On the Use of Some Misspecified Models of Customer Choice in Revenue Management

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Abstract

Much of the recent revenue management literature takes customer behavior into account. A number of parametric models that incorporate customer choice have been developed. In some cases, these models closely approximate reality and provide high quality solutions. Nevertheless, most studies of revenue management models do not consider the possibility that the model used to generate decisions is different from reality. Such analyses also typically do not address effects of forecasting or how forecasts evolve when the model being used is misspecified. (A model for which there is no parameter setting that makes the model a correct description of reality is called a misspecified model). In this dissertation, we study some models of customer choice in revenue management and test their performance when implemented in settings where their assumptions are violated; i.e., when they are misspecified.

First, we study a model based on the notion of "buy-up" that considers the dependency of the customers who are willing to purchase low-fare tickets and those who prefer high-fare tickets. To implement this parametric model, a decision maker (revenue manager) needs to observe some data to estimate its parameters (buy-up rate and demand distributions), and make decisions (booking limits) using the model. Meanwhile, the choices of booking limits will affect customers' behavior and thus affect the following observed data. We study the above dynamics and show the convergence of booking limits when the buy-up model is misspecified and customer arrivals are actually deterministic. Numerical studies are also provided to show the performance of the model.

Second, we continue the study of the "buy-up" model and consider more complicated actual customers' behavior in which the numbers of different types of customers

are stochastic. We present a general necessary condition for the convergence of booking limits and buy-up rate estimates. We provide sufficient conditions for convergence using two different approaches. Comparisons among the optimal revenue, the revenue associated with convergence in the "buy-up" model, and the revenue obtained from the Littlewood rule are presented in the end.

Third, we study the performance of the Littlewood rule when it is used to manage bookings for substitutable flights. We show the convergence of booking limits under some assumptions, and make some conjectures about the limiting behavior of booking limits in other settings. Some numerical studies are performed to enlighten future work.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Background

Firms in a variety of industries use quantitative models to help manage the price and availability of their products to improve revenues. Such revenue management techniques have proved to be particularly important in the airline, hotel, and car rental industries. There is now a large and growing literature on revenue management. See, for example, the textbooks by Talluri and van Ryzin (2004b) or Phillips (2005) for an overview. Before we begin the discussion of our work, we will briefly provide some backgrounds on revenue management.

In the field of revenue management (sometimes called yield management), there are two main areas: overbooking and fare class management. In the airline industry, overbooking refers to the practice of selling more tickets than the capacity of a particular flight. This action helps to protect the airline against no-shows and cancelations. Without overbooking, many such no-shows and cancelations would lead to empty seats, which would in turn lead to a loss of revenue if there were other travelers who wanted to purchase those seats. Fare class management (sometimes called capacity allocation) is the process of controlling how many seats (or hotel rooms or rental cars) to allow low-fare customers to book while there is a chance that high-fare customers will arrive to the system later. The key tradeoff is as follows. If many seats are made available for sale at a low price, customers who are willing to purchase at a high price may instead purchase a low-fare ticket or might be rejected because all seats are filled (with low price). On the other hand, if few seats are made available at a as low price, then a low number of customers who want to purchase high fare will make the flight depart with many empty seats.

Prior to 1978 both fares and schedules for domestic and international flights in US were heavily controlled by the Civil Aeronautics Board, see Kole and Lehn (2004). Therefore, much of the early research on revenue management focused on overbooking control in airline and hotel management. See, for example, Rothstein (1971) and Rothstein (1974). The passage of the Airline Deregulation Act in 1978 opened up the potential use of fare class management techniques, see Morrison and Winston (1986), and helped spark interest in yield management within the airline industry and also academia.

Some of the earliest research work in revenue management was presented by an analyst at British Overseas Airways Company (now, British Airways) in 1972. Littlewood (1972) described a method for setting a booking limit on low-fare ticket sales on a single-leg flight. This booking control policy is now known as the Littlewood rule. The core idea is to compare the guaranteed revenue obtained from selling a seat as a discount ticket with the expected revenue obtained from rejecting a discount ticket request and reserving that seat for potential sale as a full-fare ticket. Subsequent research produced many variations and extensions related to Littlewood's original work. One important generalization of the Littlewood rule was presented by Belobaba (1987a), Belobaba (1987b), and Belobaba (1989). In his work, Belobaba presented the concepts of EMSR-a and EMSR-b, which are designed to determine booking limits in single-leg problems with an arbitrary number of fare classes. The two methods differ in how they estimate expected marginal seat revenue, but both involve solving multiple suitably defined two-class problems similar to those found in Littlewood's work and then using these solutions to obtain booking limits for the general problem. Although it has been more than 20 years since the introduction of the concept of EMSR, it is still used in practice. Often EMSR is combined with other heuristics such as virtual nesting as described in Kimes (1989) and Williamson (1992).

One important and influential success story of revenue management occurred when American Airlines announced its "Ultimate Super Saver Fares" program in January 1985. By providing the concepts of "14-day purchase in advance" and "Saturday night stay", American Airlines successfully segmented the market between leisure and business customers, and thus allowed different prices of tickets to be charged to different travelers. The team that developed the program and associated system was awarded the 1991 Edelman Prize for best application of management science. More detailed discussions of the origins of revenue management can be found in Belobaba (1987b), Smith et al. (1992), Dunleavy (1995), Vinod (1995), and Jenkins (1995).

Much early literature focused on the variations and extensions of the Littlewood rule; see Bhatia and Parekh (1973), Richter (1982), Wollmer (1992), Brumelle and McGill (1993), and Robinson (1995). Dynamic programming approaches in the single-leg booking control problem appear in Lee and Hersh (1993), Zhao (1999), and Subramanian et al. (1999). An important assumption in this literature is that demands for different fare classes are assumed to be independent of each other. That is, the availability of tickets in any fare class will not affect the sales of tickets in any other fare classes. In practice, this is typically not the case, because for example, the availability of discounted tickets will make some customers who are willing to buy a full-fare ticket instead purchase a discounted ticket. The independent demand assumption was perhaps close to reality prior to the advent of the internet and online booking. However, in today's environment, such an assumption seems to be quite inappropriate.

Researchers have developed various approaches to modeling customer choice. In Brumelle et al. (1990), the authors studied problems with dependent demand for two classes of tickets, and worked with the concept of "buy-up". That is, customers whose request for discount tickets were rejected might subsequently "buy-up" to full-fare tickets with a fixed probability. See also in Belobaba (1989). A recent paper by Talluri and van Ryzin (2004a) considered a problem in which customers' choices were made from a selected set of products, and the authors concluded the optimal policy is that the sets given to customers was nested. Several recent papers taking customer behavior into account include van Ryzin and Vulcano (2008), Liu and van Ryzin (2008), and Bront et al. (2009). Customer behavior in substitution problems of network revenue management were studied in Zhang and Cooper (2005) and Zhang and Cooper (2009). Recently there have been considerable developments in the literature on customer choice model in revenue management. However, it is unclear the extent to which such models have been adapted in practice.

1.2 Motivation and Overview

In almost any operations research application, practitioners use models to make decisions with the hope of optimizing some objective. In an airline revenue management context, it is often the case that an airline will use a model to select booking limits with the hope of maximizing the expected revenue from ticket sales. The model will typically have expected revenue (as a function of, e.g., the booking limits) as its objective function. The model will also take some parameters and demand distributions as input. However, in practice, some of these demand distributions and parameters are unknown, and need to be estimated. Revenue managers typically collect data and make parameter/distribution estimates according to these data using some statistical method. The revenue manager will use these estimated distributions and parameters in the model to select a booking limit. In practice the airline often repeatedly operates a particular flight on, e.g., a weekly basis. It is necessary for the revenue manager to make decisions for each flight. When doing so, it is typical to revise estimates of parameters over time. In such a setting, the revenue manager continues collecting new data, updating the parameter estimates, and making new decisions. In summary, the process to use a model in practice often follows the pattern *data collection - estimation - optimization - data collection - estimation - optimization*, and so on.

Although mathematical models have been used widely to improve revenue in practice, it is typically understood that these models are not exact reflections of reality. Indeed, models necessarily ignore some aspects of the real-world problems they are designed to address. In an airline context, the revenue management model's objective function will typically not accurately reflect how expected revenue *actually* depends upon the booking limits; i.e., there are no values of the model's parameters for which the model's objective function coincides with the actual expected revenue as a function of the booking limits. We say such a model is *misspecified.*

Supposing that the airline uses an estimation procedure for a misspecified model that would be reasonable if the assumptions of the model were valid, what happens? Although one might hope that expected revenues would improve as additional data are collected, the recent paper by Cooper et al. (2006) — hereafter CHK — shows that this is generally not the case. In CHK's paper, they study the so-called spiral-down effect for a simple two-class revenue management problem. If the revenue manager uses the well-known Littlewood rule to control seat allocation, their results show that the performance may become systematically worse over time. The model that yields the Littlewood rule is not accurate because it does not consider customer choice.

At this point it will be helpful to explain briefly the setting of CHK, and also the well-known Littlewood rule, which is a focus of that paper. CHK consider an airline that repeatedly operates a particular flight. On each instance of the flight, the airline sells discount tickets and full-fare tickets. The airline seeks to maximize its expected revenue by setting a booking limit on discount bookings using the Littlewood rule, which sets the low-fare booking limit η according to the formula $p_1\text{Prob}[\text{High-fare demand} > c - \eta] \approx p_2$ where p_1 (respectively, p_2) is the price of a full-fare (resp., discount) ticket and c is the number of seats on the flight. The rule is based on the assumption that this distribution of "high-fare demand" is not affected by the availability of discount tickets. The airline must estimate the distribution of high-fare demand, which is assumed to be exogenous. Notwithstanding the model's assumption of exogenous demand, the customers in CHK actually do make choices between the available ticket types. By considering a setting in which the actual behavior of customers is different from the assumptions underlying the selection of booking limits, CHK are able to study effects of modeling errors (i.e., misspecification) and the interaction of such errors with reasonable estimation methods.

Several questions naturally arise from CHK. For example, what if the revenue manager uses a more accurate model rather than that which yields the Littlewood rule? The Littlewood rule assumes high-fare demand and low-fare demand are independent (high-fare demand exogenous), but in practice, this is typically not the case. The buy-up model in Brumelle et al. (1990) considers the fact that absence of the low-fare tickets will lead some customers to buy up to high-fare tickets, and is considered as a modification of the Littlewood rule. Does use of such an apparently better model lead to improved performance?

In chapter 2, we study some effects of using "buy-up" to model customer choice in revenue management when the actual arrival process is deterministic. This can be viewed as an extension to CHK's work. We develop a mathematical framework to study an airline's use of the buy-up model for setting booking limits over a sequence of flights. We consider natural methods for estimating demand distributions and parameters from data. We study the limiting behavior of the booking limits given by the buy-up model. In some cases with deterministic arrivals, we demonstrate that the sequence of booking limits converges to a limit, which can be characterized as the solution to a particular equation. We also compare the expected revenue in the limit with that which would be obtained if the decision maker knew the true behavior of customers.

In chapter 3, we continue our study of the effect of using "buy-up" to model customer choice, and again analyze the limiting behavior of the booking limits given by the buy-up model. Here, however, we consider more complicated actual customer behavior. In particular, we focus on stochastic arrivals. We provide necessary conditions for the convergence of booking limits. We also present two methods for proving the convergence of the booking limits and thus provide sufficient conditions for convergence. Through a numerical study, we also compare the expected revenue in the limit with that which would be obtained if the decision maker knew the true behavior of customers. We also compare the expected revenue with that which would be obtained using the Littlewood rule.

In chapter 4, we study some effects of using the Littlewood rule to set booking limits for substitutable flights. Under some assumptions regarding customers' actual behavior, we show that booking limits converge. We make some conjectures about the long-run behavior of the booking limits under various other assumptions on actual customer behavior. We provide some numerical studies to enlighten future work.

Before we proceed, we shall next present a brief high-level technical description of model misspecification (modeling error). Suppose there exists a true objective function, say $f(x)$, where x is the decision variable. Suppose a decision maker's job is to maximize the objective function; i.e., to solve $\max_x f(x)$. If the decision maker selects decision $x = x_0$, the decision maker will receive a noisy observation of $f(x_0)$; say $f(x_0) + \epsilon$ where ϵ is random. This may happen, for example, if $f(x) = E[g(x, Y)]$ for some random variable Y in which case the noisy observation of $f(x_0)$ is a realization of $g(x_0, Y)$. In practice, it is virtually always the case that the decision maker does not know exactly the form of this true objective function, $f(x)$. Therefore, it is not possible for the decision maker to simply maximize $f(x)$ over x. It is often the case that the decision maker will use a parametric family of models $\{m(x, \theta)\}$ in an attempt to optimize the unknown objective, where x is the decision variable and θ is a parameter. In a typical application setting, the decision maker makes an estimate $\hat{\theta}$ of θ and then solves $\max_x m(x, \hat{\theta})$. Modeling error or misspecification occurs if there is no θ such that $m(\cdot, \theta) = f(\cdot)$.

Consider now a multi-period setting, where the problem (i.e., maximize the objective function $f(x)$) is faced repeatedly at times $k = 1, 2 \ldots$ That is, at time k, the decision maker seeks to choose x^k to maximize $f(x)$. Again, the decision maker does not know $f(\cdot)$. In this case, data are collected each time to produce a sequence of estimates $\{\hat{\theta}^k\}$ and the k-th decision determined by $\max_x m(x, \hat{\theta}^k)$. If there is no modeling error; i.e., if $f(\cdot) = m(\cdot, \theta^*)$ for some θ^*, then the decision maker might hope that $\hat{\theta}^k \to \theta^*$, $m(x, \theta^*) \to f(x)$ and $\max_x m(x, \hat{\theta}^k) \to \max_x f(x)$ as $k \to \infty$ if a "reasonable" estimation method for generating the $\{\hat{\theta}^k\}$ is used. However, if there is modeling error, what happens to decisions based upon "apparently reasonable" estimation procedures? The remainder of this dissertation focuses on these issues for some revenue management problems.

1.3 Organization of the Dissertation

The dissertation is organized as follows. Chapter 2 studies the use of a buy-up model to control seat allocation in revenue management and focuses on the long-run dynamics of the booking limits from such a model under different deterministic actual customer behaviors. Chapter 3 discusses necessary and sufficient conditions for convergence of portions of the booking limits with stochastic arrivals. Chapter 2 and chapter 3 contain material from a working paper with my advisor, see Cooper and Li (2009). Chapter 4 discusses the performance of the Littlewood rule for setting booking limits for substitutable flights. Chapter 5 summarizes and points out several future research directions.

Chapter 2

On the Use of "Buy-Up" as a Model of Customer Choice in Revenue Management with Deterministic Arrivals

2.1 Introduction

The study and practice of revenue management can be traced back to the 1980's with the airline industry. The success of revenue management in the airline industry led to a broader study of revenue management in a wide range of industries such as car rental and hotel reservation. Both practitioners and academics have developed a large amount of literature on revenue management. Much of this work is reviewed in Talluri and van Ryzin (2004b), Bitran and Caldentey (2003), Boyd and Bilegan (2003) and van Ryzin and McGill (2000).

In much of this literature, researchers assume that there is a stochastic or deterministic model of demand and that they know the model exactly or that they can estimate its parameters correctly. Based on these assumptions, researchers analyze the formulated model and derive policies. However, the situation faced by revenue managers in practice is different. For the purpose of simplification for analytical tractability, revenue managers often use a model with incorrect assumptions. Furthermore, a revenue manager may know of a modeling error, but may not fully understand its consequences. CHK studied the so-called spiral-down effect to demonstrate some possible consequences of using a flawed model that does not account for customer choice. If a revenue manager decides how many seats to protect for sale at a high fare based on past high-fare sales, then high-fare sales will decrease, resulting in lower future estimates of high-fare demand. This subsequently results in lower protection levels for high-fare tickets, greater availability of low-fare tickets, and even lower high-fare ticket sales. Boyd et al. (2001) used simulation to show this spiral-down effect.

In revenue management practice, there is a process whereby controls are implemented, sales occur, flights depart, new data are observed, and parameter estimates are updated. The updated estimates are then used to choose new controls for the next set of flights, and so on. CHK introduced a generic framework for the study of this iterative data collection-forecasting-optimization processes. They also defined what is regarded as a good forecasting method and gave conditions under which spiral-down occurs. Even if the data are collected properly and a good forecasting method is used, the problem remains that parameters are being estimated for an inappropriate model, and in some cases there exist no parameter values that make the revenue manager's model correct.

In CHK's paper, they analyzed the dynamic behavior of the Littlewood (1972) rule, which is widely used in practice. One key assumption in Littlewood rule is that the probability distribution of high-fare demand is exogenously determined, independent of anything else. Nevertheless, the observed historical data do depend on the past values of the protection levels, and this dependence is not captured by the model that underlies the Littlewood rule. That is, the revenue manager makes a modeling error.

The Littlewood rule dates to Littlewood (1972). Richter (1982) proposed a marginal analysis to support Littlewood's result and provided an intuitive explanation of the optimal policy for the two fare class problem. Belobaba (1989) discussed the possible impact of demand dependencies and formulated a seat allocation problem in which demand dependency arises from upgrades. Pfeifer (1989) showed a similar result using different methods. Brumelle et al. (1990) studied the case that the dependency between high-fare and low-fare arises from the tendency for a portion of low-fare customers to buy up to high-fare if no more low-fare tickets are available, and showed the optimal policy for this case. Since this buy-up, or upgrade, model provides an apparently reasonable description of customer behavior which is intuitively understandable and has uncomplicated form, it is widely studied and used in both academia and industry.

Discrete choice models of customer purchase have been studied and have become appealing alternatives to the independent demand models. Talluri and van Ryzin (2004a) considered such a discrete choice model in which sales are controlled by offering subsets of fare products. They proved structural properties that simplify the computation of an optimal policy. Boyd and Kallesen (2004) showed the effect of considering demand models for price-sensitive customers in which customer behavior is not perfectly captured and thus the model may allow customers to buy a fare product less than they are willing to pay. A recent paper discussing how to estimated independent demand is written by Vulcano et al., in which they provided an approach for estimating demand for substitutable products in retail.

Our work can be considered as an extension of CHK's spiral-down paper. Without dealing with Littlewood's model, we assume that the revenue manager uses a buy-up model to manage seat availability. We want to understand revenue management performance if the revenue manager decides to use this seemingly "better" model to capture customer behavior. To test this, we analyze some forms of arrivals that are not consistent with the model's assumption to examine model's behavior. We develop a mathematical framework wherein the revenue manager uses a buy-up model to control bookings over a sequence of flights. Over the sequence of flights, the revenue manager uses data to estimate demand distributions and buy-up rates. However, there is no setting of the parameters of the model that makes it correct. We study the limiting behavior of the booking limits obtained from this process.

The remainder of the chapter is organized as follows. Section 2.2 describes our framework, including the data observed by the revenue manager, the quantities estimated by the revenue manager, and the method used by the revenue manager to make decisions. Section 2.3 contains convergence results for two particular patterns of deterministic customer arrivals. Section 2.4 contains results from a simulation study. Section 2.5 summarizes the chapter and provides future directions. Most of the proofs are shown in Section 2.6.

2.2 Framework

Consider a particular flight with $c > 0$ seats and two classes of tickets. High-fare (class-1) tickets sell for p_1 and low-fare (class-2) tickets sell for p_2, where $p_1 > p_2 > 0$. There is a revenue manager, whose job it is to control availability of the tickets to maximize the airline's expected revenue by setting a booking limit on low-fare ticket sales. The revenue manager does not know precisely how customers behave or how they will respond to his availability decisions. Nevertheless, the revenue manager uses a mathematical model to make such decisions.

Below we describe a setup in which the revenue manager faces a sequence of instances of the flight. As the sequence of flights unfolds, the revenue manager collects data that is used to estimate the parameters of his mathematical model. Our aim is to understand the effects of using an incorrectly specified model; that is, a model in which there is no setting of its parameters that correctly expresses the revenue manager's objective (expected revenue from ticket sales) as a function of the revenue manager's decision (the booking limit).

We are particularly interested in situations in which the revenue manager understands that there is customer choice behavior, and therefore uses a model that attempts to incorporate such behavior. In the next section we describe the model and its solution.

2.2.1 The Revenue Manager's Model

In this section, we describe a model from Brumelle et al. (1990) that we assume the revenue manager uses to set booking limits. The model assumes that there are independent random exogenous demands for class-1 and class-2 tickets, D_1 and D_2. The class-2 demand is assumed to arrive before the class-1 demand. [It is important to keep in mind that the entirety of this section is a description of the model used by the revenue manager and the assumptions that underly it. Recall that our objective is to study misspecified models, and hence the model used by the revenue manager will have quantities and assumptions that do not and cannot describe how customers actually behave in the framework of our study. We will later describe the actual behavior of customers.]

The revenue manager's model incorporates customer choice via the notion of "buy up". Specifically, the model assumes that each class-2 customer (i.e., each unit of class-2 demand) who arrives to find no low-fare tickets for sale, will, with probability γ buy a full-fare ticket, independent of everything else. Let η denote the booking limit, and $U(\eta)$ denote the number of such customers that want to buy up. Hence, in the revenue manager's model, the number of class-2 customers who seek to buy up to the full fare is conditionally binomially distributed with parameters n and γ, given that (say) $n = [D_2 - \eta]^+$ class-2 customers arrive after the booking limit is reached. Then, the assumption of the revenue manager's model is that total demand for full fare tickets is $D_1(\eta) = D_1 + U(\eta)$ when the booking limit is η [and D_1 is independent of $(D_2, U(\eta))$]. The objective function in the revenue manager's model is

$$r_{\mathrm{M}}(\eta) = E[p_1 \min\{D_1(\eta), c - \min(D_2, \eta)\} + p_2 \min\{D_2, \eta\}]. \tag{2.1}$$

Here we should again emphasize that while the preceding expression is the objective function in the revenue manager's (incorrectly specified) model, it is *not* the actual expected revenue from implementing a booking limit η.

Let $\alpha = p_2/p_1$. Then the booking limit specified by the model is

$$\eta_{\mathrm{M}} = \max\{\eta \in [0, c] : \ P[D_1(\eta) > c - \eta \mid D_2 \geq \eta] < \rho(\gamma)\} \tag{2.2}$$

where

$$\rho(\gamma) = \frac{\alpha - \gamma}{1 - \gamma}. \tag{2.3}$$

We use the convention $\max \varnothing = 0$, so that $\eta_{\mathrm{M}} = 0$ if $P[D_1(\eta) > c - \eta \mid D_2 \geq \eta] \geq \rho(\gamma)$ for all $\eta \in [0, c]$. See Brumelle et al. (1990) for complete details. To clarify, the booking limit in (2.2) maximizes the objective function (2.1) in the model used by the revenue manager; it need not, however, maximize the actual expected revenue. Note also that if $\gamma = 0$ (there is no buy up), then (2.2) simplifies to $\eta_{\mathrm{M}} = \max\{\eta \in [0, c] : p_1 P[D_1 > c - \eta] < p_2\}$, which is the Littlewood rule.

We close this section by noting that, upon defining $g_1(\cdot)$ and $g_2(\cdot)$ to be the mass functions of D_1 and D_2, we can express the term $P[D_1(\eta) > c - \eta \mid D_2 \geq \eta]$ in (2.2) as

$$P[D_1(\eta) > c - \eta \mid D_2 \geq \eta] = \frac{H(g_1(\cdot), g_2(\cdot), \gamma, \eta)}{\sum_{y>\eta} g_2(y)} \tag{2.4}$$

where the mapping $H(\cdot)$ is defined to be

$$H(g_1(\cdot), g_2(\cdot), \gamma, \eta) = \sum_{d_2 \geq \eta} \left[\sum_{d_1 \geq 0} g_1(d_1) \sum_{i=c-\eta-d_1+1}^{[d_2-\eta]^+} \binom{[d_2-\eta]^+}{i} \gamma^i (1-\gamma)^{[d_2-\eta]^+-i} \right] g_2(d_2) \,. \tag{2.5}$$

A derivation of the above can be found in Section 2.6. At this point, we should note that although (2.4) and (2.5) may look rather complicated, they do reveal how $P[D_1(\eta) > c - \eta \mid D_2 \geq \eta]$ — which is used in (2.2) — depends upon the basic parameters of the revenue manager's model, namely the mass functions $g_1(\cdot)$ and $g_2(\cdot)$ of demand and the buyup probability γ. The expressions (2.4) and (2.5) will prove useful below, when describing how the revenue manager translates from data to parameter estimates to decisions about booking limits.

2.2.2 The Revenue Manager's Data Collection

Once the revenue manager has decided to use the model in the previous section, the distributions of D_1 and D_2 as well as the buy-up rate γ need to be estimated based upon data. In practice, these data typically include historical values of both class-1 and class-2 tickets sales, possibly after some so-called unconstraining to remove effects caused by censoring and/or truncation. In this paper, estimates will be based upon the following quantities.

S = sales of class-2 tickets;

T = requests for class-1 tickets from customers who do not first ask for class-2 tickets;

Q = requests for class-2 tickets when class-2 tickets are not available;

R = requests for class-2 tickets that are denied, but that subsequently lead to requests for class-1 tickets;

At this point, a natural objection is that a revenue manager may not be able to learn values of T, Q, and R in practice. There is certainly merit to such an objection; however, our focus is not on data collection difficulties, but rather on the effects of model misspecification. To separate these two distinct issues, we give the revenue manager the "benefit of doubt" and allow him to obtain data he would ideally want to estimate the parameters of his model. Another interpretation is that, for example, T is obtained from data by the revenue manager using an unconstraining process. Unconstraining is common in revenue management practice; see Queenan et al. (2007). Here again, however, we are allowing the revenue manager to be an unrealistically good unconstrainer in the interest of separating the issues of model misspecification and data collection.

2.2.3 Dynamics of the Forecasting and Optimization Process

As mentioned earlier, we consider a sequence of instances of a particular flight. We index the instances $k = 1, 2, 3, \ldots$. The revenue manager selects a booking limit η^0 for the initial flight (flight instance 1). Subsequently, customers arrive and book tickets on flight instance 1. The revenue manager then records (Q^1, R^1, S^1, T^1), where the superscript 1 indicates that the data corresponds to the first flight instance. (For example, S^1 represents the number of class-1 tickets sold on instance 1 of the flight.) The distribution of the vector $X^1 = (Q^1, R^1, S^1, T^1)$ depends upon booking limit η^0. Based upon the vector, the revenue manager generates estimates of the parameters in the model in the previous section, and sets a booking limit η^1 for flight instance 2. More precisely, the revenue manager generates estimates of $P[D_1(\eta) >$

$c - \eta \mid D_2 \geq \eta]$ and γ, which we denote by $H^1(\eta)$ and γ^1, and sets the booking limit $\eta^1 = \max\{\eta \in [0, c]: \; H^1(\eta) < \rho(\gamma^1)\}$, consistent with (2.2). (We will provide details of the estimation procedure shortly.)

Next, bookings realize for flight instance 2, and the revenue manager records data $X^2 = (Q^2, R^2, S^2, T^2)$ corresponding to instance 2. The distribution of this random vector depends upon the implemented booking limit η^1. Based upon the data from the first two instances, the revenue manager produces updated estimates $H^2(\eta)$ and γ^2 of $P[D_1(\eta) > c - \eta \mid D_2 \geq \eta]$ and γ, and sets booking limit $\eta^2 = \max\{\eta \in [0, c]:$ $H^2(\eta) < \rho(\gamma^2)\}$ for flight instance 3. The process continues in this manner with

$$\eta^k = \max\{\eta \in [0, c]: \; H^k(\eta) < \rho(\gamma^k)\} \quad k = 1, 2, \ldots \tag{2.6}$$

where estimates $H^k(\cdot)$ and γ^k are functions of $(X^1, \ldots, X^k)$, and $X^k = (Q^k, R^k, S^k, T^k)$ is the data obtained from implementing booking limit η^{k-1} in flight instance k. Note that (2.6) is simply (2.2), but with the revenue manager's estimates in place of the supposedly "true" parameters. It appears to be common in revenue management practice (and, more generally, operations research practice) to employ a model by solving it as if point estimates of its parameters were exactly correct.

We next explain the procedure for generating the estimates $H^k(\cdot)$ and γ^k. After observing data from k flight instances, we assume that the revenue manager's estimate of γ is

$$\gamma^k = \frac{\sum_{i=1}^k R^i}{\sum_{i=1}^k Q^i}. \tag{2.7}$$

After k flight instances, the revenue manager's estimates of $g_1(\cdot)$ and $g_2(\cdot)$ are $g_1^k(x)$, $g_2^k(x)$ where

$$g_1^k(x) = \frac{1}{k} \sum_{i=1}^k \mathbb{I}_{\{T^i = x\}} \tag{2.8}$$

$$g_2^k(x) = \frac{1}{k} \sum_{i=1}^k \mathbb{I}_{\{S^i + Q^i = x\}} \tag{2.9}$$

The revenue manager can then use (2.4) and (2.5) to estimate the conditional probability in (2.2) by

$$H^k(\eta) = \frac{H(g_1^k(\cdot), g_2^k(\cdot), \gamma^k, \eta)}{\sum_{y>\eta} g_2^k(y)}\,. \tag{2.10}$$

At this point it is important to note that if the buy-up model of Section 2.2.1 correctly described customer behavior (i.e., if the buy-up model were correct), then the estimators above would be consistent in the usual statistical sense under mild conditions. That is, if the buy-up model were correct, then we would have $\gamma^k \to \gamma$ and $g_i^k(\cdot) \to g_i(\cdot)$ with probability one as $k \to \infty$. Moreover, the associated booking limits would converge to the optimal booking limit ($\eta^k \to \eta_{\mathrm{M}}$ with probability one) were the model correct. Hence, it is quite reasonable for a revenue manager who uses the model in Section 2.2.1 to also use these estimators.

Finally, we should note that we can also study the Littlewood rule using the above framework. In particular, if we assume that the revenue manager chooses booking limits according to (2.6), but with (2.7) replaced by $\gamma^k = 0$, then we have a revenue manager who uses the Littlewood rule and who estimates the high-fare demand distribution using the empirical distribution of $\{T^k\}$.

2.3 Deterministic Arrival Processes

In this section we provide an analysis of the sequence of booking limits $\{\eta^k\}$ as defined in (2.6). For such an analysis, we must describe the actual behavior of customers. Before proceeding, it is important to emphasize that different actual behaviors of customers will generally lead to different of the properties of sequence $\{\eta^k\}$.

In our first pass analysis, the actual behavior of the customers is deterministic. There are two types of customers: type L and type F. Type-L customers purchase only low-fare tickets, and type-F customers are willing to purchase either low- or high-fare tickets but prefer low fares. A type-F customer will buy a high-fare ticket when only high-fare tickets are available for purchase. A type-F customer will first request a low-fare ticket. If the request is denied, then the customer will request a high-fare ticket. There are $a > 0$ type-L customers and $b > 0$ type-F customers with $a + b > c$ and $a \leq c$. Before proceeding, it is important to emphasize that different actual behaviors of customers will generally lead to different properties of the sequence $\{\eta^k\}$. One might object that the actual behavior described above is itself not realistic, in which case one might undertake a study such as the one below under different assumptions about the actual behavior of customers. Complex behaviors may not lend themselves to analytical results and may require simulation studies.

2.3.1 Discount Customers Arrive First

Suppose in this sub-section that the a type-L customers arrive first, followed by the b type-F customers. This is consistent with the typical scenario in which customers who are willing to buy high-fare tickets arrive later. If the revenue manager sets booking limit $\eta \in [0, c]$ then a revenue of

$$r(\eta) = p_1 \min\{b - (\eta - a)^+, c - \eta\} + p_2 \eta \tag{2.11}$$

is earned. Note that $r(\eta)$ in (2.11) is different from the objective function $r_{\mathrm{M}}(\eta)$ in the revenue manager's decision model (2.11). This reflects the common situation whereby the revenue manager does not know exactly how his decisions will affect the actual expected revenue. It is not difficult to see that the actual optimal booking limit [the booking limit that maximizes (2.11)] is $\eta = [c - b]^+$. That is, if the revenue manager knew the actual behavior of customers, then the revenue manager would set the booking limit for each instance to $\eta^k = [c - b]^+$ rather than using (2.6).

