Axisymmetric Dislocation and Dislocation Dipole (DD) Singularities in an Elastic Half-Space

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This notebook provides expressions for ring dislocation and ring dislocation dipole solutions and for the involved generalized Lipshitz-Hankel integrals. The notebook includes an example of computation of the displacement field around a cone DD element in an infinite space. To access a *Mathematica* notebook containing expressions for the ring dislocation solutions relevant to a cylindrical discontinuity surface, suitable for evaluation along a cylindrical cut, visit http://purl.umn.edu/97344.

Reference:
[1] E. Gordeliy and E. Detournay. "Displacement Discontinuity Method for Modeling Axisymmetric Cracks in an Elastic Half-Space." International Journal of Solids and Structures (2011), doi:10.1016/j.ijsolstr.2011.05.009.

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Influence Functions

- **Generalised Lipshitz-Hankel integrals (independent of cut orientation)**

$$k[\rho\_, \xi\_] := \frac{2\,\rho^{1/2}}{\left((1+\rho)^2+\xi^2\right)^{1/2}}$$

$$k1[\rho\_, \xi\_] := \left(\frac{(1-\rho)^2+\xi^2}{(1+\rho)^2+\xi^2}\right)^{1/2}$$

$$h[\rho\_] := \frac{4\,\rho}{(1+\rho)^2};$$

$$P000[\rho\_, \xi\_] := \frac{1}{\pi}\,k[\rho, \xi]\,\rho^{-1/2}\,\text{EllipticK}\left[k[\rho, \xi]^2\right]$$

$$\mathbf{P001}[\rho\_, \xi\_] := \frac{\mathbf{k}[\rho, \xi]^3\, \xi}{4\,\pi\,\rho^{3/2}\,\mathbf{k1}[\rho, \xi]^2}\ \mathbf{EllipticE}\left[\mathbf{k}[\rho, \xi]^2\right]$$

$$\mathbf{P002}[\rho\_, \xi\_] := \frac{\mathbf{k}[\rho, \xi]^5}{16\,\pi\,\rho^{5/2}\,\mathbf{k1}[\rho, \xi]^2}\left(\left(\frac{2\left(1+\mathbf{k1}[\rho, \xi]^2\right)\xi^2}{\mathbf{k1}[\rho, \xi]^2} - \frac{4\,\rho}{\mathbf{k}[\rho, \xi]^2}\right)\mathbf{EllipticE}\left[\mathbf{k}[\rho, \xi]^2\right] - \xi^2\,\mathbf{EllipticK}\left[\mathbf{k}[\rho, \xi]^2\right]\right)$$

$$\mathbf{P003}[\rho\_, \xi\_] := \frac{\mathbf{k}[\rho, \xi]^5\, \xi}{64\,\pi\,\rho^{5/2}\,\mathbf{k1}[\rho, \xi]^4}\left(\left(12\,\mathbf{k1}[\rho, \xi]^2 - \frac{4\,\mathbf{k}[\rho, \xi]^2\left(1+\mathbf{k1}[\rho, \xi]^2\right)\xi^2}{\rho}\right)\mathbf{EllipticK}\left[\mathbf{k}[\rho, \xi]^2\right] - \left(24\left(1+\mathbf{k1}[\rho, \xi]^2\right) - \frac{\mathbf{k}[\rho, \xi]^2\,\xi^2\left(8+7\,\mathbf{k1}[\rho, \xi]^2+8\,\mathbf{k1}[\rho, \xi]^4\right)}{\rho\,\mathbf{k1}[\rho, \xi]^2}\right)\mathbf{EllipticE}\left[\mathbf{k}[\rho, \xi]^2\right]\right)$$

$$\mathbf{P101}[\rho\_, \xi\_] := \frac{\mathbf{k}[\rho, \xi]}{2\,\pi\,\rho^{1/2}}\left(\frac{\mathbf{k}[\rho, \xi]^2\left(1-\rho^2-\xi^2\right)}{4\,\rho\,\mathbf{k1}[\rho, \xi]^2}\mathbf{EllipticE}\left[\mathbf{k}[\rho, \xi]^2\right] + \mathbf{EllipticK}\left[\mathbf{k}[\rho, \xi]^2\right]\right)$$

$$\mathbf{P102}[\rho\_, \xi\_] := \frac{\mathbf{k}[\rho, \xi]^3\, \xi}{8\,\pi\,\rho^{3/2}\,\mathbf{k1}[\rho, \xi]^2}\left(\left(\frac{\mathbf{k}[\rho, \xi]^4\left(1-\left(\rho^2+\xi^2\right)^2\right)}{4\,\rho^2\,\mathbf{k1}[\rho, \xi]^2}+3\right)\mathbf{EllipticE}\left[\mathbf{k}[\rho, \xi]^2\right] - \frac{\mathbf{k}[\rho, \xi]^2\left(1-\rho^2-\xi^2\right)}{4\,\rho}\mathbf{EllipticK}\left[\mathbf{k}[\rho, \xi]^2\right]\right)$$

$$\mathbf{P103}[\rho\_,\ \xi\_] := \frac{\mathbf{k}[\rho,\ \xi]^3}{8\,\pi\,\rho^{3/2}\,\mathbf{k1}[\rho,\ \xi]^2}$$
$$\left(\left(\left(\frac{\mathbf{k}[\rho,\ \xi]^4\,\left(1-\left(\rho^2+\xi^2\right)^2\right)}{4\,\rho^2\,\mathbf{k1}[\rho,\ \xi]^2}+3\right)\left(\frac{\mathbf{k}[\rho,\ \xi]^2\,\left(2-\mathbf{k}[\rho,\ \xi]^2\right)\,\xi^2}{2\,\rho\,\mathbf{k1}[\rho,\ \xi]^2}-1\right)+\right.\right.$$
$$\frac{\mathbf{k}[\rho,\ \xi]^4\,\xi^2}{16\,\rho^2\,\mathbf{k1}[\rho,\ \xi]^2}\left(\frac{2\,\mathbf{k}[\rho,\ \xi]^2\,\left(2-\mathbf{k}[\rho,\ \xi]^2\right)\,\left(1-\left(\rho^2+\xi^2\right)^2\right)}{\rho\,\mathbf{k1}[\rho,\ \xi]^2}+\right.$$
$$\left.\left.\left.17\,\left(\rho^2+\xi^2\right)-1\right)\right)\mathbf{EllipticE}\left[\mathbf{k}[\rho,\ \xi]^2\right]+$$
$$\left.\frac{\mathbf{k}[\rho,\ \xi]^2}{4\,\rho}\left(1-\rho^2-2\,\xi^2\left(\frac{\mathbf{k}[\rho,\ \xi]^4\,\left(1-\left(\rho^2+\xi^2\right)^2\right)}{4\,\rho^2\,\mathbf{k1}[\rho,\ \xi]^2}+3\right)\right)\mathbf{EllipticK}\left[\mathbf{k}[\rho,\ \xi]^2\right]\right)$$

$$\mathbf{P011}[\rho\_,\ \xi\_] :=$$
$$\frac{\mathbf{k}[\rho,\ \xi]^3\,\left(\rho^2-1-\xi^2\right)}{8\,\pi\,\mathbf{k1}[\rho,\ \xi]^2\,\rho^{5/2}}\mathbf{EllipticE}\left[\mathbf{k}[\rho,\ \xi]^2\right]+\frac{1}{2\,\pi}\,\frac{\mathbf{k}[\rho,\ \xi]}{\rho^{3/2}}\mathbf{EllipticK}\left[\mathbf{k}[\rho,\ \xi]^2\right]$$

$$\mathbf{P012}[\rho\_,\ \xi\_] :=$$
$$\frac{\mathbf{k}[\rho,\ \xi]^3\,\xi}{8\,\pi\,\mathbf{k1}[\rho,\ \xi]^2\,\rho^{5/2}}\left(\left(\frac{\mathbf{k}[\rho,\ \xi]^4\,\left(\rho^4-\left(1+\xi^2\right)^2\right)}{4\,\rho^2\,\mathbf{k1}[\rho,\ \xi]^2}+3\right)\mathbf{EllipticE}\left[\mathbf{k}[\rho,\ \xi]^2\right]+\right.$$
$$\left.\frac{\mathbf{k}[\rho,\ \xi]^2\,\left(1-\rho^2+\xi^2\right)}{4\,\rho}\mathbf{EllipticK}\left[\mathbf{k}[\rho,\ \xi]^2\right]\right)$$

$$\mathbf{P013}[\rho\_,\ \xi\_] := \frac{\mathbf{k}[\rho,\ \xi]^3}{8\,\pi\,\mathbf{k1}[\rho,\ \xi]^2\,\rho^{5/2}}$$
$$\left(\left(\left(\frac{\mathbf{k}[\rho,\ \xi]^4\,\left(\rho^4-\left(1+\xi^2\right)^2\right)}{4\,\mathbf{k1}[\rho,\ \xi]^2\,\rho^2}+3\right)\left(\frac{\mathbf{k}[\rho,\ \xi]^2\,\left(2-\mathbf{k}[\rho,\ \xi]^2\right)\,\xi^2}{2\,\mathbf{k1}[\rho,\ \xi]^2\,\rho}-1\right)+\right.\right.$$

