Effects of Extra Degrees of Freedom on Decay Rate of Metastable Topological Objects A DISSERTATION

SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Adviser Dr. Mikhail Voloshin

August 2010

Acknowledgments

I am grateful to my adviser Dr. M.B. Voloshin for permanent concern and support, elucidating and interesting discussions, and patience in answering my questions.

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Introduction

1.1 Where the problem comes from

Everyone is familiar with decays of particles: whether it is an elementary particle like muon, or it is the composite one like neutron. One can even pose the question of stability of particle-like solitons (solutions of classical equations of motion) like magnetic monopole or dyon (field configuration which possesses both magnetic and electric charges). All these objects in certain circumstances can be described as point-like ones (0-dimensional), therefore the only parameter in this case characterizing them is the mass, while the dynamics is defined by the action which is the worldline length in Euclidean space-time.

The natural generalization is to consider 1(or even more)-dimensional objects [1]. In the case of 1-dimension the object is called string. It is somewhat similar to a regular guitar string but differs in that its dynamics and its mass are determined by only one parameter the tension of the string. The Euclidean action is the area spanned by the string when moving.

The string-like configuration can be found in different parts of physics. For example, in theoretical models of non-Abelian dynamics: such as QCD strings, which are responsible for the confinement of quarks, providing the potential which grows with distance, or as topological defects in models with spontaneously broken gauge or global symmetry. The condition for spontaneously broken theory ($G \to H$) to support the existence of the string is that the vacuum manifold should be non-simply connected, meaning that its fundamental group is non-zero

$$\pi_1(M_{vac}) = \pi_1\left(G/H\right) \neq 0 \qquad (1.1)$$

The most renowned example of such string configuration is the ANO (Abrikosov-Nilsen-Olesen) string (the vertex in $2+1$ dimensions). The ANO vortices appear, for instance, when studying superconductors. Another example of linear topological defects is the cosmic strings [2, 3]. They can be formed in the very early universe during the phase transition. These strings are formed at the very large mass scale and can provide a possible explanation for the existence of large structures in the universe: galaxies and clusters. Closed loops of cosmic strings can serve as a seeds attracting the particles to form the structure.

Analogously one can consider 2-dimensional objects which naturally called walls. The examples are the well-known D-branes or axion wall [4, 5, 6].

Topological configurations[1] of fields with the geometry of a string or a wall are classically stable, because there is no continuous classical trajectory which takes from one topological solution into another with different topological number. But it may be the case that a classically forbidden trajectory for such deformations exists and such configurations are unstable quasiclassically and decay due to quantum tunneling. As an illustration of such decay process one can consider the example of a metastable wall investigated in Refs. [7, 8] which arises in a model with a scalar field potential shown in Fig.1.1. The wall solution corresponds to the interpolation between the same vacuum state at two spatial infinities, e.g. at $z=-\infty$ and $z=+\infty$ with the field winding around the 'peg' in the potential. Such configuration is classically stable in the sense that the solution with certain winding number can not evolve classically to a solution with different topological number. However, such a transition can proceed either due to a thermal fluctuation, where the path, corresponding to the solution, is lifted over the barrier or due to quantum tunneling (the motion in a classically forbidden region).

The topological configurations can be metastable with respect to either a complete breaking, or to a transition to an object of lower tension emerging instead of the initial one [9, 10, 11, 12].

[1] The configurations whose properties do not depend on small variations of fields but rather on geometry or topology of a system.

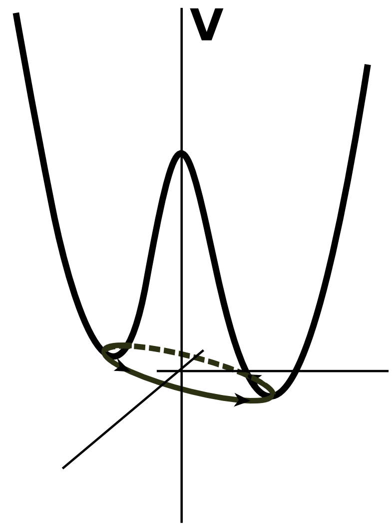

Figure 1.1: Potential

Consider the QCD string stretched between two heavy quarks. The energy of the string is increasing with the distance between the quarks. Therefore at some point it is energetically more favorable to break the string with the formation of light quarks (in order to preserve the flux).

The general mechanism for the decay of string-like topological defects is as follows. Consider the theory with spontaneous symmetry breaking at two sufficiently different scales $SU(2) \to U(1) \to 1$. The $U(1)$ low energy effective theory satisfies the mentioned before condition (1.1). Therefore it possesses a string-like solution, which is stable. However, $SU(2)$ theory (since $\pi_1(SU(2)) = 0$) doesn't have solutions of the type. Hence, the string found at low energy is metastable from the microscopic theory point of view.

ANO type strings can break with the formation of a monopole-antimonopole pair at the ends. The infinite cosmic strings due to the conservation of the topological charge (winding number) are stable. However, the closed loops of the cosmic strings are not protected by the argument and can decay [2, 3]. Even the fundamental strings can undergo the splitting [13]. It is more favorable for a long string to split, and the split can occur at any point, therefore the expected rate is proportional to the length of the string. As one can see the metastable strings or walls appear in many different contexts, therefore it is important to find the rate of the breaking.

Generally a string can decay either by emitting light particles or by splitting into two strings. For example, the dominant decay channel for the loops of cosmic string is trough the emission of the gravitational radiation. In the situation of breaking into two long strings, the two outgoing strings are highly excited string states which admit a classical description. The decay rate in this case can be estimated by semiclassical arguments. In the present study we investigate the breaking of the string and wall using the quasiclassical approximation.

In the next section we describe the similar processes and outline the whole study presented.

1.2 Quasiclassical treatment

The class of problems involving spontaneous and induced semiclassical processes with quantum fields invariably presents an interesting case study in nonperturbative calculations. The most discussed examples of such processes are the Schwinger e^+e^- pair creation in an electric field [14, 15, 16], the decay of false vacuum [17, 18, 19] and the breakup of metastable strings and walls [20]. A semiclassical treatment of such processes can be formulated in terms of a configuration in the Euclidean space time, which is a solution to classical field equations and is called the bounce [18, 21, 22, 23]. The considered processes can either proceed spontaneously or be induced by the presence of particles and by their collisions, such as in the false vacuum decay [24, 25, 26], in the decay of metastable strings and walls [20, 27], and in the photon-stimulated Schwinger process [28, 29]. In terms of a Euclidean-space treatment the case of induced processes requires one to consider the interaction of the particles with the semiclassical configuration.

A metastable wall decay is analogous to the well-known spontaneous false vacuum decay process [17, 18, 30] in $2+1$ dimensions. Indeed, if a hole of area $\mathcal{A}_c$ is created in a wall with tension ϵ, the gain in the energy is $\epsilon\,\mathcal{A}_c$. The barrier that inhibits the process is created by the energy $\sigma\mathcal{P}$, with $\mathcal{P}$ being the perimeter of the hole and σ being a tension associated with the interface. Thus, the 'area' energy gain exceeds the barrier energy only starting from a critical size of the hole created, i.e. starting with a round hole of radius $R_w = 2\mu/\epsilon$. Once a critical bubble of the lower phase has nucleated due to tunneling, it expands, converting the metastable wall. Therefore the probability of the transition is given by the rate of nucleation of the critical holes in the wall.

The same argument also applies to the string decay which is similar to a metastable vacuum decay process in $1+1$ dimensions. The critical size of the gap in the initial string is then $\ell_c = 2R_s = \mu/\varepsilon$. Moreover due to the geometry of the problem such decay also bears a great similarity to the well known Schwinger process of production of pair of charged particles by an external electromagnetic field [14].

Despite the similarity with the false vacuum decay the breaking of the strings and walls involves an essential difference associated with the transverse waves propagating on the latter objects, corresponding to massless bosons, which in fact are the Goldstone bosons of the spontaneously broken translational invariance. The effect of the soft modes of the field of these bosons enters the probability of breaking the topological defects at the preexponential level [8, 12].

The process of a metastable string transition from a state with tension ε_1 to a state with tension ε_2 was considered in [9, 10, 11]. Using the analogy with the false vacuum decay the decay rate was found with exponential accuracy in terms of the rate of nucleation of critical gaps per unit length of the string, $\gamma_s = d\Gamma/d\ell$ [2],

$$\gamma_s = C \, \exp\left(-\frac{\pi \, \mu^2}{\varepsilon_1 - \varepsilon_2}\right) \;, \tag{1.2}$$

where μ is the mass associated with the interface between the two phases of the string. The decay rate of an axion wall was considered at the level of the semiclassical exponent in Ref. [7] by adapting the expression for metastable vacuum decay in $2+1$ dimensions.

The preexponential factors for the spontaneous decay of string and walls were calculated in [8, 12]. For a string the rate of the transition between the states with tension ε_1 and ε_2 was found to be

$$\gamma_s = \frac{\varepsilon_1 - \varepsilon_2}{2\pi} \, \left[F\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\right]^{d-2} \, \exp\left(-\frac{\pi \, \mu_R^2}{\varepsilon_1 - \varepsilon_2}\right) \;, \tag{1.3}$$

[2] It should be mentioned that in $d = 2$ there is no transverse motion for the string, and the preexponential factor can be also copied from the known result of spontaneous metastable vacuum decay

$$C_{d=2} = \frac{\varepsilon_1 - \varepsilon_2}{2\pi} \;.$$

with the dimensionless factor F given by

$$F\left(\frac{\varepsilon_2}{\varepsilon_1}\right) = \sqrt{\frac{\varepsilon_1+\varepsilon_2}{2\,\varepsilon_1}}\;\Gamma\!\left(\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1-\varepsilon_2}+1\right)\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)^{\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1-\varepsilon_2}}\exp\left(\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1-\varepsilon_2}\right)\left(2\pi\,\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1-\varepsilon_2}\right)^{-1/2}, \tag{1.4}$$

while the decay rate of a wall is formulated in terms of the rate γ_w of nucleation of critical holes per unit area, $\gamma_w = d\Gamma/dA$,

$$\gamma_w = \frac{\tilde{\mathcal{C}}}{\epsilon^{7/3}}\,\exp\left(-\frac{16\,\pi\,\sigma_R^3}{3\,\epsilon^2}\right)\,, \tag{1.5}$$

where μ_R and σ_R are renormalized mass and tension parameters associated with the interface of the strings and walls respectively, and $\tilde{\mathcal{C}}$ is a constant that does not depend on ϵ but does depend on other dimensional parameters in the underlying field theory.

The thermal effects in the decay of metastable strings [31] further expose the difference from the false vacuum decay and the Schwinger process. Namely, as long as the temperature T is lower than the inverse of the critical length, $T < 1/\ell_c$ the thermal effects in the latter processes are exponentially suppressed as $\exp(-m/T)$ with m being the lowest scale for particle masses in the theory, and these effects are very small due to the strongly suppressed presence of massive particles in the thermal equilibrium[32]. In the case of a string, however, the transverse waves on the string are massless so that their excitation has no suppression by the mass at arbitrarily low temperature. The thermal excitations of these waves create fluctuations in the distribution of the energy in the string which catalyze the nucleation of the critical gap. Clearly, at $T \ll 1/\ell_c$ the typical wavelength of the thermal waves $\sim 1/T$ is large in comparison with the critical length ℓ_c, and the thermal effect in the rate is quite small, although not exponentially small. The leading low temperature correction in the nucleation rate is given by the thermal catalysis factor[31]

$$\mathcal{K}_s = 1 + (d-2)\,\frac{\pi^8}{450}\left(\frac{\varepsilon_1-\varepsilon_2}{3\varepsilon_1-\varepsilon_2}\right)^2\left(\frac{\ell_c T}{2}\right)^8 + O\left[(\ell_c T)^{12}\right]\,, \tag{1.6}$$

while as T approaches $1/\ell_c$, the catalysis factor develops a singularity at $\ell_c T = 1$. At still higher temperatures the considered string transition behaves, in a sense, similarly to the false vacuum decay[32], namely the regime of the transition changes to a different tunneling trajectory, so that the temperature dependence appears in the semiclassical exponential factor in Eq.(1.3), rather than in the preexponential term.

The effect of the thermally excited waves in the decay rate can in fact be considered [33] as an additional contribution to the probability of the decay due to collisions of the Goldstone bosons that are present in the thermal bath. As it turns out, it is possible to identify in the thermal catalysis factor the contribution of individual such collisions and thus calculate the probability of the destruction of metastable string by colliding particles. In particular, the first temperature dependent term in the expansion (1.6) is entirely due to the process of the critical gap creation in a collision of two particles in the limit, where their center of mass energy $E = \sqrt{s}$ is much smaller than $1/\ell_c$. In order to separate the terms in the decay probability originating from collisions of different number of the bosons, the standard thermal approach to calculating the decay rate at finite temperature [31] is modified [33] by formally introducing a negative chemical potential for the bosons. This allows to find the dependence of the probability of string destruction in an n-boson collision for an arbitrary relation between ℓ_c and the energy of the bosons. One of the results of such a consideration is that the string destruction takes place only in collisions of even number of bosons and is absent at odd n. The explicit expression for the (dimensionless) probability W_2 of the critical gap creation in two-boson collisions at arbitrary values of the parameter $E\,\ell_c$ has especially simple form for the decay of the string into 'nothing' i.e. at $\varepsilon_2 = 0$:

$$W_2 = 2\,\pi^2\,\ell_c^2\,\gamma_s\,\left[I_3\left(\frac{E\ell_c}{2}\right)\right]^2 ~, \tag{1.7}$$

with $I_3(x)$ being the standard notation for the modified Bessel function of the third order. This expression is applicable as long as the energy E is small in comparison with the string energy scale: $E \ll \sqrt{\epsilon}$, which condition still allows for the parameter $E\,\ell_c$ to be large. At $E\,\ell_c \gg 1$ the energy dependent factor in Eq.(1.7) has the exponential behavior $\exp(E\ell_c)$ which matches the semiclassical approximation for the energy-dependent factor[34], corresponding to tunneling at the energy E rather than at zero energy. In this sense the discussed energy dependence of the collision-induced probability of the decay describes the onset of the semiclassical behavior.

One can also notice that the thermal factor $\mathcal{K}_s$, is singular [31] at $\ell_c T = 1$. However the two-boson production rate does not exhibit any singularity at any value of the parameter $E\ell_c$, and a similar smooth energy behavior is also true for individual n-boson processes. Thus the 'explosion' of the thermal rate at $\ell_c T = 1$ is a result of infinite number of processes becoming important at this point, rather than due to a finite set of processes with a limited range of n developing large probability at the energy per particle of the order of $1/\ell_c$.

In the paper [20] the generalization of the approach used for analyzing the thermal and collision induced effects in the decay of metastable strings was made to the case of a metastable wall . We found the general expression for the thermal catalysis factor $\mathcal{K}_w$. In particular, the leading term in this factor at low temperature (in d dimensions) is given by

$$\mathcal{K}_w = 1 + 12\,(d-3)\,\zeta^2(5)\,(R_w\,T)^{10} + \ldots . \tag{1.8}$$

where R_w is the radius of the critical hole in the metastable wall, and $\zeta(x)$ is the standard notation for the Riemann ζ function, $\zeta(5) \approx 1.037$.

In the present paper we also address the problem of induced e^+e^- pair production in external electromagnetic field. It has been theoretically understood since long ago [15] that a static electric field E can spontaneously produce electron-positron pairs due to quantum tunneling. The probability of this phenomenon, usually called the Schwinger process, can be found from the imaginary part of the one-loop effective action in an external field [14, 16], and the pair production rate per unit volume in a constant field is given by the well known formula

$$\Gamma = \frac{(eE)^2}{4\pi^3} \sum_{n=1}^{+\infty} \frac{1}{n^2} \exp\left(- \frac{\pi\, m^2}{eE}\, n \right) \;, \tag{1.9}$$

with e and m being respectively the charge and the mass of the electron. The generalizations of this formula to varying external field include the special cases of a spatially constant field with the time dependence $E \sim 1/\cosh^2(\Omega t)$ [35] and of the field pointing along the z axis and arbitrarily depending on the light cone variable $E_z(t \pm x)$ [36], a complete review of the topic can be found in Ref. [37, 38].

Being thoroughly investigated theoretically, the Schwinger process has no experimental evidence so far. The reason is that any practically available strength of the electric field is much smaller than the critical value $E_c = m^2/e \sim 10^{16}\,\mathrm{V/cm}$ at which the probability described by Eq.(1.9) would not be exponentially suppressed.

It has been suggested recently [28] that the pair creation can be significantly stimulated by superimposing a relatively weak photon beam with a (quasi)static electric field. It was shown [29] that in the presence of an external photon the barrier for the tunneling is effectively lowered and the negative exponential power in the pair production rate is decreased in absolute value. In particular, for a photon with energy ω propagating perpendicularly to the field E the negative exponential power in the pair production rate at the threshold $\omega = 2m$ is modified from $-\pi\, m^2/(eE)$ to $-(\pi - 2)\, m^2/(eE)$, leading to a large exponential enhancement of the rate [3].

In the paper [40] we considered the photon induced pair creation in an external electric field in the realistic limit $E \ll E_c$ and at lower photon energies $\omega \ll m$ for which higher beam intensity can be practical. Under this condition the leading effects are described by the so-called Keldysh parameter $\gamma_\theta = m\,\omega\, \sin\theta/(eE)$, with θ being the angle between the photon momentum and the electric field E. We did not assume the Keldysh parameter to be small and found the exact in this parameter expression for the attenuation rate κ for the photon beam intensity due to the pair production in the form

$$\kappa_\parallel(\vec{k}) = 2\, \frac{\alpha\, m^2}{\omega} e^{-\frac{\pi\, m^2}{eE}} \left[I_1\left(\gamma_\theta\right)\right]^2 \ , \tag{1.10}$$

with $I_1(x)$ being the standard notation for the modified Bessel function. Our consideration was restricted to the lowest order in the ratio $\omega\, \sin\theta/m$, and in this order we found that only the photons whose polarization is parallel to $\vec{E}$ stimulate the pair production (hence the notation $\kappa_\parallel$), while the effect for the orthogonal polarization, $\kappa_\perp$, arises only in a higher order in this ratio. The exponential behavior of the Bessel function in Eq.(1.10) at large argument matches the low ω limit of the exponential expression found in Ref. [29], so that our result describes both the exponential and the pre-exponential factors in this limit.

The photon-induced pair creation can be calculated in a standard way in terms of the imaginary part of the electromagnetic vacuum polarization function Π in external field:

$$\kappa = -\frac{1}{\omega}\, \Im\Pi \tag{1.11}$$

starting from the known [38] general formulas for Π. In fact the expression (1.10) was found [29] in this way by considering a small ω expansion of the contour integral representation of the imaginary part of the polarization tensor for photons in a constant E field. In the paper [40] we didn't use this approach but rather found the result (1.10)

[3]Recently a more complex arrangement has also been considered [39], where in addition to a strong low-frequency laser field and a weak high-frequency photon beam there is a field of a heavy nucleus.

by considering the Schwinger pair creation as a semiclassical tunneling process[4], and using the Euclidean-space description of the tunneling trajectory [22, 23, 41]. In order to find the pair creation rate stimulated by photons we used the extension of the approach described in previous chapter.

In the paper [42] we employed another technique, namely, we considered the on-shell imaginary part of the vacuum polarization operator in external field within a purely semiclassical approach. We believe that such an approach provides some additional insight in the physical origin of various terms describing the photon-stimulated pair production, which may be useful in further studies of this subject. In terms of the specific results we reproduce the formulas of Refs. [28, 40], and additionally calculate a contribution to the attenuation rate arising from the electron magnetic moment, which contribution dominates, in a situation where ω^2 is larger than eE, over that recently found [43] for photons polarized orthogonally to the external field.

There are two dimensionless parameters in the considered problem (apart from fine structure constant $\alpha = e^2/4\pi$), namely eE/m^2 and ω/m. The polarization operator can be split into two parts $\Pi(q) = \Pi_\| + \Pi_\perp$, corresponding to the photons having their polarization along or orthogonal to the field. In the leading order in eE/m^2 only the parallel part of the polarization operator develops imaginary part $\Im\Pi_\| = \pi_\|$ [40, 43],

$$\pi_\| = -2\alpha m^2 I_1^2(\gamma_\theta) \exp\left[-\frac{\pi m^2}{eE}\right], \tag{1.12}$$

with $\gamma_\theta = m\omega \sin\theta/eE$ being the well-known Keldysh parameter and θ is the angle between the field and the photon propagation direction. Finally, $I_n(z)$ is the standard notation for the modified Bessel function. The formula in Eq.(1.12) is valid for arbitrary values of γ_θ as long as the photon energy is small as compared to the electron mass, $\omega^2 \ll m^2$.

The orthogonal part of the polarization tensor develops imaginary part only in the next order in eE/m^2. We found here the following explicit expression for $\Im\Pi_\perp = \pi_\perp$ in this order in eE/m^2, at arbitrary γ_θ and small ω:

$$\pi_\perp = -\frac{\alpha}{\pi} eE \left(1 - I_0^2(\gamma_\theta)\right) \exp\left[-\frac{\pi m^2}{eE}\right]. \tag{1.13}$$

We also argued that there is another potentially important contribution to the $\pi_\perp$

[4] Which treatment in fact goes back to the original idea [15]

coming from the interaction of the photon with the electron magnetic moment

$$\pi_\perp^m = -\frac{\alpha}{4}\,(\omega\,\sin\theta)^2\, I_1^2(\gamma_\theta)\exp\left[-\frac{\pi m^2}{eE}\right] \tag{1.14}$$

The suppression of this term in comparison with the parallel part (1.12) is governed by the parameter ω^2/m^2, rather than eE/m^2 as is the case for the term in Eq.(1.13). Thus the magnetic contribution is more important if $\omega^2 \gg eE$, which situation is quite compatible with both ω^2 and eE being much smaller than m^2.

Our treatment is based on considering the semiclassical tunneling trajectory for the electron in the electric field. Within this approach the parallel part (1.12) and the magnetic one (1.14) arise as purely classical correlators of respectively the classical current and the current associated with the magnetic moment on an unperturbed semiclassical trajectory. The term (1.13) arises from small fluctuations, 'zitterbewegung', orthogonal to the electric field around the same unperturbed trajectory. The results in the equations (1.12) - (1.14) are derived assuming that the deformation by the photon of the trajectory is small. At large photon energy this approximation is no longer valid and the deformation of the tunneling path should be taken into account. We found the semiclassical trajectory for an arbitrary energy of the photon and calculated the action on the resulting trajectory by purely geometrical means. As usual, the (Euclidean) action S describes the exponential factor e^{-S} in the rate. Our result reads as

$$\pi_{\|,\perp} \sim \exp\left[\gamma_\theta - \left(\frac{2m^2}{eE} + \frac{eE}{2m^2}\,\gamma_\theta^2\right)\arctan\frac{2m^2}{eE\,\gamma_\theta}\right] \tag{1.15}$$

and reproduces that found in Ref. [43].

We also used the same approach for solving the induced vacuum decay problem. We considered two interacting scalar fields in the following setup. The potential for one of the fields φ is chosen as having the shape shown in Fig.4.1. It is well known that the system with such potential admits quantum tunneling from the metastable to the true vacuum, therefore one can discuss the spontaneous decay of the false vacuum [17, 18, 19, 21]. Choosing the parameter of the potential asymmetry to be small and the mass of the corresponding field φ to be much larger than that of the second field χ, $M \gg m$, greatly simplifies the treatment. Indeed, for a small difference of the vacuum energy density ε one can employ the so called thin wall approximation [17, 18]. In this case the tunneling configuration is given by a disk of the radius

$$R = \frac{\mu}{\varepsilon}, \tag{1.16}$$

with the false vacuum outside of the disk and the true vacuum inside. The extent of the transition region between the two vacua is assumed to be much shorter than the radius R (the thin wall approximation), which is ensured by the condition $M\,R \gg 1$. The quantity μ in Eq.(1.16) is the mass associated with the transition region, $\mu \gg M$. The smallness of the mass m of the second field χ compared to M guarantees that in the leading order there is no contribution to the interaction of the bubble (bounce) with other objects due to the self-action of the field φ but only due to the exchange of the light particle χ.

The problem that is addressed here is that of the false vacuum decay stimulated either by the presence of one particle of the field χ or by a collision between two such particles. Previously a similar problem was considered [24, 26, 34] for the particles of the field φ, which is always strongly coupled to the bounce in the sense that there are always zero and soft modes for this field localized on the bounce. Our present problem is different in that we consider the catalysis of the process by particles of a second field χ that is only weakly coupled to the master field φ so that no anomalously soft modes for the field χ arise on the bounce. As a result we find closed formulas for the probability of an induced vacuum decay, which describe the onset of the expected semiclassical behavior similar to that the catalysis by the particles of the master field [24, 26, 34], where the energy of the initial particle(s) is transferred to the tunneling degree of freedom.

Chapter 2

Decays of strings and metastable walls

In present chapter we aim at providing a systematic and self-contained presentation of the calculations leading to our final results. To this end we recapitulate in detail in Section 1.1 the technically simpler calculations for the processes with metastable strings, and then in Section 1.2 expand the method to the case of a metastable wall. Although the calculations in both cases are very much similar, the results differ in some essential details. In particular the destruction of a metastable wall is induced by a collision of any number of Goldstone bosons, whereas a string can be destroyed by a collision of only even number of the bosons. Another difference relates to the thermal catalysis of the decay: for a string the low temperature expansion for the enhancement of the quantum tunneling is applicable everywhere where the series is convergent, i.e. up to the temperature $T = 1/\ell_c$, while for a wall the thermal fluctuations start dominating[32] at a lower temperature ($T = 3/(16\ell_w)$ with ℓ_w being the diameter of the critical hole in the wall), at which the thermal effects in the quantum tunneling probability are still very small numerically.

2.1 String-like configuration

2.1.1 Spontaneous decay of a metastable string

The tunneling trajectory can be described in the Euclidean space by a configuration called the 'bounce'[18], which is a solution to (Euclidean) classical equations of motion. The general expression for the effective Euclidean space action for the string with the two considered phases can be written in the familiar Nambu-Goto form:

$$S = \mu P + \varepsilon_1 A_1 + \varepsilon_2 A_2 \ , \tag{2.1}$$

where A_1 and A_2 are the areas of the world sheet for the two phases, and P (the perimeter) is the length of the world line for the interface between them.