It will be convenient to express the quantities from Section 2.2.2 as functions of the booking limit $\eta \in [0, c]$. We have

$$Q(\eta) = a + b - \eta \tag{2.12}$$

$$R(\eta) = b - [\eta - a]^+ \tag{2.13}$$

$$S(\eta) = \eta \tag{2.14}$$

$$T(\eta) = 0 \tag{2.15}$$

so that $S^k + Q^k = S(\eta^{k-1}) + Q(\eta^{k-1}) = a + b$ and $T^k = T(\eta^{k-1}) = 0$ for $k = 1, 2, \ldots$. Therefore, $g_1^k(0) = 1$ and $g_2^k(a + b) = 1$, and thus by (2.5) and (2.10) we have $H^k(\eta) = \Psi(\eta, \gamma^k)$ where

$$\Psi(\eta, \gamma) = \sum_{i=c-\eta+1}^{a+b-\eta} \binom{a+b-\eta}{i} \gamma^i (1-\gamma)^{a+b-\eta-i} \qquad \eta \in [0, c]. \tag{2.16}$$

Similarly, we have $Q^k = Q(\eta^{k-1})$ and $R^k = R(\eta^{k-1})$, and so the estimated buy-up rate can expressed, using (2.7) and (2.12)–(2.13) as

$$\gamma^k = \frac{\sum_{i=1}^k R(\eta^{k-1})}{\sum_{i=1}^k Q(\eta^{k-1})} = \frac{\sum_{i=1}^k b - [\eta^{k-1} - a]^+}{\sum_{i=1}^k a + b - \eta^{k-1}}. \tag{2.17}$$

At this stage it is important to point out why the revenue manager's model is not correct. Recall from Section 2.2.1 that the buy-up probability γ in that model does not depend upon the booking limit. In the actual behavior described above, if the revenue manager sets booking limit η, then $Q(\eta) = a + b - \eta$ customers ask for a low-fare ticket and are turned down. Among those $Q(\eta)$ customers, $R(\eta) = b - [\eta - a]^+$ subsequently request to purchase a high-fare ticket. Hence, a fraction $R(\eta)/Q(\eta) = (b - [\eta - a]^+)/(a + b - \eta)$ of rejected low-fare booking requests subsequently yields high-fare booking requests. Note that $R(\eta)/Q(\eta)$ depends upon η, whereas γ in the revenue manager's model does not. Moreover, the number of buy-up requests $U(\eta)$ in the revenue manager's model is assumed to be conditionally binomial, whereas $R(\eta) = b - [\eta - a]^+$ is deterministic.

Finally, we should mention again that the revenue manager does not know how the customers actually behave, and hence these "flaws" in the buy-up model are unknown to the revenue manager. The revenue manager might be able to identify some of the flaws. However, even if the revenue manager were to recognize the presence of misspecification, then short of the revenue manager discovering the exact behavior of the customers, it is unclear whether the misspecification would be deemed severe enough to replace the buy-up model with something else. In fact, an analysis of the sort we perform below would be helpful in making such an evaluation.

Returning to the analysis, the booking limit (2.6) chosen by the revenue manager for the flight instance k can now be expressed as

$$\eta^k = \eta(\gamma^k) \quad \text{where} \quad \eta(\gamma) = \max\{\eta \in [0, c] : \Psi(\eta, \gamma) < \rho(\gamma)\}. \tag{2.18}$$

Let

$$\gamma(\eta) = \frac{R(\eta)}{Q(\eta)} = \frac{b - [\eta - a]^+}{a + b - \eta} \tag{2.19}$$

$$\widehat{\rho}(\eta) = \rho(\gamma(\eta)) = \frac{\alpha - \gamma(\eta)}{1 - \gamma(\eta)}. \tag{2.20}$$

We will use the following lemma repeatedly.

Lemma 1 *(A) $\rho(\gamma)$ is decreasing in γ;*

(B) For fixed $\eta \in [0, c]$, $\Psi(\eta, \gamma)$ is increasing in γ;

(C) For fixed $\gamma \in (0, 1)$, $\Psi(\eta, \gamma)$ is increasing in η;

(D) $\gamma(\eta)$ in (2.19) is non-decreasing in η;

In preparation for the main result of this section. define

$$\eta^* := \min\{\eta \in [0, c] : \Psi(\eta, \gamma(\eta)) \geq \widehat{\rho}(\eta)\} \tag{2.21}$$

Note that $\widehat{\rho}(c) = -\infty$ and $\Psi(c, \gamma(c)) = 1$. Hence, $\Psi(c, \gamma(c)) \geq \widehat{\rho}(c)$. Therefore, $\eta^* \leq c-1$. Note also that if $\Psi(0, \gamma(0)) \geq \widehat{\rho}(0)$ then $\Psi(\eta, \gamma(\eta)) \geq \widehat{\rho}(\eta)$ for all $\eta \in [0, c]$ by Lemma 1; in this case $\eta^* = 0$ by our convention that $\max \varnothing = 0$.

Lemma 2 *Consider η^* defined in (2.21), and suppose $\eta^* \neq 0$.*

1. *If $\Psi(\eta^*, \gamma(\eta^* - 1)) < \widehat{\rho}(\eta^* - 1)$, then there exist unique γ^*, $\overline{\gamma}$, and $\underline{\gamma}$, such that*

$$\Psi(\eta^*, \gamma^*) = \rho(\gamma^*), \qquad \Psi(\eta^* - 1, \overline{\gamma}) = \rho(\overline{\gamma}), \qquad \Psi(\eta^* + 1, \underline{\gamma}) = \rho(\underline{\gamma})\,. \quad (2.22)$$

Moreover, γ^*, $\overline{\gamma}$, and $\underline{\gamma}$ satisfy $\gamma^* \in (\gamma(\eta^* - 1), \gamma(\eta^*)]$, $\gamma^* \in (\underline{\gamma}, \overline{\gamma})$ and

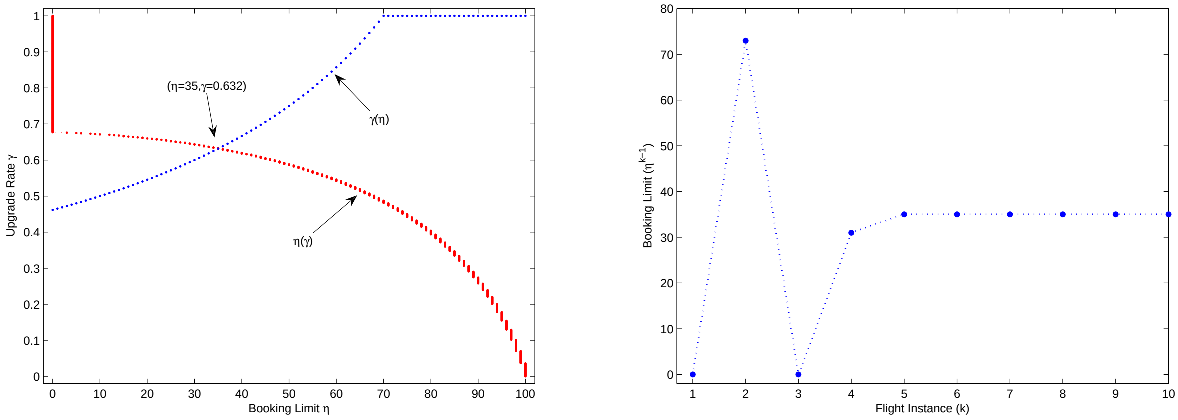
$$\eta(\gamma) = \begin{cases} \eta^* & \text{if } \underline{\gamma} < \gamma < \gamma^* \\ \eta^* - 1 & \text{if } \gamma^* \leq \gamma < \overline{\gamma}\,. \end{cases} \quad (2.23)$$

2. *If $\Psi(\eta^*, \gamma(\eta^* - 1)) \geq \widehat{\rho}(\eta^* - 1)$, then there exist unique $\overline{\gamma}$ and $\underline{\gamma}$, such that*

$$\Psi(\eta^* - 1, \overline{\gamma}) = \rho(\overline{\gamma}), \qquad \Psi(\eta^*, \underline{\gamma}) = \rho(\underline{\gamma}). \quad (2.24)$$

Moreover, $\overline{\gamma}$ and $\underline{\gamma}$ satisfy $\gamma(\eta^* - 1) \in [\underline{\gamma}, \overline{\gamma})$ and $\eta(\gamma) = \eta^* - 1$ for $\gamma \in [\underline{\gamma}, \overline{\gamma})$.

The booking limit η^* in (2.21) is a potential "equilibrium" of the optimization and estimation process. To see this, suppose for the moment that $\Psi(\eta^* + 1, \gamma(\eta^*)) \geq \widehat{\rho}(\eta^*) = \rho(\gamma(\eta^*))$ and note that if we set a booking limit of η^* then the resulting data imply an estimated buy-up probability of $\gamma(\eta^*)$ if we were to ignore previously obtained data. [Another interpretation is that if we set a booking limit of η^* many times in a row, then the estimate of the buy-up probability will be roughly $\gamma(\eta^*)$.] If we subsequently choose booking limit $\eta(\gamma(\eta^*))$, then we obtain η^* because $\eta^* = \eta(\gamma(\eta^*))$ by (2.18) and (2.21). That is, η^* is a fixed point of the function $\eta(\gamma(\cdot))$, where one may view $\eta(\cdot)$ in (2.18) as mapping from parameter estimates to decisions, and $\gamma(\cdot)$ in (2.19) as mapping from decisions to parameter estimates. When $\Psi(\eta^* +$



(a) The functions $\gamma(\eta)$ and $\eta(\gamma)$

(b) Dynamics of Booking Limits

Figure 2.1: Dynamics of Booking Limits with One Fixed Point

$1, \gamma(\eta^*)) < \widehat{\rho}(\eta^*)$, the situation is moderately more complicated and the booking limits will eventually equilibrate "around" η^* as we will see below.

The fixed point condition is depicted in Figure 2.1a. The red dotted curve shows the buy-up probability $\gamma(\eta)$ as a function of the booking limit η. The piecewise constant curve shows the booking limit $\eta(\gamma)$ as a function of the buy-up probability γ. The booking limit (35 in the example shown) where the curves coincide is a fixed point of $\eta(\gamma(\cdot))$. Note also that the associated buy-up probability $(0.632 = \gamma(35))$ is a fixed point of $\gamma(\eta(\cdot))$.

To provide some intuition into the limiting behavior of the booking limits $\{\eta^k\}$, we next discuss out how the estimates $\{\gamma^k\}$ of γ evolve. Let

$$\gamma_k \;=\; \gamma(\eta^{k-1}) \;=\; \frac{R^k}{Q^k} \;=\; \frac{R(\eta^{k-1})}{Q(\eta^{k-1})} \;=\; \frac{b - [\eta^{k-1} - a]^+}{a + b - \eta^{k-1}} \tag{2.25}$$

so that $\gamma^k = \gamma^{k-1} \oplus \gamma_k$, where $\frac{x}{y} \oplus \frac{v}{w} = \frac{x+v}{y+w}$. An important property of the operator $\oplus$ is that for $x, v \geq 0$ and $y, w > 0$ such that $\frac{x}{y} < \frac{v}{w}$, we have $\frac{x}{y} < \frac{x}{y} \oplus \frac{v}{w} < \frac{v}{w}$. Note that it is something of an abuse of notation to write $\gamma^k = \gamma^{k-1} \oplus \gamma_k$, because the quantity $\gamma^{k-1} \oplus \gamma_k$ depends upon the magnitudes of both the numerator the denominator in

(2.7) and (2.25).

Returning to the dynamics of $\{\gamma_k\}$, if $\gamma^k < \gamma(\eta^* - 1)$, then $\Psi(\eta^* - 1, \gamma^k) < \Psi(\eta^* - 1, \gamma(\eta^* - 1)) < \widehat{\rho}(\eta^* - 1)$, and thus $\eta^k \geq \eta^* - 1$ and $\gamma_{k+1} \geq \gamma(\eta^* - 1)$. Therefore, $\gamma^{k+1} = \gamma^k \oplus \gamma_{k+1} \geq \gamma^k \oplus \gamma(\eta^* - 1) > \gamma^k$. Similarly, if $\gamma^k > \gamma(\eta^*)$, then $\Psi(\eta^*, \gamma^k) > \Psi(\eta^*, \gamma(\eta^*)) \geq \widehat{\rho}(\eta^*)$, and thus $\eta^k < \eta^*$ and $\gamma_{k+1} < \gamma(\eta^*)$. Therefore, $\gamma^{k+1} = \gamma^k \oplus \gamma_{k+1} < \gamma^k \oplus \gamma(\eta^*) < \gamma^k$. To summarize, if the estimated buy-up rate γ^k is less than $\gamma(\eta^* - 1)$, then the chosen booking limit η^k will be at least $\eta^* - 1$ and the associated new buy-up data γ_{k+1} will be at least $\gamma(\eta^* - 1)$, causing the next estimate of the buy-up rate γ^{k+1} to increase. If the estimated buy-up rate γ^k is greater than $\gamma(\eta^*)$, then the chosen booking limit η^k will be smaller than η^* and the associated new buy-up data γ_{k+1} will be less than $\gamma(\eta^*)$, causing the next estimate of the buy-up rate γ^{k+1} to decrease. The following result gives the limiting behavior of the booking limits $\{\eta^k\}$ and buy-up rate estimates $\{\gamma^k\}$.

Proposition 1 *Consider η^* defined in (2.21).*

1. *If $\eta^* = 0$, then $\eta^k = 0$ for all $k \geq 1$ and $\gamma^k \to \gamma(0)$ as $k \to \infty$.*
2. *If $\eta^* \neq 0$ and $\Psi(\eta^*, \gamma(\eta^* - 1)) < \widehat{\rho}(\eta^* - 1)$, then there exists K such that $\eta^k \in \{\eta^* - 1, \eta^*\}$ for all $k > K$, moreover, $\gamma^k \to \gamma^*$ as $k \to \infty$, where γ^* is defined in (2.22).*
3. *If $\eta^* \neq 0$ and $\Psi(\eta^*, \gamma(\eta^* - 1)) \geq \widehat{\rho}(\eta^* - 1)$, then there exists K such that $\eta^k = \eta^* - 1$ for all $k > K$, moreover, $\gamma^k \to \gamma(\eta^* - 1)$ as $k \to \infty$.*

Proof. We consider parts 1, 2, and 3 separately.

Part 1. By Lemma 1(D), $\gamma(\eta)$ is non-decreasing in η, so we know that $\gamma(0) \leq \gamma(\eta)$ for all $\eta \in [0, c]$. Combining this observation with the definition of γ^k in (2.7), we have $\gamma^k \geq \gamma(0)$ and hence $\Psi(0, \gamma^k) \geq \Psi(0, \gamma(0)) \geq \rho(\gamma(0)) \geq \rho(\gamma^k)$, where the inequalities follow from Lemma 1(B), the condition that $\eta^* = 0$, and Lemma 1(A). Therefore, no η satisfies $\Psi(\eta, \gamma^k) < \rho(\gamma^k)$, and hence $\eta^k = 0$ for all $k \geq 1$. If follows easily that $\gamma^k \to \gamma(0)$.

Part 2. The main idea of the proof is to show that when k is large enough, the estimated buy-up rates $\{\gamma^k\}$ remain in the interval $(\underline{\gamma}, \overline{\gamma})$, from which it follows that the booking limits satisfy $\eta^k \in \{\eta^* - 1, \eta^*\}$.

Let γ^*, $\overline{\gamma}$, and $\underline{\gamma}$ be defined in (2.22) in Lemma 2. From (2.23) it follows that if $\gamma^k \in (\underline{\gamma}, \overline{\gamma})$, then η^k is either η^* or $\eta^* - 1$, and

$$\gamma^{k+1} = \gamma^k \oplus \gamma_{k+1} = \gamma^k \oplus \gamma(\eta^k) = \begin{cases} \gamma^k \oplus \gamma(\eta^*) & \text{if } \underline{\gamma} < \gamma^k < \gamma^* \\ \gamma^k \oplus \gamma(\eta^* - 1) & \text{if } \gamma^* \leq \gamma^k < \overline{\gamma}\,. \end{cases} \tag{2.26}$$

Applying Lemma 3 of Section 2.6, we see that there exists B_0, such that if $B > B_0$ and $\gamma^k = \frac{A}{B} \in (\underline{\gamma}, \overline{\gamma})$ then $\gamma^{k+1} \in (\underline{\gamma}, \overline{\gamma})$. Since $R(\eta) > 0$ and $Q(\eta) > 0$ are bounded away from zero uniformly in $\eta \in [0, c]$, it follows that there exists K such that $\gamma^k = \frac{R(\eta^0)+\cdots+R(\eta^{k-1})}{Q(\eta^0)+\cdots+Q(\eta^{k-1})}$ satisfies $Q(\eta^0) + \cdots + Q(\eta^{k-1}) \geq B_0$ for all $k > K$. Hence, if $\gamma^k \in (\underline{\gamma}, \overline{\gamma})$ and $k > K$, then $\gamma^\ell \in (\underline{\gamma}, \overline{\gamma})$ for all $\ell \geq k$, and moreover, $\eta^\ell \in \{\eta^*, \eta^* - 1\}$ for all $\ell \geq k$. In addition, Lemma 4 shows $\gamma^k \to \gamma^*$.

We have now established that if the estimated buy-up rate γ^k falls in the interval $(\underline{\gamma}, \overline{\gamma})$ for some $k > K$, then it will remain there, and moreover, the booking limit is thereafter either $\eta^* - 1$ or η^*. It remains to prove that γ^k does eventually fall into $(\underline{\gamma}, \overline{\gamma})$ for some $k > K$. Lemma 5 of Section 2.6 shows that there indeed exists $k > K$ such that $\gamma^k \in (\underline{\gamma}, \overline{\gamma})$ and thereby completes the proof of Part 2.

Part 3. The proof is similar to that of Part 2, so we provide only a brief sketch. Let $\overline{\gamma}$ and $\underline{\gamma}$ be defined in (2.24). From part 2 of Lemma 2 of Section 2.6, it follows that if $\gamma^k \in [\underline{\gamma}, \overline{\gamma})$, then $\eta^k = \eta^* - 1$ and $\gamma_{k+1} = \gamma(\eta^* - 1)$. Therefore, $\gamma^{k+1} = \gamma^k \oplus \gamma_{k+1} \in [\underline{\gamma}, \overline{\gamma})$. Hence, if $\gamma^k \in [\overline{\gamma}, \underline{\gamma})$ for some k, then $\gamma^\ell \in [\overline{\gamma}, \underline{\gamma})$ and $\eta^\ell = \eta^* - 1$ for all $\ell \geq k$. As in the proof of Part 2 above, it can be shown that there does indeed exist some k such that $\gamma^k \in [\overline{\gamma}, \underline{\gamma})$ for some k. An argument as in Lemma 4 shows that $\gamma^\ell \to \gamma(\eta^* - 1)$ $\square$

Figure 2.1b depicts the convergence described in part 3 of the proposition. Observe that the limit ($\eta^* = 35$) in Figure 2.1b is indeed the fixed point shown in Figure 2.1a. It can bee seen that, in this example, convergence to η^* occurs quickly; the limit of 35 is reached in just five instances.

2.3.2 Flexible Customers Arrive First

In this section, we again suppose that there are $a > 0$ type-L customers and $b > 0$ type-F customers, for a total of $a + b$ customers. We assume that $a + b > c$, and $a \leq c$. However, we now consider a setting in which the b type-F customers arrive first, followed by the a type-L customers. For any booking limit $\eta \in [0, c]$ we again have $Q(\eta) = a + b - \eta$, $S(\eta) = \eta$, and $T(\eta) = 0$ as in the previous section. Now, however, the number of requests for class-2 tickets that are denied, but that subsequently lead to requests for class-1 tickets, is given by

$$R(\eta) = [b - \eta]^+ \tag{2.27}$$

rather than by (2.13). The actual revenue as a function of booking limit $\eta \in [0, c]$ is given by $r(\eta) = p_1[\min\{b, c\} - \eta]^+ + p_2\eta$, and the optimal booking limit (i.e., the booking limit that maximizes revenue) is $\eta = 0$ if $b \geq \alpha c$ and $\eta = c$ if $b \leq \alpha c$.

As before, booking limits are chosen according to (2.6) and estimates are made according to (2.7)–(2.10). The estimate of the buy-up rate can be expressed as a function of the past booking limits according to

$$\gamma^k = \frac{\sum_{i=1}^k R(\eta^{k-1})}{\sum_{i=1}^k Q(\eta^{k-1})} = \frac{\sum_{i=1}^k [b - \eta^{k-1}]^+}{\sum_{i=1}^k a + b - \eta^{k-1}}\,. \tag{2.28}$$

We have $S^k + Q^k = S(\eta^{k-1}) + Q(\eta^{k-1}) = a + b$ and $T^k = T(\eta^{k-1}) = 0$ for all $k \geq 1$. Therefore, as before, we have $g_1^k(0) = 1$ and $g_2^k(a+b) = 1$ for all $\eta \in [0, c]$, and consequently $H^k(\eta) = \Psi(\eta, \gamma^k)$ where γ^k is given by (2.28). Likewise, the booking limit η^k is given by (2.18) with γ^k given by (2.28). Finally, in view of (2.27) we replace (2.19) in the previous section with

$$\gamma(\eta) = \frac{R(\eta)}{Q(\eta)} = \frac{[b-\eta]^+}{a+b-\eta} \tag{2.29}$$

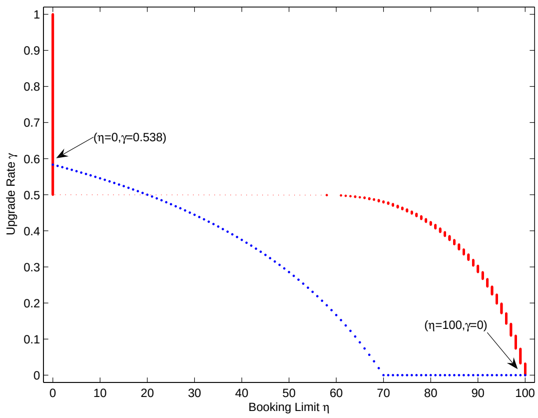
and let $\widehat{\rho}(\eta) = \rho(\gamma(\eta))$ where $\gamma(\eta)$ is defined by (2.29). With $R(\eta)$ and $\gamma(\eta)$ defined by (2.27) and (2.29), we have an analog of part (D) of Lemma 2. In particular, it is easy to see that $\gamma(\eta)$ in (2.29) is non-increasing in η; $R(\eta)$ in (2.27) is non-increasing in η, and $Q(\eta)$ is decreasing in η. Note that parts (A)–(C) of the lemma apply to our current setting without modification, because the expressions for $\rho(\gamma)$ and $\Psi(\eta, \gamma)$ in (2.3) and (2.16) do not depend upon the order of arrivals of customers.

The following is the main result of this sub-section. Interestingly, the limit of the sequence of booking limits may depend upon the initial condition.

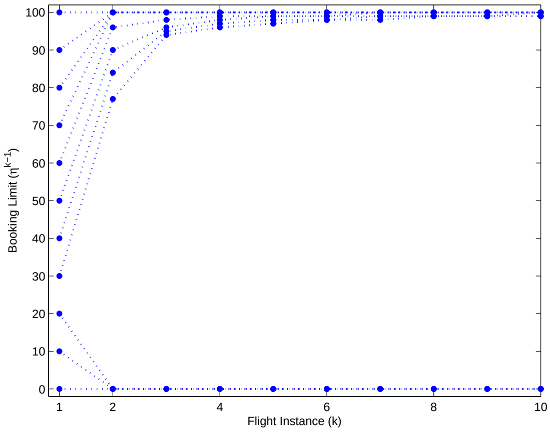
Proposition 2 *The sequence of booking limits converges; i.e., $\eta^\circ := \lim_{k\to\infty} \eta^k$ exists. The limit η° equals 0 or c, or else satisfies $\Psi(\eta^\circ, \gamma(\eta^\circ)) \leq \widehat{\rho}(\eta^\circ)$ and $\Psi(\eta^\circ + 1, \gamma(\eta^\circ)) \geq \widehat{\rho}(\eta^\circ)$. In addition, $\lim_{k\to\infty} \gamma^k = \gamma(\eta^\circ)$.*

Figure 2.2a shows the curves $\gamma(\eta)$ and $\eta(\gamma)$ in the setting of Proposition 2 when flexible customers arrive before low-fare customers. In this case, there are two fixed points at $\eta = 0$ and $\eta = 100$. In addition, the curves cross each other, but do not meet, at $\eta = 20$. More precisely, $\gamma(20) > \sup\{\gamma : \eta(\gamma) = 20\}$ and $\gamma(21) < \inf\{\gamma : \eta(\gamma) = 21\}$. In this example, the limit of the sequence of booking limits (which is guaranteed to exist by Proposition 2) is $\eta^\circ = 0$ if $\eta^0 \leq 20$, but is $\eta^\circ = 100$ if $\eta^0 \geq 21$. Figure 2.2b shows $\{\eta^k\}$ from different starting positions.

It is possible that booking limits converge to values other than 0 and c. Figure 2.3a depicts an example in which there are three fixed points of $\eta(\gamma(\cdot))$: 0, 6, and 10. Figure 2.3b shows the evolution of the booking limits from different initial values. If



(a) The functions $\gamma(\eta)$ and $\eta(\gamma)$

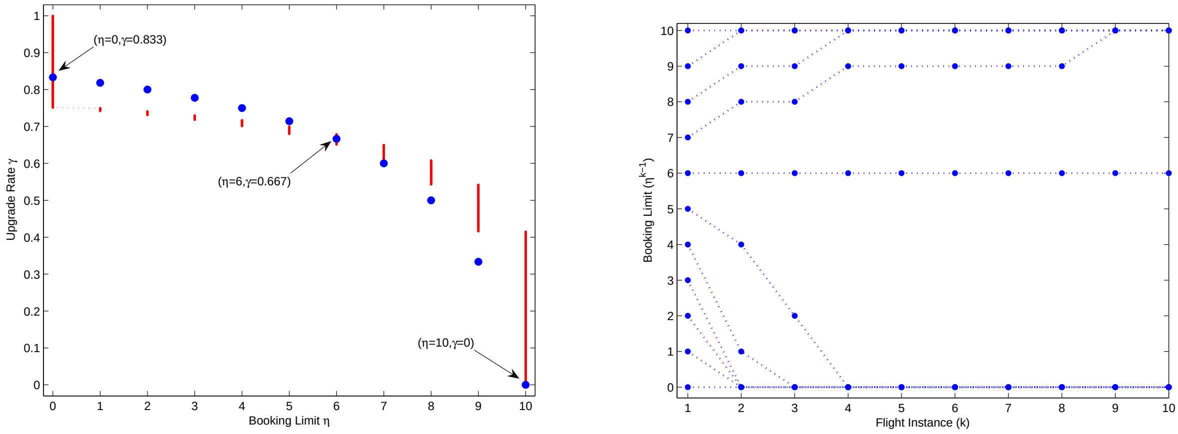


(b) Dynamics of Booking Limits

Figure 2.2: Dynamics of Booking Limits with Two Fixed Points

$\eta^0 = 6$, then $\eta^k = 6$ for all k. If $\eta^0 < 6$, then $\eta^k \to 0$. If $\eta^0 > 6$, then $\eta^k \to c = 10$. Here, $\eta = 6$ is an "unstable" fixed point in the sense that a slight perturbation away from it causes the sequence to converge to a different fixed point; 0 or 10.

An important distinction between the systems studied in this section and those studied in the previous section is that here the limiting booking limit may depend upon the choice of initial booking limit. In the previous section we saw no such behavior. The source of this difference is the shape of the function $\gamma(\eta)$, which maps from booking limits to buy-up rate estimates. In the preceding section where flexible customers arrived late $\gamma(\eta)$ was non-decreasing, whereas here (where flexible customers arrive early) it is non-increasing. When $\gamma(\eta)$ is non-decreasing it may cross the curve $\eta(\gamma)$ just once. However, when $\gamma(\eta)$ is non-increasing, it may cross the curve $\eta(\gamma)$ multiple times, allowing multiple different limits. As mention earlier, it is rather typical to study how parameter estimates affect decisions. The preceding analysis helps emphasize that in the presence of model misspecification, it is perhaps equally important to understand how decisions affect parameter estimates.



(a) The functions $\gamma(\eta)$ and $\eta(\gamma)$ (b) Dynamics of Booking Limits

Figure 2.3: Dynamics of Booking Limits with Three Fixed Points

2.3.3 Comparisons and Sensitivity Analysis

We continue to consider the case in which there are $a > 0$ type-L customers and $b > 0$ type-F customers, and thus a total of $a+b$ customers with $a+b > c$, and $a \leq c$. We use a subscript l to indicate notation for the setting in which all type-L customers arrive before type-F. We use a subscript f to indicate notion for the setting in which all type-F customers arrive before type-L. To be specific, let $\{\eta_l^0, \eta_l^1, ...\}$ and $\{\gamma_l^1, \gamma_l^2, ...\}$ be the sequences of the chosen booking limits and estimated buy-up rates for the case type-L customers arrive first; we let $\{\eta_f^0, \eta_f^1, ...\}$ and $\{\gamma_f^1, \gamma_f^2, ...\}$ be the sequences of chosen booking limits and estimated buy-up rates for the case type-F customers arrive first. The following proposition demonstrates the relationship between the booking limits at any particular time.

Proposition 3 *$\eta_l^i \leq \eta_f^i$ for all $i \in \mathbb{N}^+$.*

Next we provide some sensitivity analysis related to the case in which all type-L customers arrive first. We consider the case there are a type-L customers who come to the system first, followed by b type-F customers, and the total capacity is c.

Moreover, in the following analysis, we maintain the restriction that $a + b > c$ and $a \leq c$. Let η^* defined be in (2.21). We have following conclusions.

Proposition 4 *Consider the following three scenarios: (1) b decreases; (2) c increases; and (3) α increases. If at least one of the scenarios happens, then η^* increases or does not change.*

A corollary follows directly from above analysis.

Corollary 1 *If both a and c increase by 1, then η^* increases by 1.*

2.4 Numerical Studies

We begin our discussion with the deterministic case from Section 2.3.1 in which type-L customers arrive first. As shown in Proposition 1, the booking limits converge (part 3 of the proposition) or else eventually oscillate between two values (part 2 of the proposition). To illustrate the behavior described in the proposition, we consider a setting in which the capacity is $c = 100$, and suppose that $a = 70$ type-L (discount-fare) customers arrive first, followed by $b = 50$ type-F (flexible) customers. Suppose that p_1 is normalized to $p_1 = 1$ and that the price ratio is $\alpha = p_2/p_1 = 0.6$. This setting satisfies the condition of part 2 in Proposition 1 with $\eta^* = 37$. Figure 2.4 shows the behavior of the booking limits starting from initial booking limit $\eta^0 = 50$. As predicted by the proposition, the booking limits eventually oscillate between $\eta^* - 1 = 36$ and $\eta^* = 37$. If instead $\alpha = 0.75$ but other parameters remain the same, then the conditions of part 3 of Proposition 1 hold. Figure 2.5 shows the behavior of the booking limits starting from 0, 50, and 100. As predicted by the proposition, the booking limits can be seen to converge to $\eta^* - 1 = 47$. We can see that the convergence point does not depend on initial conditions.