$$\frac{k[\rho,\xi]^4\,\xi^2}{16\,k1[\rho,\xi]^2\,\rho^2}\left(\frac{2\,k[\rho,\xi]^2\left(2-k[\rho,\xi]^2\right)\left(\rho^4-\left(1+\xi^2\right)^2\right)}{k1[\rho,\xi]^2\,\rho}+17\left(1+\xi^2\right)-\rho^2\right)\text{EllipticE}\left[k[\rho,\xi]^2\right]+\frac{k[\rho,\xi]^2}{4\rho}\left(\rho^2-1-2\,\xi^2\left(\frac{k[\rho,\xi]^4\left(\rho^4-\left(1+\xi^2\right)^2\right)}{4\,k1[\rho,\xi]^2\,\rho^2}+3\right)\right)\text{EllipticK}\left[k[\rho,\xi]^2\right]$$

```
P110[ρ_, ξ_] :=
  2/(π k[ρ, ξ] ρ^(1/2)) ((1 - 1/2 k[ρ, ξ]^2) EllipticK[k[ρ, ξ]^2] - EllipticE[k[ρ, ξ]^2])

P111[ρ_, ξ_] :=
  k[ρ, ξ] ξ/(2 π ρ^(3/2)) ((2 - k[ρ, ξ]^2)/(2 k1[ρ, ξ]^2) EllipticE[k[ρ, ξ]^2] - EllipticK[k[ρ, ξ]^2])

P112[ρ_, ξ_] :=
  k[ρ, ξ]/(2 π ρ^(3/2)) (k[ρ, ξ]^2/(4 ρ k1[ρ, ξ]^2) (k[ρ, ξ]^4 ξ^2/k1[ρ, ξ]^2 - 1 - ρ^2) * EllipticE[k[ρ, ξ]^2] +
    (1 - k[ρ, ξ]^2 ξ^2 (2 - k[ρ, ξ]^2)/(8 ρ k1[ρ, ξ]^2)) EllipticK[k[ρ, ξ]^2])

P113[ρ_, ξ_] := k[ρ, ξ]^3 ξ/(16 π k1[ρ, ξ]^4 ρ^(5/2))
  ((k[ρ, ξ]^2 (2 - k[ρ, ξ]^2) (8 k[ρ, ξ]^4 + 3 k1[ρ, ξ]^2) ξ^2)/(4 k1[ρ, ξ]^2 ρ) -
      6 (1 - k[ρ, ξ]^2 k1[ρ, ξ]^2)) EllipticE[k[ρ, ξ]^2] +
    k[ρ, ξ]^2/(2 ρ) (3 k1[ρ, ξ]^2 (1 + ρ^2) - 2 k[ρ, ξ]^4 ξ^2) EllipticK[k[ρ, ξ]^2])
P202[ρ_, ξ_] := 2 P101[ρ, ξ] - P002[ρ, ξ]
P203[ρ_, ξ_] := 2 P102[ρ, ξ] - P003[ρ, ξ]
P211[ρ_, ξ_] := 2 P110[ρ, ξ] - P011[ρ, ξ]
P212[ρ_, ξ_] := 2 P111[ρ, ξ] - P012[ρ, ξ]
P213[ρ_, ξ_] := 2 P112[ρ, ξ] - P013[ρ, ξ]
```

```
P004[ρ_, ξ_] = -∂ξ P003[ρ, ξ];
P014[ρ_, ξ_] = -∂ξ P013[ρ, ξ];
P104[ρ_, ξ_] = -∂ξ P103[ρ, ξ];
P114[ρ_, ξ_] = -∂ξ P113[ρ, ξ];
```

■ Generalised Lipshitz-Hankel integrals (dependent on cut orientation)

```
P11m1[ρ_, ξ_, α_] :=
  1/(2 π) ξ ρ^(-1/2) (2 k[ρ, ξ]^-1 EllipticE[k[ρ, ξ]^2] -
      (k[ρ, ξ] ρ/2 + k[ρ, ξ] ρ^-1/2 + (2 - k[ρ, ξ]^2)/k[ρ, ξ]) EllipticK[
        k[ρ, ξ]^2] + k[ρ, ξ]/2 (1 - ρ)^2 ρ^-1 EllipticPi[h[ρ], k[ρ, ξ]^2]) +
  1/2 If[α == 0, Sign[ξ], If[α == π/2, -1, Sign[ArcTan[ξ/Abs[ρ - 1]] - α]]]
   If[ρ > 1, 1/ρ, ρ];
P10m1[ρ_, ξ_, α_] := 1/(2 π) ρ^(-1/2) (4 ρ k[ρ, ξ]^-1 EllipticE[k[ρ, ξ]^2] +
      (1 - ρ^2) k[ρ, ξ] EllipticK[k[ρ, ξ]^2] +
      k[ρ, ξ] ξ^2 (1 - ρ)/(1 + ρ) EllipticPi[h[ρ], k[ρ, ξ]^2]) -
  ξ If[α == 0, Sign[ξ], If[α == π/2, -1, Sign[ArcTan[ξ/Abs[ρ - 1]] - α]]]
   If[ρ > 1, 0, If[ρ < 1, 1, 1/2]];
P010[ρ_, ξ_, α_] := -1/(2 π) k[ρ, ξ] ξ ρ^(-3/2)
    (EllipticK[k[ρ, ξ]^2] + (ρ - 1)/(1 + ρ) EllipticPi[h[ρ], k[ρ, ξ]^2]) +
  If[α == 0, Sign[ξ], If[α == π/2, -1, Sign[ArcTan[ξ/Abs[ρ - 1]] - α]]]
   If[ρ > 1, 1/ρ, If[ρ < 1, 0, 1/2]];
P100[ρ_, ξ_, α_] := -1/(2 π) k[ρ, ξ] ξ ρ^(-1/2)
    (EllipticK[k[ρ, ξ]^2] + (1 - ρ)/(1 + ρ) EllipticPi[h[ρ], k[ρ, ξ]^2]) +
```

```
    If[α == 0, Sign[ξ], If[α == π / 2, -1, Sign[ArcTan[ξ / Abs[ρ - 1]] - α]]]
     If[ρ > 1, 0, If[ρ < 1, 1, 1 / 2]];
P01m1[ρ_, ξ_, α_] := 1/(2 π) ρ^(-3/2) (4 ρ k[ρ, ξ]^-1 EllipticE[k[ρ, ξ]^2] -
      (1 - ρ^2) k[ρ, ξ] EllipticK[k[ρ, ξ]^2] -
      k[ρ, ξ] ξ^2 (1 - ρ)/(1 + ρ) EllipticPi[h[ρ], k[ρ, ξ]^2]) -
    ξ If[α == 0, Sign[ξ], If[α == π / 2, -1, Sign[ArcTan[ξ / Abs[ρ - 1]] - α]]]
     If[ρ > 1, ρ^-1, If[ρ < 1, 0, 1 / 2]];
P20m1[ρ_, ξ_, α_] := -ρ P11m1[ρ, ξ, α] - ξ P10m1[ρ, ξ, α]
P200[ρ_, ξ_, α_] := 2 P10m1[ρ, ξ, α] - P000[ρ, ξ]
P201[ρ_, ξ_, α_] := 2 P100[ρ, ξ, α] - P001[ρ, ξ]
P210[ρ_, ξ_, α_] := 2 P11m1[ρ, ξ, α] - P010[ρ, ξ, α]
```

- **Papkovich-Neuber displacement potentials for dislocation**

- **b_z (cylindrical cut : $\alpha = \pi/2$; interior disc cut : $\alpha = 0$)**

```
ϕz0[r_, z_, r1_, z1_, α_] := -E1/4 P100[r/r1, (z - z1)/r1, α];
Δϕz[r_, z_, r1_, z1_, α_] :=
  E1/4 (-P100[r/r1, (-z - z1)/r1, Abs[α]] + 2 z1/r1 P101[r/r1, (-z - z1)/r1]);
ψz0[r_, z_, r1_, z1_, α_] :=
 r1 E1/4 ((1 - 2 ν) P10m1[r/r1, (z - z1)/r1, α] + z1/r1 P100[r/r1, (z - z1)/r1, α]);
Δψz[r_, z_, r1_, z1_, α_] := r1 E1/4 (-(1 - 2 ν) P10m1[r/r1, (-z - z1)/r1, Abs[α]] +
     (3 - 4 ν) z1/r1 P100[r/r1, (-z - z1)/r1, Abs[α]]);
ϕz[r_, z_, r1_, z1_, α_] := ϕz0[r, z, r1, z1, α] + Δϕz[r, z, r1, z1, α];
ψz[r_, z_, r1_, z1_, α_] := ψz0[r, z, r1, z1, α] + Δψz[r, z, r1, z1, α];
```