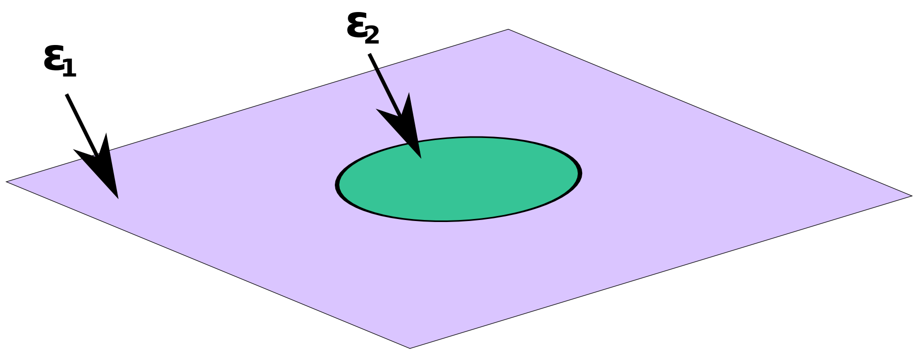

Figure 2.1: The bounce configuration, describing the semiclassical tunneling trajectory.

The action (2.1) is an effective low-energy expression in the sense that it only describes the 'stringy' variables and is applicable as long as the effects of thickness of the string and of its internal structure can be neglected. Denoting M_0 the mass scale at which such approach becomes invalid (e.g. the thickness of the string $r_0 \sim 1/M_0$), one can write the condition for the applicability of the effective action (2.1) in terms of the length scale, $\ell \gg 1/M_0$, and the momentum scale $k \ll M_0$. Assuming that the initial very long string with tension ε_1 is located along the x axis, one can readily find that the action (2.1) has a nontrivial stationary configuration, the bounce, namely, that of a disk in the (t,x) plane occupied by the phase 2, as shown in Figure 2.1, with the radius

$$R_s = \frac{\mu}{\varepsilon_1 - \varepsilon_2} \ , \tag{2.2}$$

which is the radius (one half of the length ℓ_c) of the critical gap. The difference between the action (2.1) on this configuration and on the trivial one is exactly the expression for the exponential power in Eq.(1.2), and the condition for applicability of the effective action (2.1) requires

$$M_0 \, R_s = \frac{M_0 \, \mu}{\varepsilon_1 - \varepsilon_2} \gg 1 \ . \tag{2.3}$$

Generally one also has $\mu \gtrsim M_0$, and for the strings in weakly coupled theories $\mu \gg M_0$, so that the power in the exponent in Eq.(1.2) is large, which justifies a semiclassical treatment.

The probability of the transition is determined[18, 44, 45] by (the imaginary part of) the ratio of the path integrals $\mathcal{Z}_{12}$ and $\mathcal{Z}_1$ calculated with the action (2.1) around respectively the bounce configuration and around the initial flat string:

$$\gamma_s = \frac{1}{A} \, \text{Im} \frac{\mathcal{Z}_{12}}{\mathcal{Z}_1} \ . \tag{2.4}$$

It can also be reminded that, as explained in great detail in Ref.[44], that the imaginary part of $\mathcal{Z}_{12}$ arises from one negative mode at the bounce configuration, and that due to two translational zero modes the numerator in Eq.(2.4) is proportional to the total space time area A in the (t, x) plane occupied by the string, so that the finite quantity is the transition probability per unit time (the rate) and per unit length of the string.

In order to evaluate the relevant path integrals with pre-exponential accuracy we use the cylindrical coordinates, with r and θ being the polar variables in the (t, x) plane (of the bounce), and z being the transverse coordinate. We consider only one transverse coordinate, since the effect of each of the extra dimensions factorizes, so that the corresponding generalization is straightforward. We further assume, for definiteness, that the space-time boundary in the (t, x) plane is a circle of large radius L, where the boundary condition for the string is $z(r = L) = 0$. The small deviations of the string configuration from the bounce, illustrated in Fig.2.2, can be parametrized by the radial (f) and transverse (ξ) shifts of the boundary between the string phases:

$$r(\theta) = R_s + f(\theta) \, , \qquad z(\theta) = \xi(\theta) \ , \tag{2.5}$$

and by the variations of the surfaces of the two string phases: $z_1(r,\theta)$ and $z_2(r,\theta)$, where, naturally,

$$z_1(R_s,\theta) = z_2(R_s,\theta) = \xi(\theta)\ . \tag{2.6}$$

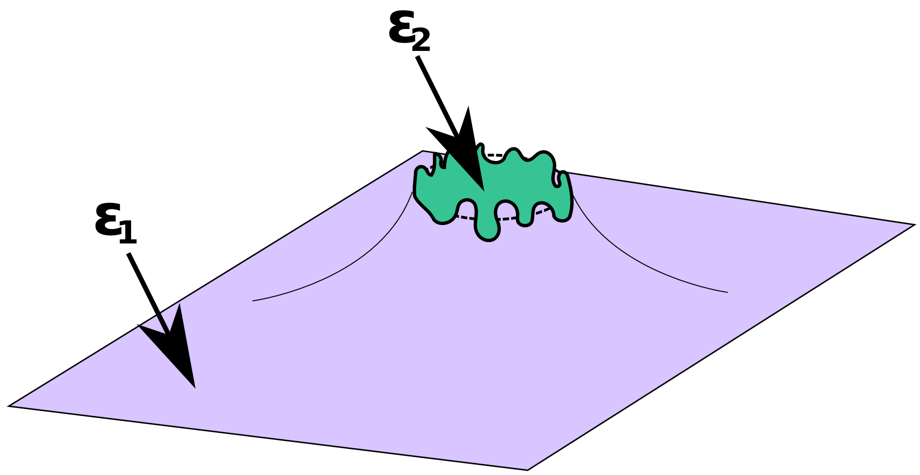

Figure 2.2: Fluctuations around the bounce configuration.

In terms of these variables the action (2.1) can be written in the quadratic approximation in the deviations from the bounce as

$$S_{12} = \varepsilon_1\,\pi\,L^2 + \frac{\pi\,\mu^2}{\varepsilon_1-\varepsilon_2} + \frac{\varepsilon_1-\varepsilon_2}{2}\int d\theta\,\left(\dot{\xi}^2 + \dot{f}^2 - f^2\right) +$$
$$\frac{\varepsilon_1}{2}\int_{R_s}^{L} r dr d\theta\,\left(z_1'^{\,2} + \frac{\dot{z}_1^{\,2}}{r^2}\right) + \frac{\varepsilon_2}{2}\int_0^{R_s} r dr d\theta\,\left(z_2'^{\,2} + \frac{\dot{z}_2^{\,2}}{r^2}\right)\ , \tag{2.7}$$

where the primed and dotted symbols stand for the derivatives with respect to r and θ correspondingly.

Finally, the action around a flat initial string configuration in the quadratic approximation takes the form

$$S_1 = \varepsilon_1\,\pi L^2 + \frac{\varepsilon_1}{2}\int_0^L r dr d\theta\,\left(z'^{\,2} + \frac{\dot{z}^2}{r^2}\right)\ , \tag{2.8}$$

with $z(r,\theta)$ parametrizing small deviations of the string in the transverse direction.

Separating variables in the path integrals

One can readily see that in the quadratic part of the action (2.7) the 'longitudinal' variation of the bounce boundary in the (t,x) plane, described by the function $f(\theta)$

completely decouples from the rest of the variables. This implies that the path integral over f can be considered independently of the integration over other variables and that it enters as a factor in $\mathcal{Z}_{12}$. On the other hand it is this integral that provides the imaginary part to the partition function, and it is also proportional to the total space-time area A. Moreover, this path integral is identical to the one entering the problem of false vacuum decay in (1+1) dimensions and we can directly apply the result of that calculation[46]:

$$\frac{1}{A}\,\mathrm{Im}\int \mathcal{D}f\,\exp\left[-\frac{\varepsilon_1-\varepsilon_2}{2}\int_0^{2\pi}\left(\dot{f}^2-f^2\right)d\theta\right]=\frac{\varepsilon_1-\varepsilon_2}{2\pi}\,. \tag{2.9}$$

The expression for the transition rate thus can be written in the form

$$\frac{d\Gamma}{d\ell}=\frac{\varepsilon_1-\varepsilon_2}{2\pi}\,\exp\left(-\frac{\pi\,\mu^2}{\varepsilon_1-\varepsilon_2}\right)\,\frac{\tilde{\mathcal{Z}}_{12}}{\mathcal{Z}_1}\,, \tag{2.10}$$

with the path integral $\tilde{\mathcal{Z}}_{12}$ running only over the transverse variables ξ, z_1 and z_2,

$$\tilde{\mathcal{Z}}_{12}=\int \mathcal{D}\xi\,\mathcal{D}z_1\,\mathcal{D}z_2\,\exp\left(-\tilde{S}_{12}\right) \tag{2.11}$$

and involving only the quadratic in these variables part of the action (2.7),

$$\tilde{S}_{12}=\frac{\varepsilon_1-\varepsilon_2}{2}\int d\theta\,\dot{\xi}^2+\frac{\varepsilon_1}{2}\int_{R_s}^{L} r dr d\theta\left({z_1'}^{\,2}+\frac{\dot{z}_1^{\,2}}{r^2}\right)+\frac{\varepsilon_2}{2}\int_0^{R_s} r dr d\theta\left({z_2'}^{\,2}+\frac{\dot{z}_2^{\,2}}{r^2}\right)\,. \tag{2.12}$$

In the same quadratic approximation the flat string partition function $\mathcal{Z}_1$ is given by

$$\mathcal{Z}_1=\int \mathcal{D}z\,\exp\left(-S_1\right) \tag{2.13}$$

with S_1 given by Eq.(2.8) and the integral running over all the functions vanishing at the space-time boundary: $z(L,\theta)=0$.

At this point there still is a coupling in the path integral (2.11) between the bulk variables z_1, z_2 and the boundary variable ξ arising from the boundary conditions (2.6). This however is a simple issue which is resolved by a straightforward shift of the integration variables z_1 and z_2. Namely, we write

$$z_1(r,\theta)=z_{1c}(r,\theta)+z_{1q}(r,\theta)\,,\qquad z_2(r,\theta)=z_{2c}(r,\theta)+z_{2q}(r,\theta)\,, \tag{2.14}$$

where z_{1q} and z_{2q} are the new integration variables in $\tilde{\mathcal{Z}}_{12}$ and these functions satisfy zero boundary conditions,

$$z_{1q}(R_s,\theta)=z_{2q}(R_s,\theta)=z_{1q}(L,\theta)=z_{2q}(L,\theta)=0\,, \tag{2.15}$$

while z_{1c} and z_{2c} are fixed (for a fixed $\xi(\theta)$) functions satisfying the boundary conditions

$$z_{1c}(R_s,\theta) = z_{2c}(R_s,\theta) = \xi(\theta)\ ,\ z_{1c}(L,\theta) = 0\ , \tag{2.16}$$

which are also harmonic, i.e. satisfying the Laplace equation $\Delta z = 0$.

After the specified shift of the variables we arrive at the expression for the action (2.12) in which the bulk and the boundary degrees of freedom are fully separated:

$$\begin{aligned}\tilde{S}_{12} &= \frac{\varepsilon_1 - \varepsilon_2}{2}\int d\theta\,\dot{\xi}^2 + R_s\int d\theta\,\left(\frac{\varepsilon_2}{2}\partial_r z_{2c}\,|_{r=R_s} - \frac{\varepsilon_1}{2}\partial_r z_{1c}\,|_{r=R_s}\right)\xi \\ &- \frac{\varepsilon_1}{2}\int d^2r\, z_{1q}\Delta z_{1q} - \frac{\varepsilon_2}{2}\int d^2r\, z_{2q}\Delta z_{2q}\ .\end{aligned} \tag{2.17}$$

Clearly, the boundary terms in the first line here, arising from the integration by parts, depend only on the transverse shift of the boundary $\xi(\theta)$. The partition function $\tilde{\mathcal{Z}}_{12}$ can thus be written as a product of the 'boundary' and the 'bulk' terms:

$$\tilde{\mathcal{Z}}_{12} = \mathcal{Z}_{12(\text{boundary})}\,\mathcal{Z}_{12(\text{bulk})}\ , \tag{2.18}$$

with the $\mathcal{Z}_{12(\text{bulk})}$ being given by the path integral over the bulk variables z_{1q} and z_{2q} only :

$$\mathcal{Z}_{12(\text{bulk})} = \int \mathcal{D}z_{1q}\,\mathcal{D}z_q\ \exp\left(\frac{\varepsilon_1}{2}\int d^2r\, z_{1q}\Delta z_{1q} + \frac{\varepsilon_2}{2}\int d^2r\, z_{2q}\Delta z_{2q}\right)\ , \tag{2.19}$$

and the boundary term

$$\mathcal{Z}_{12(\text{boundary})} = \int \mathcal{D}\xi\ \exp\left[-\frac{\varepsilon_1 - \varepsilon_2}{2}\int d\theta\,\dot{\xi}^2 - R_s\int d\theta\,\left(\frac{\varepsilon_2}{2}\partial_r z_{2c}\,|_{r=R_s} - \frac{\varepsilon_1}{2}\partial_r z_{1c}\,|_{r=R_s}\right)\xi\right] \tag{2.20}$$

involving integration over only the boundary values.

The subsequent calculation of the ratio of the partition functions in Eq.(2.10) can in fact be reduced to a calculation of $\mathcal{Z}_{12(\text{boundary})}$ only. In order to achieve this reduction one should organize the partition function $\mathcal{Z}_1$ in the denominator of Eq.(2.10) in a form similar to Eq.(2.18) as follows. Although the flat string configuration makes no reference to a circle of the radius R, the partition function $\mathcal{Z}_1$ can be calculated by first fixing the transverse variable z at $r = R_s$: $z(R_s,\theta) = \xi(\theta)$ and separating the integration over the bulk variables. Then the flat string partition function factorizes in the form similar to Eq.(2.18):

$$\tilde{\mathcal{Z}}_1 = \mathcal{Z}_{1(\text{boundary})}\,\mathcal{Z}_{1(\text{bulk})}\ , \tag{2.21}$$

with $\mathcal{Z}_{1(\mathrm{bulk})}$ given a similar path integral as in Eq.(2.19),

$$\mathcal{Z}_{1(\mathrm{bulk})} = \int \mathcal{D}z_{1q}\, \mathcal{D}z_q \, \exp\left(\frac{\varepsilon_1}{2} \int d^2r \, z_{1q} \Delta z_{1q} + \frac{\varepsilon_1}{2} \int d^2r \, z_{2q} \Delta z_{2q} \right) , \tag{2.22}$$

where, as in Eq.(2.19), z_{1q} and z_{2q} are respectively the outer (i.e. at $r > R_s$) and the inner ($r < R_s$) transverse fluctuations with zero boundary conditions. The difference in the coefficient in the expressions (2.19) and (2.22) for the contribution of the inner part, ε_2 vs. ε_1, is not essential, since the overall coefficient of the quadratic part of the action is absorbed in the measure of integration, as can be seen by rescaling to the corresponding canonically normalized variables $\phi = \sqrt{\varepsilon}\, z$.

One therefore finds that $\mathcal{Z}_{1(\mathrm{bulk})} = \mathcal{Z}_{12(\mathrm{bulk})}$, and the ratio of the partition functions in Eq.(2.10) is in fact determined by the ratio of the boundary terms.

Regularization

The boundary factor $\mathcal{Z}_{1(\mathrm{boundary})}$ for the flat string is somewhat different from the one given by Eq.(2.20) and reads as

$$\mathcal{Z}_{1(\mathrm{boundary})} = \int \mathcal{D}\xi \, \exp\left[-R_s \frac{\varepsilon_1}{2} \int d\theta \, \left(\partial_r z_{2c} \,|_{r=R_s} - \partial_r z_{1c} \,|_{r=R_s} \right) \xi \right] , \tag{2.23}$$

where the functions z_{1c} and z_{2c} are defined in the same way as in Eq.(2.20).

The latter functions can be readily found by expanding the boundary function $\xi(\theta)$ in angular harmonics:

$$\xi(\theta) = \frac{a_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} [a_n \, \cos(n\theta) + b_n \, \sin(n\theta)] \tag{2.24}$$

with a_n and b_n being the amplitudes of the harmonics. Then at $n \neq 0$ the harmonics of the discussed functions are found as

$$z_{1c}^{(n)}(r,\theta) = \frac{1}{\sqrt{\pi}} \, [a_n \, \cos(n\theta) + b_n \, \sin(n\theta)] \, \frac{R_s^n}{r^n} \, , \;\; z_{2c}^{(n)}(r,\theta) = \frac{1}{\sqrt{\pi}} \, [a_n \, \cos(n\theta) + b_n \, \sin(n\theta)] \, \frac{r^n}{R_s^n} \, , \tag{2.25}$$

and for n=0 these are given by

$$z_{1c}^{(0)}(r,\theta) = a_0 \, \frac{\ln(r/L)}{\ln(R_s/L)} \, , \qquad z_{2c}^{(0)}(r,\theta) = a_0 \, . \tag{2.26}$$

Substituting these expressions for the harmonics in the equations (2.20) and (2.23) and performing the Gaussian integration over the amplitudes a_n and b_n we find the boundary factors in the form

$$\mathcal{Z}_{12(\text{boundary})} = \mathcal{N}\sqrt{\varepsilon_1 \, \ln \frac{L}{R_s}} \prod_{n=1}^{\infty} \frac{1}{(\varepsilon_1 - \varepsilon_2)\, n^2 + (\varepsilon_1 + \varepsilon_2)\, n} \tag{2.27}$$

and

$$\mathcal{Z}_{1(\text{boundary})} = \mathcal{N}\sqrt{\varepsilon_1 \, \ln \frac{L}{R_s}} \prod_{n=1}^{\infty} \frac{1}{2\, \varepsilon_1 \, n} \tag{2.28}$$

with $\mathcal{N}$ being a normalization factor.

Clearly, each of the formal expressions (2.27) and (2.28) contains a divergent product, and their ratio is also ill defined, so that our calculation requires a regularization procedure that would cut off the contribution of harmonics with large n. A regularization of high harmonics is also required on general grounds. Indeed, as previously mentioned, our consideration using the effective string action (2.1) is only valid for smooth deformations of the string, i.e. as long as the relevant momenta are smaller than the mass scale M_0 for excitation of the internal degrees of freedom within the thickness of the string. For an n-th harmonic the relevant momentum is $k \sim n/R_s$ so that the applicability of the effective low energy action requires a cutoff at $n \ll M_0 R_s$. In order to perform such regularization we use the standard Pauli-Villars method and introduce a regulator field Z with negative norm and the action corresponding to the quadratic part of the Nambu-Goto expression (2.1) for small z:

$$S_R = \frac{\varepsilon_1 - \varepsilon_2}{2} \int d\theta \dot{\xi}_R^{\,2} + \frac{\varepsilon_1}{2} \int_{A_1} d^2 r \left[(\partial_\mu Z)^2 + M^2 \, Z^2 \right] + \frac{\varepsilon_2}{2} \int_{A_2} d^2 r \left[(\partial_\mu Z)^2 + M^2 \, Z^2 \right] \tag{2.29}$$

with M being the regulator mass, which physically should be understood as satisfying the condition $M \ll M_0$ and still being much larger than the relevant scale in the discussed problem, in particular $M R_s \gg 1$.

The regularized expression for the ratio of the boundary terms in $\mathcal{Z}_{12}$ and $\mathcal{Z}_1$ thus takes the form

$$\frac{\mathcal{Z}_{12(\text{boundary})}}{\mathcal{Z}_{1(\text{boundary})}} \longrightarrow \mathcal{R} = \left[\frac{\mathcal{Z}_{12(\text{boundary})}}{\mathcal{Z}^{(R)}_{12(\text{boundary})}} \right] \left[\frac{\mathcal{Z}_{1(\text{boundary})}}{\mathcal{Z}^{(R)}_{1(\text{boundary})}} \right]^{-1} , \tag{2.30}$$

where we introduced the notation $\mathcal{R}$ for the regularized ratio, and the regulator partition functions $\mathcal{Z}^{(R)}_{12(\text{boundary})}$ and $\mathcal{Z}^{(R)}_{1(\text{boundary})}$ are determined by the same expressions as in Eqs.(2.20) and (2.23) with the 'outer' and 'inner' functions z_{1c} and z_{2c} being replaced by their regulator counterparts Z_{1c} and Z_{2c} which still satisfy the boundary conditions similar to (2.16):

$$Z_{1c}(R_s, \theta) = Z_{2c}(R_s, \theta) = \xi_R(\theta) \ , \tag{2.31}$$

but are the solutions of the Helmholtz rather than Laplace equation: $(\Delta - M^2) Z = 0$.

The solutions of the latter equation fall off exponentially at the scale determined by M, and for our purposes only the behavior near the circle $r = R_s$ is needed. For this reason we write the equation for the radial part of the n-th angular harmonic $Z_n(r)$,

$$Z_n'' + \frac{1}{r} Z_n' - \frac{n^2}{r^2} Z_n - M^2 Z_n = 0 \ , \tag{2.32}$$

and parametrize the radial coordinate as $r = R_s + x$, and treat the parameter (x/R_s) as small, since the scale for the variation of the solution is $x \sim 1/\sqrt{M^2 + n^2/R_s^2}$. This approach yields an expansion of the regulator action associated with the boundary at $r = R$ in powers of $1/\sqrt{(MR_s)^2 + n^2}$. With the accuracy required in the present calculation, the (normalized to one at $r = R_s$) solution to Eq.(2.32) is found in the first order of expansion in (x/R_s) as

$$Z_n(R_s + x) = \left(1 - \frac{1}{2} \frac{(MR_s)^2}{(MR_s)^2 + n^2} \frac{x}{R_s} \right) \exp \left(-\sqrt{(MR_s)^2 + n^2} \, \frac{|x|}{R_s} \right) \ . \tag{2.33}$$

Using the form of the solutions for the harmonics of the regulator field given by Eq.(2.33) and the expressions (2.27) and (2.28), one can write the regularized ratio of the boundary partition functions (2.30) as

$$\begin{aligned} \mathcal{R} \ = \ & \sqrt{\frac{\varepsilon_1 + \varepsilon_2}{2\varepsilon_1}} \left[\prod_{n=1}^{\infty} \frac{n^2 + b\sqrt{(MR_s)^2 + n^2}}{n^2 + b\, n} \right] \left[\prod_{n=1}^{\infty} \frac{n}{\sqrt{(MR_s)^2 + n^2}} \right] \times \\ & \prod_{n=1}^{\infty} \left\{ 1 + \frac{1}{2} \frac{(MR_s)^2}{[(MR_s)^2 + n^2] \left[n^2 + b\sqrt{(MR_s)^2 + n^2} \right]} \right\} \ , \end{aligned} \tag{2.34}$$

where we introduced the notation

$$b = \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 - \varepsilon_2} \ , \tag{2.35}$$

and the last product factor in Eq.(2.34) arises from the first term of expansion in (x/R) in the pre-exponential factor in Eq.(2.33)

Calculating the products

Each of the products in Eq.(2.34) is finite at a finite M and can be calculated separately. We start with the most straightforward one, which is directly given by an Euler's formula:

$$\prod_{n=1}^{\infty} \frac{n}{\sqrt{(MR_s)^2 + n^2}} = \sqrt{\frac{\pi M R_s}{\sinh(\pi M R_s)}} \longrightarrow \sqrt{2\pi M R_s}\, \exp\left(-\frac{\pi M R_s}{2}\right) \;, \tag{2.36}$$

where the last transition corresponds to the limit $MR_s \gg 1$.

The other two factors in Eq.(2.34), the first and the last, generally depend on the relation between the parameters b and MR_s, or equivalently between $(\varepsilon_1 + \varepsilon_2)$ and μM. We find however that the latter product is equal to one at $MR_s \gg 1$ independently of b. In particular in the nontrivial case of $b \ll MR_s$ we find

$$\ln \prod_{n=1}^{\infty} \left\{ 1 + \frac{1}{2} \frac{(MR_s)^2}{[(MR_s)^2 + n^2] \left[n^2 + b\sqrt{(MR_s)^2 + n^2} \right]} \right\} \Bigg|_{MR_s \to \infty} \to$$
$$\left\{ \frac{(MR_s)^2}{2} \int_{n_0}^{\infty} dn \left[(MR_s)^2 + n^2 \right]^{-1} \left[n^2 + b\sqrt{(MR_s)^2 + n^2} \right]^{-1} \right\} \Bigg|_{MR_s \to \infty} \to 0 \,, \tag{2.37}$$

where the lower limit in the integral is any number n_0 that is finite in the limit $MR_s \to \infty$.

The dependence of the first product factor in Eq.(2.34) on the ratio $(\varepsilon_1 + \varepsilon_2)/(\mu M) = b/(MR_s)$ is essential and we consider two limiting cases when this ratio is much bigger than one and when it is very small. In the former case, i.e. for $b \gg MR_s$, the first product in Eq.(2.34) becomes reciprocal of the second, and one finds

$$\mathcal{R}|_{b \gg MR_s \gg 1} = 1 \;. \tag{2.38}$$

(Clearly one can also safely make the replacement $(\varepsilon_1 + \varepsilon_2)/(2\varepsilon_1) \to 1$ at $b \gg 1$.)

The behavior of $\mathcal{R}$ in the opposite limit, i.e. at $b \ll MR_s$, turns out to be significantly more interesting. Using the Euler-Maclaurin summation formula for the logarithm of the first product in Eq.(2.34) we find in the limit $MR_s \gg 1$ and $MR_s \gg b$:

$$\prod_{n=1}^{\infty} \frac{n^2 + b\sqrt{(MR_s)^2 + n^2}}{n^2 + b\,n} = \frac{\Gamma(b+1)}{2\pi\sqrt{bMR_s}} \exp\left[\pi\sqrt{bMR_s} - b\,\ln(MR_s) - b\,(1 - \ln 2) \right] \;. \tag{2.39}$$

Being combined with the expression (2.37) this yields the formula

$$\mathcal{R} = \sqrt{\frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_1}}\,\frac{\Gamma(b+1)}{\sqrt{2\pi\, b}}\,\exp\left[-\frac{\pi}{2}\,MR_s+\pi\,\sqrt{bMR_s}-b\,\ln(MR_s)-b\,(1-\ln 2)\right]~, \tag{2.40}$$

which contains an essential dependence on the regulator mass parameter M. We will show however that all such dependence in the phase transition rate can be absorbed in renormalization of the parameter μ in the leading semiclassical term.

