Logical Basis of Dimensionality

David J. Krus
Arizona State University

The isolation of dimensions from a data matrix has been traditionally formulated in terms of an algebraic or geometric model. Order analysis was developed as a method of multidimensional analysis and scaling based on the theory of Boolean algebra. The order analytic algorithm utilizes functions of the propositional calculus in lieu of eigenvalues and eigenvectors of the general linear model. Also, the graphic presentation of latent space in coordinates of the Euclidian space is paralleled in ordering-theoretic models by dendrograms of the test space. A conceptual outline of order analysis is presented, followed by an empirical comparison of factor and order analysis solutions of a sample data problem. Resulting factor and order analytic structures are evaluated in terms of meeting criteria of simple structure and correct reflection of broad cognitive categories. In addition, the relations of proximity and dominance are discussed from the perspectives of both Cartesian and Leibnitzian theories of dimensionality as pertaining to problems of multivariate analysis and scaling.

Mathematicians of ancient Greece visualized a variable as a length of some line segment, the product of two variables as an area of a plane, and the product of three variables as a volume of some object. The analytic geometry of Fermat and Descartes provided for geometric representations not only for cubics, but also for equations of higher degree. Most contemporary models of multivariate analysis were derived from postulates of Cartesian geometry describing combinations of subspaces within a hyperspace. In psychology, the hyperspaces most frequently analyzed correspond to cognitive structures of an individual or a group of individuals.

There has been a parallel development in attempts to describe cognitive structures along the lines of formal logic. Thus Leibnitz's (1677/1951) *characteristica universalis* and an algebra of reasoning (*calculus ratiocinator*) represented early attempts to develop a formal model of human cognitive space. Leibnitz's ideas experienced a rebirth in the writings of Augustus De Morgan (1847/1968) and George Boole (1854/1958); they were further elaborated by Russell and Whitehead (1910/1925) and by numerous contemporary logicians. In Russell's writings, Leibnitz's first axiom of transitivity (*A* contains *B* and *B* contains *C*; therefore *A* contains *C*; Leibnitz, 1679/1951, p. 26) was employed as a basis for the definition of an order relation. The order relation also constituted a foundation of the dimensionality concept within the formal logic framework. Discussing the properties of an order relation, Russell (1919/1971, p. 29) concluded that "dimensions . . . are a development of order" and that "order depends upon transitive asymmetrical relations" (Russell, 1903/1938, p. 219).

APPLIED PSYCHOLOGICAL MEASUREMENT
Vol. 2, No. 3 Summer 1978 pp. 321–329

Some implications of Leibnitz's first axiom for the theory of dimensionality, as developed within the area of psychological measurement, can be considered. In formal logic, the implication function (→) is defined as false if, and only if, the conclusion is false. The conjunctive function (&) is defined as true if, and only if, both arguments are true. Applying these two functions to a binary data matrix containing all possible response patterns for three items, the implicative part of Leibnitz's first axiom can be written as in Table 1. The data matrix (first three columns) was evaluated by the implicative functions in columns 4 and 6, and the overall truth value of the implicative chain was evaluated by the conjunctive function in column 5. The response pattern compatible with the proposed logical structure (i.e., having the truth values in column 5 equal to one) resulted in a response pattern, as shown in the last three columns of Table 1. This response pattern also underlies a unidimensional Guttman scale (Guttman, 1941).

A recent revival of interest in the formal logic-based methods of data analysis can be observed (e.g., Ducamp & Falmagne, 1969; Airasian & Bart, 1973; van Leeuwe, 1974; Cliff, 1977). Among various methods developed within this context, order analysis (Krus & Bart, 1974; Krus, 1977) attempts to isolate the logical orders among variables in a hyperspace and to model formal structures in a way analogous to cognitive structuring of data by a set of primitive logical processes.

The computational algorithm of the probabilistic model of order analysis has been described elsewhere (Krus, 1977). The purpose of the present article is to present an illustration of the procedure on a set of real data and a comparison with factor analysis, the main representative of the analytic-geometry-based series of models for multivariate data analysis.

Table 1
Construction of Unidimensional Response Pattern, Compatible with the Order Relation
(A→B) & (B→C)

Subjects	Data Matrix			Logical Structure			Unidimensional Scale		
	A	B	C	(A→B)	&	(B→C)	A	B	C
a	1	1	1	1	1	1	1	1	1
b	1	1	0	1	0	0			
c	1	0	1	0	0	1			
d	1	0	0	0	0	1			
e	0	1	1	1	1	1	0	1	1
f	0	1	0	1	0	0			
g	0	0	1	1	1	1	0	0	1
h	0	0	0	1	1	1	0	0	0

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Method

A checklist labeled the Marital Adjustment Inventory was constructed from items developed by Knapp (1960) for his Conscience scale. This checklist of concepts (see Tables 2 and 3) was administered to 33 subjects participating in a marital counseling program (mean age 26 years, SD=6.2).

