-\mathbf{x}^*\|^2 \sum_{t=0}^{\infty}(1-2\delta)\|e(\mathbf{x}^t,\mathbf{z}^t;\gamma)\|^2 + \sum_{t=0}^{\infty}\lambda_t^2 a_0. \qquad (4.19)$$

Since $\sum_{t=0}^{\infty} \lambda_t^2 < \infty$, we can denote $\sum_{t=0}^{\infty} \lambda_t^2 = b_0$. Thus,

$$(1-2\delta)\sum_{t=0}^{\infty} \|e(\mathbf{x}^t, \mathbf{z}^t; \gamma)\|^2 \leq \|\mathbf{x}^0 - \mathbf{x}^*\|^2 + a_0 b_0. \tag{4.20}$$

This means that

$$\lim_{t\to\infty} e(\mathbf{x}^t, \mathbf{z}^t; \gamma) = \lim_{t\to\infty} e(\mathbf{x}^t; \gamma) = \lim_{t\to\infty} \left\{\mathbf{x}^t - \mathrm{Pr}_\Omega[\mathbf{x}^t - \gamma c(\mathbf{x}^t)]\right\} = 0. \tag{4.21}$$

This limitation implies that the sequence $\{\mathbf{x}^t\}$ is bounded and it has a cluster point. Since the zero point of the residual function $e(\mathbf{x}^t; \gamma)$ satisfies the UE principle as shown by Patriksson (1994), the UE flow pattern $\mathbf{x}^*$ is a cluster point of $\{\mathbf{x}^t\}$. Therefor, sequence $\{\mathbf{x}^t\}$ has a subsequence $\{\mathbf{x}^{t_j}\}$ converging to $\mathbf{x}^*$. Because $e(\mathbf{x}^t; \gamma)$ is continuous,

$$e(\mathbf{x}^*; \gamma) = \lim_{j\to\infty} e(\mathbf{x}^{t_j}; \gamma) = 0. \tag{4.22}$$

In the following we prove that sequence $\{\mathbf{x}^t\}$ has exactly one cluster point. Assume that $\tilde{\mathbf{x}}$ is another cluster point, and $\|\tilde{\mathbf{x}} - \mathbf{x}^*\| = \eta > 0$. Because $\mathbf{x}^*$ is a cluster point of $\{\mathbf{x}^t\}$, there exists a $k_0 > 0$ such that $\|\mathbf{x}^{k_0} - \mathbf{x}^*\| < \frac{\eta}{\sqrt{8}}$. At the same time, due to $\sum_{t=0}^{\infty} \|\lambda_t \mathbf{z}^t\|^2 \leq a_0 b_0$, there exists a $k_1 > 0$ such that $\sum_{t=k_1}^{\infty} \|\lambda_t \mathbf{z}^t\|^2 < \frac{\eta^2}{8}$. Without loss of generality, assume $k_1 < k_0$. Summing up (4.18) from $t = k_0$ to $t = k$ gives us

$$\begin{aligned}
\|\mathbf{x}^k - \mathbf{x}^*\|^2 &\leq \|\mathbf{x}^{k_0} - \mathbf{x}^*\|^2 - \sum_{t=k_0}^{k}(1-2\delta)\|e(\mathbf{x}^t, \mathbf{z}^t; \gamma)\|^2 + \sum_{t=k_0}^{k} \|\lambda_t \mathbf{z}^t\|^2 \\
&< \|\mathbf{x}^{k_0} - \mathbf{x}^*\|^2 + \frac{\eta^2}{8} \\
&< \frac{\eta^2}{4} \qquad \forall k \geq k_0.
\end{aligned}$$

This inequality implies $\|\mathbf{x}^k - \mathbf{x}^*\| < \frac{\eta}{2}$, $\forall k \geq k_0$. By applying triangle inequality, it follows that

$$\|\mathbf{x}^k - \tilde{\mathbf{x}}\| \geq \|\tilde{\mathbf{x}} - \mathbf{x}^*\| - \|\mathbf{x}^k - \mathbf{x}^*\| \geq \frac{\eta}{2}, \qquad \forall k \geq k_0.$$

This is a contradiction to the assumption that $\tilde{\mathbf{x}}$ is a cluster point of sequence $\{\mathbf{x}^t\}$. Thus $\mathbf{x}^t$ globally converges to the UE flow pattern $\mathbf{x}^*$ as $t \to \infty$. In other words, UE flow pattern $\mathbf{x}^*$ is a globally attractive point of traffic dynamic (4.7). $\square$

Remark 4.1 *Theorem 4.1 provides sufficient conditions for the global attractiveness property of the UE flow pattern for discrete-time traffic dynamic (4.7). One specific sufficient condition is that of* $\sum_{t=0}^{\infty} \lambda_t^2 < \infty$. *This sufficient condition can easily be satisfied, e.g.,* $\lambda_t = \frac{1}{t}$. *However, day-to-day dynamic (4.7) may not converge to the UE flow pattern if* λ_t *is a constant. Instead, it may stabilize at a point that differs from the UE flow pattern, when the impact of prediction does not disappear quickly enough.*

Remark 4.2 *A second sufficient condition for the global attractiveness is the requirement that* $\gamma < \frac{4\kappa}{3L^2}$. *Notice* $\gamma = \frac{\beta}{2(1-\beta)}$. *This condition implies* $\beta < \frac{8\kappa}{3L^2+8\kappa}$, *which offers a threshold value of* β *to guarantee attractiveness. As we will see, a large* β *may result in divergent trajectories of day-to-day dynamic (4.7). However, due to the difficulty of estimating scalars* κ *and* L, *this sufficient condition is generally difficult to verify.*

Remark 4.3 *From the proof, we can see that link flow traffic dynamic (4.7) may not be stable, due to existence of the predicted traffic pattern* $\lambda_t \mathbf{z}^t$ *in travelers' perception. Even though the initial traffic flow pattern* $\mathbf{x}^0$ *is* $\mathbf{x}^*$, *traffic pattern* $\mathbf{x}^t$ *may be diverted away from* $\mathbf{x}^*$. *At the time immediately after an unexpected network disruption,* $\lambda_t \mathbf{z}^t$ *has a significant impact on the system, such that the inequality* $\|\mathbf{x}^{t+1} - \mathbf{x}^*\| \leq \|\mathbf{x}^t - \mathbf{x}^*\|$ *may not be valid. However, the stability property of the discrete model can be developed easily if the prediction process is removed, as shown by the following corollary.*

Corollary 4.1 *Assume* $\mathbf{x}^*$ *is a UE flow pattern and link performance function* $c(\cdot)$ *is Lipschitz continuous and strongly monotone. Let* L *denote the Lipschitz constant, and* κ *denote the strong monotonicity constant. Also assume that* $\gamma < \frac{2\kappa}{L^2}$. *If* $\hat{\mathbf{x}}^t = \mathbf{x}^t$ *or* $\lambda \equiv 0$ *then discrete-time link flow traffic dynamic (4.7) is asymptotically stable.*

Proof. Without the prediction process, the asymptotic stability of the discrete-time link flow traffic dynamic (4.7) is guaranteed due to Theorem 3.2. $\square$

Remark 4.4 *Corollary 4.1 indicates that, without the prediction process, the traffic flow pattern will approach the new UE flow pattern asymptotically, i.e., the distance to the UE flow pattern is monotonically decreasing. The existence of the prediction raises the possibility that traffic flow may divert away from the UE flow pattern (i.e., the distance to the UE flow pattern is increasing) at the very beginning, although the traffic flow will gradually stabilize at the UE pattern as time passes by. This provision enables the discrete-time model to capture the traffic recovery characteristics shown in the field observations.*

4.5 Solution Approach

After analytically developing a discrete-time day-to-day model and discussing its stability, we are ready to further address the practical challenges presented by large scale day-to-day traffic assignment problems. As shown in Table 4.1, a link-based day-to-day traffic assignment model following the "prediction-correction" structure needs to (1) generate predicted link costs, (2) update cost perceptions, (3) identify target flow patterns, and then (4) adjust link flows. As we can see from the formulation, the main calculations for updating link flow patterns occur during Step 1, **Congestion Prediction**, Step 3, **Target Flow Determination**. We will discuss the technical issues of these two steps.

4.5.1 Congestion Prediction

After network disruption traveler make predictions about the congestion they will encounter on diverted routes. As discussed in Section 4.3.1, generating the predicted link flows $\hat{\mathbf{x}}^{t_0+1}$ on the day immediately after network disruption is critical. After the first day's link cost patterns are predicted, the patterns can then be easily updated by a weighted average. The computation of determining initial predicted flow pattern

$\hat{\mathbf{x}}^{t_0+1}$ involves assigning the impacted link flows to alternative links. Depending on the assumptions used, this process could be implemented in various ways.