Renormalization of μ

The parameter μ is defined in the action (2.1) as the coefficient in front of the length of the boundary between the world sheets for two phases of the string. Generally this parameter gets renormalized by the quantum corrections, and in order to find such renormalization at the level of first quantum corrections, one needs to perform the path integration using the quadratic part of the action around a configuration, in which the length of the interface is an arbitrary parameter. For a practical calculation of this effect we consider a Euclidean space configuration, shown in Fig.2.3, with the string lying flat along the x axis, and the interface between the two phases being at $x=0$. The length of the world line for the boundary is thus the total size T of the world sheet in the time direction. It should be mentioned that such configuration with different string tension on each side of the boundary is not stationary for the action (2.1). However it can be 'stabilized' by a source term (external force) depending on the coordinate $x(t)$ of the boundary: $\int J(t)x(t)\,dt$, which term does not affect the quadratic in fluctuations part of the action.

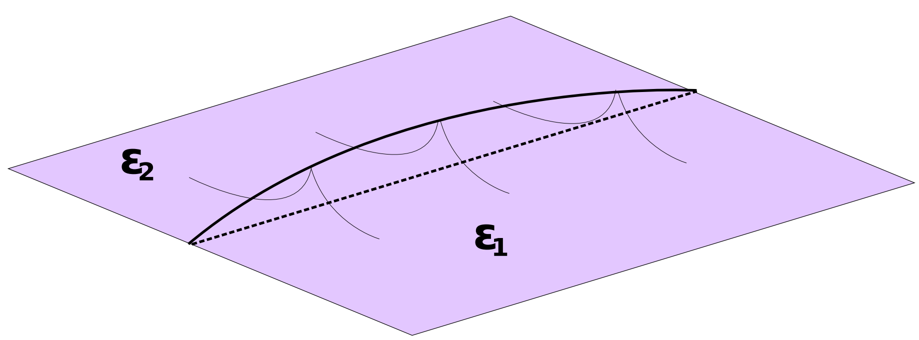

Figure 2.3: The configuration for the calculation of the renormalization of μ.

The Gaussian path integral over the transverse coordinates $z(x,t)$ is then to be calculated with zero boundary conditions at the edges of the total world sheet. Using the notation $\xi(t)$ for the transverse shift of the boundary, this condition implies $\xi(0) = \xi(T) = 0$, so that the function $\xi(t)$ has the Fourier expansion of the form

$$\xi(t) = \sqrt{\frac{2}{T}} \sum_{n=1}^{\infty} a_n \, \sin\left(\frac{\pi \, n \, t}{T}\right) \; , \tag{2.41}$$

and a similar expansion applies to the regulator boundary function $\xi_R(t)$. The part of the effective action associated with the boundary is determined by the functions $z_c(x,t)$ for the transverse shift of the string and the corresponding regulator functions $Z_c(x,t)$ that satisfy the equations

$$\Delta z_c = 0 \quad \text{and} \quad (\Delta - M^2) \, Z_c = 0 \; , \tag{2.42}$$

and the boundary conditions

$$z_c(0,t) = \xi(t) \; , \qquad Z_c(0,t) = \xi_R(t) \tag{2.43}$$

as well as zero boundary conditions at the edges of the total world sheet. One can readily find these functions for each harmonic of the boundary values ξ and ξ_R, using the solutions for z_c and Z_c in each harmonic:

$$z_c^{(n)}(x,t) = \exp\left(-|x| \, \frac{\pi \, n}{T}\right) \, \sin\left(\frac{\pi \, n \, t}{T}\right) \; , \tag{2.44}$$

and

$$Z_c^{(n)}(x,t) = \exp\left[-|x| \sqrt{\left(\frac{\pi \, n}{T}\right)^2 + M^2}\,\right] \, \sin\left(\frac{\pi \, n \, t}{T}\right) \; . \tag{2.45}$$

In order to separate the boundary effect in the path integral around the considered configuration from the bulk effects, we again divide it by the path integral around the configuration where the whole world sheet is occupied by the same phase of the string. The latter phase can be chosen with either of the tensions, or with an arbitrary tension ε. Such division results, as previously, in the cancellation of the bulk contributions, and the remaining part of the effective action associated with the boundary is written in terms of the regularized path integral over the boundary function ξ as

$$\mu_R \, T = \mu \, T -$$

$$\ln \frac{\int \mathcal{D}\xi \, \exp\left\{-\frac{1}{2} \int dt \left[\mu \dot{\xi}^2 + \xi(t) \, \left(\varepsilon_2 \, z_c'(x,t)|_{x\to -0} - \varepsilon_1 \, z_c'(x,t)|_{x\to +0}\right)\right]\right\}}{\int \mathcal{D}\xi_R \, \exp\left\{-\frac{1}{2} \int dt \left[\mu \dot{\xi_R}^2 + \xi_R(t) \, \left(\varepsilon_2 \, Z_c'(x,t)|_{x\to -0} - \varepsilon_1 \, Z_c'(x,t)|_{x\to +0}\right)\right]\right\}} +$$
$$\ln \frac{\int \mathcal{D}\xi \, \exp\left\{-\varepsilon \int dt \xi(t) \, z_c'(x,t)|_{x\to -0}\right\}}{\int \mathcal{D}\xi_R \, \exp\left\{-\varepsilon \int dt \xi_R(t) \, Z_c'(x,t)|_{x\to -0}\right\}} \,, \tag{2.46}$$

where $\mu_R = \mu + \delta\mu$ is the renormalized mass parameter. The correction to μ can thus be written in the form

$$\delta\mu = -\frac{1}{2T} \ln \left\{ \prod_{n=1}^{\infty} \frac{n^2 + \tilde{b} \sqrt{(MT/\pi)^2 + n^2}}{n^2 + \tilde{b}\, n} \prod_{n=1}^{\infty} \frac{n}{\sqrt{(MT/\pi)^2 + n^2}} \right\} \,, \tag{2.47}$$

where

$$\tilde{b} = \frac{\varepsilon_1 + \varepsilon_2}{\mu} \frac{T}{\pi} \,. \tag{2.48}$$

In the limit $\varepsilon_1 + \varepsilon_2 \ll \mu M$ one can directly apply the results in Eqs.(2.36) and (2.39) for evaluation of the expression (2.47) and write

$$\begin{aligned} \delta\mu &= -\frac{1}{2T} \left(\tilde{b} \ln \tilde{b} - \tilde{b} - \frac{MT}{2} + \pi \sqrt{\tilde{b} \, \frac{MT}{\pi}} - \tilde{b} \ln \frac{MT}{\pi} - \tilde{b} \, (1 - \ln 2) \right) \\ &= \frac{M}{4} - \frac{1}{2} \sqrt{\frac{(\varepsilon_1 + \varepsilon_2) \, M}{\mu}} - \frac{\varepsilon_1 + \varepsilon_2}{2\pi\mu} \ln \frac{2 \, (\varepsilon_1 + \varepsilon_2)}{\mu \, M} \,, \end{aligned} \tag{2.49}$$

where a use is made of the Stirling formula

$$\ln \frac{\Gamma(\tilde{b} + 1)}{\sqrt{2\pi \tilde{b}}} \to \tilde{b} \, (\ln \tilde{b} - 1) \,,$$

considering that $\tilde{b}$ is proportional to large T.

In the limiting case where $\varepsilon_1 + \varepsilon_2 \gg \mu M$ the correction $\delta\mu$ vanishes, so that the renormalization effect is negligible.

The result

We can now assemble all the relevant elements into a formula for the rate of the considered transition. Clearly, the path integration over the variables z factorizes for each of the $(d - 2)$ transverse dimensions, so that the expression for the decay rate takes the form

$$\gamma_s = \frac{\varepsilon_1 - \varepsilon_2}{2\pi} \, \mathcal{R}^{d-2} \, \exp\left(- \frac{\pi \, \mu^2}{\varepsilon_1 - \varepsilon_2} \right) \,, \tag{2.50}$$

where μ is the zeroth order mass parameter. In the case of large string tension, $\varepsilon_1 + \varepsilon_2 \gg \mu\, M_0$, the 'bare' μ coincides with the renormalized one, and the factor $\mathcal{R}$ is equal to one. It can be noted that the resulting obvious expression for the rate is also correctly given by Eq.(1.3) as soon as the factor F in Eq.(1.4) is taken in the limit $\varepsilon_1 - \varepsilon_2 \ll \varepsilon_1 + \varepsilon_2$: $F \to 1$, which limit, as previously discussed, is mandated in this case.

In the opposite limit of heavy μ, $\varepsilon_1 + \varepsilon_2 \ll \mu\, M_0$, both the expression (2.40) depends on the regulator mass M and the M-dependent renormalization of μ is essential. Taking into account that each of the transverse dimensions contributes additively to $\delta\mu$ and expressing in Eq.(2.50) the bare μ through the renormalized one: $\mu = \mu_R - \delta\mu$, one readily finds that the dependence on the regulator mass M cancels in the transition rate, and one arrives at the formula given by Eq.(1.3) and Eq.(1.4).

The formula (1.4) is applicable for arbitrary ratio of the string tensions $\varepsilon_2/\varepsilon_1$. In particular it can be also applied at $\varepsilon_2 = 0$, in which case the considered transition describes a complete breaking of the string.

It can be also noted that numerically the factor F depends very moderately on the ratio of the tensions and changes approximately linearly between $F(0) = e/\sqrt{4\pi} = 0.7668\ldots$ and $F(1) = 1$. We thus conclude that the two-dimensional expression $(\varepsilon_1 - \varepsilon_2)/(2\pi)$ for the pre-exponential factor in the transition rate provides a fairly accurate approximation in higher dimensions as well, as long as the exponential factor is expressed in terms of the physical renormalized mass μ_R.

There is however an interesting methodical point pertaining to the considered here problem. Indeed, as was already mentioned, the difference from the problem of particle creation by external electric field is in that the motion of the ends of the string involves in addition to the mass μ also an adjacent part of the string. In terms of the calculation of the path integrals around the bounce the difference is in that the spectrum of soft modes in the particle creation problem (as well as in that of the two-dimensional false vacuum decay) consists only of the modes associated with one-dimensional world line of the boundary of the bounce. The entire pre-exponential factor can then be found using the effective low energy action for these modes[46]. In the considered here string transition there are also low modes in the bulk of the world sheet of the string, and there is no parametric separation of their eigenvalues from those of the modes associated with fluctuations of the boundary. In the presented calculation the separation of the boundary and bulk variables is achieved through an 'artificial' organization of the normalization partition function for a flat string into boundary and bulk factors $\mathcal{Z}_{\rm boundary}$ and $\mathcal{Z}_{\rm bulk}$. The bulk contribution then cancels in the ratio of the partition functions near the bounce and near a flat string, so that the remaining calculation is reduced to considering the integrals over the boundary functions only. One can also readily notice that the additional contribution to the action from the boundary terms as e.g. those with the functions z_{1c} and z_{2c} in Eq.(2.20) corresponds to precisely the effect of 'dragging' of the string by its end.

2.1.2 String transition at non-zero temperature

The formula in Eq.(2.4) corresponds to a calculation of the decay rate as the imaginary part of the energy of the initial string. At a finite temperature T the corresponding relevant quantity is the imaginary part of the free energy[47], which one can calculate in the Euclidean space by considering periodic in time configurations with the period $\beta = T^{-1}$. In other words the thermal calculation corresponds to the path integration in the Euclidean space-time having the topology of a cylinder. The nucleation rate is then described by the same formula (2.4) with the action and the area being calculated over one period, $A = X\,\beta$, where X is the length of the string along the spatial dimension.

We consider only sufficiently small temperatures $\beta > \ell_c$, at which temperatures we show the thermal effects behaving as powers of T, which distinguish the string process from the decay of metastable vacuum[32]. We also treat the length X of the metastable string as the largest length parameter in the problem, so that $\beta \ll X$. Under these conditions the bounce corresponding to the action (2.1) is the same as at zero temperature, except that it is placed on a long cylinder rather than on a large flat plane (Fig.2.4).

As above we aim at calculating the path integral over the variations of the string around the bounce configuration as illustrated in Fig. 2.5. The coordinates on the cylinder (or on the plane where all the points separated by $n\beta$ along Euclidean time are identified) are t, x, with t being the periodic time coordinate, and the coordinate orthogonal to the surface of the cylinder is z. The boundary conditions for the

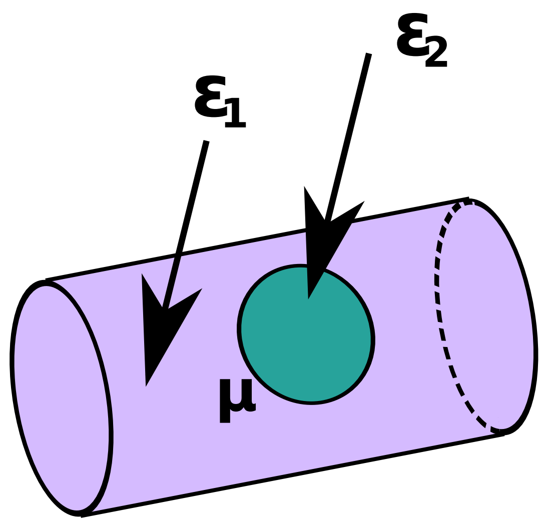

Figure 2.4: Bounce configuration for nonzero temperature

configurations over which we integrate are

$$z\left(t, x = \pm\frac{X}{2}\right) = 0, \qquad z(t+\beta, x) = z(t, x)\ . \tag{2.51}$$

In polar coordinates (r, θ) in the (t, x)-plane one finds the variation of the bounce action (2.1) similar to (2.7) while for the flat string one has the expression analogous to that in Eq.(2.8).

As it was shown for the case of zero temperature the partition functions factorize, therefore the expression for a decay rate is given by (2.10), with the only difference for a nonzero temperature case that the boundary conditions for outer solution are

$$z_{1c}(t+\beta, x) = z_{1c}(t, x), \tag{2.52}$$

and

$$z_{1c}\left(t, x = \pm\frac{X}{2}\right) = 0 \tag{2.53}$$

while the inner solution z_{2c} is required to be regular inside the disk.

In order to do the path integrals as before one can expand the boundary function $\xi(\theta)$ in angular harmonics. For the inner solution z_{2c} to the Laplace equation, i.e. at $r \leq R_s$ one finds no difficulty in finding the harmonics matching the boundary function at the interface (2.25). For the outer solution z_{1c} however there is a difficulty due to the mismatch between the symmetry of the boundary and of the periodicity

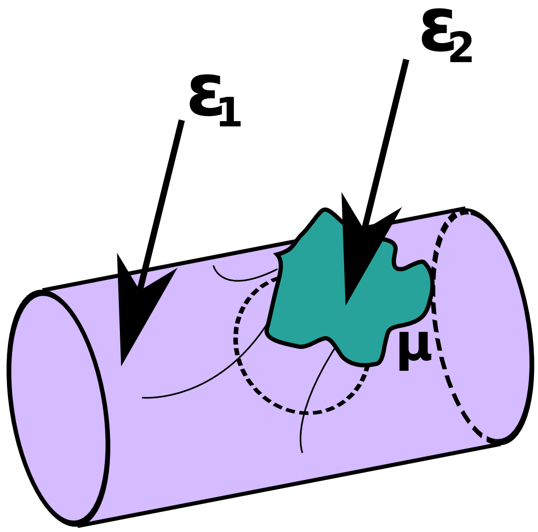

Figure 2.5: Variation of the bounce configuration

conditions. It is impossible to choose the solution to the Laplace equation for outer string bulk variable to be $z^{(n)}_{1c}(r,\theta) \sim r^{-n}$, since it is not periodic in time. In this situation in order to have a periodic solution one can perform a periodic mapping of the cylinder on the plane and consider the outer solution of the Laplace equation as the sum of the solutions produced by a "source" at each period as illustrated in Fig. 2.6. Introducing the complex variable $w = t + i\,x$, we construct the solutions for the functions $z_{1c}(r,\theta)$ using the harmonic real and imaginary parts of the following basis set of periodic functions, satisfying the boundary condition (2.53) at large $|x|$,

$$\begin{aligned} g_0(w) &= \ln\left[\sin\left(\frac{\pi\, w}{\beta}\right)\right] - \frac{\pi\, X}{2\beta} - i\,\frac{\pi x}{X} + \ln 2\;, \\ g_1(w) &= \frac{\pi\, R_s}{\beta}\,\cot\left(\frac{\pi\, w}{\beta}\right) + i\,\frac{2\,\pi\, R_s\, x}{\beta\, X}\;, \\ g_k(w) &= \frac{R_s^k}{w^k} + \sum_{n=1}^{\infty}\left[\frac{R_s^k}{(w-n\beta)^k} + \frac{R_s^k}{(w+n\beta)^k}\right], \quad \text{for } k > 1\;. \end{aligned} \tag{2.54}$$

Clearly, these functions are periodic in the Euclidean time by construction, with the period β. Also the functions $g_k(w)$ with $k > 1$ are analytic complex functions, so that their real and imaginary parts are harmonic. In the functions $g_0(w)$ and $g_1(w)$ the explicit dependence on x, introducing non-analyticity, is linear and is thus also

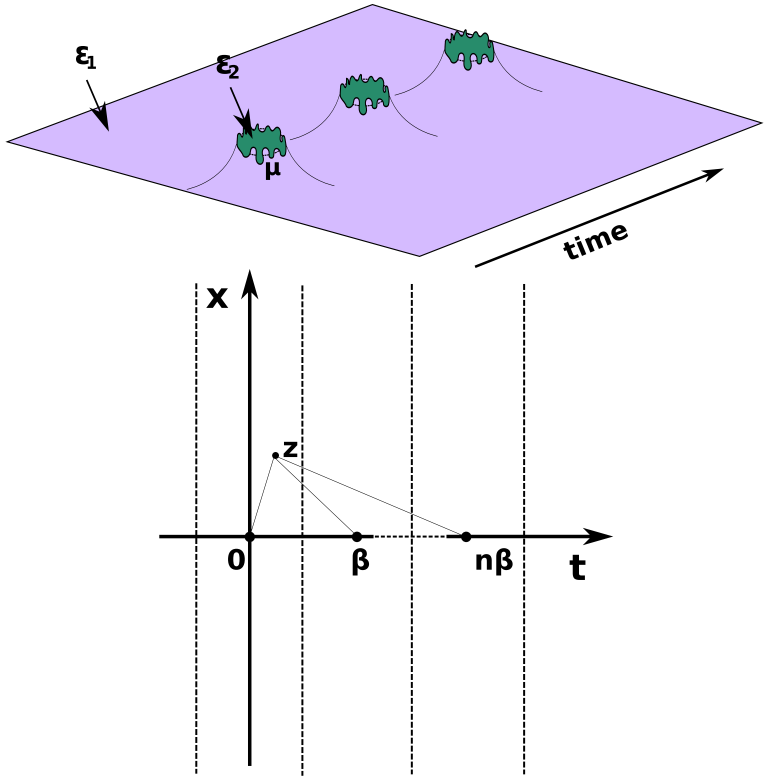

Figure 2.6: Periodic configuration on the plane

harmonic, so that their real and imaginary parts do satisfy the Laplace equation.

An arbitrary periodic outer solution to the Laplace equation, satisfying the boundary condition (2.53) at large $|x|$ can be expanded in the series

$$z_{1c}(r,\theta) = A_0 \mathrm{Re}\, g_0(w) + \sum_{k=1}^{\infty} A_k \mathrm{Re}\, g_k(w) + B_k \mathrm{Im}\, g_k(w), \tag{2.55}$$

The disadvantage of this set of solutions (hence of such an expansion) is that it is not orthogonal, so that the expression for the action up to quadratic terms is not diagonal in this basis. Therefore, the calculation of the integral is not just a calculation of the product of eigenvalues. To find the integral over the amplitudes of the Fourier harmonics one has to express the amplitudes of the solutions chosen A_k,B_k in terms of the amplitudes a_k, b_k. The relation between the coefficients A_k,B_k and a_k, b_k can be found from the matching condition (first relation in 2.16) on the interface

$$\frac{a_0}{\sqrt{2\pi}}+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left[a_n\,\cos(n\theta)+b_n\,\sin(n\theta)\right]=A_0\,g_0(w)\Big|_{r=R_s}+\sum_{k=1}^{\infty}\left[A_k\mathrm{Re}g_k(w)+B_k\mathrm{Im}g_k(w)\right]\Big|_{r=R_s}. \tag{2.56}$$

One can readily notice that the function g_0 contains a large constant term, proportional to X, which totally dominates the matching condition for the a_0 mode at $r = R_s$, so that

$$\frac{a_0}{\sqrt{2\pi}}=-A_0\,\frac{\pi\,X}{2\beta}\,\left[1+O(R/X)\right]. \tag{2.57}$$

For this reason the effect of the mixing between a_0 and higher modes is suppressed by inverse powers of X and can be ignored in the limit of a long string. For this reason in considering the mixing of the modes in the following calculation we keep only $k \neq 0$. Furthermore the linear in x terms in the functions g_0 and g_1 are suppressed at $r = R_s$ by the factor R_s/X and can also be disregarded.

In what follows we consider the expansion of the functions g_k at $r = R_s$ in powers of R_s/β, which expansion, as will be seen later, converges at $R_s < \beta/2$. Using

$$(1+x)^{-k}=1+\sum_{p=1}^{\infty}(-1)^p x^p C^p_{p+k-1}, \tag{2.58}$$

where C^p_{p+k-1} are the binomial coefficients,

$$C^p_{p+k-1}=\frac{(p+k-1)!}{p!(k-1)!} \tag{2.59}$$

and also a definition of the Riemann ζ-function

$$\zeta(k)=\sum_{n=1}^{\infty}n^{-k}\ , \tag{2.60}$$

we find for $k \neq 0$

$$g_k(w)=\frac{R_s^k}{w^k}+\sum_{p=1}^{\infty}d_{pk}\left(\frac{w}{R_s}\right)^p, \tag{2.61}$$

with

$$d_{pk}=\left[(-1)^k+(-1)^p\right]\left(\frac{R}{\beta}\right)^{k+p}\,\zeta(p+k)\,C^p_{p+k-1}. \tag{2.62}$$

We have omitted in the expression (2.61) a constant term, which describes the mixing with the a_0 mode as well as a term explicitly proportional to R_s/X. Using the expansion (2.61) for $g_k(w)$ and considering the real part of the functions $g_k(w)$, one can find the coefficients a_l in terms of A_k

$$a_l = \frac{1}{\pi}\sum_k A_k \int_0^{2\pi} \mathrm{Re} g_k \cos(l\theta) d\theta = A_l + \sum_{k=1}^{\infty} A_k\, d_{lk} \ , \tag{2.63}$$

or in the matrix form

$$A = (1+D)^{-1}\, a \ , \tag{2.64}$$

where matrix D has elements d_{lk}. Similarly for the imaginary part one gets

$$b_l = \frac{1}{\pi}\sum_k B_k \int_0^{2\pi} \mathrm{Im} g_k \cos(l\theta) d\theta = B_l - \sum_k B_k d_{lk} \tag{2.65}$$

and

$$B = (1-D)^{-1}\, b \ . \tag{2.66}$$

As usual, the contributions from the cos and sin modes are independent, and we consider the contribution to the boundary term from the even (cos) harmonics first

$$-\int_0^{2\pi} \left(R_s g^{(R)} \partial_r g^{(R)}\right)\Big|_{R_s} d\theta = \int_0^{2\pi} d\theta \left[\sum_l a_l\ cos(l\theta)\right] \sum_k \left[k\left(\frac{R_s}{r}\right)^k \cos(k\theta) - \sum_{p=1}^{\infty} p d_{pk}\right] =$$
$$\sum_k A_k \left[k a_k - \sum_p p a_p d_{pk}\right] = \sum_{k,p} a_p \left[k\delta_{pk} - p d_{pk}\right] A_k \ . \tag{2.67}$$

Introducing the matrix

$$\hat{N} = \mathrm{diag}(1, 2, \ldots, n, \ldots) \ , \tag{2.68}$$

one can rewrite the expression (2.67) in the matrix form

$$-\int_0^{2\pi} \left(R_s g^{(R)} \partial_r g^{(R)}\right)\Big|_{R_s} d\theta = a\hat{N}(1-D)A \ . \tag{2.69}$$

A substitution in this expression of the solution for A in terms of a (2.64) leads to

$$-\int_0^{2\pi} \left(R_s g^{(R)} \partial_r g^{(R)}\right)\Big|_{R_s} d\theta = a\hat{N}(1-D)(1+D)^{-1} a \ . \tag{2.70}$$

Clearly, for the odd (sin) modes one gets the same expression with the replacement $D \to -D$. Collecting all the terms together one can write the result for the boundary partition functions (2.20) and (2.23) as

$$\begin{aligned} \mathcal{Z}_{12(\mathrm{boundary})} &= \mathrm{Det}\left[(\varepsilon_1-\varepsilon_2)\hat{N}^2 + \varepsilon_2\hat{N} + \varepsilon_1\hat{N}(1-D)\frac{1}{1+D}\right]^{-1/2} \cdot \{D \to -D\} \\ \mathcal{Z}_{1(\mathrm{boundary})} &= \mathrm{Det}\left[\varepsilon_1\hat{N} + \varepsilon_1\hat{N}(1-D)\frac{1}{1+D}\right]^{-1/2} \cdot \{D \to -D\}, \end{aligned} \tag{2.71}$$

where $\{D \to -D\}$ means that one should take the expression and make the replacement $D \to -D$.

The zero temperature limit for the probability rate γ_s formally corresponds to setting $D \to 0$. Thus one can use the result for the zero temperature decay rate in Eq.(2.50), and concentrate on a calculation of the thermal catalysis factor K defined as

$$\left.\frac{d\Gamma}{d\ell}\right|_T = \mathcal{K}_s\, \gamma_s\,. \tag{2.72}$$

In a d-dimensional theory, i.e. with $d-2$ transverse dimensions, the catalysis factor can be written as $\mathcal{K}_s = G^{d-2}$, where G is the factor per each transverse direction given by

$$G = \frac{\mathcal{Z}_{12(\text{boundary})}}{\mathcal{Z}^{D=0}_{12(\text{boundary})}}\, \frac{\mathcal{Z}^{D=0}_{1(\text{boundary})}}{\mathcal{Z}_{1(\text{boundary})}}\,. \tag{2.73}$$

According to Eq.(2.71) it is a matter of simple algebra to express the factor G in terms of the matrix D:

$$G = \text{Det}\left[1 - \left(\frac{\hat{N}-1}{\hat{N}+b}\, D\right)^2\right]^{-1/2} \tag{2.74}$$

with b defined in Eq.(2.35).