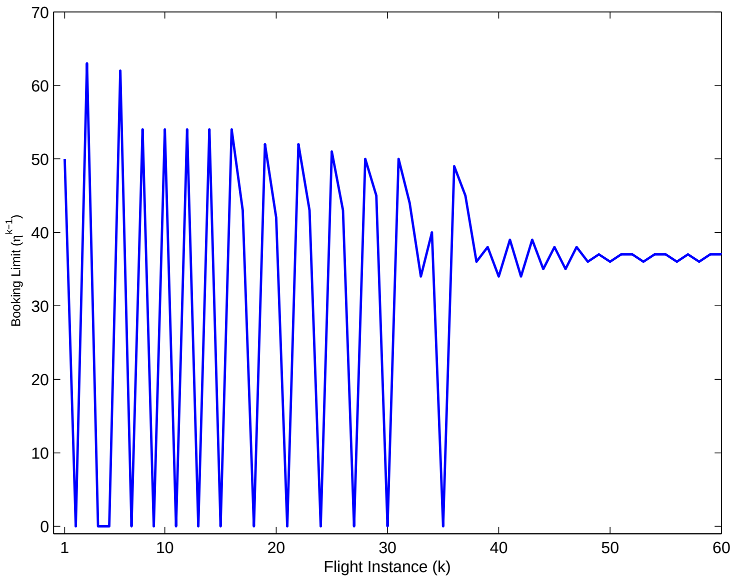


Figure 2.4: Dynamics of Booking Limits with One Fixed Point

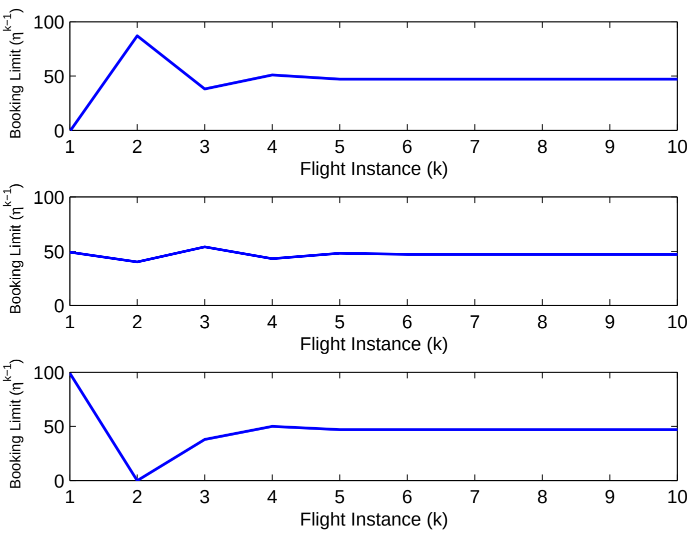


Figure 2.5: Dynamics of Booking Limits with No Fixed Point (Oscillation)

Next, we study how the price ratio $\alpha = p_1/p_2$ affects the limiting behavior of the booking limits and we also compare the actual revenue corresponding to the limiting booking limits with the actual optimal revenue. We let $\eta^\infty = \lim_{k\to\infty} \eta^k$ (when the limit exists). The actual revenue associated with the limiting booking limit is $r^\infty = r(\eta^\infty)$. Note that in settings where part 2 of the proposition applies, $\lim_{k\to\infty} \eta^k$ does not exist, so we define $\eta^\infty = \eta^* - 1$ and set $r^\infty = r(\eta^* - 1)$. As described in Section 2.3.1, the actual optimal booking limit is $\eta_{\text{OPT}} = [c-b]^+ = 50$, which does not depend on $\alpha \in (0,1)$. Setting the booking limit to be $\eta_{\text{OPT}} = 50$ allows the airline to sell 50 low-fare tickets to type-L customers and 50 high-fare tickets to type-F customers. The optimal booking limit yields an actual optimal revenue of $r_{\text{OPT}} = r(50) = 50 + 50\alpha$. Table 2.1 compares limiting and optimal booking limits for different values of the ratio α.

Table 2.1: Revenue Comparison with Type-L Customers Arrive First

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Optimal BL	50	50	50	50	50	50	50	50	50
Revenue, r_{OPT}	55	60	65	70	75	80	85	90	95
η^∞	0	0	0	0	0	36	45	48	51
r^∞	50	50	50	50	50	71.6	81.5	89.4	94.9
$100 \times \frac{(r_{\text{OPT}} - r^\infty)}{r_{\text{OPT}}}$	9.1	16.7	23.1	28.6	33.3	10.5	4.1	0.7	0.1
$\eta^\infty_{\text{L(T)}}$	100	100	100	100	100	100	100	100	100
$r^\infty_{\text{L(T)}}$	10	20	30	40	50	60	70	80	90
$100 \times \frac{(r_{\text{OPT}} - r^\infty_{\text{L(T)}})}{r_{\text{OPT}}}$	81.8	66.7	53.8	42.9	33.3	25	17.6	11.1	5.2
$\eta^\infty_{\text{L(T+R)}}$	50	50	50	50	50	50	50	50	50
$r^\infty_{\text{L(T+R)}}$	55	60	65	70	75	80	85	90	95
$100 \times \frac{(r_{\text{OPT}} - r^\infty_{\text{L(T+R)}})}{r_{\text{OPT}}}$	0	0	0	0	0	0	0	0	0

From the table, we see that if α is small enough then the limit η^∞ is zero. To understand this, note that when α is low, then it is natural to set a low booking limit on low-fare tickets, and a high booking limit η yields a high value of $\gamma(\eta)$. High buy-up estimates subsequently lead to lower booking limits. These combined effects eventually push the booking limit to zero in the limit. Although $\eta^\infty = 0$ is not close to the optimal $\eta_{\text{OPT}} = 50$, it does lead to near optimal revenue when α is very small since $r_{\text{OPT}} = r(50) = 50 + 50\alpha \approx 50 = r(0) = r^\infty$ for $\alpha \approx 0$. However, even for quite low values of α, the difference in revenues can be significant. For example, when $\alpha = 0.1$ so that a high-fare ticket costs ten times as much as a low-fare ticket, η^∞ from the buy-up model yields a revenue that is 9.1% below optimal. Somewhat higher values of α, cause the buy-up to preform quite poorly in the limit. For instance, when $\alpha = 0.5$, then the limiting booking limit for the buy-up model gives revenue that is 33.3% below optimal. The poor performance with moderate price ratio is because with an incorrect (high) estimate of the buy-up rate and low booking limit for the low-fare tickets, the buy-up model turns away many type-L customers while on the other hand there are not enough type-F customers to buy high-fare tickets. As shown in the table, when α is large enough, we get strictly positive values of η^∞. This is to be expected, because when α is high, the price of a low-fare ticket is close to the price of a high-fare ticket. As the price ratio gets larger, the buy-up model begins to protect less for high-fare tickets and the booking limit gets larger. Another issue to be pointed out is that when the price ratio is high, the difference between a high-fare and a low-fare ticket is small, thus it is not necessary to protect many for high-fare tickets.

The table also shows comparisons with booking limits specified by the Littlewood rule. The Littlewood rule sets the booking limit to be

$$\eta_{\text{L}} = \max\{\eta \in [0, c] : \text{Prob[High-fare demand} > c - \eta] < p_2/p_1\}\,. \tag{2.30}$$

To implement the rule, it is necessary to estimate Prob[High-fare demand $> x$]. In the table, we consider two estimation schemes. In one scheme, we assume the revenue manager estimates the distribution of high-fare demand using the empirical distribution of $\{T^i\}$, namely $k^{-1}\sum_{i=1}^{k}\mathbb{I}_{\{T^i>x\}}$. T is the observed quantity; i.e., the quantity that the revenue manager believes (erroneously) to be the exogenous high-fare demand. The revenue manager then uses (2.30) with the estimated distribution in place of the supposed true distribution to set the k-th booking limit; that is the k-th booking limit is

$$\eta^k_{\mathrm{L(T)}} = \max\{\eta \in [0,c] : k^{-1}\sum_{i=1}^{k}\mathbb{I}_{\{T^i>c-\eta\}} < p_2/p_1\}\,. \tag{2.31}$$

The table shows the limit of the booking limits $\{\eta^k_{\mathrm{L(T)}}\}$, which we denote by $\eta^\infty_{\mathrm{L(T)}}$, as well as the revenue $r^\infty_{\mathrm{L(T)}} = r(\eta^\infty_{\mathrm{L(T)}})$ associated with the limit. In the second estimation scheme we assume the revenue manager estimates the distribution of high-fare demand using the empirical distribution of $\{T^i + R^i\}$. In this case (2.31) is replaced by $\eta^k_{\mathrm{L(T+R)}} = \max\{\eta \in [0,c] : k^{-1}\sum_{i=1}^{k}\mathbb{I}_{\{T^i+R^i>c-\eta\}} < p_2/p_1\}$. Using $\{T^i + R^i\}$ yields what are perhaps an unrealistically high estimates of high-fare demand, but it is nevertheless instructive to consider such situations to help understand the limits of the performance of the Littlewood rule. The table shows $\eta^\infty_{\mathrm{TR}} = \lim_{k\to\infty}\eta^k_{\mathrm{TR}}$ and $r^\infty_{\mathrm{TR}} = r(\eta^\infty_{\mathrm{TR}})$. Convergence proofs for booking limits generated by the Littlewood rule can be constructed as in CHK.

In Table 2.1 we see that $\eta^\infty_{\mathrm{L(T)}} = 100$. This occurs because $T^i = 0$ for all $i = 1, 2, \ldots$, which causes the distribution of the observed quantity to be a point mass at zero. In such a situation it makes sense to the revenue manager to set a low-fare booking limit of 100. This leads to very poor revenues as seen in the table. On the other hand, we have $\eta^\infty_{\mathrm{L(T+R)}} = 50$, which is the optimal value. Hence, even with the erroneous assumptions underlying the use of the Littlewood rule, we obtain convergence to the optimal booking limit. When the revenue manager uses $T +$

R as the observed quantity, his/her estimate of the high-fare demand distribution eventually becomes a point mass at 50, which leads to a booking limit of $50 = 100-50$.

To summarize, in this setting the buy-up model yields good performance in the limit if α is high or very low. For moderate values of α the limiting performance is poor. The Littlewood rule using observed quantity of $T+R$ yields the optimal booking limit in the limit. One of the motivations for considering a buy-up model is the hope that it does not lead to the undesirable spiral-down effects by using the Littlewood rule. This example suggests that a buy-up model may not be an improvement. As we will see, however, there are many situations in which the buy-up model does indeed offer an improvement.

Next we turn to the case covered in Section 2.3.2 where the flexible customers arrive first, followed by those customers who want only low-fare tickets, namely type-F before type-L. As shown in Proposition 2, the booking limits from the buy-up model converge to a limit in this case. However, as we have seen in Figures 2.2 and 2.3, the limit may depend upon the initial booking limit, η^0.

We again consider a setting in which there are $c = 100$ seats, $b = 70$ type-F customers, $a = 50$ type-L customers. Table 2.2 shows optimal and limiting values of booking limits for different values of α. Note that in the table, there are two values listed for η^∞ and r^∞ for each $\alpha = 0.1, \ldots, 0.5$, because for these values of α there were two different limiting booking limits depending upon the initial booking limit. For $\alpha = 0.1, \ldots, 0.5$, the booking limits from the buy-up model converge to either 0 or 100. Note also that the optimal booking limit is zero in this range. We point out that while the revenue associated with 0 is indeed optimal, the revenue associated with 100 is poor. For higher values of α, the booking limits converge to 100 from all starting points, and for those values of α, 100 is optimal. This shows that in many cases the buy-up model eventually leads to optimal decisions even though the model itself is not correct. The table also shows that the booking limits from the Littlewood rule converge to 100, regardless of the starting position. Hence, the Littlewood rule has good limiting behavior when α is high, but poor limiting behavior when α is low.

Table 2.2: Revenue Comparison with Type-F Customers Arrive First

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Optimal BL	0	0	0	0	0/100	100	100	100	100
Revenue, r_{OPT}	50	50	50	50	50	60	70	80	90
η^{∞}	0/100	0/100	0/100	0/100	0/100	100	100	100	100
r^{∞}	50/10	50/20	50/30	50/40	50/50	60	70	80	90
$100 \times \frac{(r_{\text{OPT}} - r^{\infty})}{r_{\text{OPT}}}$	0/80	0/60	0/40	0/20	0/0	0	0	0	0
$\eta^{\infty}_{\text{L(T)}}$	100	100	100	100	100	100	100	100	100
$\eta^{\infty}_{\text{L(T+R)}}$	100	100	100	100	100	100	100	100	100
r^{∞}_{L}	10	20	30	40	50	60	70	80	90
$100 \times \frac{(r_{\text{OPT}} - r^{\infty}_{\text{L}})}{r_{\text{OPT}}}$	80	60	40	20	0	0	0	0	0

We close this section with an example comparing the limiting booking limits between different arrival orders. Suppose that $c = 50$, $\alpha = 0.6$, and there are $a = 35$ type-L customers and $b = 25$ type-F customers. We use dotted (blue) curve to describe chosen booking limits when type-F customers arrive first, and (red) curve to describe chosen booking limits when type-L customers arrive first, and if some part of these two curves overlaps, then we make it as a solid curve. As predicted by Proposition 3, Figure 2.6 shows that beginning with same value of initial booking limit, the dotted (blue) curve is always on or above the other (red) curve.

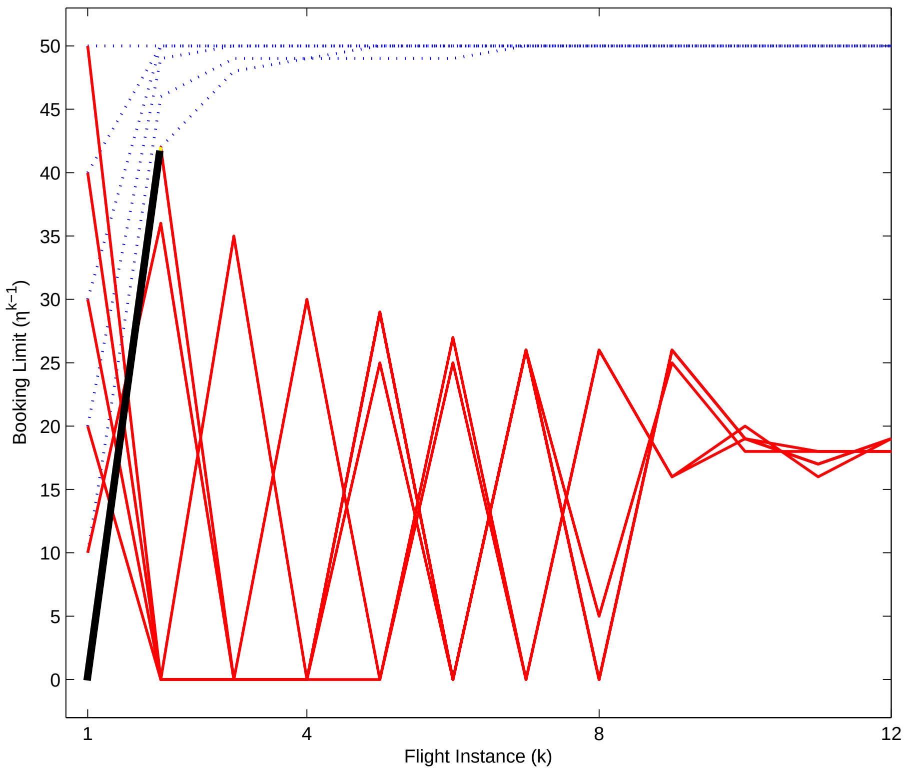


Figure 2.6: Comparison with Dynamics of Booking Limits

2.5 Summary and Research Directions

In this chapter, we considered a problem in which the revenue manager used a seemingly reasonable model of customer choice to control seat allocation in a single-leg revenue management problem. This "buy-up" model is a modification of the well-known Littlewood rule. The model yielding the Littlewood rule assumed the high-fare demand was exogenous and all high-fare demand appeared at the end the horizon, which was generally not consistent with actual customer behavior in practice. The buy-up model took the customer behavior of buy-up into account, and was considered to capture customer behavior more precisely.

We assumed the revenue manager used some basic statistical methods to estimate parameters in the buy-up model and controlled the seat allocation in practice by implementing this buy-up model. We tested the performance of using this buy-up model under different assumptions about actual customer behavior. We intentionally considered customer behavior that violated the assumptions underlying the buy-up model.

In two deterministic arrival settings, we proved some convergence properties of the chosen booking limits and also tested the revenue performance of using this buy-up model. In one setting, the buy-up model performed poorly when the price ratio was moderate. This was because the buy-up model had an incorrect (high) estimate of the buy-up rate and protected too many seats for the high-fare tickets. In the other setting, the booking limits given by the buy-up model converged to two different values depending on the initial chosen booking limit. While the revenue associated with one convergence value was indeed optimal, the revenue associated with the other convergence value was poor. For comparison, we also provided the revenue associated with the limiting booking limit obtained when using the Littlewood rule.

We also provide some future directions. In Section 2.3 with fixed demand, we only considered cases with two different types of customers. More complicated customer behavior with deterministic can be studied.

2.6 Appendix: Proofs

Derivation of Expressions (2.4) and (2.5). We have

$$P[D_1(\eta) > c - \eta \mid D_2 \geq \eta] = P[D_2 \geq \eta > c - D_1(\eta)]/P[D_2 \geq \eta]. \tag{2.32}$$

and

$$P[D_2 \geq \eta] = \sum_{y \geq \eta} g_2(y) \tag{2.33}$$

Working with the numerator gives us

$$\begin{aligned}
&P[D_2 \geq \eta > c - D_1(\eta)] \\
&= \sum_{d_2 \geq 0} P[d_2 \geq \eta > c - D_1 - U(\eta) \,|\, D_2 = d_2] P[D_2 = d_2] \\
&= \sum_{d_2 \geq \eta} P[\eta > c - D_1 - U(\eta) \,|\, D_2 = d_2] P[D_2 = d_2] \\
&= \sum_{d_2 \geq \eta} \left(\sum_{d_1 \geq 0} P[D_1 = d_1 \,|\, D_2 = d_2] P[U(\eta) > c - \eta - d_1 \,|\, D_1 = d_1, D_2 = d_2] \right) P[D_2 = d_2] \\
&= \sum_{d_2 \geq \eta} \left(\sum_{d_1 \geq 0} P[D_1 = d_1] P[U(\eta) > c - \eta - d_1 \,|\, D_2 = d_2] \right) P[D_2 = d_2]
\end{aligned}$$

where the final equality follows from the independence of D_1 and $(D_2, U(\eta))$. Expressions (2.4) and (2.5) now follow by combining the final expression above with (2.32), (2.33), with the fact that $U(\eta)$ is conditionally binomial with parameters $[d_2 - \eta]^+$ and γ given that $D_2 = d_2$. $\square$

Proof of Lemma 1. Part (A) holds because $\rho(\gamma) = \frac{\alpha - \gamma}{1 - \gamma}$ is a decreasing function of γ. For part (B) note that $\Psi(\eta, \gamma) = P(B > c - \eta)$ where B is a binomial random variable with parameters $a+b-\eta$ and γ. Part (B) then follows from the fact that the binomial distribution is stochastically increasing in the "success probability". Part (D) is true because $\gamma(\eta) = \frac{b-[\eta-a]^+}{a+b-\eta}$, which is non-decreasing in η.

It remains to show part (C). For this it suffices to show that $P(B > m) < P(C > m-1)$ where $m = c - \eta$ and B (respectively, C) is binomially distributed with parameters $n = a+b-\eta$ (resp., $n-1 = a+b-\eta-1$) and γ. Let $\{X_i : i = 1, \ldots, n\}$ be be i.i.d. Bernoulli random variables, each with parameter γ. It suffices to prove that $P(X_1 + \cdots + X_n > m) < P(X_1 + \cdots + X_{n-1} > m-1)$. We have

$$
\begin{aligned}
P(X_1 + &\cdots + X_n > m) \\
&= P(X_1 + \cdots + X_n > m, X_n = 0) + P(X_1 + \cdots + X_n > m, X_n = 1) \\
&= P(X_1 + \cdots + X_{n-1} > m, X_n = 0) + P(X_1 + \cdots + X_{n-1} > m-1, X_n = 1) \\
&= P(X_1 + \cdots + X_{n-1} > m)(1-\gamma) + P(X_1 + \cdots + X_{n-1} > m-1)\gamma \\
&< P(X_1 + \cdots + X_{n-1} > m-1)(1-\gamma) + P(X_1 + \cdots + X_{n-1} > m-1)\gamma \\
&= P(X_1 + \cdots + X_{n-1} > m-1)\,.
\end{aligned}
$$

This completes the proof. □

Proof of Lemma 2. For part 1, we first show that γ^*, $\overline{\gamma}$, and $\underline{\gamma}$ exist.

We have $\Psi(\eta^*, \gamma(\eta^*)) \geq \rho(\gamma(\eta^*))$ and $\Psi(\eta^*, \gamma(\eta^*-1)) < \rho(\gamma(\eta^*-1))$. Also, $\rho(\gamma)$ is decreasing and continuous in γ and $\Psi(\eta^*, \gamma)$ is increasing and continuous in γ. Therefore, there exists $\gamma^* \in (\gamma(\eta^*-1), \gamma(\eta^*)]$ that satisfies $\Psi(\eta^*, \gamma^*) = \rho(\gamma^*)$.

Turning to $\overline{\gamma}$, we have $\Psi(\eta^*-1, \gamma^*) < \Psi(\eta^*, \gamma^*) = \rho(\gamma^*)$. For $\widehat{\gamma} \in (\alpha, 1)$ we have that $\Psi(\eta^*-1, \widehat{\gamma}) > 0 > \rho(\widehat{\gamma})$. In addition, $\rho(\gamma)$ is decreasing and continuous in γ and $\Psi(\eta^*-1, \gamma)$ is increasing and continuous in γ; hence, there exists $\overline{\gamma} \in (\gamma^*, \widehat{\gamma}) \subset (\gamma^*, 1)$ that satisfies $\Psi(\eta^*-1, \overline{\gamma}) = \rho(\overline{\gamma})$.

For $\underline{\gamma}$, we have $\Psi(\eta^*+1, \gamma^*) > \Psi(\eta^*, \gamma^*) = \rho(\gamma^*)$ and $\Psi(\eta^*+1, 0) = 0 < \rho(0)$. In addition, $\rho(\gamma)$ is decreasing and continuous in γ and $\Psi(\eta^* + 1, \gamma)$ is increasing and continuous in γ. Hence, there exists $\underline{\gamma} \in (0, \gamma^*)$ with $\Psi(\eta^* + 1, \underline{\gamma}) = \rho(\underline{\gamma})$.

To prove uniqueness, we argue by contradiction. Suppose there exist $\gamma_1^* < \gamma_2^*$, both satisfying $\Psi(\eta^*, \gamma_i^*) = \rho(\gamma_i^*)$. Then $\Psi(\eta^*, \gamma_1^*) > \Psi(\eta^*, \gamma_2^*) = \rho(\gamma_2^*) > \rho(\gamma_1^*)$, which is a contradiction. Hence, γ^* is unique. We can use an analogous method to prove uniqueness of $\overline{\gamma}$ and $\underline{\gamma}$.

Next, we will prove (2.23). Note that if $\gamma \in (\underline{\gamma}, < \gamma^*)$, then $\Psi(\eta^*, \gamma) < \Psi(\eta^*, \gamma^*) = \rho(\gamma^*) < \rho(\gamma)$ and $\Psi(\eta^* + 1, \gamma) > \Psi(\eta^*, \gamma) > \Psi(\eta^*, \underline{\gamma}) = \rho(\underline{\gamma}) > \rho(\gamma)$; thus $\eta(\gamma) = \eta^*$. Likewise, if $\gamma \in [\gamma^*, \overline{\gamma})$ then $\Psi(\eta^*, \gamma) > \Psi(\eta^*, \gamma^*) = \rho(\gamma^*) > \rho(\gamma)$ and $\Psi(\eta^* - 1, \gamma) < \Psi(\eta^* - 1, \overline{\gamma}) = \rho(\overline{\gamma}) < \rho(\gamma)$; thus $\eta(\gamma) = \eta^* - 1$. This completes the proof of part 1.

We now prove the existence of $\overline{\gamma}$ and $\underline{\gamma}$ in part 2. We have $\Psi(\eta^* - 1, \gamma(\eta^* - 1)) < \rho(\gamma(\eta^* - 1))$. For $\widehat{\gamma} \in (\alpha, 1)$, we have $\Psi(\eta^* - 1, \widehat{\gamma}) > 0 > \rho(\widehat{\gamma})$; therefore there exists $\overline{\gamma} \in (\gamma(\eta^* - 1), \widehat{\gamma}) \subset (\gamma(\eta^* - 1), 1)$ that satisfies $\Psi(\eta^* - 1, \overline{\gamma}) = \rho(\overline{\gamma})$. To show the existence of $\underline{\gamma}$, note that $\Psi(\eta^*, \gamma(\eta^* - 1)) \geq \rho(\gamma(\eta^* - 1))$ and $\Psi(\eta^*, 0) = 0 < \rho(0)$. Hence, there exists $\underline{\gamma} \in (0, \gamma(\eta^* - 1)]$ that satisfies $\Psi(\eta^*, \underline{\gamma}) = \rho(\underline{\gamma})$. The remainder of the proof is similar to part 1, and hence omitted. □

Lemma 3 *Suppose $\eta^* \neq 0$ and $\Psi(\eta^*, \gamma(\eta^* - 1)) < \widehat{\rho}(\eta^* - 1)$. Then, there exists B_0 such that if $B > B_0$ and $\gamma^k = \frac{A}{B} \in (\underline{\gamma}, \overline{\gamma})$, then $\gamma^{k+1} \in (\underline{\gamma}, \overline{\gamma})$.*

Proof. Note that $\gamma(\eta) = 1$ for all $\eta \geq a$. Hence, $\widehat{\rho}(\eta) = -\infty$ for all $\eta \geq a$, which implies that $\Psi(\eta, \gamma(\eta)) \geq \widehat{\rho}(\eta)$ for all $\eta \geq a$. It then follows from the definition of η^* that $\eta^* \leq a$. Therefore, $\gamma(\eta^*) = \frac{b}{a+b-\eta^*}$ and $\gamma(\eta^* - 1) = \frac{b}{a+b-\eta^*+1}$.

Suppose first that $\gamma^k = \frac{A}{B} \in (\underline{\gamma}, \gamma^*)$. Then $\gamma^{k+1} = \gamma^k \oplus \gamma_{k+1} = \gamma^k \oplus \gamma(\eta^*) = \frac{A}{B} \oplus \frac{b}{a+b-\eta^*} = \frac{A+b}{B+a+b-\eta^*} < \frac{\gamma^* B+b}{B+a+b-\eta^*} < \overline{\gamma}$ if $B > \frac{b-\overline{\gamma}(a+b-\eta^*)}{\overline{\gamma}-\gamma^*}$. From Lemma 2 we have $\gamma(\eta^*) > \gamma^* > \underline{\gamma}$, and thus $\underline{\gamma} < \gamma^k \oplus \gamma(\eta^*) = \gamma^{k+1}$. Hence, $\gamma^{k+1} \in (\underline{\gamma}, \overline{\gamma})$ if $B > \frac{b-\overline{\gamma}(a+b-\eta^*)}{\overline{\gamma}-\gamma^*}$.

Suppose next that $\gamma^k = \frac{A}{B} \in [\gamma^*, \overline{\gamma})$. Then $\gamma^{k+1} = \gamma^k \oplus \gamma(\eta^* - 1) = \frac{A}{B} \oplus \frac{b}{a+b-\eta^*+1} = \frac{A+b}{B+a+b-\eta^*+1} \geq \frac{\gamma^* B+b}{B+a+b-\eta^*+1} > \underline{\gamma}$ if $B > \frac{\underline{\gamma}(a+b-\eta^*+1)-b}{\gamma^*-\underline{\gamma}}$. From Lemma 2 we have $\gamma(\eta^*-1) < \gamma^* < \overline{\gamma}$, and thus $\gamma^{k+1} = \gamma^k \oplus \gamma(\eta^*-1) < \overline{\gamma}$. Therefore, $\gamma^{k+1} \in (\underline{\gamma}, \overline{\gamma})$ if $B > \frac{\underline{\gamma}(a+b-\eta^*+1)-b}{\gamma^*-\underline{\gamma}}$.

Taking $B_0 = \max\{\frac{\underline{\gamma}(a+b-\eta^*+1)-b}{\gamma^*-\underline{\gamma}}, \frac{b-\overline{\gamma}(a+b-\eta^*)}{\overline{\gamma}-\gamma^*}\}$, we see that $\gamma^{k+1} \in (\underline{\gamma}, \overline{\gamma})$.

$\square$

Lemma 4 *Suppose $\eta^* \neq 0$ and $\Psi(\eta^*, \gamma(\eta^*-1)) < \widehat{\rho}(\eta^*-1)$, and consider γ^k defined in (2.17). Suppose there exists K_0 such that $\gamma^k \in (\underline{\gamma}, \overline{\gamma})$ for all $k \geq K_0$. Then $\lim_{k\to\infty} \gamma^k = \gamma^*$.*

Proof. To simplify notation let $\gamma^k = \frac{A_k}{B_k}$ for $k \geq 1$. Let $Q = a + b - \eta^*$ so that $\gamma(\eta^*-1) = \frac{b}{Q+1}$ and $\gamma(\eta^*) = \frac{b}{Q}$. And from Lemma 2, $Q\gamma^* \leq b < (Q+1)\gamma^*$.

For $k \geq K_0$, we also have:

$$\gamma^{k+1} = \gamma^k \oplus \gamma_{k+1} = \begin{cases} \frac{A_k}{B_k} \oplus \frac{b}{Q} & \text{if } \gamma^k < \gamma^* \\ \frac{A_k}{B_k} \oplus \frac{b}{Q+1} & \text{if } \gamma^k \geq \gamma^*. \end{cases}$$

Since A_k and B_k are strictly increasing. It is clear that for any $M > 0$, there exists K_1 such that $B_k > M$ for any $k > K_1$.

One one hand, we show: for any $\epsilon > 0$, there exists $K_\epsilon \geq K_0$ such that if $k \geq K_\epsilon$ and $\gamma^k < \gamma^*$, then $\gamma^{k+1} < \gamma^* + \epsilon$. From our definition, we just need to show that for any $\epsilon > 0$, there exists $K_\epsilon \geq K_0$ such that if $\frac{A_k}{B_k} < \gamma^*$, then $\frac{A_k+b}{B_k+Q} < \gamma^* + \epsilon$ for $k \geq K_\epsilon$. We know for any ϵ, there exists K_ϵ such that $B_k > B_{K_\epsilon} > 1/\epsilon$ for all $k > K_\epsilon$ and thus $\epsilon B_k - \gamma^* \geq \epsilon B_k - 1 > 0$. Since $A_k + b < B_k\gamma^* + (1+Q)\gamma^* < B_k\gamma^* + (1+Q)\gamma^* + \epsilon B_k - \gamma^* + \epsilon Q = (B_k + Q)(\gamma^* + \epsilon)$, where the second inequality follows $Q\gamma^* \leq b < (Q+1)\gamma^*$,we know $\frac{A_k+a}{B_k+b} < \gamma^* + \epsilon$. Moreover, if $k \geq K_0$ and $\gamma^k \in [\gamma^*, \gamma^* + \epsilon)$, then $\gamma_{k+1} = \gamma(\eta^*-1) < \gamma^* \leq \gamma^k$ by Lemma 2, and $\gamma^{k+1} = \gamma^k \oplus \gamma_{k+1} < \gamma^k$. Therefore, if $k \geq K_0$ and $\gamma^k \in [\gamma^*, \gamma^* + \epsilon)$ then there exists l such that $\gamma^k > \gamma^{k+1} > ... > \gamma^{k+l-1} \geq \gamma^* > \gamma^{k+l}$. Combine these, for any $j \geq K_\epsilon + l$, we have $\gamma^j < \gamma^* + \epsilon$. Therefore, $\limsup \gamma^k \leq \gamma^* + \epsilon$. Since $\epsilon > 0$ is arbitrary, it then follows that $\limsup_{k\to\infty} \gamma^k \leq \gamma^*$.

Similar arguments show that if $\gamma^k \geq \gamma^*$, then (a) $\gamma^{k+1} > \gamma^* - \epsilon$; and (b) if $\gamma^k \in (\gamma^* - \epsilon, \gamma^*)$, then there exists $l \geq 1$ such that $\gamma^k < \gamma^{k+1} < ... < \gamma^{k+l-1} < \gamma^* \leq \gamma^{k+l}$; or $\gamma^m < \gamma^*$ for all $m \geq k$. It follows $\liminf_{k\to\infty} \gamma^k \geq \gamma^*$.

Therefore, $\lim_{k\to\infty} \gamma^k = \gamma^*$. □

Lemma 5 *Suppose $\eta^* \neq 0$ and $\Psi(\eta^*, \gamma(\eta^* - 1)) < \widehat{\rho}(\eta^* - 1)$, and consider γ^k defined in (2.17). For any $K_0 \geq 0$ there exists some $k \geq K_0$ such that $\gamma^k \in (\underline{\gamma}, \overline{\gamma})$.*

Proof. Suppose for a contradiction that the lemma is not true. Then there are three possibilities: (I) $\gamma^k \geq \overline{\gamma}$ for all $k \geq K_0$ for some K_0; (II) $\gamma^k \leq \underline{\gamma}$ for $k \geq K_0$ for some K_0; or (III) γ^k "jumps over" area $(\underline{\gamma}, \overline{\gamma})$, after any $k \geq K_0$. We show that (I), (II), and (III) each yields a contradiction.

I. Suppose that there exists K_0 such that $\gamma^k \geq \overline{\gamma}$ for all $k \geq K_0$.

   We first show that $\gamma_k < \overline{\gamma}$ for all $k \geq K_0 + 1$. Consider $k \geq K_0$. From the supposition $\gamma^k \geq \overline{\gamma}$ and the definition of η^k, we have that $\Psi(\eta^* - 1, \gamma^k) \geq \Psi(\eta^* - 1, \overline{\gamma}) = \rho(\overline{\gamma}) \geq \rho(\gamma^k) \geq \Psi(\eta^k, \gamma^k)$. Thus, we have $\eta^* - 1 \geq \eta^k$, furthermore, $\overline{\gamma} > \gamma^* > \gamma(\eta^* - 1) \geq \gamma(\eta^k) = \gamma_{k+1}$.