If the origin-destination (OD) and route information of the travelers using the disrupted link can be estimated, then all path flows that traverse the disrupted link on day t_0, denoted by $\tilde{\mathbf{f}}^{t_0}$, are known. Thus, we can determine the affected link flows by $\tilde{\mathbf{x}}^{t_0} = \Delta\tilde{\mathbf{f}}^{t_0}$, and the link flows that are not affected by the disruption, denoted by $\bar{\mathbf{x}}^{t_0} = \mathbf{x}^{t_0} - \tilde{\mathbf{x}}^{t_0}$. Then, on day $t_0 + 1$, the predicted link flows $\hat{\mathbf{x}}^{t_0+1}$ can be determined by a combination of two link flow patterns. The first pattern $\hat{\mathbf{x}}_1^{t_0+1}$ is determined by reassigning the impacted path flows $\tilde{\mathbf{f}}^{t_0}$ to paths with shortest free flow travel times in the new network. Mathematically, $\hat{\mathbf{x}}_1^{t_0+1}$ is the optimal solution of the minimization problem (4.1) that actually represents an all-or-noting (AON) assignment of the impacted path flows. The rest link flows in the prediction are not impacted by network disruption, i.e., $\bar{\mathbf{x}}^{t_0+1} = \bar{\mathbf{x}}^{t_0}$. Therefore, $\hat{\mathbf{x}}^{t_0+1} = \hat{\mathbf{x}}_1^{t_0+1} + \bar{\mathbf{x}}^{t_0}$. The representation of predicted link flows $\hat{\mathbf{x}}^{t_0+1}$ is based on the assumption that the users of the disrupted link(s) will change to the new shortest routes after the disruption, while non-affected drivers would retain their current routes.

Currently, it is not practical to identify origins and destinations of all users of a network. However, with travel demand models used in the transportation planning analysis and historical data (demographical information, drivers' activity diaries, travel behavioral surveys, etc.), we can achieve a good understanding of the impacted users due to the disruption, in terms of their origins and destinations, so that the All-or-Nothing assignment for all the impacted users can be performed.

If no OD and route information of link users is available in the prediction process, we may determine the predicted link flows $\hat{\mathbf{x}}^{t_0}$ based on alternative assumptions. One reasonable assumption is that a fixed proportion of the users of disrupted link(s) would use a predetermined alternative bypass. This bypass and a small segment of route that contains the disrupted link share the same origin and destination. In this case, the predicted link flows $\hat{\mathbf{x}}^{t_0+1}$ on day $t_0 + 1$ is determined by reassigning the trips to the bypass route. Mathematically, this reassignment can be formulated by (4.1) as well. AON assignment is applied on a two-route network with demand proportional to the impacted link flows. Solving the minimization problem (4.1) is trivial in this case since it simply reassigns all impacted flow to the bypass, while maintaining all other link flows.

4.5.2 Target Flow Determination

To turn the solution process presented in Table 4.1 into a practical implementation for studying traffic evolution after network disruption, we need to focus on solving subproblem (4.6), to determine the target flows $\mathbf{y}$ for each day. Since this problem is a convex optimization problem with a continuous differentiable objective function, it is suitable to use traditional gradient-based line search algorithms, e.g., Quasi-Newton methods (see Chapter 6 in Nocedal and Wright 1999), to solve it. However, when this minimization problem is applied to large transportation networks containing thousands of links, computing the gradient of the objective function is extremely burdensome.

To help simplify the calculation of gradient, we can turn to the algorithm developed by Frank and Wolfe (1956). This algorithm is the classic method for solving quadratic programming problems with linear constraints. Also known as the convex combination algorithm, the Frank-Wolfe algorithm updates the solution in every step by linearly combining an optimal solution of a linearized subproblem. LeBlanc et al. (1975) introduced the Frank-Wolfe algorithm to solving the user's equilibrium problem, with a nonlinear programming formulation (2.1).

We can use the Frank-Wolfe approach to our non-linear traffic assignment problem

(4.6), and calculate target flow $\mathbf{y}$ in our discrete-time day-to-day assignment model. Following the Frank-Wolfe algorithm as summarized by Sheffi (1985), we need to determine an auxiliary traffic flow pattern in solving the nonlinear problem (4.6).

First, let us review the auxiliary subproblem in the Frank-Wolfe algorithm. Denote the objective function in UE assignment by

$$z(\mathbf{x}) = \sum_{a \in \mathcal{L}} \int_0^{x_a^n} t_a(\omega) \mathrm{d}\omega. \tag{4.23}$$

Then, any feasible search direction $\mathbf{d}$ must satisfy $\langle \nabla z, \mathbf{d} \rangle < 0$ (see Chapter 4 in Sheffi 1985). The search direction in the Frank-Wolfe algorithm is the steepest feasible descent direction satisfying

$$\min_{\mathbf{d} \in \Omega} \nabla z(\mathbf{x})^T \mathbf{d}. \tag{4.24}$$

LeBlanc et al. (1975) realized that $\partial z(\mathbf{x})/\partial x_a = t_a(x_a)$, and its subproblem (4.24) can be replaced by $\min_{\mathbf{d} \in \Omega} \mathbf{t}(\mathbf{x})^T \mathbf{d}$. Thus, the optimal solution to subproblem (4.24) actually provides an all-or-nothing assignment on shortest paths calculated using the travel times $t_a(x_a)$.

Now, denote the objective function of (4.6) in the proposed day-to-day traffic assignment by

$$z(\mathbf{y}) = \beta \left(\mathbf{P}^{t+1}\right)^T \mathbf{y} + (1-\beta) \|\mathbf{x}^t - \mathbf{y}\|^2. \tag{4.25}$$

Its derivative, denoted by $\nabla z(\mathbf{y}) = (\partial z(\mathbf{y})/\partial y_a)$, contains

$$\frac{\partial z(\mathbf{y})}{\partial y_a} = \beta P_a^{t+1} + 2(1-\beta)(y_a - x_a^t) \quad \forall a \in \mathcal{L}, \tag{4.26}$$

in which y_a and x_a^t are link flows and P_a^{t+1} is perceived link cost.

With the derivative formula (4.26), we can apply the convex combination algorithm for determining target flows $\mathbf{y}$ in day-to-day traffic assignment. Each update step of the convex combination algorithm contains an AON assignment that requires a shortest path search for all OD pairs. Notice that, in solving the UE problem, the derivative of the objective function is a vector of travel times that are always positive. However, in our model, the value of $\frac{\partial z(\mathbf{y})}{\partial y_a}$ can be negative if $y_a - x_a^t < 0$ and β is small enough. This means that AON assignment needs to be implemented on a graph with negative link weights which will induce cyclic flows in the assignment and pose a potentially unreasonable link flow patterns for our model.

In static traffic assignment, a fundamental property is that it is impossible for equilibrium traffic flows from a common origin or to a common destination to jointly form a complete cycle (Newell 1980; Janson and Zozaya-Gorostiza 1987), provided the link costs are always positive. A similar observation was made by Hagstrom and Tseng (1999) for general asymmetric traffic assignment problems. However, when negative link costs exist, this property is no longer valid. Therefore, cyclic flows could severely degrade the convergence of the traffic assignment and impede the applicability of our proposed discrete-time model. Interestingly, only a few studies in the literature have investigated this cyclic flow problem and their insights for our case are limited. For example, Janson and Zozaya-Gorostiza (1987) illustrated the convergence difficulty imposed by cyclic flows in the Frank-Wolfe algorithm, and also proposed methods to avoid generating two-arc cyclic flows, which they found are the most common forms of the cyclic flows.

In addition, Bellman (1958) reported using the Bellman-Ford algorithm to perform a shortest path search on graphs with negative weights. However, we cannot identify shortest paths if negative cycles exist in the graph, since the route cost (based on the cost values defined by $\frac{\partial z(\mathbf{y})}{\partial y_a}$) decreases each time flows traverse such cycles. To avoid negative cycles in the shortest path search, we use an alternative approach to identify acyclic "shortest" paths for all OD pairs. We summarize this algorithm in Table 4.2.

The basic structure of this algorithm is similar to Dijkstra's algorithm (Dijkstra 1959), with two main differences: First, it allows revisiting the nodes that have been

Table 4.2: Pseudocode of Acyclic Path Determination

```
1  for each node v in graph G:          // Initializations
2      cost[v] := ∞                     // Unknown cost from origin to node v
3      predecessor[v] :=null            // Precedent node in optimal path from origin
4  cost[origin] := 0                    // Cost from origin to origin
5  Q := the node set in G               // Nodes not yet visited
6  while Q is not empty:                // The main loop
7      u := node in Q with the smallest cost[ ]
8      remove u from Q
9      for each neighbor v of u:        // Where v may have been visited
10         alt := cost[u]+ cost_between(u, v)
11         if alt < 0:                  // Avoid negative cycle
12             alt := 0
13         if alt < cost[v]:            // Update the path to node v
14             cost[v] := alt
15             predecessor[v] := u
16 return predecessor[ ]                // return the precedent node train
```

removed from set $\mathcal{Q}$ (which collects the nodes not yet visited). Therefore, a visited node may be connected by an alternative route containing links with negative weights in later iterations of the path search. Second, the approach resets the travel cost (represented by cost$[v]$ in Table 4.2) to zero whenever it detects a negative route cost to the currently visiting node v, as shown by lines 11 and 12 in Table 4.2. If link weights (or costs) are all nonnegative, this approach is Dijkstra's algorithm. If a negative cycle exists in the graph, the algorithm is able to provide an acyclic path with low cost. However, this approach does not guarantee that the final output will contain the shortest path when negative link costs are present in the graph.

By applying the path search algorithm, we can implement AON assignment to provide an auxiliary link flow pattern $\mathbf{d}^n$ as a feasible search direction. A line search step is performed to find an optimal step size ρ_n which is a solution to

$$\min_{0 \le \rho \le 1} \beta \left(\mathbf{P}^t\right)^T (\mathbf{y}^n + \rho(\mathbf{d}^n - \mathbf{y}^n)) + (1-\beta)\|\mathbf{x}^t - (\mathbf{y}^n + \rho(\mathbf{d}^n - \mathbf{y}^n))\|^2. \tag{4.27}$$

Finally, the link flow pattern can be updated by

$$\mathbf{y}^{n+1} = \mathbf{y}^n + \rho_n(\mathbf{d}^n - \mathbf{y}^n). \tag{4.28}$$

When a convergence criterion is met, “target” flow pattern $\mathbf{y}$ is found and one day’s traffic assignment is complete.