Analysis of the general formula

In this section we consider in some detail the temperature effect in the string transition rate described by our general formula (2.74) in the situation where the inverse temperature is larger then the diameter of the classical configuration (bounce): $\beta > 2R_s$. We first notice that due to the presence of the factor $(\hat{N}-1)$ the first elements from the first row and the first column of the matrix D, d_{1k} and d_{p1}, enter the expression $[(\hat{N}-1)\, D]^2$ with zero coefficients, so that the final result (2.74) does not depend on them. Furthermore, one can see from Eq.(2.62) that the matrix element d_{pk} is not equal to zero only if the indices p and k have the same parity. Hence, there is no mixing between the amplitudes with even (a_{2l}, b_{2l}) and odd (a_{2l+1}, b_{2l+1}) indices. Therefore the determinant in the (2.74) can be written as a product of determinants corresponding to the even and odd amplitudes

$$\mathrm{Det}\left[1-\left(\frac{\hat N-1}{\hat N+b}\,D\right)^2\right]=\mathrm{Det}\left[1-\left(\frac{2\,\hat N-1}{2\,\hat N+b}\,U\right)^2\right]\,\mathrm{Det}\left[1-\left(\frac{2\,\hat N}{2\,\hat N+1+b}\,V\right)^2\right], \tag{2.75}$$

where the matrix elements of the matrices U and V are

$$U_{pk}=d_{2p\,2k}\,,\qquad V_{pk}=d_{2p+1,\,2k+1}\ , \tag{2.76}$$

and the indices p and k take values $1, 2, \ldots$.

For practical calculations it is also convenient to write the expressions for the elements of the matrices entering in Eq.(2.75) in terms of their indices:

$$\begin{aligned}\left(\frac{2\,\hat N-1}{2\,\hat N+b}\,U\right)_{pk} &= 2\,\frac{2p-1}{2p+b}\,\frac{(2p+2k-1)!}{(2p)!\,(2k-1)!}\,\zeta(2p+2k)\left(\frac{R}{\beta}\right)^{2p+2k}, \\ -\left(\frac{2\,\hat N}{2\,\hat N+1+b}\,V\right)_{pk} &= \frac{4p}{2p+1+b}\,\frac{(2p+2k+1)!}{(2p+1)!\,(2k)!}\,\zeta(2p+2k+2)\left(\frac{R}{\beta}\right)^{2p+2k+2}\end{aligned} \tag{2.77}$$

with $p, k = 1, 2, \ldots$.

In order to find the first thermal correction at low temperature one can expand the matrices in power series using well known formula for the determinant

$$\mathrm{Det}(1-M)=\exp\left[\mathrm{Tr}\ln\left(1-M\right)\right]=\exp\left[-\mathrm{Tr}\sum_{l=1}^{\infty}\frac{M^l}{l}\right]=1-\mathrm{Tr}M+O(M^2). \tag{2.78}$$

In our case

$$\left(\frac{2\,\hat N-1}{2\,\hat N+b}\,U\right)^2=\begin{pmatrix}\frac{36\,\zeta^2(4)}{(2+b)^2}\left(\frac{R_s}{\beta}\right)^8+\frac{600\,\zeta(6)^2}{(2+b)\,(4+b)}\left(\frac{R_s}{\beta}\right)^{12} & \frac{120\,\zeta(4)\zeta(6)}{(2+b)^2}\left(\frac{R_s}{\beta}\right)^{10} & \cdots \\ \frac{180\,\zeta(4)\zeta(6)}{(2+b)\,(4+b)}\left(\frac{R_s}{\beta}\right)^{10} & \frac{600\,\zeta(6)^2}{(2+b)\,(4+b)}\left(\frac{R_s}{\beta}\right)^{12} & \cdots \\ \vdots & \vdots & \ddots\end{pmatrix}, \tag{2.79}$$

$$\left(\frac{2\,\hat N}{2\,\hat N+1+b}\,V\right)^2=\begin{pmatrix}\frac{1600\,\zeta(6)^2}{(3+b)^2}\left(\frac{R_s}{\beta}\right)^{12} & \cdots \\ \vdots & \ddots\end{pmatrix}. \tag{2.80}$$

Therefore the first correction to the zero temperature value of the rate is proportional to R_s^8/β^8 and is given by Eq.(1.6).

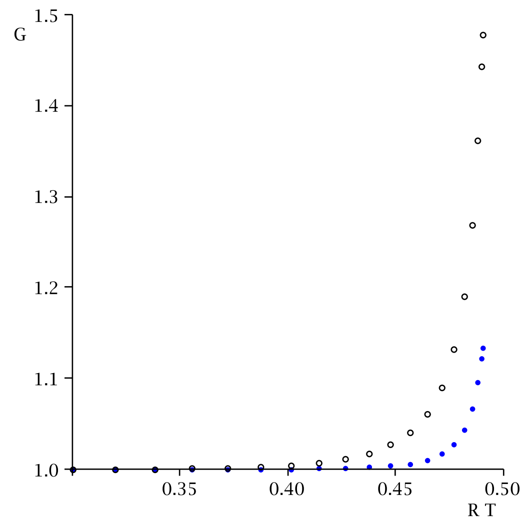

Figure 2.7: The thermal catalysis factor per each transverse dimension vs RT for $b = 1$ (open circles) and for $b = 10$ (solid circles)

The series for the function $G(T)$ diverges when $\beta = 2\,R_s$. The corrections for the temperature close to $(2R_s)^{-1}$ can be found numerically. The proximity to this point defines the number of terms which should be taken into account in the series for $G(T)$. The plot for the function $G(T)$ vs the parameter $R_s/\beta = R_s\,T$ calculated numerically with 50 first rows and columns retained in each of the matrices U and V is shown in Fig. 2.7.

2.1.3 Decay of a string induced by two-particle collisions

As we have seen in the previous sections, the only difference between the calculation of the decay at finite temperature T and at $T = 0$ arises at the level of calculating the pre-exponential factor due to the functional determinant of the quadratic part of the action and amounts to the standard treatment of the boundary conditions in (Euclidean) time for the fluctuations: zero boundary conditions at large time for considering zero temperature and periodic boundary conditions with period $\beta = 1/T$ at finite temperature.

The formula (2.4) is in fact the unitarity relation [45] between the decay rate and the imaginary part of the amplitude of the transition amplitude from the false vacuum to the false vacuum $\langle (\text{vacuum } 1)_{\text{out}} | \text{vacuum } 1)_{\text{in}} \rangle$. Similarly, one can treat the probability of the string breakup by an excited state in terms of the bounce contribution to the imaginary part of its forward scattering amplitude. Proceeding to discussion of the string decay induced by the Goldstone bosons, we readily notice that a state with just one Goldstone boson cannot induce the destruction of the string. Indeed the total probability of such induced process is Lorentz invariant and thus can depend only on the (Lorentz) square of the particle momentum k^2. The Goldstone bosons are massless, so that for them $k^2 = 0$ and is fixed. If a single massless boson produced an effect on the decay, this effect would thus be not depend on the energy ω of the boson. In the limit $\omega \to 0$ the Goldstone boson is indistinguishable from the vacuum. (In other words the limit $\omega \to 0$ corresponds to an overall shift of the string in transverse direction.) Thus the decay rate of a single - boson state is the same as that of the vacuum, and the presence of a single boson with any energy produces no effect.

The simplest excited state, contributing to the string destruction, is that with two particles. The probability W_2 of creation of the critical gap in a collision of two particles with the two-momenta k_1 and k_2 can depend only on the Lorentz invariant $s = (k_1 + k_2)^2$. Obviously, for two particles colliding on a string $s = 4\omega_1\omega_2$ (and $s = 0$ for two particles moving in the same direction, i.e. non-colliding). Using the unitarity relation, this probability can in principle be found in terms of the imaginary part of the forward scattering amplitude $A(k_1, k_2; k_1, k_2)$:

$$W_2 = C \frac{\text{Im} A(k_1, k_2; k_1, k_2)}{\omega_1 \omega_2} \tag{2.81}$$

where the factor $\omega_1\omega_2$ is the usual flux factor, and the constant C does not depend on either of the energies, and is determined by specific convention on the normalization of the amplitude.

The dynamics of the Goldstone bosons on the string, including their scattering, can be considered in terms of the transverse shift $z_i(x)$ treated as a two-dimensional field described by the Nambu-Goto action (2.1) as

$$S = \varepsilon_1 \int_{A_1} \sqrt{1 + (\partial_\alpha z_i)^2}\, d^2x + \varepsilon_2 \int_{A_2} \sqrt{1 + (\partial_\alpha z_i)^2}\, d^2x + \mu \int_P \sqrt{1 + (\partial z_i / \partial l)^2}\, dl \ , \tag{2.82}$$

where dl is the element of the length of the interface P between the phases.

Clearly, at low energy of the Goldstone bosons one can make use of the expansion in Eq.(2.82) in powers of (∂z) which generates the expansion of the scattering amplitudes in the momenta of the particles with each one entering the amplitude with (at least) one power of its energy ω, as is mandatory for Goldstone bosons. For the scattering in the metastable state this generates expansion in powers of $\omega/\sqrt{\varepsilon_1}$, so that in the lowest order in this ratio, that we are discussing here, it is sufficient to retain only the quadratic in (∂z) terms in the action (2.82). It should be noted that in spite of retaining only the quadratic terms, the multi-boson scattering amplitudes do not vanish, since the necessary non-linearity arises from the bounce configuration. In other words, the bosons scatter 'through the bounce'. The energy expansion for these amplitudes is determined by the bounce scale ℓ_c, so that the parameter of such expansion is $(\omega \ell_c)$ which we do not assume to be small. Clearly, the condition for applicability of the present approach where the terms of order $\omega/\sqrt{\varepsilon_1}$ are dropped, while those with the parameter $\omega \ell_c$ are retained is that

$$\frac{\mu^2}{\varepsilon_1 - \varepsilon_2} \gg 1 - \frac{\varepsilon_2}{\varepsilon_1} \ , \tag{2.83}$$

which is always true if the semiclassical tunneling can be applied at all to the string decay.

We shall now show that in the on-shell scattering through the bounce each external leg enters with at least two powers of its energy. Let us start, for the simplicity of illustration, with the binary scattering. The general two $\to$ two scattering amplitude $A(k_1, k_2; k_3, k_4)$ can be related in the standard application of the reduction formula to the connected 4-point Green's function $\langle \text{vacuum } 1 | T\{z(x_1) z(x_2) z(x_3) z(x_4)\} | \text{vacuum } 1 \rangle$, which in turn is an analytical continuation of the Euclidean connected correlator

$$\langle \text{vacuum } 1 | z(x_1) z(x_2) z(x_3) z(x_4) | \text{vacuum } 1 \rangle = \mathcal{Z}_0^{-1} \, \frac{\delta^4 \, \mathcal{Z}_b[j]}{\delta j(x_1) \, \delta j(x_2) \, \delta j(x_3) \, \delta j(x_4)} \, \Big|_{j=0} \ . \tag{2.84}$$

The latter expression for the correlator assumes the conventional procedure of introducing in the action the source term $\int j(x) \, z(x) \, d^2 x$ for the Goldstone variable $z(x)$ and $\mathcal{Z}_b[j]$ is the path integral around the bounce in the presence of the source.

The low-energy limit of the on-shell scattering amplitude is determined by the correlator at widely separated points $x_1, \ldots, x_4$. Weak δ-function sources 'prop' the string in the transverse direction at those points as shown in Fig. 2.8, and generally

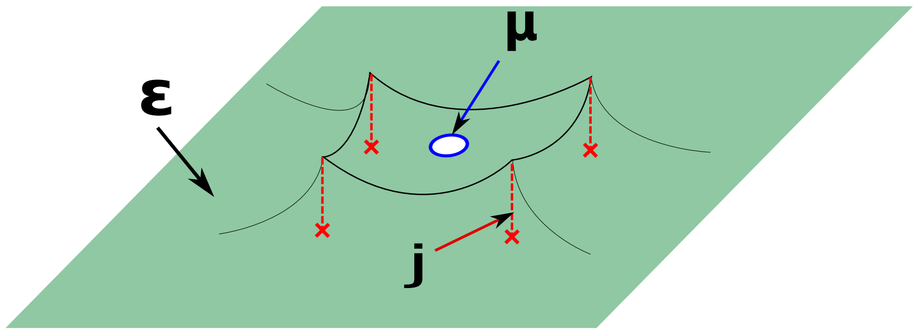

Figure 2.8: Bounce configuration distorted by the sources

distort the bounce located between those points. The correlator (2.84) is determined by the distortion of the bounce by all four sources, so that at large separation between the points the bounce is located far (in the scale of its size ℓ_c) from the sources, where the overall distortion of the world sheet for the string is small and is slowly varying on the scale ℓ_c. One can therefore expand the background field z_s generated by the sources at the bounce location in the Taylor series around an arbitrarily chosen inside the bounce point $x = 0$. Clearly the first term of this expansion $z_s(0)$ corresponds to an overall transverse shift of the string and does not alter the bounce shape and the action. Moreover the linear term in this expansion, proportional to the gradient $\partial z_s(0)$ does not change the bounce action either. Indeed this term corresponds to a linear incline of the string in the transverse coordinates, and can be eliminated by an overall rotation of the string in the d dimensional space. In other words the absence of the linear in the gradient of z_s term in the action is a direct consequence of the d-dimensional Lorentz invariance of the string. We thus arrive at the conclusion that the expansion for the distortion of the bounce starts from the second order in the derivatives of z_s (the curvature of the background world sheet), which for a connected correlator implies that in the expansion in the energy each external source enters with at least two powers of the energy. This conclusion clearly applies to the on-shell amplitudes with arbitrary number of external legs, since the generalization to multiparticle scattering is straightforward.

In particular, the binary scattering amplitude $A(k_1, k_2; k_3, k_4)$ at low energy scales as the eighth power of the energy, and the related to it (by the Eq.(2.81)) probability of the string destruction by collision of two particles is proportional to the sixth power of the energy scale, or equivalently, to s^3 at small s. Applying in the same manner the unitarity condition to the forward scattering amplitude of a general n-particle state, one readily concludes that the corresponding probability W_n of the induced string breakup scales with the overall energy scale ω as $W_n \propto \omega^{3n}$.

Thermal decay and the string destruction by particle collisions

The described procedure for calculating the bounce-induced scattering amplitudes in terms of the Euclidean correlators runs into the technical difficulty of calculating the bounce distortion by the background created by the sources. Furthermore, this procedure obviously involves a great redundancy, if the final purpose is to calculate the probabilities W_n, i.e. only the absorptive parts of the forward scattering on-shell amplitudes are the quantities of interest. We find that in fact one can directly calculate the probabilities W_n by an appropriate interpretation of the more readily calculable thermal decay rate in terms of the collision-induced probabilities. Such an approach is technically more tractable due to the fact that in the thermal calculation the bounce is not distorted, i.e. it is still a flat disk, and only the boundary conditions for the modes of fluctuations are modified. We first illustrate this approach by using the first nontrivial term of the low temperature expansion in Eq.(1.6) for calculation of the low energy limit of the binary probability W_2.

Indeed, the temperature dependent factors in the catalysis factor $\mathcal{K}_s$ arise from the contribution to the critical gap nucleation of the boson collision, weighed with the thermal number density distribution for the massless Goldstone bosons

$$d\,n(k) = \frac{1}{e^{\omega/T} - 1} \, \frac{d\,k}{2\pi} \ , \tag{2.85}$$

where k is the spatial momentum, $|k| = \omega$, running from $-\infty$ to $+\infty$. Given the low energy behavior $W_n \propto \omega^{3n}$ found in the previous section, one readily concludes that the contribution of n-particle string destruction starts with the term T^{4n} in the low T expansion for K. Thus the first term written in Eq.(1.6) can arise only from $n = 2$ i.e. from the binary production of the critical gap. Writing the expansion in s of the probability W_2 as $W_2 = c_3\, s^3 + \ldots$, one determines the coefficient c_3 of the s^3 term by comparing the result in Eq.(1.6) with the one calculated in terms of the two-particle collision rate using the number density distribution (2.85):

$$(d-2)\,\gamma_s\,\frac{\pi^8}{450}\left(\frac{\varepsilon_1-\varepsilon_2}{3\varepsilon_1-\varepsilon_2}\right)^2\left(\frac{\ell_c T}{2}\right)^8 =$$
$$(d-2)\,c_3\int_0^\infty \frac{d\omega_1}{e^{\omega_1/T}-1}\int_0^\infty \frac{d\omega_2}{e^{\omega_2/T}-1}\,\frac{s^3}{4\pi^2} = (d-2)\,c_3\,\frac{16\,\pi^6}{225}\,T^8\;, \quad (2.86)$$

where the relation $s = 4\omega_1\omega_2$ is used and the integration is done only for the bosons with opposite signs of their momenta, and avoiding the double-counting for the identical particles. The factor $(d-2)$ counting the number of the transverse dimension, corresponds to the summation over the polarizations of the Goldstone bosons. The expression for the coefficient c_3 following from Eq.(2.86) thus determines the first term in the expansion for W_2

$$W_2 = \gamma_s\,R_s^2\left[\frac{\pi^2}{32}\left(\frac{\varepsilon_1-\varepsilon_2}{3\varepsilon_1-\varepsilon_2}\right)^2 R_s^6\,s^3+\ldots\right]\;. \quad (2.87)$$

(One can notice that at $\varepsilon_2 = 0$ the s^3 term coincides with the first term of expansion of the expression in Eq.(1.7).)

Thermal bath with a chemical potential

The discussed procedure for extracting the coefficients of the energy expansion for the probability of collision-induced string decay is obviously limited to only the first term in W_2. In the higher terms in the temperature expansion for $\mathcal{K}_s$ the contribution of the energy expansion for W_n with different n generally gets entangled. This happens because the temperature is the only parameter and terms originating from different n can contribute in the same power in T. In order to disentangle the terms of higher order in energy in W_n with low n from similar terms originating from higher n we introduce a *negative* chemical potential ν for the Goldstone bosons. Generally such procedure would be impossible, since the number of these bosons is not conserved. However in our application this procedure is fully legitimate. Indeed the thermal state of the string that we study here is that of *collisionless* bosons, in which their number *is* conserved. The string decay, resulting in a change in this number, is a very weak process that we consider only in the first order, which justifies averaging the rate of this process over the unperturbed state with conserved number of particles.

At negative ν the number density distribution of the bosons (2.85) is replaced by

$$d\,n(k) = \frac{1}{e^{\frac{\omega+|\nu|}{T}} - 1} \, \frac{d\,k}{2\pi} \;, \tag{2.88}$$

and by tuning the parameter $|\nu|/T$ one can readily resolve between the contribution of n-particle processes with different n.

The introduction of the chemical potential requires us to modify our previous thermal calculation (see Sec. 2.1.2). The modification of this calculation for a thermal state with a negative chemical potential, where the number density of the bosons be given by Eq.(2.88), is achieved by introducing a 'damping factor' in the periodic sums for the outer functions in Eq.(2.54):

$$g_k(w) \to g_k^{(\nu)}(w) = \frac{R_s^k}{w^k} + \sum_{n=1}^{\infty} \left[\frac{R_s^k}{(w-n\beta)^k} + \frac{R_s^k}{(w+n\beta)^k} \right] \exp\left(-n\,|\nu|\,\beta\right) \;. \tag{2.89}$$

One can readily see that the only net result of the ν dependent factor in the calculation is a modification of the matrix coefficients d_{pk} amounting to the replacement of the factors $\zeta(q)$ by the standard polylogarithm function,

$$\mathrm{Li}_q(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^q} \;,$$

$\zeta(q) \to \mathrm{Li}_q(e^{-|\nu|/T})$. In other words, the catalysis factor for a thermal state with a negative chemical potential is given by the expression

$$\mathcal{K}_s(\nu, T) = \mathrm{Det}\left[1 - \mathcal{D}^2(\nu, T)\right]^{-(d-2)/2} \;, \tag{2.90}$$

with the elements of the matrix $\mathcal{D}(\nu, T)$ having the form

$$\left[\mathcal{D}(\nu,T)\right]_{pk} = -\left[(-1)^k + (-1)^p\right] (R_s T)^{k+p+2} \, \frac{p}{p+b} \, \frac{(p+k+1)!}{(p+1)!\,k!} \, \mathrm{Li}_{p+k+2}\left(e^{-|\nu|/T}\right) \;. \tag{2.91}$$

The dependence on two 'tunable' parameters ν and T in Eq.(2.90) makes it possible to disentangle the contribution of processes with different number of particles from the energy behavior in each of these processes. Such a separation becomes straightforward if one notices that each factor with the polylogarithm Li arises from the integration over the distribution (2.88):

$$\int_0^{\infty} \frac{\omega^q}{e^{\frac{\omega+|\nu|}{T}} - 1} \, d\omega = q!\, T^{q+1} \, \mathrm{Li}_{q+1}\left(e^{-|\nu|/T}\right) \;. \tag{2.92}$$

One therefore concludes that the number of the 'Li factors' in each term of the expansion of the catalysis factor in Eq.(2.90) directly gives the number of particles in the process, while the indices of these polylogarithmic factors give the power of the energy for each particle. Given that the matrix $\mathcal{D}(\nu, T)$ is linear in the 'Li factors', one can count the number of particles contributing to each term of the expansion for $\mathcal{K}(\nu, \mathcal{T})_s$ by counting instead the power of $\mathcal{D}(\nu, T)$. The latter counting is simplified if one rewrites the equation (2.90) in the equivalent form, suitable for the expansion in powers of $\mathcal{D}(\nu, T)$:

$$\begin{aligned}\mathcal{K}_s(\nu, T) &= \exp\left\{-\frac{d-2}{2}\,\mathrm{Tr}\ln\left[1-\mathcal{D}(\nu,T)^2\right]\right\} = \qquad (2.93)\\ 1+\frac{d-2}{2}\,\mathrm{Tr}\left[\mathcal{D}(\nu,T)^2\right] &+ \frac{d-2}{4}\,\mathrm{Tr}\left[\mathcal{D}(\nu,T)^4\right]+\frac{(d-2)^2}{8}\,\left\{\mathrm{Tr}\left[\mathcal{D}(\nu,T)^2\right]\right\}^2+O\left(\mathcal{D}^6\right).\end{aligned}$$

The latter expression merits some observations. The first is that all the terms in the expansion in powers of $\mathcal{D}(\nu, T)$ are positive, which is certainly the necessary condition for the consistency of our interpretation of these terms as corresponding to the probability of the destruction of the string by n-particle collisions. The second is that the string is destroyed only in collisions of *even* number of particles, since the expansion goes in the even powers of $\mathcal{D}(\nu, T)$. Finally, the third observation is related to the dependence in Eq.(2.93) on the number of the transverse dimensions $(d-2)$. Namely, the quadratic in $\mathcal{D}(\nu, T)$ term is proportional to $(d-2)$. This corresponds to that in two-particle collisions only the Goldstone bosons with the same transverse polarization do destroy the string. On the contrary, the quartic in $\mathcal{D}(\nu, T)$ term has one contribution proportional to $(d-2)$ and one proportional to $(d-2)^2$. The linear in $(d-2)$ part corresponds to all the bosons in the collision having the same polarization, while the quadratic in $(d-2)$ part is necessarily contributed by the collisions, where the colliding bosons have different polarization.

Destruction of the string in two-particle collisions at arbitrary $R_s\sqrt{s}$

The expression (2.93) for the catalysis factor $\mathcal{K}_s(\nu, T)$ together with the formulas (2.91) and (2.92) reduce the calculation of the probability of the string breakup by a collision of an arbitrary (even) number n of particles to straightforward, although not necessarily short, algebraic manipulations. In this section we consider in full the most physically transparent case of two-particle collisions. The probability in this case is found from the term in Eq.(2.93) with the trace $\mathrm{Tr}[\mathcal{D}(\nu,T)^2]$. Using Eq.(2.91), this trace can be written as a double sum:

$$\mathrm{Tr}\left[\mathcal{D}(\nu,T)^2\right] = \tag{2.94}$$
$$4\sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(p+b+1)\,(k+b+1)}\,\frac{\left[(p+k+1)!\,(R_sT)^{p+k+2}\,\mathrm{Li}_{p+k+2}\left(e^{-|\nu|/T}\right)\right]^2}{(p-1)!\,(p+1)!\,(k-1)!\,(k+1)!}$$

One can readily recognize the expression in the straight braces here as the integral (2.92) with the power of the energy q given by $(p+k+1)$, and thus identify the coefficient of the same power of $s = 4\omega_1\omega_2$ in the expansion of the probability $W_2(s)$ in s. In this way we find the following formula for $W_2(s)$ in terms of this expansion,

$$\begin{aligned} W_2(s) &= 8\,\pi^2\,\gamma_s\,R_s^2\sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(p+b+1)\,(k+b+1)}\,\frac{(sR_s^2/4)^{p+k+1}}{(p-1)!\,(p+1)!\,(k-1)!\,(k+1)!} \\ &= 8\,\pi^2\,\gamma_s\,R_s^2\left[\Phi_b(\sqrt{s}\,R_s)\right]^2 \ , \end{aligned} \tag{2.95}$$

where the function $\Phi_b(x)$ expands in a single series as

$$\Phi_b(x) = \frac{x}{2}\sum_{p=1}^{\infty}\frac{1}{p+b+1}\,\frac{x^{2p}}{(p-1)!\,(p+1)!} \ . \tag{2.96}$$

It can be noted that the two-particle probability depends only on the odd powers of s. This in fact is consequence of the binary forward scattering amplitude being even in s, as required by the Bose symmetry.