■ b_r^r (cylindrical cut : $\alpha = \pi/2$; in halfspace the cut is from the dislocation origin to infinity)

```
ϕrr0[r_, z_, r1_, z1_] := -E1/4 (P000[r/r1, (z - z1)/r1] - P10m1[r/r1, (z - z1)/r1, π/2]);
Δϕrr[r_, z_, r1_, z1_] := -E1/4 (-P000[r/r1, (-z - z1)/r1] - 2 z1/r1 P001[r/r1, (-z - z1)/r1] +
    2 z1/r1 P100[r/r1, (-z - z1)/r1, π/2] + (1 - 4 ν) P10m1[r/r1, (-z - z1)/r1, π/2]);
ψrr0[r_, z_, r1_, z1_] := -r1 E1/4 ((1 - 2 ν) P20m1[r/r1, (z - z1)/r1, π/2] - P00m1[
    r/r1, (z - z1)/r1, π/2] - z1/r1 (P000[r/r1, (z - z1)/r1] - P10m1[r/r1, (z - z1)/r1, π/2]));
Δψrr[r_, z_, r1_, z1_] := -r1 E1/4
  ((3 - 4 ν) z1/r1 (P10m1[r/r1, (-z - z1)/r1, π/2] - P000[r/r1, (-z - z1)/r1]) -
    P00m1[r/r1, (-z - z1)/r1, π/2] +
    (1 - 2 ν) (P20m1[r/r1, (-z - z1)/r1, π/2] - 4 ν P10m2[r/r1, (-z - z1)/r1, π/2]));
ϕrr[r_, z_, r1_, z1_] := ϕrr0[r, z, r1, z1] + Δϕrr[r, z, r1, z1];
ψrr[r_, z_, r1_, z1_] := ψrr0[r, z, r1, z1] + Δψrr[r, z, r1, z1];
```

■ b_r^z (interior disc cut : $\alpha = 0$)

```
ϕrz0[r_, z_, r1_, z1_] := E1/4 P200[r/r1, (z - z1)/r1, 0];
dϕrz[r_, z_, r1_, z1_] :=
  -E1/4 (P200[r/r1, (-z - z1)/r1, 0] + 2 z1/r1 P201[r/r1, (-z - z1)/r1, 0]);
ψrz0[r_, z_, r1_, z1_] :=
  -r1 E1/4 (2 (1 - ν) P20m1[r/r1, (z - z1)/r1, 0] + z1/r1 P200[r/r1, (z - z1)/r1, 0]);
dψrz[r_, z_, r1_, z1_] := -r1 E1/4
  (2 (1 - ν) P20m1[r/r1, (-z - z1)/r1, 0] + (3 - 4 ν) z1/r1 P200[r/r1, (-z - z1)/r1, 0]);
ϕrz[r_, z_, r1_, z1_] := ϕrz0[r, z, r1, z1] + dϕrz[r, z, r1, z1];
ψrz[r_, z_, r1_, z1_] := ψrz0[r, z, r1, z1] + dψrz[r, z, r1, z1];
```

■ Dislocation influence functions

```
κ := 3 - 4 ν
```

■ b_z (cylindrical cut : $\alpha = \pi/2$; interior disc cut : $\alpha = 0$)

```
urz0[r_, z_, r1_, z1_, α_] :=
  1/(κ + 1) ((κ - 1)/2 P110[r/r1, (z - z1)/r1] - (z - z1)/r1 P111[r/r1, (z - z1)/r1]);
Δurz[r_, z_, r1_, z1_, α_] := 1/(κ + 1) (-(κ - 1)/2 P110[r/r1, (-z - z1)/r1] -
    (z - κ z1)/r1 P111[r/r1, (-z - z1)/r1] + 2 z z1/r1^2 P112[r/r1, (-z - z1)/r1]);
uzz0[r_, z_, r1_, z1_, α_] := -1/(κ + 1)
  ((κ + 1)/2 P100[r/r1, (z - z1)/r1, α] + (z - z1)/r1 P101[r/r1, (z - z1)/r1]);
Δuzz[r_, z_, r1_, z1_, α_] := -1/(κ + 1) ((κ + 1)/2 P100[r/r1, (-z - z1)/r1, Abs[α]] -
    (z + κ z1)/r1 P101[r/r1, (-z - z1)/r1] + 2 z z1/r1^2 P102[r/r1, (-z - z1)/r1]);
urz[r_, z_, r1_, z1_, α_] := urz0[r, z, r1, z1, α] + Δurz[r, z, r1, z1, α];
uzz[r_, z_, r1_, z1_, α_] := uzz0[r, z, r1, z1, α] + Δuzz[r, z, r1, z1, α];
```

```
σrrz0[r_, z_, r1_, z1_] :=
  1/r1 E1/4 (P101[r/r1, (z - z1)/r1] - (z - z1)/r1 P102[r/r1, (z - z1)/r1] +
    (z - z1)/r P111[r/r1, (z - z1)/r1] - (κ - 1)/(2 r/r1) P110[r/r1, (z - z1)/r1]);
Δσrrz[r_, z_, r1_, z1_] := 1/r1 E1/4
  (-P101[r/r1, (-z - z1)/r1] - (z - 3 z1)/r1 P102[r/r1, (-z - z1)/r1] +
    (z - κ z1)/r P111[r/r1, (-z - z1)/r1] + (κ - 1)/(2 r/r1) P110[r/r1, (-z - z1)/r1] +
    2 z z1/r1^2 (P103[r/r1, (-z - z1)/r1] - r1/r P112[r/r1, (-z - z1)/r1]));
σrrz[r_, z_, r1_, z1_] := σrrz0[r, z, r1, z1] + Δσrrz[r, z, r1, z1];

σzzz0[r_, z_, r1_, z1_] :=
  1/r1 E1/4 (P101[r/r1, (z - z1)/r1] + (z - z1)/r1 P102[r/r1, (z - z1)/r1]);
Δσzzz[r_, z_, r1_, z1_] := 1/r1 E1/4 (-P101[r/r1, (-z - z1)/r1] +
    (z + z1)/r1 P102[r/r1, (-z - z1)/r1] - 2 z z1/r1^2 P103[r/r1, (-z - z1)/r1]);
σzzz[r_, z_, r1_, z1_] := σzzz0[r, z, r1, z1] + Δσzzz[r, z, r1, z1];

σrzz0[r_, z_, r1_, z1_] := 1/r1 E1/4 (z - z1)/r1 P112[r/r1, (z - z1)/r1];
Δσrzz[r_, z_, r1_, z1_] :=
  1/r1 E1/4 (-(z - z1)/r1 P112[r/r1, (-z - z1)/r1] + 2 z z1/r1^2 P113[r/r1, (-z - z1)/r1]);
σrzz[r_, z_, r1_, z1_] := σrzz0[r, z, r1, z1] + Δσrzz[r, z, r1, z1];
```

- **b_r^r (cylindrical cut : $\alpha = \pi/2$; in halfspace the cut is from the dislocation origin to infinity)**

Composite solution

$$\text{urr0}[r\_, z\_, r1\_, z1\_] := \frac{\kappa - 1}{\kappa + 1}\,\text{urri0}\left[r, z, r1, z1, \frac{\pi}{2}\right] + \frac{2}{\kappa + 1}\,\text{urro0}\left[r, z, r1, z1, \frac{\pi}{2}\right] - \frac{3 - \kappa}{\kappa + 1}\,\text{urK0}\left[r, z, r1, z1, \frac{\pi}{2}\right];$$

$$\Delta\text{urr}[r\_, z\_, r1\_, z1\_] := \frac{\kappa - 1}{\kappa + 1}\,\Delta\text{urri}\left[r, z, r1, z1, \frac{\pi}{2}\right] + \frac{2}{\kappa + 1}\,\Delta\text{urro}\left[r, z, r1, z1, \frac{\pi}{2}\right] - \frac{3 - \kappa}{\kappa + 1}\,\Delta\text{urK}\left[r, z, r1, z1, \frac{\pi}{2}\right];$$

$$\text{urr}[r\_, z\_, r1\_, z1\_] := \text{urr0}[r, z, r1, z1] + \Delta\text{urr}[r, z, r1, z1];$$

$$\text{uzr0}[r\_, z\_, r1\_, z1\_] := \frac{\kappa - 1}{\kappa + 1}\,\text{uzri0}\left[r, z, r1, z1, \frac{\pi}{2}\right] + \frac{2}{\kappa + 1}\,\text{uzro0}\left[r, z, r1, z1, \frac{\pi}{2}\right] - \frac{3 - \kappa}{\kappa + 1}\,\text{uzK0}\left[r, z, r1, z1, \frac{\pi}{2}\right];$$