4.6 An Illustrative Example

One of the features of our “prediction-correction” structure is modeling flexibility. In this section, we will demonstrate this flexibility using a simple network with disruption. Suppose that we have a small network, shown in Figure 4.7, with 8 nodes, 11 links, and 2 origin-destination (OD) pairs: (1, 3) and (2, 4). Both OD pairs have the same OD demand of 1000 trips. All links in this network have the same capacity of 100, and their link performance function follows the Bureau of Public Roads (BPR) function:

$$t_a^0 \left(1 + 0.15 \left(\frac{x_a}{c_a}\right)^4\right),$$

where t_a^0 denotes the free flow travel time, x_a denotes the link flow, and c_a denotes the link capacity on link a. Links 4, 5, 6, 7, 8 each has a free flow travel time of 5, while other links have a free flow travel time of 20.

Assume that the link flow pattern follows UE before link 9 is removed at the end of day 1. After the removal of link 9, link chain 7→ 6→ 8 becomes the designated detour

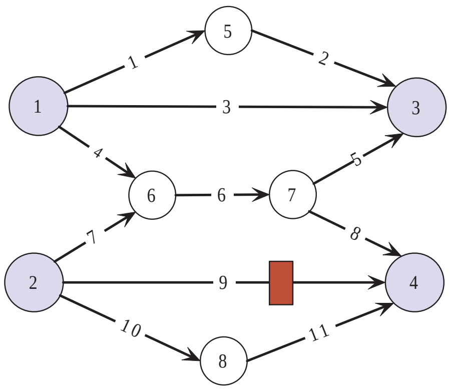


Figure 4.7: A Small Network

since this route now has the shortest free flow travel time. The prediction process then redistributes impacted link 9 users to the designated detour. The output of this process is a predicted traffic pattern $\hat{\mathbf{x}}^2$. As mentioned earlier, we set link flow update parameter ν in Eq. (4.5) to 1; otherwise, some traffic will remain on link 9 on day 2 even though the link has been removed. We also set parameter $\lambda_t = \frac{1}{t}$, according to Eq. (4.2), to capture the damping effect of prediction.

For comparison purposes, we test the proposed model under different settings of α and β. In the first trial, we investigated the impact of parameter α on the day-to-day evolution of traffic flow after network disruption for a single link. Recall that α is a measure associated with predicted traffic pattern. In this example, we set different values of α, while maintaining a constant cost sensitivity of $\beta = 0.3$. These different values of α quantified the weights of prediction in drivers' perception. Figure 4.8 shows that the flow on link 6 gradually stabilizes in all cases. However, we also see that the higher the value of α, the more significant the drop in link flow on the day immediately following the disruption. The reason is that the prediction process in the proposed model puts more weight on drivers' anticipation of potential congestion on the designated detour route, as a consequence of the removal of link 9.

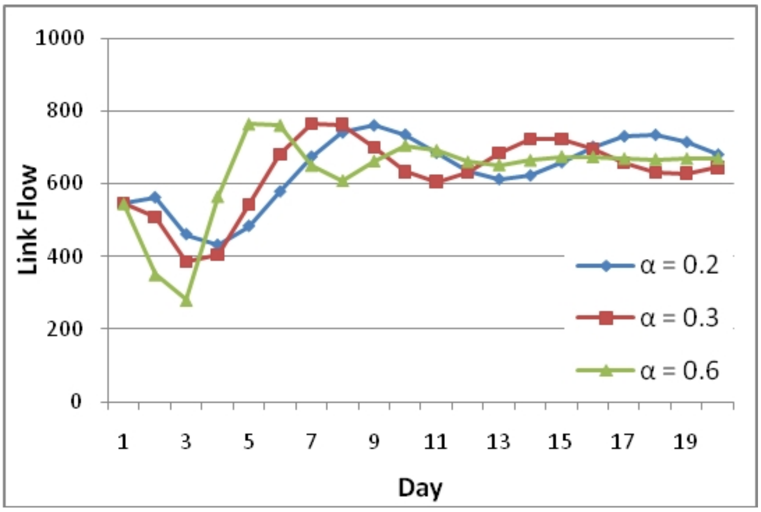


Figure 4.8: Flow Evolutions on Link 6 with Different Perception Updating Weights

In the second trial, we confirmed the necessity of including the prediction process when modeling traffic evolution of a disrupted network. This example is an illustration of Corollary 4.1. Specifically, we removed the prediction process from the day-to-day traffic assignment process by letting $\lambda_t \equiv 0$ in Eq. (4.2), such that the process coincides with traditional day-to-day assignment models. By doing so, drivers' link cost perception on day 2 (i.e., immediately after the removal of link 9) is the same as the pattern before the link removal. As shown in Figure 4.9, when drivers are not making predictions, traffic volume on link 6 on day 2 always increases, since users of link 9 would always consider the detour, i.e., the shortest route, as a main alternative. A decreased link flow on link 6 would never happen. Of course, this result does not account for drivers' ability to anticipate congestion on the new detour route. As we saw after the I-35W Bridge collapse, drivers indeed chose to avert travel on designated detour routes based on their predictions, not their actual experience of the route. Therefore, including a prediction component in our model offers us more flexibility to capture the characteristics of traffic flow after disruption.

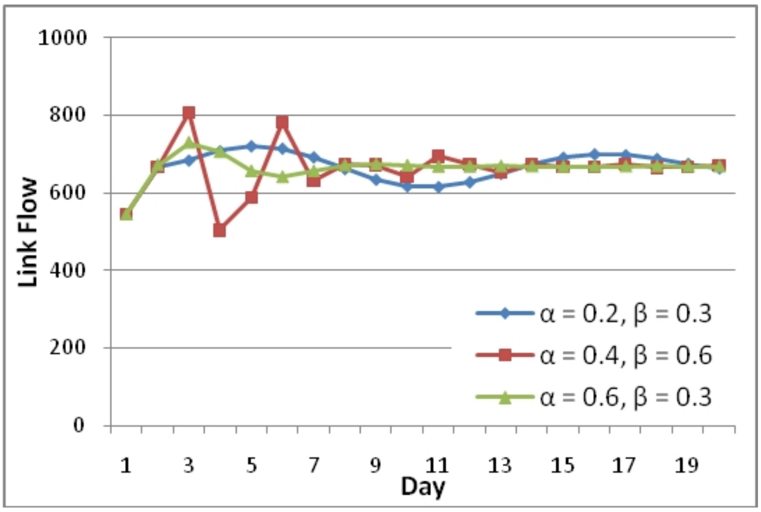


Figure 4.9: Flow Evolutions on Link 6 without Prediction Process

The final trial illustrates that the cost sensitivity parameter β plays an important role in the stability of our model. We kept α as a constant ($\alpha = 0.6$) and let traffic evolve by using different values of β. The link flow evolution patterns shown in Figure 4.10 provide evidence of the system stability. As can be seen, a larger value of β (e.g., $\beta = 0.6$ in this trial) widens the variation of link flows, because higher sensitivity to travel cost would result in more drastic changes of drivers' daily travel choice. This is also a demonstration of Remark 4.2. Whenever β violates the threshold value, e.g., $\beta < \frac{8\kappa}{(3L^2+8\kappa)}$ shown in Theorem 4.1, the attractiveness property of the link flow dynamic can no longer be guaranteed.

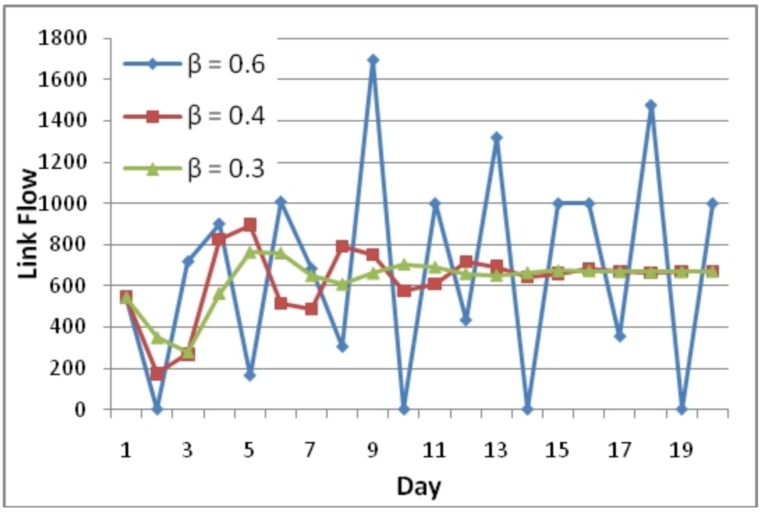


Figure 4.10: Flow Evolutions on Link 6 with Different Cost Sensitivity Weights

Chapter 5

Model Calibration and Validation

Due to the lack of data, very few studies have compared the outcomes of a day-to-day traffic assignment model against reality, and thus the quality of existing models has not been verified. As noted by Friesz and Shah (2001), what is needed now is not delicate mathematical formulas or computational tools, but rather real world data which allows construction and calibration of day-to-day travel adjustment dynamics. There clearly exists a gap between day-to-day traffic flow evolution models and their practical applications, especially for disrupted networks, the scenario of greatest concern for traffic management authorities.

5.1 Network Topology and Demands

To demonstrate the applicability of the proposed discrete-time day-to-day traffic assignment model, we validated it against the field data collected before and after the I-35W bridge collapse. Before the collapse, more than 140,000 vehicles crossed the I-35W Bridge daily. As a major entry point of the city of Minneapolis, this bridge carried nearly 25% of the traffic accessing the downtown area.