For integer values of the parameter b, the function $\Phi_b(x)$ has a simple expression in terms of the modified Bessel functions $I_q(x)$ of the order q up to $q = b+2$. This expression is especially simple for $b = 1$, i.e. for the case of the string decay into 'nothing': $\Phi_1(x) = I_3(x)$, so that one arrives at the formula (1.7). We also write here, for an illustration, the corresponding expressions for the next two integer values of b:

$$\Phi_2(x) = \frac{1}{x}\,\left[I_5(x) + 6I_4(x)\right] \ ; \qquad \Phi_3(x) = \frac{1}{x^2}\,\left[(48+x^2)\,I_5(x) + x\,I_6(x)\right] \ . \tag{2.97}$$

2.2 Metastable wall

2.2.1 Spontaneous decay of a metastable wall

In the following sections we will generalize the obtained previously results for the decay of a metastable wall. We denote the tension of a wall by ϵ and the tension of a string associated with the edge of the wall by σ, so that there should not be any confusion with previously used notations for a string. For a complete decay of a wall the low energy effective action is similar to (2.1) and is given by Nambu-Goto action for two and tree dimensional objects

$$S = \sigma \mathcal{A} + \epsilon \mathcal{V}, \tag{2.98}$$

with $\mathcal{V}$ being the world volume of the wall, while $\mathcal{A}$ is the world area of interface. As before this action does not take into account the inner structure of the wall or the interface. Nontrivial classical solution (bounce) in this case is empty sphere with the radius

$$R_w = \frac{2\sigma}{\epsilon}, \tag{2.99}$$

surrounded by the metastable phase (Fig. 2.9). The appropriate quantity to be found

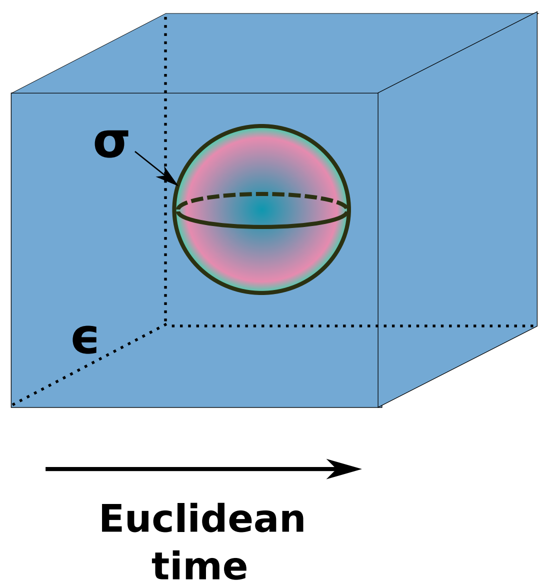

Figure 2.9: Bounce configuration

is γ_w: the nucleation rate of critical holes per unit area (not length as it was for a string transition).

Following the same steps as in calculating the decay rate of a string we find[1]

$$\gamma_0^w = \frac{\mathcal{C}}{\epsilon^{7/3}} \exp\left(-\frac{16\,\pi\,\sigma^3}{3\,\epsilon^2}\right) \frac{\mathcal{Z}_{12}}{\mathcal{Z}_1} \;, \tag{2.100}$$

with boundary partition functions given by

$$\mathcal{Z}_{12(\text{boundary})} = \int \mathcal{D}\xi \, \exp\left[-\frac{\sigma}{2} \int d\Omega\, \xi\, \Delta^{(2)}\xi + \frac{\epsilon R_w^2}{2} \int d\Omega\, \xi\, \partial_r z_c\Big|_{r=R_w}\right], \tag{2.101}$$

and

$$\mathcal{Z}_{1(\text{boundary})} = \int \mathcal{D}\xi \, \exp\left[\frac{\epsilon R_w^2}{2} \int d\Omega\, \xi\, \partial_r z_c\Big|_{r=R}\right], \tag{2.102}$$

with the function $z_c(r,\theta,\varphi)$ satisfying the Laplace equation $\Delta\, z_c = 0$ with the boundary conditions

$$z_c(R_w,\theta,\varphi) = \xi(\theta,\varphi), \qquad z_c(r=L) = 0. \tag{2.103}$$

The operator $\Delta^{(2)}$ is the angular part of the Laplace operator in 3d (the Laplace operator on a sphere). Expanding the boundary function $\xi(\theta,\varphi)$ in the series in spherical harmonics

$$\xi(\theta,\varphi) = \sum_{l,m} A_{lm} Y_{lm}(\theta,\varphi) \;, \tag{2.104}$$

we find a complete set of functions

$$z_{lm}(r,\theta,\varphi) = A_{lm} \frac{R_w^{l+1}}{r^{l+1}}\, Y_{lm}(\theta,\varphi) \;. \tag{2.105}$$

Substituting these functions to the equations (2.101) and (2.102) and performing integration over the amplitudes A_{lm} yields

$$\mathcal{Z}_{12(\text{boundary})} = \mathcal{N} \prod_{l=0}^{\infty} \left(\frac{1}{\sigma\, l\,(l+1) + \epsilon\, R_w\,(l+1)}\right)^{(2l+1)/2} \tag{2.106}$$

and

$$\mathcal{Z}_{1(\text{boundary})} = \mathcal{N} \prod_{l=0}^{\infty} \left(\frac{1}{\epsilon\, R_w\,(l+1)}\right)^{(2l+1)/2} \;. \tag{2.107}$$

[1]The factor $\frac{\mathcal{C}}{\epsilon^{7/3}} \exp\left(-\frac{16\,\pi\,\mu^3}{3\,\varepsilon^2}\right)$ corresponds to the decay rate of a metastable vacuum in $2+1$ dimensions [30].

Regularization

Each of the expressions (2.101) and (2.102) contains a divergent product (compare to Eqs.(2.27) and (2.28)). As previously, introducing Pauli-Villars regulators with masses M_α, we find the ratio of the boundary partition functions to be equal to

$$\begin{aligned}\mathcal{R}_w \;=\; & \prod_{l=0}^{\infty}\left[\frac{l\,(l+1)+2\sqrt{M_\alpha^2\,R_w^2+\left(l+\frac{1}{2}\right)^2}+1}{(l+1)\,(l+2)}\right]^{(2l+1)\,C_\alpha/2} \times \\ & \prod_{l=0}^{\infty}\left[\frac{l+1}{\sqrt{M_\alpha^2\,R_w^2+\left(l+\frac{1}{2}\right)^2}+\frac{1}{2}}\right]^{(2l+1)\,C_\alpha/2} \times \\ & \prod_{l=0}^{\infty}\left[1+\frac{M_\alpha^2R_w^2}{\left(M_\alpha^2\,R_w^2+\left(l+\frac{1}{2}\right)^2\right)\left(l\,(l+1)+2\sqrt{M_\alpha^2\,R_w^2+\left(l+\frac{1}{2}\right)^2}+1\right)}\right]^{(2l+1)\,C_\alpha/2} \qquad (2.108)\end{aligned}$$

where we took into account that $R_w = 2\,\sigma/\epsilon$.

Now, each of the products in Eq.(2.34) is finite at a finite M_α and can be calculated separately. Instead of calculating the product directly, we can use the relation

$$\prod_l F_l = \exp\left(\sum_l \ln F_l\right), \qquad (2.109)$$

and calculate the sum. We start from the third product. The expression under the sign of product is of the form $1 + g(l)$, with $g(l)$ close to 0 for any l, since it behaves as M_α^{-1}. Hence we can leave only the first term in the expansion of the logarithm $\ln(1+x) = x + O(x^2)$. Thus, we need to find the sum

$$S_3 = \frac{1}{2}\sum_{l=0}^{\infty}\sum_\alpha C_\alpha\,(2l{+}1)\,M_\alpha^2R_w^2\left[M_\alpha^2\,R_w^2+\left(l+\frac{1}{2}\right)^2\right]^{-1}\left[l\,(l+1)+2\sqrt{M_\alpha^2\,R_w^2+\left(l+\frac{1}{2}\right)^2}+1\right]^{-1}. \qquad (2.110)$$

The sums associated with the three products are readily calculable with the help of the Euler-Maclaurin summation formula and the result for the regularized ratio has the form

$$\mathcal{R}_w = \exp\left[\frac{1}{2}M^2R_w^2\ln MR_w + MR_w\ln MR_w + \ln MR_w\right] \qquad (2.111)$$

where $M^n\ln M = \sum_\alpha C_\alpha M_\alpha^n \ln M_\alpha$, for any n. The expression for $\mathcal{R}_w$ contains an essential dependence on the regulator mass parameter M. However, similarly to the previously considered case all such dependence in the phase transition rate can be absorbed in renormalization of the parameter σ in the leading semiclassical term.

Using the same technique as employed in Section 2.1 one can find the renormalized parameter σ to be given by

$$\sigma_R = \sigma + \delta\sigma \tag{2.112}$$

with

$$\begin{aligned} \delta\sigma \;=\; & \frac{1}{2}\int \frac{d^2\,k}{(2\pi)^2}\left[\ln\left(k^2 + \frac{\varepsilon}{\mu}\sqrt{M^2+k^2}\right) - \ln\left(k^2 + \frac{\epsilon}{\sigma}\sqrt{k^2}\right)\right] - \\ & \frac{1}{4}\int \frac{d^2\,k}{(2\pi)^2}\left[\ln\left(M^2+k^2\right) - \ln k^2\right] = \\ & \frac{M^2}{8\pi}\ln MR_w + \frac{M\varepsilon}{8\pi\mu}\ln MR_w + \frac{\varepsilon^2}{16\pi\mu^2}\ln MR_w. \end{aligned} \tag{2.113}$$

Results

Collecting all terms together we find the rate of the process. It is clear that the result for each $d-3$ transverse dimensions factorizes, thus we have the expression for the rate in the form

$$\gamma_w = \frac{\mathcal{C}}{\epsilon^{7/3}}\,\mathcal{R}_w^{d-3}\,\exp\left(-\frac{16\,\pi\,\sigma^3}{3\,\epsilon^2}\right), \tag{2.114}$$

where σ is the bare (non-renormalized) tension, and the regularized ratio $\mathcal{R}$ is given by (2.111). Taking into account that each of the transverse dimensions contributes additively to $\delta\sigma$ and the interface is a sphere with area

$$\mathcal{A} = 4\pi R_w^2, \tag{2.115}$$

and expressing the bare σ through the renormalized one: $\sigma = \sigma_R - \delta\sigma$, one readily finds that the dependence on the regulator mass M cancels in the transition rate, and one arrives at the formula given by Eq.(1.5).

Thus we obtained the result similar to the decay of a string, where the effect of all extra transverse dimensions results only in the renormalization of parameter σ associated with the interface. It should be mentioned, however, that the result for a string was, actually, obtained for a transition between two states of a string with different tensions. Here we considered decay of a wall (transition into nothing). For the calculation used it is impossible to preserve finite terms, but only proportional to some power of the regulator mass parameter M.

Having calculated the probability rate for a decay of one- and two- dimensional objects, e.g. string and metastable wall, it is tempting to assume that the situation is somewhat similar for the decay of an object of arbitrary dimensionality. But it is not true already for the decay of tree- and four- dimensional objects, where the dependence of the result on regulator mass is substantial even after renormalization of a parameter associated with an interface. That dependence demands introduction of new terms into initial action, which corresponds to nonrenormalizibility of the effective 'low-energy' theory.

2.2.2 Thermally induced decay of a metastable wall

The thermal effects in decay of a metastable wall involve one important difference from those in the decay of a string. Namely, the previously considered expansion of the catalysis factor $\mathcal{K}_s$ is valid up to the temperature $T = 1/\ell_c = 1/(2R_s)$, at which point the whole calculation breaks down due to a change in the mechanism of the transition: the thermal fluctuations of the string start dominating over the quantum ones. Simultaneously, at this temperature the thermal factor in Eq.(2.74) develops a singularity, and the low-temperature expansion diverges. In the case of a wall decay at low temperature the thermal effects in the quantum tunneling result in a catalysis factor $\mathcal{K}_w$ multiplying the semiclassical exponential factor in γ_w:

$$\gamma_w(T) = \mathcal{K}_w \gamma_w \ . \tag{2.116}$$

On the other hand, the static potential for a bubble, a round hole in the wall, depends on the radius R of the bubble as

$$V(R) = 2\pi\,\sigma\,R - 4\pi\,\varepsilon\,R^2 \ , \tag{2.117}$$

and this potential has a maximum at $R_m = \sigma/(4\varepsilon)$ where its value is

$$V_m = V(R_m) = \frac{\pi}{4}\frac{\sigma^2}{\varepsilon} \ . \tag{2.118}$$

The probability of classical thermal fluctuations over the barrier V_m is proportional to the activation factor $\exp(-V_m/T)$, and this factor becomes larger than the exponential term in γ_w (Eq.(1.5)) starting from $T = 3/(32R_w)$, at which temperature the classical fluctuations over the barrier start to dominate over the considered here quantum tunneling. In terms of the Euclidean space formulation of the problem the periodic replication with the period β of the spherical bounce (Fig.2.10a) gives a larger action per period[32] than the cylindrical configuration in Fig.2.10b as soon as $T > 3/(32R_w)$. For this reason our calculation of the thermal factor $\mathcal{K}_w$ makes physical sense only as long as the temperature is lower than this value.

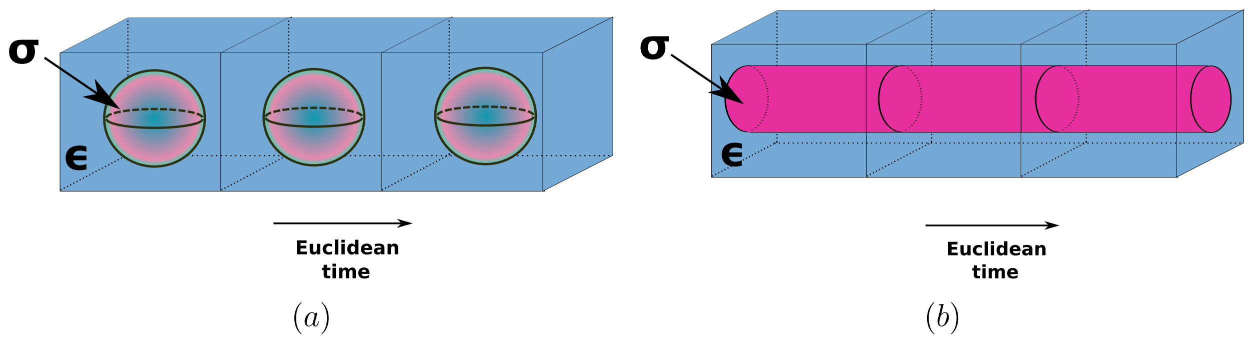

Figure 2.10: Bounce configuration for different temperatures

In calculating the preexponential factor arising in the periodically replicated bounce configuration, we encounter the same problem of a mismatch between the symmetry of the boundary conditions and the symmetry of the bounce. However one can cope with this problem in a way similar to the previously discussed approach to calculating the decay rate of the string at finite temperature. Namely, one can consider the solution to the Laplace equation as one produced by the sources placed at each period (see Fig. 2.11). Thus the solution z_{lm} (the periodic analog of the functions in Eq.(2.105)) is given by

$$z_{lm} = \frac{R_w^{l+1}}{r^{l+1}} Y_{lm}(\theta, \varphi) + \sum_{n \neq 0} \frac{R_w^{l+1}}{r_n^{l+1}} Y_{lm}(\theta_n, \varphi), \tag{2.119}$$

with r_n and θ_n reading as

$$\begin{aligned} r_n &= \sqrt{r^2 + (\beta n)^2 - 2\,\beta n\, r \cos\theta}, \\ \sin\theta_n &= \frac{r \sin\theta}{r_n}. \end{aligned} \tag{2.120}$$

The solution (2.119) can be expanded in a power series

$$z_{lm} = \frac{R_w^{l+1}}{r^{l+1}} Y_{lm}(\theta, \varphi) + \sum_{l'} \sum_{n>0} \frac{R_w^{l'} r^{l+1}}{(n\beta)^{l+l'+1}} C_{ll'}^{m} Y_{l'm}(\theta, \varphi)$$

$$= \frac{R_w^{l+1}}{r^{l+1}} Y_{lm}(\theta,\varphi) + \sum_{l'} \frac{R_w^{l'} r^{l+1}}{\beta^{l+l'+1}} \zeta(l+k+1) C_{ll'}^m Y_{l'm}(\theta,\varphi). \tag{2.121}$$

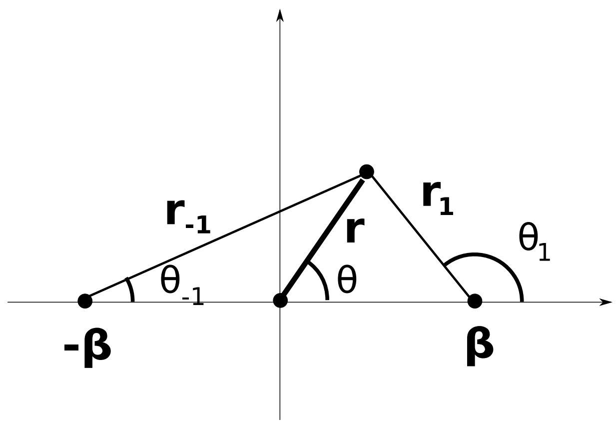

Figure 2.11: Periodic configuration

In order to find the constants $C_{ll'}^m$ one can notice that the expression in Eq.(2.119) has the same structure for each n. Therefore, for the purposes at hand we can consider only two adjacent periods ($\pm\beta$). Then integrating the *r.h.s.* of the (2.119) with $Y_{lm}^*(\theta,\varphi)$ we find the result to be

$$C_{ll'}^m = 2(-1)^{l+l'+m} \sqrt{\frac{2l+1}{2l'+1}} \frac{(l+l')!}{\sqrt{(l+m)!\,(l-m)!\,(l'+m)!\,(l'-m)!}} \tag{2.122}$$

It should be mentioned here that obviously

$$\sum_{m=-l}^{l} C_{ll}^m = 0. \tag{2.123}$$

This relation can be understood even without actual calculation. Indeed, the expression

$$\int \sum_m Y_{lm}^*(\theta,\varphi) Y_{lm}(\theta_1,\varphi) \sin\theta \, d\theta \, d\varphi \tag{2.124}$$

is proportional to the electric potential at point β created by the set of 2^l-poles placed on a sphere with its center in the origin. Obviously, such a potential is equal to zero.

Once we have found the coefficients (2.122) of the expansion (2.121) we can follow the same routine as used in the section (2.1.2). Namely, we can express any solution of the Laplace equation with periodic boundary conditions as a series

$$z_c = \sum_{lm} z_{lm} B_{lm}. \tag{2.125}$$

After that using the expansion of a boundary function (2.104) and the relation (2.103), we express coefficients B_{lm} through A_{lm}

$$A_{lm} = \left[\delta_{ll'} + \left(\frac{R_w}{\beta} \right)^{l+l'+1} C^m_{ll'} \, \zeta(l + l' + 1) \right] B_{l'm}, \tag{2.126}$$

or

$$A_m = (1 + D_m) B_m, \tag{2.127}$$

where D_m is a matrix with elements

$$[D_m]_{ll'} = \left(\frac{R_w}{\beta} \right)^{l+l'+1} C^m_{ll'} \, \zeta(l + l' + 1), \tag{2.128}$$

and A_m (similarly B_m) is the row type object with elements

$$[A_m]_l = A_{lm}. \tag{2.129}$$

It should be mentioned that since the solution z_{lm} exists only for $l \geq m$, the row A_m has nonzero elements A_{lm} only for $l \geq m$.

Substituting the solution (2.125) to the expressions for the partition functions (2.101) and (2.102) and integrating over the amplitudes A_{lm} (with a help of relation (2.127)) we find the catalysis factor in the following form

$$\mathcal{K}_w = \prod_m \det \left(1 + \frac{N\,(N-1)}{(N+1)\,(N+2)} \, D_m \right)^{-\frac{d-3}{2}}, \tag{2.130}$$

with matrix N defined in (2.68). Equivalently, one can rewrite the expression in the following form

$$\mathcal{K}_w = \det \left(1 + \bar{\mathcal{D}} \right)^{-\frac{d-3}{2}}, \tag{2.131}$$

where $\mathcal{D}$ stands for the block diagonal matrix defined as

$$\bar{\mathcal{D}} = \begin{pmatrix} \mathcal{D}_0 & & & & \\ & \mathcal{D}_1 & & & \\ & & \ddots & & \\ & & & \mathcal{D}_m & \\ & & & & \ddots \end{pmatrix} \tag{2.132}$$

with elements

$$\mathcal{D}_m = \frac{N\,(N-1)}{(N+1)\,(N+2)}\,D_m. \tag{2.133}$$

The first nontrivial term of expansion of the catalysis factor $\mathcal{K}_w$ at small temperatures is determined by the coefficients in Eq. (2.128) at $l = l' = 2$ and the result for this term is given in Eq.(1.8).

One can notice that the low-temperature expansion for the factor $\mathcal{K}_w$ generated by Eq.(2.131) is well behaved at $TR_w = 3/32$, beyond which temperature, as previously discussed, the regime of the transition changes from quantum tunneling to a classical thermal activation. Moreover the calculated thermal effect at this point is numerically extremely small: $\mathcal{K}_w - 1 \sim 10^{-9}$. Thus the explicit form of higher terms in the temperature expansion is of a 'practical' interest only inasmuch as the thermal calculation is used as a preliminary step for finding the generating function for the rate of the wall destruction in collisions of Goldstone bosons.

2.2.3 Destruction of a metastable wall in binary collisions

Using the arguments from the section 2.1.3 it is possible to relate the catalysis factor $\mathcal{K}_w$ for a decay rate of the wall at finite temperature calculated in the previous section, to the effective length of a particle collision (the analog of the probability in $1+1$ dimensional case).

As before we consider the distribution function for the Goldstone bosons in the following form (with negative chemical potential $-|\nu|$)

$$dn(\vec{k}) = \frac{1}{e^{\frac{\omega+|\nu|}{T}} - 1}\frac{d^2k}{(2\pi)^2}. \tag{2.134}$$

Such an introduction of the chemical potential modifies the result for the catalysis factor $\mathcal{K}_w$ in the way that the solutions (2.121) are modified as

$$\begin{aligned} z_{lm} &= \frac{R_w^{l+1}}{r^{l+1}} Y_{lm}(\theta,\varphi) + \sum_{l'}\sum_{n>0} \frac{R_w^{l'} r^{l+1}}{(n\beta)^{l+l'+1}} C_{ll'}^m Y_{l'm}(\theta,\varphi)\, e^{-|\nu|\beta} & (2.135)\\ &= \frac{R_w^{l+1}}{r^{l+1}} Y_{lm}(\theta,\varphi) + \sum_{l'} \frac{R_w^{l'} r^{l+1}}{\beta^{l+l'+1}} \zeta(l+k+1) C_{ll'}^m Y_{l'm}(\theta,\varphi)\, \mathrm{Li}_{l+l'+1}(e^{-|\nu|\beta}) & (2.136)\end{aligned}$$

This modification can be accounted for by the substitution $\zeta(l+l'+1) \to \mathrm{Li}_{l+l'+1}(e^{-|\nu|/T})$. Thus, the catalysis factor becomes (with $d-3$ transverse dimensions)

$$\mathcal{K}_w = \det\left[1 + \bar{\mathcal{D}}(T,\nu)\right]^{-\frac{d-3}{2}}, \tag{2.137}$$

where the matrix $\bar{\mathcal{D}}(T,\nu)$ has a block diagonal form with elements

$$[\mathcal{D}_m]_{ll'}(T,\nu) = \frac{l\,(l-1)}{(l+1)\,(l+2)}\, C^m_{ll'}\, (RT)^{l+l'+1}\ \mathrm{Li}_{l+l'+1}(e^{-|\nu|/T}). \tag{2.138}$$

In order to separate the contribution of binary collisions we expand the catalysis factor in powers of $\mathcal{D}$ to the second order:

$$\mathcal{K}_w = 1 + \frac{d-3}{4}\mathrm{Tr}\,\bar{\mathcal{D}}^2(T,\nu) = \frac{d-3}{4}\sum_m \mathrm{Tr}\,\mathcal{D}^2_m, \tag{2.139}$$

where we have taken into account that the linear term vanishes:

$$\mathrm{Tr}\,\bar{\mathcal{D}}(T,\nu) = \sum_m \mathrm{Tr}\,\mathcal{D}_m = 0, \tag{2.140}$$

due to the relation (2.123). Similarly to the case of string decay the physical reason for the absence of the first power is the Lorentz invariance of the system (there is no destruction of the wall by presence of one massless particle). It should be also noted that unlike in the decay of a string the destruction of a metastable wall can occur in collisions of an odd number of particles, since the expansion (2.139) generally contains any larger then one power of matrix $\bar{\mathcal{D}}$.

For an actual calculation of binary processes one can now follow the same steps as in the calculation of the section 2.1.3. The only difference from the (1+1) dimensional string geometry is that now the kinematical invariant s also depends on the angle θ between the two particles' momenta,

$$s(\vec{k}_1,\vec{k}_2) = 2\,\omega_1\,\omega_2\,(1+\cos\theta)\ . \tag{2.141}$$

Therefore an analog of the integral (2.92) has an angular part, and one should make use of the relation

$$\int n(\vec{k}_1)\, n(\vec{k}_2) s^{n-1}(\vec{k}_1,\vec{k}_2)\frac{d^2k_1\, d^2k_2}{(2\pi)^4} = \frac{4^{n-1}}{(2\pi)^3}\,(n!)^2\, T^{2(n+1)}\,\mathrm{Li}^2_{n+1}\left(e^{-|\nu|/T}\right)\, B(n-1/2,1/2), \tag{2.142}$$

where $B(a,b)$ is the Euler beta function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2.143}$$

The appropriate quantity describing the probability of a binary process on a (2+1) dimensional wall is the effective length λ, which is an analog of the cross section in

(3+1) dimensions and of the dimensionless probability W_2 on a (1+1) dimensional string.

Using the expression (2.142) and the quoted in Eq.(1.8) low temperature behavior of the thermal catalysis factor one readily finds the low energy limit for the effective length of destruction of the wall in a collision of two Goldstone bosons:

$$\lambda = \frac{d-3}{5}\,\pi^2\,\gamma_w\,s^3\,R_w^{10} + \ldots \; . \tag{2.144}$$

The general formula for the effective length at arbitrary values of $\sqrt{s}\,R_w$ is found using Eq.(2.139) and the expression (2.138) for the elements of the matrix $\tilde{\mathcal{D}}$. In this way we arrive at the following expression for the effective length of a two particle collision in the form of a triple sum

$$\begin{aligned} \lambda &= \sum_m \lambda_m = \frac{16\pi^3\,(d-3)\,R_w^2\gamma_w}{s} \sum_m \sum_{p,q\geq|m|} \frac{q\,(q-1)\,p\,(p-1)}{(q+1)\,(q+2)\,(p+1)\,(p+2)} \\ &\times \left(\frac{R_w\sqrt{s}}{2}\right)^{2p+2q} B^{-1}(p+q-1/2,1/2)\,\frac{1}{(q+m)!\,(q-m)!\,(p+m)!\,(p-m)!} \end{aligned} \tag{2.145}$$

where γ_w is a decay rate per unit area of the wall at zero temperature.