The subjects were administered the Marital Adjustment Inventory with the following instructions:

> "Below are a number of images which might be used to describe your spouse. Please indicate, by pressing the 1 or 0 key, the capacity of those images to describe some of your feelings toward your marital partner."

The check list was administered via a computer CRT terminal with responses directly recorded on the mass data storage device.

Results

As a first step in the analysis, the binary data matrix of responses to the Marital Adjustment Inventory was intercorrelated using phi coefficients. The resulting correlation matrix was factor analyzed by the principal factors method. Communalities were estimated by Roff's (1936) method of squared multiple correlations; the initial estimate was improved by iterations. The number of factors to be extracted was determined from a preliminary principal components solution by Kaiser's (1960) rule. The factor matrix was rotated by Kaiser's Varimax (1958), as reported in Table 2. Eight retained eigenvectors of the preliminary principal components solution accounted for 80% of the total variation of the correlation matrix. The final eigenvalues of the principal factor solution ranged from 7.06 to .69, accounting for 39.7, 16.8, 12.8, 9.3, 6.7, 6.1, 4.7, and 3.9% of variance of the reduced variance space of the correlation matrix.

Inspection of Table 2 reveals two factors indexing positive and negative feelings toward one's spouse; the third factor suggests protective and stability feelings; and the remaining five factors are open to varied interpretations, which will not be attempted here.

The parallel order analysis of the same data was commenced in the second step by computing the dominance matrices for both the attributes (rated concepts) and entities (subjects) of the data matrix. The analysis of the dendrogram adjacent to the dominance matrix for entities allowed for isolation of eight dimensions, accounting for 27.3, 18.2, 18.2, 12.1, 9.1, 6.1, 6.1, and 3.0% of the variance of their corresponding dominance matrix.

Reconstruction of the total logical space was attempted at the α = .70 level[1], resulting in reduction of the logical space to five dimensions (as reported in Table 3). As in factor analysis, this solution was rotated by Kaiser's Varimax method. Moreover, the matrix of order loadings was rescaled relative to the point of highest information density. Thus, when inspecting Table 3, it is important to keep in mind that order loadings are only structural analogues of factor loadings. For example, the order loading of 1.00 for the rated concept of "lighthouse" on the first dimension indexes the highest information density point (isolated by order analysis as several thousand bits); this does not mean that 100% of variance was accounted for by this variable, as would be signified by a factor loading.

Interpretation of the resulting dimensions was attempted as follows: The first dimension indexed the positive, and the second dimension the negative, feelings toward the marital partner. The third dimension suggested the feelings of being tied down and captured by the spouse, while the fourth dimension hinged on feelings of marital security and protection. As in the case of

[1]Order analysis at a specific α level selects for analysis only order with probability equal to or greater than α of being in a certain direction. Thus, for example, order analysis at the .70 level includes in the analysis orders with probabilities greater than or equal to .70 to extend in the direction suggested by branches of their corresponding dendrogram (cf. Krus, 1977).

Table 2
Factor Analytic Structure of the
Marital Adjustment Inventory

Concept	Factor 1	2	3	4	5	6	7	8
straight jacket	.821	-.181	.136	-.072	.227	.021	.167	-.097
threatening father	.817	-.229	-.127	.104	.196	-.047	-.067	-.088
whipping post	.794	.131	.020	-.044	.491	-.201	.128	.158
vicious bully	.777	.073	-.241	-.017	-.131	.114	.025	.035
scolding mother	.587	-.176	-.142	-.030	-.014	-.110	.185	-.079
hampering burden	.526	-.353	.158	-.151	.070	.071	.351	-.504
dam in a river	.518	-.321	-.349	.079	-.053	.171	-.081	.023
swarm of flies	.402	-.523	-.100	-.043	.287	.195	.353	.038
tedious sermon	.178	-.786	-.125	.034	.132	-.051	.258	-.012
generous provider	-.049	.781	-.103	.046	.061	.125	-.053	.053
treasured book	.141	.635	.222	.075	-.103	.226	.010	.127
lighthouse	-.246	.586	.398	.132	-.061	.088	.033	.307
uncomfortable bed	.251	-.564	.100	-.257	-.377	.076	.418	-.011
probing searchlight	-.218	.383	.052	.359	-.178	-.044	.185	-.112
harbor buoy	.001	.095	.751	.438	.038	.023	-.039	.064
compass needle	-.266	.073	.714	.359	.026	.242	.261	.097
protective armour	-.083	-.085	.706	-.025	-.132	.069	-.066	.099
just judge	-.089	.428	.501	-.164	-.113	.117	.002	.061
pillar of a temple	.072	.026	.085	.775	.069	-.006	.039	-.086
accurate compass	-.345	.264	.444	.544	.019	.193	-.237	-.123
buried splinter	.184	-.126	-.045	-.060	.859	.067	.237	.024
secret map	-.017	-.039	.069	.213	-.390	.328	.137	.174
hidden lamp	-.008	.212	.213	.017	.019	.943	-.044	-.048
entoiling net	.250	-.372	-.030	.131	.218	-.060	.685	-.129
secure fortress	-.014	.074	.231	-.144	.015	.004	-.040	.846