   We next derive a contradiction by showing that there exists K' such that

$$\gamma^{K_0+K'} = \gamma^{K_0} \oplus \gamma_{K_0+1} \oplus \cdots \oplus \gamma_{K_0+K'} < \overline{\gamma}\,. \tag{2.34}$$

   Suppose $\gamma^{K_0} = \frac{A}{B}$. Since $\gamma_{K_0+1}, \ldots, \gamma_{K_0+K'} < \overline{\gamma}$ and each $\gamma_k = \frac{R^k}{Q^k}$ is an element of the finite set $\{\gamma(0), \ldots, \gamma(c)\}$, it follows that $0 < \epsilon := \min\{\epsilon_k = \overline{\gamma}Q^k - R^k : k > K_0\}$. Hence, $R^k \leq \overline{\gamma}Q^k - \epsilon$, and consequently

$$\gamma^{K_0} \oplus \gamma_{K_0+1} \oplus \cdots \oplus \gamma_{K_0+K'} = \frac{A + R^{K_0+1} + \cdots + R^{K_0+K'}}{B + Q^{K_0+1} + \cdots + Q^{K_0+K'}} \leq \frac{A + \overline{\gamma}C_{K'} - K'\epsilon}{B + C_{K'}}\,,$$

where $C_{K'} = Q^{K_0+1} + \cdots + Q^{K_0+K'}$. Then $\gamma^{K_0+K'} \leq \frac{A+\overline{\gamma}C_{K'}-K'\epsilon}{B+C_{K'}} < \overline{\gamma}$ for $K' > \frac{A-\overline{\gamma}B}{\epsilon}$, and hence (2.34) holds for $K' > \frac{A-\overline{\gamma}B}{\epsilon}$.

II. Suppose that there exists K_0 such that $\gamma^k \leq \underline{\gamma}$ for all $k \geq K_0$.

We will first show that $\gamma_{k+1} > \underline{\gamma}$ for all $k \geq K_0$. Consider $k \geq K_0$. If $\eta^* = c$ then $\eta^k \leq \eta^*$ and therefore $\underline{\gamma} < \gamma^* \leq \gamma(\eta^*) \leq \gamma(\eta^k) = \gamma_{k+1}$. If $\eta^* < c$ then $\Psi(\eta^*+1, \underline{\gamma}) = \rho(\underline{\gamma}) \leq \rho(\gamma^k) \leq \Psi(\eta^k+1, \gamma^k)$. Thus, we have $\eta^*+1 \leq \eta^k+1$, which implies $\eta^* \leq \eta^k$. Consequently, $\underline{\gamma} < \gamma^* \leq \gamma(\eta^*) \leq \gamma(\eta^k) = \gamma_{k+1}$.

Similar to I above, it follows that there exists K' such that

$$\gamma^{K_0+K'} = \gamma^{K_0} \oplus \gamma_{K_0+1} \oplus \cdots \oplus \gamma_{K_0+K'} > \underline{\gamma}\,,$$

which is a contradiction.

III. Suppose that there exists K_0 such that $\gamma^k \notin (\overline{\gamma}, \underline{\gamma})$ for all $k \geq K_0$, and for every $K_1 \geq K_0$ there exist $k', k'' \geq K_1$ such that $\gamma^{k'} \leq \underline{\gamma}$ and $\gamma^{k''} \geq \overline{\gamma}$.

All possible buy-up rates are in the set $\omega = \{\frac{b}{a+b}, \frac{b}{a+b-1}, \ldots, \frac{b}{b}, \frac{b-1}{b-1}, \ldots, \frac{a+b-c}{a+b-c}\}$. For notational convenience let $\gamma^k = \frac{A_k}{B_k}$. We will show there exists a B_0, if $B > B_0$, then for any $\widetilde{\gamma} \in R$.

$$\begin{cases} \gamma^k \oplus \widetilde{\gamma} < \overline{\gamma}, & \text{while } \gamma^k < \underline{\gamma} \\ \gamma^k \oplus \widetilde{\gamma} > \underline{\gamma}, & \text{while } \gamma^k > \overline{\gamma} \end{cases}$$

We let $B_l = \frac{(1-\overline{\gamma})b}{\overline{\gamma}-\underline{\gamma}}$ and $B_h = \frac{\underline{\gamma}(a+b)-b}{\overline{\gamma}-\underline{\gamma}}$. If $\gamma^k = \frac{A_k}{B_k} < \underline{\gamma}$ and $B_k > B_l = \frac{(1-\overline{\gamma})b}{\overline{\gamma}-\underline{\gamma}}$, then $\gamma^k \oplus \widetilde{\gamma} \leq \frac{A_k+b}{B_k+b} < \frac{\underline{\gamma}B_k+b}{B_k+b} < \overline{\gamma}$; if $\gamma^k = \frac{A_k}{B_k} > \overline{\gamma}$ and $B_k > B_h = \frac{\underline{\gamma}(a+b)-b}{\overline{\gamma}-\underline{\gamma}}$, then $\gamma^k \oplus \widetilde{\gamma} \geq \frac{A_k+b}{B_k+a+b} > \frac{\overline{\gamma}B_k+b}{B_k+a+b} > \underline{\gamma}$ or $\gamma^k \oplus \widetilde{\gamma} \geq \frac{A_k+a+b-c}{B_k+a+b-c} > \frac{\overline{\gamma}B_k+a+b-c}{B_k+a+b-c} > \overline{\gamma} > \underline{\gamma}$. Let $B_0 = \max\{B_l, B_h\} = \max\{\frac{(1-\overline{\gamma})b}{\overline{\gamma}-\underline{\gamma}}, \frac{\underline{\gamma}(a+b)-b}{\overline{\gamma}-\underline{\gamma}}\}$, which completes the proof.

□

Proof of Proposition 2. Fix k and consider the booking limit η^k and buy-up rate estimate γ^k. Below, we consider two separate cases, $\gamma^{k+1} \geq \gamma^k$ and $\gamma^{k+1} < \gamma^k$, which correspond, respectively, to an increase in the buy-up estimate and a decrease in the buy-up estimate.

Case 1 ($\gamma^{k+1} \geq \gamma^k$): Let $\gamma^k = \frac{A}{B}$ so that $\gamma^{k+1} = \gamma^k \oplus \gamma(\eta^k) = \frac{A+R^{k+1}}{B+Q^{k+1}}$. Since $\gamma^{k+1} \geq \gamma^k$ by assumption, we have $\frac{A+R^{k+1}}{B+Q^{k+1}} \geq \frac{A}{B}$, which implies that $\frac{R^{k+1}}{Q^{k+1}} \geq \frac{A}{B}$ and thus $\gamma(\eta^k) \geq \gamma^k$. Note that if $\eta^k = c$ then $\eta^{k+1} \leq \eta^k$ and $\gamma(\eta^{k+1}) \geq \gamma(\eta^k)$. If $\eta^k < c$ then from (**??**) we have $\Psi(\eta^k+1, \gamma^k) \geq \rho(\gamma^k)$, and therefore $\Psi(\eta^k+1, \gamma^{k+1}) \geq \Psi(\eta^k+1, \gamma^k) \geq \rho(\gamma^k) \geq \rho(\gamma^{k+1})$. This implies that $\eta^{k+1} < \eta^k+1$, and hence $\eta^{k+1} \leq \eta^k$ and $\gamma(\eta^{k+1}) \geq \gamma(\eta^k)$.

Moreover, upon recalling that $\gamma_{k+2} = \gamma(\eta^{k+1}) = \frac{R^{k+2}}{Q^{k+2}}$, we see that $\frac{R^{k+2}}{Q^{k+2}} \geq \frac{R^{k+1}}{Q^{k+1}} \geq \frac{A}{B}$. It is easy to check that $\frac{A+R^{k+1}+R^{k+2}}{B+Q^{k+1}+Q^{k+2}} \geq \frac{A+R^{k+1}}{B+Q^{k+1}}$, and thus $\gamma^{k+2} = \gamma^k \oplus \gamma(\eta^k) \oplus \gamma(\eta^{k+1}) = \frac{A+R^{k+1}+R^{k+2}}{B+Q^{k+1}+Q^{k+2}} \geq \frac{A+R^{k+1}}{B+Q^{k+1}} = \gamma^k \oplus \gamma(\eta^k) = \gamma^{k+1}$. In summary, if $\gamma^{k+1} \geq \gamma^k$, then $(\eta^{k+1}, \gamma^{k+1})$ are such that $\eta^{k+1} \leq \eta^k$ and $\gamma^{k+2} \geq \gamma^{k+1}$.

It now follows easily by induction that if $\gamma^{k+1} \geq \gamma^k$, then $\{\eta^\ell : \ell \geq k\}$ is a non-increasing sequence. Moreover, the sequence is bounded below by 0, so it follows that $\eta^\circ := \lim_{\ell\to\infty} \eta^\ell$ exists. Booking limits are integer-valued; therefore $\eta^k = \eta^\circ$ for all k large enough. It then follows that $\lim_{\ell\to\infty} \gamma^\ell = \gamma(\eta^\circ)$. Suppose now that $\eta^\circ \neq 0, c$. For large enough k, $\Psi(\eta^\circ, \gamma^k) = \Psi(\eta^k, \gamma^k) < \rho(\gamma^k)$. Upon taking limits on both sides of the preceding expression, we conclude that $\Psi(\eta^\circ, \gamma(\eta^\circ) \leq \widehat{\rho}(\eta^\circ)$. A similar argument shows that $\Psi(\eta^\circ + 1, \gamma(\eta^\circ)) \geq \widehat{\rho}(\eta^\circ)$.

Case 2 ($\gamma^{k+1} < \gamma^k$): As above, let $\gamma^k = \frac{A}{B}$ so that $\gamma^{k+1} = \gamma^k \oplus \gamma(\eta^k) = \frac{A+R^{k+1}}{B+Q^{k+1}}$. We have assumed that $\gamma^{k+1} < \gamma^k$, and therefore $\frac{A+R^{k+1}}{B+Q^{k+1}} < \frac{A}{B}$. This implies $\frac{R^{k+1}}{Q^{k+1}} < \frac{A}{B}$; i.e., $\gamma(\eta^k) < \gamma^k$. Observe that $\eta^k \neq 0$. [To see this, if it were the case that $\eta^k = 0$, then the preceding argument shows that $\gamma(0) < \gamma^k$; however, we know that $\gamma(0) \geq \gamma(\eta^\ell)$ for all $\ell \geq 1$ so that $\gamma^k = \gamma(\eta^1) \oplus \cdots \oplus \gamma(\eta^k) \leq \gamma(0)$ which is a contradiction.] From the definition of η^k we have $\Psi(\eta^k, \gamma^k) < \rho(\gamma^k)$. Therefore, $\Psi(\eta^k, \gamma^{k+1}) < \Psi(\eta^k, \gamma^k) < \rho(\gamma^k) < \rho(\gamma^{k+1})$. Hence, $\eta^{k+1} \geq \eta^k$, from which it follows that $\gamma(\eta^{k+1}) \leq \gamma(\eta^k)$. Therefore, $\gamma(\eta^{k+1}) \leq \gamma^k \oplus \gamma(\eta^k)$ and consequently, $\gamma^{k+2} = \gamma^k \oplus \gamma(\eta^k) \oplus \gamma(\eta^{k+1}) < \gamma^k \oplus \gamma(\eta^k) = \gamma^{k+1}$. To summarize, if $\gamma^{k+1} < \gamma^k$, then $(\eta^{k+1}, \gamma^{k+1})$ are such that $\eta^{k+1} \geq \eta^k$ and $\gamma^{k+2} < \gamma^{k+1}$.

Hence, if $\gamma^{k+1} < \gamma^k$, then $\{\eta^\ell : \ell \geq k\}$ is a non-decreasing sequence that is bounded above by c. Therefore, $\eta^\circ := \lim_{\ell\to\infty} \eta^\ell$ exists. Hence, $\lim_{t\to\infty} \gamma^t = \gamma(\eta^\circ)$ as well. The remainder of the argument is the same as in case 1. $\square$

Lemma 6 *For any $\eta_1, \eta_2 \in [0, c]$, we have $\gamma_l(\eta_1) \geq \gamma_f(\eta_2)$. Furthermore, $\gamma_l^k \geq \gamma_f^k$ for any given initial booking limits η_f^0 and η_l^0.*

Proof. From Table 2.3, we know for any $\eta_1, \eta_2 \in [0, c]$, we have $\gamma_l(\eta_1) \geq \frac{b}{a+b} \geq \gamma_f(\eta_2)$. For the second part, we will use induction. It suffices to show if $\gamma_l^{k-1} \geq \frac{b}{a+b} \geq \gamma_f^{k-1}$, then $\gamma_l^k \geq \frac{b}{a+b} \geq \gamma_f^k$. By the property of $\oplus$, $\gamma_l^k = \gamma_l^{k-1} \oplus \gamma_{l,k} \geq \frac{b}{a+b} \geq \gamma_f^{k-1} \oplus \gamma_{f,k} = \gamma_f^k$.

$\square$

Lemma 7 *Given γ_1 and γ_2 with $\gamma_1 \geq \gamma_2$, let η_i, $i = 1, 2$ be defined by $\eta_i = \max\{0 \leq \eta \leq c : \Psi(\eta_i, \gamma_i) < \rho(\gamma_i))\}$. Then, $\eta_1 \leq \eta_2$.*

Proof. If there is a $\eta \in [0, c]$ that satisfies $\Psi(\eta_1, \gamma_1) < \rho(\gamma_1)\}$, then $\Psi(\eta_1, \gamma_2) \leq \Psi(\eta_1, \gamma_1) < \rho(\gamma_1) \leq \rho(\gamma_2)$, and thus from the definition $\eta_1 \leq \eta_2$. If $\Psi(\eta, \gamma_1) \geq \rho(\gamma_1)$ for all $\eta \in [0, c]$, then $\eta_1 = 0 \leq \eta_2$. $\square$

Proof of Proposition 3. It follows by Lemma 6 and 7.

$\square$

Table 2.3: $\gamma_l(\eta)$ and $\gamma_f(\eta)$ with $a < b$

η	$Q_l(\eta)$	$R_l(\eta)$	$\gamma_l(\eta)$	$\gamma_f(\eta)$	$Q_f(\eta)$	$R_f(\eta)$
0	$a+b$	b	$\frac{b}{a+b}$	$\frac{b}{a+b}$	$a+b$	b
1	$a+b-1$	b	$\frac{b}{a+b-1}$	$\frac{b-1}{a+b-1}$	$a+b$	$b-1$
.	.	.	.	.	.	.
.	.	.	.	.	.	.
a	b	b	$\frac{b}{b}$	$\frac{b-a}{b}$	b	$b-a$
$a+1$	$b-1$	$b-1$	$\frac{b-1}{b-1}$	$\frac{b-a-1}{b-1}$	$b-1$	$b-a-1$
.	.	.	.	.	.	.
.	.	.	.	.	.	.
b	a	a	$\frac{a}{a}$	$\frac{0}{a}$	a	0
$b+1$	$a-1$	$a-1$	$\frac{a-1}{a-1}$	$\frac{0}{a-1}$	$a-1$	0
.	.	.	.	.	.	.
.	.	.	.	.	.	.
c	$a+b-c$	$a+b-c$	$\frac{a+b-c}{a+b-c}$	$\frac{0}{a+b-c}$	$a+b-c$	0

Proof of Proposition 4. We append a prime to indicate quantities after either (1) a decrease from b to $b-1$; (2) an increases from c to $c+1$; or (3) fare ratio changes from α to $\alpha' > \alpha$.

We consider (1), (2) and (3) separately. In each case, we need to show $\Psi'(\eta^*-1, \gamma'(\eta^*-1)) < \widehat{\rho'}(\eta^*-1)$.

(1) We have $\gamma(\eta) = \frac{b-[\eta-a]^+}{a+b-\eta} \geq \frac{b-1-[\eta-a]^+}{a+b-1-\eta} = \gamma'(\eta)$, so $\Psi'(\eta^*-1, \gamma'(\eta^*-1)) \leq \Psi'(\eta^*-1, \gamma(\eta^*-1)) < \Psi(\eta^*-1, \gamma(\eta^*-1)) < \hat{\rho}(\eta^*-1) = \rho(\gamma(\eta^*-1)) < \rho(\gamma'(\eta^*-1)) = \widehat{\rho'}(\eta^*-1)$. The first and last inequality hold because of Lemma 1. For the second inequality it suffices to show that $P(B > m) < P(C > m)$ where $m = c - \eta^* + 1$ and B (respectively, C) is binomially distributed with parameters $n = a + b - \eta^*$ (resp., $n + 1 = a + b - \eta^* + 1$) and $\gamma(\eta^*-1)$.

(2) We have $\gamma(\eta) = \gamma'(\eta)$, so $\Psi'(\eta^*-1, \gamma'(\eta^*-1)) = \Psi'(\eta^*-1, \gamma(\eta^*-1)) \leq \Psi(\eta^*-1, \gamma(\eta^*-1)) < \hat{\rho}(\eta^*-1) = \widehat{\rho'}(\eta^*-1)$. For the first inequality it suffices to show that $P(B > m) < P(B > m-1)$ where $m = c - \eta^* + 1$ and B is binomially distributed with parameters $n = a + b - \eta^* + 1$ and $\gamma(\eta^*-1)$.

(3) We only have $\alpha < \alpha'$, so $\Psi'(\eta^*-1, \gamma'(\eta^*-1)) = \Psi(\eta^*-1, \gamma(\eta^*-1)) < \hat{\rho}(\eta^*-1) = \frac{\alpha-\gamma(\eta^*-1)}{1-\gamma(\eta^*-1)} < \frac{\alpha'-\gamma'(\eta^*-1)}{1-\gamma'(\eta^*-1)} = \widehat{\rho'}(\eta^*-1)$.

If more than one scenario happens, we only consider one update at a time, and consider one another update based on the previous update. Therefore, we combine all the above analysis to finish the proof.

□

Proof of Corollary 1. We append a prime to indicate quantities after both a and c increases by 1.

We have $\gamma(\eta-1) = \frac{b-[(\eta-1)-a]^+}{a+b-(\eta-1)} = \frac{b-[\eta-(a+1)]^+}{(a+1)+b-\eta} = \gamma'(\eta)$, so $\Psi'(\eta^*, \gamma'(\eta^*)) = \Psi'(\eta^*, \gamma(\eta^*-1)) = \Psi(\eta^*-1, \gamma(\eta^*-1)) < \hat{\rho}(\eta^*-1) = \rho(\gamma(\eta^*-1)) = \rho(\gamma'(\eta^*)) = \widehat{\rho'}(\eta^*)$. The second equality is true because they are both $P(B > m)$ where $m = c - \eta^* + 1$ and

B is binomially distributed with parameters $n = a + b - \eta^* + 1$ and $\gamma(\eta^* - 1)$. Moreover, $\Psi'(\eta^* + 1, \gamma'(\eta^* + 1)) = \Psi'(\eta^* + 1, \gamma(\eta^*)) = \Psi(\eta^*, \gamma(\eta^*)) \geq \hat{\rho}(\eta^*) = \rho(\gamma(\eta^*)) = \rho(\gamma'(\eta^* + 1)) = \widehat{\rho'}(\eta^* + 1)$, which completes the proof.

$\square$

Chapter 3

On the Use of "Buy-Up" as a Model of Customer Choice in Revenue Management with Stochastic Arrivals

3.1 Introduction

In the previous chapter, we constructed a framework of how the revenue manager uses the buy-up model to control the seat allocation, and analyzed the performance of the model assuming actual arrivals are deterministic. A question that arises is what if customers' behavior is more complicated than the deterministic arrival patterns considered in the previous chapter. For instance, what if customers come to the system randomly. In this chapter, we focus our attention on actual arrival process that are stochastic.

A key step in our analysis is the identification of the fixed points of certain composite functions. In particular, the potential limits are those booking limits that have the property that the data they generate lead to buy-up estimates that yield back the same booking limits. We showed in the previous chapter the convergence points are exactly those fixed points when arrivals are deterministic. In this chapter, we analyze the relationship between convergence in the stochastic setting and such fixed points. Under some mild conditions, we show that the existence of fixed points is necessary for convergence, and all convergence points need to be fixed points. We also provide sufficient conditions for convergence of booking limits and buy-up rate estimates.

Some of our convergence results employ results from the theory of stochastic approximation algorithms. Stochastic approximation methods are a family of iterative stochastic optimization algorithms which are designed to minimize or maximize an unknown function when only noisy function evaluations are possible. The first and prototypical algorithms can be traced back to Robbins and Monro (1951). In their paper, a stochastic approximation method was studied by placing conditions on iterative step sizes, and convergence of the method was proved under mild conditions. However, the method has some requirements on the gradient of the function. Kiefer and Wolfowitz (1952) developed a finite difference version of the Robbins-Monro method that maintains the convergence properties, while not requiring the knowledge of the analytic form of the gradient.

Since then, a large amount of literature has grown up around these algorithms. This literature discusses conditions for convergence, rates of convergence, proper choices of step size, possible noise models, multivariate and other generalizations, and so on. Benveniste et al. (1990) and Kushner and Yin (2003) provide overviews of stochastic approximation and stochastic adaptive recursive algorithms. We shall use a convergence result for stochastic approximation algorithms from Bertsekas and

Tsitsiklis (1996) that centers around the notion of a pseudo-contraction mapping.

There have been several recent papers in revenue management that have used results from stochastic approximation theory. van Ryzin and McGill (2000) used stochastic approximation theory to prove the convergence of an adaptive algorithm of the optimal protection level in seat control problem. Topaloglu (2008) implemented a stochastic approximation method to compute bid prices in network revenue management problems, in which the total expected revenue was formulated as a function of the bid prices. van Ryzin and Vulcano (2008) computed virtual vesting controls for network revenue management by stochastic approximation, and a stochastic steepest ascent algorithm was constructed by the stochastic gradient. Other papers that use stochastic approximation approaches to study specification error in revenue management include Cooper et al. (2006), Cooper et al. (2009), and Lee et al. (2009). Moreover, Lee et al. (2009) used pseudo-contraction to prove the convergence. Stochastic approximation theory has also played an important role in the study of model misspecification and learning in economics; see, e.g. Evans and Honkapohja (2001).

The remainder of the chapter is organized as follows. Section 3.2 describes our framework in the stochastic setting. Section 3.3 contains general necessary conditions for convergence of booking limits and buy-up rate estimates. Section 3.4 provides two specific sufficient conditions for the convergence. Section 3.5 contains numerical results that make comparisons among optimal solutions, solutions from the buy-up model, and the Littlewood rule. Section 3.6 summarizes the chapter and provides future directions. Most of the proofs are shown in Section 3.7.

3.2 Framework

In this chapter, we consider a setting in which the actual arrival of customers is random. We also assume in this chapter that the revenue manager uses a common variation on the formula (2.2).

We begin by describing the actual customer arrivals. For each problem instance, we assume that customers arrive over a continuous span of time of length τ. We further assume that there are three different types of customers, type H, type L, and type F. Type-L and type-F customers have been defined in previous sections. We suppose type-H customers are willing to buy only high-fare tickets. If a type-H customer arrives to the system to find no high-fare ticket available, then he will depart without purchase. Arrivals of each of the three types of customers are independent Poisson processes over the booking horizon of length τ. Therefore, we know the total number of each type is Poisson distributed. We denote their respective means by λ_H, λ_L and λ_F. Since we want to test the performance with the presence of all three types of customers, we assume λ_H, λ_L and λ_F are all strictly positive. We also assume that customer arrival processes are independent across problem instances.

We next describe the revenue manager's procedure to select booking limits. Recall from Section 2.2.1 that the revenue manager's model relies on both high-fare and low-fare demand distributions. In this chapter we suppose that the revenue manager decides to approximate the distribution of high-fare demand with a normal distribution (with mean μ and variance σ^2). This appears to be common in practice. Then, a simple approximation to (2.2) is to set

$$\eta_{\mathrm{N}} = \max\{\eta \in [0, c] : \ 1 - \Phi_{(\mu,\sigma^2)}(c - \eta) < \rho(\gamma)\}, \tag{3.1}$$

where $\Phi_{(\mu,\sigma^2)}(\cdot)$ denotes the normal cumulative distribution function with mean μ and variance σ^2. The formula (3.1) can be motivated with an informal marginal analysis; see, e.g., Belobaba (1989), Talluri and van Ryzin (2004b), or Phillips (2005).

To use (3.1) the revenue manager needs to estimate μ and σ^2, as well as the buy-up rate γ. Upon recalling from Section 2.2.2 that T represents the requests for high-fare tickets from customers who do not first ask for low-fare tickets, it is natural that the revenue manager would estimate the mean and variance of the high-fare demand distribution as the sample average and sample variance of the observations of T. Note that in the current setting, T is simply the number of type-H customers that arrive. Hence, we assume that after k instances, the revenue manager estimates μ by

$$M^k = \frac{1}{k} \sum_{i=1}^{k} T^i \tag{3.2}$$

and σ^2 by

$$V^k = \frac{1}{k-1} \sum_{i=1}^{k} (T^i - M^k)^2 \,. \tag{3.3}$$

Given the revenue manager's "normal approximation of high-fare demand", these are the usual estimators for μ and σ^2 that can be found in any standard statistics reference. As before, we assume the revenue manager estimates γ by γ^k in (2.7). Plugging the estimates into (3.4) gives the following expression for the chosen booking limits:

$$\eta^k = \max\{\eta \in [0, c] : \ 1 - \Phi_{(M^k, V^k)}(c - \eta) < \rho(\gamma^k)\} \,. \tag{3.4}$$

The booking limit is implemented as before: requests to purchase low-fare tickets are granted until the total number of low-fare tickets sold reaches the booking limit, at which point only full-fare tickets can be sold.

3.3 Necessary Conditions for Convergence

Does the sequence $\{\eta^k\}$ converge to a limit as $k \to \infty$? If so, what conditions need to be satisfied by the limit? Before we provide proofs of convergence under some specific conditions, we identify a general necessary condition needed for a convergence point. In many cases, simulations suggest that there is indeed convergence if the necessary condition holds. Below, we will describe the necessary condition.

Recall we have assumed that for each problem instance (each k) the actual arrival processes of type-H, L, and F customers are independent Poisson processes over a time horizon of length τ. This implies that the total numbers of type-H, L, and F customers that arrive over the time horizon in a single instance are Poisson-distributed, and λ_H is the parameter of the Poisson-distributed total number of type-H customers. Recall the quantity T^k is the total number of type-H customers that arrive in instance k, so the expectation and variance of T^k are both λ_H. It then follows that $M^k = k^{-1}\sum_{i=1}^k T^i \to \lambda_H$ and $V^k = \frac{1}{k-1}\sum_{i=1}^k (T^i - M^k)^2 \to \lambda_H$ as $k \to \infty$ with probability one by the strong law of large numbers. Consequently,

$$\Phi_{(M^k,V^k)}(c-\eta) \to \Phi_{(\lambda_H,\lambda_H)}(c-\eta) \text{ as } k \to \infty \tag{3.5}$$

for all $\eta \in [0, c]$ with probability one.

For $\gamma \in [0, 1]$, define

$$B(\gamma) = \max\{\eta \in [0, c] : 1 - \Phi_{(\lambda_H,\lambda_H)}(c-\eta) < \rho(\gamma)\}\,. \tag{3.6}$$

Intuitively, $B(\gamma)$ represents the booking limit (as a function of γ) that would be chosen in (3.4) if γ^k were equal to γ and if M^k and V^k were at their limiting values. Note also that $B(\gamma)$ is a non-increasing step function.

For $\eta \in [0, c]$ define

$$G(\eta) = \frac{E[R^1|\eta^0 = \eta]}{E[Q^1|\eta^0 = \eta]}\,. \tag{3.7}$$

To understand the quantity $G(\eta)$, recall that the revenue manager is using (2.7) to estimate the buy-up probability. If the revenue manager sets the booking limit to η for every instance k, then $\gamma^k \to G(\eta)$ as $k \to \infty$ w.p.1. That is, if the booking limits $\{\eta^k\}$ were to converge to η, then the buy-up probabilities would converge to $G(\eta)$.

Proposition 5 *Suppose that $1 - \Phi_{\lambda_H,\lambda_H}(c-\eta) \neq \rho(G(\eta))$ for all $\eta \in [0, c]$, and suppose that η^∞ is a random variable. Then $\eta^\infty = B(G(\eta^\infty))$ on almost all sample paths for which $\eta^k \to \eta^\infty$. In particular, if $\eta^k \to \eta^\infty$ w.p.1, then $\eta^\infty = B(G(\eta^\infty))$ w.p.1.*

On an intuitive level, if $\eta^k \to \eta^\infty$, then eventually $\eta^k = \eta^\infty$ (booking limits are integer valued) and consequently the estimate γ^k will be very close to $G(\eta^\infty)$. When k is large, the booking limit η^k will be $B(\gamma^k) \approx B(G(\eta^\infty))$. In the limit, it must be that $\eta^\infty = B(G(\eta^\infty))$; the booking limit yields a buy-up probability that in turn yields back the same booking limit. Note also that when $\eta^k \to \eta^\infty$, then $\gamma^k \to \gamma^\infty := G(\eta^\infty) = G(B(G(\eta^\infty)) = G(B(\gamma^\infty))$; that is if $\eta^k \to \eta^\infty$, then γ^k converges to γ^∞, which is a fixed point of the function $G(B(\cdot))$.

The fixed point condition is depicted in Figure 3.1. The dotted (blue) curve shows the buy-up probability $G(\eta)$ as a function of the booking limit η. The piecewise constant (red) curve shows the booking limit $B(\gamma)$ as a function of the buy-up probability γ. The booking limit, 61 in the example, where the curves coincide is a fixed point of $B(G(\cdot))$. The associated buy-up probability (0.709) is a fixed point of $G(B(\cdot))$. Figure 3.2 shows one sample path of the behavior of the booking limits and buy-up estimates for this example. The figure suggests that in this example, the booking limits and buy-up estimates converge to 61 and 0.709, respectively. We observe that the apparent convergence of $\{\eta^k\}$ occurs more quickly than that of $\{\gamma^k\}$. The occurs because there is range of values of γ that yield a booking limit of 61. This can be seen in Figure 3.1.

It is also possible that there is no fixed point of $B(G(\cdot))$ or multiple fixed points of $B(G(\cdot))$. If there is no fixed point, then it still may be that the curves $B(\gamma)$ and $G(\eta)$ "cross" each other; i.e., there may be an η' such that $\inf\{\gamma : B(\gamma) = \eta'-1\} > G(\eta'-1)$ and $\sup\{\gamma : B(\gamma) = \eta'\} \leq G(\eta')$. We refer to such an η' as a crossing point. Such a crossing point is shown in Figure 3.3. In this particular case, as shown in Figure 3.4, it appears the estimated buy-up rates γ^k converge. However, the booking limits do not converge. We know this because in this example the necessary condition for convergence of the booking limits, existence of a fixed point of $B(G(\cdot))$, does not hold.