The behavior at large energy, $R_w\,\sqrt{s} \gg 1$, can be found using saddle point approximation to estimate the sum over p and q, which gives

$$\lambda \sim \exp\left(-\frac{16\,\pi\,\sigma^3}{3\,\epsilon^2} + 4\sqrt{s}\,\frac{\sigma}{\epsilon}\right), \tag{2.146}$$

which matches the semiclassical expression for the tunneling exponent at energy $\sqrt{s}$ [48].

2.3 Summary of chapter 1

In this chapter we considered the decays of metastable topological configurations such as string and metastable wall. For each process we found the rate at zero temperature and calculated catalysis factor at finite temperature. Using the relation between catalysis factor and probability (effective) length of a collision of the Goldstone bosons we found probability of a string (wall) decay in a collision of two particles.

One can readily notice that our calculation of the collision-induced rate through a thermal ensemble with an artificially introduced chemical potential is quite indirect.

In a sense, this approach is reminiscent of the treatment in Ref. [49], where finite volume effects in the energy of a two-particle state are related to the binary scattering amplitude. The obvious difference is that we calculate the free energy of a statistical ensemble rather than of a particular quantum state. In relation to this part of our calculation it would certainly be illuminating to have a more direct, and possibly simpler method for calculating the probability of creating semiclassical objects, such as the bubbles of the stable phase, in collisions of particle. However, lacking such a method at present, we have to resort to the indirect calculation.

Chapter 3

Pair production in external field stimulated by photon

Previously we considered the decays of string and wall in a collision of several particles. In this chapter we consider the similar situations, when the presence of the particle in the initial state catalyzes the process comparing to the spontaneous one.

In what follows we provide the actual calculation leading to the expression (1.10). First, in Section 2.1.1 we briefly recapitulate the quasiclassical method of calculating the probability rate, then in Section 2.1.2 we describe the treatment of the problem in terms of Euclidean space effective action, then we derive the expression for the rate at nonzero temperature T in terms of expansion in powers of $mT/(eE)$. In Section 2.1.3. we relate the thermal result to the contribution of the one-photon induced process and thus we find the probability described by Eq.(1.10). Also we present a calculation of the parallel and orthogonal absorptive parts $\pi_{\|}$ and $\pi_{\perp}$ of the vacuum polarization due to the 'electric' interaction of the electron in Section 2.2.2. Section 2.2.3. is dedicated to the calculation of the contribution of the electron magnetic moment to the effective action and calculate its contribution to the 'magnetic' part of $\pi_{\perp}$, while in the Section 2.2.4. we consider the tunneling trajectory deformed by the incident photon at arbitrary energy and calculate the exponential factor in the rate of the process.

3.1 Temperature approach

3.1.1 Euclidean-space tunneling

The Euclidean space approach [18] to tunneling is based on constructing a localized solution to the classical equations of motion. The production rate is related to the imaginary part of the vacuum energy $\Gamma = -2\mathrm{Im}E_{\mathrm{vac}}$, which can be found in terms of the (Euclidean) path integral $\mathcal{Z}$ in the theory as

$$e^{-E_{\mathrm{vac}}T} = \mathcal{Z} \equiv \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\mathcal{D}A_\mu e^{-S[\psi,A_\mu]}. \tag{3.1}$$

It can be also mentioned that if in the problem there is a separation of scales such that some degrees of freedom can be considered as soft on the scale of the size of the bounce, both the exponential factor [18] and the pre-exponential one [19] can be treated within an effective theory of those soft variables.

The contribution of configurations with e^+e^- pairs, giving rise to the instability of the vacuum in the presence of an external electric field, can be rewritten in terms of an integral over closed trajectories of the electron [22, 23, 41] with the effective action

$$S = m\int d\ell - e\int A_\mu dx_\mu, \tag{3.2}$$

with x_μ being the coordinate of the particle and $d\ell$ is the element of the length of the particle trajectory. The actual integration is then done semiclassically using the saddle point method. The stationary configuration (trajectory) for the action in this case is called the bounce and is given by a circle of a fixed radius $R = m/(eE)$ (see Fig.3.1). The leading exponent and the determinant around the bounce combine to give the semiclassical result for the process rate per unit time and unit volume V:

$$\frac{\Gamma}{V} = 2\mathrm{Im}\int \mathcal{D}\gamma e^{-S[\gamma]} = \frac{(eE)^2}{4\pi^3}\exp\left[-\frac{\pi m^2}{eE}\right]. \tag{3.3}$$

The value of the action on this trajectory is $S_B = \frac{\pi m^2}{eE}$, in a complete agreement with the leading exponent in the exact expression (1.9) under the condition $E \ll E_c$, which ensures applicability of the semiclassical treatment.

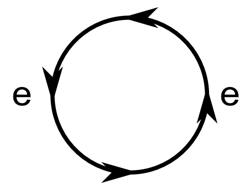

Figure 3.1: The bounce configuration for e^+e^- pair creation.

3.1.2 Pair creation in electric field in a thermal bath.

As mentioned, we eventually find the photon-induced pair production rate by extracting the corresponding one-particle contribution from an expression for the Schwinger process at a finite temperature. Therefore we start with calculating the probability rate per unit volume at nonzero temperature. For a sufficiently small temperature, namely $T \ll m$, one can still employ the same effective Euclidean one particle action (3.2), except that now the system lives on a cylinder with a periodic Euclidean time with the period equal to the inverse temperature $\beta = 1/T$. Equivalently one can consider the system on the (t, x) plane with periodic in t boundary conditions (see Fig.3.2). Moreover for the purpose of the present calculation it is sufficient to

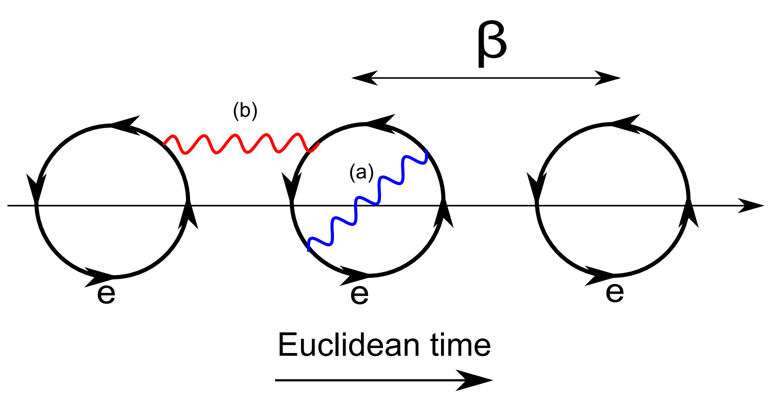

Figure 3.2: Periodic plane. Two types of correction: (a) is the correction due to the self interaction of the particle on a circle; (b) is the correction due to the interaction of the circles separated in Euclidean time.

consider arbitrarily low but non vanishing temperature, and we thus also impose the condition $T < (2R)^{-1} = eE/(2m)$ which even more justifies the applicability of the effective action and also ensures that the circular bounce fully fits within one period.

The action for the bounce and thus the probability of pair creation does not change in this limit if the electromagnetic field is considered as an external object without dynamics of its own. We however are interested in the effects produced by the photons in the thermal bath and we should thus consider the dynamics of the electromagnetic field by adding to the low energy action the kinetic term for the field A_μ. In order to exclude the contribution of the energy of the external field to the action one can make the shift in the definition of the electromagnetic potential: $A_\mu \to A_\mu^{ext} + A_\mu$ and write the effective action as

$$S[x, A] = m \int dl - eEA - e \int A_\mu dx^\mu - \frac{1}{4} \int d^4x \, F_{\mu\nu}^2 \ , \tag{3.4}$$

where A_μ is the shifted potential with the corresponding field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Generally, the rate is found [18, 19, 22, 37] by calculating the partition function around the one bounce configuration

$$\Gamma = \frac{2}{VT} \mathrm{Im} \int \mathcal{D}x_\mu \, \mathcal{D}A_\mu \exp -S[x, A]. \tag{3.5}$$

with effective one particle Euclidean action $S[x, A]$ given by the expression (3.4), and VT being the space-time volume of the system.

The zero temperature result corresponds to the limit $\beta \to \infty$, and, without any corrections from exchange of the photons, yields the well known expression for the rate

$$\frac{\Gamma}{V} = \frac{(eE)^2}{4\pi^3} \exp\left(-\frac{\pi \, m^2}{eE}\right), \tag{3.6}$$

which corresponds to the first term in the sum (1.9). The self interaction of the particle (one-loop correction: the correction of the type (a) in Fig. 2) was taken into account in paper [22]. It amounts to a finite additive term in the exponent, $e^2/4 = \pi\alpha$. The thermal effect can in fact be viewed as a thermal distortion of this self-interaction term due to the modification of the photon propagator on the cylinder as compared to an infinite space-time. In the equivalent periodic picture of Fig. 2 this modification can be considered as an interaction between the periodic copies of the current loops with the photon propagator being that in an infinite space-time (the corrections of type (b)).

Before proceeding to a calculation of this latter effect we note that the corrections due to the thermal fluctuations of the shape of the bounce, i.e. deviations from the circle, are of a higher order in eE/m^2 and are entirely neglected in the present treatment.

The contribution to the action due to such interaction has the following form

$$\Delta S_{tot} = -\int d^4\, x \left(\frac{1}{4}F_{\mu\nu}^2 + A_\mu\, j_\mu\right) = -\frac{1}{2}\int d^4\, x A_\mu\, j_\mu. \tag{3.7}$$

with j_μ being the total current in the circles

$$j_\mu = \sum_n e\, n_\mu^{(n)} \delta(r_n - R)\, \delta(y)\, \delta(z), \tag{3.8}$$

and $n_\mu^{(n)}$ is the tangential unit vector to the $n-$th circle:

$$n_\mu^{(n)} = (-\sin\theta_n, \cos\theta_n)\ , \tag{3.9}$$

where (r_n, θ_n) are the polar coordinates with the origin at the center of the n-th copy of the circle located at $(x, t) = (0, n\,\beta)$. A_μ is the field produced by all those circles

$$A_\mu = \sum_n A_\mu^{(n)}, \tag{3.10}$$

where all $A_\mu^{(n)}$ in turn are the the solutions of the Laplace equation

$$\Delta A_\mu^{(n)} = j_\mu^{(n)}, \tag{3.11}$$

$$A_\mu^{(n)}(r_n, \theta_n, y = 0, z = 0) = \frac{eR^2}{2\pi}\,\frac{n_\mu^{(n)}}{r_n\,(r_n^2 - R^2)}, \tag{3.12}$$

Omitting the contribution from $A_\mu^{(0)}$, which corresponds to self interaction, we get the correction to the action per one period β

$$\begin{aligned}\Delta S &= -\sum_{n=1}^{+\infty}\int d^4\, x\, A_\mu^{(n)}\, j_\mu^{(0)} = -\frac{e^2}{2}\sum_{n=1}^{+\infty}\left(\frac{1-2(RT/n)^2}{\sqrt{1-(2\,RT/n)^2}} - 1\right)\\ &= -\frac{e^2}{2}\sum_{p=2}^{\infty} 2^{2p-1}(RT)^{2p}\,\frac{(p-1)\,\Gamma(p-1/2)}{\sqrt{\pi}\,\Gamma(p+1)}\,\zeta(2p)\ ,\end{aligned} \tag{3.13}$$

where the sum runs over only positive n and it is taken into account that the contribution of the terms with negative n is the same as that from $n > 0$. Finally, ζ is the standard Riemann zeta function

$$\zeta(q) = \sum_{n=1}^{\infty} n^{-q}\ .$$

A remark is in order concerning the apparent 'extra' factor of one half in Eq.(3.13). In the treatment in 'flat' space time with periodic copies this factor arises for the following reason. Each term $-A_\mu^{(n)}\, j_\mu^{(0)}$ in the sum corresponds to the additional action within *the pair* of the $n-th$ and $0-th$ current loops. Thus the additional action per one loop, i.e. per one period β, is one half of that. In the picture of a current loop on the cylinder the equivalent explanation of this factor is that the self-interaction of the loop through n windings of the photon propagator around the cylinder does not contain any notion of the sign of n. Therefore summing over the positive and negative values of n would be double counting.

Using the pre-exponential factor from Eq.(3.6) and the expression (3.13) for ΔS, one can write the rate of pair creation at finite temperature in the form

$$\frac{d\Gamma_T}{dV} = \frac{(eE)^2}{4\pi^3} \exp\left(-\frac{\pi\, m^2}{eE} - \Delta S\right) \, , \tag{3.14}$$

so that the thermal enhancement factor is $\exp(-\Delta S)$.

In particular, the first temperature dependent term of expansion at low temperature resulting from Eqs.(3.13) and (3.14),

$$\frac{d\Gamma_T}{dV} = \frac{(eE)^2}{4\pi^3} \exp\left(-\frac{\pi\, m^2}{eE}\right) \left\{1 + \alpha\, \frac{2\pi^5}{45}\, \frac{m^4\, T^4}{(eE)^4} + O\left[\left(\frac{m\, T}{eE}\right)^6\right]\right\} \, , \tag{3.15}$$

agrees with the same term resulting from the expansion of the general two-loop effective QED Lagrangian [50] in external electromagnetic field at finite temperature.

3.1.3 Pair production induced by a photon.

In a microscopic description of the thermal effects, the enhancement of pair creation in a bath at finite temperature arises through the stimulation of the process by the photons present in the bath. The dependence of the photon induced process on the photon energy ω then translates into the dependence on the temperature T after averaging over the thermal distribution of the photons with the standard density function

$$n(\vec{k}) = \frac{1}{e^{\omega\beta} - 1} \tag{3.16}$$

with $\omega = |\vec{k}|$. The number of photons involved in each of these microscopic processes can be readily identified by the power of the factor e^2. Since the thermal correction

(3.13) in the action is proportional to e^2, the one-photon contribution to the thermal rate is given by the linear in ΔS term in the expansion of the factor $\exp(-\Delta S)$ in the expression (3.14). Namely, the one-photon contribution to the pair creation rate in a thermal state is given by [1]

$$\Gamma_{1\gamma}/V = \frac{(eE)^2}{4\pi^3} \exp\left(-\frac{\pi\, m^2}{eE}\right) (-\Delta S) \, . \tag{3.17}$$

On the other hand the same contribution can be found in terms of the averaged over the photon polarizations probability $\bar{\kappa}$ rate of pair production induced by a photon, which is the same as the absorption rate for the photons per unit length. The latter can be expanded in a power series of $\omega \sin\theta$, with yet to be defined coefficients C_n as [29]

$$\bar{\kappa}(\vec{k}) = \frac{1}{\omega} \sum_{p=2}^{\infty} C_p(\omega\, \sin\theta)^{2p-2}. \tag{3.18}$$

The functional form of κ in fact follows from its relation (1.11) to the vacuum polarization in electric field and the dependence of the on-shell imaginary part $\Im\Pi$ on $\omega \sin\theta$ [38].

The thermal probability is then found in terms of the coefficients C_p by integrating over the photon momentum with the weight given by the distribution (3.16):

$$\frac{\Gamma_{1\gamma}}{V} = 2 \int \frac{d^3\, k}{(2\pi)^3} \, \frac{\bar{\kappa}(\vec{k})}{e^{\omega/T} - 1} = \sum_{p=2}^{\infty} C_p \, \frac{2}{(2\pi)^2} \, T^{2n} \, \Gamma(2p) \, \frac{\sqrt{\pi}\Gamma(p)}{\Gamma(p+1/2)} \, \zeta(2p), \tag{3.19}$$

where the factor of two accounts for two polarizations of the photon.

The expression in Eq.(3.19) can now be compared with the one resulting from the equations (3.17) and (3.13) thus determining the coefficients C_p and yielding the expansion for $\bar{\kappa}$ in the form

$$\begin{aligned} \bar{\kappa}(\vec{k}) &= \frac{\alpha\, m^2}{4\,\omega} \exp\left(-\frac{\pi\, m^2}{eE}\right) \sum_{n=1}^{\infty} \frac{2^{2n+4}}{\pi} \, \frac{\Gamma^2(n+3/2)}{(2n+1)\,(n-1)!\,(n+1)!\,(2n+1)!} \, \gamma_\theta^{2n} \\ &= \frac{\alpha\, m^2}{\omega} \exp\left(-\frac{\pi\, m^2}{eE}\right) \sum_{n=1}^{\infty} \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,(n-1)!\,n!\,(n+1)!} \, \gamma_\theta^{2n} \end{aligned}$$

[1] In Refs. [27, 20] the contribution of processes with different number of massless bosons in a thermal bath was separated by formally introducing a negative chemical potential. In the problem discussed here this is not necessary, since the power of the coupling e^2 automatically 'tags' the number of photons.

$$
\begin{aligned}
&= \frac{\alpha\, m^2}{4\,\omega}\, \exp\left(-\frac{\pi\, m^2}{eE}\right) \left(\gamma_\theta^2 + \frac{1}{4}\gamma_\theta^4 + \frac{5}{192}\gamma_\theta^6 + \ldots\right) \\
&= \frac{\alpha\, m^2}{\omega}\, \exp\left(-\frac{\pi\, m^2}{eE}\right) [I_1(\gamma_\theta)]^2 \qquad (3.20)
\end{aligned}
$$

with γ_θ being the Keldysh parameter $m\,\omega\, \sin\theta/(eE)$. The latter form of the result in Eq.(3.20) in terms of the square of the Bessel function can be verified by squaring the standard Taylor expansion and collecting terms with the same power of the argument:

$$
\begin{aligned}
[I_1(x)]^2 &= \left[\sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!\,(k+1)!}\right]^2 = \sum_{n=1}^{\infty} x^{2n} \left[2^{-2n} \sum_{p=0}^{n-1} \frac{1}{p!\,(p+1)!\,(n-p-1)!\,(n-p)!}\right] \\
&= \sum_{n=1}^{\infty} x^{2n}\, \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,(n-1)!\,n!\,(n+1)!}\;. \qquad (3.21)
\end{aligned}
$$

The coefficients in the latter expansion clearly coincide with those in the second line of Eq.(3.20).

3.2 Semiclassical approach

For the case at hand we use the approach similar to one discussed above, except that we calculate the correlator of currents rather than the vacuum energy. The interaction between the electromagnetic field and the current is described by the familiar Lagrangian $\mathcal{L}_{int} = A_\mu j_\mu$, so that the vacuum polarization operator is given by the standard expression

$$
\Pi_{\mu\nu}(x) = \langle j_\mu(x) j_\nu(0)\rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\,\bar{\psi}\, j_\mu(x) j_\nu(0)\, e^{-S[A,\psi]}. \qquad (3.22)
$$

Due to the instability (metastability) of the vacuum in an external electric field, the vacuum polarization develops an imaginary part $\pi(x) = \Im\Pi(x)$. In a semiclassical approach this imaginary part is evaluated similarly to the imaginary part of the vacuum energy by calculating the correlator of the currents on the bounce configuration. In terms of the effective action for the electron trajectories γ the correlator then can be found by integrating over γ the product of the currents on each of the trajectories:

$$
\langle j_\mu(x) j_\nu(0)\rangle = \int \mathcal{D}\gamma\, j_\mu^\gamma(x) j_\nu^\gamma(0) e^{-S[\gamma]}\;, \qquad (3.23)
$$

where the explicit form of the action $S[\gamma]$ in the external potential $A_\mu^{\rm ext}$ has the standard form

$$S = \oint_\gamma \left(m\sqrt{\dot{X}_\mu^2} - eA_\mu^{\rm ext}\dot{X}_\mu \right) ds, \tag{3.24}$$

with $X_\mu(s)$ describing the trajectory γ in terms of the position X_μ as a function of the length parameter s, and the dot standing for the derivative $\dot{X} = dX/ds$.

It should be noted that the vacuum in a constant electric field is invariant under space-time translations. As a result the probability of the spontaneous pair creation is proportional to the the space time volume. This is reflected in Eq.(3.3) in that a finite physical quantity is the rate per unit time and per unit volume. In terms of a path integration around the bounce configuration the translational invariance corresponds to the existence of four translational zero modes of the action at the bounce. The integration over these modes then corresponds to an integral over the space-time position $(x_\mu)_B$ of the bounce, and the proper measure for the imaginary part of the integral (arising from the exponential $\exp(-S_B)$ and the integration over the nonzero modes) can be read directly off Eq.(3.3). For the bounce contribution to the imaginary part $\pi_{\mu\nu}$ of the vacuum polarization $\Pi_{\mu\nu}$ the proper expression takes the form

$$\pi_{\mu\nu} = -\frac{\Gamma}{2V}\int \langle j_\mu(x) j_\nu(0)\rangle_{x_B}\, d^4x_B = -\frac{(eE)^3}{8\pi^3}\exp\left[-\frac{\pi m^2}{eE}\right]\int \langle j_\mu(x) j_\nu(0)\rangle_{x_B}\, d^4x_B \ , \tag{3.25}$$

where the $\langle \dots \rangle_{x_B}$ stands for the correlator calculated with a fixed position x_B of the bounce.

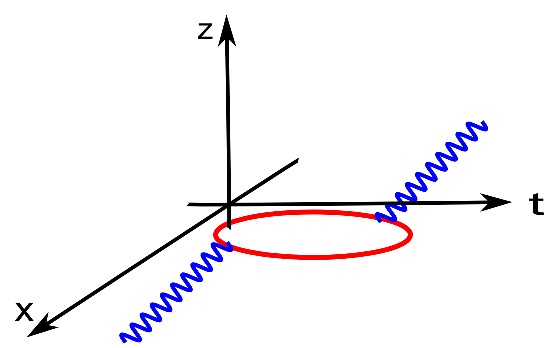

Figure 3.3: Undisturbed electron trajectory.

3.2.1 Calculating $\pi_{\parallel}$ and $\pi_{\perp}$.

In this and the next section we consider the situation when the energy of the photon is small compared to the mass of the electron $\omega \ll m$. This limit allows one to use the same bounce configuration as in the spontaneous pair production case. To find the polarization operator in the leading order in the small parameter eE/m^2 one can neglect all the spinor structure of the current j^{γ}_{μ} and take it in the form

$$j^{\gamma}_{\mu}(x) = e \int \dot{X}_{\mu}(s) \delta^4(x - X(s))\, ds \ . \tag{3.26}$$

In the saddle point approximation one finds the tunneling trajectory, the bounce, which is a solution to the classical equations of motion corresponding to the action (3.24) $X^B_{\mu}(s)$. In order to take into account small fluctuations around the stationary path one can parametrize the trajectory $X(s)$ as

$$X_{\mu}(s) = X^B_{\mu}(s) + \xi_{\mu}(s). \tag{3.27}$$

Therefore the correlator (3.23) takes the form

$$\langle j_{\mu}(x) j_{\nu}(0) \rangle = \\ e^2 \int \mathcal{D}\xi\, ds_1\, ds_2 \left[\dot{X}^B_{\mu}(s_1) \dot{X}^B_{\nu}(s_2) + \dot{\xi}^B_{\mu}(s_1) \dot{\xi}^B_{\nu}(s_2) \right] \delta^4 \left(x - X^B(s_1) \right) \delta^4 \left(X^B(s_2) \right) e^{-S_B - \delta S[\xi]} \tag{3.28}$$

The first term in the straight braces in (3.28) corresponds to the calculation of the correlator using undisturbed electron trajectory (see Fig. 3.3), while the second term corresponds to the fluctuations of the bounce shown in Fig. 3.4.

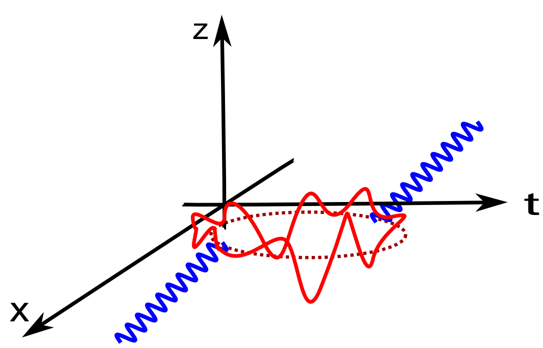

Figure 3.4: Fluctuation of the electron trajectory.

We choose the coordinate axes in such a way that the external electric field is in the x direction, so that the bounce is a circle in the (t, x) plane, and start with calculating the leading part in the correlator (3.28). For a bounce centered at the origin, the current can be expressed in the cylindrical coordinates (r, φ, y, z) as

$$j_\mu^B(x) = e \int \dot{X}_\mu(s)\, \delta^4(x - X(s))\, ds = e n_\mu(\theta) \delta(r - R) \delta(y) \delta(z), \tag{3.29}$$

where $n_\mu = (-\sin\varphi, \cos\varphi, 0, 0)$ is a tangential unit vector to the circle. As previously argued, the leading contribution to the polarization tensor comes from the parallel part. Indeed, the current from the bounce has no components in the transverse to the plane (t, x) directions. Therefore it is obvious that only $j_a^B j_b^B$ with $a, b = 0, 1$ produce nonzero result. Integration over the zero modes x_B (the position of the bounce) can be readily performed giving

$$\int d^4x_B\, j_a^B(x) j_b^B(0) = e^2 \frac{4R^2 r_a r_b - \delta_{ab} r^4}{r^3 \sqrt{4R^2 - r^2}} \delta(y)\delta(z), \tag{3.30}$$

where r is the length of the vector in the plane (t, x), $r = \sqrt{t^2 + x^2}$ (some details of the calculation can be found in Appendix A). Performing also the Fourier transform of the polarization operator

$$\pi_{ab}(q) = \int d^4x\, \pi_{ab}(x)\, e^{iqx} \tag{3.31}$$

and inserting the proper integration measure from Eq.(3.25) one finally finds

$$\pi_{ab}(q^2) = -2\alpha m^2 I_1^2\left(\sqrt{-q^2}\, R\right) \left(\delta_{ab} - \frac{q_a q_b}{q^2}\right) \exp\left(-\frac{\pi m^2}{eE}\right), \tag{3.32}$$

where $q^2 = q_0^2 + q_1^2$. Clearly, the dependence on $\sqrt{-q^2}\, R$ in terms of the Euclidean variables translates into the dependence on γ_θ for an on-shell photon propagating in the Minkowski space. Indeed for a photon with four-momentum k one has $-q^2 = \omega^2 - k_x^2 = \omega^2 \sin^2\theta$, so that

$$\sqrt{-q^2}\, R \to \gamma_\theta\ , \tag{3.33}$$

given that $R = m/(eE)$. With this substitution the expression in (3.32) reduces to that in Eq.(1.12), which coincides with the recently obtained result [40, 43].