Table 3
Order Analytic Structure of the Marital Adjustment Inventory

Concept	Dimension 1	2	3	4	5
lighthouse	1.000	.169	-.104	.253	.106
probing searchlight	.904	.104	.501	.220	.208
harbor buoy	.871	.093	.080	.215	.026
compass needle	.856	.029	.020	.310	.141
secret map	.461	.137	.279	.389	.181
hidden lamp	.405	.105	.095	.387	.343
accurate compass	.394	.072	.104	.017	.026
treasured book	.390	.067	.113	.061	.081
generous provider	.327	.055	.139	-.005	.000
threatening father	.087	.398	.265	-.032	.067
uncomfortable bed	.111	.367	.021	.183	.297
strait jacket	.248	.351	.038	.025	.142
whipping post	.043	.287	.074	.014	.039
scolding mother	.158	.272	.022	.067	.191
buried splinter	.035	.261	.076	.040	.028
vicious bully	-.013	.234	-.005	.021	.016
pillar of a temple	.293	.101	.578	.090	.040
tedious sermon	.250	.433	.462	.295	.387
dam in a river	.113	.339	.378	.096	.165
entoiling net	.059	.167	.343	.068	.221
protective armour	.094	-.033	.077	.291	.160
secure fortress	.066	.057	.003	.210	.022
just judge	.141	-.001	.051	.166	.050
hampering burden	.082	.267	.243	.241	.494
swarm of flies	.040	.093	.035	.063	.132

factor analysis, interpretation of the fifth dimension based on only two items was not attempted.

As observed in these analyses, matrices of factor and order loadings revealed considerable structural similarities. Both solutions conformed to the majority of Thurstone's (1947, pp. 319–346) criteria for simple structure. The complexity of each factor or dimension was less than the complexity of the analyzed data matrix as a whole; and the rows and columns of both factor and order matrices followed a pattern of high vs. low factor and order loadings, indicating presence of sets of linearly independent variables and underlying simple orders.

Of interest is the percentage of variance accounted for by the eigenvalues of the factor analytic solution and by the extraction indices of order analysis. Although both methods originally suggested the presence of eight dimensions, the highest order analytic index of extraction was 27.3%, while the highest eigenvalue of the factor analytic solution accounted for 39.7% of variance. The more equally distributed variance of the order analytic solution suggests the presence of an oblique simple order, as compared with the orthogonal simple structure of the factor analytic solution.

Inspection of order loadings shows that nearly all order loadings were positive and, therefore, that all dimensions were defined by a positive hyperplane. This finding, together with the distribution of the extraction indices, indicated the presence of the oblique positive manifold. On the other hand, the frequent occurrence of negative factor loadings in the factor analytic solution indicated the absence of the orthogonal positive manifold, which presents some problems with the interpretation of factors (e.g., in Factor 5, what does the opposite of a "secret map" stand for?).

Discussion

To date, several comparisons of factor and order analytic structures have been attempted. Parallel factor and order analysis of Thurstone's (1947, pp. 140–143) "box problem" suggested that the results of these methods are nearly identical. The largest discrepancies between order and factor analytic solutions were found on reanalysis of Armstrong and Soelberg's (1968) "random data experiment" in which the resulting structures were not comparable at all. In the latter case, discrepancies between the analyses were ascribed to inferential routines used by order analysis (cf. Krus & Weiss, 1976).

Results of the present analysis suggest partial comparability of order and factor analytic solutions. The question which arises is whether or not the factor and order analytic structures are comparable at all; and if they are, why are there variations between the particular comparisons?

An immediate question in this respect pertains to the orthogonal vs. oblique types of extracted dimensions obtained in the present analyses. In a typical factor analytic solution, the extracted principal components (or factors) are orthogonal and are rotated into an orthogonal or oblique structure, depending upon the researcher's perception of the structural requirements of the solution. In order analysis, the extracted dimensions are intrinsically neither orthogonal nor oblique. Since the dimensionality of a data set is determined in order analysis on the basis of connected, asymmetric, and transitive relations as observed among the data elements, the order analytic solution can be either orthogonal or oblique, depending on the character of the data analyzed. Due to the prevalence of "positive manifold" data sets in social research, most order analytic solutions originally will be oblique. However, a researcher can stress the orthogonality or obliqueness of the final solution by selection of an appropriate method of rotation. As Harman commented, (1967, p. 314) "an orthogonal (or near-orthogonal) solution may result out of the more general oblique conditions if, in fact, the 'best' solution should tend toward orthogonality" and, as can be reasoned, vice-versa.