$$\Delta\text{uzr}[r\_, z\_, r1\_, z1\_] := \frac{\kappa - 1}{\kappa + 1}\,\Delta\text{uzri}\left[r, z, r1, z1, \frac{\pi}{2}\right] + \frac{2}{\kappa + 1}\,\Delta\text{uzro}\left[r, z, r1, z1, \frac{\pi}{2}\right] - \frac{3 - \kappa}{\kappa + 1}\,\Delta\text{uzK}\left[r, z, r1, z1, \frac{\pi}{2}\right];$$

$$\text{uzr}[r\_, z\_, r1\_, z1\_] := \text{uzr0}[r, z, r1, z1] + \Delta\text{uzr}[r, z, r1, z1];$$

$$\sigma\text{rrr0}[r\_, z\_, r1\_, z1\_] := \frac{\kappa - 1}{\kappa + 1}\,\sigma\text{rrri0}\left[r, z, r1, z1, \frac{\pi}{2}\right] + \frac{2}{\kappa + 1}\,\sigma\text{rrro0}\left[r, z, r1, z1, \frac{\pi}{2}\right] - \frac{3 - \kappa}{\kappa + 1}\,\sigma\text{rrK0}\left[r, z, r1, z1, \frac{\pi}{2}\right];$$

$$\Delta\sigma\text{rrr}[r\_, z\_, r1\_, z1\_] := \frac{\kappa - 1}{\kappa + 1}\,\Delta\sigma\text{rrri}\left[r, z, r1, z1, \frac{\pi}{2}\right] + \frac{2}{\kappa + 1}\,\Delta\sigma\text{rrro}\left[r, z, r1, z1, \frac{\pi}{2}\right] - \frac{3 - \kappa}{\kappa + 1}\,\Delta\sigma\text{rrK}\left[r, z, r1, z1, \frac{\pi}{2}\right];$$

$$\sigma\text{rrr}[r\_, z\_, r1\_, z1\_] := \sigma\text{rrr0}[r, z, r1, z1] + \Delta\sigma\text{rrr}[r, z, r1, z1];$$

$$\sigma zzr0[r\_, z\_, r1\_, z1\_] := \frac{\kappa - 1}{\kappa + 1}\, \sigma zzri0\left[r, z, r1, z1, \frac{\pi}{2}\right] + \frac{2}{\kappa + 1}\, \sigma zzro0\left[r, z, r1, z1, \frac{\pi}{2}\right] - \frac{3 - \kappa}{\kappa + 1}\, \sigma zzK0\left[r, z, r1, z1, \frac{\pi}{2}\right];$$

$$\Delta\sigma zzr[r\_, z\_, r1\_, z1\_] := \frac{\kappa - 1}{\kappa + 1}\, \Delta\sigma zzri\left[r, z, r1, z1, \frac{\pi}{2}\right] + \frac{2}{\kappa + 1}\, \Delta\sigma zzro\left[r, z, r1, z1, \frac{\pi}{2}\right] - \frac{3 - \kappa}{\kappa + 1}\, \Delta\sigma zzK\left[r, z, r1, z1, \frac{\pi}{2}\right];$$

$$\sigma zzr[r\_, z\_, r1\_, z1\_] := \sigma zzr0[r, z, r1, z1] + \Delta\sigma zzr[r, z, r1, z1];$$

$$\sigma rzr0[r\_, z\_, r1\_, z1\_] := \frac{\kappa - 1}{\kappa + 1}\, \sigma rzri0\left[r, z, r1, z1, \frac{\pi}{2}\right] + \frac{2}{\kappa + 1}\, \sigma rzro0\left[r, z, r1, z1, \frac{\pi}{2}\right] - \frac{3 - \kappa}{\kappa + 1}\, \sigma rzK0\left[r, z, r1, z1, \frac{\pi}{2}\right];$$

$$\Delta\sigma rzr[r\_, z\_, r1\_, z1\_] := \frac{\kappa - 1}{\kappa + 1}\, \Delta\sigma rzri\left[r, z, r1, z1, \frac{\pi}{2}\right] + \frac{2}{\kappa + 1}\, \Delta\sigma rzro\left[r, z, r1, z1, \frac{\pi}{2}\right] - \frac{3 - \kappa}{\kappa + 1}\, \Delta\sigma rzK\left[r, z, r1, z1, \frac{\pi}{2}\right];$$

$$\sigma rzr[r\_, z\_, r1\_, z1\_] := \sigma rzr0[r, z, r1, z1] + \Delta\sigma rzr[r, z, r1, z1];$$

Outer cut

$$urro0[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{\kappa + 1}\left(-\frac{\kappa + 1}{2}\, P010\left[\frac{r}{r1}, \frac{z - z1}{r1}, \alpha\right] + \frac{z - z1}{r1}\, P011\left[\frac{r}{r1}, \frac{z - z1}{r1}\right]\right);$$

$$\Delta urro[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{\kappa + 1}\left(-\frac{\kappa + 1}{2}\, P010\left[\frac{r}{r1}, \frac{-z - z1}{r1}, Abs[\alpha]\right] - \frac{z + \kappa\, z1}{r1}\, P011\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right] - 2\, \frac{z\, z1}{r1^2}\, P012\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right]\right);$$

$$uzro0[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{\kappa + 1}\left(\frac{\kappa - 1}{2}\, P000\left[\frac{r}{r1}, \frac{z - z1}{r1}\right] + \frac{z - z1}{r1}\, P001\left[\frac{r}{r1}, \frac{z - z1}{r1}\right]\right);$$

$$\Delta uzro[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{\kappa + 1}\left(-\frac{\kappa - 1}{2}\, P000\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right] + \frac{z - \kappa\, z1}{r1}\, P001\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right] + 2\, \frac{z\, z1}{r1^2}\, P002\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right]\right);$$

$$\sigma rrro0[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{r1}\frac{E1}{4}\left(-2\,P001\left[\frac{r}{r1}, \frac{z-z1}{r1}\right] + \frac{z-z1}{r1}\,P002\left[\frac{r}{r1}, \frac{z-z1}{r1}\right] - \frac{z-z1}{r}\,P011\left[\frac{r}{r1}, \frac{z-z1}{r1}\right] + \frac{\kappa+1}{2\,r/r1}\,P010\left[\frac{r}{r1}, \frac{z-z1}{r1}, \alpha\right]\right);$$

$$\Delta\sigma rrro[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{r1}\frac{E1}{4}\left(-2\,P001\left[\frac{r}{r1}, \frac{-z-z1}{r1}\right] - \frac{z+3\,z1}{r1}\,P002\left[\frac{r}{r1}, \frac{-z-z1}{r1}\right] + \frac{z+\kappa\,z1}{r}\,P011\left[\frac{r}{r1}, \frac{-z-z1}{r1}\right] + \frac{\kappa+1}{2\,r/r1}\,P010\left[\frac{r}{r1}, \frac{-z-z1}{r1}, Abs[\alpha]\right] - 2\,\frac{z\,z1}{r1^2}\left(P003\left[\frac{r}{r1}, \frac{-z-z1}{r1}\right] - \frac{1}{r/r1}\,P012\left[\frac{r}{r1}, \frac{-z-z1}{r1}\right]\right)\right);$$

$$\sigma zzro0[r\_, z\_, r1\_, z1\_, \alpha\_] := \frac{1}{r1}\frac{E1}{4}\frac{z-z1}{r1}\,P002\left[\frac{r}{r1}, \frac{z-z1}{r1}\right];$$

$$\Delta\sigma zzro[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{r1}\frac{E1}{4}\left(\frac{z-z1}{r1}\,P002\left[\frac{r}{r1}, \frac{-z-z1}{r1}\right] + 2\,\frac{z\,z1}{r1^2}\,P003\left[\frac{r}{r1}, \frac{-z-z1}{r1}\right]\right);$$

$$\sigma rzro0[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{r1}\frac{E1}{4}\left(P011\left[\frac{r}{r1}, \frac{z-z1}{r1}\right] - \frac{z-z1}{r1}\,P012\left[\frac{r}{r1}, \frac{z-z1}{r1}\right]\right);$$

$$\Delta\sigma rzro[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{r1}\frac{E1}{4}\left(-P011\left[\frac{r}{r1}, \frac{-z-z1}{r1}\right] - \frac{z+z1}{r1}\,P012\left[\frac{r}{r1}, \frac{-z-z1}{r1}\right] - 2\,\frac{z\,z1}{r1^2}\,P013\left[\frac{r}{r1}, \frac{-z-z1}{r1}\right]\right);$$

