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DIRECT AND INDIRECT ESTIMATION OF HEIGHT DISTRIBUTIONS IN EVEN-AGED STANDS*

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ABSTRACT

Height distribution of a 32 year old red pine stand was described using two mathematical models. First, a Weibull distribution was fitted directly to tree height observations. Second, a Weibull distribution was derived indirectly from a Weibull distribution for diameter and the relationship between tree height and diameter. Both height distributions agreed with the actual distribution at a 95% significance level using Chi-square tests.

INTRODUCTION

For an even-aged stand of a single species, stand structure is usually described by the distribution of tree size. Diameter and height distributions are two of the important distributions used for describing this structure. Discussion of diameter distributions has been more extensive in the forestry literature because diameters are more easily measured. It is possible, however, to develop distributions for other stand characteristics such as height that are closely related to diameter in an even-aged stand (Chen, 1976).

This paper describes research aimed at deriving tree height distributions by (1) directly fitting a distribution model to height observations and (2) indirectly deriving such a distribution from a diameter distribution based on the relationship between tree height and diameter. Data from an unthinned red pine plantation near Star Lake, Wisconsin, was used (site index 65, age 32, initial spacing 6' x 6') (see Wilson, 1963). The data was provided by the University of Wisconsin Department of Forestry and the Wisconsin Department of Natural Resources.

TWO METHODS FOR FITTING TREE HEIGHT DISTRIBUTIONS

If a relationship exists between two variables and a distribution is known for one of the variables, then a distribution can be derived for the other variable provided that the function describing the relationship can be inverted. Tree height and diameter are correlated in immature dense even-aged stands of single species. For a given site and stand age, an indirect height distribution can be generated from a diameter distribution based on the relationship between tree height and diameter. One useful model describing this relationship is:

$$H = aD^b \tag{1}$$

where H = mean or expected total tree height in feet for given D;
D = tree dbh in inches;
a and b are positive regression coefficients related to species, age, site and density.

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Suppose the random variable D follows a Weibull distribution (Johnson and Kotz, 1970) with scale parameter and shape parameter:

$$F_D(D) = 1 - \exp\{-D/\alpha)^{\beta}\} \tag{2}$$

Because of its capability to describe a wide range of distributional forms, this distribution is extremely well suited to fit diameter distributions of both uneven-aged, as well as even-aged stands (Dailey and Bell, 1973).

The height distribution can be estimated in the following way:

$$\begin{aligned} F_H(h) &= P(H \leq h) \\ &= P(aD^b \leq h) \text{ by equation (1)} \\ &= P(D \leq w) \text{ for } W = (h/a)^{1/b} \\ &= F_D(w) \\ &= 1 - \exp\{-w/\alpha)^{\beta}\} \end{aligned} \tag{3}$$

By the chain rule for differentiation of a function which itself is a function of another variable, the probability density function of tree height associated with diameter is:

$$\begin{aligned} f_H(h) &= \frac{d}{dh} F_H(h) = \frac{d}{dh} F_D(w) \\ &= \frac{\partial}{\partial w} F_D(w) \frac{\partial w}{\partial h} \\ &= (1/a)^{(1/b)} (1/b)\ h^{(1/b-1)} f_D(w) \end{aligned} \tag{4}$$

If observations on tree heights are available, however, it is possible to estimate a tree height distribution directly from the two parameter Weibull distribution after substituting height for diameter in equation (2).

RESULTS AND DISCUSSION

The correlation coefficient between tree height and diameter (equation 1) is strong ($r = 0.88$, $SE = 0.07$). This supports a finding by Curtis (1967) that this equation is well suited for young stands.

For this case study, a Newton-Raphson iteration (an efficient procedure to find approximate solutions for nonlinear equations) was used to solve the normal equations resulting from the maximum likelihood estimate of the Weibull parameters (see Thoman, Bain, and Antle, 1969). The initial value for the iteration was based on Menon's (1963) estimator. The convergence was fast (2 steps for the direct estimation and 4 steps for the indirect estimation). The rule used for stopping the iteration was:

$$|\hat{\beta}_k - \hat{\beta}_{k+1}| \leq 0.002$$

Table 1 compares the results of fitting height directly with a Weibull distribution and of deriving a height distribution indirectly from a Weibull function fitted to a diameter distribution. Both approaches give height distributions which are not significantly different from that observed according to Chi-square tests. The null hypotheses were accepted at the 95% significance level. Reduction of the computed value of Chi-square may be possible if a location parameter (three parameter Weibull) or a truncated value is used, or if an alternative height diameter equation were employed. If a low thinning is used or the stand is old, inclusion of the location parameter would be advisable.

Once the distributions of both height and diameter have been established it is possible to study the distributional relationship of the two variables simultaneously. Further study might be directed towards examination of the relationship between the height distributional parameters and species, age, site, and stand density.

LITERATURE CITED

Bailey, R. L., and T. R. Dell. 1973. Quantifying diameter distributions with the Weibull function. For. Sci. 19(2):97-104.

Chen, C. M. 1976. Dynamics of an even-aged stand-structure, mortality, competition and growth. Ph.D. Thesis. Univ. of Minnesota, College of Forestry, St. Paul. Univ. Microfilm No. 77-12, 885. Ann Arbor, Michigan.

Curtis, R. O. 1967. Height-diameter and height-diameter-age equations for second growth Douglas fir. For. Sci. 13:365-375.

Johnson, N.I., and S. Kotz. 1970. Continuous Univariate Distributions (Vol. I). Houghton Mifflin Comp., Boston. p. 300-310.

Menon, M. V. 1963. Estimates of the shape and scale parameters of the Weibull distribution. Technometrics 5:175-182.

Thomas, D. R., L. J. Gain and C. E. Antle. 1969. Inferences on the parameters of the Weibull distribution. Technometrics 11:445-460.

Wilson, F. G. 1963. Fifty years from seed--The Star Lake plantation. Wisconsin Conserv. Dept. Tech. Bull. 27. 24 p.

Table 1. Comparison between observed and expected frequencies of tree height of a red pine plantation (site index 65, age 32).

Height Class (m)	Height Class (feet)	Observed Frequency	Expected Frequencies: Direct Estimation#	Expected Frequencies: Indirect Estimation##
9.1	30	14	13	12
10.7	35	19	22	24
12.2	40	33	27	29
13.7	45	8	11	9
Total		74	73	74
Computed Chi-Square Values:			2.638	2.037

Note: The null hypotheses of no difference between observed and predicted height distributions were accepted at a 95% confidence level using a Chi-square test ($\chi^2_{2;0.95} = 5.99$).

#Direct fit of Weibull distribution model to height observations.

##Indirectly derived height distribution obtained from a diameter distribution and the height-diameter relationship: $H = aD^b$.

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