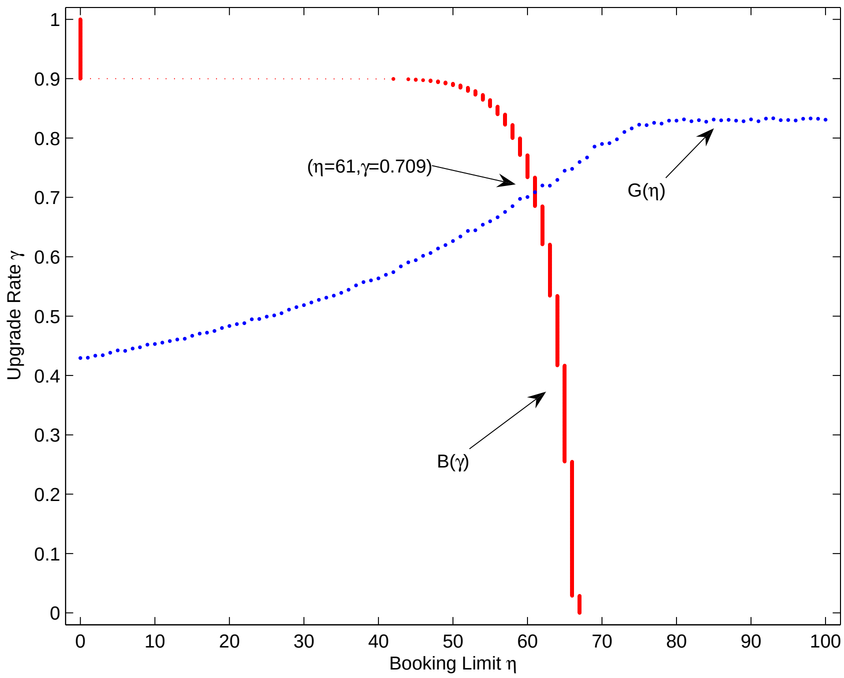
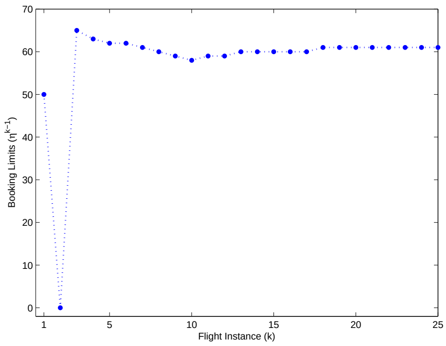
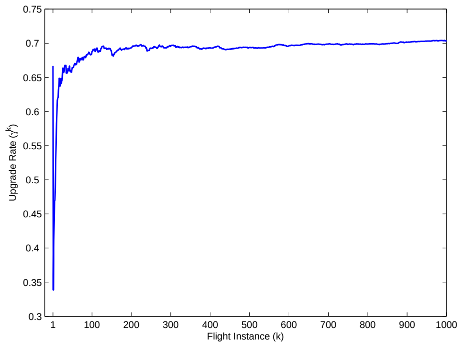


Figure 3.1: $B(\gamma)$ and $G(\eta)$ with One Fixed Point

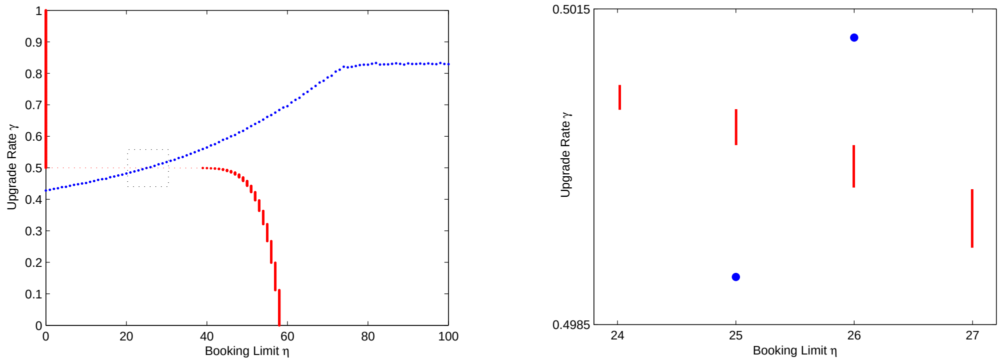


(a) Dynamics of Booking Limits



(b) Dynamics of Buy-up Rate Estimates

Figure 3.2: Dynamics of Booking Limits and Buy-up Rate Estimates with One Fixed Point



(a) $B(\gamma)$ and $G(\eta)$ with No Fixed Point

(b) Boxed Area of Enlarged to Show Crossing Point (Not to Scale)

Figure 3.3: $B(\gamma)$ and $G(\eta)$ with No Fixed Point

Nevertheless, we conjecture that the booking limits eventually oscillate between $\eta' - 1$ and η'. This is similar to the behavior described in part 2 of Proposition 1.

When multiple fixed points are present, the situation is more complex. Figure 3.5 displays an example in which there are two fixed points as well as one crossing point. Figures 3.6 and 3.7 show that different sample paths can converge to different limits (both are fixed points) even when starting from the same initial condition of $\eta^0 = 100$. Figures 3.6 and 3.7 also suggest the estimated buy-up rates γ^k converge to the fixed points of $G(B(\cdot))$. In this particular example, simulations suggest that the dynamics of the booking limits are such that they are "pushed away" from the crossing point, and we do not see sample paths with the just-mentioned oscillation around the crossing point. This can be viewed as a stochastic version of the behavior in the example shown in Figure 2.3.

A comparison of Figures 3.1 and 3.2 with Figures 3.5, 3.6 and 3.7 again emphasizes the important role played by the mapping from booking limits to parameter estimates. Here, this mapping, $G(\eta)$ is analogous to the mapping $\gamma(\eta)$ from Section 2.3. With an increasing $G(\eta)$ in Figure 3.1, there is a single fixed point of

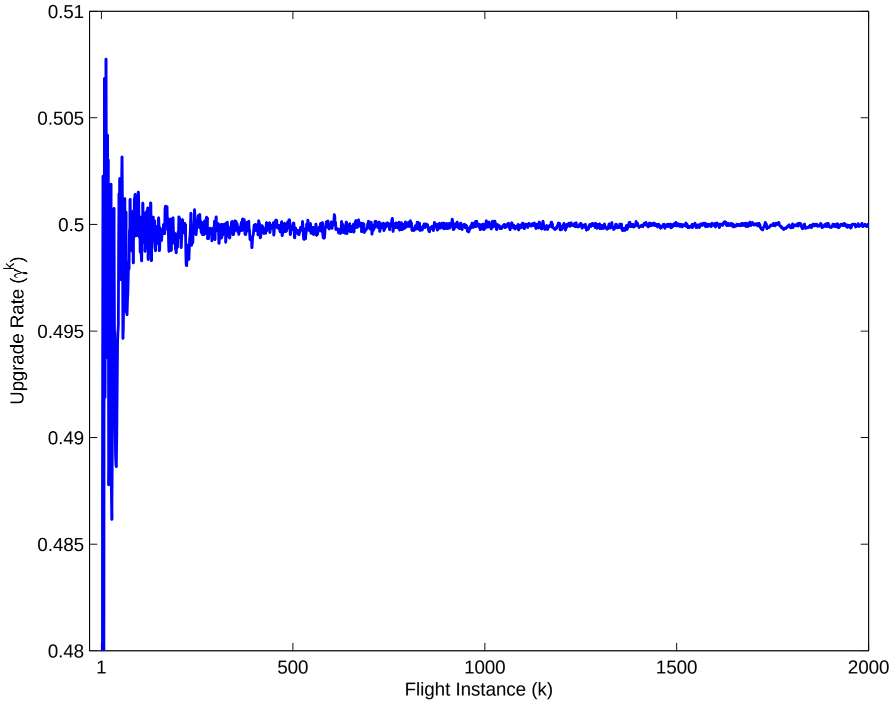


Figure 3.4: Dynamics of Buy-up Rate Estimates with No Fixed Point

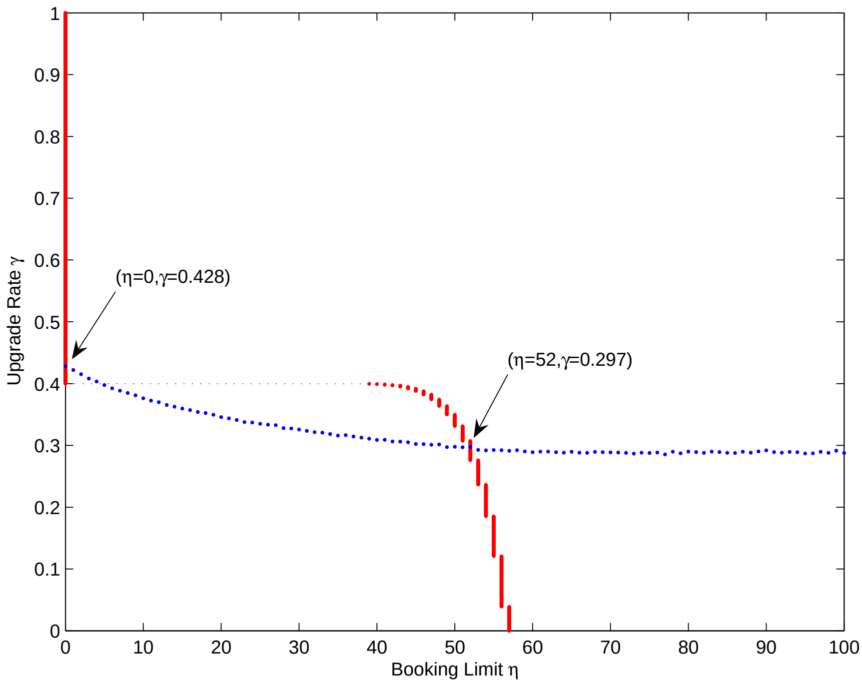
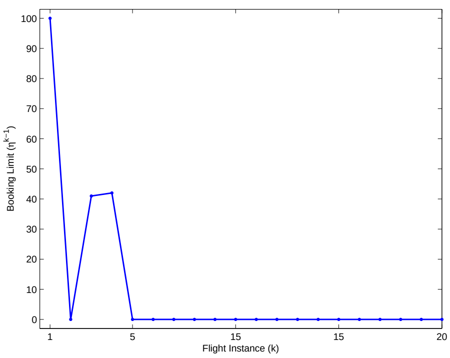
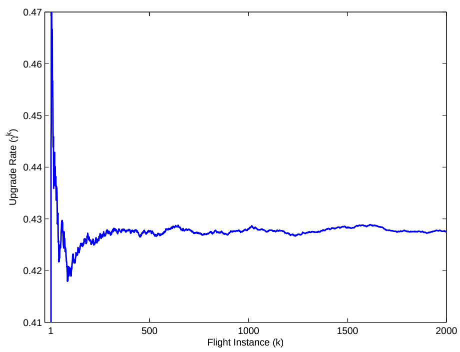


Figure 3.5: $B(\gamma)$ and $G(\eta)$ with Two Fixed Points

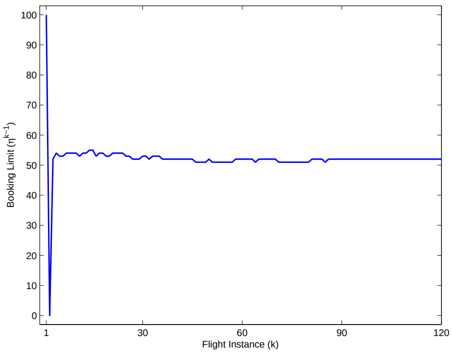


(a) Dynamics of Booking Limits

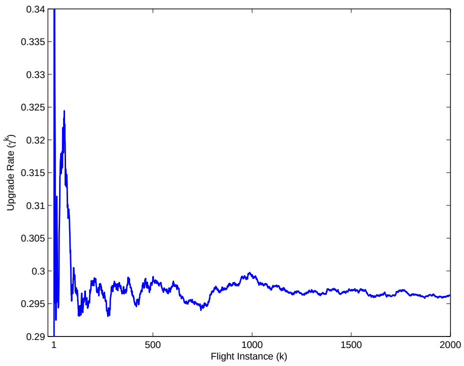


(b) Dynamics of Buy-up Rate Estimates

Figure 3.6: Dynamics of Booking Limits and Buy-up Rate Estimates with Two Fixed Points



(a) Limiting Behavior of Booking Limits



(b) Limiting Behavior of Gamma

Figure 3.7: Dynamics of Booking Limits and Buy-up Rate Estimates with Two Fixed Points (Another Sample Path)

$B(G(\cdot))$ and a single limiting booking limit, similar to what we saw in Figure 2.1 and in Proposition 1 for deterministic systems. One the other hand, Figure 3.5 shows a decreasing $G(\eta)$, in which case there can be different limits on different sample paths (see Figure 3.6 and 3.7). This is analogous to the behavior for deterministic systems shown in Figures 2.2 and 2.3 and described in Proposition 2. As in Section 2.3, the shape of the mapping $G(\eta)$ can be traced back to the pattern of arrivals. In Figures 3.1 and 3.2 the non-homogeneous Poisson processes that govern customer arrivals are such that flexible customers tend to arrive evenly throughout the booking horizon and customers willing only to buy low-fare tickets tend to arrive early in the horizon. Hence, most arrivals of flexible customers are late relative to the arrivals of inflexible low-fare-only customers. In Figures 3.5, 3.6 and 3.7 the flexible customers again arrive evenly, but inflexible low-fare-only customers tend to arrive late in the horizon. Hence, arrivals of flexible customers come early relative to the arrivals of low-fare customers. Complete descriptions of the arrival processes will be given shortly. Figures 3.1 and 3.2 correspond to $\alpha = 0.9$ in Table 3.1. Figures 3.5, 3.6 and 3.7 correspond to $\alpha = 0.4$ in Table 3.2.

3.4 Sufficient Conditions for Convergence

In this section, we develop some sufficient conditions for convergence of $\{\eta^k\}$ and $\{\gamma^k\}$. We consider two approaches. The first one uses results of the convergence of stochastic approximation algorithms and relies on the notion of a pseudo-contraction mapping. The second approach is based on a simple geometric argument in which the ranges of the mappings $G(\eta)$ and $B(\gamma)$ "shrink".

3.4.1 Stochastic Approximation Algorithms

Our goal now is to obtain sufficient conditions for convergence of $\{\eta^k\}$ and $\{\gamma^k\}$. To do so, we shall employ a convergence result from the theory of stochastic approximation algorithms. For simplicity, in this section (Stochastic Approximation Algorithms) we use $V^k = \frac{1}{k}\sum_{i=1}^{k}(T^i - M^k)^2$ instead of $V^k = \frac{1}{k-1}\sum_{i=1}^{k}(T^i - M^k)^2$ shown in (3.3). Let $L^k = (T^k)^2$ and

$$\widetilde{Q}^k = \frac{1}{k}\sum_{i=1}^{k} Q^i \tag{3.8}$$

$$\widetilde{R}^k = \frac{1}{k}\sum_{i=1}^{k} R^i \tag{3.9}$$

$$M^k = \frac{1}{k}\sum_{i=1}^{k} T^i \tag{3.10}$$

$$N^k = \frac{1}{k}\sum_{i=1}^{k} L^i \tag{3.11}$$

Then we have

$$\widetilde{Q}^{k+1} = (1 - \frac{1}{k+1})\widetilde{Q}^k + \frac{1}{k+1}Q^{k+1} \tag{3.12}$$

$$\widetilde{R}^{k+1} = (1 - \frac{1}{k+1})\widetilde{R}^k + \frac{1}{k+1}R^{k+1} \tag{3.13}$$

$$M^{k+1} = (1 - \frac{1}{k+1})M^k + \frac{1}{k+1}T^{k+1} \tag{3.14}$$

$$N^{k+1} = (1 - \frac{1}{k+1})N^k + \frac{1}{k+1}L^{k+1} \tag{3.15}$$

Let us define

$$B'(x, y, m, n) = \max\{\eta \in [0, c] :\ 1 - \Phi_{(m,(n-m^2))}(c - \eta) < \rho(y/x)\}.$$

Notice that $B'(x, y, \lambda_H, \lambda_H + \lambda_H^2) = B(y/x)$. We also define

$$q(x, y, m, n) = E[Q^{k+1}|\eta^k = B'(x, y, m, n)] \tag{3.16}$$

$$r(x, y, m, n) = E[R^{k+1}|\eta^k = B'(x, y, m, n)] \tag{3.17}$$

$$m(x, y, m, n) = E[T^{k+1}|\eta^k = B'(x, y, m, n)] = E[T^{k+1}] = \lambda_H \tag{3.18}$$

$$n(x, y, m, n) = E[L^{k+1}|\eta^k = B'(x, y, m, n)] = E[L^{k+1}] = \lambda_H + \lambda_H^2 \tag{3.19}$$

Notice $E[Q^{k+1}|\eta^k = \eta_0] = E[Q^1|\eta^0 = \eta_0]$ and $E[R^{k+1}|\eta^k = \eta_0] = E[R^1|\eta^0 = \eta_0]$ for any $\eta_0 \in [0, 1, ..., c]$ and any $k \geq 1$, i.e., these quantities do not depend upon k. Thus, $q(x, y, m, n)$ and $r(x, y, m, n)$ are well defined. Observe also that $E[T^k] = E[T^0]$ and $E[L^k] = E[L^0]$ for all $k \geq 1$, thus the functions $m(\cdot)$ and $n(\cdot)$ are constants. Note that $V^k = \frac{1}{k}\sum_{i=1}^k (T^i - M^k)^2 = \frac{1}{k}\{\sum_{i=1}^k (T^i)^2 - 2(\sum_{i=1}^k T^i) \times M^k + \sum_{i=1}^k (M^k)^2\} = \frac{1}{k}\{k \times N^k - 2k \times M^k \times M^k + k \times (M^k)^2\} = N^k - (M^k)^2$. Therefore

$$\begin{aligned}\eta^k &= \max\{\eta \in [0, c] :\ 1 - \Phi_{(M^k,V^k)}(c - \eta) < \rho(\gamma^k)\} \\ &= B'(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k).\end{aligned}$$

Thus $q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k) = E[Q^{k+1}|\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k] = E[Q^{k+1}|\gamma^k, M^k, N^k]$. Similarly, $r(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k) = E[R^{k+1}|\gamma^k, M^k, N^k]$.

Then, we can rewrite (3.12) – (3.15) as

$$\widetilde{Q}^{k+1} = (1 - \frac{1}{k+1})\widetilde{Q}^k + \frac{1}{k+1}\left\{q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k) + [Q^{k+1} - q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k)]\right\} \tag{3.20}$$

$$\widetilde{R}^{k+1} = (1 - \frac{1}{k+1})\widetilde{R}^k + \frac{1}{k+1}\left\{r(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k) + [R^{k+1} - r(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k)]\right\} \tag{3.21}$$

$$M^{k+1} = (1 - \frac{1}{k+1})M^k + \frac{1}{k+1}\left\{m(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k) + [T^{k+1} - m(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k)]\right\} \tag{3.22}$$

$$N^{k+1} = (1 - \frac{1}{k+1})N^k + \frac{1}{k+1}\left\{n(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k) + [L^{k+1} - n(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k)]\right\} \tag{3.23}$$

We will write the above expressions in a vector form. Let

$$\widetilde{W}^k = (\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k) \text{ and } W^k = (Q^k, R^k, T^k, L^k) \tag{3.24}$$

and define

$$\mathcal{H}(x, y, m, n) = (q(x, y, m, n), r(x, y, m, n), m(x, y, m, n), n(x, y, m, n)) \tag{3.25}$$

The recursions (3.12) - (3.15) can now be written as

$$\widetilde{W}^{k+1} = (1 - \frac{1}{k+1})\widetilde{W}^k + \frac{1}{k+1}\left(H(\widetilde{W}^k) + [W^{k+1} - \mathcal{H}(\widetilde{W}^k)]\right), \quad k = 1, 2, ... \tag{3.26}$$

The recursion (3.26) has the standard form of a stochastic approximation recursion: see, e.g., Bertsekas and Tsitsiklis (1996). Under some conditions that we describe below, we will show that the four-dimensional vector $\widetilde{W}^k$ converges w.p.1 to a fixed point of the mapping $H(\cdot)$.

Next, we review the property of *weighted maximum norm pseudo-contraction.* Given any positive vector $\xi \in \Re^n$, let the *weighted maximum norm* $||\cdot||_\xi$ of a vector

$r \in \Re^n$ be

$$||r||_\xi = \max_i \frac{|r(i)|}{\xi_i}. \tag{3.27}$$

When all components of ξ are equal to 1, the resulting norm is the usual *maximum norm*, typically denoted by $||\cdot||_\infty$. A function $F : \Re^n \mapsto \Re^n$ is a weighted maximum norm pseudo-contraction if there exists some $r^* \in \Re^n$, a positive vector $\xi = (\xi_1, \xi_2 \ldots \xi_n) \in \Re^n$, and a constant $\beta \in [0, 1)$ such that

$$||Fr - r^*||_\xi \leq \beta ||r - r^*||_\xi, \ \ \forall\, r. \tag{3.28}$$

Observe that if such an r^* exists, then it is the unique fixed point of F. See Lee et al. (2009) for example of a proof of convergence using pseudo-contraction.

In componential notation, the pseudo-contraction condition (3.28) can be written as

$$\frac{|(Fr)(i) - r^*(i)|}{\xi_i} \leq \beta \max_j \frac{|r(j) - r^*(j)|}{\xi_j}, \ \ \forall\, i, r. \tag{3.29}$$

Above Fr denotes the function $f(\cdot)$ evaluated at r, and $(Fr)(i)$ denotes the i-th component of Fr.

In our particular case, the mapping $\mathcal{H}(\cdot)$ in (3.25) is weighted norm pseudo-contraction if there is $\widetilde{W}^* = (\widetilde{Q}^*, \widetilde{R}^*, M^*, N^*)$, $\xi \in \Re^4$, and $\beta \in [0, 1)$ such that for any (x, y, m, n) in $\Re^4$, we have

$$\frac{|q(x,y,m,n) - \widetilde{Q}^*|}{\xi_1} \leq \beta \max\{\frac{|x - \widetilde{Q}^*|}{\xi_1}, \frac{|y - \widetilde{R}^*|}{\xi_2}, \frac{|m - M^*|}{\xi_3}, \frac{|n - N^*|}{\xi_4}\} \tag{3.30}$$

$$\frac{|r(x,y,m,n) - \widetilde{R}^*|}{\xi_2} \leq \beta \max\{\frac{|x - \widetilde{Q}^*|}{\xi_1}, \frac{|y - \widetilde{R}^*|}{\xi_2}, \frac{|m - M^*|}{\xi_3}, \frac{|n - N^*|}{\xi_4}\} \tag{3.31}$$

$$\frac{|m(x,y,m,n) - M^*|}{\xi_3} \leq \beta \max\{\frac{|x - \widetilde{Q}^*|}{\xi_1}, \frac{|y - \widetilde{R}^*|}{\xi_2}, \frac{|m - M^*|}{\xi_3}, \frac{|n - N^*|}{\xi_4}\} \tag{3.32}$$

$$\frac{|n(x,y,m,n) - N^*|}{\xi_4} \leq \beta \max\{\frac{|x - \widetilde{Q}^*|}{\xi_1}, \frac{|y - \widetilde{R}^*|}{\xi_2}, \frac{|m - M^*|}{\xi_3}, \frac{|n - N^*|}{\xi_4}\} \tag{3.33}$$

To clarify, (3.30) – (3.33) are exactly (3.29), or equivalently (3.28), for our problem.

Notice that it must be that $M^* = m(x,y,m,n) = \lambda_H$ and $N^* = n(x,y,m,n) = \lambda_H^2 + \lambda_H$. This can be proved using the following argument. If $M^* \neq \lambda_H$, then with $(x,y,m,n) = (\widetilde{Q}^*, \widetilde{R}^*, M^*, N^*)$, we know that the right-hand side of (3.32) is zero but the left-hand side of (3.32) is strictly positive, which would give a contradiction.

Now we state the main result of this section.

Proposition 6 *Suppose $\mathcal{H}(\cdot) : \Re^4 \to \Re^4$ defined in (3.25) is a weighted norm pseudo-contraction with a fixed point $\widetilde{W}^* = (\widetilde{Q}^*, \widetilde{R}^*, \lambda_H, \lambda_H^2 + \lambda_H)$. Then $\widetilde{W}^k \to \widetilde{W}^*$ w.p.1. Moreover, $\gamma^k \to \gamma^*$ w.p.1, where $\gamma^* = \widetilde{R}^*/\widetilde{Q}^*$. Furthermore, if $B(\gamma)$ is continuous at γ^*, then $\eta^k \to \eta^\infty := B(G(\gamma^*))$ w.p.1.*

Proof. We will first show $\widetilde{W}^k$ converges to $\widetilde{W}^*$. In view of $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$, $\sum_{k=1}^{\infty} (\frac{1}{k})^2 < \infty$, it suffices by Proposition 4.4 of Bertsekas and Tsitsiklis (1996) (see below for a statement of the proposition) to prove that the noise terms $W^{k+1} - \mathcal{H}(\widetilde{W}^k)$ satisfy:

(1) For every i and k, we have $E[W^{k+1}(i) - \mathcal{H}\widetilde{W}^k(i) \mid \mathcal{F}_k] = 0$.

(2) There exist constants A and B such that $E[(W^{k+1} - \mathcal{H}(\widetilde{W}^k))^2(i) \mid \mathcal{F}_k] \leq A + B||\widetilde{W}^k||^2$, $\forall\, i, k$ for some norm $||\cdot||$ on $\Re^4$.

We will now show that (1) and (2) above hold.

Since T^i is the number of type-H customers (which is Poisson distributed with parameter λ_H) in instance i, we have $E[T^{k+1}|\mathcal{F}_k] = \lambda_H$ and $E[L^{k+1}|\mathcal{F}_k] = E[(T^{k+1})^2|\mathcal{F}_k] = \lambda_H + \lambda_H^2$. Furthermore, from (3.16) and (3.17) we know $E[R^{k+1}|\mathcal{F}_k] = r(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k)$ and $E[Q^{k+1}|\mathcal{F}_k] = q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k)$, thus $E[W^{k+1}(i) - \mathcal{H}\widetilde{W}^k(i) \mid \mathcal{F}_k] = 0$. Therefore condition (1) holds.

Turning to condition (2), we have $E[(T^{k+1} - \lambda_H)^2|\mathcal{F}_k] = \lambda_H < \infty$ and $E[((T^{k+1})^2 -$

$\lambda_H^2 - \lambda_H)^2|\mathcal{F}_k] \leq E[(T^{k+1})^4] := A^0 < \infty$. Let $A' = \max\{\lambda_H, A^0\}$. We also have

$$
\begin{aligned}
& E[(Q^{k+1} - q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k))^2|\mathcal{F}_k] \\
& = E[(Q^{k+1})^2 - 2Q^{k+1}q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k) + q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k)^2|\mathcal{F}_k] \\
& = E[(Q^{k+1})^2|\mathcal{F}_k] - 2q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k)E[Q^{k+1}|\mathcal{F}_k] + q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k)^2 \\
& = E[(Q^{k+1})^2|\mathcal{F}_k] - q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k)^2 \\
& \leq E[(Q^{k+1})^2|\mathcal{F}_k].
\end{aligned}
$$

As Q^{k+1} is the number of customers who ask for low-fare tickets but are rejected, the value of Q^{k+1} cannot be bigger than the total number of type-L and type-F customers that arrive in instance $k+1$, say N_L^{k+1} and N_F^{k+1}. Therefore, $E[(Q^{k+1})^2|\mathcal{F}_k] \leq E[(N_L^{k+1} + N_F^{k+1})^2|\mathcal{F}_k] = E[(N_L^{k+1} + N_F^{k+1})^2] = (\lambda_L + \lambda_F) + (\lambda_L + \lambda_F)^2 := A''$; that is $E[(Q^{k+1} - q(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k))^2|\mathcal{F}_k] \leq A''$. Thus,, $E[(R^{k+1} - r(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k))^2|\mathcal{F}_k] \leq A'''$ for some large number A''' analogously. Thus, $E[(W^{k+1} - \mathcal{H}(\widetilde{W}^k))^2(i) \mid \mathcal{F}_k] \leq A := \max\{A', A'', A'''\}$, and condition (2) holds. Therefore, from Proposition 4.4 of Bertsekas and Tsitsiklis (1996), we know that $\widetilde{W}^k$ converges to $\widetilde{W}^*$ w.p.1.

We next show that $\widetilde{Q}^* > 0$. Suppose for a contradiction that $\widetilde{Q}^* = 0$. Since $R^k \leq Q^k$ for all k, we have $\widetilde{R}^* \leq \widetilde{Q}^* = 0$, that is $\widetilde{W}^* = (0, 0, \lambda_H, \lambda_H^2 + \lambda_H)$. We will show $(0, 0, \lambda_H, \lambda_H^2 + \lambda_H)$ cannot be a fixed point of the pseudo-contraction $H(\cdot)$. Since $\lambda_F > 0$, we can find ϵ such that $0 < \epsilon < \lambda_F$. Consider the point $\widetilde{W} = (\epsilon, \epsilon, \lambda_H, \lambda_H^2 + \lambda_H)$. By definition of $B'(\cdot)$ we have that $B'(\epsilon, \epsilon, \lambda_H, \lambda_H^2 + \lambda_H) = 0$; that is the booking limit given by $(\epsilon, \epsilon, \lambda_H, \lambda_H^2 + \lambda_H)$ is 0 because $\rho(\epsilon/\epsilon) = -\infty$ and $\max \varnothing = 0$. Thus we know $r(\epsilon, \epsilon, \lambda_H, \lambda_H^2 + \lambda_H) = \lambda_F > \epsilon$ and $q(\epsilon, \epsilon, \lambda_H, \lambda_H^2 + \lambda_H) = \lambda_F + \lambda_H > \epsilon$. Moreover,

$$\max_j \frac{|\widetilde{W}(j) - \widetilde{W}^*(j)|}{\xi(j)} = \max_j\{\frac{\epsilon}{\xi_1}, \frac{\epsilon}{\xi_2}, \frac{\lambda_H - \lambda_H}{\xi_3}, \frac{\lambda_H + \lambda_H^2 - \lambda_H - \lambda_H^2}{\xi_4}\} = \max\{\frac{\epsilon}{\xi_1}, \frac{\epsilon}{\xi_2}\};$$

while

$$\frac{|(q(\epsilon, \epsilon, \lambda_H, \lambda_H^2 + \lambda_H) - 0|}{\xi_1} > \frac{\epsilon}{\xi_1} \text{ and } \frac{|(r(\epsilon, \epsilon, \lambda_H, \lambda_H^2 + \lambda_H) - 0|}{\xi_2} > \frac{\epsilon}{\xi_2}.$$

Therefore, there are no ξ_1 and ξ_2, such that

$$\begin{aligned}
&\frac{|q(\epsilon,\epsilon,\lambda_H,\lambda_H^2+\lambda_H)-0|}{\xi_1} \le \beta\max\{\frac{\epsilon}{\xi_1},\frac{\epsilon}{\xi_2}\} \\
\text{and}\quad &\frac{|r(\epsilon,\epsilon,\lambda_H,\lambda_H^2+\lambda_H)-0|}{\xi_2} \le \beta\max\{\frac{\epsilon}{\xi_1},\frac{\epsilon}{\xi_2}\},
\end{aligned}$$

where $0 \le \beta < 1$. Thus, (3.30) – (3.33) cannot hold, giving us a contradiction. Hence, we have proved that $\widetilde{Q}^* > 0$.

Let us define $\gamma^* = \widetilde{R}^*/\widetilde{Q}^*$. Since $R^k \to \widetilde{R}^*$ and $Q^k \to \widetilde{Q}^*$, it follows that $\gamma^k \to \gamma^*$. Since $M^k \to \lambda_H$ and $N^k \to \lambda_H^2+\lambda_H$, then $1-\Phi_{(M^k,(N^k-(M^k)^2)}(c-\eta) \to 1-\Phi_{(\lambda_H,\lambda_H)}(c-\eta)$ for all $\eta \in [0,c]$. Moreover, define $\eta^\infty = B(\gamma^*)$. Since $\gamma^k \to \gamma^*$ w.p.1 and $B(\gamma)$ is continuous at γ^*, we know $\eta^k \to \eta^\infty$ w.p.1. It follows from Proposition 5 (necessary condition) that $\gamma^* = G(B(\gamma^*))$ and $\eta^\infty = B(G(\eta^\infty))$.

□

For convenience, we provide the following from Bertsekas and Tsitsiklis (1996). The result and proof was adapted from Bertsekas (1982) and Tsitsiklis (1994), and an informal proof of the convergence was provided in Jaakkola et al. (1994).

Proposition 7 (Proposition 4.4 in Bertsekas and Tsitsiklis (1996)) *Let r_k be the sequence generated by the iteration*

$$r_{k+1}(i) = (1-\gamma_k(i))r_k(i) + \gamma_k(i)((\mathcal{H}r_k)(i)+\omega_k(i)), \quad k=0,1,...; i=1,2,... \tag{3.34}$$

where $r_k(i)$ is the ith component of r_k. Assume that:

(a) The stepsizes $\gamma_k(i)$ are nonnegative and satisfy

$$\sum_{k=0}^{\infty}\gamma_k(i) = \infty, \qquad \sum_{k=0}^{\infty}\gamma_k^2(i) < \infty.$$

(b) The noise terms $\omega_k(i)$ satisfy:

(b.1) For every i and k, we have $E[\omega_k(i) \mid \mathcal{F}_k] = 0$.

(b.2) Given norm $||\cdot||$ on $\Re^n$, there exist constants A and B such that $E[\omega_k^2(i) \mid \mathcal{F}_k] \leq A + B||r_k||^2$, $\forall\, i, k$

(c) The mapping $\mathcal{H}$ is a weighted maximum norm pseudo-contraction with point r^*. Then, r_k converges to r^*, w.p.1.

Next, we will develop sufficient conditions for $\mathcal{H}(\cdot)$ to be a weighted maximum norm pseudo-contraction. These sufficient conditions are easier to check than (3.30) – (3.33). The basic idea is to reduce checking (3.30) – (3.33) for all (x, y, m, n) to checking some simple conditions for all (x, y).