Inner cut

```
urri0[r_, z_, r1_, z1_, α_] :=
  1/(κ + 1) (-(κ + 1)/2 P210[r/r1, (z - z1)/r1, α] + (z - z1)/r1 P211[r/r1, (z - z1)/r1]);
Δurri[r_, z_, r1_, z1_, α_] := 1/(κ + 1) (-(κ + 1)/2 P210[r/r1, (-z - z1)/r1, Abs[α]] -
    (z + κ z1)/r1 P211[r/r1, (-z - z1)/r1] - 2 z z1/r1^2 P212[r/r1, (-z - z1)/r1]);
uzri0[r_, z_, r1_, z1_, α_] := 1/(κ + 1)
  ((κ - 1)/2 P200[r/r1, (z - z1)/r1, α] + (z - z1)/r1 P201[r/r1, (z - z1)/r1, α]);
Δuzri[r_, z_, r1_, z1_, α_] := 1/(κ + 1) (-(κ - 1)/2 P200[r/r1, (-z - z1)/r1, Abs[α]] +
    (z - κ z1)/r1 P201[r/r1, (-z - z1)/r1, Abs[α]] + 2 z z1/r1^2 P202[r/r1, (-z - z1)/r1]);
```

```
σrrri0[r_, z_, r1_, z1_, α_] :=
  1/r1 E1/4 (-2 P201[r/r1, (z - z1)/r1, α] + (z - z1)/r1 P202[r/r1, (z - z1)/r1] -
    (z - z1)/r P211[r/r1, (z - z1)/r1] + (κ + 1)/(2 r/r1) P210[r/r1, (z - z1)/r1, α]);
Δσrrri[r_, z_, r1_, z1_, α_] := 1/r1 E1/4
  (-2 P201[r/r1, (-z - z1)/r1, Abs[α]] - (z + 3 z1)/r1 P202[r/r1, (-z - z1)/r1] +
    (z + κ z1)/r P211[r/r1, (-z - z1)/r1] + (κ + 1)/(2 r/r1) P210[r/r1, (-z - z1)/r1, Abs[α]] -
    2 z z1/r1^2 (P203[r/r1, (-z - z1)/r1] - 1/(r/r1) P212[r/r1, (-z - z1)/r1]));

σzzri0[r_, z_, r1_, z1_, α_] := -1/r1 E1/4 (z - z1)/r1 P202[r/r1, (z - z1)/r1];
Δσzzri[r_, z_, r1_, z1_, α_] :=
  1/r1 E1/4 ((z - z1)/r1 P202[r/r1, (-z - z1)/r1] + 2 z z1/r1^2 P203[r/r1, (-z - z1)/r1]);

σrzri0[r_, z_, r1_, z1_, α_] :=
  -1/r1 E1/4 (-P211[r/r1, (z - z1)/r1] + (z - z1)/r1 P212[r/r1, (z - z1)/r1]);
Δσrzri[r_, z_, r1_, z1_, α_] := -1/r1 E1/4
  (P211[r/r1, (-z - z1)/r1] + (z + z1)/r1 P212[r/r1, (-z - z1)/r1] + 2 z z1/r1^2 P213[r/r1, (-z - z1)/r1]);
```

Axial traction

$$\text{urK0}[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{\varkappa + 1}\,\frac{z - z1}{r1}\,\text{P110}\left[\frac{r}{r1}, \frac{z - z1}{r1}\right];$$

$$\Delta\text{urK}[r\_, z\_, r1\_, z1\_, \alpha\_] :=$$
$$-\frac{1}{\varkappa + 1}\left(\varkappa\,\frac{z - z1}{r1}\,\text{P110}\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right] - 2\,\frac{z\,z1}{r1^2}\,\text{P111}\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right] +$$
$$\frac{\varkappa^2 - 1}{2}\,\text{P11m1}\left[\frac{r}{r1}, \frac{-z - z1}{r1}, \text{Abs}[\alpha]\right]\right);$$

$$\text{uzK0}[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{\varkappa + 1}$$
$$\left(\varkappa\,\text{P10m1}\left[\frac{r}{r1}, \frac{z - z1}{r1}, \alpha\right] + \frac{z - z1}{r1}\,\text{P100}\left[\frac{r}{r1}, \frac{z - z1}{r1}, \alpha\right]\right);$$

$$\Delta\text{uzK}[r\_, z\_, r1\_, z1\_, \alpha\_] := -\frac{1}{\varkappa + 1}\left(\frac{\varkappa^2 + 1}{2}\,\text{P10m1}\left[\frac{r}{r1}, \frac{-z - z1}{r1}, \text{Abs}[\alpha]\right] -\right.$$
$$\left.\varkappa\,\frac{z + z1}{r1}\,\text{P100}\left[\frac{r}{r1}, \frac{-z - z1}{r1}, \text{Abs}[\alpha]\right] + 2\,\frac{z\,z1}{r1^2}\,\text{P101}\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right]\right);$$

```
σrrK0[r_, z_, r1_, z1_, α_] := -1/r1 E1/4 (-(3 - κ)/2 P100[r/r1, (z - z1)/r1, α] +
    (z - z1)/r1 P101[r/r1, (z - z1)/r1] - (z - z1)/r P110[r/r1, (z - z1)/r1]);
ΔσrrK[r_, z_, r1_, z1_, α_] := -1/r1 E1/4
   ((3 κ - 1)/2 P100[r/r1, (-z - z1)/r1, Abs[α]] + (κ z - 3 z1)/r1 P101[r/r1, (-z - z1)/r1] -
    κ (z - z1)/r P110[r/r1, (-z - z1)/r1] - (κ^2 - 1)/(2 r/r1) P11m1[r/r1, (-z - z1)/r1, Abs[α]] +
    2 (z z1)/(r r1) P111[r/r1, (-z - z1)/r1] - 2 (z z1)/r1^2 P102[r/r1, (-z - z1)/r1]);

σzzK0[r_, z_, r1_, z1_, α_] :=
  -1/r1 E1/4 (-(κ + 1)/2 P100[r/r1, (z - z1)/r1, α] - (z - z1)/r1 P101[r/r1, (z - z1)/r1]);
ΔσzzK[r_, z_, r1_, z1_, α_] := -1/r1 E1/4 ((κ + 1)/2 P100[r/r1, (-z - z1)/r1, Abs[α]] -
    (κ z + z1)/r1 P101[r/r1, (-z - z1)/r1] + 2 (z z1)/r1^2 P102[r/r1, (-z - z1)/r1]);

σrzK0[r_, z_, r1_, z1_, α_] :=
  -1/r1 E1/4 (-(κ - 1)/2 P110[r/r1, (z - z1)/r1] - (z - z1)/r1 P111[r/r1, (z - z1)/r1]);
ΔσrzK[r_, z_, r1_, z1_, α_] := -1/r1 E1/4 ((κ - 1)/2 P110[r/r1, (-z - z1)/r1] +
    (κ z - z1)/r1 P111[r/r1, (-z - z1)/r1] - 2 (z z1)/r1^2 P112[r/r1, (-z - z1)/r1]);
```

- **b_r^z (interior disc cut : $\alpha = 0$)**

```
urrz0[r_, z_, r1_, z1_] := urri0[r, z, r1, z1, 0];
uzrz0[r_, z_, r1_, z1_] := uzri0[r, z, r1, z1, 0];
urrz[r_, z_, r1_, z1_] := urrz0[r, z, r1, z1] + Δurri[r, z, r1, z1, 0];
uzrz[r_, z_, r1_, z1_] := uzrz0[r, z, r1, z1] + Δuzri[r, z, r1, z1, 0];
```

```
σrrrz0[r_, z_, r1_, z1_] := σrrri0[r, z, r1, z1, 0];
σrrrz[r_, z_, r1_, z1_] :=
  σrrrz0[r, z, r1, z1] + Δσrrri[r, z, r1, z1, 0];

σzzrz0[r_, z_, r1_, z1_] := σzzri0[r, z, r1, z1, 0];
σzzrz[r_, z_, r1_, z1_] :=
  σzzrz0[r, z, r1, z1] + Δσzzri[r, z, r1, z1, 0];

σrzrz0[r_, z_, r1_, z1_] := σrzri0[r, z, r1, z1, 0];
σrzrz[r_, z_, r1_, z1_] :=
  σrzrz0[r, z, r1, z1] + Δσrzri[r, z, r1, z1, 0];
```

■ Papkovich-Neuber displacement potentials for DD

■ D_{rr}

```
Φrr0[r_, z_, r1_, z1_] := E1/(4 r1) (P001[r/r1, (z - z1)/r1] - P100[r/r1, (z - z1)/r1, π/2]);
ΔΦrr[r_, z_, r1_, z1_] := E1/(4 r1)
    (-3 P001[r/r1, (-z - z1)/r1] + 2 z1/r1 (P101[r/r1, (-z - z1)/r1] - P002[r/r1, (-z - z1)/r1]) +
      (3 - 4 ν) P100[r/r1, (-z - z1)/r1, π/2]);
Ψrr0[r_, z_, r1_, z1_] := E1/4 ((3 - 4 ν) P10m1[r/r1, (z - z1)/r1, π/2] - (3 - 2 ν)
      P000[r/r1, (z - z1)/r1] + z1/r1 (P100[r/r1, (z - z1)/r1, π/2] - P001[r/r1, (z - z1)/r1]));
ΔΨrr[r_, z_, r1_, z1_] := E1/4 (-(5 - 6 ν) P000[r/r1, (-z - z1)/r1] -
      (3 - 4 ν) z1/r1 (P001[r/r1, (-z - z1)/r1] - P100[r/r1, (-z - z1)/r1, π/2]) +
      ((3 - 4 ν)^2 + 1)/2 P10m1[r/r1, (-z - z1)/r1, π/2]);
Φrr[r_, z_, r1_, z1_] := Φrr0[r, z, r1, z1] + ΔΦrr[r, z, r1, z1];
Ψrr[r_, z_, r1_, z1_] := Ψrr0[r, z, r1, z1] + ΔΨrr[r, z, r1, z1];
```