The calculation of the second term in (3.28) can also be done in a standard way. It is obvious that the contribution to the orthogonal part of the polarization tensor comes from the fluctuations of the trajectory in transverse directions, and for the purpose of calculation it is sufficient to consider one of the orthogonal directions, e.g. that along the z axis. It then follows from (3.28) that the orthogonal part of the polarization operator is determined by the path integral

$$\Pi_\perp = \langle j_3(x) j_3(0)\rangle = e^2 \int \mathcal{D}\xi \, ds_1 \, ds_2 \, \dot{\xi}_3^B(s_1)\dot{\xi}_3^B(s_2)\delta^4\left(x - X^B(s_1)\right)\delta^4\left(X^B(s_2)\right) e^{-S_B - \delta S[\xi]} \, . \tag{3.34}$$

Expanding the $\xi_3(s)$ in a Fourier series, integrating over the amplitude of each mode and making the Fourier transformation we find the expression (1.13), with the details of the calculation described in Appendix B.

3.2.2 The magnetic moment term $\pi_\perp^m$

As it was mentioned before, there is another contribution to the orthogonal vacuum polarization $\pi_\perp$ that is suppressed by the parameter ω^2/m^2 rather than by (eE/m^2). This contribution arises from the spinor structure of the current. The electron has a magnetic moment $\mu = e/(2m)$. The interaction between the field and the magnetic moment in the rest frame of electron is given by $-\vec{\mu}\vec{B}$. The relativistic expression for such an interaction has the form

$$\mathcal{L}_{int} = \frac{1}{4m}\,\epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu} f_\lambda j_\sigma^\gamma = \frac{1}{2m}\,\epsilon_{\mu\nu\lambda\sigma}\partial_\mu A_\nu \mu_\lambda j_\sigma^\gamma \to -\frac{1}{2m}\,A_\nu \partial_\mu \epsilon_{\mu\nu\lambda\sigma} f_\lambda j_\sigma^\gamma, \tag{3.35}$$

where f_λ is a relativistic generalization of a unit vector in the direction of polarization of the electron. Namely, if $\vec{f}$ with $|\vec{f}| = 1$ describes the polarization of the electron in its rest frame, in a frame where the electron four-momentum is $(\epsilon, \vec{p})$ the four-vector f_λ has the form: $[(\vec{p}\cdot\vec{f})/m, \vec{f} + \vec{p}(\vec{p}\cdot\vec{f})/(\epsilon+m)m]$. The current j_σ^γ in Eq.(3.35) is the same as given by the expression (3.26). Thus a calculation of $\pi_\perp^m$ is reduced to an evaluation of the correlator of the currents

$$j_\nu^m = -\frac{1}{2m}\,\epsilon_{\mu\nu\lambda\sigma} f_\lambda \, \partial_\mu j_\sigma^\gamma. \tag{3.36}$$

The required expression can in fact be found using the formula for the correlator of the currents from Eq.(3.26). Indeed, the correlator can be reduced to the calculation of $\langle j_a^\gamma(x) j_b^\gamma(0)\rangle$ as

$$\langle j_\nu^m(x) j_\rho^m(0)\rangle = \frac{1}{4m^2}\,\varepsilon_{\mu\nu\lambda\sigma}\, f_\lambda \, \varepsilon_{\alpha\rho\beta\tau}\, f_\beta \, \partial_\mu \, \partial_\alpha \, \langle j_\sigma^\gamma(x) j_\tau^\gamma(0)\rangle. \tag{3.37}$$

We then find that after averaging over the transverse indices 2 and 3 the Fourier transform of this expression takes a simple form:

$$\pi_\perp^m = \frac{1}{2}\left(\pi_{22} + \pi_{33}\right) = -\frac{1}{8m^2}\, f_\mu^2\, q^2 \pi_\parallel = -\frac{1}{8m^2}\, q^2\, \pi_\parallel(q^2) \tag{3.38}$$

with q^2, as previously, being the square of the two dimensional Euclidean vector related to the photon momentum as $-q^2 = \omega^2 \sin^2\theta$. Clearly, the formula (3.38) gives the relation between the expressions in Eq.(1.14) and Eq.(1.12).

3.2.3 Polarization operator at arbitrary ω.

In order to find the vacuum polarization tensor for arbitrary frequency of the photon one should take into account the deformation of the tunneling trajectory by the photon energy and momentum (see Fig. 3.5). This deformation is driven by two opposing factors. On one hand the photon supplies the energy to the pair while on the other hand it transfers its momentum to the e^+e^- pair which increases the energy barrier for tunneling. A proper treatment of these effects amounts to finding the new configuration and calculating the action on it. The notation takes a simple form if

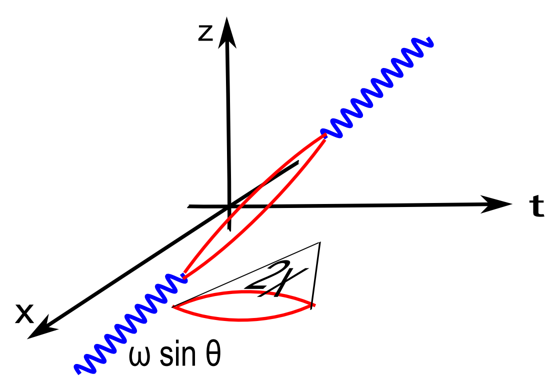

Figure 3.5: Deformation of the electron trajectory for large ω.

one first performs a Lorentz transformation in the direction of the electric field such that sets the longitudinal momentum of the photon to zero, $k_x = 0$, and also the transverse axes are chosen so that $k_y = 0$. In terms of the initial energy ω and the incident angle θ of the photon the four-momentum of the photon in the new frame has the form $(\omega \sin\theta,\, 0,\, 0, \omega \sin\theta)$. The proper trajectory is then found by solving the classical (Euclidean) equations of motion corresponding to the modified action

$$S = m \int \sqrt{x'^2_\mu} d\varphi - e \int A_\mu x'_\mu d\varphi - T\,\omega \sin\theta + Z\,\omega \sin\theta, \tag{3.39}$$

where φ is the angular coordinate of the projection of the trajectory on the (t, x) plane, and $x' = dx/d\varphi$. The expression (3.39) allows for the electron to have nonzero momentum in the z direction (it should be mentioned that in the Euclidean version this momentum is imaginary ip). T and Z are the sizes of the loop along the t and z directions correspondingly, and the last two terms in (3.39) are given by the negative of the Euclidean action of the photon, corresponding to the gap in the photon propagation over the duration and the extent in the z direction of the e^+e^-loop. Such an approach is similar to the one described in [24, 51]. As usually, the equations of motion for the particle in external electromagnetic field are

$$m \frac{d}{d\varphi} \frac{x'_\mu}{\sqrt{x'^2}} = -eF_{\mu\nu} x'_n. \tag{3.40}$$

Solving these one gets

$$\begin{aligned} t &= \frac{\sqrt{p^2 + m^2}}{eE} \cos\varphi + c_0, \\ x &= \frac{\sqrt{p^2 + m^2}}{eE} \sin\varphi + c_1, \\ z &= \frac{p}{eE}\varphi + c_3, \end{aligned} \tag{3.41}$$

where c_μ are constants. One can see from the expressions (3.41) that the projection of the trajectory on the (t, x) plane consists of two arcs of a circle with the radius $\sqrt{p^2 + m^2}/eE$. Introducing the angle spanned by each of the arcs as 2χ and substituting the solution back to Eq.(3.39), one finds

$$S = \frac{4m^2\chi}{eE} - \frac{2\chi(m^2 + p^2)}{eE} + \frac{m^2 + p^2}{eE} \sin 2\chi - \frac{2\omega \sin\theta \sqrt{p^2 + m^2}}{eE} \sin\chi + \frac{2\omega \sin\theta\, p\chi}{eE}. \tag{3.42}$$

Minimizing the action with respect to χ and p gives the expression (1.15), which coincides with the one derived by another technique in [43] [2].

[2] One can readily notice that the derived in this way values of χ and p correspond to enforcing the energy and momentum conservation in the creation and annihilation of the pair by the photon

3.3 Discussion of the results

One can readily notice that the thermal expression (3.13) is applicable only at a low temperature $T < 1/(2R)$. However the resulting formula (3.20) for the one-photon rate is valid at arbitrary values of the Keldysh parameter γ_θ (as long as the assumed bounds, $\omega \ll m$ and $eE \ll m^2$ are satisfied). This behavior, where the thermal expression is singular at the critical temperature, while the rates for individual processes are smooth functions, is similar to the one observed in analogous calculations in Refs. [20, 27].

It can be also readily argued that the absorption rate in Eq.(3.20) is in fact related only to the photon polarization parallel to the electric field, so that for the photons with that particular polarization the absorption rate is twice larger than the average:

$$\kappa_\| = 2\,\bar{\kappa}\ , \tag{3.43}$$

while for the photons with polarization orthogonal to the external field there is no absorption: $\kappa_\perp = 0$. Indeed, our Euclidean space calculation would not be affected if we considered the system, including the external electric field $E = E_x$, in a flat capacitor with small distance Δ between the plates (but still $\Delta \gg R$), i.e. if we imposed zero boundary condition on the components A_y and A_z of the vector potential at $x = \pm\Delta/2$: $A_{y,z}(x = \pm\Delta/2) = 0$. Clearly, such an arrangement leaves the components A_x and A_t intact, so that the potential created by the loop currents in the (x, t) plane is still given by our Eq.(3.12), and one would arrive at the same result for the action per period ΔS. On the other hand the boundary conditions at the plates of the capacitor introduce an energy gap π/Δ in the spectrum of the photons with polarization in the y and z direction and thus their presence in the thermal bath is suppressed. The absence of dependence of the thermal rate on the boundary conditions for the transversal to the external field E polarizations implies that no absorption rate $\kappa_\perp$ arises for the perpendicular polarization as long as only the expansion in the leading parameter γ_θ is concerned. Such absorption would however arise in the next order of expansion in the ratio ω/m. Within the described here technique the terms of that order originate from the effects that are left beyond our essentially classical treatment of the electron Euclidean trajectory and of the field that it creates. In particular, the terms of higher order in ω/m would arise if one also includes the magnetic interaction of the current loops due to the spin of the electron.

Using the asymptotic expression for the Bessel function

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}}, \text{ for } z \gg 1, \tag{3.44}$$

one can find the exponential behavior of the probability rate (3.20)

$$\bar{\kappa} \sim \exp\left[-\frac{m^2}{eE}\left(\pi - \frac{2\omega}{m}\sin\theta\right)\right], \tag{3.45}$$

which agrees with the $\omega \ll m$ limit of the exponential expression recently found in Ref. [29].

We have found the exponential factor of the polarization operator Eq.(1.15), which is valid at an arbitrary energy ω of the photon. The exponential power in this factor can be expanded in γ_θ as

$$\pi_{\|,\perp} \sim \exp\left\{-\frac{\pi\, m^2}{eE} + 2\,\gamma_\theta - \frac{\pi}{4}\,\gamma_\theta^2\,\frac{eE}{m^2} + O\left[\frac{\gamma_\theta^3\,(eE)^2}{m^4}\right]\right\}\ , \tag{3.46}$$

which shows that indeed the parameter for the expansion is $\gamma_\theta\, eE/m^2 = \omega \sin_\theta /m$, and the higher terms are small as long as $\omega \ll m$. It is also satisfying to notice that the linear in γ_θ term in Eq.(3.46) agrees with the large γ_θ asymptotic behavior of the Bessel functions in the equations (1.12) - (1.14). In other words, those expressions describe, including the preexponential factors, the matching between the low energy and the high energy behavior of the pair creation rate. Clearly the overlap region for these two types of expressions is determined by the condition $1 \ll \gamma_\theta \ll m^2/(eE)$.

Chapter 4

Induced false vacuum decay

4.1 Vacuum tunneling in Euclidean space time

In this section we consider an application of both approaches discussed earlier to the case of interacting scalar particles in two dimensions. Since the main purpose of the present section is an illustration of an application of the two approaches to treatment of an induced semiclassical process, we limit ourselves, for simplicity, to a model in (1+1) space-time dimensions, while a generalization to higher dimensional models is quite straightforward.

4.2 Model

We consider a model describing two real scalar fields in $1+1$ dimensions with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi\right)^2 + \frac{1}{2}\left(\partial_\mu \chi\right)^2 - \frac{1}{2} m^2 \chi^2 - V(\varphi) - V_{\rm int}(\varphi, \chi), \tag{4.1}$$

where the potential $V(\phi)$ has asymmetric form shown in Fig.1. For further uses this potential can be chosen as

$$V(\varphi) = \frac{1}{4}\lambda\left(\varphi^2 - v^2\right)^2 - \frac{\varepsilon}{2v}\left(\varphi + v\right). \tag{4.2}$$

We also choose the interaction potential between the two scalar fields in such a way that it is not equal to zero only where the field φ differs from its vacuum values, for example

$$V_{\rm int} = \alpha \left(\varphi^2 - v^2 \right) \chi = \alpha \rho(\varphi) \chi \,, \tag{4.3}$$

with α being a constant. As argued previously, the parameters of the Lagrangian are assumed to be such that the mass of the field φ is much greater then that of the field χ, namely

$$m \ll M = v\sqrt{2\lambda} \,. \tag{4.4}$$

In the thin wall limit, when the size of the bounce is much greater than the thickness

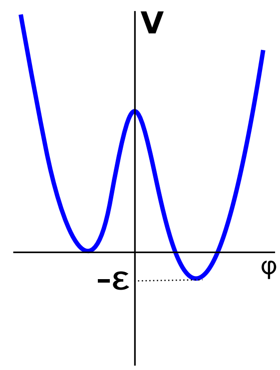

Figure 4.1: The asymmetric potential for the master field φ.

of the wall (the inverse mass of the field φ) $R = \mu/\varepsilon \gg 1/M$ the dynamics is described in purely geometrical terms, namely the shape of the wall, which can be represented in polar coordinates by the function $r(\theta)$ (closed trajectory corresponding to the distorted bounce configuration). Therefore the Euclidean action of the system can be written in the following form

$$S = \mu \ell - \varepsilon \mathcal{A} + \int d^2x \left[\frac{1}{2} \left(\partial_\mu \chi \right)^2 + \frac{1}{2} m^2 \chi^2 + \rho^r \chi \right] , \tag{4.5}$$

where $\ell = \int d\theta \sqrt{\dot r^2 + r^2}$, $\mathcal{A} = \frac{1}{2} \int d\theta r^2$ are the length of the bounce boundary and the area it encircles, and the dot stands for the derivative with respect to θ. The density $\rho^{(r)}$ is proportional to the coefficient α and it plays the role of the source for the χ field. It is obvious that in the discussed approximation the area, where the value of field φ substantially differs from the vacuum, is very thin and the support of the function $\rho^{(r)}$ coincides with the shape of the boundary of the bounce. Hence, it has the form of a surface δ-function

$$\rho^{(r)}(t,x) = g\,\delta(\sqrt{t^2+x^2} - r(\theta)), \qquad g \propto \alpha. \tag{4.6}$$

As a result the partition function can be expressed as an integral over closed trajectories describing the shape of the boundary

$$\frac{\Gamma}{L} = \frac{2}{LT}\mathrm{Im}\int \mathcal{D}r\mathcal{D}\chi e^{-S[r,\chi]}, \tag{4.7}$$

with the action S derived from Eq.(4.5). At a small constant g the latter expression can be expanded in a power series in g. Therefore at a small coupling between the scalar fields one can find the decay rate of the false vacuum in terms of an expansion in powers of g. The decay induced by n particles of the field χ arises in the order g^{2n}, while the spontaneous vacuum transition proceeds at $g \to 0$. The rate of the spontaneous decay is found from calculating the action on the bounce configuration (1.16) and the determinant corresponding to the integration over small fluctuations around the tunneling trajectory, and is given by [19]

$$\frac{\Gamma}{L} = \frac{\varepsilon}{2\pi}\exp\left[-\frac{\pi\mu^2}{\varepsilon}\right]. \tag{4.8}$$

4.2.1 False vacuum decay in thermal bath

As is already mentioned, one can use a thermal approach to extract [20, 27, 40] from the expression for the total process rate at a small but finite temperature the contribution of the processes induced by a specific number of particles present in the initial state. Therefore we start with calculating the total rate of the false vacuum decay in a thermal bath. It is quite clear that at a sufficiently low temperature only the contribution of states containing particles of the field χ are relevant, while the contribution from the states with the φ excitations are suppressed by the Gibbs factor $e^{-M/T}$.

At a low temperature one can consider the same effective Euclidean action (4.5), except that now the system lives on the cylinder with the period equal to the inverse temperature $\beta = 1/T$. In other words, one may consider the system living on the (t,x) plane with periodic boundary conditions for the Euclidean time coordinate $t \to t+\beta$ (see Fig.2.7). It is worth noting that the use of the action from Eq.(4.5) is justified

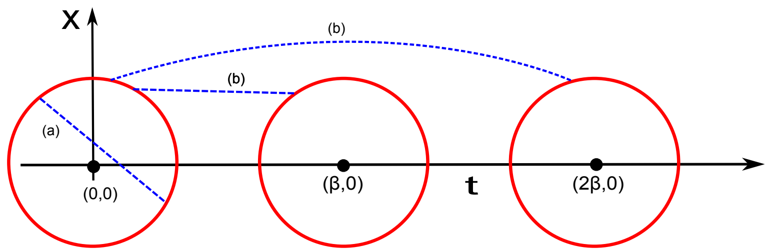

Figure 4.2: Periodic copies of the bounce and their interactions through the field χ. The self interaction (a) within one copy does not depend on the period β, while the interaction (b) between different copies does depend on β and describes the thermal effects.

as long as the temperature satisfies the condition $T \ll M$. However, we consider still much lower temperatures, such that a stronger condition $T \ll \varepsilon/\mu$ is satisfied, which drastically simplifies the calculations. At such a small temperature (equivalently, at a large period β) one still may use the same bounce configuration to calculate the partition function (4.7). The zero temperature result corresponds to the leading order contribution at $\beta \to \infty$ and is given by Eq.(4.8). The next to leading order corrections arise in fact from the exchange of the χ particle, since at $T < 2\,R$ there is no bounce deformation [52]. The effect of the exchange of χ can be classified into two types. One is the self action of the bounce independent of the temperature, it is labeled by (a) in the Fig.4.2. The second, labeled as (b), is the interaction of the two copies of the bounce separated along the Euclidean time by β, 2β, and so on. In fact, it is still a part of the self interaction of one and the same bounce, living on a cylinder, and this part clearly depends on the period of the cylinder and disappears when β goes to infinity. We restrict our interest to the second type of contribution, since this is the only one that produces a nontrivial temperature dependence.

The contribution to the action due to the interaction of two copies of the bounce separated by β can be found as usual by using the expression for the propagator of the scalar particle with mass m. In $1+1$ dimensions the propagator D satisfies the equation

$$-\Delta D(t,x) + m^2 D(t,x) = \delta^{(2)}(t,x), \tag{4.9}$$

and is given by

$$D(t,x) = \frac{1}{2\pi} K_0(mr), \tag{4.10}$$

where $K_0(mr)$ is the modified Bessel function of the second kind, and $r = \sqrt{t^2 + x^2}$. The expression for the part of the action due to the interaction between two copies of the bounce then takes the form

$$\Delta S_{(1)} = -\frac{1}{2} \int \rho^B(t_1, x_1) D(t_2 - t_1, x_2 - x_1) \rho^B(t_2 - \beta, x_2) dt_1 dx_1 \, dt_2 dx_2, \tag{4.11}$$

where ρ^B is the density (4.6) for the bounce located at the origin

$$\rho^B(t,x) = g\, \delta(\sqrt{t^2 + x^2} - R). \tag{4.12}$$

The shift in the argument of the second function $\rho^B(t_2 - \beta, x_2)$ clearly corresponds to the second copy of the bounce placed at $(\beta, 0)$. The extra factor of one half in the expression (4.11) is due to the apparent double counting: the expression without this factor corresponds to the interaction between the two copies of the circle, while we need to consider the change in the action per one bounce (per one period).

In order to find the integral in Eq.(4.11) one can readily notice that the function

$$\chi(t_2, x_2) = \int D_0(t_2 - t_1, x_2 - x_1) \rho^B(t_1, x_1) dt_1 dx_1 \tag{4.13}$$

satisfies the equation

$$\left(-\Delta + m^2\right) \chi(t,x) = g\delta(\sqrt{t^2 + x^2} - R). \tag{4.14}$$

Let (r_i, θ_i) be the polar coordinates corresponding to the Cartesian ones (t_i, x_i). The solution to the equation (4.14) can then be found as

$$\chi(r) = \begin{cases} gR\, I_0(mR)\, K_0(mr), & r > R, \\ gR\, I_0(mr)\, K_0(mR), & r < R, \end{cases} \tag{4.15}$$

and the integration over the first circle then reduces to the following

$$\int K_0\left(m\sqrt{r_2^2 + R^2 - 2r_2 R\cos\theta_1}\right) d\theta_1 = 2\pi\, I_0(mR)\, K_0(mr_2)\,, \tag{4.16}$$

where $I_0(mR)$ is the modified Bessel function of the first kind. Performing the further integration over r_2 and θ_2 in Eq.(4.11) and summing over the contributions from the interaction with all copies one finally finds

$$\frac{\Gamma_T}{L} = \frac{\varepsilon}{2\pi} \exp\left[-\frac{\pi\mu^2}{\varepsilon} - \Delta S\right], \tag{4.17}$$

with the total temperature dependent part of the action given by the following expression

$$\Delta S = -2\pi g^2 R^2 I_0^2(mR) \sum_{n=1}^{\infty} K_0(m\beta n), \tag{4.18}$$

Expanding the expression (4.17) in ΔS at large β one can find the low temperature behavior of the thermal corrections to the decay rate:

$$\frac{\Gamma_T}{L} = \frac{\varepsilon}{2\pi} \exp\left[-\frac{\pi\mu^2}{\varepsilon}\right] \left(1 + 2\pi g^2 R^2 I_0^2(mR) \sqrt{\frac{\pi}{2m\beta}}\, e^{-m\beta} + \dots\right). \tag{4.19}$$

As expected, these corrections are exponentially suppressed in the parameter m/T.

4.2.2 Induced false vacuum decay

The enhancement of the tunneling at a finite temperature arises through the stimulation of the process by the particles present in the bath. The dependence of the rate of the process induced by the χ particles on their energies then translates into the dependence on the temperature T after averaging over the thermal distribution of the particles with the standard density function

$$n(\vec{k}) = \frac{1}{e^{E_k \beta} - 1}\,, \tag{4.20}$$

where the energy is given by $E_k = \sqrt{\vec{k}^2 + m^2}$. The number of particles involved in each of the microscopic processes can be readily identified by the power of the factor g^2. Since the thermal correction (4.18) in the action is proportional to g^2, the n-particle contribution to the thermal rate is given by the n-th power of ΔS in the expansion of the factor $\exp(-\Delta S)$ in the expression (4.17). In particular, the one and two-particle contributions to the decay rate in a thermal state are given by

$$\begin{aligned} \frac{\Gamma_{1\chi}}{L} &= \frac{\varepsilon}{2\pi} \exp\left[-\frac{\pi\mu^2}{\varepsilon}\right] (-\Delta S)\,, \\ \frac{\Gamma_{2\chi}}{L} &= \frac{\varepsilon}{2\pi} \exp\left[-\frac{\pi\mu^2}{\varepsilon}\right] \frac{1}{2} (\Delta S)^2\,. \end{aligned} \tag{4.21}$$

On the other hand, the one-particle contribution can be found in terms of the width γ_χ of the particle χ associated with the destruction of the false vacuum by the particle's presence:

$$\frac{\Gamma_{1\chi}}{L} = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{\gamma_\chi \, m}{\sqrt{k^2+m^2}}\, n(k) = \frac{m\gamma_\chi}{\pi} \sum_{n=1}^{\infty} \int_{m\beta}^{\infty} \frac{dx}{\sqrt{x^2-(m\beta)^2}}\, e^{-nx} = \frac{m\gamma_\chi}{\pi} \sum_{n=1}^{\infty} K_0(m\beta n). \tag{4.22}$$

The origin of the Lorentz factor $m/\sqrt{\vec{k}^2+m^2}$ is rather straightforward: the width γ_χ is measured in its rest frame, while we consider the process in the laboratory frame. Comparing the two results (4.21) and (4.22) we find the width

$$\gamma_\chi = \frac{\pi\,\varepsilon}{m}\, g^2 R^2 I_0^2(mR)\, \exp\left[-\frac{\pi\mu^2}{\varepsilon}\right]. \tag{4.23}$$

Similarly, one can express the two-particle contribution from (4.21) through the rate $w(k_1,k_2)$ of the metastable vacuum destruction by collisions of two particles. Namely, the relation is as follows

$$\frac{\Gamma_{2\chi}}{L} = L \int \frac{dk_1 dk_2}{(2\pi)^2}\, w(k_1,k_2)\, n(\vec{k}_1) n(\vec{k}_2)\,. \tag{4.24}$$

The two-particle rate function w is related to the dimensionless and Lorentz invariant $1+1$ dimensional analog σ of a cross section in $3+1$-dimensions:

$$w(k_1,k_2) = \sigma(k_1,k_2) \,\times\, \text{Flux} = \sigma\, \frac{v_{rel}}{L}\,, \tag{4.25}$$

where v_{rel} is the relative velocity of the two particles, which is expressed in terms of the energies of the particles and of the standard kinematical invariant $I = \sqrt{(k_1\cdot k_2)^2 - m^4}$ as

$$v_{rel} = \frac{I}{E_{k_1}E_{k_2}}\,. \tag{4.26}$$

Due to the Lorentz invariance, the function $\sigma(k_1,k_2)$ can depend only on the invariant I, so that one can consider it in a form of a power series in I:

$$\sigma(I) = \sum_n c_n I^n. \tag{4.27}$$

In order to write the integral over the momenta of the two particles with proper integration limits one should take into account that the particles are identical and also that for the collision to take place they should move toward each other. Let particle '1' be the one on the left and the particle '2' on the right. Also let the spatial momentum k be considered as positive when the particle moves from left to right.