For $\epsilon > 0$, we let

$$S'(\epsilon) = \{(x, y, m, n) : |m - \lambda_H| < \epsilon \text{ and } |n - \lambda_H - \lambda_H^2| < \epsilon^2\}.$$

If $|m - \lambda_H| < \epsilon$ and $|n - \lambda_H - \lambda_H^2| < \epsilon^2$, then $-\lambda_H^2 - \epsilon^2 - 2\lambda_H\epsilon < -m^2 < -\lambda_H^2 - \epsilon^2 + 2\lambda_H\epsilon$ and $\lambda_H + \lambda_H^2 - \epsilon^2 < n < \lambda_H + \lambda_H^2 + \epsilon^2$. Thus $\lambda_H - 2\epsilon^2 - 2\epsilon\lambda_H < n - m^2 < \lambda_H + 2\epsilon\lambda_H < \lambda_H + 2\epsilon^2 + 2\epsilon\lambda_H$, which implies $|n - m^2 - \lambda_H| < 2\epsilon^2 + 2\epsilon\lambda_H$. That is, if $(x, y, m, n) \in S'(\epsilon)$, then $|m - \lambda_H| < \epsilon$ and $|n - m^2 - \lambda_H| < 2\epsilon^2 + 2\epsilon\lambda_H$.

Lemma 8 *There exists $\epsilon > 0$ (that does **not** depend on (x, y, m, n)) such that*

$$|B'(x, y, \lambda_H, \lambda_H + \lambda_H^2) - B'(x, y, m, n)| \leq 1 \text{ for any } x \text{ and } y \tag{3.35}$$

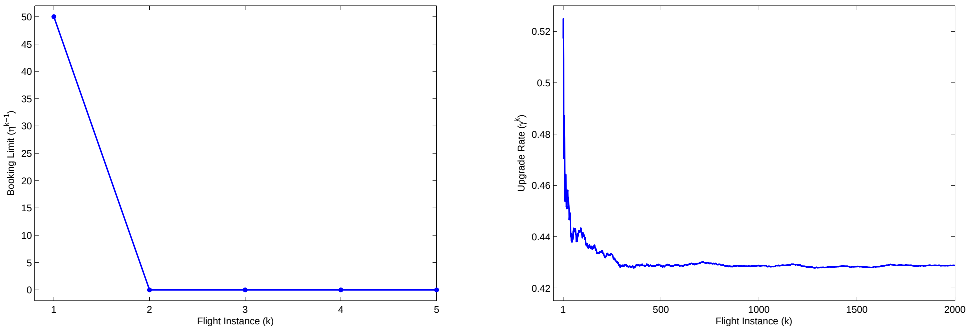
if $|m - \lambda_H| < \epsilon$ and $|n - m^2 - \lambda_H| < 2\epsilon^2 + 2\epsilon\lambda_H$.

Let us define

$$\hat{q}(x, y) = q(x, y, \lambda_H, \lambda_H^2 + \lambda_H) \tag{3.36}$$

$$\hat{r}(x, y) = r(x, y, \lambda_H, \lambda_H^2 + \lambda_H) \tag{3.37}$$

Proposition 8 *Suppose there exists ξ_1 and ξ_2, such that for all $(x, y, m, n) \in S'(\epsilon)$ where ϵ is defined in Lemma 8, we have*



(a) Dynamics of Booking Limits

(b) Dynamics of Buy-up Rate Estimates

Figure 3.8: Dynamics of Booking Limits and Buy-up Rate Estimates with One Fixed Point

(a) $\frac{|\hat{q}(x,y)-\widetilde{Q}^*|}{\xi_1} \leq \beta \max\left(\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}\}\right) and \frac{|\hat{r}(x,y)-\widetilde{R}^*|}{\xi_2} \leq \beta \max\left(\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}\}\right)$

if $|B'(x,y,\lambda_H,\lambda_H+\lambda_H^2) - B'(x,y,m,n)| = 0$*;*

(b) $\frac{|\hat{q}(x,y)-\widetilde{Q}^*|+1}{\xi_1} \leq \beta \max\left(\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}\}\right) and \frac{|\hat{r}(x,y)-\widetilde{R}^*|+1}{\xi_2} \leq \beta \max\left(\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}\}\right)$

if $|B'(x,y,\lambda_H,\lambda_H+\lambda_H^2) - B'(x,y,m,n)| = 1$.

Then $\mathcal{H}(\cdot)$ *is a weighted maximum norm pseudo-contraction with a fixed point* $\widetilde{W}^* = (\widetilde{Q}^*, \widetilde{R}^*, \lambda_H, \lambda_H^2 + \lambda_H)$.

Now, we show an example in which the conditions in Proposition 8 hold. Suppose again that there are $c = 100$ seats and that for each fixed k, customers of each of the three types arrive according to non-homogeneous Poisson processes on $[0, \tau]$ with $\tau = 100$. Type-H customers have arrival rate $\lambda_{\mathrm{H}}(t) = 0.008t$, type-L customers have arrival rate $\lambda_{\mathrm{L}}(t) = 0.8 - 0.008t$, and type-F have arrival rate $\lambda_{\mathrm{F}}(t) = 0.3$ for $t \in [0, \tau]$. Suppose the ratio of prices satisfies $\alpha = 0.1$. With parameters $\xi_1 = 1$ and $\xi_2 = 2.5$, it can be verified that the conditions in Proposition 8 hold. As guaranteed by Proposition 6, Figure 3.8 shows the convergence of η^k and γ^k.

To close this section, we should point out the limitations of the preceding ideas. In particular, the pseudo-contraction mapping approach is applicable only when there is a single fixed point. This simply follows from the fact that if a mapping is a pseudo-contraction, then it has at most one fixed point, as we have already mentioned earlier.

3.4.2 Another Approach to Prove Convergence

We begin our study in this section with a specific example. Suppose there are $c = 100$ seats. For each fixed k, customers of each of the three types arrive according to non-homogeneous Poisson processes on $[0, \tau]$ with $\tau = 100$. Type-L customers have arrival rate $\lambda_{\mathrm{L}}(t) = 0.8 - 0.008t$, type-H customers have arrival rate $\lambda_{\mathrm{H}}(t) = 0.008t$, and type-F customers have arrival rate $\lambda_{\mathrm{F}}(t) = 0.3$, and the ratio of prices satisfies $\alpha = 0.9$. Here, type-L customers tend to arrive early in the horizon, type-H customers tend to arrive late in the horizon. Arrivals of type-F customers are late relative to arrivals of type-L customers.

As before, we let

$$B_{(m,v)}(\gamma) = \max\{\eta \in [0, c] : 1 - \Phi_{(m,v)}(c - \eta) < \rho(\gamma)\} \qquad \text{for } \gamma \in [0, 1] \tag{3.38}$$

$$B(\gamma) = B_{(\lambda_H,\lambda_H)}(\gamma) = \max\{\eta \in [0, c] : 1 - \Phi_{(\lambda_H,\lambda_H)}(c - \eta) < \rho(\gamma)\} \qquad \text{for } \gamma \in [0, 1] \tag{3.39}$$

$$G(\eta) = \frac{E[R^1|\eta^0 = \eta]}{E[Q^1|\eta^0 = \eta]} \qquad \text{for } \eta \in [0, c]\,. \tag{3.40}$$

The explanations of $B(\gamma)$ and $G(\beta)$ can be found after (3.6) and (3.7). Moreover, $B(\gamma)$ is non-increasing in γ, and in this specific setting $G(\eta)$ is non-decreasing in η. In Figure 3.9, the (red) step function shows $B(\gamma)$, and the dotted curve shows $G(\eta)$. We will first show that in this special case, the booking limit will converge to 61, which is the point at which the two curves $B(\cdot)$ and $G(\cdot)$ meet each other in Figure 3.9.

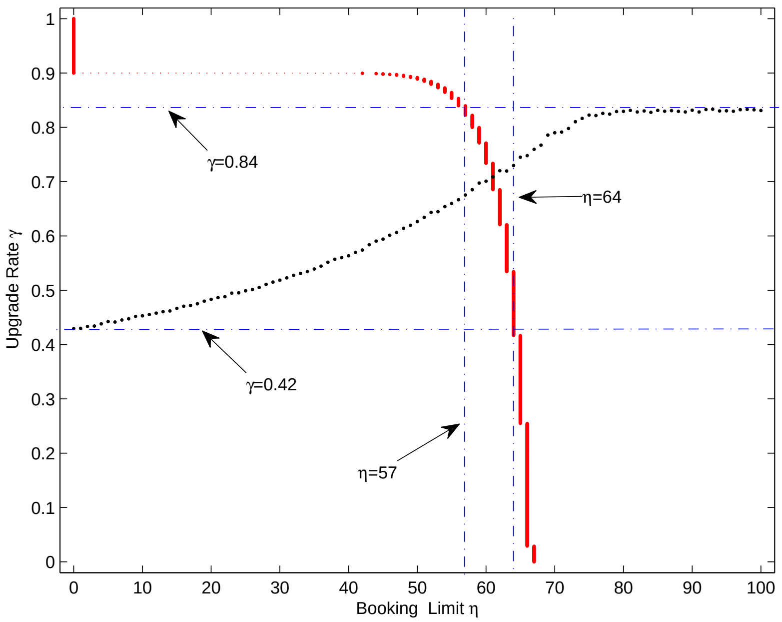
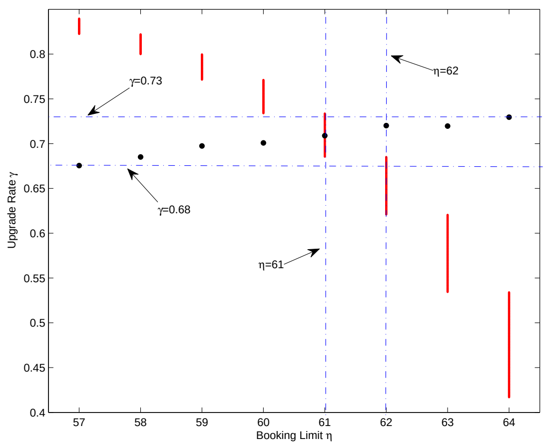
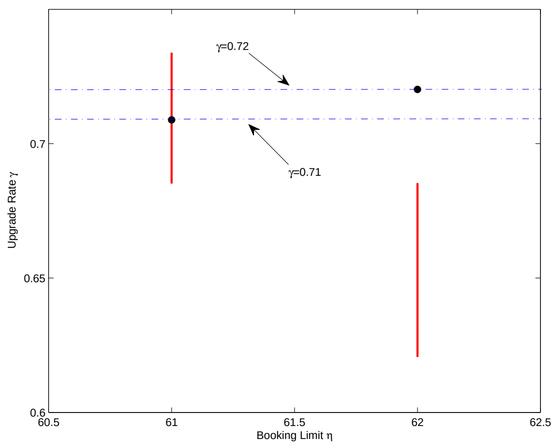


Figure 3.9: $B(\gamma)$ and $G(\eta)$ with One Fixed Point



(a) Second "Iteration"



(b) Final "Iteration"

Figure 3.10: "Iterations" of $B(\gamma)$ and $G(\eta)$ with One Fixed Point

Before we prove above convergence, we first show that if M^k and V^k are close to their limiting points λ_H and λ_H, then under some mild conditions, the booking limits given by (M^k, V^k) and (λ_H, λ_H) are identical.

Lemma 9 *Suppose $\gamma \in [0,1]$ is such that there is no $\eta \in [0,c]$ that $1 - \Phi_{(\lambda_H,\lambda_H)}(c - \eta) = \rho(\gamma)$. Then, there exists $\epsilon > 0$ such that if $|m - \lambda_H| < \epsilon$ and $|v - \lambda_H| < \epsilon$, then $B_{(m,v)}(\gamma) = B_{(\lambda_H,\lambda_H)}(\gamma)$.*

Now, we provide the general idea for the proof of convergence by considering the example depicted in Figure 3.9. We know that $M^k \to \lambda_H$ and $V^k \to \lambda_H$ as $k \to \infty$ w.p.1. By Lemma 9, under some mild conditions, in the long run the booking limits given by $B_{(M^k,V^k)}(\gamma)$ and by $B(\gamma)$ will be the same. Therefore, by analyzing $B(\gamma^k)$, we will know the long run behavior of the booking limits. We will show the booking limits will always be selected from a certain set, say $S_1 = [0, 1, \ldots, c]$. Since the booking limits are selected from this set, the buy-up rate estimates will eventually belong to a particular set, say T_1. We know that the booking limits will be selected roughly by a function of this set T_1, composing another set, say S_2; and this $S_2 \subseteq S_1$ will suggest another range of upgrade rates, say T_2. This pattern continues. If these sets $\{S_i\}$ satisfy $S_{i+1} \subseteq S_i$ and eventually shrink to one point, say η^*, then the booking limits will also converge to this η^*.

Now we discuss the convergence of the booking limits in the specific example above in more detail. Since $G(\eta)$ is non-decreasing in η and booking limits can only be chosen from [0,1,...,100], as shown in figure 3.9, we know that $\{\gamma^k\}$ will satisfy that $0.42 = G(0) \leq \liminf \gamma^k \leq \limsup \gamma^k \leq G(100) = 0.82$ w.p.1. Therefore, w.p.1, for any sequence of $\{(\eta^{k-1}, \gamma^k)\}$, there exists K depending on the sequence $\{(\eta^{k-1}, \gamma^k)\}$ such that $\gamma^k \in [0.419, 0.841]$ for all $k \geq K$. In this case, we also have $1 - \Phi_{(\lambda_H,\lambda_H)}(c - \eta) \neq \rho(0.419)$ and $1 - \Phi_{(\lambda_H,\lambda_H)}(c - \eta) \neq \rho(0.821)$ for all η. Recall that

$$\eta^k = B'(\widetilde{Q}^k, \widetilde{R}^k, M^k, N^k) = B_{(M^k,V^k)}(\gamma^k)$$

where $V^k = N^k - (M^k)^2$ and $\gamma^k = \widetilde{Q}^k/\widetilde{R}^k$. Note that $B_{(m,v)}(\gamma)$ is non-increasing in γ. Hence for k large enough, we have

$$B_{(M^k,V^k)}(0.821) \le \eta^k \le B_{(M^k,V^k)}(0.419). \tag{3.41}$$

It follows from (3.41), Lemma 9, and the fact that $M^k \to \lambda_H$, $V^k \to \lambda_H$ w.p.1 that

$$57 = B_{(\lambda_H,\lambda_H)}(0.821) \le \eta^k \le B_{(\lambda_H,\lambda_H)}(0.419) = 64$$

for k large enough w.p.1. That is $\eta^k \in [57, 58, \ldots, 64]$ for k sufficiently large.

In the next step, we will apply the same logic above. Since booking limits eventually can only be chosen from [57, 58, 59, 60, 61, 62, 63, 64] as shown in figure 3.10a, it follows that $\gamma^k \in [G(57) - \epsilon, G(64) + \epsilon] = [0.68 - \epsilon, 0.73 + \epsilon]$ for any $\epsilon > 0$ and k sufficiently large. As above, with $\epsilon = 0.001$ we know that eventually the booking limits can only be chosen from $\{B(\gamma) : 0.679 \le \gamma \le 0.731\} = [61, 62]$.

In the end, since booking limits will eventually be chosen from [61, 62], as shown in figure 3.10b, $\gamma^k \in [G(61) - \epsilon, G(62) + \epsilon] = [0.71 - \epsilon, 0.72 + \epsilon]$ for any $\epsilon > 0$ and k large enough. We know that there exists $\epsilon > 0$, such that, by Lemma 9, $\eta^k = B_{(M^k,V^k)}(0.71 - \epsilon) = B_{(M^k,V^k)}(0.72 + \epsilon) = B(0.71 - \epsilon) = B(0.72 + \epsilon) = 61$ for k large enough because $M^k \to \lambda_H$ and $V^k \to \lambda_H$. This proves that $\eta^k \to 61$ w.p.1. The convergence of the buy-up rates γ^k to $G(61) = 0.71$ follows from the convergence of booking limits to 61.

Proposition 9 *Consider two sequences* $(\underline{\beta}_1, \overline{\beta}_1), (\underline{\beta}_2, \overline{\beta}_2), \ldots$ *and* $(\underline{\delta}_1, \overline{\delta}_1), (\underline{\delta}_2, \overline{\delta}_2), \ldots$ *such that*

(i) $\underline{\beta}_1 = 0$ *and* $\overline{\beta}_1 = c$*;*

(ii) $\underline{\delta}_i = \min\{G(\eta) : \eta \in [\underline{\beta}_i, \overline{\beta}_i]\}$ *and* $\overline{\delta}_i = \max\{G(\eta) : \eta \in [\underline{\beta}_i, \overline{\beta}_i]\}$ *for* $i = 1, 2 \ldots$*;*

(iii) $\underline{\beta}_{i+1} = B(\overline{\delta}_i)$ *for* $i = 1, 2 \ldots$;

(iv) $\overline{\beta}_{i+1} = \lim_{\delta \to \overline{\delta}_i -} B(\delta)$ *for* $i = 1, 2 \ldots$;

If there exists an n *such that* $\underline{\beta}_{n+1} = B(\overline{\delta}_n) = B(\underline{\delta}_n) = \overline{\beta}_{n+1}$*, then* $\lim_{k\to\infty} \eta^k = \underline{\beta}_{n+1} = \overline{\beta}_{n+1}$ *w.p.1.*

Proof. Before we prove the result, we first point out some facts which will be important to understand the proof. That is, $B(\gamma)$ is a piecewise constant function that is right-continuous and has left limits. There are finitely many jumps of $B(\gamma)$ and all are of size one.

We have $\underline{\beta}_1 = 0$ and $\overline{\beta}_1 = c$. Therefore, we know that $\{\gamma^k\}$ will satisfy that $\underline{\delta}_1 \leq \liminf \gamma^k \leq \limsup \gamma^k \leq \overline{\delta}_1$ w.p.1. Therefore, w.p.1, for any sequence of $\{(\eta^{k-1}, \gamma^k)\}$ and for any $\epsilon', \epsilon'' > 0$, there exists K depending on ϵ', ϵ'' and the sequence $\{(\eta^{k-1}, \gamma^k)\}$ such that $\gamma^k \in [\underline{\delta}_1 - \epsilon', \overline{\delta}_1 + \epsilon'']$ for all $k \geq K$. We will show the booking limits given from $[\underline{\delta}_1 - \epsilon', \overline{\delta}_1 + \epsilon'']$ are bounded by some functions of $\underline{\delta}_1$ and $\overline{\delta}_1$.

As above, we know for $\forall\ \epsilon' > 0$, $\epsilon'' > 0$ and for k large enough, we have

$$\underline{\delta}_1 - \epsilon' \leq \gamma^k \leq \overline{\delta}_1 + \epsilon''.$$

From the definition of η^k, we also know that, for k large enough

$$B_{(M^k,V^k)}(\overline{\delta}_1 + \epsilon'') \leq \eta^k \leq B_{(M^k,V^k)}(\underline{\delta}_1 - \epsilon') \tag{3.42}$$

Consider the left-hand side of (3.42). Suppose ϵ'' is such that $1 - \Phi_{(\lambda_H,\lambda_H)}(c - \eta) \neq \rho(\overline{\delta}_1 + \epsilon'')$ for all $\eta \in [0, c]$ and $B(\overline{\delta}_1 + \epsilon'') = B(\overline{\delta}_1)$. Then by (3.42), Lemma 9, and the fact that $M^k \to \lambda_H$, $V^k \to \lambda_H$ w.p.1, for k large enough, we have

$$\eta^k \geq B_{(M^k,V^k)}(\overline{\delta}_1 + \epsilon'') = B(\overline{\delta}_1 + \epsilon'') = B(\overline{\delta}_1) := \underline{\beta}_2.$$

Similarly, consider the right-hand side of (3.42). Suppose ϵ' is such that $1 - \Phi_{(\lambda_H,\lambda_H)}(c - \eta) \neq \rho(\underline{\delta}_1 - \epsilon')$ for all $\eta \in [0, c]$ and $B(\underline{\delta}_1 - \epsilon') = \lim_{\epsilon' \to 0} B(\underline{\delta}_1 - \epsilon')$. Then by (3.42), Lemma 9, and the fact that $M^k \to \lambda_H$, $V^k \to \lambda_H$ w.p.1, for k large enough, we have

$$\eta^k \le B_{(M^k,V^k)}(\underline{\delta}_1 - \epsilon') = B(\underline{\delta}_1 - \epsilon') = \lim_{\delta \to \overline{\delta}_1 -} B(\delta) := \overline{\beta}_2$$

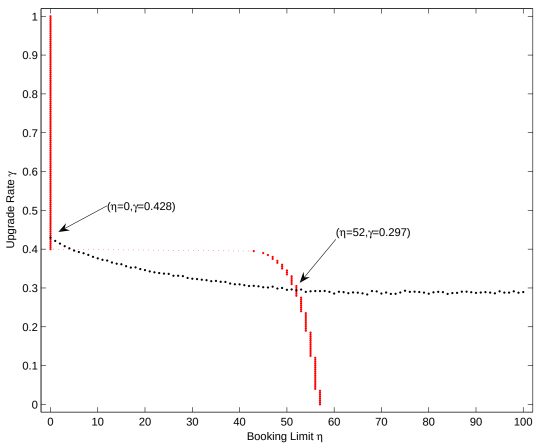
In summary, $\eta^k \in [\underline{\beta}_2, \ldots, \overline{\beta}_2]$ for k large enough w.p.1. It follows that $\underline{\delta}_2 \le \liminf \gamma^k \le \limsup \gamma^k \le \overline{\delta}_2$.

Continuing inductively, we have for each i that $\eta^k \in [\underline{\beta}_i, \ldots, \overline{\beta}_i]$ for k large enough and $\underline{\delta}_i \le \liminf \gamma^k \le \limsup \gamma^k \le \overline{\delta}_i$.

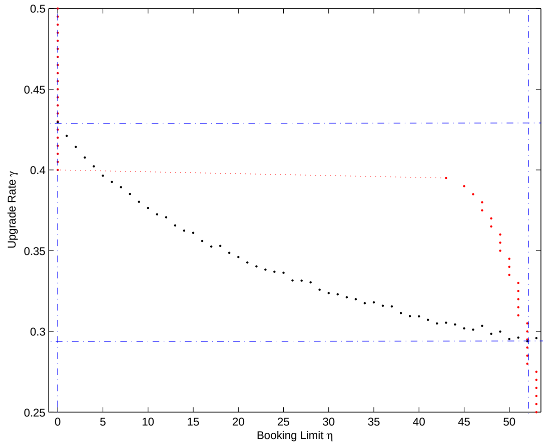
If there exists an n such that $\underline{\beta}_{n+1} = B(\overline{\delta}_n) = B(\underline{\delta}_n) = \overline{\beta}_{n+1}$, then eventually booking limit can only be $\eta^* = \underline{\beta}_{n+1} = \overline{\beta}_{n+1}$ and $\lim_{k\to\infty} \gamma^k = G(\eta^*)$.

$\square$

We close this section by illuminating the limitations of the approach in Proposition 9. The approach cannot be used if there are multiple fixed points, because by the proposition, if there is a convergence, then the limit must be the same on almost all sample paths. Figure 3.11 shows an example with multiple fixed points. Figure 3.11a is identical to Figure 3.5, and the two convergence points in different sample paths have been depicted in Figure 3.6 and Figure 3.7. Figure 3.11b illustrates that the procedure of shrinking ends in the dotted range, and thus does not lead to a convergence. It also may be that there is a unique fixed point, but we cannot find sequences with properties required to apply Proposition 9. There is only one fixed point in Figure 3.11a, hence Proposition 9 might be applicable here. However, Figure 3.12b illustrates that we cannot apply Proposition 9. After the first shrinkage, the range of the buy-up rates will be $(0.42 - \epsilon, 0.68 + \epsilon)$, which suggests a set of booking limits of $[0, 1 \ldots 57]$. However, this set of booking limits in turn provides the same range of buy-up rates as before, $(0.42 - \epsilon, 0.68 + \epsilon)$.

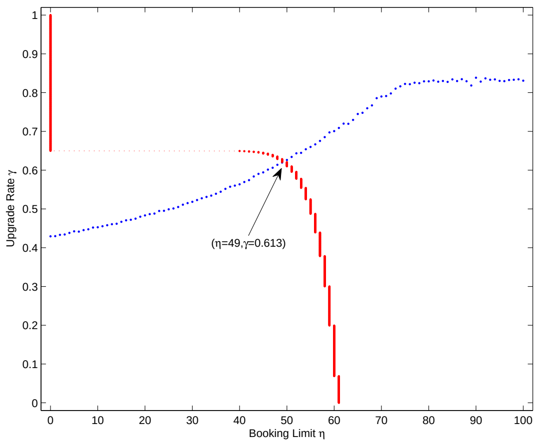


(a) $B(\gamma)$ and $G(\eta)$ with One Fixed Point

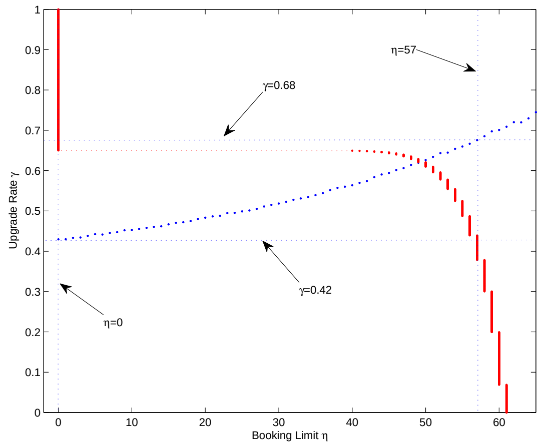


(b) Demonstration with "Unsuccessful Shrinkage"

Figure 3.11: $B(\gamma)$ and $G(\eta)$ with One Fixed Point



(a) $B(\gamma)$ and $G(\eta)$ with Two Fixed Points



(b) Demonstration with "Unsuccessful Shrinkage"

Figure 3.12: $B(\gamma)$ and $G(\eta)$ with Two Fixed Points

3.5 Numerical Studies

In this section, we compare the optimal revenue with the revenue obtained from the buy-up model and from the Littlewood rule. With the two models, we need to find a final booking limit (say, convergence point) which can be used to calculate the associated revenue. In this section, instead of using $\frac{1}{k}$ in (3.3), we use $\frac{1}{k-1}$ to estimate the variance.

We begin our study by explaining how the Littlewood rule is implemented. It is again necessary to estimate Prob[High-fare demand $> x$]. As in the case of deterministic arrivals, we consider two estimation schemes, one in which the high-fare demand is estimated from $\{T^i\}$ and the other in which it is estimated from $\{T^i + R^i\}$. As before, we assume the revenue manager uses the Normal distribution $\Phi_{(\mu,\sigma^2)}(\cdot)$ to approximate the high-fare demand distribution. In this scheme, the revenue manager uses the standard statistical methods and estimates μ by $M^k = \frac{1}{k}\sum_{i=1}^{k} T^i$ and σ^2 by $V^k = \frac{1}{k-1}\sum_{i=1}^{k}(T^i - M^k)^2$.

Plugging the estimates into (2.30) gives the following expression for the chosen booking limits:

$$\eta^k_{\mathrm{L(T)}} = \max\{\eta \in [0, c] :\ 1 - \Phi_{(M^k,V^k)}(c - \eta) < p_2/p_1 = \alpha\}\,. \tag{3.43}$$

As before, $M^k = k^{-1}\sum_{i=1}^{k} T^i \to \lambda_H$ and $V^k = \frac{1}{k-1}\sum_{i=1}^{k}(T^i - M^k)^2 \to \lambda_H$ as $k \to \infty$ with probability one. Therefore, $\Phi_{(M^k,V^k)}(x) \to \Phi_{(\lambda_H,\lambda_H)}(x)$ as $k \to \infty$ for all $x \in [0, c]$ with probability one.

Let us define

$$\eta^\dagger_{\mathrm{L(T)}} = \max\{\eta \in [0, c] :\ 1 - \Phi_{(\lambda_H,\lambda_H)}(c - \eta) < \alpha\}\,. \tag{3.44}$$

It is easy to show that $\eta^k_{\mathrm{L(T)}} \to \eta^\dagger_{\mathrm{L(T)}}$, so that $\eta^k_{\mathrm{L(T)}} = \eta^\dagger_{\mathrm{L(T)}}$ for all k large enough.

In the second estimation scheme the revenue manager still uses the Normal approximation, but estimates high-fare demand from $\{T^i + R^i\}$ rather than $\{T^i\}$. Correspondingly, the revenue manager estimates μ by

$$M_{\mathrm{L}}^k = \frac{1}{k}\sum_{i=1}^{k}\{T^i + R^i\}$$

and σ^2 by

$$V_{\mathrm{L}}^k = \frac{1}{k-1}\sum_{i=1}^{k}(T^i + R^i - M_{\mathrm{L}}^k)^2\,.$$

.

Note that although T^i does not depend on η^{i-1}, R^i does depend on η^{i-1}. Therefore, both M_{L}^k and V_{L}^k depend on the sequence $\{\eta^0, \eta^1, ..., \eta^{k-1}\}$. Plugging the estimates into (2.30) gives the following expression for the chosen booking limits:

$$\eta_{\mathrm{L(T+R)}}^k = \max\{\eta \in [0, c] :\ 1 - \Phi_{(M_{\mathrm{L}}^k, V_{\mathrm{L}}^k)}(c - \eta) < p_2/p_1 = \alpha\}\,.$$

Let $m(\eta) = E[R^1|\eta^0 = \eta]$ and $M_{\mathrm{L}}(\eta) = \lambda_H + m(\eta)$. If the revenue manager sets $\eta^k \equiv \eta$ for all k, then $M_{\mathrm{L}}^k = \frac{1}{k}\sum_{i=1}^{k}\{T^i + R^i\} \to \lambda_H + m(\eta) = M_{\mathrm{L}}(\eta)$, moreover, $V_{\mathrm{L}}^k \to V_{\mathrm{L}}(\eta) := E[(T^1 + R^1 - M_{\mathrm{L}}(\eta))^2|\eta^0 = \eta])]$.

If $\eta_{\mathrm{L(T+R)}}^k \to \eta$, then it follows that $M_{\mathrm{L}}^k \to M_{\mathrm{L}}(\eta)$ and $V_{\mathrm{L}}^k \to V_{\mathrm{L}}(\eta)$ as $k \to \infty$ with probability one. Consequently, $\Phi_{(M_{\mathrm{L}}^k, V_{\mathrm{L}}^k)}(x) \to \Phi_{(M_{\mathrm{L}}(\eta), V_{\mathrm{L}}(\eta))}(x)$ as $k \to \infty$, for all $x \in [0, c]$ with probability one.

If $\eta_{\mathrm{L(T+R)}}^k \to \eta^\dagger$, then $\eta^\dagger$ must satisfy

$$\eta^\dagger = \max\{\eta \in [0, c] :\ 1 - \Phi_{(M_{\mathrm{L}}(\eta^\dagger), V_{\mathrm{L}}(\eta^\dagger))}(c - \eta) < \alpha\}\,. \tag{3.45}$$

We use $\eta_{\mathrm{L(T+R)}}^\dagger$ to denote any point satisfying (3.45); that is, $\eta_{\mathrm{L(T+R)}}^\dagger = \max\{\eta \in [0, c] :\ 1 - \Phi_{(M_{\mathrm{L}}(\eta_{\mathrm{L(T+R)}}^\dagger), V_{\mathrm{L}}(\eta_{\mathrm{L(T+R)}}^\dagger))}(c - \eta) < \alpha\}$. Note that there may be more than one fixed point. In simulations with multiple fixed points (i.e., when $\eta_{\mathrm{L(T+R)}}^\dagger$ is not unique), we have observed convergence to each of the fixed points in different simulations (i.e., on different sample paths).

In the following tables, we show three examples with different price ratios α. In each setting, we consider each of the three methods discussed above to set booking limits, namely the buy-up model, the Littlewood rule with high-fare demand estimated from T (hereafter, the pure Littlewood rule), and the Littlewood rule with high-fare demand estimated from $T+R$ (hereafter, the mixed Littlewood rule). The tables show $\eta^\dagger$ in the buy-up model, $\eta^\infty_{\mathrm{L(T)}}$ in the pure Littlewood rule, and $\eta^\dagger_{\mathrm{L(T+R)}}$ in the mixed Littlewood rule. Note that it is easy to prove that $\eta^\infty_{\mathrm{L(T)}}$ is the limit of the booking limits in the pure Littlewood rule. However, we do not have proofs that the fixed points of $B(G(\cdot))$ and of (3.45) are limits of the booking limits in the buy-up model and the mixed Littlewood rule, respectively. In the buy-up model, if there is no fixed point, the tables show the crossing point; and if there are multiple fixed points, the tables show all of them. Similarly, if there are multiple fixed points in the mixed Littlewood rule, the tables show all of them. We also calculate the optimal booking limit and the optimal expected revenue. By "the optimal booking limit", we mean the booking limit with the highest actual expected revenue. The tables show the expected revenues $r^\dagger := r(\eta^\dagger)$, $r^\infty_{\mathrm{L(T)}} := r(\eta^\infty_{\mathrm{L(T)}})$, and $r^\dagger_{\mathrm{L(T+R)}} := r(\eta^\dagger_{\mathrm{L(T+R)}})$ as well. In the end, we calculate the difference in percentage of the optimal revenue.