The network used in this research is the Twin Cities Seven-County conflated planning network, whose skeleton structure is shown as Figure 5.1. This network contains 22,476 links and 8,618 nodes, of which 1201 are traffic analysis zones (TAZs) that generate and attract trips.

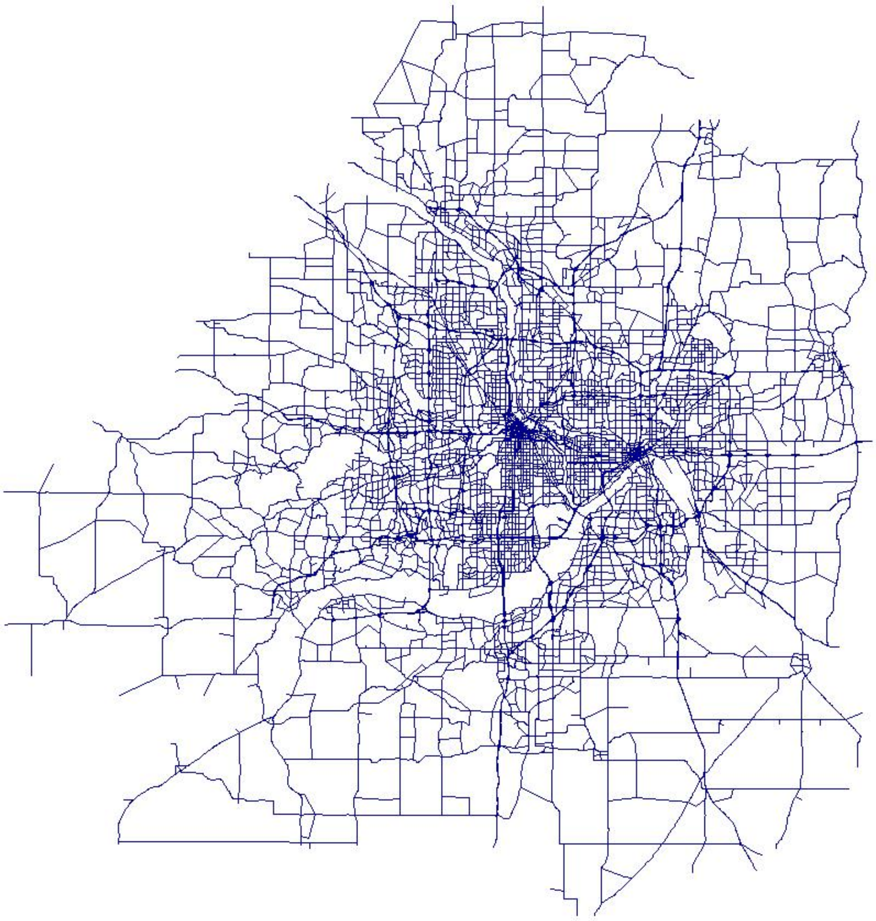


Figure 5.1: The Conflated Seven-County Planning Network

The transportation links in the network were categorized into ten road facility types. Each link of the network has its free flow travel speed and link capacity according to the type the link belongs to and the area where it is located. Table 5.1 and Table 5.2 summarize the free flow travel speeds and capacities of all link facilities. These data are

Table 5.1: Free Flow Speeds in Mile Per Hour

Area Type (AT)	**AT1: Rural**	**AT2: Developing**	**AT3: Developed**	**AT4: Residential Core**	**AT5: Business Core**	**AT6: Outlying Business Concentration**
0 – HOV Facility	74	74	72	71	70	78
1 – Metered Freeway	74	74	72	71	70	78
2 – Unmetered Freeway	74	74	72	71	70	78
3 – Metered Ramp	37	37	36	35	35	39
4 – Unmetered Ramp	37	37	36	35	35	39
5 – Divided Arterial	59	42	30	27	22	31
6 – Undivided Arterial	56	31	30	23	22	30
7 – Collector	51	44	33	23	23	28
8 – HOV Ramp	37	37	36	35	35	39
9 – Centroid Connector	23	23	23	23	23	23

Source: The Metropolitan Council of the Twin Cities

Table 5.2: Network Capacities

Area Type (AT)	**AT1: Rural**	**AT2: Developing**	**AT3: Developed**	**AT4: Residential Core**	**AT5: Business Core**	**AT6: Outlying Business Concentration**
0 – HOV Facility	1400	1400	1400	1400	1400	1400
1 – Metered Freeway	1950	1950	1950	1950	1950	1950
2 – Unmetered Freeway	1750	1750	1750	1750	1750	1750
3 – Metered Ramp	750	725	675	625	600	600
4 – Unmetered Ramp	1500	1450	1350	1250	1200	1200
5 – Divided Arterial	1000	950	850	750	700	700
6 – Undivided Arterial	900	850	750	650	600	600
7 – Collector	600	550	500	450	400	400
8 – HOV Ramp	NA	1450	1350	1250	1250	1250
9 – Centroid Connector	0	0	0	0	0	0

Source: The Metropolitan Council of the Twin Cities from the Metropolitan Council of the Twin Cities.

Since the focus of this research is link flow evolution, we apply link performance functions and origin-destination demands given in other studies. The link performance functions used in our research are conical volume-delay functions (Spiess 1990), defined as follows.

$$t(x_a) = t_a^0 \left[2 + \sqrt{A^2 \left(1 - \frac{x_a}{c_a}\right)^2 + B^2} - A\left(1 - \frac{x_a}{c_a}\right) - B \right], \tag{5.1}$$

where t_a^0 denotes the free flow travel time, x_a denotes the link flow, and c_a denotes the link capacity on link a; A and B in Eq. (5.1) are parameters satisfying:

$$B = \frac{2A - 1}{2A - 2}. \tag{5.2}$$

The parameters A and B for different types of facilities in this study are summarized in Table 5.3.

Table 5.3: Parameters Used in Conical Volume-Delay Function*

Facility Type	A	B
0 – HOV Facility	4	1.167
1 – Metered Freeway	4	1.167
2 – Unmetered Freeway	4	1.167
3 – Metered Ramp	4	1.167
4 – Unmetered Ramp	4	1.167
5 – Divided Arterial	4	1.167
6 – Undivided Arterial	5	1.125
7 – Collector	6	1.100
8 – HOV Ramp	4	1.167
9 – Centroid Connector	NA	NA

* Source: The Metropolitan Council of the Twin Cities

The link performance functions used for centroid connectors were set as free flow travel times. This was reasonable because conical volume-delay function (5.1) guarantees that the travel time on a centroid connector is equivalent to its free flow travel time due to its infinite link capacity and the relationship Eq. (5.2).

Since we assume that traffic volume variation after the I35W Bridge collapse was mainly due to drivers' route choice adjustment, as we discussed in Chapter 4, we used a fixed demand table to study the day-to-day traffic evolution of this adjustment. A trip table, derived from the 2006 Longitudinal Employer-Household Dynamics (LEHD) database[1], was adopted as the origin-destination demand data.

To reflect trip-preferences for AM peak hours in 2007, projected demand was scaled by a demand multiplier. The demand multiplier was determined by assuming that freeway system usage is a constant proportion of total number of trips. We first summed up the traffic volumes entering the freeway system through on-ramps based on loop detector data at AM peak hours. We also calculated freeway usage by a static traffic assignment implemented with the gradient projection algorithm (Chen et al. 2002) on the seven-county planning network using the derived LEHD trip table. The demand multiplier for 2007 was the rate of actual freeway usage to the assigned freeway usage.

5.2 Parameter Estimation

To verify our model for disrupted networks, we focused the study period on the month when the I-35W Bridge collapsed. Parameters are determined to minimize the difference between assignment results and observations during the study period. In detail, the study period chosen for calibration and validation purposes was from July 30, 2007 to August 31, 2007. Only weekdays were considered. During this period, a series of

[1]Downloaded from: http://www.vrdc.cornell.edu/onthemap/data

network topology changes took place. On August 1, 2007, the I-35W Bridge collapsed. Freeway ramps around the bridge were closed on that day and the Cedar Ave Bridge (an urban arterial parallel to the I-35W Bridge) remained closed for investigation purpose until the end of the month. On August 2, TH-280 was approved as a detour route and converted to a freeway by blocking off all side-street access. On August 13, the ramps connecting TH-280 and I-94 were expanded to two lanes. Finally, on August 20, one additional lane was added in each direction of I-94 for the three-mile stretch between TH-280 and I-35W to accommodate re-routed traffic.

5.2.1 Parameters in the Model

Here we first give a summary of the model parameters to facilitate the presentation of the model applications in subsequent sections. In its most general form, our "prediction-correction" structure requires four parameters: a prediction damping parameter λ_t in Eq. (4.2), the cost updating weight α in Eq. 4.4, the link flow changing rate ν in Eq. (4.5), and the cost sensitivity parameter β in Eq. (4.6). In our experiments, two of these parameters will be predetermined: link flow changing rate ν will be set to 1 to guarantee that a removed link will not carry any flow right after its removal; and prediction damping parameter λ_t will have some given functional form as discussed earlier. Therefore, in the subsequent model calibration and validation, we shall focus on two parameters only, namely cost updating weight α and cost sensitivity β.

5.2.2 Calibration Objective

We calibrated the our model by applying the first 15 weekdays' (July 30, 2007 to August 17, 2007) loop detector data collected from ten detector stations, whose locations are shown as red dots in Figure 5.2. The ten selected detector stations provided us reliable data with less detector errors. Further, their dispersed locations reduced the data correlation and thus gave us more independent information of the traffic evolution during the study period. Theoretically, the number of detector stations is enough for parameter estimation since we have only two parameters to be estimated.