D_{zz}

```
Φzz0[r_, z_, r1_, z1_] := -E1/(4 r1) P001[r/r1, (z - z1)/r1];
ΔΦzz[r_, z_, r1_, z1_] :=
  E1/(4 r1) (-P001[r/r1, (-z - z1)/r1] + 2 z1/r1 P002[r/r1, (-z - z1)/r1]);
Ψzz0[r_, z_, r1_, z1_] :=
  E1/4 ((1 - 2 ν) P000[r/r1, (z - z1)/r1] + z1/r1 P001[r/r1, (z - z1)/r1]);
ΔΨzz[r_, z_, r1_, z1_] :=
  E1/4 (-(1 - 2 ν) P000[r/r1, (-z - z1)/r1] + (3 - 4 ν) z1/r1 P001[r/r1, (-z - z1)/r1]);
Φzz[r_, z_, r1_, z1_] := Φzz0[r, z, r1, z1] + ΔΦzz[r, z, r1, z1];
Ψzz[r_, z_, r1_, z1_] := Ψzz0[r, z, r1, z1] + ΔΨzz[r, z, r1, z1];
```

D_{rz}

```
Φrz0[r_, z_, r1_, z1_] := E1/(4 r1) P101[r/r1, (z - z1)/r1];
ΔΦrz[r_, z_, r1_, z1_] :=
  -E1/(4 r1) (P101[r/r1, (-z - z1)/r1] + 2 z1/r1 P102[r/r1, (-z - z1)/r1]);
Ψrz0[r_, z_, r1_, z1_] :=
  -E1/4 (2 (1 - ν) (P100[r/r1, (z - z1)/r1, 0] - Sign[(z - z1)/r1]) + z1/r1 P101[r/r1, (z - z1)/r1]);
ΔΨrz[r_, z_, r1_, z1_] :=
  -E1/4 (2 (1 - ν) (P100[r/r1, (-z - z1)/r1, 0] - Sign[(-z - z1)/r1]) +
     (3 - 4 ν) z1/r1 P101[r/r1, (-z - z1)/r1]);
Φrz[r_, z_, r1_, z1_] := Φrz0[r, z, r1, z1] + ΔΦrz[r, z, r1, z1];
Ψrz[r_, z_, r1_, z1_] := Ψrz0[r, z, r1, z1] + ΔΨrz[r, z, r1, z1];
```

D_{zr}

$$\Phi zr0[r\_, z\_, r1\_, z1\_] := \frac{E1}{4\,r1}\,P101\left[\frac{r}{r1}, \frac{z - z1}{r1}\right];$$

$$\Delta\Phi zr[r\_, z\_, r1\_, z1\_] := -\frac{E1}{4\,r1}\left(P101\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right] + 2\,\frac{z1}{r1}\,P102\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right]\right);$$

$$\Psi zr0[r\_, z\_, r1\_, z1\_] := -\frac{E1}{4}\left(2\,(1 - \nu)\,P100\left[\frac{r}{r1}, \frac{z - z1}{r1}, \frac{\pi}{2}\right] + \frac{z1}{r1}\,P101\left[\frac{r}{r1}, \frac{z - z1}{r1}\right]\right);$$

$$\Delta\Psi zr[r\_, z\_, r1\_, z1\_] := -\frac{E1}{4}\left(2\,(1 - \nu)\,P100\left[\frac{r}{r1}, \frac{-z - z1}{r1}, \frac{\pi}{2}\right] + (3 - 4\,\nu)\,\frac{z1}{r1}\,P101\left[\frac{r}{r1}, \frac{-z - z1}{r1}\right]\right);$$

```
Φzr[r_, z_, r1_, z1_] := Φzr0[r, z, r1, z1] + ΔΦzr[r, z, r1, z1];
Ψzr[r_, z_, r1_, z1_] := Ψzr0[r, z, r1, z1] + ΔΨzr[r, z, r1, z1];
```

DD influence functions

$$\kappa := 3 - 4\,\nu$$

D_{rr}

```
Urrr0[r_, z_, r1_, z1_] :=
  -1/(κ+1) 1/r1 (-κ P110[r/r1, (z-z1)/r1] + (κ+3)/2 P011[r/r1, (z-z1)/r1] +
    (z-z1)/r1 (P111[r/r1, (z-z1)/r1] - P012[r/r1, (z-z1)/r1]));
ΔUrrr[r_, z_, r1_, z1_] := 1/(κ+1) 1/r1 (-(3κ+1)/2 P011[r/r1, (-z-z1)/r1] +
    κ z1/r1 (P111[r/r1, (-z-z1)/r1] - P012[r/r1, (-z-z1)/r1]) +
    (κ^2+1)/2 P110[r/r1, (-z-z1)/r1] - 3 z/r1 P012[r/r1, (-z-z1)/r1] + 2 z z1/r1^2
     (P112[r/r1, (-z-z1)/r1] - P013[r/r1, (-z-z1)/r1]) + κ z/r1 P111[r/r1, (-z-z1)/r1]);

Uzrr0[r_, z_, r1_, z1_] := -1/(κ+1) 1/r1
   ((3-κ)/2 P001[r/r1, (z-z1)/r1] + (z-z1)/r1 (P101[r/r1, (z-z1)/r1] - P002[r/r1, (z-z1)/r1]));
ΔUzrr[r_, z_, r1_, z1_] := 1/(κ+1) 1/r1 (-(3κ-1)/2 P001[r/r1, (-z-z1)/r1] +
    κ z1/r1 (P101[r/r1, (-z-z1)/r1] - P002[r/r1, (-z-z1)/r1]) +
    (κ^2-1)/2 P100[r/r1, (-z-z1)/r1, π/2] + 3 z/r1 P002[r/r1, (-z-z1)/r1] - 2 z z1/r1^2
     (P102[r/r1, (-z-z1)/r1] - P003[r/r1, (-z-z1)/r1]) - κ z/r1 P101[r/r1, (-z-z1)/r1]);

Urrr[r_, z_, r1_, z1_] := Urrr0[r, z, r1, z1] + ΔUrrr[r, z, r1, z1];
Uzrr[r_, z_, r1_, z1_] := Uzrr0[r, z, r1, z1] + ΔUzrr[r, z, r1, z1];
```

```
Grrrr0[r_, z_, r1_, z1_] := -1/r1^2 E1/4
    (3 P002[r/r1, (z - z1)/r1] - (z - z1)/r1 P003[r/r1, (z - z1)/r1] - 2/(r/r1) P011[r/r1, (z - z1)/r1] +
      (z - z1)/r P012[r/r1, (z - z1)/r1] - 2 P101[r/r1, (z - z1)/r1] + (z - z1)/r1 P102[r/r1, (z - z1)/r1] +
      (κ + 1)/(2 r/r1) (P110[r/r1, (z - z1)/r1] - (z - z1)/r1 P111[r/r1, (z - z1)/r1]));
ΔGrrrr[r_, z_, r1_, z1_] := -1/r1^2 E1/4
    (5 P002[r/r1, (-z - z1)/r1] - 3 (-z - z1)/r1 P003[r/r1, (-z - z1)/r1] +
      2 z z1/r1^2 P004[r/r1, (-z - z1)/r1] - (3 κ + 1)/(2 r/r1) P011[r/r1, (-z - z1)/r1] -
      (3 z + κ z1)/r P012[r/r1, (-z - z1)/r1] - 2 z z1/(r r1) P013[r/r1, (-z - z1)/r1] -
      1/2 (3 κ + 1) P101[r/r1, (-z - z1)/r1] - (κ z + 3 z1)/r1 P102[r/r1, (-z - z1)/r1] -
      2 z z1/r1^2 P103[r/r1, (-z - z1)/r1] + (κ^2 + 1)/(2 r/r1) P110[r/r1, (-z - z1)/r1] -
      (-z - z1)/r κ P111[r/r1, (-z - z1)/r1] + 2 z z1/(r r1) P112[r/r1, (-z - z1)/r1]);
Grrrr[r_, z_, r1_, z1_] := Grrrr0[r, z, r1, z1] + ΔGrrrr[r, z, r1, z1];

Gzzrr0[r_, z_, r1_, z1_] :=
  -1/r1^2 E1/4 (-P002[r/r1, (z - z1)/r1] + (z - z1)/r1 P003[r/r1, (z - z1)/r1] -
      (z - z1)/r1 P102[r/r1, (z - z1)/r1] + (κ - 1)/2 P101[r/r1, (z - z1)/r1]);
```

```
ΔGzzrr[r_, z_, r1_, z1_] := -1/r1^2 E1/4
    (P002[r/r1, (-z - z1)/r1] - (3 z - z1)/r1 P003[r/r1, (-z - z1)/r1] -
      2 z z1/r1^2 P004[r/r1, (-z - z1)/r1] - 1/2 (-1 + κ) P101[r/r1, (-z - z1)/r1] -
      (z1 - z κ)/r1 P102[r/r1, (-z - z1)/r1] + 2 z z1/r1^2 P103[r/r1, (-z - z1)/r1]);
Gzzrr[r_, z_, r1_, z1_] := Gzzrr0[r, z, r1, z1] + ΔGzzrr[r, z, r1, z1];

Grzrr0[r_, z_, r1_, z1_] :=
  -1/r1^2 E1/4 (-2 P012[r/r1, (z - z1)/r1] + (z - z1)/r1 P013[r/r1, (z - z1)/r1] +
      (κ + 1)/2 P111[r/r1, (z - z1)/r1] - (z - z1)/r1 P112[r/r1, (z - z1)/r1]);
ΔGrzrr[r_, z_, r1_, z1_] := -1/r1^2 E1/4
    (2 P012[r/r1, (-z - z1)/r1] + (3 z + z1)/r1 P013[r/r1, (-z - z1)/r1] +
      2 z z1/r1^2 P014[r/r1, (-z - z1)/r1] - 1/2 (1 + κ) P111[r/r1, (-z - z1)/r1] -
      (κ z + z1)/r1 P112[r/r1, (-z - z1)/r1] - 2 z z1/r1^2 P113[r/r1, (-z - z1)/r1]);
Grzrr[r_, z_, r1_, z1_] := Grzrr0[r, z, r1, z1] + ΔGrzrr[r, z, r1, z1];
```