We next consider several examples in greater detail. Suppose again that there are $c = 100$ seats and that for each fixed k, customers of each of the three types arrive according to non-homogeneous Poisson processes on $[0, \tau]$ with $\tau = 100$. Type-H customers have arrival rate $\lambda_{\mathrm{H}}(t) = 0.008t$, type-L customers have arrival rate $\lambda_{\mathrm{L}}(t) = 0.8 - 0.008t$, and type-F have arrival rate $\lambda_{\mathrm{F}}(t) = 0.3$ for $t \in [0, \tau]$. Hence, the average numbers of type-H, L, and F customers are 40, 40, and 30 respectively. In this problem, type-L customers tend to arrive early in the horizon, type-H customers tend to arrive late in the horizon, and type-F customers arrive evenly throughout the horizon. Table 3.1 shows that buy-up model performs quite well. The expected revenue $r^\dagger$ is quite close to the optimal revenue. Although the customers' behavior is different from that underlying in the assumptions of buy-up model, the buy-up model provides a good approximation of the actual behavior. However, the worst performance occurs when $\alpha = 0.4$. The estimate of buy-up rate equilibrates at a value that is relatively high, and thus causes the model to protect more seats for high-fare tickets. If $\alpha = 0.5$, then the booking limit does not converge. The associated revenue is calculated with $\eta' = 26$, which is the crossing point shown in Figure 3.3b. The pure Littlewood rule fails to account for upgrades. Therefore, if α is low, then the Littlewood rule protects too few seats for high-fare tickets and provides poor revenue performance. As α goes up, the cap to protect more or less shrinks, and the revenue performance gets better. Although the mixed Littlewood rule accounts for some upgrades, the estimates of high-fare demand are still low. Therefore, if α is low, then the mixed Littlewood rule still protects too few seats for high-fare tickets and provides poor revenue performance. However, it is much better than that of the pure Littlewood rule. The revenue performance also gets better as α becomes larger. Multiple fixed points $\eta^\dagger_{\mathrm{L(T+R)}}$ and associated revenue $r^\dagger_{\mathrm{L(T+R)}}$ are shown separately in the table. For instance, if $\alpha = 0.3$, two fixed points 38 and 39 of (3.45) are observed as the limiting booking limits in certain sample paths, and the table shows the expected revenues of both these fixed points.

In Table 3.2, we consider an example identical to the above, except that type-H customers have arrival rate $\lambda_{\mathrm{H}}(t) = 0.8 - 0.008t$, type-L customers have arrival rate $\lambda_{\mathrm{L}}(t) = 0.008t$, and type-F have arrival rate $\lambda_{\mathrm{F}}(t) = 0.3$. Here, type-H customers tend to arrive early in the horizon, type-L customers tend to arrive late in the horizon. The performance of buy-up model, the pure Littlewood rule and the mixed Littlewood rule are comparable to that in Table 3.1. Observe that in the buy-up model with $\alpha = 0.4$, there are two fixed points. They are both observed as the limiting booking

Table 3.1: Revenue Comparison with L before F Arrival

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Optimal BL	0	0	15	28	35	41	45	50	56
Revenue, r_{OPT}	69.75	70.43	70.12	72.59	75.69	79.31	83.74	88.49	93.87
$\eta^{\dagger}$	0	0	0	0	26*	45	52	57	61
$r^{\dagger}$	69.75	70.43	70.12	70.25	74.95	78.70	82.59	87.51	92.98
$100 \times \frac{(r_{\mathrm{OPT}} - r^{\dagger})}{r_{\mathrm{OPT}}}$	0	0	0	3.69	0.98	0.77	1.37	1.11	0.95
$\eta^{\infty}_{\mathrm{L(T)}}$	50	53	55	57	59	60	62	64	67
$r^{\infty}_{\mathrm{L(T)}}$	53.04	55.88	60.20	64.63	69.58	75.21	80.67	86.47	92.63
$100 \times \frac{(r_{\mathrm{OPT}} - r^{\infty}_{\mathrm{L(T)}})}{r_{\mathrm{OPT}}}$	23.96	20.65	14.15	10.97	8.07	5.17	3.37	2.28	1.32
$\eta^{\dagger}_{\mathrm{L(T+R)}}$	26	31	35	38/39	43	46	50	55/56	63
$r^{\dagger}_{\mathrm{L(T+R)}}$	64.68	65.6	68.42	71.38/71.22	74.62	78.71	83.02	87.69/87.55	92.91
$100 \times \frac{(r_{\mathrm{OPT}} - r^{\dagger}_{\mathrm{L(T+R)}})}{r_{\mathrm{OPT}}}$	6.98	7.28	2.42	1.67/1.89	1.41	0.86	0.99	0.90/1.06	1.02

Asterisks denote crossing points.

limits in certain sample paths, as shown in Figure 3.6a and 3.7a. While convergence to a booking limit of 0 gives the optimal revenue, convergence to a booking limit of 52 provides rather poor revenue. The booking limit of 52 allows most of the flexible customers to purchase low-fare tickets. Therefore, observations of buy-ups $\{R^k\}$ are low and thus the estimate of the buy-up rate is also low, which results in a high booking limit and in the end bad revenue. This phenomenon occurs when α is moderate because if α is lower, there is a big incentive to protect all seats for high-fare tickets; and if α is higher, there is little difference between protecting many or few seats.

In the end, we consider an example for which type-H, L, and F customers have arrival rates $\lambda_{\mathrm{H}}(t) = 0.2$, $\lambda_{\mathrm{L}}(t) = 1 - 0.01t$ and $\lambda_{\mathrm{F}}(t) = 0.01t$, respectively. In Table 3.3, we see the mixed Littlewood rule yields good revenue. Since most of the flexible customers arrive late in the horizon, allowing the mixed Littlewood rule to count most of the potential high-fare requests, the mixed Littlewood rule provides good performance. The pure Littlewood rule again fails to account for buy-ups, and thus does not protect enough seats for high-fare tickets, and has poor revenue performance. In this setting, the buy-up model has relatively poor performance compared to previous settings. The worst cases occur when α is relatively low or moderate. Since most of the flexible customers come late in the horizon, the buy-up model makes a high estimate of buy-up rate and thus protects too many seats for high-fare tickets. This results in low revenue because not enough tickets are sold to type-L customers. In this setting, the performance of the buy-up model is poor mostly because that there is no buy-up rate in the model reflecting the actual customers' behavior and the estimated high buy-up rate leads to booking limits that are too low.

Table 3.2: Revenue Comparison with F before L

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Optimal BL	0	0	0	0	0	71	78	79	76
Revenue, r_{OPT}	69.66	69.98	69.71	70.20	69.90	75.67	81.88	87.76	93.92
$\eta^{\dagger}$	0	0	0	0/52	55	57	60	62	65
$r^{\dagger}$	69.66	69.98	69.71	70.20/64.37	69.71	75.25	81.05	86.87	92.95
$100 \times \frac{(r_{\mathrm{OPT}} - r^{\dagger})}{r_{\mathrm{OPT}}}$	0	0	0	0/8.30	0.27	0.56	1.01	1.01	1.03
$\eta^{\infty}_{\mathrm{L(T)}}$	50	53	55	57	59	60	62	64	67
$r^{\infty}_{\mathrm{L(T)}}$	49.80	53.82	58.81	63.89	69.47	75.14	81.06	87.14	93.17
$100 \times \frac{(r_{\mathrm{OPT}} - r^{\infty}_{\mathrm{L(T)}})}{r_{\mathrm{OPT}}}$	29.81	23.09	15.64	8.99	0.61	0.70	1.01	0.71	0.80
$\eta^{\dagger}_{\mathrm{L(T+R)}}$	40	44/45	48	51	54	56	59	62	65/66
$r^{\dagger}_{\mathrm{L(T+R)}}$	53.06	56.30/56.15	60.05	64.61	69.76	75.04	80.87	86.83	93.01/92.94
$100 \times \frac{(r_{\mathrm{OPT}} - r^{\dagger}_{\mathrm{L(T+R)}})}{r_{\mathrm{OPT}}}$	23.83	19.55/19.76	13.86	7.97	0.20	0.83	1.14	1.06	0.97/1.04

Table 3.3: Revenue Comparison with Bad Performance

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Optimal BL	9	17	23	27	34	39	44	50	51
Revenue, r_{OPT}	70.78	71.98	73.67	76.12	79.31	82.28	86.18	90.24	94.95
$\eta^{\dagger}$	0	0	0	0	1*	21*	41*	62*	76
$r^{\dagger}$	70.08	70.03	70.07	70.01	70.67	79.84	85.42	87.35	92.27
$100\frac{(r_{\mathrm{OPT}}-r^{\dagger})}{r_{\mathrm{OPT}}}$	0.99	2.71	4.89	8.03	10.68	2.97	0.88	3.20	2.82
$\eta^{\infty}_{\mathrm{L(T)}}$	73	75	76	77	79	80	81	82	84
$r^{\infty}_{\mathrm{L(T)}}$	34.15	39.91	46.66	53.75	60.49	68.06	75.780	83.65	91.65
$100 \times \frac{(r_{\mathrm{OPT}}-r^{\infty}_{\mathrm{L(T)}})}{r_{\mathrm{OPT}}}$	51.75	44.55	36.66	29.399	23.73	17.28	12.07	7.30	3.48
$\eta^{\dagger}_{\mathrm{L(T+R)}}$	20	24/25	28/29	32	35	37/38	41/42/43	47	53/54/55
$r^{\dagger}_{\mathrm{L(T+R)}}$	69.66	71.15/70.68	73.19/73.02	75.68	78.60	81.88/81.66	85.53/85.45/85.27	89.41	94.04/93.98/93.98
$100 \times \frac{(r_{\mathrm{OPT}}-r^{\dagger}_{\mathrm{L(T+R)}})}{r_{\mathrm{OPT}}}$	1.58	1.15/1.81	0.65/0.88	0.58	0.57	0.49/0.75	0.75/0.85/1.06	0.92	1.06/1.02/1.02

3.6 Summary and Research Direction

In this chapter, we considered more complicated customer behavior, in particular stochastic arrivals. With three types of heterogeneous Poisson distributed arrivals, we first developed necessary conditions for convergence of the chosen booking limits by introducing an idea of fixed points and provided numerical results. In one stochastic setting, we showed in simulation that the booking limits converged to two different points on different sample paths even with identical initial booking limits.

We also provided two different sufficient conditions for the convergence of booking limits. One of the approaches was based upon stochastic approximations and pseudo-contraction mappings. In the second approach, we used the idea that the ranges of both booking limits and buy-up rates might reduce. If the range of the booking limits shrink to a single point, (buy-up rates shrink to a certain interval respectively), then there is a convergence.

In the numerical section, we tested the revenue performance of using this buy-up model and compared with that of the Littlewood rule. In general, the revenue performance of using the buy-up model was good. However, we also described a setting in which the performance was poor. This occurred because the customer behavior heavily violated the assumptions underlying the buy-up model.

In the discussion of each of the two sufficient conditions for convergence, we pointed out the limitations. Neither of the methods is applicable to a multi-fixed point setting. Additional theoretical analysis might allow us to prove that the booking limits converge when there are multiple fixed points.

Although we assume three different types of customer behavior and introduce the heterogeneous Poisson distributed arrivals, more complicated customer behavior, for example, customers who might "buy down", may be taken into consideration in future analysis.

3.7 Appendix: Proofs

Proof of Proposition 5. If $M^k \to \lambda_H$ and $V^k \to \lambda_H$, then it follows that

$$\Phi_{(M^k,V^k)}(c-\eta) \to \Phi_{(\lambda_H,\lambda_H)}(c-\eta) \;\; \text{as} \;\; k \to \infty \tag{3.46}$$

for all $\eta \in [0, c]$.

Under the condition that $1 - \Phi_{(\lambda_H,\lambda_H)}(c-\eta) \neq \rho(G(\eta))$ for all $\eta \in [0, c]$, it follows from (3.46) that if $\eta^k \to \eta^\infty$ on some sample path, then $\eta^k \to \max\{\eta \in [0, c] : 1 - \Phi_{(\lambda_H,\lambda_H)}(c-\eta) < \rho(G(\eta^\infty))\} = B(G(\eta^\infty))$ on that sample path, provided that $M^k \to \lambda_H$, $V^k \to \lambda_H$, and $\gamma^k \to G(\eta^\infty)$ on that sample path. By the strong law of large numbers, we have that $M^k \to \lambda_H$ and $V^k \to \lambda_H$ w.p.1 and moreover $\gamma^k \to G(\eta^\infty)$ for almost all sample paths on which $\eta^k \to \eta^\infty$. Hence, the intersection of the events $\{\eta^k \to \eta^\infty\}$ and $\{\eta^\infty \neq B(G(\eta^\infty))\}$ has probability zero, and therefore $\eta^\infty = B(G(\eta^\infty))$ on almost all sample paths on which $\eta^k \to \eta^\infty$. It follows immediately that if $\eta^k \to \eta^\infty$ w.p.1, then $\eta^\infty = B(G(\eta^\infty))$ w.p.1.

□

Proof of Lemma 8 . To prove the existence of this ϵ, we will identify such ϵ by providing a specific value of it (i.e., we let $\epsilon < X_0$, where X_0 does not depend on (x, y, m, n)).

If $1 - \rho(y/x) \geq 1$, then

$$B'(x, y, \lambda_H, \lambda_H + \lambda_H^2) = B'(x, y, m, n) = 0.$$

If $1 - \rho(y/x) < 0$, then

$$B'(x, y, \lambda_H, \lambda_H + \lambda_H^2) = B'(x, y, m, n) = c.$$

If $0 \leq 1 - \rho(y/x) < 1$, then note that

$$\Phi_{(m,n-m^2)}(c-\eta) = \Phi_{(0,1)}(\frac{c-\eta-m}{\sqrt{n-m^2}}).$$

Let $p_0 = \Phi^{-1}_{(0,1)}(1 - \rho(y/x))$. Note that p_0 depends on x and y. Then

$$\begin{aligned} B'(x, y, m, n) &= \max\{\eta \in [0, c] :\ 1 - \Phi_{(m,n-m^2)}(c - \eta) < \rho(y/x)\} \\ &= \max\{\eta \in [0, c] :\ 1 - \Phi_{(0,1)}(\frac{c - \eta - m}{\sqrt{n - m^2}}) < \rho(y/x)\} \\ &= \max\{\eta \in [0, c] :\ \Phi^{-1}_{(0,1)}(1 - \rho(y/x)) < \frac{c - \eta - m}{\sqrt{n - m^2}}\} \\ &= \max\{\eta \in [0, c] :\ p_0 < \frac{c - \eta - m}{\sqrt{n - m^2}}\} \\ &= \max\{\eta \in [0, c] :\ c - \eta > m + \sqrt{n - m^2}p_0\}. \end{aligned}$$

Similarly,

$$B'(x, y, \lambda_H, \lambda_H + \lambda_H^2) = \max\{\eta \in [0, c] :\ c - \eta > \lambda_H + \sqrt{\lambda_H}p_0\}$$

According to the definition of $B'(\cdot)$, we know

$$B'(x, y, m, n) = \begin{cases} 0 & \text{if } \lfloor m + \sqrt{n - m^2}p_0 \rfloor \geq c \\ c - \lfloor m + \sqrt{n - m^2}p_0 \rfloor - 1 & \text{if } -1 \leq \lfloor m + \sqrt{n - m^2}p_0 \rfloor \leq c - 1 \\ c & \text{if } \lfloor m + \sqrt{n - m^2}p_0 \rfloor \leq -2\,. \end{cases}$$

Likewise,

$$B'(x, y, \lambda_H, \lambda_H + \lambda_H^2) = \begin{cases} 0 & \text{if } \lfloor \lambda_H + \sqrt{\lambda_H}p_0 \rfloor \geq c \\ c - \lfloor \lambda_H + \sqrt{\lambda_H}p_0 \rfloor - 1 & \text{if } -1 \leq \lfloor \lambda_H + \sqrt{\lambda_H}p_0 \rfloor \leq c - 1 \\ c & \text{if } \lfloor \lambda_H + \sqrt{\lambda_H}p_0 \rfloor \leq -2\,. \end{cases}$$

We will show there is $\epsilon > 0$ (not depending on (x, y, m, n)), such that if $|m - \lambda_H| < \epsilon$ and $|n - m^2 - \lambda_H| < 2\epsilon^2 + 2\epsilon\lambda_H$, then for any x and y, we have

(1) if $\lfloor m + \sqrt{n - m^2}p_0 \rfloor \geq c$, then $\lfloor \lambda_H + \sqrt{\lambda_H}p_0 \rfloor \geq c - 1$;

(2) if $\lfloor \lambda_H + \sqrt{\lambda_H}p_0 \rfloor \geq c$, then $\lfloor m + \sqrt{n - m^2}p_0 \rfloor \geq c - 1$;

(3) if $\lfloor m + \sqrt{n - m^2}p_0 \rfloor \leq -2$, then $\lfloor \lambda_H + \sqrt{\lambda_H}p_0 \rfloor \leq -1$;

(4) if $\lfloor \lambda_H + \sqrt{\lambda_H}p_0 \rfloor \leq -2$, then $\lfloor m + \sqrt{n - m^2}p_0 \rfloor \leq -1$;

(5) if $-1 \le \lfloor \lambda_H + \sqrt{\lambda_H} p_0 \rfloor \le c-1$, and $1 \le \lfloor m + \sqrt{n-m^2} p_0 \rfloor \le c-1$, then $|\lfloor \lambda_H + \sqrt{\lambda_H} p_0 \rfloor - \lfloor m + \sqrt{n-m^2} p_0 \rfloor| \le 1$.

Then, for such $\epsilon > 0$, it follows by the above expressions for $B'(x, y, \lambda_H, \lambda_H + \lambda_H^2)$ and $B'(x, y, m, n)$ that $|B'(x, y, \lambda_H, \lambda_H + \lambda_H^2) - B'(x, y, m, n)| \le 1$ for all x and y if $|m - \lambda_H| < \epsilon$ and $|n - m^2 - \lambda_H| < 2\epsilon^2 + 2\epsilon\lambda_H$.

The general idea is that for each case (i), we will identify a specific value $A_i > 0$ such that if $0 < \epsilon < A_i$, then case (i) holds for $|m - \lambda_H| < \epsilon$ and $|n - m^2 - \lambda_H| < 2\epsilon^2 + 2\epsilon\lambda_H$. Taking $A = \min_i A_i$, the Lemma then holds for any $\epsilon \in (0, A)$.

Now we begin with the case (1), define A_1 as follows

$$A_1 = \begin{cases} \min\{\frac{1}{8(c-1-\lambda_H)}, \lambda_H, \frac{1}{2}\} & \text{if } c - 1 - \lambda_H > 0; \\ \min\{-\frac{1}{8(c-1-\lambda_H)}, \lambda_H, \frac{1}{2}\} & \text{if } c - 1 - \lambda_H < 0; \\ \min\{\lambda_H, \frac{1}{2}\} & \text{if } c - 1 - \lambda_H = 0\,. \end{cases} \tag{3.47}$$

Now we will show that for $\epsilon \in (0, A_1)$, if $|m - \lambda_H| < \epsilon$ and $|n - m^2 - \lambda_H| < 2\epsilon^2 + 2\epsilon\lambda_H$ and $\lfloor m + \sqrt{n-m^2} p_0 \rfloor \ge c$, then $\lfloor \lambda_H + \sqrt{\lambda_H} p_0 \rfloor \ge c - 1$. Suppose for contradiction $\lfloor \lambda_H + \sqrt{\lambda_H} p_0 \rfloor \le c - 2$; that is $\lambda_H + \sqrt{\lambda_H} p_0 < c - 1$. Then

$$p_0 < \frac{c - 1 - \lambda_H}{\sqrt{\lambda_H}}. \tag{3.48}$$

Moreover, given $\lfloor m + \sqrt{n-m^2} p_0 \rfloor \ge c$, we know $m + \sqrt{n-m^2} p_0 \ge c$. Thus

$$p_0 \ge \frac{c - m}{\sqrt{n - m^2}}. \tag{3.49}$$

Furthermore, we have $\lambda_H + \sqrt{\lambda_H} p_0 < c - 1 \le m + \sqrt{n-m^2} p_0 - 1$. Thus

$$1 - m + \lambda_H < (\sqrt{n - m^2} - \sqrt{\lambda_H}) p_0. \tag{3.50}$$

For any $\epsilon \in (0, \lambda_H)$, we have $2\epsilon^2 < 2\epsilon\lambda_H$, and thus $2\epsilon^2 + 2\epsilon\lambda_H < 4\epsilon\lambda_H$. From this, it follows that if $|n - m^2 - \lambda_H| < 2\epsilon^2 + 2\epsilon\lambda_H$, then

$$|\sqrt{n - m^2} - \sqrt{\lambda_H}| < \frac{2\epsilon^2 + 2\epsilon\lambda_H}{\sqrt{n - m^2} + \sqrt{\lambda_H}} < \frac{2\epsilon^2 + 2\epsilon\lambda_H}{\sqrt{\lambda_H}} < 4\sqrt{\lambda_H}\epsilon.$$

For $\epsilon \in (0, \frac{1}{2})$, if $|m - \lambda_H| < \epsilon < \frac{1}{2}$, then $\frac{3}{2} > 1 - m + \lambda_H > \frac{1}{2}$.

(i) If $\sqrt{n - m^2} - \sqrt{\lambda_H} > 0$, then from (3.50) and $\sqrt{n - m^2} - \sqrt{\lambda_H} < 4\sqrt{\lambda_H}\epsilon$, we know

$$p_0 > \frac{1 - m + \lambda_H}{\sqrt{n - m^2} - \sqrt{\lambda_H}} > \frac{1}{2(\sqrt{n - m^2} - \sqrt{\lambda_H})} > \frac{1}{8\sqrt{\lambda_H}\epsilon} > 0. \tag{3.51}$$

If $c - 1 - \lambda_H \leq 0$, then from (3.48) we know p_0 is negative. But this contradicts (3.51).

If $c - 1 - \lambda_H > 0$, we let $\epsilon = \min\{\frac{1}{8(c-1-\lambda_H)}, \lambda_H, \frac{1}{2}\}$, then $p_0 > \frac{1}{8\sqrt{\lambda_H}\epsilon} > \frac{c-1-\lambda_H}{\sqrt{\lambda_H}}$, which contradicts (3.48).

(ii) If $\sqrt{n - m^2} - \sqrt{\lambda_H} < 0$, then from (3.50) we know $p_0 < \frac{1-m+\lambda_H}{\sqrt{n-m^2}-\sqrt{\lambda_H}} < 0$, since $1 - m + \lambda_H > \frac{1}{2}$. We also know that $\sqrt{n - m^2} - \sqrt{\lambda_H} > -4\sqrt{\lambda_H}\epsilon$.

If $c - 1 - \lambda_H \geq 0$, then

$$\begin{aligned}
\sqrt{n - m^2}(1 - m + \lambda_H) &= (\sqrt{n - m^2} - \sqrt{\lambda_H})(1 - m + \lambda_H) + \sqrt{\lambda_H}(1 - m + \lambda_H) \\
&> (\sqrt{n - m^2} - \sqrt{\lambda_H})(1 - m + \lambda_H) + (\sqrt{n - m^2} - \sqrt{\lambda_H})(c - 1 - \lambda_H) \\
&= (\sqrt{n - m^2} - \sqrt{\lambda_H})(c - m),
\end{aligned}$$

where the inequality holds since $\sqrt{\lambda_H}(1 - m + \lambda_H) > 0$ and $(\sqrt{n - m^2} - \sqrt{\lambda_H})(c - 1 - \lambda_H) \leq 0$. It follows that $p_0 < \frac{1-m+\lambda_H}{\sqrt{n-m^2}-\sqrt{\lambda_H}} < \frac{c-m}{\sqrt{n-m^2}}$, which contradicts (3.49).

If $c - 1 - \lambda_H < 0$, then

$$\begin{aligned}
\sqrt{n - m^2}(1 - m + \lambda_H) &= (\sqrt{n - m^2} - \sqrt{\lambda_H})(1 - m + \lambda_H) + \sqrt{\lambda_H}(1 - m + \lambda_H) \\
&> (\sqrt{n - m^2} - \sqrt{\lambda_H})(1 - m + \lambda_H) + \frac{1}{2}\sqrt{\lambda_H} \\
&> (\sqrt{n - m^2} - \sqrt{\lambda_H})(1 - m + \lambda_H) + (-4\sqrt{\lambda_H}\epsilon)(c - 1 - \lambda_H) \\
&> (\sqrt{n - m^2} - \sqrt{\lambda_H})(1 - m + \lambda_H) + (\sqrt{n - m^2} - \sqrt{\lambda_H})(c - 1 - \lambda_H) \\
&= (\sqrt{n - m^2} - \sqrt{\lambda_H})(c - m).
\end{aligned}$$

The first inequality is true since $1 - m + \lambda_H > \frac{1}{2}$; the second follows from $\frac{1}{2} > -4\epsilon(c - 1 - \lambda_H)$; the last holds because $\sqrt{n - m^2} - \sqrt{\lambda_H} > -4\sqrt{\lambda_H}\epsilon$. It follows that $p_0 < \frac{1-m+\lambda_H}{\sqrt{n-m^2}-\sqrt{\lambda_H}} < \frac{c-m}{\sqrt{n-m^2}}$, which contradicts (3.49).

(iii) If $\sqrt{n-m^2}-\sqrt{\lambda_H}=0$, then $1-m+\lambda_H<0$, which is a contradiction.

This completes the proof that if (x,y,m,n) is such that $\lfloor m+\sqrt{n-m^2}p_0\rfloor \geq c$, then (3.35) holds if $|m-\lambda_H|<\epsilon$ and $|n-m^2-\lambda_H|<2\epsilon^2+2\epsilon\lambda_H$ with $\epsilon\in(0,A_1)$ where A_1 is defined in (3.47).

Similarly, we can analyze cases (2) – (4) by an analogous method. In particular we have

$$A_2=\begin{cases} \min\{\frac{1}{8(c-\lambda_H)},\lambda_H,\frac{1}{2}\} & \text{if } c-\lambda_H>0;\\ \min\{-\frac{1}{8(c-\lambda_H)},\lambda_H,\frac{1}{2}\} & \text{if } c-\lambda_H<0;\\ \min\{\lambda_H,\frac{1}{2}\} & \text{if } c-\lambda_H=0\,. \end{cases} \tag{3.52}$$

$$A_3=\min\{\frac{1}{8\lambda_H},\lambda_H,\frac{1}{2}\}. \tag{3.53}$$

$$A_4=\min\{\frac{1}{8(1+\lambda_H)},\lambda_H,\frac{1}{2}\}. \tag{3.54}$$

Finally, we consider case (5); suppose $-1\leq\lfloor\lambda_H+\sqrt{\lambda_H}p_0\rfloor\leq c-1$, and $1\leq\lfloor m+\sqrt{n-m^2}p_0\rfloor\leq c-1$. Let $\eta_1=B'(x,y,m,n)$ and $\eta_2=B'(x,y,\lambda_H,\lambda_H+\lambda_H^2)$. If $c-\eta_1=\lfloor m+\sqrt{n-m^2}p_0\rfloor+1$ and $c-\eta_2=\lfloor\lambda_H+\sqrt{\lambda_H}p_0)\rfloor+1$, then $\eta_1-\eta_2=\lfloor\lambda_H+\sqrt{\lambda_H}p_0\rfloor-\lfloor m+\sqrt{n-m^2}p_0\rfloor$. Moreover, since $c-\eta_2=\lfloor\lambda_H+\sqrt{\lambda_H}p_0\rfloor+1$, we know $\lambda_H+\sqrt{\lambda_H}p_0<c-\eta_2\leq c$ and $\lambda_H+\sqrt{\lambda_H}p_0+1\geq c-\eta_2\geq 0$, thus

$$-\frac{1+\lambda_H}{\sqrt{\lambda_H}}\leq p_0<\frac{c-\lambda_H}{\sqrt{\lambda_H}}.$$

Since p_0 is bounded on both sides (the bounds do not depend on (x,y,m,n)), it is easy to find an A_5 such that if $\epsilon\in(0,A_5)$, then $||\lfloor\lambda_H+\sqrt{\lambda_H}p_0\rfloor-\lfloor m+\sqrt{n-m^2}p_0\rfloor|\leq 1$ for all (m,n) such that $|m-\lambda_H|<\epsilon$ and $|n-m^2-\lambda_H|<2\epsilon^2+2\epsilon\lambda_H$.

□

Proof of Proposition 8. Consider arbitrary (x, y, m, n). Observe that for the third and fourth components (m, n), we have

$$\begin{aligned}
&\frac{|m(x,y,m,n)-\lambda_H|}{\xi_3} = 0 \le \beta\frac{|m-\lambda_H|}{\xi_3} \\
&\le \beta\max\{\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}, \frac{|m-\lambda_H|}{\xi_3}, \frac{|n-\lambda_H-\lambda_H^2|}{\xi_4}\} && (3.55)\\
&\frac{|n(x,y,z,w)-\lambda_H^2-\lambda_H|}{\xi_4} = 0 \le \beta\frac{|n-\lambda_H-\lambda_H^2|}{\xi_4} \\
&\le \beta\max\{\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}, \frac{|m-\lambda_H|}{\xi_3}, \frac{|n-\lambda_H-\lambda_H^2|}{\xi_4}\} && (3.56)
\end{aligned}$$

for any $\xi_3, \xi_4 > 0$ and $\beta \in [0, 1)$.

Now we consider the first and second components (x, y) of (x, y, m, n). We know both $E(R^k) \le \lambda_F$ and $E(Q^k) \le \lambda_F + \lambda_L$, thus $|q(x, y, m, n) - \widetilde{Q}^*| \le \lambda_F + \lambda_L + \widetilde{Q}^*$ and $|r(x, y, m, n) - \widetilde{R}^*| \le \lambda_F + \widetilde{R}^*$.

Let $A := \max\{\lambda_F + \widetilde{R}^*, \lambda_F + \lambda_L + \widetilde{Q}^*\}$ and ϵ be defined as in Lemma 8. We will show that (3.29) holds with ξ_1 and ξ_2 as in the statement of the proposition and with $\xi_3 = \min\{\beta\epsilon\xi_1/A, \beta\epsilon\xi_2/A\}$ and $\xi_4 = \min\{\beta\epsilon^2\xi_1/A, \beta\epsilon^2\xi_2/A\}$. We will now consider three cases separately: (i) $|m-\lambda_H| \ge \epsilon$; (ii) $|n-\lambda_H-\lambda_H^2| \ge \epsilon^2$; and (iii) $|m-\lambda_H| < \epsilon$ and $|n-\lambda_H-\lambda_H^2| < \epsilon^2$. For case (i), we know $\frac{1}{\xi_1} \le \frac{1}{A\xi_3/(\beta\epsilon)}$ and $\frac{1}{\xi_2} \le \frac{1}{A\xi_3/(\beta\epsilon)}$, then

$$\begin{aligned}
\frac{|q(x,y,m,n)-\widetilde{Q}^*|}{\xi_1} &\le \frac{A}{\xi_1} \le \frac{A}{A\xi_3/(\beta\epsilon)} = \beta\frac{\epsilon}{\xi_3} \le \beta\frac{|m-\lambda_H|}{\xi_3} \\
&\le \beta\max\{\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}, \frac{|m-\lambda_H|}{\xi_3}, \frac{|n-\lambda_H-\lambda_H^2|}{\xi_4}\} \quad (3.57)\\
\frac{|r(x,y,m,n)-\widetilde{R}^*|}{\xi_2} &\le \frac{A}{\xi_2} \le \frac{A}{A\xi_3/(\beta\epsilon)} = \beta\frac{\epsilon}{\xi_3} \le \beta\frac{|m-\lambda_H|}{\xi_3} \\
&\le \beta\max\{\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}, \frac{|m-\lambda_H|}{\xi_3}, \frac{|n-\lambda_H-\lambda_H^2|}{\xi_4}\}. \quad (3.58)
\end{aligned}$$

For case (ii), similarly (3.29) holds for all x and y.