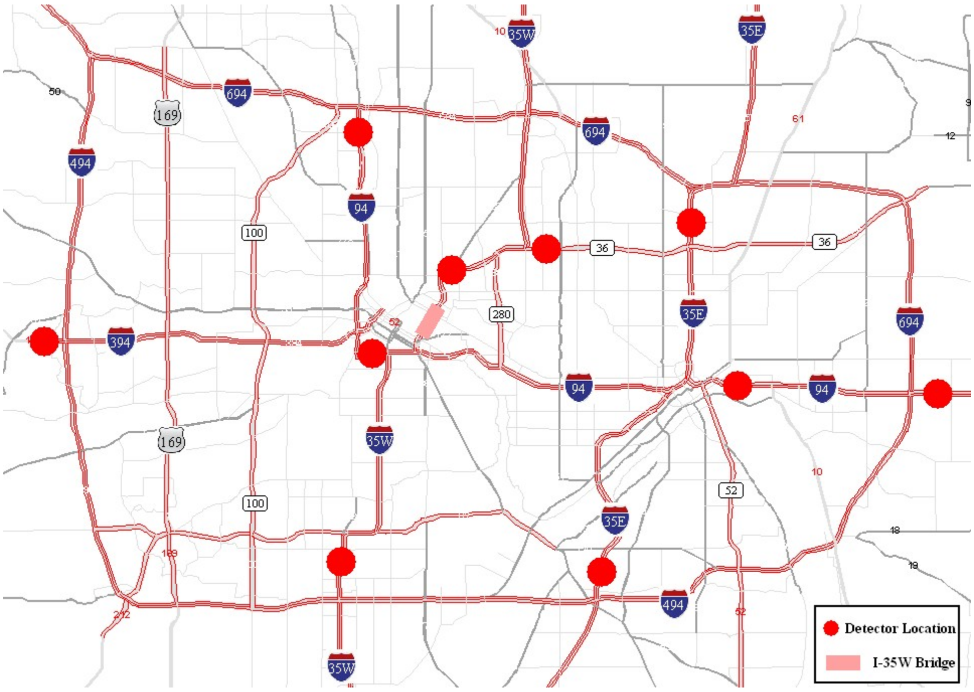


Figure 5.2: Detector Locations for Calibration

The objective of the parameter calibration was to minimize the root mean square percent error (RMSPE), defined as:

$$\min_{\hat{\theta}} \sqrt{\frac{1}{n} \sum_{t} \sum_{a \in \hat{\mathcal{L}}} \left(z_a^t - \hat{z}_a^t(\hat{\theta}) \right)^2}, \tag{5.3}$$

where $z_a^t = (x_a^t - x_a^0)/x_a^0$ represents the observed link-flow-change percentage w.r.t day 0 (i.e., July 30, 2007), at link a. The predicted link-flow-change percentage $\hat{z}_a^t = (\hat{x}_a^t - \hat{x}_a^0)/\hat{x}_a^0$ was determined by the proposed day-to-day traffic assignment model under given parameters $\hat{\theta} = \left(\hat{\alpha}, \hat{\beta}\right)$. The total number of observations was $n = 140$, including data from 10 detector locations, each of which provided 14 days' link-flow-change percentages.

The reason for applying the method of root mean square percent error is because we are modeling the day-to-day evolution of link flow pattern, not the the absolute values of link flows. The link flow change percentage can better achieve this goal, since it considers the relative change of flow patterns and mitigates the impact from the difference between the initial link flow pattern and field observations.

5.2.3 Estimation Results

Figure 5.3 shows the software interface we developed in C#.Net, for estimating the parameters. We applied a mesh method to estimate the parameters in the model,

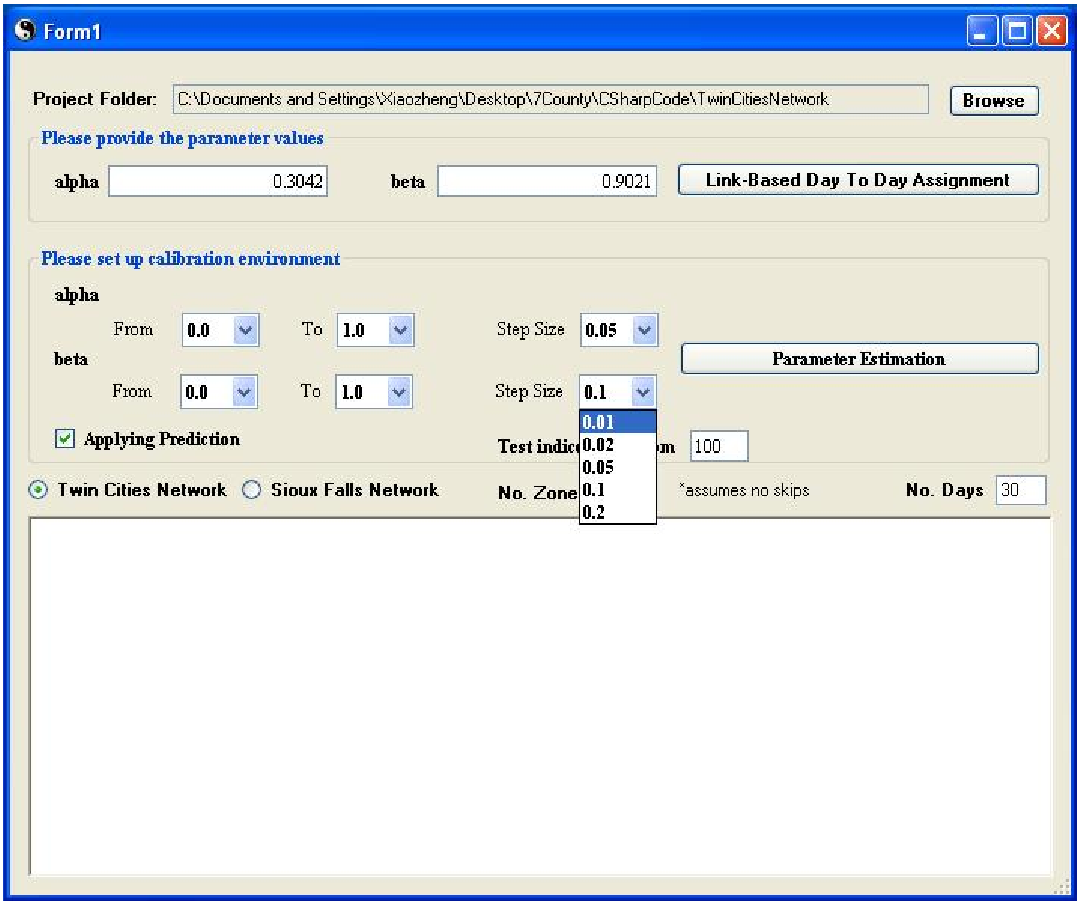


Figure 5.3: Calibration Software Interface

setting $\lambda_t = 0$ when $t \leq 3$ (i.e., no prediction before the collapse) and $\lambda_t = \frac{1}{(t-3)}$ when $t > 3$. First, we divided the two-dimensional parameter space into small bins, using a step size of 0.1 along each axis. Thus, we had ten bins along each parameter domain and a grid of 100 bins. As can be seen in the middle of Figure 5.3, our mesh approach allows us to reduce the step sizes, e.g., from 0.1 to 0.01, for further refinement.

We randomly generated one sample pair of parameters $(\hat{\alpha}, \hat{\beta})$ inside each bin for a total of 100 pairs. The 100 sampled parameter pairs are shown in Figure 5.4. Each pair of parameters $(\hat{\alpha}, \hat{\beta})$ was applied with the proposed day-to-day traffic assignment method to generate daily traffic flow patterns.