Then the collision takes place if and only if $k_1 > k_2$, where both the spatial momenta can be of either sign, and the integral in Eq.(4.24) takes the form

$$\frac{\Gamma_{2\chi}}{L} = \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi\, E_{k_1}}\, n(k_1) \int_{-\infty}^{k_1} \frac{dk_2}{2\pi\, E_{k_2}}\, n(k_2)\, \sigma(I)\, I\ . \tag{4.28}$$

This integral can reproduce the two-particle thermal term in Eq.(4.21) only if the product $\sigma(I)I$ is constant, i.e. only if the expansion in Eq.(4.27) reduces to a single term $\sigma(I) = c_{-1}/I$. The integrand in Eq.(4.28) then factorizes into terms each depending on the absolute value of the momentum, so that the limits of integration can be rearranged as

$$\begin{aligned} \frac{\Gamma_{2\chi}}{L} &= c_{-1} \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi\, E_{k_1}}\, n(k_1) \int_{-\infty}^{k_1} \frac{dk_2}{2\pi\, E_{k_2}}\, n(k_2) \\ &= \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi\, E_{k_1}}\, n(k_1) \int_{0}^{\infty} \frac{dk_2}{2\pi\, E_{k_2}}\, n(k_2) = 2c_{-1} \left[\sum_{n=1}^{\infty} K_0(m\beta n) \right]^2\ . \end{aligned} \tag{4.29}$$

Upon a comparison with the expressions (4.18) and (4.21) one thus arrives at the final result for the dimensionless invariant ‘cross section’ for the destruction of the false vacuum in collision of two χ-particles:

$$\sigma(I) = \frac{\pi\varepsilon}{2I} \left[2\pi g^2 R^2 I_0^2(mR) \right]^2 \exp\left[-\frac{\pi\mu^2}{\varepsilon} \right]\ . \tag{4.30}$$

4.2.3 Semiclassical calculation

In this section we consider the second approach to calculating the rate of the false vacuum decay induced by one particle. Interpreting the particle-induced decay rate as the width of the particle χ, one can find it from the propagator of the χ field in the background of the bounce as the imaginary part of the density-density correlator[1]:

$$\Pi(t, x) = \langle \rho(t, x)\rho(0, 0) \rangle. \tag{4.31}$$

Using the formulas (4.5) and (4.7) we can rewrite the leading semiclassical term of the correlator in the following form

$$\begin{aligned} \mathrm{Im}\Pi(t, x) &= \mathrm{Im} \int \mathcal{D}\varphi \rho(t, x)\rho(0, 0)\, e^{-S[\varphi]} = \mathrm{Im} \int \mathcal{D}r \rho^{(r)}(t, x)\rho^{(r)}(0, 0) e^{-S[r]} \\ &= \frac{\Gamma}{2L} \int \rho^B(t - t_B, x - x_B)\rho^B(-t_B, -x_B) d^2x_B, \end{aligned} \tag{4.32}$$

[1] This is in a one to one correspondence with QED, where the attenuation rate of the photon is given by the imaginary part of current-current correlator.

with $\rho^B(t,x)$ from Eq.(4.12).

The integration in Eq(4.32) is performed in the following manner. We choose the coordinate system such that the coordinate x in the integral is equal to zero. One can always do so by an appropriate rotation of the coordinates. Due to the O(2) symmetry, the full final result can be restored in an arbitrary coordinate system by simply promoting t to $\sqrt{t^2+x^2}$ in the final expression for the correlator. After this

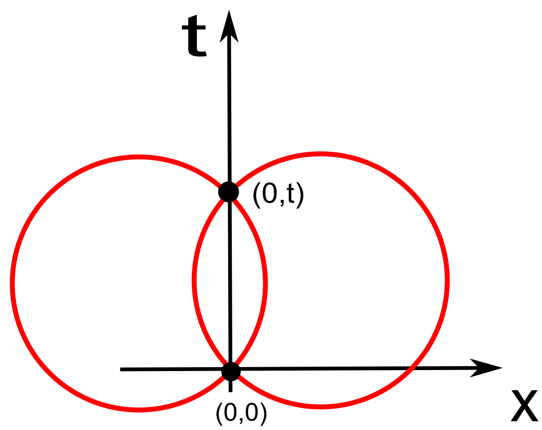

Figure 4.3: The two positions of the bounce for which the density correlator is not equal to zero.

choice of orientation we make the change of the variables of integration

$$\begin{aligned} z_1 &= \sqrt{t_B^2+x_B^2}, \\ z_2 &= \sqrt{(t_B-t)^2+x_B^2}, \end{aligned} \tag{4.33}$$

and rewrite the integral in the following form

$$\begin{aligned} & g^2 \int dt_B dx_B \delta\left(\sqrt{t_B^2+x_B^2}-R\right)\delta\left(\sqrt{(t_B-t)^2+x_B^2}-R\right) \\ &= 2g^2 \int dz_1 dz_2 \delta(z_1-R)\delta(z_2-R) J\left(\frac{t_B x_B}{z_1 z_2}\right) dz_1 dz_2 = \frac{4R^2 g^2}{t\sqrt{4R^2-t^2}}, \end{aligned} \tag{4.34}$$

where $J\left(\frac{t_B x_B}{z_1 z_2}\right)$ is the Jacobian of the transformation (4.33). The factor of two in the second expression in Eq.(4.34) comes from two possible configurations shown in Fig.4.3. After promoting $t \to \sqrt{t^2+x^2}$ and making the Fourier transformation one finds

$$\int dq_0 dq_1 \frac{4R^2 g^2}{\sqrt{t^2+x^2}\sqrt{4R^2-t^2-x^2}} e^{iq_0 t + iq_1 x} = -4\pi^2 g^2 R^2 I_0^2(mR), \tag{4.35}$$

where the two-momentum q is set on the mass shell, $q^2 = -m^2$. Using the formula (4.32) one thus finds:

$$\gamma_\chi = -\frac{\mathrm{Im}\Pi}{m} = \frac{\pi\,\varepsilon}{m}\, g^2 R^2 I_0^2(mR)\, \exp\left[-\frac{\pi\mu^2}{\varepsilon}\right], \tag{4.36}$$

which coincides with Eq.(4.23).

4.3 Summary of the chapter

The formulas for the false vacuum decay rate induced by one (Eq.(4.23) or Eq.(4.36)) and by a collision of two (Eq.(4.30)) light particles of the weakly interacting field χ merit some remarks. Firstly, one can readily notice that the width of the particle associated with the destruction of the vacuum is singular in the limit $m \to 0$, which is a consequence of the infrared behavior in this limit in 1+1 dimensions. The singular growth at low m however does not result in any unphysical artifacts, since the rate of decay for a state with any fixed finite energy E is finite in the massless limit due to the time dilation factor m/E. Secondly, at large mR the exponential asymptotic behavior of the Bessel function correctly reproduces the known [34, 24] semiclassical exponential factor in both the one-particle and two-particle processes:

$$\gamma_\chi \sim \exp\left[-\frac{\pi\mu^2}{\varepsilon} + 2mR\right], \qquad w \sim \exp\left[-\frac{\pi\mu^2}{\varepsilon} + 4mR\right]. \tag{4.37}$$

It is worth noting that the enhancement factor in the collision is determined by the mass of the particles rather than by their c.m. energy $\sqrt{s}$. This behavior, which is in agreement with the geometrical calculation of the exponent in Refs. [34, 24] is a consequence of the assumed linear in χ interaction in Eq.(4.3). Had we assumed instead a quadratic in χ interaction, the exponential factor would be determined by $\sqrt{s}\,R$ at $\sqrt{s} \gg m$, as is the case for the similar factor in the rate of destruction of metastable strings and walls by Goldstone bosons [20]. Moreover, the absence of the dependence on the energy of the exponent in σ is true only as long as the energy is small in comparison with the mass scale M of the master field φ, $\sqrt{s} \ll M$. The reason for this behavior is that iteration of the interaction in Eq.(4.3) through the short-distance propagator of φ gives rise to terms with higher powers of χ, which however are suppressed by inverse powers of M.

Chapter 5

Conclusion

The main conclusions that can be drawn from the presented here study can be formulated as follows

i. The effect on the preexponential factor in the tunneling decay rate arising from the Goldstone modes living on strings and walls is expressed in terms of boundary terms in the path integral around the bounce configuration. This effect is fully calculable within the low-energy effective action of the Nambu-Goto type, and essentially reduces to a renormalization of the tension of the interface between the two phases of the topological defect. We have found that this property is unique for the strings and walls and is generally lost in a description of decay of topological objects with more dimensions, where one would have to go beyond the effective Nambu-Goto description.

ii. The thermal effects in the tunneling rate are also fully calculable and are described by an expansion in powers of the temperature. The power-like, rather than exponential, behavior of the thermal terms is due to the presence of the massless Goldstone bosons.

iii. The expansion for the thermal catalysis factor converges as long as the temperature is less than the inverse diameter of the bounce, so that in the Euclidean calculation of the bounce configuration to (the imaginary part of) the free energy there is no obstruction to periodically replicating the bounce with the Matsubara period $\beta = 1/T$. For the decay of strings this range of temperatures corresponds to the dominance of the (thermally enhanced) quantum tunneling over the classical thermal effects, and at the critical temperature $1/\ell_c$ the regime of the transition changes and the classical thermal effects take over. For the wall decay the classical thermal activation starts dominating at a significantly lower temperature, where the thermal enhancement of the quantum tunneling is still very small.

iv. The thermal enhancement of the tunneling rate can be viewed as and additional contribution to the decay probability arising from collisions of the Goldstone bosons which are present in the thermal state. Such an interpretation in fact allows to determine the rate of destruction of the considered topological objects in collisions of the Goldstone bosons. In order to unambiguously identify in the thermal expansion the terms corresponding to the contribution of collisions between a given number of the bosons, we have modified the considered statistical ensemble by introducing a (fictitious) negative chemical potential ν for the Goldstone bosons. The calculated dependence on ν and T of the preexponential factor in the decay rate thus serves as a generating function for the rates of the decay induced by collisions of particles, including the dependence of those rates on the energies in the collisions.

v. We find that a destruction of a string takes place only in collisions of even number of the Goldstone bosons, while a metastable wall can be destroyed in collisions of any number of the bosons, $n \geq 2$.

vi. We have calculated in a closed form the energy behavior of the probability of destruction of a string in a collision of two Goldstone bosons, which is valid at an arbitrary relation between the energy $\sqrt{s}$ and the infrared scale in the problem R_s. The found expression takes an especially simple form of Eq.(1.7) for the decay of a string into 'nothing'

vii. At large values of $R\sqrt{s}$ the collision-induced decay rate contains an exponential factor $\exp(2R\sqrt{s})$, which has been previously argued[34, 48] within the leading semiclassical approach and corresponds to a full transfer of the energy from the colliding particles to the tunneling degrees of freedom.

viii. We have calculated the rate of the photon-induced Schwinger process in the limit $\omega \ll m$ and $eE \ll m^2$ for arbitrary value of the Keldysh parameter γ_θ. Our calculation differs from the one [29] based on the vacuum polarization operator in external field in that we use an extension of a semiclassical treatment of the process to finite temperatures.

ix. The thermal rate is calculated in a standard way by considering the tunneling trajectory on a cylinder, i.e. periodic in the Euclidean time. The leading contribution of the photons, present in the thermal bath then arises from the the classical self-interaction of the electron current on the tunneling trajectory with itself on the cylinder. The contribution of stimulation of the pair creation by one photon is then determined from the term with appropriate power of the coupling e^2 in the thermal expression. In this way we reproduce the nontrivial behavior in Eq.(1.10) of the calculated rate. We believe that the considered method is of interest and can be used in other applications of tunneling processes induced by quantum particles.

x. We have considered the photon-stimulated Schwinger process in terms of a calculation of the vacuum polarization operator by purely semiclassical means.

xi. At the value of the electric field much smaller than E_{crit} the radius $R = m/(eE)$ is large in comparison with the Compton wavelength $1/m$, so that an expansion in eE/m^2 is in fact an expansion in the curvature of the semiclassical trajectory divided by m^2. A small curvature at each point of the tunneling path implies that one can employ a nonrelativistic expansion in the co-moving frame, which in fact is a way of interpreting the described treatment.

xii. The dominant in the parameter eE/m^2 effect arises for the photons with polarization along the external electric field. As can be seen from the discussed here calculation this effect is described by essentially the classical current of the charged particles on the tunneling trajectory interacting with field of the incident photon. Such an origin of this leading contribution is also in full agreement with the behavior obtained from the thermal treatment where the same effect originates from a classical interaction. Clearly, this part is not sensitive to the electron spin and has exactly the same form for creation of pairs of spinless particles (modulo the statistical spin factor of two). In terms of a nonrelativistic expansion in the co-moving frame this contribution is that of the electrostatic interaction.

xiii. For photons with polarization orthogonal to the electric field, however, there are two types of contribution to the stimulated pair production. One source, also independent of the spin, arises from the interaction of the incident photon with the current due to the fluctuations of the tunneling trajectory in the direction perpendicular to the external field. This effect, described by Eq.(1.13), in fact requires an equivalent of a one-loop calculation as discussed in the Appendix B. In the nonrelativistic interpretation this contribution is that of the interaction of a particle having velocity $\vec{v}$ with the vector potential $\vec{A}$. It can be noted in connection with the calculation of this effect that the photon energy dependence required by the gauge invariance is ensured by the subtraction in Eq.(1.13) of the constant term, which automatically arises as a result of the summation in Eq.(B.9). This constant term is analogous to the well known 'sea gull' type graph in QED of scalar particles.

xiv. An effect of the electron spin in the discussed process arises for photons with polarization orthogonal to the external field through the interaction in the co-moving frame of the electron magnetic moment with the photon. As discussed in the Section 4, once the 'quantum' origin of this contribution is absorbed in the magnetic interaction, the rest of the calculation is again essentially classical. Formally, the effect of this term is of a higher (second) order in the expansion in the ratio eE/m^2 as compared to that given by Eq.(1.13), and is suppressed at small values of γ_θ. However at γ_θ of order one and larger the ratio of the magnetic contribution to that in Eq.(1.13) is of the order of $(\omega \sin\theta)^2/(eE)$, which is an independent dimensionless parameter in the problem, and there is generally no reason to assume that this parameter is small.

xv. A deformation of the tunneling trajectory due to the interaction with the incident photon can be neglected as long as the photon energy is small: $\omega \ll m$. In terms of the expansion in the co-moving frame this corresponds to appearance of only a weak 'kink' in the trajectory at the point of interaction with the photon, which does not invalidate the expansion in small curvature. However at larger ω the kink is 'strong' and the approximation of an unperturbed tunneling path is not valid. Although at present a full calculation in this situation is not available, the exponential factor in the stimulated rate of pair creation can be found using the effective action approach, previously applied [24, 51] in similar problems related to an induced decay of metastable vacuum.

xvi. We have considered an illustrative 1+1 dimensional model of interacting scalar fields and demonstrated an application of the semiclassical thermal method to calculation of the rate of destruction of the false vacuum by one- and two- particle states of a light scalar field. For the one-particle state a direct calculation of the rate is also done in terms of the imaginary part of the particle propagator in the tunneling background. In the latter case the direct method is possibly simpler, while in the case of two particles the thermal approach appears to be more straightforward. Furthermore, the formula for the thermal decay rate produces a generating function for the probability of the process induced by an arbitrary number of particles.

In a few words the present study can be described as follows. We started from the process of spontaneous string (and wall) breaking, which is crucial for understanding the theoretical models of non-Abelian dynamics or the behavior of cosmic strings, because not only it is important which objects can decay (break in this case) but also it is important to know how long living these metastable objects are. For example, the loops of the cosmic strings must survive for more then one Hubble expansion time to serve as the seeds for structure (galaxies or clusters) formation. Only the actual calculation can provide us with the answer to the question whether the rate of the process sufficiently small or not.

As the next step we investigated how temperature affects the rate of the process, because it is even more natural to have the system with temperature. The results can be applied to the dynamics of the strings (walls, branes) in the early universe.

The treatment of the thermal enhancement as the combined contributions to the decay rate of the breaking in collisions of thermally excited states, allowed us to extract from the decay rate at non-zero temperature the probability of the breaking induced by arbitrary number of excitations (Goldstone bosons). We used the same technique to find the attenuation rate of the photons in external field.

One can state that although there is no direct experiment where one can measure the decay rate of the string or wall, the method developed investigating these processes can be successfully utilized for investigating the processes which can be observed (or will be observed in near future), like charged pair production in external field stimulated by photon.

The calculations discussed in the present study can be readily generalized to other models of interaction and to models in higher dimensions.

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Appendix A

$\pi_{\|}$

Let us consider the circular electron trajectory (the bounce) with the center located at $x_B = (t_B, x_B, y_B, z_B)$. The current at points $(0, x, y, z)$ and $(0, 0, 0, 0)$ (we can always choose such a coordinate system) is found using the expression (3.29)

$$\begin{aligned} j_a^B(x) &= e\,\frac{(x_B - x, -t_B)}{r_2}\,\delta(r_2 - R)\,\delta(y - y_B)\,\delta(z - z_B), \\ j_b^B(0) &= e\,\frac{(x_B, -t_B)}{r_1}\,\delta(r_1 - R)\,\delta(y_B)\,\delta(z_B), \end{aligned} \tag{A.1}$$

where $r_1 = \sqrt{t_B^2 + x_B^2}$, $r_2 = \sqrt{t_B^2 + (x_B - x)^2}$, and a and b are the two-dimensional indices in the (t, x) plane. Therefore one finds that the l.h.s. of the (3.30) can be rewritten as

$$e^2 \int \frac{dt_B\,dx_B}{r_1 r_2} \begin{pmatrix} (x_B - x)x_B & -(x_B - x)t_B \\ -x_B t_B & t_B^2 \end{pmatrix} \delta(r_1 - R)\delta(r_2 - R)\delta(y)\delta(z). \tag{A.2}$$

One can integrate over r_1 and r_2 instead of t_B and x_B, taking into account the Jacobian of the transformation,

$$\left|\frac{\partial(t_B, x_B)}{\partial(r_1, r_2)}\right| = \frac{r_1 r_2}{x|t_B|}. \tag{A.3}$$

The integration of the delta functions reduces to calculation of the expression in the integral for two configurations shown in Fig. A.1. As a result one gets the

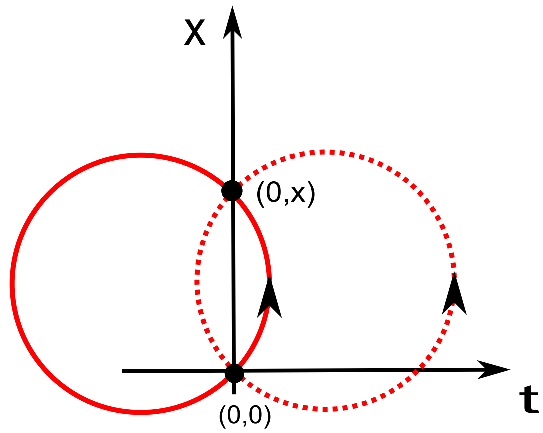

Figure A.1: The contributing configurations

l.h.s. of Eq.(3.30) in the form

$$e^2 \begin{pmatrix} -x^2 & 0 \\ 0 & 4R^2 - x^2 \end{pmatrix} \frac{\delta(y)\delta(z)}{x\sqrt{4R^2 - x^2}}. \tag{A.4}$$

As follows from the symmetries in the problem the integral can be generally written in the form

$$\int d^4x_B \, j_a^B(x) j_b^B(0) = r_a r_b B + A\delta_{ab}, \tag{A.5}$$

where yet unknown coefficient functions A and B depend only on the SO(2) invariant length $r = \sqrt{t^2 + x^2}$. Comparing (A.4) with (A.5) we find these coefficient functions and arrive at the following expression

$$\int d^4x_B \, j_a^B(x) j_b^B(0) = e^2 \, \frac{4R^2 r_a r_b - \delta_{ab} r^4}{r^3 \sqrt{4R^2 - r^2}} \delta(y)\delta(z). \tag{A.6}$$

As a cross check it can be readily verified that the found expression satisfies the current conservation $\partial_\mu \langle j_\mu(x) j_\nu(0)\rangle = 0$. Due to this property the Fourier transform of this correlator has the standard orthogonal form

$$\pi_{ab}(q^2) = \left(\delta_{ab} - \frac{q_a q_b}{q^2}\right) \pi(q^2), \tag{A.7}$$

where $\pi(q^2) = \pi_{aa}(q^2)$ (the summation over a is implied).

$$\pi(q) \;\; = \;\; \frac{1}{2} \, \frac{\Gamma}{V} \, e^2 \int \frac{4R^2 - 2r^2}{\sqrt{4R^2 - r^2}} \delta(y)\delta(z) e^{iqr\cos\varphi} r \, dr \, d\varphi \, dy \, dz$$

$$
\begin{aligned}
&= 2\pi e^2 \frac{1}{2}\frac{\Gamma}{V}\int \frac{4R^2-2r^2}{\sqrt{4R^2-r^2}} J_0(qr)\,dr \\
&= \frac{1}{2}\frac{\Gamma}{V} 4\pi^2 R^2 e^2\, J_1^2(qR). \qquad \text{(A.8)}
\end{aligned}
$$

Taking into account that $q^2 = -\omega^2 \sin^2\theta$, we find the expression in Eq.(1.12) for $\pi_{\parallel}$.

Appendix B

The 'electric' part of $\pi_\perp$

The part of the action $\delta S[\xi]$ arising from the fluctuations of the electron trajectory in the transverse direction $\delta z = \xi_3$ has the form

$$\delta S[\xi] = S_3[\xi_3] - e\int A_3\dot{\xi}_3 ds, \tag{B.1}$$

with

$$S_3[\xi_3] = \frac{m}{2}\int ds\,\dot{\xi}_3^2\ , \tag{B.2}$$

and where s is an arbitrary parameter on the trajectory, we choose it to be the length parameter of the unperturbed bounce, so that $s \in [0, 2\pi R]$. For the bounce centered at the origin on can write the additional current associated with the fluctuation as

$$j_3^\xi = e\,\dot{\xi}_3(s)\delta(r-R)\delta(y)\delta(z). \tag{B.3}$$

In order to find the correlator

$$\langle j_3^\xi(x) j_3^\xi(0)\rangle = \int \mathcal{D}\xi j_3^\xi(x) j_3^\xi(0) e^{-S_B - S[\xi]}, \tag{B.4}$$

we again choose the coordinate system where $x = (0, x, y, z)$. Then

$$\begin{aligned} j_3^\xi(x) &= e\,\dot{\xi}_3(s_1)\,\delta(r_1 - R)\,\delta(y_B - y)\,\delta(z_B - z), \\ j_3^\xi(0) &= e\,\dot{\xi}_3(s_2)\,\delta(r_2 - R)\,\delta(y_B)\,\delta(z_B). \end{aligned} \tag{B.5}$$

According to Eq.(3.25) the contribution of this current to the imaginary part of the polarization operator can then be written as

$$\pi_{33}(x) = -\frac{1}{2}\frac{\Gamma}{V}\int d^4x_B\,\mathcal{D}\xi_3\,j_3^{\xi B}(x)\,j_3^{\xi B}(0)\,e^{-S_3}/\int \mathcal{D}\xi_3\,e^{-S_3}, \tag{B.6}$$

where the $j^{\xi B}$ is the current j^{ξ} corresponding to the bounce centered at x_B. Integrating over the position of the bounce one gets

$$\pi_{33}(x) = -\frac{1}{2}\frac{\Gamma}{V}\, e^2 \frac{4R^2}{x\sqrt{4R^2-x^2}}\,\delta(y)\delta(z) \int \mathcal{D}\xi_3\, \xi_3(\bar{s})\xi_3(-\bar{s}) e^{-S_3} / \int \mathcal{D}\xi_3\, e^{-S_3}, \tag{B.7}$$

where $\bar{s} = R\, \arctan(x/\sqrt{4R^2-x^2})$. An expansion of the fluctuation $\xi_3(s)$ in the Fourier series

$$\xi_3(s) = \frac{a_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left[\frac{a_n}{\sqrt{\pi}}\cos\frac{n\,s}{R} + \frac{b_n}{\sqrt{\pi}}\sin\frac{n\,s}{R}\right], \tag{B.8}$$

reduces the expression (B.7) to the following

$$\begin{aligned}
\pi_{33}(x) &= -\frac{1}{2}\frac{\Gamma}{V}\, e^2 \frac{4R^2}{\pi x\sqrt{4R^2-x^2}} \\
&\times \int [da\, db] \sum_{n=1}^{\infty} n^2 \left(b_n^2 \cos^2\frac{n\bar{s}}{R} - a_n^2 \sin^2\frac{n\bar{s}}{R}\right) e^{-\sum_l \frac{ml^2}{2R}(a_l^2+b_l^2)} \\
&\times \left(\int [da\, db] e^{-\sum_l \frac{ml^2}{2R}(a_l^2+b_l^2)}\right)^{-1} \delta(y)\delta(z) \\
&= -\frac{1}{2}\frac{\Gamma}{V}\, e^2 \frac{4R}{m\pi x\sqrt{4R^2-x^2}} \sum_{n=1}^{\infty} \cos\frac{2n\bar{s}}{R}\, \delta(y)\delta(z) \\
&= -\frac{1}{2}\frac{\Gamma}{V}\, e^2 \frac{4R}{2\pi m x\sqrt{4R^2-x^2}} \left[\frac{\pi}{2}\delta\left(\frac{\bar{s}}{R}\right) - 1\right] \delta(y)\delta(z).
\end{aligned} \tag{B.9}$$

The component π_{33} of the vacuum polarization is a scalar with respect to rotation in the (t, x) plain. Therefore it is a function of the invariant $r = \sqrt{t^2+x^2}$, so that the invariant expression for the correlator in an arbitrary coordinate system can be obtained substituting $x \to r$ in Eq.(B.9). Proceeding in this manner and performing the Fourier transform, we find

$$\pi_{33} = \pi_{\perp} = \frac{\alpha}{\pi} eE \left(1 - I_0^2(\gamma_\theta)\right) \exp\left[-\frac{\pi m^2}{eE}\right]. \tag{B.10}$$