Finally, we consider case (iii). Note that $|m-\lambda_H| < \epsilon$ and $|n-\lambda_H-\lambda_H^2| < \epsilon^2$ implies $|m-\lambda_H| < \epsilon$ and $|n-m^2-\lambda_H| < 2\epsilon^2+2\epsilon\lambda_H$. From Lemma 8, we know $|B'(x, y, m, n)-$

$B'(x,y,\lambda_H,\lambda_H+\lambda_H^2)| \leq 1$. Moreover, from the fact $E[Q|\eta=\eta_0]-E[Q|\eta=\eta_0+1] \leq 1$ and $E[R|\eta=\eta_0]-E[R|\eta=\eta_0+1] \leq 1$ for any $\eta_0 \in [0,c-1]$, we know that if $|B'(x,y,m,n)-B'(x,y,\lambda_H,\lambda_H+\lambda_H^2)|=1$, then $|q(x,y,m,n)-\widetilde{Q}^*| \leq |q(x,y,\lambda_H,\lambda_H^2+\lambda_H)-\widetilde{Q}^*|+1$ and $|r(x,y,m,n)-\widetilde{R}^*| \leq |r(x,y,\lambda_H,\lambda_H^2+\lambda_H)-\widetilde{R}^*|+1$. Moreover, if $|B'(x,y,m,n)-B'(x,y,\lambda_H,\lambda_H+\lambda_H^2)|=0$, then $q(x,y,m,n)=q(x,y,\lambda_H,\lambda_H^2+\lambda_H)$ and $r(x,y,m,n)=r(x,y,\lambda_H,\lambda_H^2+\lambda_H)$.

If there exists ξ_1 and ξ_2, such that

$$\frac{|q'(x,y)-\widetilde{Q}^*|}{\xi_1} \leq \beta \max\left(\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}\}\right) \text{and} \frac{|r'(x,y)-\widetilde{R}^*|}{\xi_2} \leq \beta \max\left(\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}\}\right)$$

if $|B'(x,y,\lambda_H,\lambda_H+\lambda_H^2)-B'(x,y,m,n)|=0$;

$$\frac{|q'(x,y)-\widetilde{Q}^*|+1}{\xi_1} \leq \beta \max\left(\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}\}\right) \text{and} \frac{|r'(x,y)-\widetilde{R}^*|+1}{\xi_2} \leq \beta \max\left(\frac{|x-\widetilde{Q}^*|}{\xi_1}, \frac{|y-\widetilde{R}^*|}{\xi_2}\}\right)$$

if $|B'(x,y,\lambda_H,\lambda_H+\lambda_H^2)-B'(x,y,m,n)|=1$.

then (3.29) holds. This completes the proof.

□

Proof of Lemma 9. For each $\eta \in [0,c]$, note that $1-\Phi_{(m,v)}(c-\eta)$ is continuous in (m,v). If there is no $\eta \in [0,c]$ such that $1-\Phi_{(\lambda_H,\lambda_H)}(c-\eta)=\rho(\gamma)$, then there exists $\epsilon > 0$ such that if $|m-\lambda_H| < \epsilon$ and $|v-\lambda_H| < \epsilon$, then for each $\eta \in [0,c]$, we have $1-\Phi_{(m,v)}(c-\eta) < \rho(\gamma)$ if and only if $1-\Phi_{(\lambda_H,\lambda_H)}(c-\eta) < \rho(\gamma)$; hence, $\max\{\eta \in [0,c] : 1-\Phi_{(m,v)}(c-\eta) < \rho(\gamma)\} = \max\{\eta \in [0,c] : 1-\Phi_{(\lambda_H,\lambda_H)}(c-\eta) < \rho(\gamma)\}$. Therefore, $|B_{(m,v)}(\gamma)-B_{(\lambda_H,\lambda_H)}(\gamma)|=0$, if $|m-\lambda_H| < \epsilon$ and $|v-\lambda_H| < \epsilon$.

□

Chapter 4

On the Use of Littlewood Rule in Substitutable Flights

4.1 Introduction

In this chapter, we consider the revenue manager's decisions for substitutable flights. In practice, we typically see many flights serving the same origin and destination on the same day. Managing seat availability for such substitutable flights provides a difficult problem to a revenue manager. If revenue managers neglect the fact that the decision of one flight affects not only that flight but also any other flights, what can happen with his/her decisions?

Modeling customer choice behavior among substitutable flights leads to a considerably more difficult revenue management problem than that associated with a single flight. Zhang and Cooper (2005) considered customer choose among substitutable flights. Their model assumed that customers chose among tickets in the same fare class on different flights, but not among fare classes themselves. Zhang and Cooper (2009) analyzed a dynamic pricing problem for multiple substitutable flights with customer choice. Although the customer choice behavior among substitutable flights has drawn some attention recently, how to capture customer behavior precisely and how to control seat allocation remains open to researchers.

As in the previous chapters, we assume that purchases of low- and high-fare tickets for a particular flight may depend on the booking limit chosen for that flight. However, the situation with substitutable flights is even more complex, because availability of tickets for other flights also has effect on customers' choices. The choice of one flight's booking limit affects not only that flight's low-fare and high-fare ticket sales, but also other flights' sales. It is also not clear what (if any) methods the revenue managers use to account for substitutes among different flights in practice. In this chapter, we suppose the revenue manager uses the well-known Littlewood rule to manage seat allocations on each individual flight while not directly accounting for customers substitution among flights. The Littlewood rule assumes high-fare demand is exogenous. However, when customers make choices, there is no exogenous quality that is high-fare demand. Hence the revenue manager again uses a flawed model – one that does not capture customer choice among flights or ticket classes. Analysis of such a situation is important because in practice, revenue managers may not have an accurate model of customer choice among flights.

In this chapter, we suppose that there are n flights serving the same origin and destination. The n flights operate repeatedly, and the revenue manager controls ticket sales by setting a booking limit for each flight. We also assume that each of these n flights has the same capacity of c (for a total capacity of $n \times c$), and the total number of customers for all flights is d in each time period. Moreover, the prices of tickets are identical for all flights, namely f_1 for high-fare tickets and f_2 for low-fare tickets, and fare ratio $\alpha = f_2/f_1$.

4.1.1 Actual Customer Behavior

As in the previous chapters, we need to describe the actual behavior of customers. Note that this behavior will be intentionally different from the behavior underlying the assumptions of the Littlewood rule, so that we may study the Littlewood rule in a setting in which it is misspecified. This will alow us to understand effects of using a model that is not an accurate reflection of reality.

We first introduce the idea of preference. By order of preference, we mean a customer's order to check different flights. For instance, suppose there are 3 substitutable flights, then a customer can check ticket availability on the flights in 6 ($= 3!$) different orders: 1-2-3, 1-3-2, 2-3-1, 2-1-3, 3-1-2 and 3-2-1. Since the flights will be operated for multiple periods, one issue that arises is whether or not these d customers behave identically in each period. We say customers have fixed and identical orders of preference if the j-th ($j \in [1, 2, ..., d]$) customer at time 1 has an order of preference, say 3-1-2, then at any time k, the j-th customer still has the same order of preference, 3-1-2 here. On the other hand, we say customers have random orders of preference if any customer has a random order of preference to check substitutable flights at any time period. Each customer with random order of preference can change his/her order of preference randomly at any time.

Another essential issue is whether a customer's preference is focused on a certain flight, or on the price of a ticket. We call a customer fare-concerned if the customer checks the availability of all low-fare tickets first (one flight by one flight according to his/her order of preference, stops with a purchase after finding the first availability). The customer considers purchasing a high-fare ticket only when no low-fare ticket is available. We call a customer flight-concerned if the customer first picks a certain flight and request a low-fare ticket on that flight, and then requests a high-fare ticket on the same flight if no low-fare ticket is available. If no ticket is available for that flight, the customer checks the availability of a low-fare ticket on the second flight in his/her order of preference. If it is not available, then the customer in turn requests a high-fare ticket on that second flight. The customer follows this pattern until s/he purchases a ticket, or departs without purchase (all tickets are sold out).

4.1.2 Observation Data and Dynamics

Suppose that the revenue manager uses the well-known Littlewood rule to control the bookings of low- and high-fare tickets. Particularly, the revenue manager chooses the booking limit for low-fare tickets (the protection level for high-fare tickets), and employs the Littlewood rule that takes as input the cumulative probability distribution H of the assumed exogenous demand for high-fare tickets. Given H, the revenue manager chooses a booking limit η that satisfies

$$\eta := c - p := c - \min\{p \in [0, c] : H(p) \geq 1 - \alpha\} \tag{4.1}$$

where $\alpha = f_2/f_1$. The quality p is called the protection level.

Once the revenue manager decides to use the above model, the distribution of H needs to be estimated based upon data. In this chapter, estimates will be based upon the following quantities.

o_i^k=requests for high-fare tickets for flight i at time k;

s_i^k=sales of low-fare tickets for flight i at time k;

We use "untruncated" quantities for high-fare requests, that is, the revenue manager counts all requests for high-fare tickets, even when all tickets are sold out. This gives the revenue manager the "benefit of the doubt" by allowing the revenue manager to observe and record ticket requests even after tickets are sold out.

We assume the revenue manger neglects the fact that other flights' booking limits may affect all flights' ticket sales, and thus makes decisions independently for each flight. We also assume that the revenue manager uses empirical distributions of o_i^k to estimate the high-fare demand for flight i. Using o_i^k to estimate high-fare demand is reasonable if one believes there is such an exogenous quality as high-fare demand.

We let η_i^{k-1} be the booking limit for flight i at time k, so that $s_i^k \leq \eta_i^{k-1}$. At the beginning of each time period k, the revenue manager implements a booking limit η_i^{k-1} for each flight, $i = 1, 2, \ldots$ Then customers arrive and make purchases. After all customers make their decisions, the data o_i^k and s_i^k are collected. Thereafter, the revenue manager revises the estimates of the high-fare demand by empirical distribution:

$$\hat{H}_i^k(x) := \frac{1}{k} \sum_{j=1}^{k} \mathrm{I}_{\{o_i^j \leq x\}} \tag{4.2}$$

For notational convenience, if $o_i^j > c$, then we set $o_i^j = c$. Therefore, we know $\hat{H}_i^k(c) = 1$ for all i and k from (4.2). Then the booking limit given by the Littlewood rule is

$$\eta_i^k := c - p_i^k := c - \min\{p \in [0, c] : \hat{H}_i^k(p) \geq 1 - \alpha\} \tag{4.3}$$

and this expression is well defined.

4.2 Preliminary Results

4.2.1 Fare-Concerned Customers

In this setting, we assume all customers are fare-concerned, namely each customer checks for low-fare tickets first, and considers purchasing a high-fare ticket only when no low-fare ticket is available. For instance, assume there are 3 flights. If a customer has an order of preference 3-1-2, then s/he checks for fare products in the order of *LF3-LF1-LF2-HF3-HF1-HF2*, where *LFi (HFj)* stands for low-fare tickets of flight i (high-fare tickets of flight j, respectively). That is, s/he first checks for low-fare tickets of flight 3, if the tickets are not available, then s/he will check for low-fare tickets of flight 1. This process continues until the customer either purchases a ticket or find all products (tickets) unavailable.

The following result describes the limiting behavior of booking limits.

Proposition 10 *Suppose all customers are fare-concerned. If $d > n \times c$, then there exists K, such that $\eta_i^k = 0$ for all $i \in [1, n]$ and $k \geq K$.*

We will use a single proof applicable to relevant results after Proposition 13. Note that the order of preference can be either fixed and identical or random.

Conjecture 1 *Suppose all customers are fare-concerned and the customers have fixed and identical orders of preference. If $d < n \times c$, then there exists K, such that $\eta_i^k = c$ for all $i \in [1, n]$ and $k \geq K$.*

We use Figure 4.1 to highlight the difficulties of the above conjecture. Suppose that we have 2 substitutable flights, each with $c = 10$ seats, and we have 19 customers and $\alpha = 0.4$. For initial booking limits of 10 and 0, Figure 4.1 shows the booking limits in the first 12 instances, we see even the total booking limit increases at time period 4, it decreases again at time period 5. Long run simulation suggests that the booking limits of both flights converge to c (not shown in the figure).

Furthermore, one natural and consequential question we may ask is what if the customers have random orders of preference and the total demand is less than the total capacity. Unfortunately, it is not clear how booking limits behave in the long run for such a setting. However, we can show that booking limits may converge to 0 rather than c. Suppose first that there are 3 substitutable flights, each with $c = 10$ seats, and there are 29 customers and $\alpha = 0.3$. Figure 4.2a shows the paths of the three booking limits: after 5 time periods, all booking limits remain 0. Figure 4.2b shows the observed high-fare requests in 10 time periods and illustrates the reason; that is, in each time period, one of the observed high-fare requests is 9 (while others

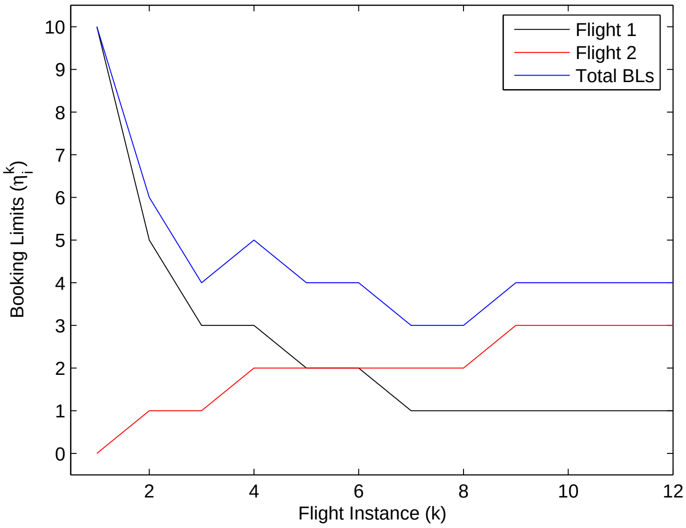
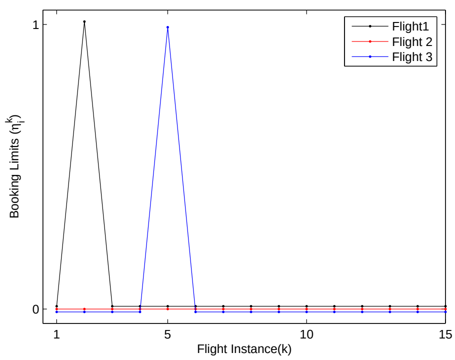


Figure 4.1: Dynamics of Booking Limits

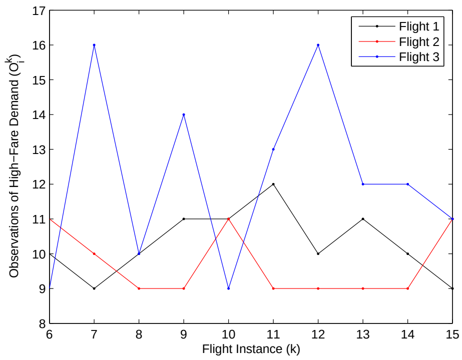
are 10 or more), but since α is relatively low, occasional observed high-fare requests of 9 cannot make the chosen booking limit increase. However, if $\alpha = 0.9$, Figure 4.3a shows all booking limits converge to c. In the end, suppose that $d = 20$ (much less than the total capacity of 30), and we also assume the fare ratio is $\alpha = 0.01$ (extremely small). Figure 4.3b shows that the booking limits converge to 0. In summary, if the demand is less than the capacity, then in all experiments we observe a convergence. Furthermore, in most cases, the booking limits converge to c; however, if the ratio of fares is low, then this convergence to c is not guaranteed, and our simulation suggests that in some cases booking limits converge to 0.

4.2.2 Flight-Concerned Customers

In this setting, we assume all customers are flight-concerned, namely each customer first picks a flight and checks for the low-fare tickets, and if the tickets are not available, s/he checks for the high-fare tickets of the same flight, and the customer checks

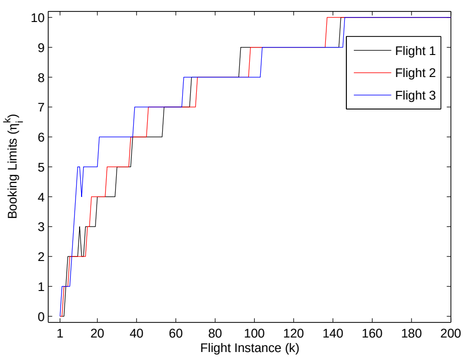


(a) Dynamics of Booking Limits with Random Orders

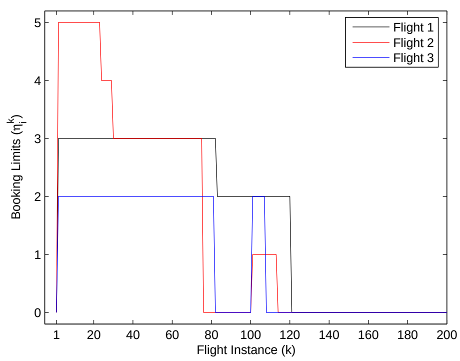


(b) Observation of High Fare Requests

Figure 4.2: Dynamics of Booking Limits with Random Orders



(a) Dynamics of Booking Limits with Random Orders



(b) Dynamics of Booking Limits with Random Orders

Figure 4.3: Dynamics of Booking Limits with Random Orders

flights according to his/her order of preference. For instance, also assume there are 3 flights. If a customer has an order of preference 3-1-2, then his/her checks for fare products in the order of *LF3-HF3-LF1-HF1-LF2-HF2*. That is, s/he first checks for the low-fare tickets of flight 3, if the tickets are not available, then s/he will check for high-fare tickets of flight 3, and so on. This process continues until the customer either purchases a ticket or find all products (tickets) unavailable.

We begin our study by assuming each customer has a fixed and identical order of preference in each period k. In this setting, we will show for any flight i and time periods j and k, that $s_i^j + o_i^j = s_i^k + o_i^k$. That is, the sum of the sales of low-fare tickets and requests for high-fare tickets remain same for each flight in any time period. Since all customers are flight-concerned, if a customer checks a flight i at time j, then s/he is counted either in s_i^j or o_i^j. This is because s/he buys a ticket (either low or high) when there is a ticket available in flight i, and even if s/he departs without purchase, s/he is counted in o_i^j. The customer switches to other flights if and only if all c tickets for this certain flight i have been booked, no matter what the booking limit is. Since each customer has the fixed and identical order of preference at any time period, we claim if a customer buys a ticket from flight j at time 1, then s/he also buys a ticket from the same flight j in any time period. We use induction to prove this. The first customer always gets a ticket from the first flight of his/her order of preference. Assume the j-th customer ($j < k$) always gets a ticket from the same flight. Since each customer coming before the k-th customer gets a ticket from the same flight as s/he got before; if a flight was full before, then it is still full now; if a flight had available seats before, then it still has available seats now. The k-th customer faces exactly same situation as before, namely if s/he got a ticket from the l-th flight of his/her order of preference before, then s/he can still get a ticket from the l-th flight of the order of preference, but not from the m-th flight for any $m < l$. Therefore, the observed quantity (the sum of the sales of low-fare tickets and requests for high-fare tickets) for any flight i just depends on the customers' preference and is independent on anything else, especially on the chosen booking limit. We have $s_i^j + o_i^j = s_i^k + o_i^k$ for all i, j and k.

Proposition 11 *Suppose all customers are flight-concerned and each customer has a fixed and identical order of preference at each period. If $d < n \times c$, then there exists K, such that $\eta_i^K = c$ for all $i \in [1, n]$ and $k \geq K$.*

Proof. We first make a stronger statement: *Let $s_i^1 + o_i^1$ be the sum of the sales of low-fare tickets and requests for high-fare tickets, at time 1. If $s_i^1 + o_i^1 > c$, then $\eta_i^k \to 0$; if $s_i^1 + o_i^1 < c$, then $\eta_i^k \to c$; and if $s_i^1 + o_i^1 = c$, then η_i^k remains η_i^0.*

Since $d < n \times c$, there is at least one i such that $s_i^1 + o_i^1 < c$; otherwise, $s_i^1 + o_i^1 \geq c$, i.e., all flights are full, which is a contradiction. Since $s_i^j + o_i^j = s_i^k + o_i^k$, it is the case identical to that in CHK's paper. The remainder of the proof follows proposition 1 and 2 in Cooper et al. (2006).

□

We are also curious about the case in which customers have random orders of preference, but analogous to the fare-concerned case, this question needs more exploration.

Finally, we consider the case that the demand exceeds the total capacity, and the customer's order of preference can be either fixed and identical or random.

Proposition 12 *Suppose all customers are flight-concerned. If $d > n \times c$, then there exists K, such that $\eta_i^k = 0$ for all $i \in [1, n]$ and $k \geq K$.*

We will use a single proof applicable to relevant results after Proposition 13.

4.2.3 Diverse Customers

In this section, we consider more complicated customer behavior. We assume customers can randomly select orders of preference. For instance, if there are $n = 2$ flights, then a customer can have 24 (= 4!) different orders to check for fare products. If a customer begins with checking for low-fare tickets of flight 1, then s/he may have 6 different orders: *LF1-HF1-LF2-HF2*, *LF1-HF1-HF2-LF2*, *LF1-LF2-HF1-HF2*, *LF1-LF2-HF2-HF1*, *LF1-HF2-HF1-LF2* and *LF1-HF2-LF2-HF1*.

Proposition 13 *Suppose all customers are diverse customers. If $d > n \times c$, then there exists K, such that $\eta_i^k = 0$ for all $i \in [1, n]$ and $k \geq K$.*

Proof. [Applicable to Propositions 10, 12, and 13.]

We first show $\eta_i^{k-1} + o_i^k \geq c$ for all positive integers k and i. Since $d > n \times c$, the last customer must depart without purchase. S/he checks the availability of high-fare tickets from all flights but gets nothing, so s/he must be counted in each o_i^k. Since all flights are full, we have $\eta_i^{k-1} + o_i^k \geq c$ for all positive integers k and i. Note that $\eta_i^{k-1} + o_i^k = c$ occurs only when $\eta_i^{k-1} = 0$ and $o_i^k = c$, since from the definition if the high-fare requests exceed the capacity c, then we set $o_i^k = c$.

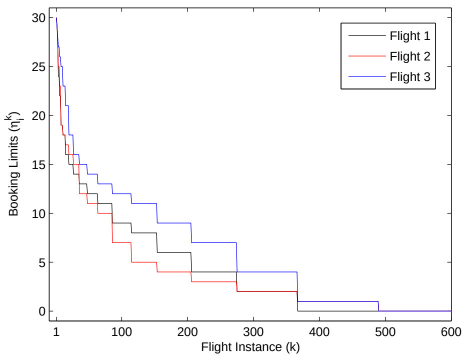
Next we will show that $\eta_i^k \leq \eta_i^{k-1}$ and $p_i^k \geq p_i^{k-1}$ for each k. Since $\eta_i^{k-1} + o_i^k \geq c$ and $\eta_i^{k-1} + p_i^{k-1} = c$, we have $o_i^k \geq p_i^{k-1}$. Furthermore, for any $x < o_i^k$, we have $\hat{H}_i^k(x) \leq \hat{H}_i^{k-1}(x)$, thus $\hat{H}_i^k(p_i^{k-1} - 1) \leq \hat{H}_i^{k-1}(p_i^{k-1} - 1) < 1 - \alpha$. From (4.3), we have $p_i^k \geq p_i^{k-1}$, and thus $\eta_i^k \leq \eta_i^{k-1}$.

η_i^k are non-increasing and bounded bellow, and p_i^k are non-decreasing and bounded above for all i, thus η_i^k and p_i^k must converge to, say η_i^* and p_i^*. Thus, there exists K, such that $\eta_i^k = \eta_i^*$ and $p_i^k = p_i^*$ for all i and $k > K$. If $\eta_i^* \neq 0$, then $\eta_i^* + o_i^k = \eta_i^k + o_i^k > c$ for all i and $k > K$. We also know the limits must satisfy $\eta_i^* = c - p_i^*$. Thus, $o_i^k > p_i^* = p_i^k$ for all i and $k > K$, which contradicts (4.3). Therefore, there must be

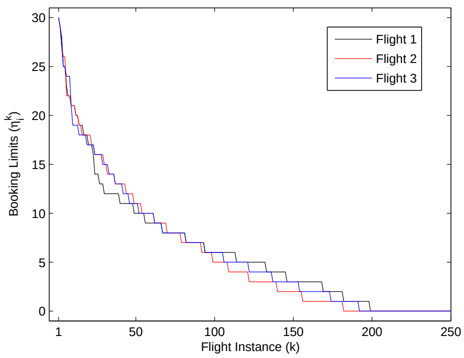
$\eta_i^* = 0$ and that is $\eta_i^k = 0$ for all i and $k > K$.

$\square$

We use some examples to illustrate the above results. In the first two examples we assume that there are $n = 3$ substitutable flights, each with $c = 30$ seats, and the total demand is just one more than the total capacity, namely $d = 91$. If all customers are fare-concerned, then Figure 4.4 shows all booking limits converge to 0. Figure 4.4a shows the limiting behavior with fixed and identical order of preference and $\alpha = 0.25$, and Figure 4.4b shows the limiting behavior with random order of preference and $\alpha = 0.3$, we observe that the latter ones converge fast. If all customers are flight-concerned, Figure 4.5 shows all booking limits converge to 0. Figure 4.5a shows the limiting behavior with fixed and identical order of preference and $\alpha = 0.2$, Figure 4.5b shows the limiting behavior with random order of preference and $\alpha = 0.3$, and we also observe that the latter ones converge fast. Some intuitive explanation is that the random order of preference makes each flight have generally comparable amount of visits (in terms of high-fare requests), while the fixed and identical order of preference makes the less favorite flight have few visits at the beginning. For instance, the booking limit of flight 3 in the blue path converges more slowly than the booking limits of the other flights, since at the beginning its observed quantities of high-fare requests are much less than those of other flights, and thus the booking limits increase slowly. Finally, we consider a case in which there are $n = 2$ substitutable flights, each with $c = 30$ seats; and there are $d = 61$ customers, 70% of them are fare-concerned (respectively, 30% flight-concerned), 50% of them select flight 1 as their first flight to check (respectively, 50% select flight 2), and $\alpha = 0.5$. Figure 4.5 shows both booking limits converge to c.

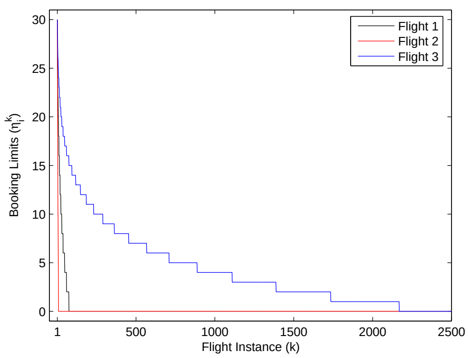


(a) Dynamics of Booking Limits with Fixed and Identical Orders

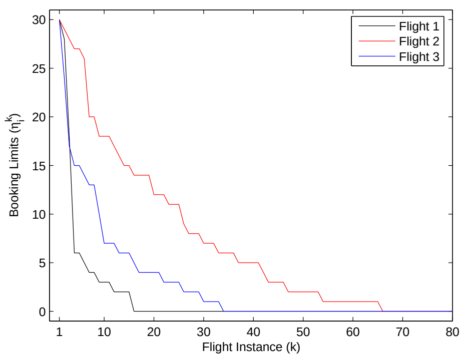


(b) Dynamics of Booking Limits with Random Orders

Figure 4.4: Dynamics of Booking Limits with Different Orders



(a) Dynamics of Booking Limits with Fixed and Identical Orders



(b) Dynamics of Booking Limits with Random Orders

Figure 4.5: Dynamics of Booking Limits with Different Orders

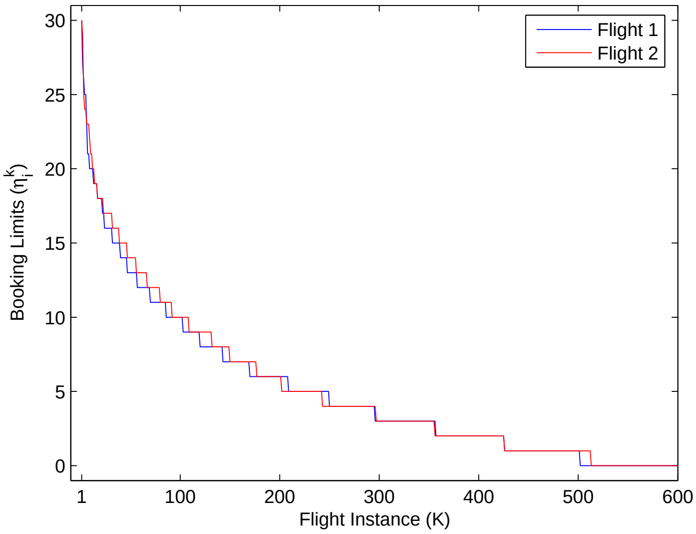


Figure 4.6: Dynamics of Booking Limits with Diverse Customers

4.3 Research Directions

As pointed out above, if the total number of customers is less than the total capacity, the limiting behavior of booking limits with fare-concerned, flight-concerned, or diverse customers is unknown and needs to be carefully studied. Moreover, we mainly considered the customers who always want to make a purchase. Thus more complicated customer behavior, such as when customers only seek certain fare classes, can be introduced and studied. We may also consider random numbers of customers.

Here, we provided some other topics not covered above. One direction is to combine the analysis of Chapters 2, 3, and Chapter 4 together; we may consider revised models of customer choice behavior (such as the buy-up model) in substitutable flights. Moreover, we may consider the case in which revenue managers are aware of substitutable flights and try to make better decisions. Revenue managers may share information across flights and may use pooling to make global decisions. Furthermore, other misspecified models can be studied in the framework of our analysis.

Chapter 5

Summary and Future Directions

5.1 Summary of Main Contributions

Revenue management has been widely and effectively used to improve revenue of services. A variety of models of customer choice behavior have been explored and studied recently, and many of them have been shown useful and helpful by revenue management practitioners. However, models have assumptions that may be violated in a specific environment. Indeed, it may be too much to ask whether a model is correct, hence, it is important to understand the performance of misspecified models.

In chapter 2, we studied the performance of a buy-up model, a variation of the well-known Littlewood rule, that appears to describe customers' behavior. We showed the convergence of booking limits for particular patterns of customer arrivals, and did some sensitivity analysis. In the numerical part, we also provided the revenue associated with the limiting booking limit obtained when using the Littlewood rule.

In chapter 3, we explored more complicated actual customers' behavior in which the numbers of different types of customers arrive to the system are Poisson distributed. We provided a general necessary condition for the convergence of booking limits and buy-up rate estimates. We also presented some sufficient conditions for convergence using two different approaches. We examined the revenue performance of using this buy-up model and compared with that of the Littlewood rule.

In chapter 4, we studied the performance of the Littlewood rule for substitutable flights. We analyzed the dynamic of use of the Littlewood rule by a revenue manager, and showed the convergence of booking limits under some assumptions and with certain customers' behavior. We made some conjectures for the limiting behavior of the booking limit under other assumptions and customer behavior.

5.2 Future Directions

One future direction is to combine our two main topics together. The performance of some other models that incorporate customer choice (such as "buy up" model) can be studied with substitutable-flight setting. In the substitutable-flight setting, the revenue manager uses the Littlewood rule independently for each flight, thus a model that considers dependency in substitutable flights can also be studied under our framework. Moreover, substitutable-flight is only one topic in network revenue management problems. Thus the usage of customer choice model in traditional multi-leg problem are quite abroad and can be studied carefully.

On a high level, in this dissertation we tested the performance of some misspecified models. Some more systematic study can be made under our framework. The core idea is that if a misspecified model (no parameter setting in a family of parametric models that makes a model precisely describe the reality) is used in practice, then the performance of this model is unclear and thus needs to be studied. Different backgrounds (single-leg and substitutable-flight in our work) and settings (actual customer behavior described in our work) will lead to the different results (convergence of booking limits and revenue difference).

Furthermore, in some models (in particular, some complicated models with multiple parameters), there could be a specific parameter setting that makes the model correct, but how to estimate these parameters correctly is another kind of open questions. In particular, in our setting, the decision made by the decision makers will affect the observation of following data. Whether the dynamics provided in our framework leads to a convergence of the parameter estimates is unknown in more complicated settings. Moreover, if there is a convergence, whether this convergence is the "correct" one is also unclear. Furthermore, if not, the performance of the model with the (incorrect) converged parameters will need some more studies.

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