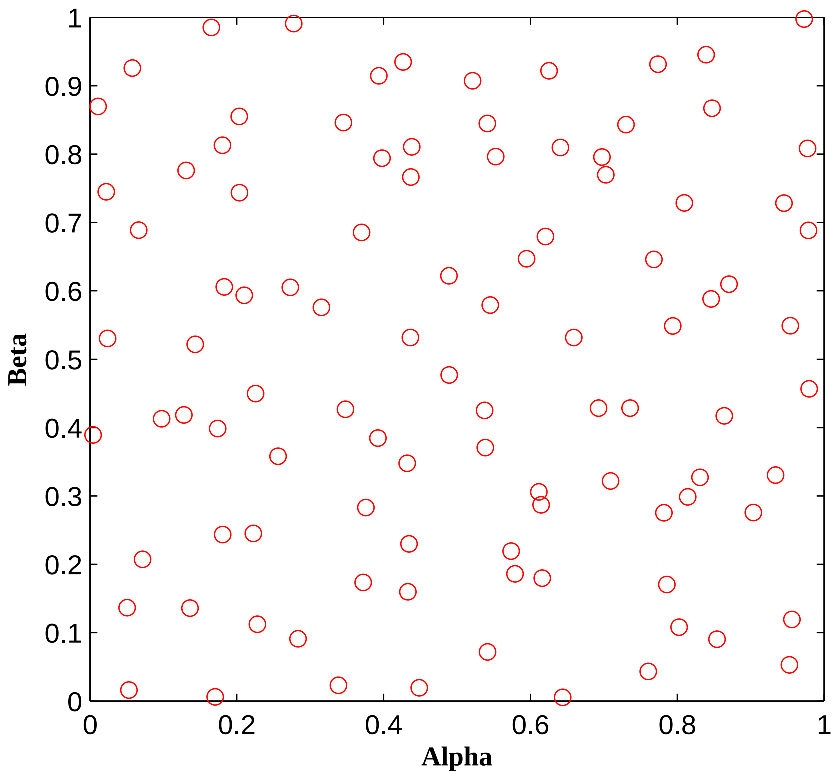
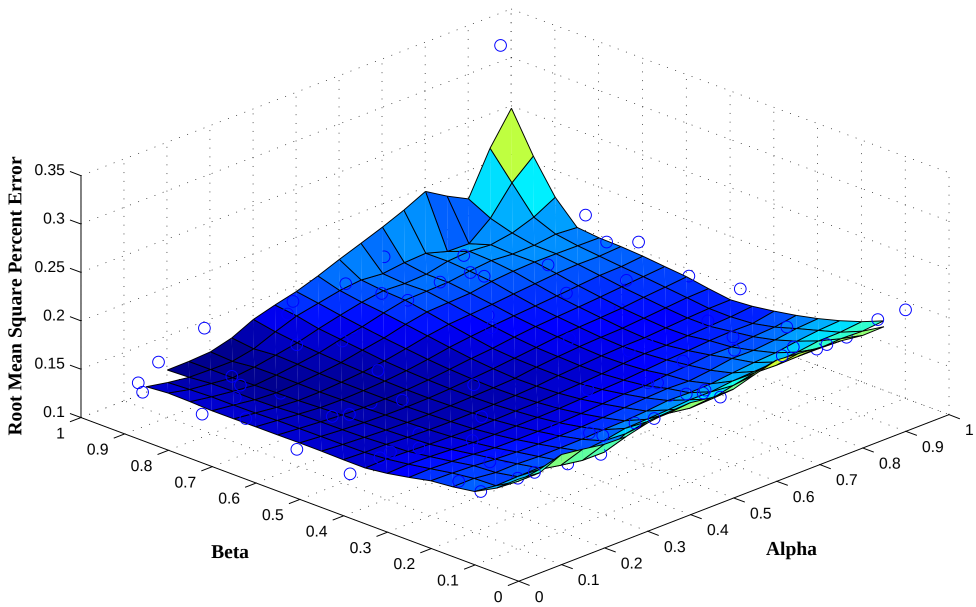
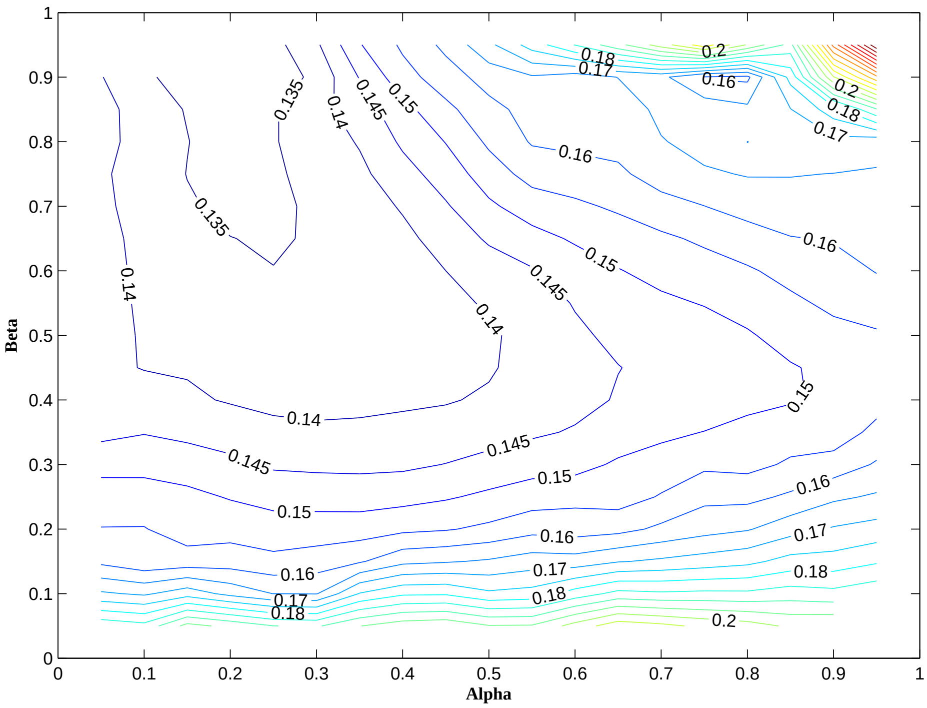


Figure 5.4: Randomly Generated Parameter Pairs

Next, we calculated the RMSPE for each day-to-day assignment result. Figure 5.5(a) shows all the RMSPE values of these 100 assignment results, and Figure 5.5(b) displays the contour map. Comparing the values of RMSPE generated by the day-to-day assignment model under the 100 sampled parameter pairs, we found that the sampled parameters $\hat{\alpha} = 0.3042$ and $\hat{\beta} = 0.9021$ provide the minimal value of the RMSPE. As we can see from Figure 5.5, the value of RMSPE around the minimum is not sensitive to the variation of these parameters. Thus our calibrated parameters are $\hat{\alpha} = 0.3042$ and $\hat{\beta} = 0.9021$.



(a) Root Mean Square Percent Errors Based on 100 Samples



(b) Contour Map of RMSPE

Figure 5.5: Calibration Results for the Model with Prediction Process

5.3 Model Validation

5.3.1 Traffic Flow Evolution on Cordons

Using $\hat{\alpha} = 0.3042$ and $\hat{\beta} = 0.9021$, we ran the day-to-day traffic assignment model over the study peroid from July 30, 2007 to August 31, 2007. Then, we compared the model results with the field observations for the inbound traffic volumes crossing the three cordons previously identified in Section 4.1.2.

Figure 5.6 shows the day-to-day traffic fluctuations from July 30, 2007 to August 31, 2007 on the three cordons. These simulated results look very similar to the actual traffic volume fluctuations presented early in Figure 4.2. In other words, our day-to-day traffic assignment with prediction replicates the traffic recovery process that occurred after the Bridge collapse.

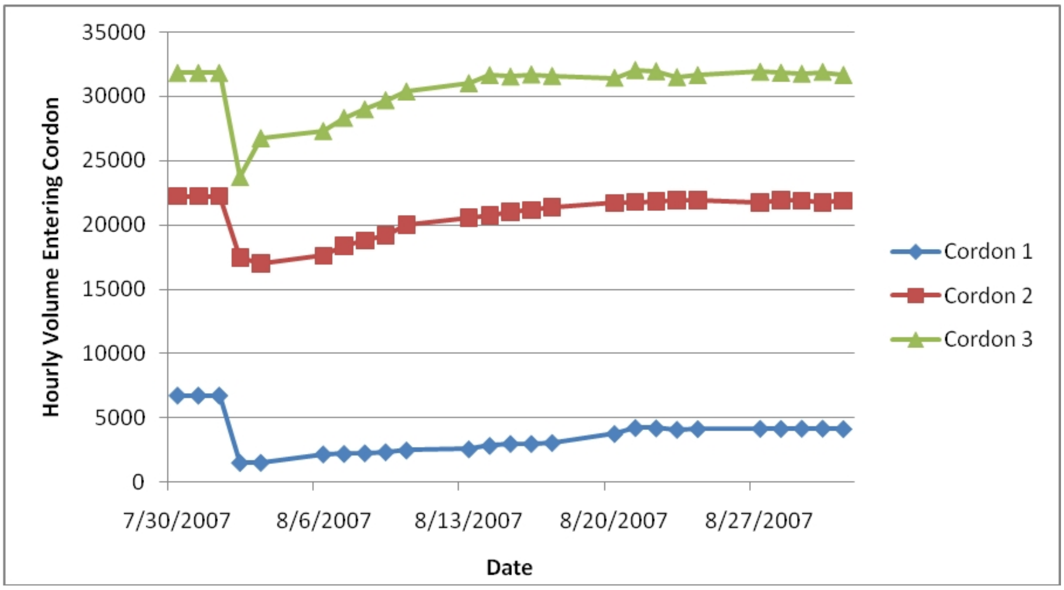


Figure 5.6: Inbound Traffic Volumes Generated by Day-to-Day Traffic Assignment with Prediction

We have argued that modeling traffic flow evolution after network disruption requires accounting for travelers' abilities to make predictions about traffic conditions. To confirm this, we also applied the day-to-day traffic assignment framework without the prediction component (i.e., "correction-only"). We set the prediction parameter λ_t to 0 and repeated the same parameter estimation methodology to calibrate the parameters in the day-to-day assignment model.