While both factor and order analysis purport to analyze data structures, factor analysis typically analyzes structures based on matrices of coefficients of correlation, while order analysis was designed primarily for analysis of matrices of dominance coefficients. For the binary case, let the frequencies of response patterns between two variables, X and Y, be written as a, b, c, and d, where a, b, c, and d stand for a frequency of the (1,1), (0,0), (1,0), and (0,1) response patterns, respectively. Using this notation, Yule's (1907) formula for the product-moment coefficient of correlation in a binary form can be written as

$$\phi = \frac{ab-cd}{[(a+d)(b+c)(c+a)(d+b)]^{\frac{1}{2}}} \qquad [1]$$

Using the same notation, the formula used in the probabilistic model of order analysis (Krus, 1977, p. 589) for estimation of dominance coefficients can be written as

$$\delta = \frac{c-d}{(c+d)^{\frac{1}{2}}} \qquad [2]$$

Unlike the phi coefficient, the delta coefficient uses only (1,0) and (0,1) response pattern frequencies.

While this difference in itself seems to preclude comparisons of factor and order analysis, one has only to recall the practice of reflection of variables in Burt's (1917) centroid method of factor analysis to realize the structural relativity of the a, b, c, and d response patterns. Since the reflection of a variable in the origin is accomplished merely by changing the signs of the correlations of that variable in the correlation matrix, in the case of binary variables this is equivalent to changing frequencies of a's and b's to frequencies of c's and d's and vice-versa.

This relative structural fluidity of a, b, c, and d types of response patterns poses one of the major problems for interpretation of comparisons between factor analysis and order analysis. On the semantic level, a and b response patterns (1,1 and 0,0) can be labeled as a proximity type; and c and d (1,0 and 0,1), as a dominance type (cf. Coombs, 1964/1976, pp.515–520). In factor analysis, the difference between proximity and dominance relational types is formed (as by Equation 1); and the resulting factorial dimension is interpreted in terms of intercluster proximities of the analyzed variables, independently of whether these proximities were generated by relations of proximity, dominance, or both.

In order analysis, a direction hypothesis is formed, and the data matrix is analyzed with respect to dominance relations only. Thus, order analysis can be either similar or dissimilar to factor analysis, depending on the character of the data analyzed.

The above observations have relevance for the theory of dimensionality. According to Harman (1967, p. 4), "the principal concern of factor analysis is the resolution of a set of variables linearly in terms of (usually) a small number of categories or 'factors.'" The categorization problem implies the search for data proximities at the factor extraction level. Nevertheless, the classification of factor analysis into a "proximity type" of model would not be quite correct, since on the item level both relationships of proximity and dominance are taken into consideration. These relations are, however, not explicitly recognized and are treated in an order-independent (i.e., nondirectional) way. The conceptual differences between dimensions as defined by a factor analytic solution and an order analytic one thus can be summarized as follows: The dimensions derived by factor analysis are based on both proximity and dominance relations at the item level and only on proximity relations at the factor extraction level. The dimensions extracted by order analysis are based solely on dominance relations at both item and factor extraction levels.

The Cartesian theory of dimensionality (defined in terms of geometric distances between points in the test space), as well as the Leibnitzian theory of dimensionality (defined in terms of order-generative connected, transitive, and asymmetric relations), could be complemented on the basis of these findings. This could potentially result in construction of more powerful methods for multivariate analysis and scaling. This type of theoretical inference would necessitate the explicit recognition of order-dependent and order-independent relations of proximity and dominance and, among other things, the development of a method, analogous to Pearson's (1901) method of principal axes, pertaining to extraction of dimensions from structures based on order-dependent dominance relations. A preliminary algorithm for a proximity-dominance type of order analysis based on the above theoretical considerations was recently proposed by Krus (1978).

The comparative analysis of Leibnitzian and Cartesian theories of dimensionality, as approached here by a series of factor and order analytic comparisons, seems to bear relevance to both theoretical and practical problems in the area of joint multivariate analysis and scaling. Only a few facets of this comparison were addressed in the present discussion. Further theoretical and experimental work in this area is needed to bring forth discoveries of new relationships and formulations of new measurement algorithms.

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Author's Address

David J. Krus, 302 Payne Hall, Arizona State University, Tempe, AZ 85281