- D_{zz}

```
Urzz0[r_, z_, r1_, z1_] :=
  1/(κ + 1) 1/r1 ((κ - 1)/2 P011[r/r1, (z - z1)/r1] - (z - z1)/r1 P012[r/r1, (z - z1)/r1]);
ΔUrzz[r_, z_, r1_, z1_] := 1/(κ + 1) 1/r1 (-(κ - 1)/2 P011[r/r1, (-z - z1)/r1] +
    (κ z1 - z)/r1 P012[r/r1, (-z - z1)/r1] + 2 z z1/r1^2 P013[r/r1, (-z - z1)/r1]);

Uzzz0[r_, z_, r1_, z1_] :=
  1/(κ + 1) 1/r1 (-(κ + 1)/2 P001[r/r1, (z - z1)/r1] - (z - z1)/r1 P002[r/r1, (z - z1)/r1]);
ΔUzzz[r_, z_, r1_, z1_] := 1/(κ + 1) 1/r1 (-(κ + 1)/2 P001[r/r1, (-z - z1)/r1] +
    (κ z1 + z)/r1 P002[r/r1, (-z - z1)/r1] - 2 z z1/r1^2 P003[r/r1, (-z - z1)/r1]);

Urzz[r_, z_, r1_, z1_] := Urzz0[r, z, r1, z1] + ΔUrzz[r, z, r1, z1];
Uzzz[r_, z_, r1_, z1_] := Uzzz0[r, z, r1, z1] + ΔUzzz[r, z, r1, z1];
```

```
Grrzz0[r_, z_, r1_, z1_] :=
  -1/r1^2 E1/4 (-P002[r/r1, (z - z1)/r1] + (z - z1)/r1 P003[r/r1, (z - z1)/r1] -
     (z - z1)/r P012[r/r1, (z - z1)/r1] + (κ - 1)/(2 r/r1) P011[r/r1, (z - z1)/r1]);
ΔGrrzz[r_, z_, r1_, z1_] := -1/r1^2 E1/4
   (P002[r/r1, (-z - z1)/r1] + (z - 3 z1)/r1 P003[r/r1, (-z - z1)/r1] -
     2 z z1/r1^2 P004[r/r1, (-z - z1)/r1] - (κ - 1)/(2 r/r1) P011[r/r1, (-z - z1)/r1] -
     (z - κ z1)/r P012[r/r1, (-z - z1)/r1] + 2 z z1/(r r1) P013[r/r1, (-z - z1)/r1]);
Grrzz[r_, z_, r1_, z1_] := Grrzz0[r, z, r1, z1] + ΔGrrzz[r, z, r1, z1];

Gzzzz0[r_, z_, r1_, z1_] :=
  -1/r1^2 E1/4 (-P002[r/r1, (z - z1)/r1] - (z - z1)/r1 P003[r/r1, (z - z1)/r1]);
ΔGzzzz[r_, z_, r1_, z1_] := -1/r1^2 E1/4 (P002[r/r1, (-z - z1)/r1] +
     (-z - z1)/r1 P003[r/r1, (-z - z1)/r1] + 2 z z1/r1^2 P004[r/r1, (-z - z1)/r1]);
Gzzzz[r_, z_, r1_, z1_] := Gzzzz0[r, z, r1, z1] + ΔGzzzz[r, z, r1, z1];

Grzzz0[r_, z_, r1_, z1_] := -1/r1^2 E1/4 (-(z - z1)/r1 P013[r/r1, (z - z1)/r1]);
ΔGrzzz[r_, z_, r1_, z1_] :=
  -1/r1^2 E1/4 ((z - z1)/r1 P013[r/r1, (-z - z1)/r1] - 2 z z1/r1^2 P014[r/r1, (-z - z1)/r1]);
Grzzz[r_, z_, r1_, z1_] := Grzzz0[r, z, r1, z1] + ΔGrzzz[r, z, r1, z1];
```

D_{zr}

```
Urzr0[r_, z_, r1_, z1_] :=
  -1/(κ+1) 1/r1 ((κ+1)/2 P111[r/r1, (z-z1)/r1] - (z-z1)/r1 P112[r/r1, (z-z1)/r1]);
ΔUrzr[r_, z_, r1_, z1_] := 1/(κ+1) 1/r1 (-(κ+1)/2 P111[r/r1, (-z-z1)/r1] -
    (κ z1+z)/r1 P112[r/r1, (-z-z1)/r1] - 2 z z1/r1^2 P113[r/r1, (-z-z1)/r1]);

Uzzr0[r_, z_, r1_, z1_] :=
  -1/(κ+1) 1/r1 (-(κ-1)/2 P101[r/r1, (z-z1)/r1] - (z-z1)/r1 P102[r/r1, (z-z1)/r1]);
ΔUzzr[r_, z_, r1_, z1_] := 1/(κ+1) 1/r1 (-(κ-1)/2 P101[r/r1, (-z-z1)/r1] -
    (κ z1-z)/r1 P102[r/r1, (-z-z1)/r1] + 2 z z1/r1^2 P103[r/r1, (-z-z1)/r1]);

Urzr[r_, z_, r1_, z1_] := Urzr0[r, z, r1, z1] + ΔUrzr[r, z, r1, z1];
Uzzr[r_, z_, r1_, z1_] := Uzzr0[r, z, r1, z1] + ΔUzzr[r, z, r1, z1];
```

```
Grrzr0[r_, z_, r1_, z1_] :=
  1/r1^2 E1/4 (-2 P102[r/r1, (z - z1)/r1] + (z - z1)/r1 P103[r/r1, (z - z1)/r1] -
    (z - z1)/r P112[r/r1, (z - z1)/r1] + (κ + 1)/(2 r/r1) P111[r/r1, (z - z1)/r1]);
ΔGrrzr[r_, z_, r1_, z1_] := 1/r1^2 E1/4
  (-2 P102[r/r1, (-z - z1)/r1] - (z + 3 z1)/r1 P103[r/r1, (-z - z1)/r1] -
    2 z z1/r1^2 P104[r/r1, (-z - z1)/r1] + (κ + 1)/(2 r/r1) P111[r/r1, (-z - z1)/r1] +
    (z + κ z1)/r P112[r/r1, (-z - z1)/r1] + 2 z z1/(r r1) P113[r/r1, (-z - z1)/r1]);
Grrzr[r_, z_, r1_, z1_] := Grrzr0[r, z, r1, z1] + ΔGrrzr[r, z, r1, z1];

Gzzzr0[r_, z_, r1_, z1_] := -1/r1^2 E1/4 (z - z1)/r1 P103[r/r1, (z - z1)/r1];
ΔGzzzr[r_, z_, r1_, z1_] :=
  1/r1^2 E1/4 ((z - z1)/r1 P103[r/r1, (-z - z1)/r1] + 2 z z1/r1^2 P104[r/r1, (-z - z1)/r1]);
Gzzzr[r_, z_, r1_, z1_] := Gzzzr0[r, z, r1, z1] + ΔGzzzr[r, z, r1, z1];

Grzzr0[r_, z_, r1_, z1_] :=
  1/r1^2 E1/4 (P112[r/r1, (z - z1)/r1] - (z - z1)/r1 P113[r/r1, (z - z1)/r1]);
ΔGrzzr[r_, z_, r1_, z1_] := 1/r1^2 E1/4 (-P112[r/r1, (-z - z1)/r1] +
    (-z - z1)/r1 P113[r/r1, (-z - z1)/r1] - 2 z z1/r1^2 P114[r/r1, (-z - z1)/r1]);
Grzzr[r_, z_, r1_, z1_] := Grzzr0[r, z, r1, z1] + ΔGrzzr[r, z, r1, z1];
```