The RMSPE values for the 100 assignments generated without prediction are shown by a contour map in Figure 5.7, with the minimal RMSPE at $\hat{\alpha} = 0.2328$ and $\hat{\beta} = 0.9224$. If we compare this contour map with the map in Figure 5.5(b), we see that the minimal

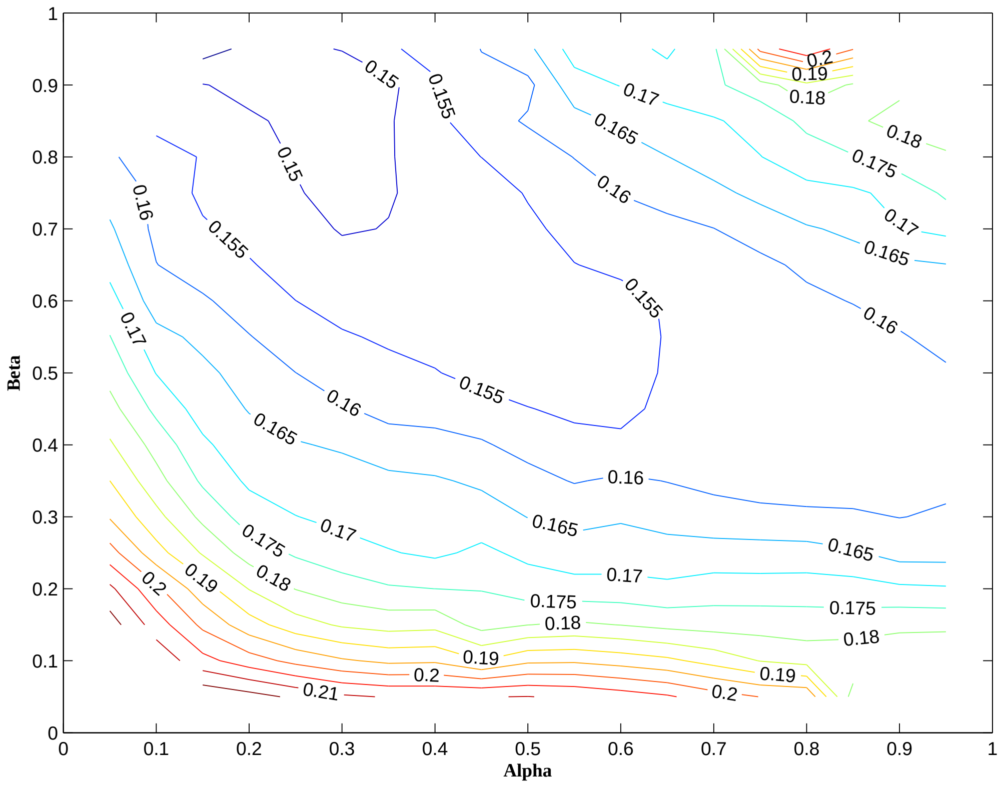


Figure 5.7: Contour Map of RMSPEs for the Model without Prediction Process

value of RMSPE is larger. In other words, the traffic assignment with prediction can better capture the link flow changes than the "correction-only" traffic assignment in the best cases.

We further performed a hypothesis test to confirm the statistical significance of the prediction component in our proposed model. The test of significance was implemented as follows. We first computed the sample mean $\bar{\mu}_1 = 0.1583$ and the sample standard deviation $s_1 = 0.02572$ for the 100 RMSPE values based on the day-to-day assignment model with prediction process; and computed the sample mean $\bar{\mu}_2 = 0.1711$ and the sample standard deviation $s_2 = 0.02029$ for the 100 RMSPE values based on the day-to-day assignment model without prediction process. Our null hypothesis is:

$$H_0 : \mu_1 = \mu_2$$

and alternative hypothesis is

$$H_1 : \mu_1 < \mu_2.$$

If H_0 is true, it means that the existence of the prediction component does not improve the assignment accuracy in terms of traffic flow change rates when comparing with the reality. Since these two sets of assignments are independent and have large sample sizes (100 samples for each model), we employed the $t-$test for testing the statistical significance. The t statistic

$$t = \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sqrt{\frac{s_1^2}{99} + \frac{s_2^2}{99}}} = -3.8876$$

whose absolute value is greater than the critical value for 99% confidence level, i.e., $|t| > z_{0.995} = 2.58$. Therefore, at 99% confidence level, we should reject H_0 and accept H_1. In other words, the prediction component in our proposed model does improve the assignment results with lower values of RMSPE.

We then applied the calibrated model without the prediction component, using estimated parameters $\hat{\alpha} = 0.2328$ and $\hat{\beta} = 0.9224$ and generated daily traffic volumes for the three cordons, shown in Figure 5.8. As can be seen, the traffic recovery process is not obvious in the assignment results, especially on Cordon 3. This confirms our argument about the necessity of prediction component in day-to-day traffic assignment. The final step in our verification process is to compare how closely the traffic volume variation simulated by our models matches actual variations recorded from the field.

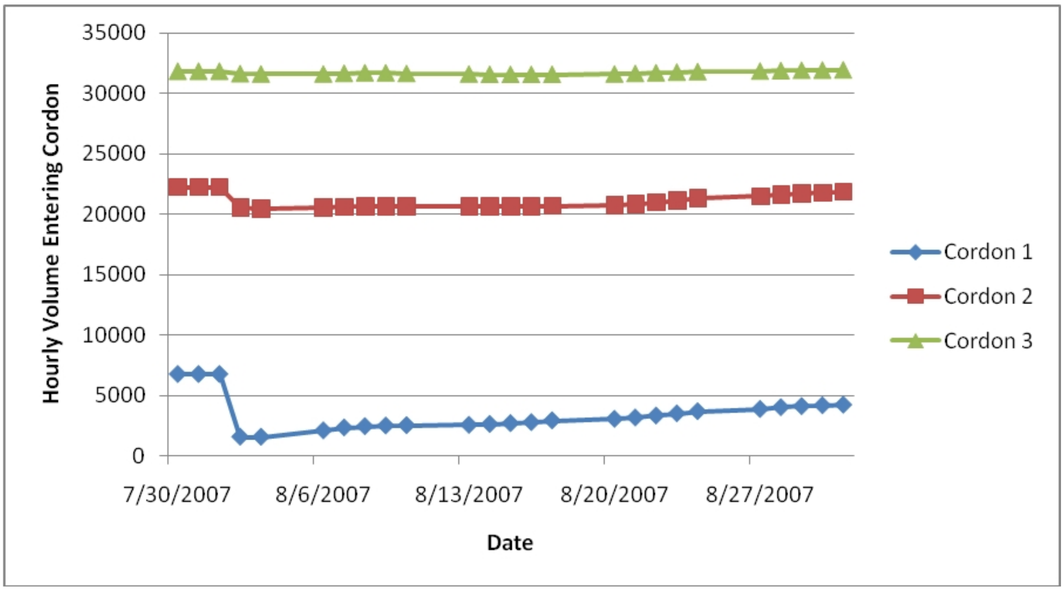
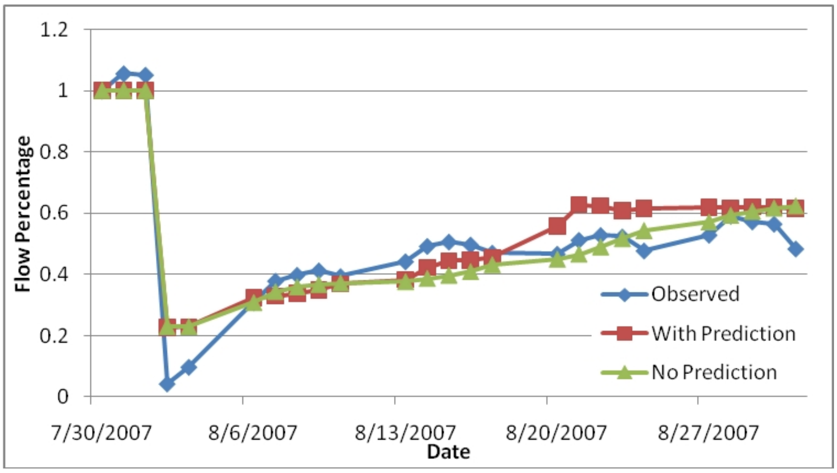


Figure 5.8: Inbound Traffic Volumes Generated by Day-to-Day Traffic Assignment without Prediction

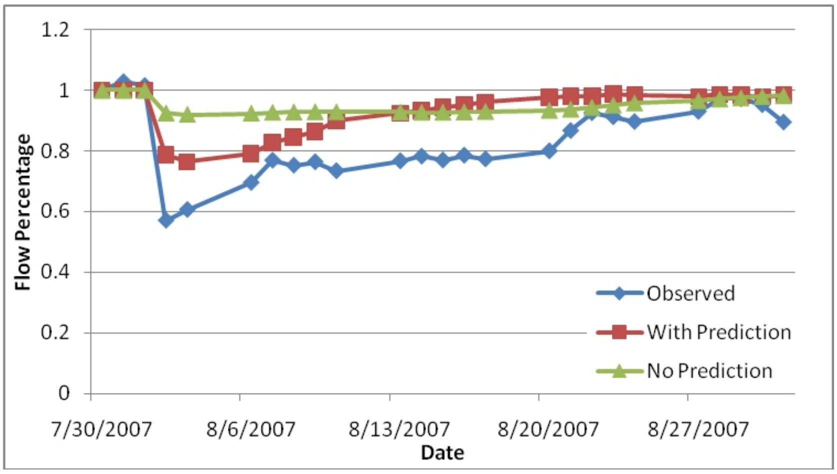
5.3.2 Comparisons

The quality of the proposed day-to-day traffic assignment model can now be shown by comparing the percentage change in traffic volume on cordons with the field observations. To do so, we denote u_i^t as the hourly volume entering cordon i on day t, u_i^0 as the hourly volume entering cordon i on the first day (July 30, 2007), and let $p_i^t = \frac{u_i^t}{u_i^0}$ represent cordon i's volume percentage with respect to the first day's cordon volume. The comparisons of volume percentage changes on cordons, between the proposed models with and without prediction together with the observations, are summarized in Figure 5.9.