■ D_{rz}

```
Urrz0[r_, z_, r1_, z1_] := Urzr0[r, z, r1, z1];
Uzrz0[r_, z_, r1_, z1_] := Uzzr0[r, z, r1, z1];
Urrz[r_, z_, r1_, z1_] := Urzr[r, z, r1, z1];
Uzrz[r_, z_, r1_, z1_] := Uzzr[r, z, r1, z1];
```

```
Grrrz0[r_, z_, r1_, z1_] := Grrzr0[r, z, r1, z1];
Grrrz[r_, z_, r1_, z1_] := Grrzr[r, z, r1, z1];

Gzzrz0[r_, z_, r1_, z1_] := Gzzzr0[r, z, r1, z1];
Gzzrz[r_, z_, r1_, z1_] := Gzzzr[r, z, r1, z1];

Grzrz0[r_, z_, r1_, z1_] := Grzzr0[r, z, r1, z1];
Grzrz[r_, z_, r1_, z1_] := Grzzr[r, z, r1, z1];
```

■ DD influence functions in rotated (normal and tangential) axes

Coordinate transformation from axes (e_1, e_2) to rotated axes $(\overline{e}_1, \overline{e}_2)$, where θ is inclination angle of $\overline{e}_1$ on the e_1-axis:

```
e1 = {1, 0};
e2 = {0, 1};
er1[θ_] := {Cos[θ], Sin[θ]};
er2[θ_] := {-Sin[θ], Cos[θ]};
L[θ_] := {{er1[θ].e1, er1[θ].e2}, {er2[θ].e1, er2[θ].e2}};
```

DD displacement and stress tensors in axes (e_1, e_2):

```
U0[r_, z_, r1_, z1_] := {{{Urrr0[r, z, r1, z1], Urrz0[r, z, r1, z1]},
     {Urrz0[r, z, r1, z1], Urzz0[r, z, r1, z1]}},
    {{Uzrr0[r, z, r1, z1], Uzrz0[r, z, r1, z1]},
     {Uzrz0[r, z, r1, z1], Uzzz0[r, z, r1, z1]}}};
U[r_, z_, r1_, z1_] := {{{Urrr[r, z, r1, z1], Urrz[r, z, r1, z1]},
     {Urrz[r, z, r1, z1], Urzz[r, z, r1, z1]}},
    {{Uzrr[r, z, r1, z1], Uzrz[r, z, r1, z1]},
     {Uzrz[r, z, r1, z1], Uzzz[r, z, r1, z1]}}};

G0[r_, z_, r1_, z1_] := {{{{Grrrr0[r, z, r1, z1], Grrrz0[r, z, r1, z1]},
      {Grrrz0[r, z, r1, z1], Grrzz0[r, z, r1, z1]}},
     {{Grzrr0[r, z, r1, z1], Grzrz0[r, z, r1, z1]},
      {Grzrz0[r, z, r1, z1], Grzzz0[r, z, r1, z1]}}},
    {{{Grzrr0[r, z, r1, z1], Grzrz0[r, z, r1, z1]},
      {Grzrz0[r, z, r1, z1], Grzzz0[r, z, r1, z1]}},
     {{Gzzrr0[r, z, r1, z1], Gzzrz0[r, z, r1, z1]},
      {Gzzrz0[r, z, r1, z1], Gzzzz0[r, z, r1, z1]}}}};
G[r_, z_, r1_, z1_] := {{{{Grrrr[r, z, r1, z1], Grrrz[r, z, r1, z1]},
      {Grrrz[r, z, r1, z1], Grrzz[r, z, r1, z1]}},
     {{Grzrr[r, z, r1, z1], Grzrz[r, z, r1, z1]},
      {Grzrz[r, z, r1, z1], Grzzz[r, z, r1, z1]}}},
    {{{Grzrr[r, z, r1, z1], Grzrz[r, z, r1, z1]},
      {Grzrz[r, z, r1, z1], Grzzz[r, z, r1, z1]}},
     {{Gzzrr[r, z, r1, z1], Gzzrz[r, z, r1, z1]},
      {Gzzrz[r, z, r1, z1], Gzzzz[r, z, r1, z1]}}}};
```

DD displacement and stress tensors in rotated axes ($\overline{e}_1, \overline{e}_2$):

```
Ur0[r_, z_, θ_, r1_, z1_, θ1_] :=
 (((L[θ].U0[r, z, r1, z1]).Transpose[L[θ1]]).{0, 1}).Transpose[L[θ1]]
Ur[r_, z_, θ_, r1_, z1_, θ1_] :=
 (((L[θ].U[r, z, r1, z1]).Transpose[L[θ1]]).{0, 1}).Transpose[L[θ1]]
```

```
σr0[r_, z_, θ_, r1_, z1_, θ1_, α_, β_, i_] :=
 Sum[(L[θ][[α]][[m]]) (L[θ][[β]][[n]])
     (G0[r, z, r1, z1][[m]][[n]][[j]][[l]])
     (L[θ1][[i]][[j]]) (L[θ1][[2]][[l]]), {m, 2}, {n, 2}, {j, 2}, {l, 2}]
σr[r_, z_, θ_, r1_, z1_, θ1_, α_, β_, i_] :=
 Sum[(L[θ][[α]][[m]]) (L[θ][[β]][[n]])
     (G[r, z, r1, z1][[m]][[n]][[j]][[l]])
     (L[θ1][[i]][[j]]) (L[θ1][[2]][[l]]), {m, 2}, {n, 2}, {j, 2}, {l, 2}]
```

DD integral kernels $\overline{G}_{\alpha i}$ in normal and tangential axes ($\overline{e}_1, \overline{e}_2$):

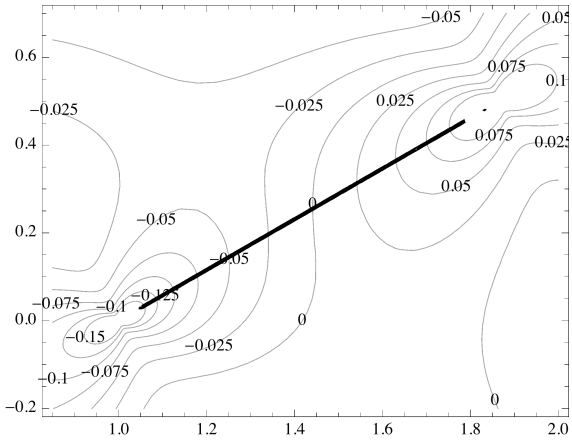
```
Gr0[r_, z_, θ_, r1_, z1_, θ1_, α_, i_] := σr0[r, z, θ, r1, z1, θ1, α, 2, i];
Gr[r_, z_, θ_, r1_, z1_, θ1_, α_, i_] := σr[r, z, θ, r1, z1, θ1, α, 2, i];
```

Example: Displacement Field Around Cone DD Element in Full Space, with $d_n = 1$

```
rp[t_] := 1 + t * Cos[θ];
zp[t_] := t * Sin[θ];
dr = {0, 1};
```

```
ur1[r_, z_] := NIntegrate[(Ur0[r, z, θ, rp[t1], zp[t1], θ].dr).{1, 0}, {t1, 0, 1}]
ur2[r_, z_] := NIntegrate[(Ur0[r, z, θ, rp[t1], zp[t1], θ].dr).{0, 1}, {t1, 0, 1}]
```

```
θ = π / 6;
ν = 0.2;
ContourPlot[ur1[r, z], {r, 0.85, 2}, {z, -0.2, 0.7},
 ContourLabels → True, Contours → Table[i, {i, -10, 10, 0.025}],
 AspectRatio → 0.9 / 1.15, ContourShading → None,
 Exclusions → {{r - 1 == z / Tan[θ], 0 < z < 0.5}}, ExclusionsStyle → {Black, Thin}]
```




```
θ = π / 6;
ν = 0.2;
ContourPlot[ur2[r, z], {r, 0.85, 2}, {z, -0.2, 0.7},
 ContourLabels → True, Contours → Table[i, {i, -10, 10, 0.05}],
 AspectRatio → 0.9 / 1.15, ContourShading → None,
 Exclusions → {{r - 1 == z / Tan[θ], 0 < z < 0.5}}, ExclusionsStyle → {Black, Thin}]
```