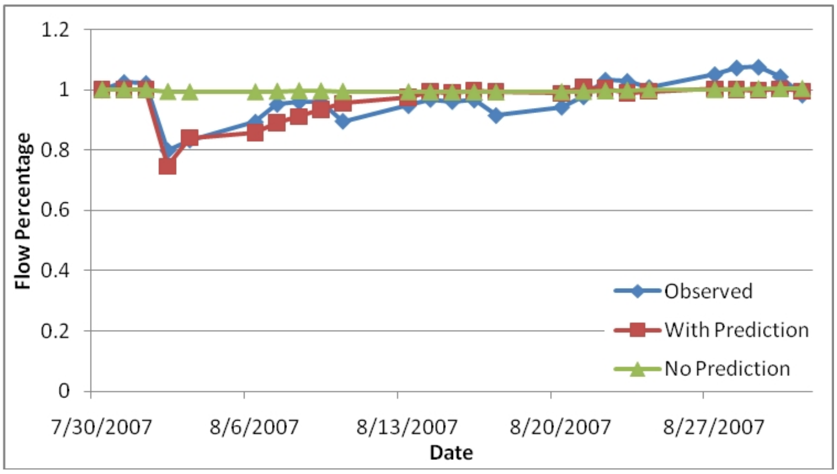
Figure 5.9 shows the percentage change in volume on each cordon for the observed data and the traffic assignment models with and without prediction. For both Cordon 2 and Cordon 3, the percentage changes given by the model with prediction fit quite well with the observations. This demonstrates that the "prediction and correction" structure we proposed for day-to-day traffic assignment has much greater power to capture traffic recovery characteristics after unexpected network disruption then "correction-only" models.



(a) Volume Percentage Changes on Cordon 1



(b) Volume Percentage Changes on Cordon 2



(c) Volume Percentage Changes on Cordon 3

Figure 5.9: Comparisons of Volume Percentage Changes on Cordons

Chapter 6

Conclusions and Future Research

6.1 Summary of This Dissertation

A good estimation of traffic flow evolution is vital for the deployment of effective traffic restoration plans for a disrupted transportation network. Due to the lack of data, very limited attention has been paid to modeling the traffic flow evolution process after network disruption, although many traffic flow evolution processes have been proposed in the literature. This dissertation attempted to bridge the gap between day-to-day traffic flow evolution models and their practical applications, especially under network disruption scenarios.

In this research, we proposed and validated new deterministic day-to-day traffic assignment models to capture the characteristics of traffic evolution after an unexpected network disruption. A review of the existing path-based models revealed two shortcomings, namely the path-flow-nonuniqueness problem and the path-overlapping problem. The first problem exists because the application of path-based models requires a given initial path flow pattern, which is typically unidentifiable in real world scenarios. The second problem arises because these models ignore the interdependence among paths and thus can provide very unreasonable results for networks with paths that overlap.

In view of the difficulty of solving the two problems within the path-based methodology, we proposed a link-based traffic dynamic model. The uniqueness of link flow trajectory is guaranteed by the uniqueness of the "target" flow pattern in the model, due to the strict convexity of the distance function; while the path overlapping problem is solved (indirectly) by minimizing the link flow deviation. Thus, the two path-based problems are effectively avoided by our link-based model.

The proposed link-based traffic dynamic model has good mathematical properties and a close relationship with existing models. It captures travelers' cost-minimization behavior as well as their inertia. In addition, stability analysis shows that the proposed link-based traffic dynamic has the classic Wardrop user equilibrium link flow as the stable point and that it is asymptotically stable under mild conditions. The proposed link-based traffic dynamic is shown to be a Rational Behavior Adjustment Process upon which most existing continuous traffic dynamical systems were built. By replacing the target flow determination subproblem as a variational inequality, we also established the relationship between the link-based model and differential variational inequality, which offers a new paradigm for modeling traffic evolution.

In addition to a continuous-time traffic dynamic model, this dissertation proposed a "prediction-correction" framework for modeling discrete-time deterministic day-to-day traffic evolution. The proposed modeling framework provides the flexibility to accommodate drivers' prediction of traffic conditions in response to a network disruption, as seen from the survey data after the I-35W Bridge collapse. It has been rigorously proven that the proposed discrete-time day-to-day link flow evolution model has the user equilibrium flow as a globally attractive point under mild conditions, although the discrete model is not asymptotically stable due to the inclusion of prediction process.

The convex combination algorithm has been adopted to solve the “target” flow determination subproblem in discrete-time day-to-day assignment model. We also discussed the computational issues in determining the predicted flows and “target” flows when dealing with real-world networks.

The proposed models have been calibrated and validated using field data collected after the I-35W Bridge collapse in Minneapolis, Minnesota. To the best of our knowledge, this is the first time that day-to-day traffic dynamic models have been verified and validated against a real-world disruption scenario. The empirical contributions of this study are extremely valuable for further understanding of traffic flow evolution after an unexpected network disruption.

Finally, we provide traffic management agencies a decision support tool for the selection of traffic restoration projects. Engineers and planners who are developing traffic restoration plans for disrupted networks may apply the results given by the proposed day-to-day traffic dynamic models to study traffic flow changes after disruption. However, we need to be careful when applying the outputs given by the models. First, the models developed in this dissertation are more suitable for unexpected network disruptions. For unexpected events, such as the collapse of I-35W Bridge studied in this dissertation, traveler avoidance of certain areas can be modeled by the prediction process as proposed; while for those planned disruptions, avoidance scenario may not take place, and the resulting traffic dynamics could be different. In addition, although the proposed models consider traveler prediction of route costs which impact route choice decision, other decisions, e.g., destination choice, may impact traffic evolution significantly. This requires researchers and practitioners to continue to collect more detailed individual behavior data and then further improve existing models.

6.2 Future Research

For future research, this research can be extended in several directions. First, we may consider developing day-to-day traffic evolution models without the strong dependence on OD demand information. Existing traffic dynamic models, including the proposed ones in this dissertation, require explicit OD trip information. This requirement has been known to be a main difficulty in transportation travel demand modeling. Although many OD demand estimation methods exist in the literature, the estimated results are also difficult to verify. Full information of OD trips may not be a problem in the future due to advances in data collection. However, it is also possible to develop day-to-day traffic evolution models that do not require full OD demand information, by modeling the competitions and complementarities between links. If link flow correlations and complementarities can be considered in our proposed link-based traffic dynamic models implicitly, then we can capture the redistribution of traffic flows without the strong assumption on OD-demand. A model with less dependence on OD information would improve our understanding of the relationship between travel demand and network topology, which could help engineers and planners develop effective controls or policies.

Second, because our link-based day-to-day dynamic is a relatively new model, it can easily be extended by relaxing some of its underlying assumptions. In this research, we only considered the deterministic day-to-day assignment model with fixed demand and single user class. Developing stochastic models could be a direct extension of this research; demand elasticity after disruption can also be considered in the "prediction-correction" framework; and multi user classes with different route choice preferences can be combined into these models as further extensions. When extending the model to stochastic perception, elastic demand, or multi-user-class, we may describe the link dynamic from different perspectives. For example, for a stochastic model we could construct the "target" flow by determining link-chosen probability instead of balancing cost minimization and inertia, as we did. Such realistic relaxations of underlying assumptions will be helpful for modeling real-world traffic dynamics after unexpected network disruption.

Third, the stability properties of the discrete-time assignment model in this study deserve further analysis. The attractiveness property explained in Chapter 4 has been restricted by letting $\alpha = 1$. A stability analysis is recommended to accommodate the more general exponential-smoothing perception update process. Furthermore, when the proposed link-based model is extended to take into account stochasticity, demand elasticity, or multi-user-class, the stability properties of these new models need to be carefully studied as well, since these properties may critically impact applicability of the models. Establishing the stability properties for these models will be challenging, especially when the prediction component is considered. However, without a good stability property, the traffic evolution models might be sensitive to perturbation and would not provide useful information in practice.

Fourth, since the traffic flow patterns predicted in the prediction process can have various representations, under different perception assumptions, all of these need empirical verifications. For example, the anticipation of traffic congestion can be assumed to be a function of the severity of the disruption and the distance from the disruption location. Although different prediction processes will result in different traffic evolution trajectories, we have already proven that the trajectories following the proposed "prediction-correction" structure will converge to user equilibrium under the given sufficient conditions.

Finally, it is possible to formulate a day-to-day network design model to help traffic engineers develop effective traffic restoration projects, incorporating the proposed "prediction-correction" traffic assignment approach. The DVI approach is a suitable methodology to model and solve dynamic network design problems, since the DVI formulation offers distinct advantages in the modeling of traffic evolution characteristics. By applying the DVI formulation shown in Section 3.5.2, link flow dynamic $\dot{\mathbf{x}}(t)$ becomes a function of current time t traffic state $\mathbf{x}(t)$ and external dynamic $\mathbf{u}(t)$. When the external dynamic $\mathbf{u}(t)$ is considered as a set of designs, then a day-to-day network design model could be established by introducing system-wide objectives. In addition, existing disequilibrium network designs were exclusively considered on continuous designs, due to the ease of dealing with continuous variables. In the future, we need methods to prioritize and schedule network designs for traffic restoration after disruption, which should be considered as discrete disequilibrium network designs.

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Appendix: Acronyms

AON	All-Or-Nothing
BPR	Bureau of Public Roads
DVI	Differential Variational Inequality
ITS	Intelligent Transport Systems
LEHD	Longitudinal Employer-Household Dynamics
LP	Linear Programming
NCP	Nonlinear Complementarity Problem
NLP	Non-Linear Programming
NTA	Network Tatonnement Adjustment
OD	Origin-Destination
ODE	Ordinary Differential Equation
PAP	Proportional-switch Adjustment Process
PDS	Projected Dynamical System
QP	Quadratic Program
RBAP	Rational Behavior Adjustment Process
RMSPE	Root Mean Square Percent Error
SUE	Stochastic User Equilibrium
TAP	Traffic Assignment Problem
TAZ	Traffic Analysis Zone
UE	User Equilibrium
VI	Variational Inequality